

A few facts on integrality

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The purpose of this note is to collect some theorems and proofs related to integrality in commutative algebra. The note is subdivided into three parts.

Part 1 (Integrality over rings) consists of known facts (Theorems 1, 4, 5) and a generalized exercise from [1] (Corollary 3) with a few minor variations (Theorem 2 and Corollary 6).

Part 2 (Integrality over ideal semifiltrations) merges integrality over rings (as considered in Part 1) and integrality over ideals (a less-known but still very useful notion; the book [2] is devoted to it) into one general notion - that of integrality over ideal semifiltrations (Definition 9). This notion is very general, yet it can be reduced to the basic notion of integrality over rings by a suitable change of base ring (Theorem 7). This reduction allows to extend some standard properties of integrality over rings to the general case (Theorems 8 and 9).

Part 3 (Generalizing to two ideal semifiltrations) continues Part 2, adding one more layer of generality. Its main result is a “relative” version of Theorem 7 (Theorem 11) and a known fact generalized one more time (Theorem 13).

This note is supposed to be self-contained (only linear algebra and basic knowledge about rings, ideals and polynomials is assumed). The proofs are constructive. However, when writing down the proofs I focussed on maximal detail (to ensure correctness) rather than on clarity, so the proofs are probably a pain to read. I think of making a short version of this note with the obvious parts of proofs left out.

Preludium

Definitions and notations:

Definition 1. In the following, “ring” will always mean “commutative ring with unity”. We denote the set $\{0, 1, 2, \dots\}$ by \mathbb{N} , and the set $\{1, 2, 3, \dots\}$ by \mathbb{N}^+ .

Definition 2. Let A be a ring, and let $n \in \mathbb{N}$. Let M be an A -module. If m_1, m_2, \dots, m_n are n elements of M , then we define an A -submodule $\langle m_1, m_2, \dots, m_n \rangle_A$ of M by

$$\langle m_1, m_2, \dots, m_n \rangle_A = \left\{ \sum_{i=1}^n a_i m_i \mid (a_1, a_2, \dots, a_n) \in A^n \right\}.$$

Also, if S is a finite set, and m_s is an element of M for every $s \in S$, then we define an A -submodule $\langle m_s \mid s \in S \rangle_A$ of M by

$$\langle m_s \mid s \in S \rangle_A = \left\{ \sum_{s \in S} a_s m_s \mid (a_s)_{s \in S} \in A^S \right\}.$$

Of course, if m_1, m_2, \dots, m_n are n elements of M , then

$$\langle m_1, m_2, \dots, m_n \rangle_A = \langle m_s \mid s \in \{1, 2, \dots, n\} \rangle_A.$$

¹**Warning:** This is a (very) old version of my note “A few facts on integrality”. Some minor mistakes have been left uncorrected here!

Definition 3. Let A be a ring, and let $n \in \mathbb{N}$. Let M be an A -module. We say that the A -module M is n -generated if there exist n elements m_1, m_2, \dots, m_n of M such that $M = \langle m_1, m_2, \dots, m_n \rangle_A$. In other words, the A -module M is n -generated if and only if there exists a set S and an element m_s of M for every $s \in S$ such that $|S| = n$ and $M = \langle m_s \mid s \in S \rangle_A$.

Definition 4. Let A and B be two rings. We say that $A \subseteq B$ if and only if

(the set A is a subset of the set B)

and (the inclusion map $A \rightarrow B$ is a ring homomorphism).

Now assume that $A \subseteq B$. Then, obviously, B is canonically an A -algebra (since $A \subseteq B$). If u_1, u_2, \dots, u_n are n elements of B , then we define an A -subalgebra $A[u_1, u_2, \dots, u_n]$ of B by

$$A[u_1, u_2, \dots, u_n] = \{P(u_1, u_2, \dots, u_n) \mid P \in A[X_1, X_2, \dots, X_n]\}.$$

In particular, if u is an element of B , then the A -subalgebra $A[u]$ of B is defined by

$$A[u] = \{P(u) \mid P \in A[X]\}.$$

Since $A[X] = \left\{ \sum_{i=0}^m a_i X^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, \dots, a_m) \in A^{m+1} \right\}$, this becomes

$$\begin{aligned} A[u] &= \left\{ \left(\sum_{i=0}^m a_i X^i \right) (u) \mid m \in \mathbb{N} \text{ and } (a_0, a_1, \dots, a_m) \in A^{m+1} \right\} \\ &\quad \left(\text{where } \left(\sum_{i=0}^m a_i X^i \right) (u) \text{ means the polynomial } \sum_{i=0}^m a_i X^i \text{ evaluated at } X = u \right) \\ &= \left\{ \sum_{i=0}^m a_i u^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, \dots, a_m) \in A^{m+1} \right\} \\ &\quad \left(\text{because } \left(\sum_{i=0}^m a_i X^i \right) (u) = \sum_{i=0}^m a_i u^i \right). \end{aligned}$$

Obviously, $uA[u] \subseteq A[u]$ (since $A[u]$ is an A -algebra and $u \in A[u]$).

1. Integrality over rings

Theorem 1. Let A and B be two rings such that $A \subseteq B$. Obviously, B is canonically an A -module (since $A \subseteq B$). Let $n \in \mathbb{N}$. Let $u \in B$. Then, the following four assertions \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are pairwise equivalent:

Assertion \mathcal{A} : There exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$.

Assertion \mathcal{B} : There exists an n -generated A -submodule U of B such that $uU \subseteq U$ and such that $v = 0$ for every $v \in B$ satisfying $vU = 0$.

Assertion \mathcal{C} : There exists an n -generated A -submodule U of B such that $1 \in U$ and $uU \subseteq U$.

Assertion \mathcal{D} : We have $A[u] = \langle u^0, u^1, \dots, u^{n-1} \rangle_A$.

Definition 5. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}$. Let $u \in B$. We say that the element u of B is n -integral over A if it satisfies the four equivalent assertions \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} of Theorem 1.

Hence, u is n -integral over A if and only if there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$.

Proof of Theorem 1. We will prove the implications $\mathcal{A} \implies \mathcal{C}$, $\mathcal{C} \implies \mathcal{B}$, $\mathcal{B} \implies \mathcal{A}$, $\mathcal{A} \implies \mathcal{D}$ and $\mathcal{D} \implies \mathcal{C}$.

Proof of the implication $\mathcal{A} \implies \mathcal{C}$. Assume that Assertion \mathcal{A} holds. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements a_0, a_1, \dots, a_{n-1} of A such that $P(X) = X^n + \sum_{k=0}^{n-1} a_k X^k$. Thus, $P(u) = u^n + \sum_{k=0}^{n-1} a_k u^k$, so that $P(u) = 0$ becomes

$$u^n + \sum_{k=0}^{n-1} a_k u^k = 0. \text{ Hence, } u^n = - \sum_{k=0}^{n-1} a_k u^k.$$

Let U be the A -submodule $\langle u^0, u^1, \dots, u^{n-1} \rangle_A$ of B . Then, U is an n -generated A -module (since u^0, u^1, \dots, u^{n-1} are n elements of U). Besides, $1 = u^0 \in U$.

Now, $u \cdot u^k \in U$ for any $k \in \{0, 1, \dots, n-1\}$ (since $k \in \{0, 1, \dots, n-1\}$ yields either $0 \leq k < n-1$ or $k = n-1$, but $u \cdot u^k = u^{k+1} \in \langle u^0, u^1, \dots, u^{n-1} \rangle_A = U$ if $0 \leq k < n-1$, and $u \cdot u^k = u \cdot u^{n-1} = u^n = - \sum_{k=0}^{n-1} a_k u^k \in \langle u^0, u^1, \dots, u^{n-1} \rangle_A = U$ if $k = n-1$, so that $u \cdot u^k \in U$ in both cases). Hence,

$$uU = u \langle u^0, u^1, \dots, u^{n-1} \rangle_A = \langle u \cdot u^0, u \cdot u^1, \dots, u \cdot u^{n-1} \rangle_A \subseteq U$$

(since $u \cdot u^k \in U$ for any $k \in \{0, 1, \dots, n-1\}$).

Thus, Assertion \mathcal{C} holds. Hence, we have proved that $\mathcal{A} \implies \mathcal{C}$.

Proof of the implication $\mathcal{C} \implies \mathcal{B}$. Assume that Assertion \mathcal{C} holds. Then, there exists an n -generated A -submodule U of B such that $1 \in U$ and $uU \subseteq U$. We have $v = 0$ for every $v \in B$ satisfying $vU = 0$ (since $1 \in U$ and $vU = 0$ yield $v \cdot \underbrace{1}_{\in U} \in vU = 0$

and thus $v \cdot 1 = 0$, so that $v = 0$). Thus, Assertion \mathcal{B} holds. Hence, we have proved that $\mathcal{C} \implies \mathcal{B}$.

Proof of the implication $\mathcal{B} \implies \mathcal{A}$. Assume that Assertion \mathcal{B} holds. Then, there exists an n -generated A -submodule U of B such that $uU \subseteq U$ and such that $v = 0$ for every $v \in B$ satisfying $vU = 0$. Since the A -module U is n -generated, there exist n elements m_1, m_2, \dots, m_n of U such that $U = \langle m_1, m_2, \dots, m_n \rangle_A$. For any $k \in \{1, 2, \dots, n\}$, we have

$$\begin{aligned} um_k &\in uU && (\text{since } m_k \in U) \\ &\subseteq U = \langle m_1, m_2, \dots, m_n \rangle_A, \end{aligned}$$

so that there exist n elements $a_{k,1}, a_{k,2}, \dots, a_{k,n}$ of A such that $um_k = \sum_{i=1}^n a_{k,i} m_i$.

Define a vector $v \in B^n$ by $v_i = m_i$ for all $i \in \{1, 2, \dots, n\}$. (Here, for any vector w and any integer x , we denote by w_x the entry of the vector w in the x -th row.)

Define a matrix $S \in A^{n \times n}$ by $S_{k,i} = a_{k,i}$ for all $k \in \{1, 2, \dots, n\}$ and $i \in \{1, 2, \dots, n\}$. (Here, for any matrix T and any integers x and y , we denote by $T_{x,y}$ the entry of the matrix T in the x -th row and the y -th column.) Then, for any $k \in \{1, 2, \dots, n\}$, we have

$u \underbrace{m_k}_{=v_k} = uv_k = (uv)_k$ and $\sum_{i=1}^n \underbrace{a_{k,i}}_{=S_{k,i}} \underbrace{m_i}_{=v_i} = \sum_{i=1}^n S_{k,i}v_i = (Sv)_k$, so that $um_k = \sum_{i=1}^n a_{k,i}m_i$

becomes $(uv)_k = (Sv)_k$. Since this holds for every $k \in \{1, 2, \dots, n\}$, we conclude that $uv = Sv$. Thus,

$$0 = uv - Sv = uI_nv - Sv = (uI_n - S)v.$$

Now, let $P \in A[X]$ be the characteristic polynomial of the matrix $S \in A^{n \times n}$. Then, P is monic, and $\deg P = n$. Besides, $P(X) = \det(XI_n - S)$, so that $P(u) = \det(uI_n - S)$. Thus,

$$\begin{aligned} P(u) \cdot v &= \det(uI_n - S) \cdot v = \underbrace{\det(uI_n - S) I_n}_{=\text{adj}(uI_n - S) \cdot (uI_n - S)} \cdot v \\ &= \text{adj}(uI_n - S) \cdot \underbrace{(uI_n - S)v}_{=0} = 0. \end{aligned}$$

Hence, for any $k \in \{1, 2, \dots, n\}$, we have

$$P(u) \cdot \underbrace{m_k}_{=v_k} = P(u) \cdot v_k = \left(\underbrace{P(u) \cdot v}_{=0} \right)_k = 0,$$

so that

$$\begin{aligned} P(u) \cdot U &= P(u) \cdot \langle m_1, m_2, \dots, m_n \rangle_A = \langle P(u) \cdot m_1, P(u) \cdot m_2, \dots, P(u) \cdot m_n \rangle_A \\ &= \langle 0, 0, \dots, 0 \rangle_A \quad (\text{since } P(u) \cdot m_k = 0 \text{ for any } k \in \{1, 2, \dots, n\}) \\ &= 0. \end{aligned}$$

This implies $P(u) = 0$ (since $v = 0$ for every $v \in B$ satisfying $vU = 0$). Thus, Assertion \mathcal{A} holds. Hence, we have proved that $\mathcal{B} \implies \mathcal{A}$.

Proof of the implication $\mathcal{A} \implies \mathcal{D}$. Assume that Assertion \mathcal{A} holds. Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements a_0, a_1, \dots, a_{n-1} of A such that $P(X) = X^n + \sum_{k=0}^{n-1} a_k X^k$. Thus, $P(u) = u^n + \sum_{k=0}^{n-1} a_k u^k$, so that $P(u) = 0$ becomes

$$u^n + \sum_{k=0}^{n-1} a_k u^k = 0. \text{ Hence, } u^n = - \sum_{k=0}^{n-1} a_k u^k.$$

Let U be the A -submodule $\langle u^0, u^1, \dots, u^{n-1} \rangle_A$ of B . As in the Proof of the implication $\mathcal{A} \implies \mathcal{C}$, we can show that U is an n -generated A -module, and that $1 \in U$ and $uU \subseteq U$.

Now, we are going to show that

$$u^i \in U \quad \text{for any } i \in \mathbb{N}. \quad (1)$$

Proof of (1). We will prove (1) by induction over i :

Induction base: The assertion (1) holds for $i = 0$ (since $u^0 \in U$). This completes the induction base.

Induction step: Let $\tau \in \mathbb{N}$. If the assertion (1) holds for $i = \tau$, then the assertion (1) holds for $i = \tau + 1$ (because if the assertion (1) holds for $i = \tau$, then $u^\tau \in U$, so

that $u^{\tau+1} = u \cdot \underbrace{u^\tau}_{\in U} \in uU \subseteq U$, so that $u^{\tau+1} \in U$, and thus the assertion (1) holds for $i = \tau + 1$). This completes the induction step.

Hence, the induction is complete, and (1) is proven.

Thus,

$$A[u] = \left\{ \sum_{i=0}^m a_i u^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, \dots, a_m) \in A^{m+1} \right\} \subseteq U$$

(since $\sum_{i=0}^m a_i u^i \in U$ for any $m \in \mathbb{N}$ and any $(a_0, a_1, \dots, a_m) \in A^{m+1}$, because $a_i \in A$ and $u^i \in U$ for any $i \in \{0, 1, \dots, m\}$ (by (1)) and U is an A -module). On the other hand, $U \subseteq A[u]$, since

$$\begin{aligned} U &= \langle u^0, u^1, \dots, u^{n-1} \rangle_A = \left\{ \sum_{i=0}^{n-1} a_i u^i \mid (a_0, a_1, \dots, a_{n-1}) \in A^n \right\} \\ &\subseteq \left\{ \sum_{i=0}^m a_i u^i \mid m \in \mathbb{N} \text{ and } (a_0, a_1, \dots, a_m) \in A^{m+1} \right\} = A[u]. \end{aligned}$$

Thus, $U = A[u]$. In other words, $\langle u^0, u^1, \dots, u^{n-1} \rangle_A = A[u]$. Thus, Assertion \mathcal{D} holds. Hence, we have proved that $\mathcal{A} \implies \mathcal{D}$.

Proof of the implication $\mathcal{D} \implies \mathcal{C}$. Assume that Assertion \mathcal{D} holds. Then, $A[u] = \langle u^0, u^1, \dots, u^{n-1} \rangle_A$.

Let U be the A -submodule $\langle u^0, u^1, \dots, u^{n-1} \rangle_A$ of B . Then, U is an n -generated A -module (since u^0, u^1, \dots, u^{n-1} are n elements of U). Besides, $1 = u^0 \in U$.

Also,

$$uU = u \cdot \langle u^0, u^1, \dots, u^{n-1} \rangle_A = u \cdot A[u] \subseteq A[u] = \langle u^0, u^1, \dots, u^{n-1} \rangle_A = U.$$

Thus, Assertion \mathcal{C} holds. Hence, we have proved that $\mathcal{D} \implies \mathcal{C}$.

Now, we have proved the implications $\mathcal{A} \implies \mathcal{D}$, $\mathcal{D} \implies \mathcal{C}$, $\mathcal{C} \implies \mathcal{B}$ and $\mathcal{B} \implies \mathcal{A}$ above. Thus, all four assertions \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} are pairwise equivalent, and Theorem 1 is proven.

Theorem 2. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}$. Let $v \in B$. Let a_0, a_1, \dots, a_n be $n + 1$ elements of A such that $\sum_{i=0}^n a_i v^i = 0$. Let

$k \in \{0, 1, \dots, n\}$. Then, $\sum_{i=0}^{n-k} a_{i+k} v^i$ is n -integral over A .

Proof of Theorem 2. Let U be the A -submodule $\langle v^0, v^1, \dots, v^{n-1} \rangle_A$ of B . Then, U is an n -generated A -module (since v^0, v^1, \dots, v^{n-1} are n elements of U). Besides, $1 = v^0 \in U$.

Let $u = \sum_{i=0}^{n-k} a_{i+k} v^i$. Then,

$$\begin{aligned}
0 &= \sum_{i=0}^n a_i v^i = \sum_{i=0}^{k-1} a_i v^i + \sum_{i=k}^n a_i v^i = \sum_{i=0}^{k-1} a_i v^i + \sum_{i=0}^{n-k} a_{i+k} \underbrace{v^{i+k}}_{=v^i v^k} \\
&\quad \text{(here, we substituted } i+k \text{ for } i \text{ in the second sum)} \\
&= \sum_{i=0}^{k-1} a_i v^i + v^k \underbrace{\sum_{i=0}^{n-k} a_{i+k} v^i}_{=u} = \sum_{i=0}^{k-1} a_i v^i + v^k u,
\end{aligned}$$

so that $v^k u = -\sum_{i=0}^{k-1} a_i v^i$.

Now, we are going to show that

$$uv^t \in U \quad \text{for any } t \in \{0, 1, \dots, n-1\}. \quad (2)$$

Proof of (2). Since $t \in \{0, 1, \dots, n-1\}$, one of the following two cases must hold:

Case 1: We have $t \in \{0, 1, \dots, k-1\}$.

Case 2: We have $t \in \{k, k+1, \dots, n-1\}$.

In Case 1, we have

$$\begin{aligned}
uv^t &= \sum_{i=0}^{n-k} a_{i+k} \underbrace{v^i \cdot v^t}_{=v^{i+t}} = \sum_{i=0}^{n-k} a_{i+k} v^{i+t} \in \langle v^0, v^1, \dots, v^{n-1} \rangle_A \\
&\quad \left(\begin{array}{l} \text{since } t \in \{0, 1, \dots, k-1\} \text{ yields } i+t \in \{0, 1, \dots, n-1\} \\ \text{and thus } v^{i+t} \in \{v^0, v^1, \dots, v^{n-1}\} \text{ for any } i \in \{0, 1, \dots, n-k\} \end{array} \right) \\
&= U.
\end{aligned}$$

In Case 2, we have $t \in \{k, k+1, \dots, n-1\}$, thus $t-k \in \{0, 1, \dots, n-k-1\}$ and hence

$$\begin{aligned}
uv^t &= u \underbrace{v^{k+(t-k)}}_{=v^k v^{t-k}} = v^k u \cdot v^{t-k} = -\sum_{i=0}^{k-1} a_i \underbrace{v^i \cdot v^{t-k}}_{=v^{i+(t-k)}} \quad \left(\text{since } v^k u = -\sum_{i=0}^{k-1} a_i v^i \right) \\
&= -\sum_{i=0}^{k-1} a_i v^{i+(t-k)} \in \langle v^0, v^1, \dots, v^{n-1} \rangle_A \\
&\quad \left(\begin{array}{l} \text{since } t-k \in \{0, 1, \dots, n-k-1\} \text{ yields } i+(t-k) \in \{0, 1, \dots, n-1\} \\ \text{and thus } v^{i+(t-k)} \in \{v^0, v^1, \dots, v^{n-1}\} \text{ for any } i \in \{0, 1, \dots, k-1\} \end{array} \right) \\
&= U.
\end{aligned}$$

Hence, in both cases, we have $uv^t \in U$. Thus, $uv^t \in U$ always holds, and (2) is proven.

Now,

$$uU = u \langle v^0, v^1, \dots, v^{n-1} \rangle_A = \langle uv^0, uv^1, \dots, uv^{n-1} \rangle_A \subseteq U \quad \text{(due to (2)).}$$

Altogether, U is an n -generated A -submodule of B such that $1 \in U$ and $uU \subseteq U$. Thus, $u \in B$ satisfies Assertion \mathcal{C} of Theorem 1. Hence, $u \in B$ satisfies the four equivalent assertions \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} of Theorem 1. Consequently, u is n -integral over A . Since $u = \sum_{i=0}^{n-k} a_{i+k}v^i$, this means that $\sum_{i=0}^{n-k} a_{i+k}v^i$ is n -integral over A . This proves Theorem 2.

Corollary 3. Let A and B be two rings such that $A \subseteq B$. Let $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$. Let $u \in B$ and $v \in B$. Let $s_0, s_1, \dots, s_\alpha$ be $\alpha + 1$ elements of A such that $\sum_{i=0}^{\alpha} s_i v^i = u$. Let t_0, t_1, \dots, t_β be $\beta + 1$ elements of A such that

$$\sum_{i=0}^{\beta} t_i v^{\beta-i} = uv^\beta. \text{ Then, } u \text{ is } (\alpha + \beta)\text{-integral over } A.$$

(This Corollary 3 generalizes Exercise 2-5 in [1].)

Proof of Corollary 3. Let $k = \beta$ and $n = \alpha + \beta$. Then, $k \in \{0, 1, \dots, n\}$. Define $n + 1$ elements a_0, a_1, \dots, a_n of A by

$$a_i = \begin{cases} t_{\beta-i}, & \text{if } i < \beta; \\ t_0 - s_0, & \text{if } i = \beta; \\ -s_{i-\beta}, & \text{if } i > \beta; \end{cases} \quad \text{for every } i \in \{0, 1, \dots, n\}.$$

Then,

$$\begin{aligned}
\sum_{i=0}^n a_i v^i &= \sum_{i=0}^{\alpha+\beta} a_i v^i = \sum_{i=0}^{\beta-1} \underbrace{a_i}_{=t_{\beta-i}, \text{ since } i < \beta} v^i + \sum_{i=\beta}^{\beta} \underbrace{a_i}_{=t_0-s_0, \text{ since } i=\beta} v^i + \sum_{i=\beta+1}^{\alpha+\beta} \underbrace{a_i}_{=-s_{i-\beta}, \text{ since } i > \beta} v^i \\
&= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + \underbrace{\sum_{i=\beta}^{\beta} (t_0 - s_0) v^i}_{=(t_0-s_0)v^\beta = t_0 v^\beta - s_0 v^\beta} + \underbrace{\sum_{i=\beta+1}^{\alpha+\beta} (-s_{i-\beta}) v^i}_{=-\sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i} \\
&= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - s_0 v^\beta - \sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i = \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - \left(s_0 v^\beta + \sum_{i=\beta+1}^{\alpha+\beta} s_{i-\beta} v^i \right) \\
&= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - \left(s_0 v^\beta + \sum_{i=1}^{\alpha} \underbrace{s_{(i+\beta)-\beta}}_{=s_i} \underbrace{v^{i+\beta}}_{=v^i v^\beta} \right) \\
&\quad \text{(here, we substituted } i + \beta \text{ for } i \text{ in the second sum)} \\
&= \sum_{i=0}^{\beta-1} t_{\beta-i} v^i + t_0 v^\beta - \left(s_0 v^\beta + \sum_{i=1}^{\alpha} s_i v^i v^\beta \right) \\
&= \sum_{i=1}^{\beta} \underbrace{t_{\beta-(\beta-i)}}_{=t_i} v^{\beta-i} + t_0 \underbrace{v^\beta}_{=v^{\beta-0}} - \left(s_0 \underbrace{v^\beta}_{=v^0 v^\beta} + \sum_{i=1}^{\alpha} s_i v^i v^\beta \right) \\
&\quad \text{(here, we substituted } \beta - i \text{ for } i \text{ in the first sum)} \\
&= \sum_{i=1}^{\beta} t_i v^{\beta-i} + t_0 v^{\beta-0} - \left(s_0 v^0 v^\beta + \sum_{i=1}^{\alpha} s_i v^i v^\beta \right) \\
&= \underbrace{\sum_{i=1}^{\beta} t_i v^{\beta-i} + t_0 v^{\beta-0}}_{=\sum_{i=0}^{\beta} t_i v^{\beta-i} = uv^\beta} - \left(\underbrace{s_0 v^0 + \sum_{i=1}^{\alpha} s_i v^i}_{=\sum_{i=0}^{\alpha} s_i v^i = u} \right) v^\beta = uv^\beta - uv^\beta = 0.
\end{aligned}$$

Thus, Theorem 2 yields that $\sum_{i=0}^{n-k} a_{i+k}v^i$ is n -integral over A . But

$$\begin{aligned}
\sum_{i=0}^{n-k} a_{i+k}v^i &= \sum_{i=0}^{n-\beta} a_{i+\beta}v^i = \sum_{i=0}^0 \underbrace{a_{i+\beta}}_{\substack{=t_0-s_0, \\ \text{since} \\ i=0 \text{ yields} \\ i+\beta=\beta}} v^i + \sum_{i=1}^{n-\beta} \underbrace{a_{i+\beta}}_{\substack{=-s_{(i+\beta)-\beta}, \\ \text{since} \\ i>0 \text{ yields} \\ i+\beta>\beta}} v^i \\
&= \underbrace{\sum_{i=0}^0 (t_0 - s_0) v^i}_{\substack{=(t_0-s_0)v^0 \\ =t_0v^0-s_0v^0 \\ =t_0-s_0v^0}} + \sum_{i=1}^{n-\beta} \left(\underbrace{-s_{(i+\beta)-\beta}}_{=s_i} \right) v^i \\
&= t_0 - s_0v^0 + \sum_{i=1}^{n-\beta} (-s_i) v^i = t_0 - s_0v^0 - \sum_{i=1}^{n-\beta} s_i v^i \\
&= t_0 - s_0v^0 - \sum_{i=1}^{\alpha} s_i v^i \quad (\text{since } n = \alpha + \beta \text{ yields } n - \beta = \alpha) \\
&= t_0 - \left(\underbrace{s_0v^0 + \sum_{i=1}^{\alpha} s_i v^i}_{= \sum_{i=0}^{\alpha} s_i v^i = u} \right) = t_0 - u.
\end{aligned}$$

Thus, $t_0 - u$ is n -integral over A . On the other hand, $-t_0$ is 1-integral over A (by Theorem 5 (a) below, applied to $a = -t_0$). Thus, $(-t_0) + (t_0 - u)$ is $n \cdot 1$ -integral over A (by Theorem 5 (b) below, applied to $x = -t_0$, $y = t_0 - u$ and $m = 1$). In other words, $-u$ is n -integral over A (since $(-t_0) + (t_0 - u) = -u$ and $n \cdot 1 = n$). On the other hand, -1 is 1-integral over A (by Theorem 5 (a) below, applied to $a = -1$). Thus, $(-1) \cdot (-u)$ is $n \cdot 1$ -integral over A (by Theorem 5 (c) below, applied to $x = -1$, $y = -u$ and $m = 1$). In other words, u is $(\alpha + \beta)$ -integral over A (since $(-1) \cdot (-u) = u$ and $n \cdot 1 = n = \alpha + \beta$). This proves Corollary 3.

Theorem 4. Let A and B be two rings such that $A \subseteq B$. Let $v \in B$ and $u \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that v is m -integral over A , and that u is n -integral over $A[v]$. Then, u is nm -integral over A .

Proof of Theorem 4. Since v is m -integral over A , we have $A[v] = \langle v^0, v^1, \dots, v^{m-1} \rangle_A$ (this is the Assertion \mathcal{D} of Theorem 1, stated for v and m in lieu of u and n).

Since u is n -integral over $A[v]$, we have $(A[v])[u] = \langle u^0, u^1, \dots, u^{n-1} \rangle_{A[v]}$ (this is the Assertion \mathcal{D} of Theorem 1, stated for $A[v]$ in lieu of A).

Let $S = \{0, 1, \dots, n-1\} \times \{0, 1, \dots, m-1\}$.

Let $x \in (A[v])[u]$. Then, there exist n elements b_0, b_1, \dots, b_{n-1} of $A[v]$ such that $x = \sum_{i=0}^{n-1} b_i u^i$ (since $x \in (A[v])[u] = \langle u^0, u^1, \dots, u^{n-1} \rangle_{A[v]}$). But for each $i \in \{0, 1, \dots, n-1\}$,

there exist m elements $a_{i,0}, a_{i,1}, \dots, a_{i,m-1}$ of A such that $b_i = \sum_{j=0}^{m-1} a_{i,j}v^j$ (because $b_i \in A[v] = \langle v^0, v^1, \dots, v^{m-1} \rangle_A$). Thus,

$$\begin{aligned} x &= \sum_{i=0}^{n-1} \underbrace{b_i}_{=\sum_{j=0}^{m-1} a_{i,j}v^j} u^i = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} a_{i,j}v^j u^i = \sum_{(i,j) \in \{0,1,\dots,n-1\} \times \{0,1,\dots,m-1\}} a_{i,j}v^j u^i = \sum_{(i,j) \in S} a_{i,j}v^j u^i \\ &\in \langle v^j u^i \mid (i,j) \in S \rangle_A \quad (\text{since } a_{i,j} \in A \text{ for every } (i,j) \in S) \end{aligned}$$

So we have proved that $x \in \langle v^j u^i \mid (i,j) \in S \rangle_A$ for every $x \in (A[v])[u]$. Thus, $(A[v])[u] \subseteq \langle v^j u^i \mid (i,j) \in S \rangle_A$. Conversely, $\langle v^j u^i \mid (i,j) \in S \rangle_A \subseteq (A[v])[u]$ (since $v^j \in A[v]$ for every $(i,j) \in S$, and thus $\underbrace{v^j}_{\in A[v]} u^i \in (A[v])[u]$ for every $(i,j) \in S$, and

therefore

$$\langle v^j u^i \mid (i,j) \in S \rangle_A = \left\{ \underbrace{\sum_{(i,j) \in S} a_{i,j}v^j u^i}_{\substack{\in (A[v])[u], \text{ since} \\ v^j u^i \in (A[v])[u] \text{ for all } (i,j) \in S \\ \text{and } (A[v])[u] \text{ is an } A\text{-module}}} \mid (a_{i,j})_{(i,j) \in S} \in A^S \right\} \subseteq (A[v])[u]$$

). Hence, $(A[v])[u] = \langle v^j u^i \mid (i,j) \in S \rangle_A$. Thus, the A -module $(A[v])[u]$ is nm -generated (since

$$|S| = |\{0, 1, \dots, n-1\} \times \{0, 1, \dots, m-1\}| = \underbrace{|\{0, 1, \dots, n-1\}|}_{=n} \cdot \underbrace{|\{0, 1, \dots, m-1\}|}_{=m} = nm$$

).

Let $U = (A[v])[u]$. Then, the A -module U is nm -generated. Besides, U is an A -submodule of B , and we have $1 = u^0 \in (A[v])[u] = U$ and

$$\begin{aligned} uU &= u(A[v])[u] \subseteq (A[v])[u] \quad (\text{since } (A[v])[u] \text{ is an } A[v]\text{-algebra and } u \in (A[v])[u]) \\ &= U. \end{aligned}$$

Altogether, we now know that the A -submodule U of B is nm -generated and satisfies $1 \in U$ and $uU \subseteq U$.

Thus, the element u of B satisfies the Assertion \mathcal{C} of Theorem 1 with n replaced by nm . Hence, $u \in B$ satisfies the four equivalent assertions \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{D} of Theorem 1, all with n replaced by nm . Thus, u is nm -integral over A . This proves Theorem 4.

Theorem 5. Let A and B be two rings such that $A \subseteq B$.

(a) Let $a \in A$. Then, a is 1-integral over A .

(b) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m -integral over A , and that y is n -integral over A . Then, $x + y$ is nm -integral over A .

(c) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m -integral over A , and that y is n -integral over A . Then, xy is nm -integral over A .

Proof of Theorem 5. (a) There exists a monic polynomial $P \in A[X]$ with $\deg P = 1$ and $P(a) = 0$ (namely, the polynomial $P \in A[X]$ defined by $P(X) = X - a$). Thus, a is 1-integral over A . This proves Theorem 5 (a).

(b) Since y is n -integral over A , there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(y) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exists a polynomial $\tilde{P} \in A[X]$ with $\deg \tilde{P} < n$ and $P(X) = X^n + \tilde{P}(X)$.

Now, define a polynomial $Q \in (A[x])[X]$ by $Q(X) = P(X - x)$. Then,

$$\begin{aligned} \deg Q &= \deg P && \text{(since shifting the polynomial } P \text{ by the constant } x \text{ does not change its degree)} \\ &= n \end{aligned}$$

$$\text{and } Q(x + y) = P((x + y) - x) = P(y) = 0.$$

Define a polynomial $\tilde{Q} \in (A[x])[X]$ by $\tilde{Q}(X) = ((X - x)^n - X^n) + \tilde{P}(X - x)$. Then, $\deg \tilde{Q} < n$ (since

$$\begin{aligned} \deg(\tilde{P}(X - x)) &= \deg(\tilde{P}(X)) \\ &\text{(since shifting the polynomial } \tilde{P} \text{ by the constant } x \text{ does not change its degree)} \\ &= \deg \tilde{P} < n \end{aligned}$$

and

$$\begin{aligned} \deg((X - x)^n - X^n) &= \deg\left(\left((X - x) - X\right) \cdot \sum_{k=0}^{n-1} (X - x)^k X^{n-1-k}\right) \\ &\leq \underbrace{\deg((X - x) - X)}_{=\deg(-x) \leq 0} + \underbrace{\deg\left(\sum_{k=0}^{n-1} (X - x)^k X^{n-1-k}\right)}_{\substack{\leq n-1, \text{ since} \\ \deg((X-x)^k X^{n-1-k}) \leq n-1 \\ \text{for any } k \in \{0, 1, \dots, n-1\}}} \\ &\leq 0 + (n - 1) = n - 1 < n \end{aligned}$$

yield

$$\begin{aligned} \deg \tilde{Q} &= \deg(\tilde{Q}(X)) = \deg\left(\left((X - x)^n - X^n\right) + \tilde{P}(X - x)\right) \\ &\leq \max\left\{\underbrace{\deg((X - x)^n - X^n)}_{< n}, \underbrace{\deg(\tilde{P}(X - x))}_{< n}\right\} < \max\{n, n\} = n \end{aligned}$$

). Thus, the polynomial Q is monic (since

$$\begin{aligned} Q(X) &= P(X - x) = (X - x)^n + \tilde{P}(X - x) && \text{(since } P(X) = X^n + \tilde{P}(X)) \\ &= X^n + \underbrace{\left(\left((X - x)^n - X^n\right) + \tilde{P}(X - x)\right)}_{=\tilde{Q}(X)} = X^n + \tilde{Q}(X) \end{aligned}$$

and $\deg \tilde{Q} < n$).

Hence, there exists a monic polynomial $Q \in (A[x])[X]$ with $\deg Q = n$ and $Q(x+y) = 0$. Thus, $x+y$ is n -integral over $A[x]$. Thus, Theorem 4 (applied to $v = x$ and $u = x+y$) yields that $x+y$ is nm -integral over A . This proves Theorem 5 (b).

(c) Since y is n -integral over A , there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(y) = 0$. Since $P \in A[X]$ is a monic polynomial with $\deg P = n$, there exist elements a_0, a_1, \dots, a_{n-1} of A such that $P(X) = X^n + \sum_{k=0}^{n-1} a_k X^k$. Thus,

$$P(y) = y^n + \sum_{k=0}^{n-1} a_k y^k.$$

Now, define a polynomial $Q \in (A[x])[X]$ by $Q(X) = X^n + \sum_{k=0}^{n-1} x^{n-k} a_k X^k$. Then,

$$\begin{aligned} Q(xy) &= \underbrace{(xy)^n}_{=x^n y^n} + \sum_{k=0}^{n-1} x^{n-k} \underbrace{a_k (xy)^k}_{\substack{=a_k x^k y^k \\ =x^k a_k y^k}} = x^n y^n + \sum_{k=0}^{n-1} \underbrace{x^{n-k} x^k}_{=x^n} a_k y^k \\ &= x^n y^n + \sum_{k=0}^{n-1} x^n a_k y^k = x^n \left(\underbrace{y^n + \sum_{k=0}^{n-1} a_k y^k}_{=P(y)=0} \right) = 0. \end{aligned}$$

Also, the polynomial $Q \in (A[x])[X]$ is monic and $\deg Q = n$ (since $Q(X) = X^n + \sum_{k=0}^{n-1} x^{n-k} a_k X^k$). Thus, there exists a monic polynomial $Q \in (A[x])[X]$ with $\deg Q = n$ and $Q(xy) = 0$. Thus, xy is n -integral over $A[x]$. Hence, Theorem 4 (applied to $v = x$ and $u = xy$) yields that xy is nm -integral over A . This proves Theorem 5 (c).

Corollary 6. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}^+$ and $m \in \mathbb{N}$. Let $v \in B$. Let b_0, b_1, \dots, b_{n-1} be n elements of A , and let $u = \sum_{i=0}^{n-1} b_i v^i$. Assume that vu is m -integral over A . Then, u is nm -integral over A .

Proof of Corollary 6. Define $n+1$ elements a_0, a_1, \dots, a_n of $A[vu]$ by

$$a_i = \begin{cases} -vu, & \text{if } i = 0; \\ b_{i-1}, & \text{if } i > 0 \end{cases} \quad \text{for every } i \in \{0, 1, \dots, n\}.$$

Then, $a_0 = -vu$. Let $k = 1$. Then,

$$\sum_{i=0}^n a_i v^i = \underbrace{a_0}_{=-vu} \underbrace{v^0}_{=1} + \sum_{i=1}^n \underbrace{a_i}_{\substack{=b_{i-1}, \\ \text{since } i > 0}} \underbrace{v^i}_{=v^{i-1}v} = -vu + \sum_{i=1}^n b_{i-1} v^{i-1} v = -vu + \underbrace{\sum_{i=0}^{n-1} b_i v^i}_{=u} v$$

(here, we substituted i for $i-1$ in the sum)

$$= -vu + uv = 0.$$

Now, $A[vu]$ and B are two rings such that $A[vu] \subseteq B$. The $n+1$ elements a_0, a_1, \dots, a_n of $A[vu]$ satisfy $\sum_{i=0}^n a_i v^i = 0$. We have $k = 1 \in \{0, 1, \dots, n\}$.

Hence, Theorem 2 (applied to the ring $A[vu]$ in lieu of A) yields that $\sum_{i=0}^{n-k} a_{i+k} v^i$ is n -integral over $A[vu]$. But

$$\sum_{i=0}^{n-k} a_{i+k} v^i = \sum_{i=0}^{n-1} \underbrace{a_{i+1}}_{=b_{(i+1)-1}, \text{ since } i+1 > 0} v^i = \sum_{i=0}^{n-1} b_{(i+1)-1} v^i = \sum_{i=0}^{n-1} b_i v^i = u.$$

Hence, u is n -integral over $A[vu]$. But vu is m -integral over A . Thus, Theorem 4 (applied to vu in lieu of v) yields that u is nm -integral over A . This proves Corollary 6.

2. Integrality over ideal semifiltrations

Definitions:

Definition 6. Let A be a ring, and let $(I_\rho)_{\rho \in \mathbb{N}}$ be a sequence of ideals of A . Then, $(I_\rho)_{\rho \in \mathbb{N}}$ is called an *ideal semifiltration* of A if and only if it satisfies the two conditions

$$\begin{aligned} I_0 &= A; \\ I_a I_b &\subseteq I_{a+b} \quad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

Definition 7. Let A and B be two rings such that $A \subseteq B$. Then, we identify the polynomial ring $A[Y]$ with a subring of the polynomial ring $B[Y]$ (in fact, every element of $A[Y]$ has the form $\sum_{i=0}^m a_i Y^i$ for some $m \in \mathbb{N}$ and $(a_0, a_1, \dots, a_m) \in A^{m+1}$, and thus can be seen as an element of $B[Y]$ by regarding a_i as an element of B for every $i \in \{0, 1, \dots, m\}$).

Definition 8. Let A be a ring, and let $(I_\rho)_{\rho \in \mathbb{N}}$ be an ideal semifiltration of A . Consider the polynomial ring $A[Y]$. Let $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ denote the A -submodule $\sum_{i \in \mathbb{N}} I_i Y^i$ of the A -algebra $A[Y]$. Then,

$$\begin{aligned} A[(I_\rho)_{\rho \in \mathbb{N}} * Y] &= \sum_{i \in \mathbb{N}} I_i Y^i \\ &= \left\{ \sum_{i \in \mathbb{N}} a_i Y^i \mid (a_i \in I_i \text{ for all } i \in \mathbb{N}), \text{ and (only finitely many } i \in \mathbb{N} \text{ satisfy } a_i \neq 0) \right\} \\ &= \{P \in A[Y] \mid \text{the } i\text{-th coefficient of the polynomial } P \text{ lies in } I_i \text{ for every } i \in \mathbb{N}\}. \end{aligned}$$

Now, $1 \in A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ (because $1 = \underbrace{1}_{\in A=I_0} \cdot Y^0 \in I_0 Y^0 \subseteq \sum_{i \in \mathbb{N}} I_i Y^i = A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$).

Also, the A -submodule $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$ of $A[Y]$ is closed under multiplication (since

$$\begin{aligned}
A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] \cdot A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] &= \sum_{i \in \mathbb{N}} I_i Y^i \cdot \sum_{i \in \mathbb{N}} I_i Y^i = \sum_{i \in \mathbb{N}} I_i Y^i \cdot \sum_{j \in \mathbb{N}} I_j Y^j \\
&\quad \text{(here we renamed } i \text{ as } j \text{ in the second sum)} \\
&= \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} I_i Y^i I_j Y^j = \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \underbrace{I_i I_j}_{\substack{\subseteq I_{i+j}, \\ \text{since } (I_\rho)_{\rho \in \mathbb{N}} \\ \text{is an ideal} \\ \text{semifiltration}}} \underbrace{Y^i Y^j}_{=Y^{i+j}} \\
&\subseteq \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} I_{i+j} Y^{i+j} \subseteq \sum_{k \in \mathbb{N}} I_k Y^k = \sum_{i \in \mathbb{N}} I_i Y^i \\
&\quad \text{(here we renamed } k \text{ as } i \text{ in the sum)} \\
&= A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]
\end{aligned}$$

). Hence, $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$ is an A -subalgebra of the A -algebra $A[Y]$. This A -subalgebra $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$ is called the *Rees algebra* of the ideal semifiltration $(I_\rho)_{\rho \in \mathbb{N}}$.

Clearly, $A \subseteq A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$, since $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] = \sum_{i \in \mathbb{N}} I_i Y^i \supseteq \underbrace{I_0}_{=A} \underbrace{Y^0}_{=1} = A \cdot 1 = A$.

Definition 9. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho \in \mathbb{N}}$ be an ideal semifiltration of A . Let $n \in \mathbb{N}$. Let $u \in B$.

We say that the element u of B is *n -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$* if there exists some $(a_0, a_1, \dots, a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^n a_k u^k = 0, \quad a_n = 1, \quad \text{and} \quad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, \dots, n\}.$$

We start with a theorem which reduces the question of n -integrality over $(A, (I_\rho)_{\rho \in \mathbb{N}})$ to that of n -integrality over a ring²:

Theorem 7. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho \in \mathbb{N}}$ be an ideal semifiltration of A . Let $n \in \mathbb{N}$. Let $u \in B$.

Consider the polynomial ring $A[Y]$ and its A -subalgebra $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$ defined in Definition 8.

Then, the element u of B is n -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$ if and only if the element uY of the polynomial ring $B[Y]$ is n -integral over the ring $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$. (Here, $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] \subseteq B[Y]$ because $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7).

²Theorem 7 is inspired by Proposition 5.2.1 in [2].

Proof of Theorem 7. In order to verify Theorem 7, we have to prove the following two lemmata:

Lemma \mathcal{E} : If u is n -integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$, then uY is n -integral over $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$.

Lemma \mathcal{F} : If uY is n -integral over $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$, then u is n -integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$.

Proof of Lemma \mathcal{E} : Assume that u is n -integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$. Then, by Definition 9, there exists some $(a_0, a_1, \dots, a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^n a_k u^k = 0, \quad a_n = 1, \quad \text{and} \quad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, \dots, n\}.$$

Note that $a_k Y^{n-k} \in A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$ for every $k \in \{0, 1, \dots, n\}$ (because $\underbrace{a_k}_{\in I_{n-k}} Y^{n-k} \in I_{n-k} Y^{n-k} \subseteq \sum_{i \in \mathbb{N}} I_i Y^i = A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$). Thus, we can define a polynomial $P \in \left(A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] \right) [X]$ by $P(X) = \sum_{k=0}^n a_k Y^{n-k} X^k$. This polynomial P satisfies $\deg P \leq n$, and its coefficient before X^n is $\underbrace{a_n}_{=1} \underbrace{Y^{n-n}}_{=Y^0=1} = 1$. Hence, this polynomial P is monic and satisfies $\deg P = n$. Also, $P(X) = \sum_{k=0}^n a_k Y^{n-k} X^k$ yields

$$P(uY) = \sum_{k=0}^n a_k Y^{n-k} (uY)^k = \sum_{k=0}^n a_k Y^{n-k} u^k Y^k = \sum_{k=0}^n a_k u^k \underbrace{Y^{n-k} Y^k}_{=Y^n} = Y^n \cdot \underbrace{\sum_{k=0}^n a_k u^k}_{=0} = 0.$$

Thus, there exists a monic polynomial $P \in \left(A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] \right) [X]$ with $\deg P = n$ and $P(uY) = 0$. Hence, uY is n -integral over $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$. This proves Lemma \mathcal{E} .

Proof of Lemma \mathcal{F} : Assume that uY is n -integral over $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$. Then, there exists a monic polynomial $P \in \left(A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] \right) [X]$ with $\deg P = n$ and $P(uY) = 0$. Since $P \in \left(A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] \right) [X]$ satisfies $\deg P = n$, there exists $(p_0, p_1, \dots, p_n) \in \left(A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] \right)^{n+1}$ such that $P(X) = \sum_{k=0}^n p_k X^k$. Besides, $p_n = 1$, since P is monic and $\deg P = n$.

For every $k \in \{0, 1, \dots, n\}$, we have $p_k \in A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] = \sum_{i \in \mathbb{N}} I_i Y^i$, and thus, there exists a sequence $(p_{k,i})_{i \in \mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum_{i \in \mathbb{N}} p_{k,i} Y^i$, such that $p_{k,i} \in I_i$ for every $i \in \mathbb{N}$, and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$. Thus, $P(X) = \sum_{k=0}^n p_k X^k$

becomes $P(X) = \sum_{k=0}^n \sum_{i \in \mathbb{N}} p_{k,i} Y^i X^k$ (since $p_k = \sum_{i \in \mathbb{N}} p_{k,i} Y^i$). Hence,

$$\begin{aligned}
P(uY) &= \sum_{k=0}^n \sum_{i \in \mathbb{N}} p_{k,i} Y^i \underbrace{(uY)^k}_{\substack{=u^k Y^k \\ =Y^k u^k}} = \sum_{k=0}^n \sum_{i \in \mathbb{N}} p_{k,i} \underbrace{Y^i Y^k}_{=Y^{i+k}} u^k \\
&= \sum_{k=0}^n \sum_{i \in \mathbb{N}} p_{k,i} Y^{i+k} u^k = \sum_{k \in \{0,1,\dots,n\}} \sum_{i \in \mathbb{N}} p_{k,i} Y^{i+k} u^k \\
&= \sum_{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}} p_{k,i} Y^{i+k} u^k = \sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} \underbrace{Y^{i+k}}_{=Y^\ell} u^k \\
&= \sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} Y^\ell u^k = \sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k Y^\ell.
\end{aligned}$$

Hence, $P(uY) = 0$ becomes $\sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k Y^\ell = 0$. In other words, the

polynomial $\underbrace{\sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k Y^\ell}_{\in B} \in B[Y]$ equals 0. Hence, its coefficient before

Y^n equals 0 as well. But its coefficient before Y^n is $\sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=n}} p_{k,i} u^k$. Hence,

$\sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=n}} p_{k,i} u^k$ equals 0.

Thus,

$$\begin{aligned}
0 &= \sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=n}} p_{k,i} u^k = \sum_{k \in \{0,1,\dots,n\}} \sum_{\substack{i \in \mathbb{N}; \\ i+k=n}} p_{k,i} u^k = \sum_{k \in \{0,1,\dots,n\}} p_{k,n-k} u^k \\
&\left(\begin{array}{l} \text{since } \{i \in \mathbb{N} \mid i+k=n\} = \{i \in \mathbb{N} \mid i=n-k\} = \{n-k\} \text{ (because } n-k \in \mathbb{N}, \\ \text{since } k \in \{0,1,\dots,n\}) \text{ yields } \sum_{\substack{i \in \mathbb{N}; \\ i+k=n}} p_{k,i} u^k = \sum_{i \in \{n-k\}} p_{k,i} u^k = p_{k,n-k} u^k \end{array} \right).
\end{aligned}$$

Note that

$$\begin{aligned}
\sum_{i \in \mathbb{N}} p_{n,i} Y^i &= p_n \quad \left(\text{since } \sum_{i \in \mathbb{N}} p_{k,i} Y^i = p_k \text{ for every } k \in \{0,1,\dots,n\} \right) \\
&= 1 = 1 \cdot Y^0
\end{aligned}$$

in $A[Y]$, and thus the coefficient of the polynomial $\sum_{i \in \mathbb{N}} p_{n,i} Y^i \in A[Y]$ before Y^0 is 1;

but the coefficient of the polynomial $\sum_{i \in \mathbb{N}} p_{n,i} Y^i \in A[Y]$ before Y^0 is $p_{n,0}$; hence, $p_{n,0} = 1$.

Define an $(n+1)$ -tuple $(a_0, a_1, \dots, a_n) \in A^{n+1}$ by $a_k = p_{k,n-k}$ for every $k \in \{0,1,\dots,n\}$. Then, $a_n = p_{n,n-n} = p_{n,0} = 1$. Besides,

$$\sum_{k=0}^n a_k u^k = \sum_{k=0}^n p_{k,n-k} u^k = \sum_{k \in \{0,1,\dots,n\}} p_{k,n-k} u^k = 0.$$

Finally, $a_k = p_{k,n-k} \in I_{n-k}$ (since $p_{k,i} \in I_i$ for every $i \in \mathbb{N}$) for every $k \in \{0, 1, \dots, n\}$. In other words, $a_i \in I_{n-i}$ for every $i \in \{0, 1, \dots, n\}$.

Altogether, we now know that

$$\sum_{k=0}^n a_k u^k = 0, \quad a_n = 1, \quad \text{and} \quad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, \dots, n\}.$$

Thus, by Definition 9, the element u is n -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$. This proves Lemma \mathcal{F} .

Combining Lemmata \mathcal{E} and \mathcal{F} , we obtain that u is n -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$ if and only if uY is n -integral over $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$. This proves Theorem 7.

The next theorem is an analogue of Theorem 5 for integrality over ideal semifiltrations:

Theorem 8. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho \in \mathbb{N}}$ be an ideal semifiltration of A .

(a) Let $u \in A$. Then, u is 1-integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$ if and only if $u \in I_1$.

(b) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$, and that y is n -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$. Then, $x + y$ is nm -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$.

(c) Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$, and that y is n -integral over A . Then, xy is nm -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$.

Proof of Theorem 8. (a) In order to verify Theorem 8 (a), we have to prove the following two lemmata:

Lemma \mathcal{G} : If u is 1-integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$, then $u \in I_1$.

Lemma \mathcal{H} : If $u \in I_1$, then u is 1-integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$.

Proof of Lemma \mathcal{G} : Assume that u is 1-integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$. Then, by Definition 9 (applied to $n = 1$), there exists some $(a_0, a_1) \in A^2$ such that

$$\sum_{k=0}^1 a_k u^k = 0, \quad a_1 = 1, \quad \text{and} \quad a_i \in I_{1-i} \text{ for every } i \in \{0, 1\}.$$

Thus, $a_0 \in I_{1-0}$ (since $a_i \in I_{1-i}$ for every $i \in \{0, 1\}$). Also,

$$0 = \sum_{k=0}^1 a_k u^k = a_0 \underbrace{u^0}_{=1} + \underbrace{a_1}_{=1} \underbrace{u^1}_{=u} = a_0 + u,$$

so that $u = - \underbrace{a_0}_{\in I_{1-0}=I_1} \in I_1$ (since I_1 is an ideal). This proves Lemma \mathcal{G} .

Proof of Lemma \mathcal{H} : Assume that $u \in I_1$. Then, $-u \in I_1$ (since I_1 is an ideal).

Set $a_0 = -u$ and $a_1 = 1$. Then, $\sum_{k=0}^1 a_k u^k = \underbrace{a_0}_{=-u} \underbrace{u^0}_{=1} + \underbrace{a_1}_{=1} \underbrace{u^1}_{=u} = -u + u = 0$. Also, $a_i \in I_{1-i}$ for every $i \in \{0, 1\}$ (since $a_0 = -u \in I_1 = I_{1-0}$ and $a_1 = 1 \in A = I_0 = I_{1-1}$). Altogether, we now know that $(a_0, a_1) \in A^2$ and

$$\sum_{k=0}^1 a_k u^k = 0, \quad a_1 = 1, \quad \text{and} \quad a_i \in I_{1-i} \text{ for every } i \in \{0, 1\}.$$

Thus, by Definition 9 (applied to $n = 1$), the element u is 1-integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$. This proves Lemma \mathcal{H} .

Combining Lemmata \mathcal{G} and \mathcal{H} , we obtain that u is 1-integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$ if and only if $u \in I_1$. This proves Theorem 8 (a).

(b) Consider the polynomial ring $A[Y]$ and its A -subalgebra $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$. Theorem 7 (applied to x and m instead of u and n) yields that xY is m -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ (since x is m -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$). Also, Theorem 7 (applied to y instead of u) yields that yY is n -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ (since y is n -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$). Hence, Theorem 5 (b) (applied to $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$, $B[Y]$, xY and yY instead of A , B , x and y , respectively) yields that $xY + yY$ is nm -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$. Since $xY + yY = (x + y)Y$, this means that $(x + y)Y$ is nm -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$. Hence, Theorem 7 (applied to $x + y$ and nm instead of u and n) yields that $x + y$ is nm -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$. This proves Theorem 8 (b).

(c) First, a trivial observation:

Lemma \mathcal{I} : Let A , A' and B' be three rings such that $A \subseteq A' \subseteq B'$. Let $v \in B'$. Let $n \in \mathbb{N}$. If v is n -integral over A , then v is n -integral over A' .

Proof of Lemma \mathcal{I} : Assume that v is n -integral over A . Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(v) = 0$. Since $A \subseteq A'$, we can identify the polynomial ring $A[X]$ with a subring of the polynomial ring $A'[X]$ (as explained in Definition 7). Thus, $P \in A[X]$ yields $P \in A'[X]$. Hence, there exists a monic polynomial $P \in A'[X]$ with $\deg P = n$ and $P(v) = 0$. Thus, v is n -integral over A' . This proves Lemma \mathcal{I} .

Now let us prove Theorem 8 (c).

Consider the polynomial ring $A[Y]$ and its A -subalgebra $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$. Theorem 7 (applied to x and m instead of u and n) yields that xY is m -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ (since x is m -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$). On the other hand, Lemma \mathcal{I} (applied to $A' = A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$, $B' = B[Y]$ and $v = y$) yields that y is n -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ (since y is n -integral over A , and $A \subseteq A[(I_\rho)_{\rho \in \mathbb{N}} * Y] \subseteq B[Y]$). Hence, Theorem 5 (c) (applied to $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$, $B[Y]$ and xY instead of A , B and x , respectively) yields that $xY \cdot y$ is nm -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$. Since $xY \cdot y = xyY$,

this means that xyY is nm -integral over $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$. Hence, Theorem 7 (applied to xy and nm instead of u and n) yields that xy is nm -integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}} \right)$. This proves Theorem 8 (c).

The next theorem imitates Theorem 4 for integrality over ideal semifiltrations:

Theorem 9. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho \in \mathbb{N}}$ be an ideal semifiltration of A .

Let $v \in B$ and $u \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

(a) Then, $(I_\rho A[v])_{\rho \in \mathbb{N}}$ is an ideal semifiltration of $A[v]$.

(b) Assume that v is m -integral over A , and that u is n -integral over $\left(A[v], (I_\rho A[v])_{\rho \in \mathbb{N}} \right)$. Then, u is nm -integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}} \right)$.

Proof of Theorem 9. (a) More generally:

Lemma \mathcal{J} : Let A and A' be two rings such that $A \subseteq A'$. Let $(I_\rho)_{\rho \in \mathbb{N}}$ be an ideal semifiltration of A . Then, $(I_\rho A')_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A' .

Proof of Lemma \mathcal{J} : Since $(I_\rho)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A , the set I_ρ is an ideal of A for every $\rho \in \mathbb{N}$, and we have

$$\begin{aligned} I_0 &= A; \\ I_a I_b &\subseteq I_{a+b} \quad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

Now, the set $I_\rho A'$ is an ideal of A' for every $\rho \in \mathbb{N}$ (since I_ρ is an ideal of A), and we have

$$\begin{aligned} I_0 A' &= A A' = A'; \\ I_a A' \cdot I_b A' &= I_a I_b A' \subseteq I_{a+b} A' \quad (\text{since } I_a I_b \subseteq I_{a+b}) \quad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

Thus, $(I_\rho A')_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A' . This proves Lemma \mathcal{J} .

Now let us prove Theorem 9 (a). In fact, Lemma \mathcal{J} (applied to $A' = A[v]$) yields that $(I_\rho A[v])_{\rho \in \mathbb{N}}$ is an ideal semifiltration of $A[v]$. This proves Theorem 9 (a).

(b) First, we will show a simple fact:

Lemma \mathcal{K} : Let A , A' and B' be three rings such that $A \subseteq A' \subseteq B'$. Let $v \in B'$. Then, $A' \cdot A[v] = A'[v]$.

Proof of Lemma \mathcal{K} : We have $\underbrace{A'}_{\subseteq A'[v]} \cdot \underbrace{A[v]}_{\substack{\subseteq A'[v], \\ \text{since } A \subseteq A'}} \subseteq A'[v] \cdot A'[v] = A'[v]$ (since $A'[v]$

is a ring). On the other hand, let x be an element of $A'[v]$. Then, there exists some $n \in \mathbb{N}$ and some $(a_0, a_1, \dots, a_n) \in (A')^{n+1}$ such that $x = \sum_{k=0}^n a_k v^k$. Thus,

$$x = \sum_{k=0}^n \underbrace{a_k}_{\in A'} \underbrace{v^k}_{\in A[v]} \in \sum_{k=0}^n A' \cdot A[v] \subseteq A' \cdot A[v] \quad (\text{since } A' \cdot A[v] \text{ is an additive group}).$$

Thus, we have proved that $x \in A' \cdot A[v]$ for every $x \in A'[v]$. Therefore, $A'[v] \subseteq A' \cdot A[v]$. Combined with $A' \cdot A[v] \subseteq A'[v]$, this yields $A' \cdot A[v] = A'[v]$. Hence, we have established Lemma \mathcal{K} .

Now let us prove Theorem 9 (b). In fact, consider the polynomial ring $A[Y]$ and its A -subalgebra $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$. We have $A[(I_\rho)_{\rho \in \mathbb{N}} * Y] \subseteq A[Y]$, and (as explained in Definition 7) we can identify the polynomial ring $A[Y]$ with a subring of $(A[v])[Y]$ (since $A \subseteq A[v]$). Hence, $A[(I_\rho)_{\rho \in \mathbb{N}} * Y] \subseteq (A[v])[Y]$. On the other hand, $(A[v])[(I_\rho A[v])_{\rho \in \mathbb{N}} * Y] \subseteq (A[v])[Y]$.

Now, we will show that $(A[v])[(I_\rho A[v])_{\rho \in \mathbb{N}} * Y] = \left(A[(I_\rho)_{\rho \in \mathbb{N}} * Y]\right)[v]$.

In fact, Definition 8 yields

$$\begin{aligned} (A[v])[(I_\rho A[v])_{\rho \in \mathbb{N}} * Y] &= \sum_{i \in \mathbb{N}} I_i A[v] \cdot Y^i = \sum_{i \in \mathbb{N}} I_i Y^i \cdot A[v] = A[(I_\rho)_{\rho \in \mathbb{N}} * Y] \cdot A[v] \\ &\quad \left(\text{since } \sum_{i \in \mathbb{N}} I_i Y^i = A[(I_\rho)_{\rho \in \mathbb{N}} * Y]\right) \\ &= \left(A[(I_\rho)_{\rho \in \mathbb{N}} * Y]\right)[v] \end{aligned}$$

(by Lemma \mathcal{K} (applied to $A' = A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ and $B' = (A[v])[Y]$)).

Note that (as explained in Definition 7) we can identify the polynomial ring $(A[v])[Y]$ with a subring of $B[Y]$ (since $A[v] \subseteq B$). Thus, $A[(I_\rho)_{\rho \in \mathbb{N}} * Y] \subseteq (A[v])[Y]$ yields $A[(I_\rho)_{\rho \in \mathbb{N}} * Y] \subseteq B[Y]$.

Besides, Lemma \mathcal{I} (applied to $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$, $B[Y]$ and m instead of A' , B' and n) yields that v is m -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ (since v is m -integral over A , and $A \subseteq A[(I_\rho)_{\rho \in \mathbb{N}} * Y] \subseteq B[Y]$).

Now, Theorem 7 (applied to $A[v]$ and $(I_\rho A[v])_{\rho \in \mathbb{N}}$ instead of A and $(I_\rho)_{\rho \in \mathbb{N}}$) yields that uY is n -integral over $(A[v])[(I_\rho A[v])_{\rho \in \mathbb{N}} * Y]$ (since u is n -integral over $(A[v], (I_\rho A[v])_{\rho \in \mathbb{N}})$). Since $(A[v])[(I_\rho A[v])_{\rho \in \mathbb{N}} * Y] = \left(A[(I_\rho)_{\rho \in \mathbb{N}} * Y]\right)[v]$, this means that uY is n -integral over $\left(A[(I_\rho)_{\rho \in \mathbb{N}} * Y]\right)[v]$. Now, Theorem 4 (applied to $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$, $B[Y]$ and uY instead of A , B and u) yields that uY is nm -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ (since v is m -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$, and uY is n -integral over $\left(A[(I_\rho)_{\rho \in \mathbb{N}} * Y]\right)[v]$). Thus, Theorem 7 (applied to nm instead of n) yields that u is nm -integral over $\left(A, (I_\rho)_{\rho \in \mathbb{N}}\right)$. This proves Theorem 9 (b).

3. Generalizing to two ideal semifiltrations

Theorem 10. Let A be a ring.

(a) Then, $(A)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A .

(b) Let $(I_\rho)_{\rho \in \mathbb{N}}$ and $(J_\rho)_{\rho \in \mathbb{N}}$ be two ideal semifiltrations of A . Then, $(I_\rho J_\rho)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A .

Proof of Theorem 10. **(a)** Clearly, $(A)_{\rho \in \mathbb{N}}$ is a sequence of ideals of A . Hence, in order to prove that $(A)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A , it is enough to verify that it satisfies the two conditions

$$\begin{aligned} A &= A; \\ AA &\subseteq A \quad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

But these two conditions are obviously satisfied. Hence, $(A)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A . This proves Theorem 10 **(a)**.

(b) Since $(I_\rho)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A , it is a sequence of ideals of A , and it satisfies the two conditions

$$\begin{aligned} I_0 &= A; \\ I_a I_b &\subseteq I_{a+b} \quad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

Since $(J_\rho)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A , it is a sequence of ideals of A , and it satisfies the two conditions

$$\begin{aligned} J_0 &= A; \\ J_a J_b &\subseteq J_{a+b} \quad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

Now, $I_\rho J_\rho$ is an ideal of A for every $\rho \in \mathbb{N}$ (since I_ρ and J_ρ are ideals of A for every $\rho \in \mathbb{N}$, and the product of any two ideals of A is an ideal of A). Hence, $(I_\rho J_\rho)_{\rho \in \mathbb{N}}$ is a sequence of ideals of A . Thus, in order to prove that $(I_\rho J_\rho)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A , it is enough to verify that it satisfies the two conditions

$$\begin{aligned} I_0 J_0 &= A; \\ I_a J_a \cdot I_b J_b &\subseteq I_{a+b} J_{a+b} \quad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

But these two conditions are satisfied, since

$$\begin{aligned} \underbrace{I_0}_{=A} \underbrace{J_0}_{=A} &= AA = A; \\ I_a J_a \cdot I_b J_b &= \underbrace{I_a I_b}_{\subseteq I_{a+b}} \underbrace{J_a J_b}_{\subseteq J_{a+b}} \subseteq I_{a+b} J_{a+b} \quad \text{for every } a \in \mathbb{N} \text{ and } b \in \mathbb{N}. \end{aligned}$$

Hence, $(I_\rho J_\rho)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A . This proves Theorem 10 **(b)**.

Now let us generalize Theorem 7:

Theorem 11. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho \in \mathbb{N}}$ and $(J_\rho)_{\rho \in \mathbb{N}}$ be two ideal semifiltrations of A . Let $n \in \mathbb{N}$. Let $u \in B$.

We know that $(I_\rho J_\rho)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 10 **(b)**).

Consider the polynomial ring $A[Y]$ and its A -subalgebra $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$.

We will abbreviate the ring $A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right]$ by $A_{[I]}$.

By Lemma \mathcal{J} (applied to $A_{[I]}$ and $(J_\tau)_{\tau \in \mathbb{N}}$ instead of A' and $(I_\rho)_{\rho \in \mathbb{N}}$), the sequence $(J_\tau A_{[I]})_{\tau \in \mathbb{N}}$ is an ideal semifiltration of $A_{[I]}$ (since $A \subseteq A_{[I]}$ and since $(J_\tau)_{\tau \in \mathbb{N}} = (J_\rho)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A).

Then, the element u of B is n -integral over $(A, (I_\rho J_\rho)_{\rho \in \mathbb{N}})$ if and only if the element uY of the polynomial ring $B[Y]$ is n -integral over $(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}})$.

(Here, $A_{[I]} \subseteq B[Y]$ because $A_{[I]} = A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7.)

Proof of Theorem 11. First, note that

$$\begin{aligned} \sum_{\ell \in \mathbb{N}} I_\ell Y^\ell &= \sum_{i \in \mathbb{N}} I_i Y^i && \text{(here we renamed } \ell \text{ as } i \text{ in the sum)} \\ &= A \left[(I_\rho)_{\rho \in \mathbb{N}} * Y \right] = A_{[I]}. \end{aligned}$$

In order to verify Theorem 11, we have to prove the following two lemmata:

Lemma \mathcal{E}' : If u is n -integral over $(A, (I_\rho J_\rho)_{\rho \in \mathbb{N}})$, then uY is n -integral over $(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}})$.

Lemma \mathcal{F}' : If uY is n -integral over $(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}})$, then u is n -integral over $(A, (I_\rho J_\rho)_{\rho \in \mathbb{N}})$.

Proof of Lemma \mathcal{E}' : Assume that u is n -integral over $(A, (I_\rho J_\rho)_{\rho \in \mathbb{N}})$. Then, by Definition 9 (applied to $(I_\rho J_\rho)_{\rho \in \mathbb{N}}$ instead of $(I_\rho)_{\rho \in \mathbb{N}}$), there exists some $(a_0, a_1, \dots, a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^n a_k u^k = 0, \quad a_n = 1, \quad \text{and} \quad a_i \in I_{n-i} J_{n-i} \text{ for every } i \in \{0, 1, \dots, n\}.$$

Note that $a_k Y^{n-k} \in A_{[I]}$ for every $k \in \{0, 1, \dots, n\}$ (because $a_k \in I_{n-k} J_{n-k} \subseteq I_{n-k}$ (since I_{n-k} is an ideal of A) and thus $a_k Y^{n-k} \in I_{n-k} Y^{n-k} \subseteq \sum_{i \in \mathbb{N}} I_i Y^i = A_{[I]}$). Thus,

we can define an $(n+1)$ -tuple $(b_0, b_1, \dots, b_n) \in (A_{[I]})^{n+1}$ by $b_k = a_k Y^{n-k}$ for every $k \in \{0, 1, \dots, n\}$. Then,

$$\begin{aligned} \sum_{k=0}^n b_k \cdot (uY)^k &= \sum_{k=0}^n a_k Y^{n-k} \cdot (uY)^k = \sum_{k=0}^n a_k Y^{n-k} u^k Y^k = \sum_{k=0}^n a_k u^k \underbrace{Y^{n-k} Y^k}_{=Y^n} = Y^n \cdot \underbrace{\sum_{k=0}^n a_k u^k}_{=0} = 0; \\ b_n &= \underbrace{a_n}_{=1} \underbrace{Y^{n-n}}_{=Y^0=1} = 1, \end{aligned}$$

and

$$b_i = \underbrace{a_i}_{\substack{\in I_{n-i} J_{n-i} \\ = J_{n-i} I_{n-i}}} Y^{n-i} \in J_{n-i} \underbrace{I_{n-i} Y^{n-i}}_{\substack{\subseteq \sum_{\ell \in \mathbb{N}} I_\ell Y^\ell \\ = A_{[I]}}} \subseteq J_{n-i} A_{[I]}$$

for every $i \in \{0, 1, \dots, n\}$.

Altogether, we now know that $(b_0, b_1, \dots, b_n) \in (A_{[I]})^{n+1}$ and

$$\sum_{k=0}^n b_k \cdot (uY)^k = 0, \quad b_n = 1, \quad \text{and} \quad b_i \in J_{n-i}A_{[I]} \text{ for every } i \in \{0, 1, \dots, n\}.$$

Hence, by Definition 9 (applied to $A_{[I]}$, $B[Y]$, $(J_\tau A_{[I]})_{\tau \in \mathbb{N}}$, uY and (b_0, b_1, \dots, b_n) instead of A , B , $(I_\rho)_{\rho \in \mathbb{N}}$, u and (a_0, a_1, \dots, a_n)), the element uY is n -integral over $(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}})$. This proves Lemma \mathcal{E}' .

Proof of Lemma \mathcal{F}' : Assume that uY is n -integral over $(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}})$. Then, by Definition 9 (applied to $A_{[I]}$, $B[Y]$, $(J_\tau A_{[I]})_{\tau \in \mathbb{N}}$, uY and (p_0, p_1, \dots, p_n) instead of A , B , $(I_\rho)_{\rho \in \mathbb{N}}$, u and (a_0, a_1, \dots, a_n)), there exists some $(p_0, p_1, \dots, p_n) \in (A_{[I]})^{n+1}$ such that

$$\sum_{k=0}^n p_k \cdot (uY)^k = 0, \quad p_n = 1, \quad \text{and} \quad p_i \in J_{n-i}A_{[I]} \text{ for every } i \in \{0, 1, \dots, n\}.$$

For every $k \in \{0, 1, \dots, n\}$, we have

$$\begin{aligned} p_k &\in J_{n-k}A_{[I]} = J_{n-k} \sum_{i \in \mathbb{N}} I_i Y^i && \left(\text{since } A_{[I]} = \sum_{i \in \mathbb{N}} I_i Y^i \right) \\ &= \sum_{i \in \mathbb{N}} J_{n-k} I_i Y^i = \sum_{i \in \mathbb{N}} I_i J_{n-k} Y^i, \end{aligned}$$

and thus, there exists a sequence $(p_{k,i})_{i \in \mathbb{N}} \in A^{\mathbb{N}}$ such that $p_k = \sum_{i \in \mathbb{N}} p_{k,i} Y^i$, such that $p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$, and such that only finitely many $i \in \mathbb{N}$ satisfy $p_{k,i} \neq 0$. Thus,

$$\begin{aligned} \sum_{k=0}^n p_k \cdot (uY)^k &= \sum_{k=0}^n \sum_{i \in \mathbb{N}} p_{k,i} Y^i \cdot \underbrace{(uY)^k}_{=u^k Y^k} && \left(\text{since } p_k = \sum_{i \in \mathbb{N}} p_{k,i} Y^i \right) \\ &= \sum_{k=0}^n \sum_{i \in \mathbb{N}} p_{k,i} \underbrace{Y^i \cdot Y^k}_{=Y^{i+k}} u^k \\ &= \sum_{k=0}^n \sum_{i \in \mathbb{N}} p_{k,i} Y^{i+k} u^k = \sum_{k \in \{0, 1, \dots, n\}} \sum_{i \in \mathbb{N}} p_{k,i} Y^{i+k} u^k \\ &= \sum_{(k,i) \in \{0, 1, \dots, n\} \times \mathbb{N}} p_{k,i} Y^{i+k} u^k = \sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0, 1, \dots, n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} \underbrace{Y^{i+k}}_{=Y^\ell} u^k \\ &= \sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0, 1, \dots, n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} Y^\ell u^k = \sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0, 1, \dots, n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k Y^\ell. \end{aligned}$$

Hence, $\sum_{k=0}^n p_k \cdot (uY)^k = 0$ becomes $\sum_{\ell \in \mathbb{N}} \sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k Y^\ell = 0$. In other words, the

polynomial $\sum_{\ell \in \mathbb{N}} \underbrace{\sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=\ell}} p_{k,i} u^k Y^\ell}_{\in B} \in B[Y]$ equals 0. Hence, its coefficient before

Y^n equals 0 as well. But its coefficient before Y^n is $\sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=n}} p_{k,i} u^k$. Hence,

$\sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=n}} p_{k,i} u^k$ equals 0.

Thus,

$$0 = \sum_{\substack{(k,i) \in \{0,1,\dots,n\} \times \mathbb{N}; \\ i+k=n}} p_{k,i} u^k = \sum_{k \in \{0,1,\dots,n\}} \sum_{\substack{i \in \mathbb{N}; \\ i+k=n}} p_{k,i} u^k = \sum_{k \in \{0,1,\dots,n\}} p_{k,n-k} u^k$$

$$\left(\begin{array}{l} \text{since } \{i \in \mathbb{N} \mid i+k=n\} = \{i \in \mathbb{N} \mid i=n-k\} = \{n-k\} \text{ (because } n-k \in \mathbb{N}, \\ \text{since } k \in \{0,1,\dots,n\}) \text{ yields } \sum_{\substack{i \in \mathbb{N}; \\ i+k=n}} p_{k,i} u^k = \sum_{i \in \{n-k\}} p_{k,i} u^k = p_{k,n-k} u^k \end{array} \right).$$

Note that

$$\sum_{i \in \mathbb{N}} p_{n,i} Y^i = p_n \quad \left(\text{since } \sum_{i \in \mathbb{N}} p_{k,i} Y^i = p_k \text{ for every } k \in \{0,1,\dots,n\} \right)$$

$$= 1 = 1 \cdot Y^0$$

in $A[Y]$, and thus the coefficient of the polynomial $\sum_{i \in \mathbb{N}} p_{n,i} Y^i \in A[Y]$ before Y^0 is 1;

but the coefficient of the polynomial $\sum_{i \in \mathbb{N}} p_{n,i} Y^i \in A[Y]$ before Y^0 is $p_{n,0}$; hence, $p_{n,0} = 1$.

Define an $(n+1)$ -tuple $(a_0, a_1, \dots, a_n) \in A^{n+1}$ by $a_k = p_{k,n-k}$ for every $k \in \{0,1,\dots,n\}$. Then, $a_n = p_{n,n-n} = p_{n,0} = 1$. Besides,

$$\sum_{k=0}^n a_k u^k = \sum_{k=0}^n p_{k,n-k} u^k = \sum_{k \in \{0,1,\dots,n\}} p_{k,n-k} u^k = 0.$$

Finally, $a_k = p_{k,n-k} \in I_{n-k} J_{n-k}$ (since $p_{k,i} \in I_i J_{n-k}$ for every $i \in \mathbb{N}$) for every $k \in \{0,1,\dots,n\}$. In other words, $a_i \in I_{n-i} J_{n-i}$ for every $i \in \{0,1,\dots,n\}$.

Altogether, we now know that

$$\sum_{k=0}^n a_k u^k = 0, \quad a_n = 1, \quad \text{and} \quad a_i \in I_{n-i} J_{n-i} \text{ for every } i \in \{0,1,\dots,n\}.$$

Thus, by Definition 9 (applied to $(I_\rho J_\rho)_{\rho \in \mathbb{N}}$ instead of $(I_\rho)_{\rho \in \mathbb{N}}$), the element u is n -integral over $\left(A, (I_\rho J_\rho)_{\rho \in \mathbb{N}} \right)$. This proves Lemma \mathcal{F}' .

Combining Lemmata \mathcal{E}' and \mathcal{F}' , we obtain that u is n -integral over $\left(A, (I_\rho J_\rho)_{\rho \in \mathbb{N}} \right)$ if and only if uY is n -integral over $\left(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}} \right)$. This proves Theorem 11.

For the sake of completeness, we mention the following trivial fact (which shows why Theorem 11 generalizes Theorem 7):

Theorem 12. Let A and B be two rings such that $A \subseteq B$. Let $n \in \mathbb{N}$.
Let $u \in B$.

We know that $(A)_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A (according to Theorem 10 (a)).

Then, the element u of B is n -integral over $(A, (A)_{\rho \in \mathbb{N}})$ if and only if u is n -integral over A .

Proof of Theorem 12. In order to verify Theorem 12, we have to prove the following two lemmata:

Lemma \mathcal{L} : If u is n -integral over $(A, (A)_{\rho \in \mathbb{N}})$, then u is n -integral over A .

Lemma \mathcal{M} : If u is n -integral over A , then u is n -integral over $(A, (A)_{\rho \in \mathbb{N}})$.

Proof of Lemma \mathcal{L} : Assume that u is n -integral over $(A, (A)_{\rho \in \mathbb{N}})$. Then, by Definition 9 (applied to $(A)_{\rho \in \mathbb{N}}$ instead of $(I_\rho)_{\rho \in \mathbb{N}}$), there exists some $(a_0, a_1, \dots, a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^n a_k u^k = 0, \quad a_n = 1, \quad \text{and} \quad a_i \in A \text{ for every } i \in \{0, 1, \dots, n\}.$$

Define a polynomial $P \in A[X]$ by $P(X) = \sum_{k=0}^n a_k X^k$. Then, $P(X) = \sum_{k=0}^n a_k X^k = \underbrace{a_n}_{=1} X^n + \sum_{k=0}^{n-1} a_k X^k = X^n + \sum_{k=0}^{n-1} a_k X^k$. Hence, the polynomial P is monic, and $\deg P = n$.

Besides, $P(u) = 0$ (since $P(X) = \sum_{k=0}^n a_k X^k$ yields $P(u) = \sum_{k=0}^n a_k u^k = 0$). Thus, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Hence, u is n -integral over A . This proves Lemma \mathcal{L} .

Proof of Lemma \mathcal{M} : Assume that u is n -integral over A . Then, there exists a monic polynomial $P \in A[X]$ with $\deg P = n$ and $P(u) = 0$. Since $\deg P = n$, there exists some $(n+1)$ -tuple $(a_0, a_1, \dots, a_n) \in A^{n+1}$ such that $P(X) = \sum_{k=0}^n a_k X^k$. Thus, $a_n = 1$ (since P is monic, and $\deg P = n$). Also, $\sum_{k=0}^n a_k X^k = P(X)$ yields $\sum_{k=0}^n a_k u^k = P(u) = 0$. Altogether, we now know that $(a_0, a_1, \dots, a_n) \in A^{n+1}$ and

$$\sum_{k=0}^n a_k u^k = 0, \quad a_n = 1, \quad \text{and} \quad a_i \in A \text{ for every } i \in \{0, 1, \dots, n\}.$$

Hence, by Definition 9 (applied to $(A)_{\rho \in \mathbb{N}}$ instead of $(I_\rho)_{\rho \in \mathbb{N}}$), the element u is n -integral over $(A, (A)_{\rho \in \mathbb{N}})$. This proves Lemma \mathcal{M} .

Combining Lemmata \mathcal{L} and \mathcal{M} , we obtain that u is n -integral over $(A, (A)_{\rho \in \mathbb{N}})$ if and only if u is n -integral over A . This proves Theorem 12.

Finally, let us generalize Theorem 8 (c):

Theorem 13. Let A and B be two rings such that $A \subseteq B$. Let $(I_\rho)_{\rho \in \mathbb{N}}$ and $(J_\rho)_{\rho \in \mathbb{N}}$ be two ideal semifiltrations of A .

Let $x \in B$ and $y \in B$. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}$. Assume that x is m -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$, and that y is n -integral over $(A, (J_\rho)_{\rho \in \mathbb{N}})$. Then, xy is nm -integral over $(A, (I_\rho J_\rho)_{\rho \in \mathbb{N}})$.

Proof of Theorem 13. First, a trivial observation:

Lemma \mathcal{I}' : Let A, A' and B' be three rings such that $A \subseteq A' \subseteq B'$. Let $(I_\rho)_{\rho \in \mathbb{N}}$ be an ideal semifiltration of A . Let $v \in B'$. Let $n \in \mathbb{N}$. If v is n -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$, then v is n -integral over $(A', (I_\rho A')_{\rho \in \mathbb{N}})$. (Note that $(I_\rho A')_{\rho \in \mathbb{N}}$ is an ideal semifiltration of A' , according to Lemma \mathcal{J} .)

Proof of Lemma \mathcal{I}' : Assume that v is n -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$. Then, by Definition 9 (applied to B' and v instead of B and u), there exists some $(a_0, a_1, \dots, a_n) \in A^{n+1}$ such that

$$\sum_{k=0}^n a_k v^k = 0, \quad a_n = 1, \quad \text{and} \quad a_i \in I_{n-i} \text{ for every } i \in \{0, 1, \dots, n\}.$$

But $(a_0, a_1, \dots, a_n) \in A^{n+1}$ yields $(a_0, a_1, \dots, a_n) \in (A')^{n+1}$ (since $A \subseteq A'$), and $a_i \in I_{n-i}$ yields $a_i \in I_{n-i} A'$ (since $I_{n-i} \subseteq I_{n-i} A'$) for every $i \in \{0, 1, \dots, n\}$. Thus, $(a_0, a_1, \dots, a_n) \in (A')^{n+1}$ and

$$\sum_{k=0}^n a_k v^k = 0, \quad a_n = 1, \quad \text{and} \quad a_i \in I_{n-i} A' \text{ for every } i \in \{0, 1, \dots, n\}.$$

Hence, by Definition 9 (applied to $B', A', (I_\rho A')_{\rho \in \mathbb{N}}$ and v instead of $B, A, (I_\rho)_{\rho \in \mathbb{N}}$ and u), the element v is n -integral over $(A', (I_\rho A')_{\rho \in \mathbb{N}})$. This proves Lemma \mathcal{I}' .

Now let us prove Theorem 13.

We have $(J_\rho)_{\rho \in \mathbb{N}} = (J_\tau)_{\tau \in \mathbb{N}}$. Hence, y is n -integral over $(A, (J_\tau)_{\tau \in \mathbb{N}})$ (since y is n -integral over $(A, (J_\rho)_{\rho \in \mathbb{N}})$).

Consider the polynomial ring $A[Y]$ and its A -subalgebra $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$. We will abbreviate the ring $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ by $A_{[I]}$. We have $A_{[I]} \subseteq B[Y]$, because $A_{[I]} = A[(I_\rho)_{\rho \in \mathbb{N}} * Y] \subseteq A[Y]$ and we consider $A[Y]$ as a subring of $B[Y]$ as explained in Definition 7.

Theorem 7 (applied to x and m instead of u and n) yields that xY is m -integral over $A[(I_\rho)_{\rho \in \mathbb{N}} * Y]$ (since x is m -integral over $(A, (I_\rho)_{\rho \in \mathbb{N}})$). In other words, xY is m -integral over $A_{[I]}$ (since $A[(I_\rho)_{\rho \in \mathbb{N}} * Y] = A_{[I]}$).

On the other hand, Lemma \mathcal{I}' (applied to $A_{[I]}, B[Y], (J_\tau)_{\tau \in \mathbb{N}}$ and y instead of $A', B', (I_\rho)_{\rho \in \mathbb{N}}$ and v) yields that y is n -integral over $(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}})$ (since y is n -integral over $(A, (J_\tau)_{\tau \in \mathbb{N}})$, and $A \subseteq A_{[I]} \subseteq B[Y]$).

Hence, Theorem 8 (c) (applied to $A_{[I]}$, $B[Y]$, $(J_\tau A_{[I]})_{\tau \in \mathbb{N}}$, y , xY , m and n instead of A , B , $(I_\rho)_{\rho \in \mathbb{N}}$, x , y , n and m respectively) yields that $y \cdot xY$ is mn -integral over $(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}})$ (since y is n -integral over $(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}})$, and xY is m -integral over $A_{[I]}$). Since $y \cdot xY = xyY$ and $mn = nm$, this means that xyY is nm -integral over $(A_{[I]}, (J_\tau A_{[I]})_{\tau \in \mathbb{N}})$. Hence, Theorem 11 (applied to xy and nm instead of u and n) yields that xy is nm -integral over $(A, (I_\rho J_\rho)_{\rho \in \mathbb{N}})$. This proves Theorem 13.

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