Radical axes revisited

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1. Introduction

In this note we are going to shed new light on some aspects of the theory of the radical axis. For a rather complete account of this theory, see Chapter VIII of [1]. We are going to make use of but the most basic properties of radical axes (the existence of the radical axis and of the radical center), and prove some possibly new facts along with a few known ones.

First we introduce several conventions and notations:

- We work in the plane. That means, the geometrical objects defined below are all meant to lie on the same plane.

- We use directed lengths. Hereby, we denote the directed length of a segment AB by $\overline{AB}$, and the non-directed (i. e. standard) length of this segment just by AB.

- For any point A and any nonnegative real x, we denote by $A(x)$ the circle with center A and radius x. [1]

- For any circle k and any point P, we define the *power of the point P with respect to the circle k* as the number $PM^2 - r^2$, where M is the center and r is the radius of the circle k. This power will be denoted by $\operatorname{pot}(P;\ k)$; thus, $\operatorname{pot}(P;\ k) = PM^2 - r^2$.
  In other words: If M is a point, and r is a number, then $\operatorname{pot}(P;\ M(r)) = PM^2 - r^2$.

- A known fact (see, e. g., [1], §421) states:
  If k and m are two circles with distinct centers, then the set of all points P satisfying $\operatorname{pot}(P;\ k) = \operatorname{pot}(P;\ m)$ is a line.
  This line is called the *radical axis of the circles k and m*, and will be denoted by $\operatorname{rad}(k;\ m)$ in the following.
  It is known that the radical axis of two circles is always perpendicular to the line joining their centers. In other words, if K and M are two points, and r and s are two numbers, then
  $$\operatorname{rad}(K(r);\ M(s)) \perp KM. \tag{1}$$

- If g is a line and P is a point, then we denote by $\operatorname{perp}(P;\ g)$ the perpendicular to the line g through the point P.

2. A theorem by Casey

Now we can start with a rather easy and known result ([1], §471):

[1] Note that this circle is defined for all $x \geq 0$, including the case $x = 0$. One can also allow x to be negative or even purely imaginary, if one departs from the usual definition of circles as point-sets and, instead, defines an "abstract circle" as an ordered pair of a point (the center) and a number (the radius) such that the square of the radius is real. All results in this note remain valid for such "abstract circles".

Theorem 1 (Casey). Let A and B be two distinct points, and let x and y be two numbers. Let P be a point, and let Q be the orthogonal projection of the point P on the line $\operatorname{rad}(A(x);\ B(y))$.

The two lines PQ and AB are parallel. If we direct these lines in the same way (that means, we direct them in such a way that equal vectors along these lines correspond to equal directed segments), then

$$\operatorname{pot}(P;\ A(x)) - \operatorname{pot}(P;\ B(y)) = 2 \cdot \overline{QP} \cdot \overline{AB}. \tag{2}$$

(See Fig. 1.)

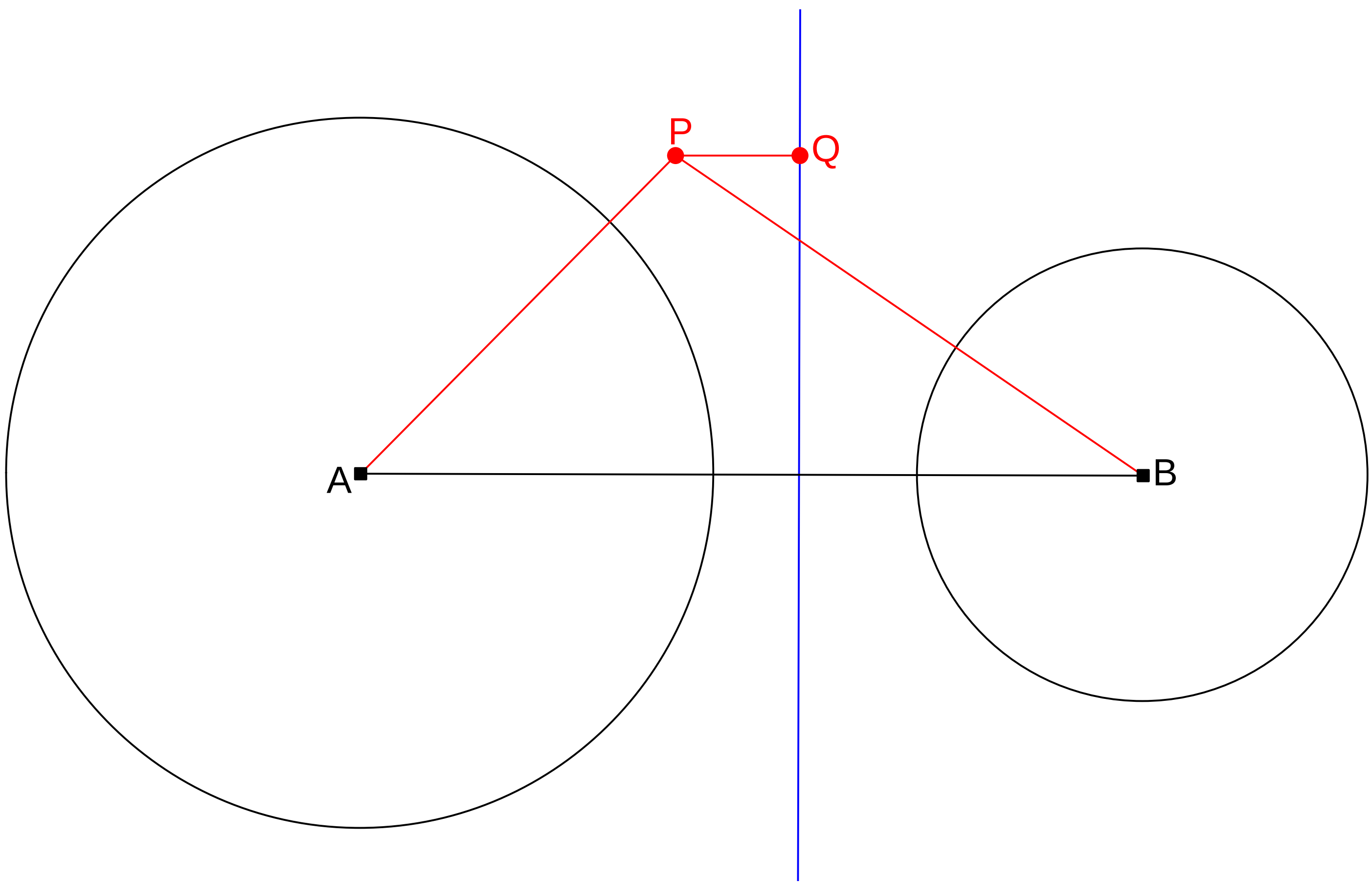

Fig. 1

Proof of Theorem 1. (See Fig. 2.) First, we have to show that the lines PQ and AB are parallel. In fact, (1) yields $\operatorname{rad}(A(x);\ B(y)) \perp AB$, but, on the other hand, $PQ \perp \operatorname{rad}(A(x);\ B(y))$ (by the construction of the point Q). Thus, $PQ \parallel AB$ follows.

As we now have shown that the lines PQ and AB are parallel, it only remains to prove the equation (2) - under the condition that the lines PQ and AB are directed in the same way.

Let P' be the orthogonal projection of the point P on the line AB. Let $Q' = \operatorname{rad}(A(x);\ B(y)) \cap AB$. Then, $\operatorname{rad}(A(x);\ B(y)) \perp AB$ yields $\measuredangle P'Q'Q = 90^\circ$. Furthermore, $\measuredangle PP'Q' = 90^\circ$ and $\measuredangle PQQ' = 90^\circ$ (by the construction of the points P' and Q). Hence, the quadrilateral $PQQ'P'$ has three right angles and thus must be a rectangle. Hence, $Q'P' = QP$. Therefore, the directed lengths $\overline{Q'P'}$ and $\overline{QP}$ have the same absolute value. On the other hand, these directed lengths have the same sign (since the lines PQ and AB were directed in the same way). Thus, $\overline{Q'P'} = \overline{QP}$.

From $Q' \in \mathrm{rad}\left(A\left(x\right);\ B\left(y\right)\right)$, it follows that $\mathrm{pot}\left(Q';\ A\left(x\right)\right) = \mathrm{pot}\left(Q';\ B\left(y\right)\right)$. Hence,

$$\overline{Q'A}^2 - x^2 = Q'A^2 - x^2 = \mathrm{pot}\left(Q';\ A\left(x\right)\right) = \mathrm{pot}\left(Q';\ B\left(y\right)\right) = Q'B^2 - y^2 = \overline{Q'B}^2 - y^2.$$

This becomes

$$\overline{Q'A}^2 - \overline{Q'B}^2 = x^2 - y^2. \tag{3}$$

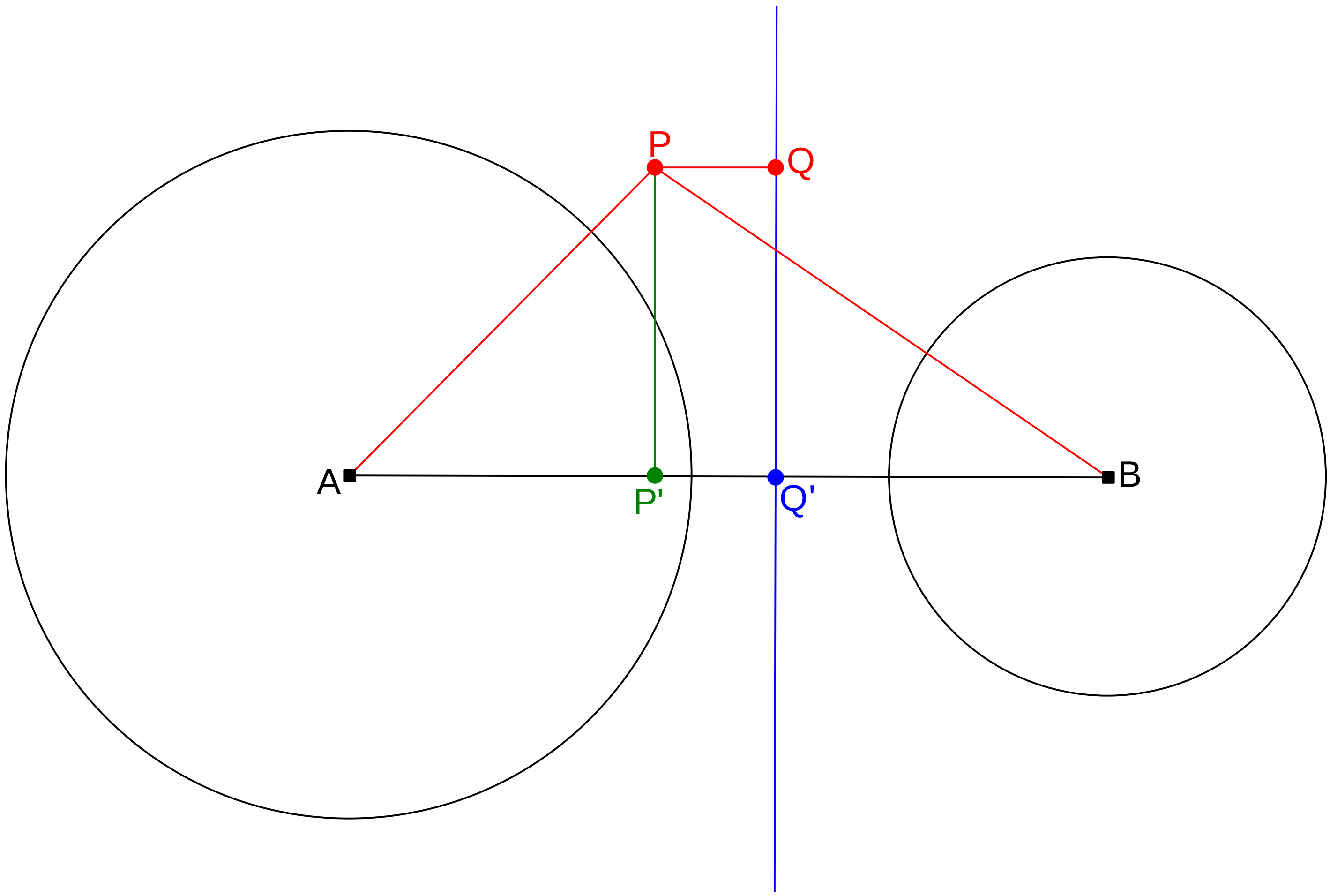

Fig. 2

From $PP' \perp AB$, it follows that $\measuredangle AP'P = 90^\circ$. This means that the triangle $AP'P$ is right-angled at P'. Thus, by the Pythagorean theorem, $PA^2 = P'P^2 + P'A^2$. Consequently,

$$\mathrm{pot}\left(P;\ A\left(x\right)\right) = PA^2 - x^2 = \left(P'P^2 + P'A^2\right) - x^2 = \left(P'P^2 + \overline{P'A}^2\right) - x^2.$$

Similarly, $\operatorname{pot}(P;\ B(y)) = \left(P'P^2 + \overline{P'B}^2\right) - y^2$. This yields

$$\begin{aligned}
&\operatorname{pot}(P;\ A(x)) - \operatorname{pot}(P;\ B(y)) \\
&= \left(\left(P'P^2 + \overline{P'A}^2\right) - x^2\right) - \left(\left(P'P^2 + \overline{P'B}^2\right) - y^2\right) = \left(\overline{P'A}^2 - \overline{P'B}^2\right) - \left(x^2 - y^2\right) \\
&= \left(\overline{P'A}^2 - \overline{P'B}^2\right) - \left(\overline{Q'A}^2 - \overline{Q'B}^2\right) \qquad \left(\text{since } x^2 - y^2 = \overline{Q'A}^2 - \overline{Q'B}^2 \text{ from (3)}\right) \\
&= \left(\overline{P'A} + \overline{P'B}\right) \cdot \left(\overline{P'A} - \overline{P'B}\right) - \left(\overline{Q'A} + \overline{Q'B}\right) \cdot \left(\overline{Q'A} - \overline{Q'B}\right) \\
&= \left(\overline{P'A} + \overline{P'B}\right) \cdot \overline{BA} - \left(\overline{Q'A} + \overline{Q'B}\right) \cdot \overline{BA} \\
&= \left(\left(\overline{P'A} + \overline{P'B}\right) - \left(\overline{Q'A} + \overline{Q'B}\right)\right) \cdot \overline{BA} \\
&= \left(\left(\overline{P'A} - \overline{Q'A}\right) + \left(\overline{P'B} - \overline{Q'B}\right)\right) \cdot \overline{BA} = \left(\overline{P'Q'} + \overline{P'Q'}\right) \cdot \overline{BA} \\
&= 2 \cdot \overline{P'Q'} \cdot \overline{BA} = 2 \cdot \left(-\overline{Q'P'}\right) \cdot \left(-\overline{AB}\right) = 2 \cdot \overline{Q'P'} \cdot \overline{AB} = 2 \cdot \overline{QP} \cdot \overline{AB}.
\end{aligned}$$

Thus, the equation (2) is proven, i. e. the proof of Theorem 1 is complete.

3. Three circles with collinear centers

Now we come to a theorem apparently new, and central to this note (Fig. 3):

Theorem 2. Let g be a line, and let A, B, C be three pairwise distinct points on this line g. Let x, y, z be three numbers. Let

$X = \operatorname{rad}(B(y);\ C(z)) \cap g; \qquad Y = \operatorname{rad}(C(z);\ A(x)) \cap g; \qquad Z = \operatorname{rad}(A(x);\ B(y)) \cap g.$

Let the line g be directed in some way. Then:

a) We have

$$\overline{YZ} = \frac{1}{2}\left(\frac{x^2}{\overline{AB} \cdot \overline{AC}} + \frac{y^2}{\overline{BC} \cdot \overline{BA}} + \frac{z^2}{\overline{CA} \cdot \overline{CB}} - 1\right) \cdot \overline{BC}. \qquad (4)$$

b) We have

$$\frac{\overline{YZ}}{\overline{BC}} = \frac{\overline{ZX}}{\overline{CA}} = \frac{\overline{XY}}{\overline{AB}}.$$

*Proof of Theorem 2 **a)**.* From (1), we have $\operatorname{rad}(A(x);\ B(y)) \perp AB$, thus $\operatorname{rad}(A(x);\ B(y)) \perp g$. In other words, $\operatorname{rad}(A(x);\ B(y)) \perp AZ$. On the other hand, $Z \in \operatorname{rad}(A(x);\ B(y))$.

Thus, the point Z is the orthogonal projection of the point A on the line $\operatorname{rad}(A(x);\ B(y))$. According to the formula (2) of Theorem 1, we thus have $\operatorname{pot}(A;\ A(x)) - \operatorname{pot}(A;\ B(y)) = 2 \cdot \overline{ZA} \cdot \overline{AB}$. Hence,

$$\begin{aligned}
\overline{ZA} &= \frac{\operatorname{pot}(A;\ A(x)) - \operatorname{pot}(A;\ B(y))}{2 \cdot \overline{AB}} = \frac{\left(AA^2 - x^2\right) - \left(AB^2 - y^2\right)}{2 \cdot \overline{AB}} \\
&= \frac{\left(0^2 - x^2\right) - \left(\overline{AB}^2 - y^2\right)}{2 \cdot \overline{AB}} = \frac{y^2 - x^2 - \overline{AB}^2}{2 \cdot \overline{AB}}.
\end{aligned}$$

Similarly,

$$\overline{YA} = \frac{z^2 - x^2 - \overline{AC}^2}{2 \cdot \overline{AC}}.$$

Therefore,

$$\begin{aligned}
\overline{YZ} &= \overline{YA} - \overline{ZA} = \frac{z^2 - x^2 - \overline{AC}^2}{2 \cdot \overline{AC}} - \frac{y^2 - x^2 - \overline{AB}^2}{2 \cdot \overline{AB}} \\
&= \frac{\left(z^2 - x^2 - \overline{AC}^2\right) \cdot \overline{AB} - \left(y^2 - x^2 - \overline{AB}^2\right) \cdot \overline{AC}}{2 \cdot \overline{AB} \cdot \overline{AC}} \\
&= \frac{\left(z^2 \cdot \overline{AB} - x^2 \cdot \overline{AB} - \overline{AC}^2 \cdot \overline{AB}\right) - \left(y^2 \cdot \overline{AC} - x^2 \cdot \overline{AC} - \overline{AB}^2 \cdot \overline{AC}\right)}{2 \cdot \overline{AB} \cdot \overline{AC}} \\
&= \frac{\left(x^2 \cdot \overline{AC} - x^2 \cdot \overline{AB}\right) - y^2 \cdot \overline{AC} + z^2 \cdot \overline{AB} - \left(\overline{AC}^2 \cdot \overline{AB} - \overline{AB}^2 \cdot \overline{AC}\right)}{2 \cdot \overline{AB} \cdot \overline{AC}} \\
&= \frac{x^2 \cdot \left(\overline{AC} - \overline{AB}\right) - y^2 \cdot \overline{AC} + z^2 \cdot \overline{AB} - \overline{AB} \cdot \overline{AC} \cdot \left(\overline{AC} - \overline{AB}\right)}{2 \cdot \overline{AB} \cdot \overline{AC}} \\
&= \frac{x^2 \cdot \overline{BC} - y^2 \cdot \overline{AC} + z^2 \cdot \overline{AB} - \overline{AB} \cdot \overline{AC} \cdot \overline{BC}}{2 \cdot \overline{AB} \cdot \overline{AC}} \\
&= \frac{1}{2} \cdot \frac{x^2 \cdot \overline{BC} - y^2 \cdot \overline{AC} + z^2 \cdot \overline{AB} - \overline{AB} \cdot \overline{AC} \cdot \overline{BC}}{\overline{AB} \cdot \overline{AC} \cdot \overline{BC}} \cdot \overline{BC} \\
&= \frac{1}{2} \cdot \left(\frac{x^2}{\overline{AB} \cdot \overline{AC}} - \frac{y^2}{\overline{BC} \cdot \overline{AB}} + \frac{z^2}{\overline{AC} \cdot \overline{BC}} - 1\right) \cdot \overline{BC} \\
&= \frac{1}{2} \cdot \left(\frac{x^2}{\overline{AB} \cdot \overline{AC}} - \frac{y^2}{\overline{BC} \cdot \left(-\overline{BA}\right)} + \frac{z^2}{\left(-\overline{CA}\right) \cdot \left(-\overline{CB}\right)} - 1\right) \cdot \overline{BC} \\
&= \frac{1}{2} \cdot \left(\frac{x^2}{\overline{AB} \cdot \overline{AC}} + \frac{y^2}{\overline{BC} \cdot \overline{BA}} + \frac{z^2}{\overline{CA} \cdot \overline{CB}} - 1\right) \cdot \overline{BC}.
\end{aligned}$$

This proves Theorem 2 **a)**.

We give two proofs of Theorem 2 **b)** now: one depending on Theorem 2 **a)**, and one not.

First proof of Theorem 2 ***b)****.* As we have already verified Theorem 2 **a)**, we can use it now: From (4), we have

$$\frac{\overline{YZ}}{\overline{BC}} = \frac{1}{2}\left(\frac{x^2}{\overline{AB} \cdot \overline{AC}} + \frac{y^2}{\overline{BC} \cdot \overline{BA}} + \frac{z^2}{\overline{CA} \cdot \overline{CB}} - 1\right).$$

Similarly,

$$\begin{aligned}
\frac{\overline{ZX}}{\overline{CA}} &= \frac{1}{2}\left(\frac{x^2}{\overline{AB} \cdot \overline{AC}} + \frac{y^2}{\overline{BC} \cdot \overline{BA}} + \frac{z^2}{\overline{CA} \cdot \overline{CB}} - 1\right); \\
\frac{\overline{XY}}{\overline{AB}} &= \frac{1}{2}\left(\frac{x^2}{\overline{AB} \cdot \overline{AC}} + \frac{y^2}{\overline{BC} \cdot \overline{BA}} + \frac{z^2}{\overline{CA} \cdot \overline{CB}} - 1\right).
\end{aligned}$$

Thus, $\frac{\overline{YZ}}{\overline{BC}} = \frac{\overline{ZX}}{\overline{CA}} = \frac{\overline{XY}}{\overline{AB}}$. This proves Theorem 2 **b)**.

Second proof of Theorem 2 ***b)****.* After (1), we have $\operatorname{rad}\left(A\left(x\right);\ B\left(y\right)\right) \perp AB$, so that $\operatorname{rad}\left(A\left(x\right);\ B\left(y\right)\right) \perp g$. In other words: $\operatorname{rad}\left(A\left(x\right);\ B\left(y\right)\right) \perp YZ$. On the other hand, $Z \in \operatorname{rad}\left(A\left(x\right);\ B\left(y\right)\right)$.

Thus, the point Z is the orthogonal projection of the point Y on the line $\operatorname{rad}(A(x);\ B(y))$. According to the formula (2) of Theorem 1, we thus have $\operatorname{pot}(Y;\ A(x)) - \operatorname{pot}(Y;\ B(y)) = 2 \cdot \overline{ZY} \cdot \overline{AB}$. By analogy, $\operatorname{pot}(Y;\ C(z)) - \operatorname{pot}(Y;\ B(y)) = 2 \cdot \overline{XY} \cdot \overline{CB}$. But $Y \in \operatorname{rad}(C(z);\ A(x))$ yields $\operatorname{pot}(Y;\ C(z)) = \operatorname{pot}(Y;\ A(x))$. Hence,

$$2 \cdot \overline{XY} \cdot \overline{CB} = \operatorname{pot}(Y;\ C(z)) - \operatorname{pot}(Y;\ B(y)) = \operatorname{pot}(Y;\ A(x)) - \operatorname{pot}(Y;\ B(y)) = 2 \cdot \overline{ZY} \cdot \overline{AB},$$

so that $\overline{XY} \cdot \overline{CB} = \overline{ZY} \cdot \overline{AB}$. Therefore,

$$\overline{XY} \cdot \overline{BC} = \overline{XY} \cdot \left(-\overline{CB}\right) = -\overline{XY} \cdot \overline{CB} = -\overline{ZY} \cdot \overline{AB} = \left(-\overline{ZY}\right) \cdot \overline{AB} = \overline{YZ} \cdot \overline{AB},$$

so that $\dfrac{\overline{XY}}{\overline{AB}} = \dfrac{\overline{YZ}}{\overline{BC}}$. Similarly, $\dfrac{\overline{YZ}}{\overline{BC}} = \dfrac{\overline{ZX}}{\overline{CA}}$. Thus, $\dfrac{\overline{YZ}}{\overline{BC}} = \dfrac{\overline{ZX}}{\overline{CA}} = \dfrac{\overline{XY}}{\overline{AB}}$, and Theorem 2 **b)** is proven once again.

As both Theorems 2 **a)** and 2 **b)** are verified now, the proof of Theorem 2 is complete.

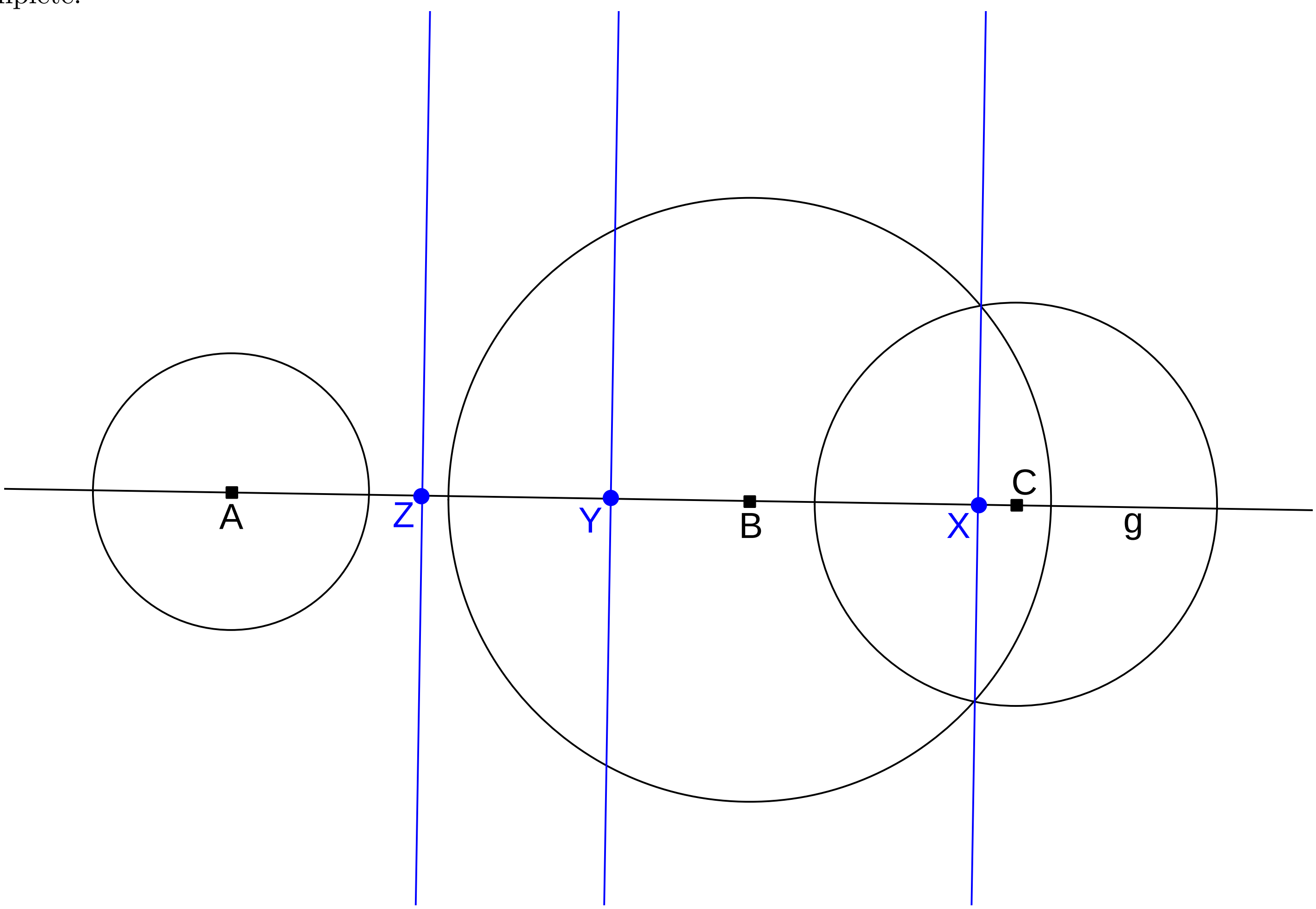

Fig. 3

4. Coaxal circles

Three circles k, m, n (with pairwise distinct centers) are said to be *coaxal* if $\operatorname{rad}(m;\ n) = \operatorname{rad}(n;\ k) = \operatorname{rad}(k;\ m)$ (that is: if the three pairwise radical axes of these three circles coincide).

One can readily show that three circles k, m, n already must be coaxal if at least two of the lines $\operatorname{rad}(m;\ n)$, $\operatorname{rad}(n;\ k)$, $\operatorname{rad}(k;\ m)$ coincide. With the aid of Theorem 2 **a)** we can obtain a more profound criterion for the coaxality of three circles:

> **Theorem 3.** Let A, B, C be three pairwise distinct points. Let x, y, z be three numbers. Then, the following two assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent:
>
> *Assertion $\mathcal{A}_1$:* The circles $A(x)$, $B(y)$, $C(z)$ are coaxal.
>
> *Assertion $\mathcal{A}_2$:* The points A, B, C lie on one line, and if we direct this line[2], then the equation $\dfrac{x^2}{\overline{AB}\cdot\overline{AC}}+\dfrac{y^2}{\overline{BC}\cdot\overline{BA}}+\dfrac{z^2}{\overline{CA}\cdot\overline{CB}}=1$ holds.

Proof of Theorem 3. In order to prove Theorem 3, we have to show that Assertion $\mathcal{A}_1$ implies Assertion $\mathcal{A}_2$, and that Assertion $\mathcal{A}_2$ implies Assertion $\mathcal{A}_1$.

First we will show that Assertion $\mathcal{A}_1$ implies Assertion $\mathcal{A}_2$:

Assume that the Assertion $\mathcal{A}_1$ holds. Then, the circles $A(x)$, $B(y)$, $C(z)$ are coaxal. Thus, $\operatorname{rad}(B(y);\ C(z)) = \operatorname{rad}(C(z);\ A(x)) = \operatorname{rad}(A(x);\ B(y))$. But (1) yields $\operatorname{rad}(B(y);\ C(z)) \perp BC$ and $\operatorname{rad}(C(z);\ A(x)) \perp CA$.

Since $\operatorname{rad}(B(y);\ C(z)) = \operatorname{rad}(C(z);\ A(x))$, the relation $\operatorname{rad}(B(y);\ C(z)) \perp BC$ becomes $\operatorname{rad}(C(z);\ A(x)) \perp BC$. Together with $\operatorname{rad}(C(z);\ A(x)) \perp CA$, this yields $BC \parallel CA$. But the lines BC and CA have a common point (namely, C), and thus can only be parallel if they coincide. Hence, $BC \parallel CA$ yields that the lines BC and CA coincide. In other words, the points A, B, C lie on one line. If we denote this line by g and direct this line, then Theorem 2 **a)** yields

$$\overline{YZ}=\frac{1}{2}\left(\frac{x^2}{\overline{AB}\cdot\overline{AC}}+\frac{y^2}{\overline{BC}\cdot\overline{BA}}+\frac{z^2}{\overline{CA}\cdot\overline{CB}}-1\right)\cdot\overline{BC}, \tag{5}$$

where $Y=\operatorname{rad}(C(z);\ A(x))\cap g$ and $Z=\operatorname{rad}(A(x);\ B(y))\cap g$. Now, $\operatorname{rad}(C(z);\ A(x)) = \operatorname{rad}(A(x);\ B(y))$ leads to $\operatorname{rad}(C(z);\ A(x))\cap g = \operatorname{rad}(A(x);\ B(y))\cap g$, so that $Y=Z$, and thus $\overline{YZ}=0$. Hence, (5) becomes

$$0=\frac{1}{2}\left(\frac{x^2}{\overline{AB}\cdot\overline{AC}}+\frac{y^2}{\overline{BC}\cdot\overline{BA}}+\frac{z^2}{\overline{CA}\cdot\overline{CB}}-1\right)\cdot\overline{BC}.$$

Since $\dfrac{1}{2}\cdot\overline{BC}\neq 0$ (because $\overline{BC}\neq 0$, since the points B and C are distinct), we can divide this equation by $\dfrac{1}{2}\cdot\overline{BC}$, and obtain

$$0=\frac{x^2}{\overline{AB}\cdot\overline{AC}}+\frac{y^2}{\overline{BC}\cdot\overline{BA}}+\frac{z^2}{\overline{CA}\cdot\overline{CB}}-1,$$

so that $\dfrac{x^2}{\overline{AB}\cdot\overline{AC}}+\dfrac{y^2}{\overline{BC}\cdot\overline{BA}}+\dfrac{z^2}{\overline{CA}\cdot\overline{CB}}=1.$

Altogether, we conclude: The points A, B, C lie on one line, and if we direct this line, then we have $\dfrac{x^2}{\overline{AB}\cdot\overline{AC}}+\dfrac{y^2}{\overline{BC}\cdot\overline{BA}}+\dfrac{z^2}{\overline{CA}\cdot\overline{CB}}=1$. Thus, Assertion $\mathcal{A}_2$ is fulfilled.

[2] Hereby it is irrelevant in which of the two possible ways we direct this line - in fact, the values of $\overline{AB}\cdot\overline{AC}$, $\overline{BC}\cdot\overline{BA}$, $\overline{CA}\cdot\overline{CB}$ don't depend on the direction of the line.

Herewith we have shown that Assertion $\mathcal{A}_1$ implies Assertion $\mathcal{A}_2$. Now it only remains to prove that Assertion $\mathcal{A}_2$ implies Assertion $\mathcal{A}_1$. This we will do as follows:

Assume that Assertion $\mathcal{A}_2$ holds. That is, the points A, B, C lie on one line, and we have $\frac{x^2}{\overline{AB}\cdot\overline{AC}}+\frac{y^2}{\overline{BC}\cdot\overline{BA}}+\frac{z^2}{\overline{CA}\cdot\overline{CB}}=1$, where the line through the points A, B, C is directed.

Let g be the line through the points A, B, C. Then, Theorem 2 **a)** yields

$$\overline{YZ}=\frac{1}{2}\left(\frac{x^2}{\overline{AB}\cdot\overline{AC}}+\frac{y^2}{\overline{BC}\cdot\overline{BA}}+\frac{z^2}{\overline{CA}\cdot\overline{CB}}-1\right)\cdot\overline{BC},$$

where $Y=\operatorname{rad}(C(z);\ A(x))\cap g$ and $Z=\operatorname{rad}(A(x);\ B(y))\cap g$. Since $\frac{x^2}{\overline{AB}\cdot\overline{AC}}+\frac{y^2}{\overline{BC}\cdot\overline{BA}}+\frac{z^2}{\overline{CA}\cdot\overline{CB}}=1$, this equation simplifies to $\overline{YZ}=\frac{1}{2}(1-1)\cdot\overline{BC}$, and thus to $\overline{YZ}=0$. Hence, $Y=Z$, so that $\operatorname{perp}(Y;\ g)=\operatorname{perp}(Z;\ g)$.

Now, $\operatorname{rad}(C(z);\ A(x))\perp g$ (in fact, this is equivalent to $\operatorname{rad}(C(z);\ A(x))\perp CA$, what follows from (1)) and $Y\in\operatorname{rad}(C(z);\ A(x))$. Hence, $\operatorname{rad}(C(z);\ A(x))=\operatorname{perp}(Y;\ g)$. Similarly, $\operatorname{rad}(A(x);\ B(y))=\operatorname{perp}(Z;\ g)$. Thus, $\operatorname{perp}(Y;\ g)=\operatorname{perp}(Z;\ g)$ yields $\operatorname{rad}(C(z);\ A(x))=\operatorname{rad}(A(x);\ B(y))$. Similarly, $\operatorname{rad}(B(y);\ C(z))=\operatorname{rad}(C(z);\ A(x))$, and hence $\operatorname{rad}(B(y);\ C(z))=\operatorname{rad}(C(z);\ A(x))=\operatorname{rad}(A(x);\ B(y))$. Thus, the circles $A(x)$, $B(y)$, $C(z)$ are coaxal; i. e., Assertion $\mathcal{A}_1$ is fulfilled. Thus we have shown that Assertion $\mathcal{A}_2$ implies Assertion $\mathcal{A}_1$. This completes our proof of Theorem 3.

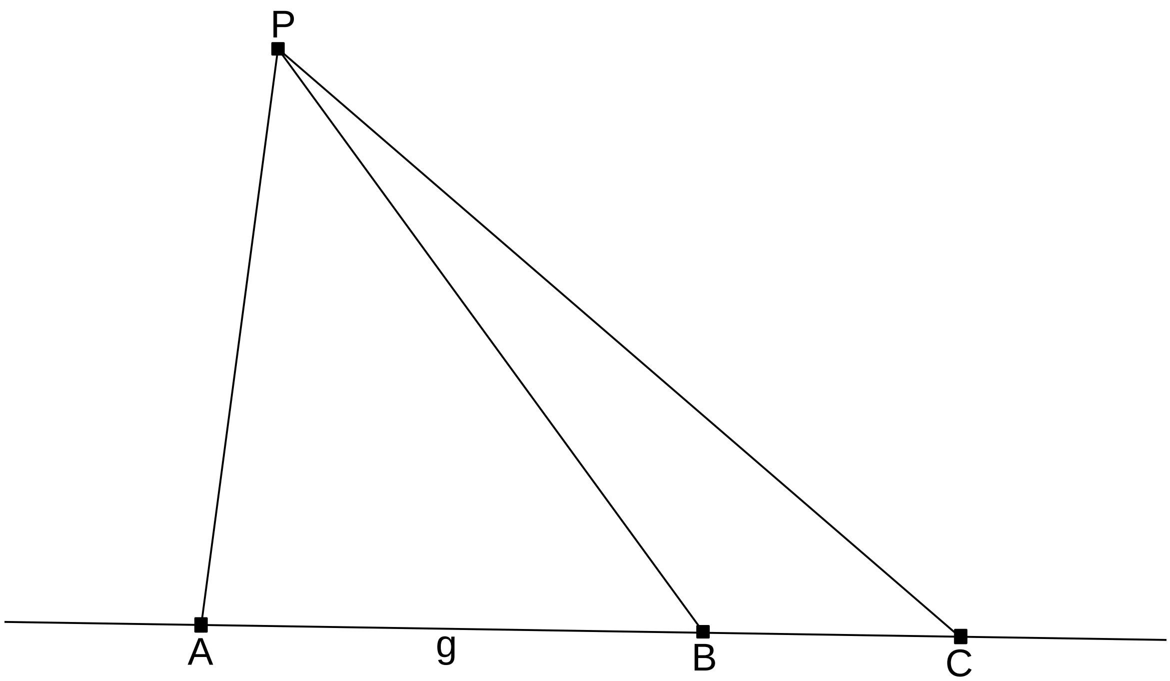

Fig. 4

From Theorem 3 we can deduce the following fact, which is a restatement of the well-known Stewart theorem ([1], §308):

> **Theorem 4.** Let A, B, C be three pairwise distinct points on a line g, and let P be a point. We direct the line g. Then,
>
> $$\frac{AP^2}{\overline{AB}\cdot\overline{AC}}+\frac{BP^2}{\overline{BC}\cdot\overline{BA}}+\frac{CP^2}{\overline{CA}\cdot\overline{CB}}=1.$$

(See Fig. 4.)

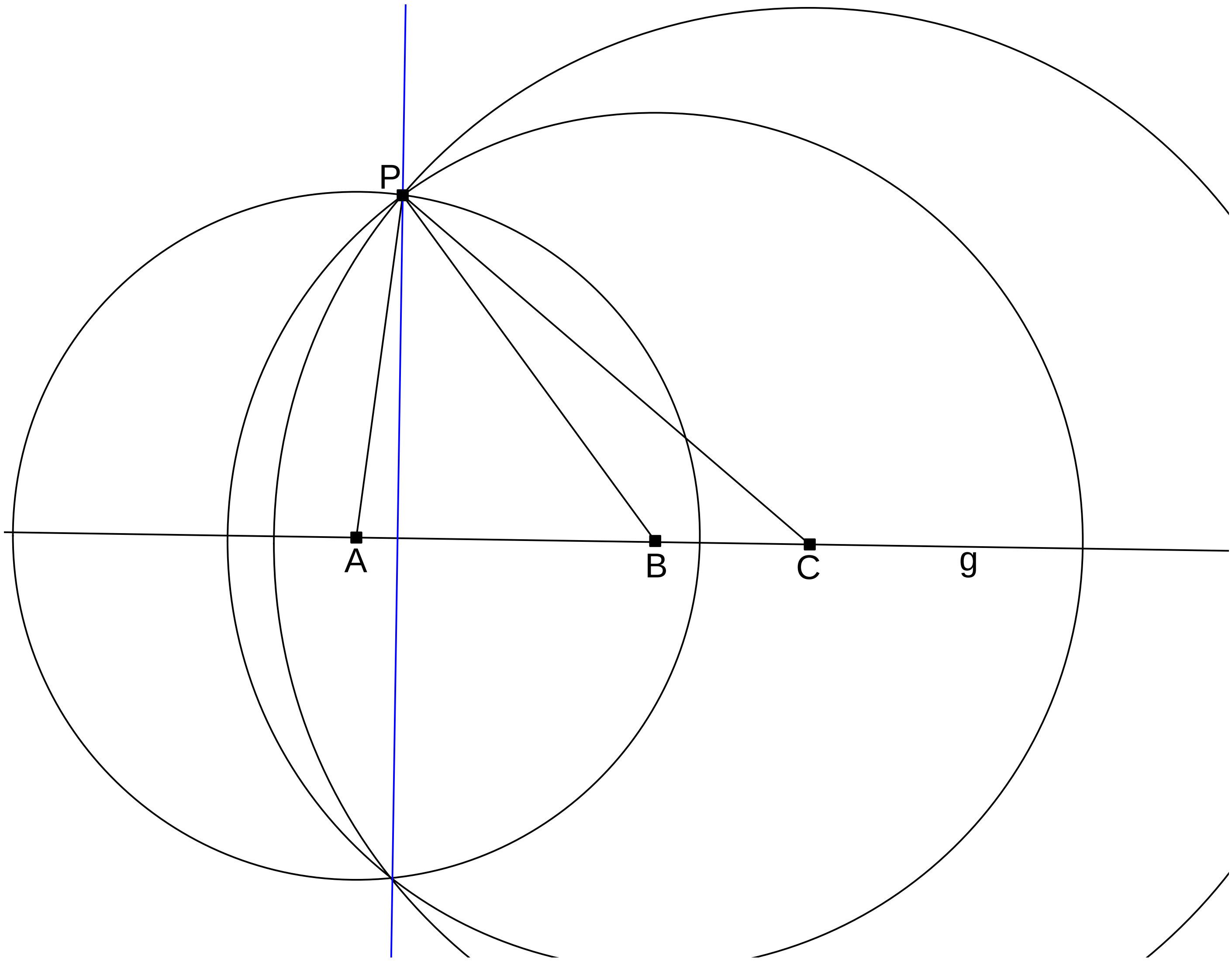

Fig. 5

Proof of Theorem 4. (See Fig. 5.) We set $x = AP$, $y = BP$, $z = CP$.

Then, $\operatorname{pot}(P;\ A(x)) = PA^2 - x^2 = AP^2 - x^2 = AP^2 - AP^2 = 0$ and similarly $\operatorname{pot}(P;\ B(y)) = 0$. Hence, $\operatorname{pot}(P;\ A(x)) = \operatorname{pot}(P;\ B(y))$, so that $P \in \operatorname{rad}(A(x);\ B(y))$. On the other hand, $\operatorname{rad}(A(x);\ B(y)) \perp AB$ (after (1)), and thus $\operatorname{rad}(A(x);\ B(y)) \perp g$.

From $\operatorname{rad}(A(x);\ B(y)) \perp g$ and $P \in \operatorname{rad}(A(x);\ B(y))$, it follows that $\operatorname{rad}(A(x);\ B(y)) = \operatorname{perp}(P;\ g)$. Similarly, $\operatorname{rad}(B(y);\ C(z)) = \operatorname{perp}(P;\ g)$ and $\operatorname{rad}(C(z);\ A(x)) = \operatorname{perp}(P;\ g)$. Thus, $\operatorname{rad}(B(y);\ C(z)) = \operatorname{rad}(C(z);\ A(x)) = \operatorname{rad}(A(x);\ B(y))$. Consequently, the circles $A(x)$, $B(y)$, $C(z)$ are coaxal. Thus, the Assertion $\mathcal{A}_1$ of Theorem 3 is fulfilled. According to Theorem 3, the Assertions $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent. Hence, Assertion $\mathcal{A}_2$ of Theorem 3 must also hold. In particular, we must therefore have $\dfrac{x^2}{\overline{AB} \cdot \overline{AC}} + \dfrac{y^2}{\overline{BC} \cdot \overline{BA}} + \dfrac{z^2}{\overline{CA} \cdot \overline{CB}} = 1$. Since $x = AP$, $y = BP$, $z = CP$, this becomes

$$\frac{AP^2}{\overline{AB} \cdot \overline{AC}} + \frac{BP^2}{\overline{BC} \cdot \overline{BA}} + \frac{CP^2}{\overline{CA} \cdot \overline{CB}} = 1.$$

Thus, Theorem 4 is proven.

5. Inversion and radical axes

Before we come to a less trivial application of Theorem 2, we prepare with a lemma (which, strangely enough, I have nowhere seen explicitely stated):

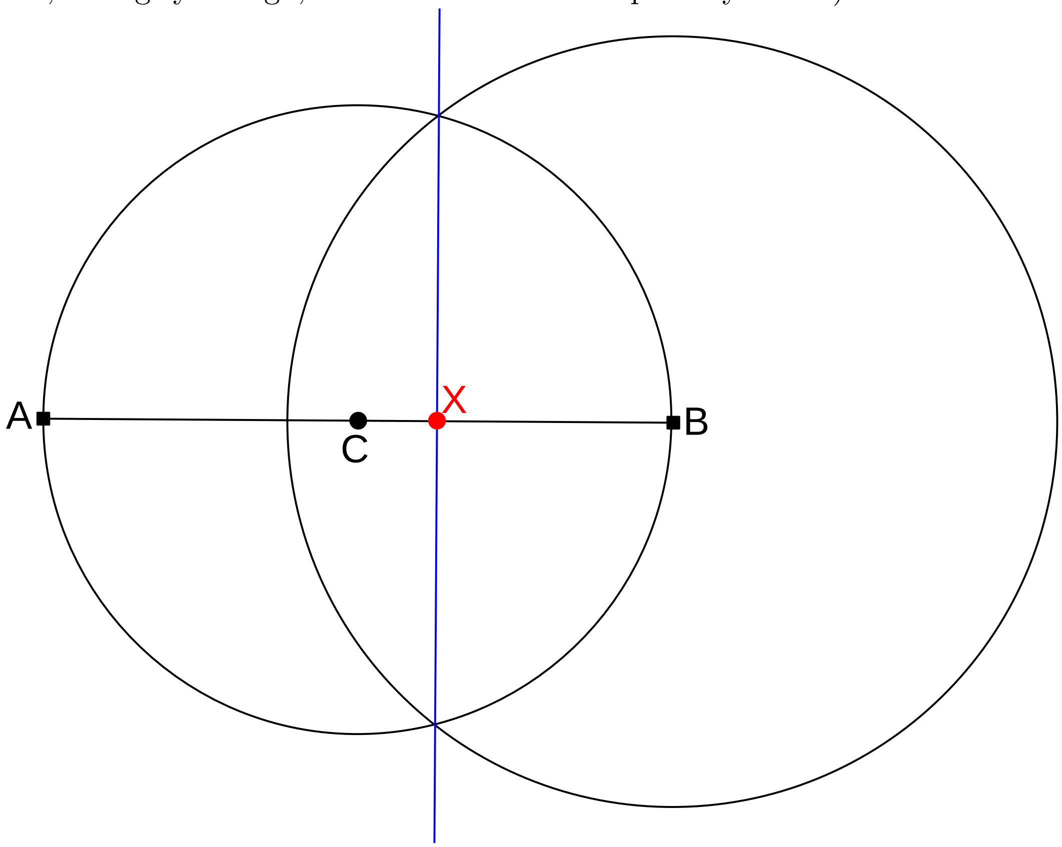

Fig. 6

Theorem 5. Let A and B be two distinct points, and let y be a number. Let C be the midpoint of the segment AB. Let X be the image of the point A under the inversion with respect to the circle $B(y)$. Then, $X \in \operatorname{rad}\left(C\left(\frac{AB}{2}\right);\ B(y)\right)$. [3] (See Fig. 6.)

Proof of Theorem 5. We direct the line BA in some way. Since the point X is the image of the point A under the inversion with respect to the circle $B(y)$, this point X must lie on the line BA and satisfy $\overline{BX} \cdot \overline{BA} = y^2$.

Since the point C is the midpoint of the segment AB, it lies on the line BA and

[3] Of course, the circle $C\left(\frac{AB}{2}\right)$ is the circle with diameter AB.

satisfies $\overline{AC} = \overline{CB} = \dfrac{1}{2} \cdot \overline{AB}$. Hence,

$$\begin{aligned}\operatorname{pot}\left(X;\ C\left(\frac{AB}{2}\right)\right) &= XC^2 - \left(\frac{AB}{2}\right)^2 = XC^2 - \frac{1}{4} \cdot AB^2 = \overline{XC}^2 - \frac{1}{4} \cdot \overline{AB}^2 \\ &= \overline{XC}^2 - \left(\frac{1}{2} \cdot \overline{AB}\right)^2 = \left(\overline{XC} - \frac{1}{2} \cdot \overline{AB}\right) \cdot \left(\overline{XC} + \frac{1}{2} \cdot \overline{AB}\right) \\ &= \left(\overline{XC} - \overline{AC}\right) \cdot \left(\overline{XC} + \overline{CB}\right) = \overline{XA} \cdot \overline{XB}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\operatorname{pot}(X;\ B(y)) &= XB^2 - y^2 = BX^2 - y^2 = \overline{BX}^2 - y^2 = \overline{BX}^2 - \overline{BX} \cdot \overline{BA} = \overline{BX} \cdot \left(\overline{BX} - \overline{BA}\right) \\ &= \overline{BX} \cdot \overline{AX} = \left(-\overline{XB}\right) \cdot \left(-\overline{XA}\right) = \overline{XA} \cdot \overline{XB}.\end{aligned}$$

Thus, $\operatorname{pot}\left(X;\ C\left(\dfrac{AB}{2}\right)\right) = \operatorname{pot}(X;\ B(y))$, so that $X \in \operatorname{rad}\left(C\left(\dfrac{AB}{2}\right);\ B(y)\right)$, and Theorem 5 is proven.

Using Theorems 2 **b)** and 5, we can give a new proof for a property of the radical axis that was published by Dave Wilson in [2] (Fig. 7):

> **Theorem 6.** Let A and B be two distinct points, and let x and y be two numbers. Let X be the image of the point A under the inversion with respect to the circle $B(y)$. Let Y be the image of the point B under the inversion with respect to the circle $A(x)$. Then, the line $\operatorname{rad}(A(x);\ B(y))$ is the perpendicular bisector of the segment XY.

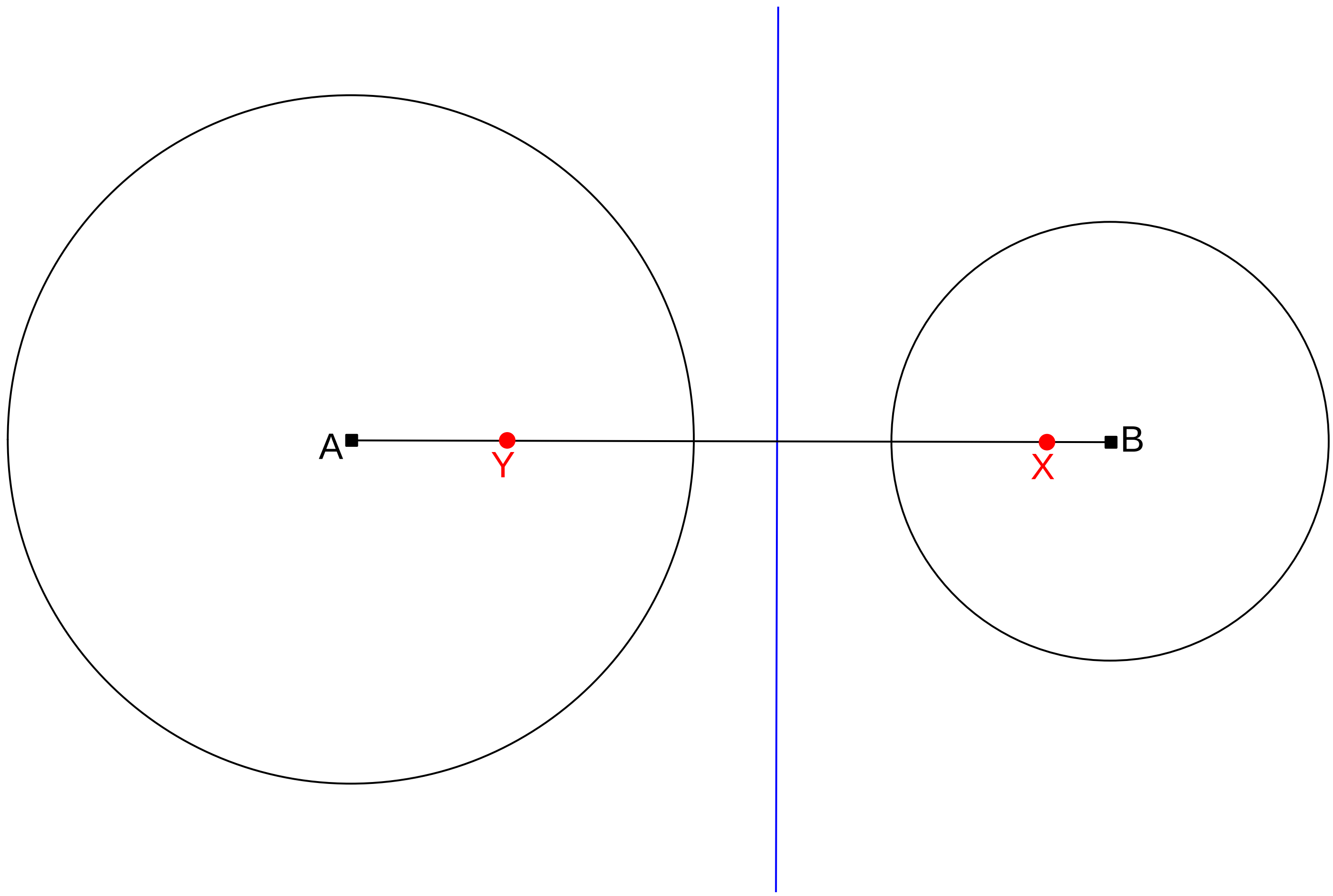

Fig. 7

Proof of Theorem 6. (See Fig. 8.) Let C be the midpoint of the segment AB. Then, Theorem 5 yields $X \in \operatorname{rad}\left(C\left(\frac{AB}{2}\right);\ B(y)\right)$. On the other hand, the point X is the image of the point A under the inversion with respect to the circle $B(y)$, and thus lies on the line AB. Therefore, $X = \operatorname{rad}\left(C\left(\frac{AB}{2}\right);\ B(y)\right) \cap AB$. Similarly, $Y = \operatorname{rad}\left(C\left(\frac{AB}{2}\right);\ A(x)\right) \cap AB$.

Now we denote the line AB by g. Then, obviously, $A \in g$ and $B \in g$, and therefore also $C \in g$ (since C is the midpoint of the segment AB).

Besides, we set $z = \frac{AB}{2}$. Then, $X = \operatorname{rad}\left(C\left(\frac{AB}{2}\right);\ B(y)\right) \cap AB$ becomes $X = \operatorname{rad}(C(z);\ B(y)) \cap g$, hence $X = \operatorname{rad}(B(y);\ C(z)) \cap g$. Also, $Y = \operatorname{rad}\left(C\left(\frac{AB}{2}\right);\ A(x)\right) \cap AB$ becomes $Y = \operatorname{rad}(C(z);\ A(x)) \cap g$.

Now let $Z = \operatorname{rad}(A(x);\ B(y)) \cap g$. Furthermore, let us direct the line g. Then, our line g, our points A, B, C, our numbers x, y, z and our points X, Y, Z satisfy all conditions of Theorem 2; hence, we can apply Theorem 2 **b)** and obtain $\frac{\overline{YZ}}{\overline{BC}} = \frac{\overline{ZX}}{\overline{CA}} = \frac{\overline{XY}}{\overline{AB}}$.

Particularly, we thus have $\frac{\overline{YZ}}{\overline{BC}} = \frac{\overline{ZX}}{\overline{CA}}$. But $\overline{BC} = \overline{CA}$ (since C is the midpoint of AB). Hence, $\overline{YZ} = \overline{ZX}$. In other words, the point Z is the midpoint of the segment XY.

According to (1), we have $\operatorname{rad}(A(x);\ B(y)) \perp AB$. In other words, $\operatorname{rad}(A(x);\ B(y)) \perp g$, or, equivalently, $\operatorname{rad}(A(x);\ B(y)) \perp XY$. Since $Z \in \operatorname{rad}(A(x);\ B(y))$, we thus have $\operatorname{rad}(A(x);\ B(y)) = \operatorname{perp}(Z;\ XY)$.

But since Z is the midpoint of the segment XY, the line $\operatorname{perp}(Z;\ XY)$ is the perpendicular bisector of the segment XY. Hence, the equality $\operatorname{rad}(A(x);\ B(y)) = \operatorname{perp}(Z;\ XY)$ means that the line $\operatorname{rad}(A(x);\ B(y))$ is the perpendicular bisector of the segment XY. This proves Theorem 6.

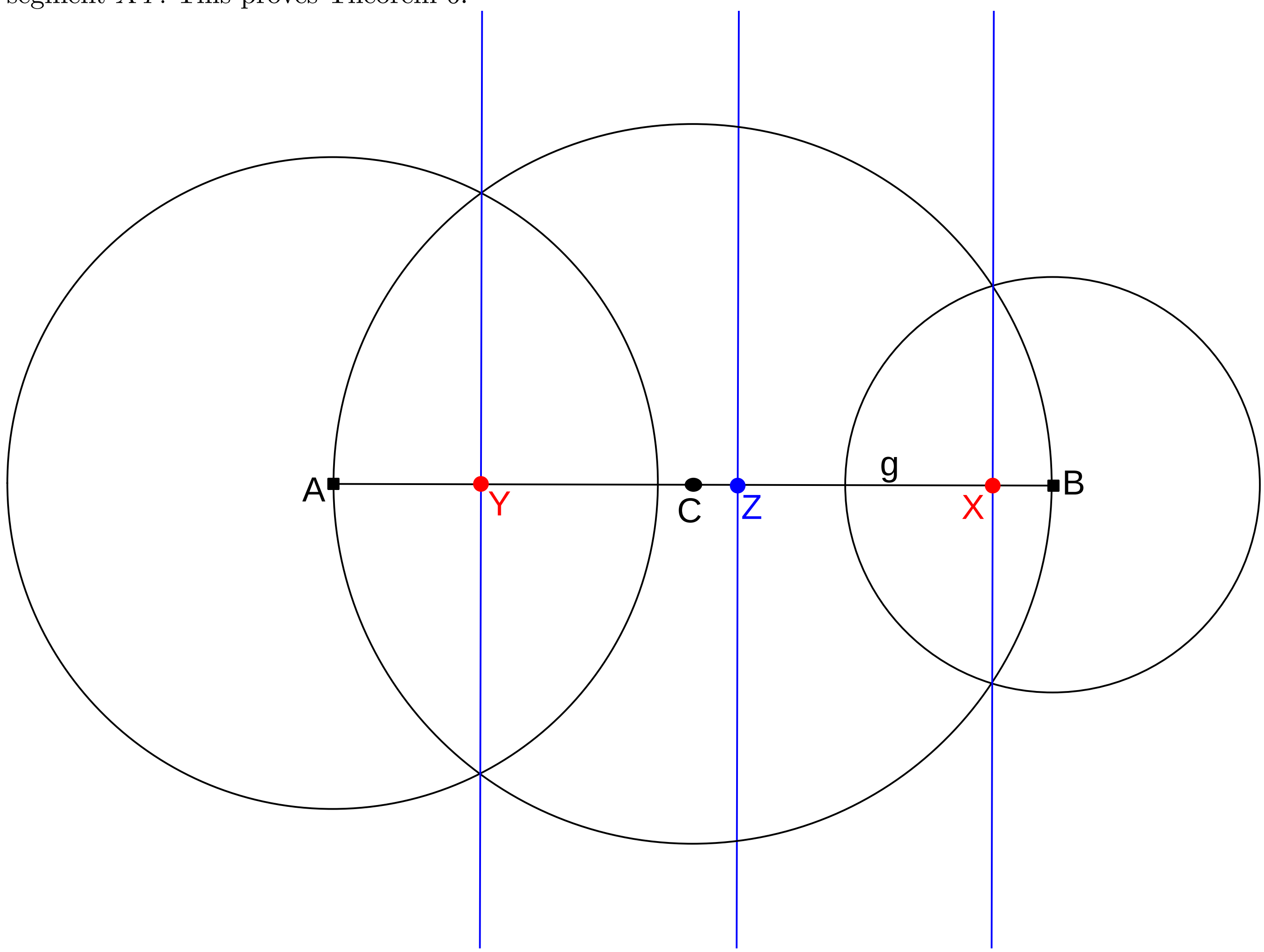

Fig. 8

6. On the center of the Taylor circle

The Theorem 6 proven above allows for some surprising applications. One of these is obvious - a rather simple construction of the radical axis of two circles suitable for a dynamical geometry macro. Another one, which we are going to elaborate on, concerns the center of the Taylor circle of a triangle. We start with a known result from triangle geometry:

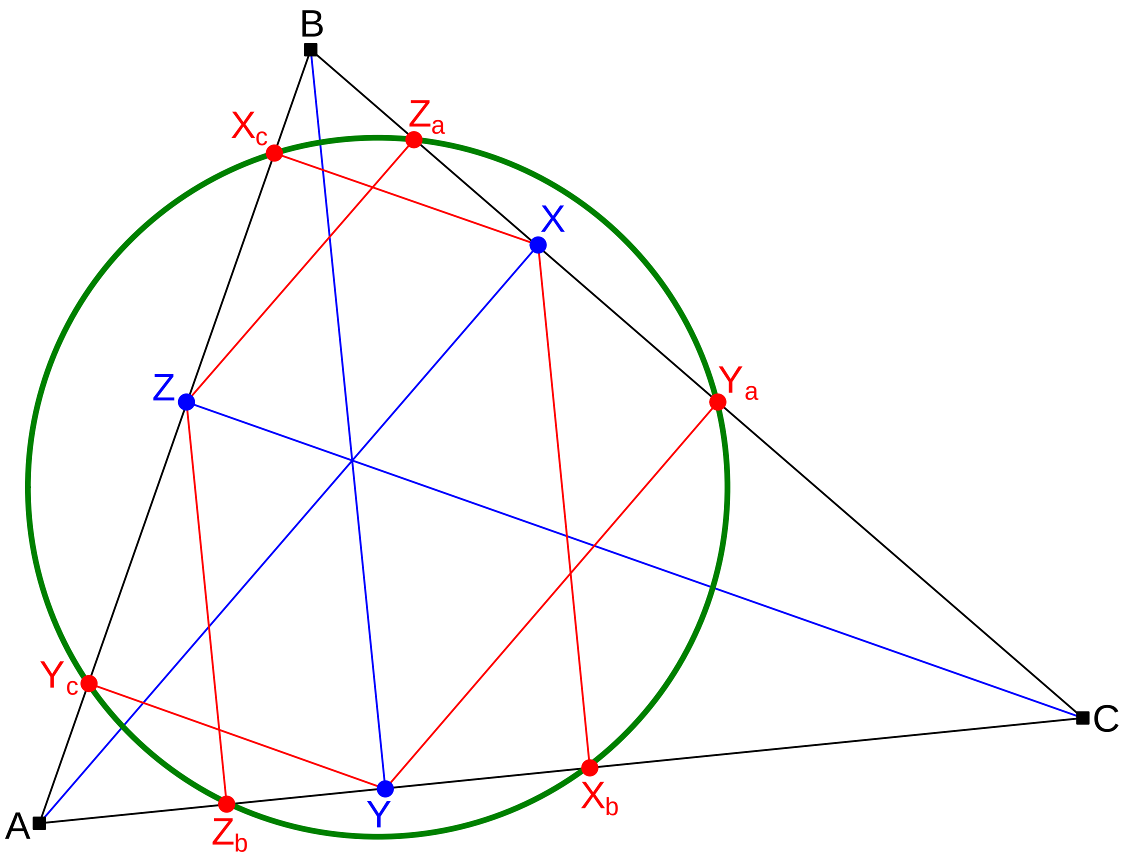

Fig. 9

Theorem 7. Let ABC be a triangle, and let X, Y, Z be the feet of the altitudes of this triangle issuing from A, B, C, respectively.

Let X_b and X_c be the orthogonal projections of the point X on the lines CA and AB.

Let Y_c and Y_a be the orthogonal projections of the point Y on the lines AB and BC.

Let Z_a and Z_b be the orthogonal projections of the point Z on the lines BC and CA.

Then, the points X_b, X_c, Y_c, Y_a, Z_a, Z_b lie on one circle.

This circle is called the *Taylor circle* of triangle ABC. (See Fig. 9.)

We will not prove this theorem here (the reader is referred to [1], §689 or [4], Chapter 9, §6 for the proof - which can, by the way, also be done by straightforward angle chasing). What we are going to do is establishing a property of the center of the Taylor circle - but first we recall a definition:

A known fact states that if k, m, n are three circles with pairwise distinct centers, then the radical axes $\operatorname{rad}(m;\ n)$, $\operatorname{rad}(n;\ k)$, $\operatorname{rad}(k;\ m)$ concur at one point (which can happen to be an infinite point). This point is called the *radical center of the three circles* k, m, n.

Now we can formulate the fact that we are going to prove (Fig. 10):

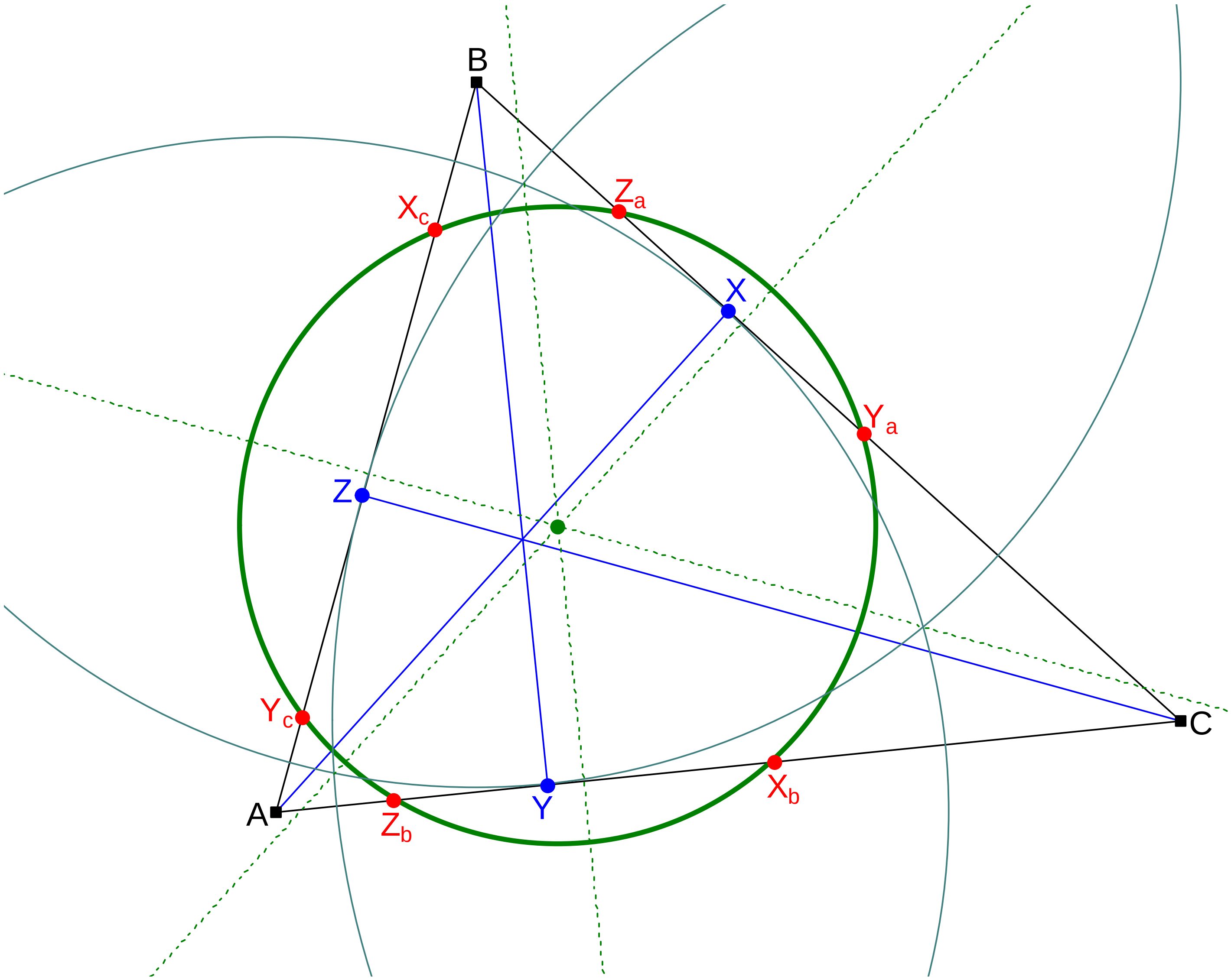

Fig. 10

Theorem 8. Let ABC be a triangle, and let X, Y, Z be the feet of the altitudes of this triangle issuing from A, B, C, respectively.

a) The circle $A\left(AX\right)$ touches the line BC at the point X.

b) The radical center of the circles $A\left(AX\right)$, $B\left(BY\right)$, $C\left(CZ\right)$ is the center of the Taylor circle of triangle ABC.

Proof of Theorem 8. First, $BC \perp AX$ (since AX is an altitude of triangle ABC).

The point X lies on the circle $A\left(AX\right)$ (since $AX = AX$). The tangent to the circle $A\left(AX\right)$ at the point X is, obviously, the perpendicular to the line AX through the point X. But the perpendicular to the line AX through the point X is the line BC (since $X \in BC$ and $BC \perp AX$). Thus, the tangent to the circle $A\left(AX\right)$ at the point X is the line BC. This means that the circle $A\left(AX\right)$ touches the line BC at the point X. This proves Theorem 8 **a)**.

The interesting part is proving Theorem 8 **b)**: (See Fig. 11.) Let the points X_b, X_c, Y_c, Y_a, Z_a, Z_b be defined as in Theorem 7.

Let T be the center of the Taylor circle of triangle ABC. Then, $TX_b = TZ_b$, since the points X_b and Z_b lie on the Taylor circle of triangle ABC. Thus, the point T lies on the perpendicular bisector of the segment X_bZ_b.

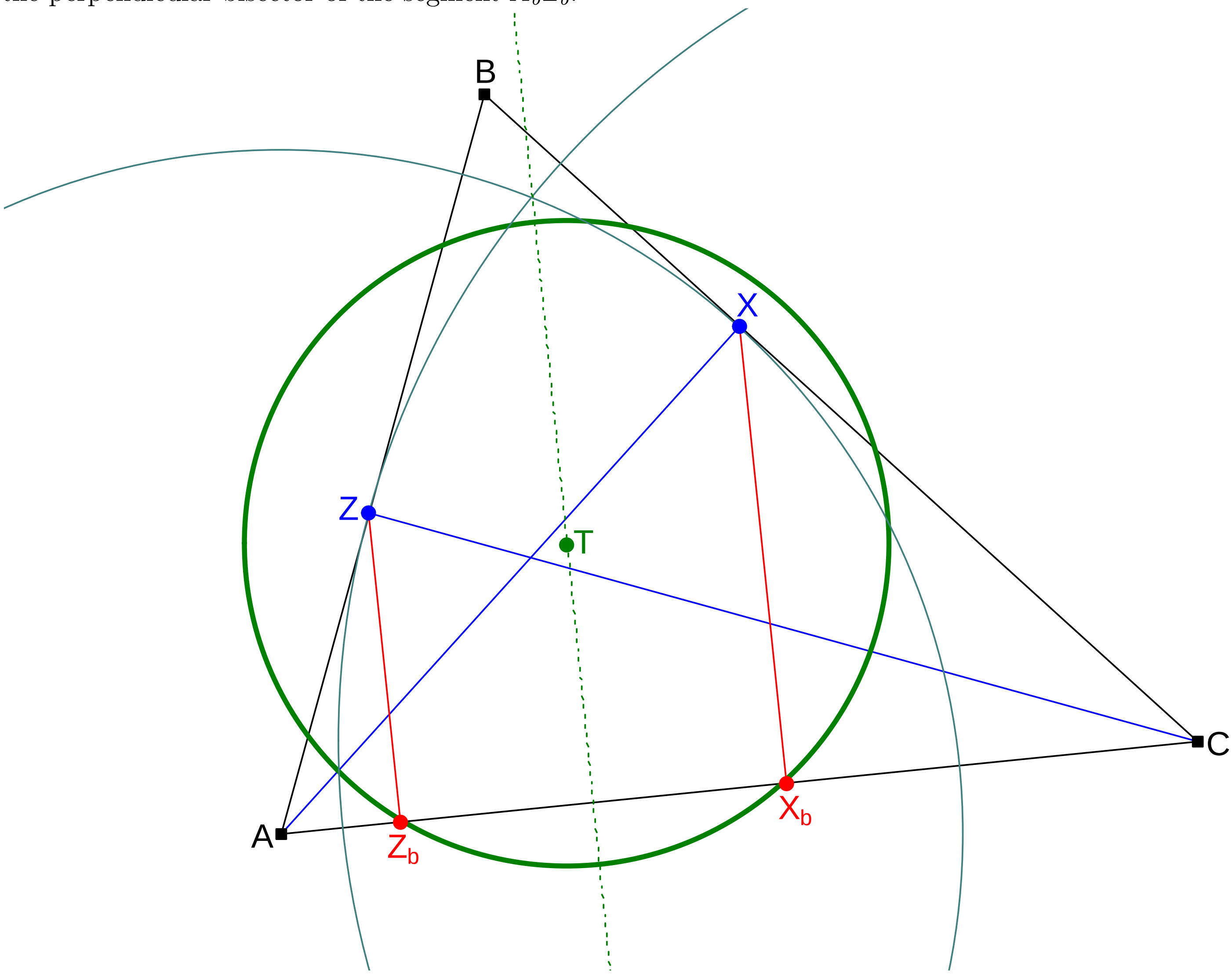

Fig. 11

We direct the line CA in some way.

We have $\measuredangle AX_bX = 90^\circ$ and $\measuredangle AXC = 90^\circ$, so that $\measuredangle AX_bX = \measuredangle AXC$. Besides, it is obvious that $\measuredangle X_bAX = \measuredangle XAC$. Thus, the triangles AX_bX and AXC are similar. Hence, $AX_b : AX = AX : AC$, what rewrites as $AX_b \cdot AC = AX^2$.

So we now know that the point X_b lies on the ray AC and satisfies $AX_b \cdot AC = AX^2$. Hence, the point X_b is the image of the point C under the inversion with respect to the circle $A(AX)$. Similarly, the point Z_b is the image of the point A under the inversion with respect to the circle $C(CZ)$. Hence, according to Theorem 6, the line $\operatorname{rad}(C(CZ);\ A(AX))$ is the perpendicular bisector of the segment X_bZ_b. Since the point T lies on the perpendicular bisector of the segment X_bZ_b, we thus obtain that the point T lies on the line $\operatorname{rad}(C(CZ);\ A(AX))$. Similarly, we can show that the point T lies on the lines $\operatorname{rad}(A(AX);\ B(BY))$ and $\operatorname{rad}(B(BY);\ C(CZ))$. Hence, the point T is the point of intersection of the lines $\operatorname{rad}(B(BY);\ C(CZ))$, $\operatorname{rad}(C(CZ);\ A(AX))$, $\operatorname{rad}(A(AX);\ B(BY))$. This yields that T is the radical center of the circles $A(AX)$,

$B(BY)$, $C(CZ)$. But since the point T was defined as the center of the Taylor circle of triangle ABC, we can conclude that the center of the Taylor circle of triangle ABC is the radical center of the circles $A(AX)$, $B(BY)$, $C(CZ)$. We have thus proven Theorem 8 **b)**, so that the proof of Theorem 8 is complete.

Theorem 8 **b)** was proven trigonometrically by myself in [3]. I don't know in how far it had been known before: The only reference I have is [5], where Stärk mentions - in passing and without proof - that the radical center of the circles $A(AX)$, $B(BY)$, $C(CZ)$ lies on the Brocard axis of triangle ABC and specifies its position on that axis more precisely. The Taylor circle appears nowhere in his result, but with some further knowledge of its properties one could see that his specification of the position of the radical center is equivalent to it being the center of the Taylor circle, i. e. to our Theorem 8 **b)**.

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[1] Nathan Altshiller-Court, *College Geometry*, second edition, New York 1952 (republication: 2007 by Dover Publications).

[2] Dave Wilson, *Radical Axis - A Generic Construction*, geometry-college posting: 12 December 2003.
`http://mathforum.org/kb/message.jspa?messageID=1072322&tstart=0`

[3] Darij Grinberg, *On the Taylor center of a triangle.*
`http://www.stud.uni-muenchen.de/~darij.grinberg/`

[4] Ross Honsberger, *Episodes in Nineteenth and Twentieth Century Euclidean Geometry*, Washington 1995 (New Mathematical Library #37, published by the Mathematical Association of America).

[5] Roland Stärk, *Ein Verfahren, Punkte der Tuckergeraden eines Dreiecks zu konstruieren*, Praxis der Mathematik 5/1992, pp. 213-215.