On the square of the antipode in a connected filtered Hopf algebra [detailed version]

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Abstract. It is well-known that the antipode S of a commutative or cocommutative Hopf algebra satisfies $S^2 = \mathrm{id}$ (where $S^2 = S \circ S$). Recently, similar results have been obtained by Aguiar, Lauve and Mahajan for connected graded Hopf algebras: Namely, if H is a connected graded Hopf algebra with grading $H = \bigoplus_{n \geq 0} H_n$, then each positive integer n satisfies $\left(\mathrm{id} - S^2\right)^n (H_n) = 0$ and (even stronger) $\left((\mathrm{id} + S) \circ \left(\mathrm{id} - S^2\right)^{n-1}\right)(H_n) = 0$. For some specific H's such as the Malvenuto–Reutenauer Hopf algebra FQSym, the exponents can be lowered.

In this note, we generalize these results in several directions: We replace the base field by a commutative ring, replace the Hopf algebra by a coalgebra (actually, a slightly more general object, with no coassociativity required), and replace both id and S^2 by "coalgebra homomorphisms" (of sorts). Specializing back to connected graded Hopf algebras, we show that the exponent n in the identity $\left(\mathrm{id} - S^2\right)^n (H_n) = 0$ can be lowered to $n - 1$ (for $n > 1$) if and only if $\left(\mathrm{id} - S^2\right)(H_2) = 0$. (A sufficient condition for this is that every pair of elements of H_1 commutes; this is satisfied, e.g., for FQSym.)

Keywords: Hopf algebra, antipode, connected graded Hopf algebra, combinatorial Hopf algebra.

MSC subject classification: 16T05, 16T30.

Consider, for simplicity, a connected graded Hopf algebra H over a field (we will soon switch to more general settings). Let S be the antipode of H. A classical result (e.g., [Sweedl69, Proposition 4.0.1 6)] or [HaGuKi10, Corollary 3.3.11] or [Abe80, Theorem 2.1.4 (vi)] or [Radfor12, Corollary 7.1.11]) says that $S^2 = \mathrm{id}$ when H is commutative or cocommutative. (Here and in the following, powers

are composition powers; thus, S^2 means $S \circ S$.) In general, $S^2 = \mathrm{id}$ need not hold. However, in [AguLau14, Proposition 7], Aguiar and Lauve showed that S^2 is still locally unipotent, and more precisely, we have

$$\left(\mathrm{id} - S^2\right)^n (H_n) = 0 \qquad \text{for each } n > 0,$$

where H_n denotes the n-th graded component of H. Later, Aguiar and Mahajan [AguMah17, Lemma 12.50] strengthened this equality to

$$\left((\mathrm{id} + S) \circ \left(\mathrm{id} - S^2\right)^{n-1}\right)(H_n) = 0 \qquad \text{for each } n > 0.$$

For specific combinatorially interesting Hopf algebras, even stronger results hold; in particular,

$$\left(\mathrm{id} - S^2\right)^{n-1} (H_n) = 0 \qquad \text{holds for each } n > 1$$

when H is the Malvenuto–Reutenauer Hopf algebra (see [AguLau14, Example 8]).

In this note, we will unify these results and transport them to a much more general setting. First of all, the ground field will be replaced by an arbitrary commutative ring; this generalization is not unexpected, but renders the proof strategy of [AguLau14, Proposition 7] insufficient[1]. Second, we will replace the Hopf algebra by a coalgebra, or rather by a more general structure that does not even require coassociativity. The squared antipode S^2 will be replaced by an arbitrary "coalgebra" endomorphism f (we are using scare quotes because our structure is not really a coalgebra), and the identity map by another such endomorphism e. Finally, the graded components will be replaced by an arbitrary sequence of submodules satisfying certain compatibility relations. We state the general result in Section 2.1 and prove it in Section 3.1. In Sections 2.2–2.4, we progressively specialize this result: first to connected filtered coalgebras with coalgebra endomorphisms (in Section 2.2), then to connected filtered Hopf algebras with S^2 (in Section 2.3), and finally to connected graded Hopf algebras with S^2 (in Section 2.4). The latter specialization covers the results of Aguiar and Lauve. (The Malvenuto–Reutenauer Hopf algebra turns out to be a red herring; any connected graded Hopf algebra H with the property that $ab = ba$ for all $a, b \in H_1$ will do.)

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[1] In fact, the proof in [AguLau14, Proposition 7] relies on the coradical filtration of H and its associated graded structure $\operatorname{gr} H$. If the base ring is a field, then $\operatorname{gr} H$ is a well-defined commutative Hopf algebra (see, e.g., [AguLau14, Lemma 1]), and thus the antipode of H can be viewed as a "deformation" of the antipode of $\operatorname{gr} H$. But the latter antipode does square to id because $\operatorname{gr} H$ is commutative. Unfortunately, this proof does not survive our generalization; in fact, even defining a Hopf algebra structure on $\operatorname{gr} H$ would likely require at least some flatness assumptions.

Remark on alternative versions

This is the detailed version of the present note. The regular version (with proofs in less detail) is available at

`http://www.cip.ifi.lmu.de/~grinberg/algebra/antipode-squared.pdf`

1. Notations

We will use the notions of coalgebras, bialgebras and Hopf algebras over a commutative ring, as defined (e.g.) in [Abe80, Chapter 2], [GriRei20, Chapter 1], [HaGuKi10, Chapters 2, 3], [Radfor12, Chapters 2, 5, 7] or [Sweedl69, Chapters I–IV]. (In particular, our Hopf algebras are **not** twisted by a $\mathbb{Z}/2$-grading as the topologists' ones are.) We use the same notations for Hopf algebras as in [GriRei20, Chapter 1]. In particular:

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- "Rings" and "algebras" are always required to be associative and have a unity.
- We fix a commutative ring $\mathbf{k}$. The symbols "$\otimes$" and "End" shall always mean "$\otimes_{\mathbf{k}}$" and "$\operatorname{End}_{\mathbf{k}}$", respectively. The unity of the ring $\mathbf{k}$ will be called $1_{\mathbf{k}}$ or just 1 if confusion is unlikely.
- The comultiplication and the counit of a $\mathbf{k}$-coalgebra are denoted by Δ and ϵ.
- "Graded" $\mathbf{k}$-modules mean $\mathbb{N}$-graded $\mathbf{k}$-modules. The base ring $\mathbf{k}$ itself is not supposed to have any nontrivial grading.
- The n-th graded component of a graded $\mathbf{k}$-module V will be called V_n. If $n < 0$, then this is the zero submodule 0.
- A graded $\mathbf{k}$-Hopf algebra means a $\mathbf{k}$-Hopf algebra that has a grading as a $\mathbf{k}$-module, and whose structure maps (multiplication, unit, comultiplication and counit) are graded maps. (The antipode is automatically graded in this case, by [GriRei20, Exercise 1.4.29 (e)].)
- If f is a map from a set to itself, and if $k \in \mathbb{N}$ is arbitrary, then f^k shall denote the map $\underbrace{f \circ f \circ \cdots \circ f}_{k \text{ times}}$. (Thus, $f^1 = f$ and $f^0 = \mathrm{id}$.)

2. Theorems

2.1. The main theorem

We can now state the main result of this note:

Theorem 2.1. Let D be a $\mathbf{k}$-module, and let $(D_1, D_2, D_3, \ldots)$ be a sequence of $\mathbf{k}$-submodules of D. Let $\delta : D \to D \otimes D$ be any $\mathbf{k}$-linear map.

Let $e : D \to D$ and $f : D \to D$ be two $\mathbf{k}$-linear maps such that

$$\operatorname{Ker} \delta \subseteq \operatorname{Ker} (e - f) \qquad \text{and} \tag{1}$$

$$(f \otimes f) \circ \delta = \delta \circ f \qquad \text{and} \tag{2}$$

$$(e \otimes e) \circ \delta = \delta \circ e \qquad \text{and} \tag{3}$$

$$f \circ e = e \circ f. \tag{4}$$

Let p be a positive integer such that

$$(e - f) \left(D_1 + D_2 + \cdots + D_p \right) = 0. \tag{5}$$

Assume furthermore that

$$\delta (D_n) \subseteq \sum_{i=1}^{n-1} D_i \otimes D_{n-i} \qquad \text{for each } n > p. \tag{6}$$

(Here, the "$D_i \otimes D_{n-i}$" on the right hand side means the image of $D_i \otimes D_{n-i}$ under the canonical map $D_i \otimes D_{n-i} \to D \otimes D$ that is obtained by tensoring the two inclusion maps $D_i \to D$ and $D_{n-i} \to D$ together. When $\mathbf{k}$ is not a field, this canonical map may fail to be injective.)

Then, for any integer $u > p$, we have

$$(e - f)^{u-p} (D_u) \subseteq \operatorname{Ker} \delta \tag{7}$$

and

$$(e - f)^{u-p+1} (D_u) = 0. \tag{8}$$

As the statement of this theorem is not very intuitive, some explanations are in order. The reader may think of the D in Theorem 2.1 as a "pre-coalgebra", with δ being its "reduced coproduct". Indeed, the easiest way to obtain a nontrivial example is to fix a connected graded Hopf algebra H, then define D to be either H or the "positive part" of H (that is, the submodule $\bigoplus\limits_{n>0} H_n$ of H), and define δ to be the map $x \mapsto \Delta (x) - x \otimes 1 - 1 \otimes x + \epsilon (x) 1 \otimes 1$ (the so-called *reduced coproduct* of H). From this point of view, $\operatorname{Ker} \delta$ can be regarded as the set of "primitive" elements of D. The maps f and e can be viewed as two commuting "coalgebra endomorphisms" of D (indeed, the assumptions (2) and (3) are essentially saying that f and e preserve the "reduced coproduct" δ). The submodules $D_1, D_2, D_3, \ldots$ are an analogue of the (positive-degree) graded components of D, while the assumption (6) says that the "reduced coproduct" δ "respects the grading" (as is indeed the case for connected graded Hopf algebras).

We stress that p is allowed to be 1 in Theorem 2.1; in this case, the assumption (5) simplifies to $(e - f) (0) = 0$, which is automatically true by the linearity of $e - f$.

We shall prove Theorem 2.1 in Section 3.1. First, however, let us explore its consequences for coalgebras and Hopf algebras, recovering in particular the results of Aguiar and Lauve promised in the introduction.

2.2. Connected filtered coalgebras

We begin by specializing Theorem 2.1 to the setting of a connected filtered coalgebra. There are several ways to define what a filtered coalgebra is; ours is probably the most liberal:

Definition 2.2. A *filtered* $\mathbf{k}$*-coalgebra* means a $\mathbf{k}$-coalgebra C equipped with an infinite sequence $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$ of $\mathbf{k}$-submodules of C satisfying the following three conditions:

- We have
$$C_{\leq 0} \subseteq C_{\leq 1} \subseteq C_{\leq 2} \subseteq \cdots . \tag{9}$$
- We have
$$\bigcup_{n \in \mathbb{N}} C_{\leq n} = C. \tag{10}$$
- We have
$$\Delta\left(C_{\leq n}\right) \subseteq \sum_{i=0}^{n} C_{\leq i} \otimes C_{\leq n-i} \qquad \text{for each } n \in \mathbb{N}. \tag{11}$$
(Here, the "$C_{\leq i} \otimes C_{\leq n-i}$" on the right hand side means the image of $C_{\leq i} \otimes C_{\leq n-i}$ under the canonical map $C_{\leq i} \otimes C_{\leq n-i} \to C \otimes C$ that is obtained by tensoring the two inclusion maps $C_{\leq i} \to C$ and $C_{\leq n-i} \to C$ together. When $\mathbf{k}$ is not a field, this canonical map may fail to be injective.)

The sequence $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$ is called the *filtration* of the filtered $\mathbf{k}$-coalgebra C.

A more categorically-minded person might replace the condition $\Delta\left(C_{\leq n}\right) \subseteq \sum_{i=0}^{n} C_{\leq i} \otimes C_{\leq n-i}$ in this definition by a stronger requirement (e.g., asking Δ to factor through a linear map $C_{\leq n} \to \bigoplus_{i=0}^{n} C_{\leq i} \otimes C_{\leq n-i}$, where the "$\otimes$" signs now signify the actual tensor products rather than their images in $C \otimes C$). However, we have no need for such stronger requirements. Mercifully, all reasonable definitions of filtered $\mathbf{k}$-coalgebras agree when $\mathbf{k}$ is a field.

The condition (10) in Definition 2.2 shall never be used in the following; we merely state it to avoid muddling the meaning of "filtered $\mathbf{k}$-coalgebra".

A graded $\mathbf{k}$-coalgebra C automatically becomes a filtered $\mathbf{k}$-coalgebra; indeed,

we can define its filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$ by setting

$$C_{\leq n} = \bigoplus_{i=0}^{n} C_i \qquad \text{for all } n \in \mathbb{N}.$$

Definition 2.3. Let C be a filtered $\mathbf{k}$-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$. Let $1_{\mathbf{k}}$ denote the unity of the ring $\mathbf{k}$.

(a) The filtered $\mathbf{k}$-coalgebra C is said to be *connected* if the restriction $\epsilon\mid_{C_{\leq 0}}$ is a $\mathbf{k}$-module isomorphism from $C_{\leq 0}$ to $\mathbf{k}$.

(b) In this case, the element $\left(\epsilon\mid_{C_{\leq 0}}\right)^{-1}(1_{\mathbf{k}}) \in C_{\leq 0}$ is called the *unity* of C and is denoted by 1_C.

Now, assume that C is a connected filtered $\mathbf{k}$-coalgebra.

(c) An element x of C is said to be *primitive* if $\Delta(x) = x \otimes 1_C + 1_C \otimes x$.

(d) The set of all primitive elements of C is denoted by $\operatorname{Prim} C$.

These notions of "connected", "unity" and "primitive" specialize to the commonly established concepts of these names when C is a graded $\mathbf{k}$-bialgebra. Indeed, Definition 2.3 **(b)** defines the unity 1_C of C to be the unique element of $C_{\leq 0}$ that gets sent to $1_{\mathbf{k}}$ by the map ϵ; but this property is satisfied for the unity of a graded $\mathbf{k}$-bialgebra as well. (We will repeat this argument in more detail later on, in the proof of Proposition 2.10.)

The following property of connected filtered $\mathbf{k}$-coalgebras will be crucial for us:

Proposition 2.4. Let C be a connected filtered $\mathbf{k}$-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$. Define a $\mathbf{k}$-linear map $\delta : C \to C \otimes C$ by setting

$$\delta(c) := \Delta(c) - c \otimes 1_C - 1_C \otimes c + \epsilon(c)\, 1_C \otimes 1_C \qquad \text{for each } c \in C.$$

Then:

(a) We have

$$\delta(C_{\leq n}) \subseteq \sum_{i=1}^{n-1} C_{\leq i} \otimes C_{\leq n-i} \qquad \text{for each } n > 0.$$

(b) If $f : C \to C$ is a $\mathbf{k}$-coalgebra homomorphism satisfying $f(1_C) = 1_C$, then we have $(f \otimes f) \circ \delta = \delta \circ f$.

(c) We have $\operatorname{Prim} C = (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$.

(d) The set $\operatorname{Prim} C$ is a $\mathbf{k}$-submodule of C.

(e) We have $\operatorname{Ker} \delta = \mathbf{k} \cdot 1_C + \operatorname{Prim} C$.

We shall prove Proposition 2.4 in Section 3.2. The map δ in Proposition 2.4 is called the *reduced coproduct* of C.

Proposition 2.4 helps us apply Theorem 2.1 to filtered $\mathbf{k}$-coalgebras, resulting in the following:

Corollary 2.5. Let C be a connected filtered $\mathbf{k}$-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$.

Let $e : C \to C$ and $f : C \to C$ be two $\mathbf{k}$-coalgebra homomorphisms such that

$$\begin{aligned} e(1_C) &= 1_C \qquad \text{and} \\ f(1_C) &= 1_C \qquad \text{and} \\ \operatorname{Prim} C &\subseteq \operatorname{Ker}(e-f) \qquad \text{and} \qquad (12) \\ f \circ e &= e \circ f. \qquad (13) \end{aligned}$$

Let p be a positive integer such that

$$(e-f)\left(C_{\leq p}\right) = 0. \qquad (14)$$

Then:

(a) For any integer $u > p$, we have

$$(e-f)^{u-p}\left(C_{\leq u}\right) \subseteq \operatorname{Prim} C. \qquad (15)$$

(b) For any integer $u \geq p$, we have

$$(e-f)^{u-p+1}\left(C_{\leq u}\right) = 0. \qquad (16)$$

Corollary 2.5 results from an easy (although not completely immediate) application of Theorem 2.1 and Proposition 2.4. The detailed proof can be found in Section 3.3.

Specializing Corollary 2.5 further to the case of $p = 1$, we can obtain a nicer result:

Corollary 2.6. Let C be a connected filtered $\mathbf{k}$-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$.

Let $e : C \to C$ and $f : C \to C$ be two $\mathbf{k}$-coalgebra homomorphisms such that

$$\begin{aligned} e(1_C) &= 1_C \qquad \text{and} \\ f(1_C) &= 1_C \qquad \text{and} \\ \operatorname{Prim} C &\subseteq \operatorname{Ker}(e-f) \qquad \text{and} \\ f \circ e &= e \circ f. \end{aligned}$$

Then:

(a) For any integer $u > 1$, we have

$$(e-f)^{u-1}\left(C_{\leq u}\right) \subseteq \operatorname{Prim} C.$$

(b) For any positive integer u, we have

$$(e-f)^{u}\left(C_{\leq u}\right) = 0.$$

See Section 3.3 for a proof of this corollary.
The particular case of Corollary 2.6 for $e = \mathrm{id}$ is particularly simple:

Corollary 2.7. Let C be a connected filtered $\mathbf{k}$-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$.
Let $f : C \to C$ be a $\mathbf{k}$-coalgebra homomorphism such that

$$f(1_C) = 1_C \qquad \text{and} \qquad \operatorname{Prim} C \subseteq \operatorname{Ker}(\mathrm{id} - f).$$

Then:
(a) For any integer $u > 1$, we have

$$(\mathrm{id} - f)^{u-1}(C_{\leq u}) \subseteq \operatorname{Prim} C.$$

(b) For any positive integer u, we have

$$(\mathrm{id} - f)^{u}(C_{\leq u}) = 0.$$

Again, the proof of this corollary can be found in Section 3.3.
Note that Corollary 2.7 **(b)** is precisely [Grinbe17, Theorem 37.1 **(a)**].

2.3. Connected filtered bialgebras and Hopf algebras

We shall now apply our above results to connected filtered bialgebras and Hopf algebras. We first define what we mean by these notions:

Definition 2.8. (a) A *filtered* $\mathbf{k}$*-bialgebra* means a $\mathbf{k}$-bialgebra H equipped with an infinite sequence $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ of $\mathbf{k}$-submodules of H satisfying the following five conditions:

- We have
$$H_{\leq 0} \subseteq H_{\leq 1} \subseteq H_{\leq 2} \subseteq \cdots.$$
- We have
$$\bigcup_{n \in \mathbb{N}} H_{\leq n} = H.$$
- We have
$$\Delta(H_{\leq n}) \subseteq \sum_{i=0}^{n} H_{\leq i} \otimes H_{\leq n-i} \qquad \text{for each } n \in \mathbb{N}.$$

(Here, the "$H_{\leq i} \otimes H_{\leq n-i}$" on the right hand side means the image of $H_{\leq i} \otimes H_{\leq n-i}$ under the canonical map $H_{\leq i} \otimes H_{\leq n-i} \to H \otimes H$ that is obtained by tensoring the two inclusion maps $H_{\leq i} \to H$ and $H_{\leq n-i} \to H$ together.)

- We have $H_{\leq i} H_{\leq j} \subseteq H_{\leq i+j}$ for any $i, j \in \mathbb{N}$. (Here, $H_{\leq i} H_{\leq j}$ denotes the $\mathbf{k}$-linear span of the set of all products ab with $a \in H_{\leq i}$ and $b \in H_{\leq j}$.)
- The unity of the $\mathbf{k}$-algebra H belongs to $H_{\leq 0}$.

The sequence $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ is called the *filtration* of the filtered $\mathbf{k}$-bialgebra H.

(b) A *filtered* $\mathbf{k}$*-Hopf algebra* means a filtered $\mathbf{k}$-bialgebra H such that the $\mathbf{k}$-bialgebra H is a Hopf algebra (i.e., has an antipode) and such that the antipode S of H respects the filtration (i.e., satisfies $S(H_{\leq n}) \subseteq H_{\leq n}$ for each $n \in \mathbb{N}$).

The $H_{\leq i} H_{\leq j} \subseteq H_{\leq i+j}$ condition in Definition 2.8 **(a)** will not actually be used in what follows. Thus, we could have omitted it; but this would have resulted in a less common (and less well-behaved in other ways) concept of "filtered bialgebra". Likewise, we have included the $S(H_{\leq n}) \subseteq H_{\leq n}$ condition in Definition 2.8 **(b)**, even though we will never use it.

Every $\mathbf{k}$-bialgebra is automatically a $\mathbf{k}$-coalgebra. Thus, every filtered $\mathbf{k}$-bialgebra is automatically a filtered $\mathbf{k}$-coalgebra. This allows the following definition:

Definition 2.9. A filtered $\mathbf{k}$-bialgebra H is said to be *connected* if the filtered $\mathbf{k}$-coalgebra H is connected.

Thus, if H is a connected filtered $\mathbf{k}$-bialgebra, then Definition 2.3 **(b)** defines a "unity" 1_H of H. This appears to cause an awkward notational quandary, since H already has a unity by virtue of being a $\mathbf{k}$-algebra (and this latter unity is also commonly denoted by 1_H). Fortunately, this cannot cause any confusion, since these two unities are identical, as the following proposition shows:

Proposition 2.10. Let H be a connected filtered $\mathbf{k}$-bialgebra. Then, the unity 1_H defined according to Definition 2.3 **(b)** equals the unity of the $\mathbf{k}$-algebra H.

Proof of Proposition 2.10. Let 1_H denote the unity 1_H defined according to Definition 2.3 **(b)**. Let $\mathbf{1}$ denote the unity of the $\mathbf{k}$-algebra H. Thus, we must show that $1_H = \mathbf{1}$.

We know that H is a filtered $\mathbf{k}$-bialgebra. Hence, the unity of the $\mathbf{k}$-algebra H belongs to $H_{\leq 0}$ (by Definition 2.8 **(a)**). In other words, $\mathbf{1}$ belongs to $H_{\leq 0}$ (since $\mathbf{1}$ is the unity of the $\mathbf{k}$-algebra H). Thus, $\left(\epsilon\mid_{H_{\leq 0}}\right)(\mathbf{1})$ is well-defined.

The axioms of a $\mathbf{k}$-bialgebra yield $\epsilon(\mathbf{1}) = 1_{\mathbf{k}}$ (since H is a $\mathbf{k}$-bialgebra with unity $\mathbf{1}$).

However, 1_H is defined to be $\left(\epsilon\mid_{H_{\leq 0}}\right)^{-1}(1_{\mathbf{k}})$ (by Definition 2.2 **(b)**, applied to $C = H$). Hence, $1_H = \left(\epsilon\mid_{H_{\leq 0}}\right)^{-1}(1_{\mathbf{k}})$. On the other hand, we have $\mathbf{1} = \left(\epsilon\mid_{H_{\leq 0}}\right)^{-1}(1_{\mathbf{k}})$ (since $\left(\epsilon\mid_{H_{\leq 0}}\right)(\mathbf{1}) = \epsilon(\mathbf{1}) = 1_{\mathbf{k}}$). Comparing these two equalities, we obtain $1_H = \mathbf{1}$. As explained above, this completes the proof of Proposition 2.10. $\square$

In Definition 2.3, we have defined the notion of a "primitive element" of a connected filtered $\mathbf{k}$-coalgebra C. In the same way, we can define a "primitive element" of a $\mathbf{k}$-bialgebra H (using the unity of the $\mathbf{k}$-algebra H instead of 1_C):

Definition 2.11. Let H be a $\mathbf{k}$-bialgebra with unity 1_H.
(a) An element x of H is said to be *primitive* if $\Delta(x) = x \otimes 1_H + 1_H \otimes x$.
(b) The set of all primitive elements of H is denoted by $\operatorname{Prim} H$.

When H is a connected filtered $\mathbf{k}$-bialgebra, Definition 2.11 **(a)** agrees with Definition 2.3 **(c)**, since Proposition 2.10 shows that the two meanings of 1_H are actually identical. Thus, when H is a connected filtered $\mathbf{k}$-bialgebra, Definition 2.11 **(b)** agrees with Definition 2.3 **(d)**. Hence, the notation $\operatorname{Prim} H$ is unambiguous.

Next we state some basic properties of the antipode in a Hopf algebra that will be used later on:

Lemma 2.12. Let H be a $\mathbf{k}$-Hopf algebra with unity $1_H \in H$ and antipode $S \in \operatorname{End} H$. Then:
(a) The map $S^2 : H \to H$ is a $\mathbf{k}$-coalgebra homomorphism.
(b) We have $S(1_H) = 1_H$.
(c) We have $S(x) = -x$ for every primitive element x of H.
(d) We have $S^2(x) = x$ for every primitive element x of H.

All parts of this lemma are proved in [Grinbe17, proof of Lemma 37.8] (at least in the case when $\mathbf{k}$ is a field; but the proof applies equally well in the general case). For the sake of completeness, we shall also give the proof in Section 3.4.

We can now state our main consequence for connected filtered Hopf algebras:

Corollary 2.13. Let H be a connected filtered $\mathbf{k}$-Hopf algebra with filtration $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ and antipode S.

Let p be a positive integer such that

$$\left(\operatorname{id} - S^2\right)\left(H_{\leq p}\right) = 0. \tag{17}$$

Then:
(a) For any integer $u > p$, we have

$$\left(\operatorname{id} - S^2\right)^{u-p}\left(H_{\leq u}\right) \subseteq \operatorname{Prim} H \tag{18}$$

and

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-p}\right)\left(H_{\leq u}\right) = 0. \tag{19}$$

(b) For any integer $u \geq p$, we have

$$\left(\operatorname{id} - S^2\right)^{u-p+1}\left(H_{\leq u}\right) = 0. \tag{20}$$

We shall derive this from Corollary 2.5 in Section 2.3.

Specializing Corollary 2.13 to $p = 1$, we can easily obtain the following:

Corollary 2.14. Let H be a connected filtered $\mathbf{k}$-Hopf algebra with filtration $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ and antipode S. Then:

(a) For any integer $u > 1$, we have

$$\left(\operatorname{id} - S^2\right)^{u-1} (H_{\leq u}) \subseteq \operatorname{Prim} H \tag{21}$$

and

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-1}\right) (H_{\leq u}) = 0. \tag{22}$$

(b) For any positive integer u, we have

$$\left(\operatorname{id} - S^2\right)^{u} (H_{\leq u}) = 0. \tag{23}$$

Corollary 2.14 **(b)** has already appeared in [Grinbe17, Theorem 37.7 **(a)**]. It can be derived from either Corollary 2.13 or Corollary 2.7; we shall show the latter derivation in Section 2.3.

2.4. Connected graded Hopf algebras

Let us now specialize our results even further to connected **graded** Hopf algebras. We have already seen that any graded $\mathbf{k}$-coalgebra automatically becomes a filtered $\mathbf{k}$-coalgebra. In the same way, any graded $\mathbf{k}$-Hopf algebra automatically becomes a filtered $\mathbf{k}$-Hopf algebra. Moreover, a graded $\mathbf{k}$-Hopf algebra H is connected (in the sense that $H_0 \cong \mathbf{k}$ as $\mathbf{k}$-modules) if and only if the filtered $\mathbf{k}$-coalgebra H is connected. (This follows easily from [GriRei20, Exercise 1.3.20 (e)].) Thus, our above results for connected filtered $\mathbf{k}$-Hopf algebras can be applied to connected graded $\mathbf{k}$-Hopf algebras. From Corollary 2.14, we easily obtain the following:

Corollary 2.15. Let H be a connected graded $\mathbf{k}$-Hopf algebra with antipode S. Then, for any positive integer u, we have

$$\left(\operatorname{id} - S^2\right)^{u-1} (H_u) \subseteq \operatorname{Prim} H \tag{24}$$

and

$$\left((\operatorname{id} + S) \circ \left(\operatorname{id} - S^2\right)^{u-1}\right) (H_u) = 0 \tag{25}$$

and

$$\left(\operatorname{id} - S^2\right)^{u} (H_u) = 0. \tag{26}$$

This is not an immediate consequence of Corollary 2.14, since the condition "u is positive" is weaker than the condition "$u > 1$" in Corollary 2.14 **(a)**; thus, deriving

Corollary 2.15 from Corollary 2.14 requires some extra work to account for the case of $u = 1$. We will give a detailed proof of Corollary 2.15 in Section 3.5.

The equality (25) in Corollary 2.15 yields [AguMah17, Lemma 12.50], whereas the equality (26) yields [AguLau14, Proposition 7]. Next, we apply Corollary 2.13 to the graded setting:

Corollary 2.16. Let H be a connected graded $\mathbf{k}$-Hopf algebra with antipode S.

Let p be a positive integer such that all $i \in \{2, 3, \ldots, p\}$ satisfy

$$\left(\mathrm{id} - S^2\right)(H_i) = 0. \tag{27}$$

Then:

(a) For any integer $u > p$, we have

$$\left(\mathrm{id} - S^2\right)^{u-p}(H_{\leq u}) \subseteq \operatorname{Prim} H \tag{28}$$

and

$$\left((\mathrm{id} + S) \circ \left(\mathrm{id} - S^2\right)^{u-p}\right)(H_{\leq u}) = 0. \tag{29}$$

(b) For any integer $u \geq p$, we have

$$\left(\mathrm{id} - S^2\right)^{u-p+1}(H_{\leq u}) = 0. \tag{30}$$

Again, the proof of Corollary 2.16 can be found in Section 3.5.

The particular case of Corollary 2.16 for $p = 2$ is the most useful, as the condition (27) boils down to the equality $\left(\mathrm{id} - S^2\right)(H_2) = 0$ in this case, and the latter equality is satisfied rather frequently. Here is one sufficient criterion (which we will prove in Section 3.5):

Corollary 2.17. Let H be a connected graded $\mathbf{k}$-Hopf algebra with antipode S. Assume that

$$ab = ba \qquad \text{for every } a, b \in H_1. \tag{31}$$

Then:

(a) We have

$$\left(\mathrm{id} - S^2\right)(H_2) = 0.$$

(b) For any integer $u > 2$, we have

$$\left(\mathrm{id} - S^2\right)^{u-2}(H_{\leq u}) \subseteq \operatorname{Prim} H \tag{32}$$

and

$$\left((\mathrm{id} + S) \circ \left(\mathrm{id} - S^2\right)^{u-2}\right)(H_{\leq u}) = 0. \tag{33}$$

(c) For any integer $u > 1$, we have

$$\left(\mathrm{id} - S^2\right)^{u-1} \left(H_{\leq u}\right) = 0. \tag{34}$$

The equality (34) in Corollary 2.17 generalizes [AguLau14, Example 8]. Indeed, if H is the Malvenuto–Reutenauer Hopf algebra[2], then the condition (31) is satisfied (since H_1 is a free $\mathbf{k}$-module of rank 1 in this case); therefore, Corollary 2.17 **(c)** can be applied in this case, and we recover [AguLau14, Example 8]. Likewise, we can obtain the same result if H is the Hopf algebra WQSym of word quasisymmetric functions[3].

It is worth noticing that the condition (31) is only sufficient, but not necessary for (34). For example, if H is the tensor algebra of a free $\mathbf{k}$-module V of rank ≥ 2, then (34) holds (since H is cocommutative, so that $S^2 = \mathrm{id}$), but (31) does not (since $u \otimes v \neq v \otimes u$ if u and v are two distinct basis vectors of V).

An example of a connected graded Hopf algebra H that does **not** satisfy (34) (and thus does not satisfy (31) either) is not hard to construct:

Example 2.18. Assume that the ring $\mathbf{k}$ is not trivial. Let H be the free $\mathbf{k}$-algebra with three generators a, b, c. We equip this $\mathbf{k}$-algebra H with a grading, by requiring that its generators a, b, c are homogeneous of degrees $1, 1, 2$, respectively. Next, we define a comultiplication Δ on H by setting

$$\begin{aligned} \Delta(a) &= a \otimes 1 + 1 \otimes a; \\ \Delta(b) &= b \otimes 1 + 1 \otimes b; \\ \Delta(c) &= c \otimes 1 + a \otimes b + 1 \otimes c \end{aligned}$$

(where 1 is the unity of H). Furthermore, we define a counit ϵ on H by setting $\epsilon(a) = \epsilon(b) = \epsilon(c) = 0$. It is straightforward to see that H thus becomes a connected graded $\mathbf{k}$-bialgebra, hence (by [GriRei20, Proposition 1.4.16]) a connected graded $\mathbf{k}$-Hopf algebra. Its antipode S is easily seen to satisfy $S(c) = ab - c$ and $S^2(c) = ba - ab + c \neq c$; thus, $\left(\mathrm{id} - S^2\right)(H_2) \neq 0$. Hence, (34) does not hold for $u = 2$.

The Hopf algebra H in this example is in fact an instance of a general construction of connected graded $\mathbf{k}$-Hopf algebras that are "generic" (in the sense that their structure maps satisfy no relations other than ones that hold in every connected graded $\mathbf{k}$-Hopf algebra). This latter construction will be elaborated upon in future work.

[2]See [Meliot17, §12.1], [HaGuKi10, §7.1] or [GriRei20, §8.1] for the definition of this Hopf algebra. (It is denoted FQSym in [Meliot17] and [GriRei20], and denoted MPR in [HaGuKi10].)

[3]See (e.g.) [MeNoTh13, §4.3.2] for a definition of this Hopf algebra.

Remark 2.19. A brave reader might wonder whether the connectedness condition in Corollary 2.15 could be replaced by something weaker – e.g., instead of requiring H to be connected, we might require that the subalgebra H_0 be commutative. However, such a requirement would be insufficient. In fact, let $\mathbf{k} = \mathbb{C}$. Then, for any integer $n > 1$ and any primitive n-th root of unity $q \in \mathbf{k}$, the Taft algebra $H_{n,q}$ defined in [Radfor12, §7.3] can be viewed as a graded Hopf algebra (with $a \in H_0$ and $x \in H_1$) whose subalgebra $H_0 = \mathbf{k}[a] / (a^n - 1)$ is commutative, but whose antipode S does not satisfy $\left(\operatorname{id} - S^2\right)^k (H_1) = 0$ for any $k \in \mathbb{N}$ (since $S^2(x) = q^{-1}x$ and therefore $\left(\operatorname{id} - S^2\right)^k (x) = \left(1 - q^{-1}\right)^k x \neq 0$ because $q^{-1} \neq 1$).

3. Proofs

We shall now prove all statements left unproved above.

We begin by stating three general facts from linear algebra, which will be used several times in what follows:

Lemma 3.1. Let D be a $\mathbf{k}$-module. Let U and V be two $\mathbf{k}$-submodules of D. Let α and β be two elements of $\operatorname{End} D$. Then,

$$(\alpha \otimes \beta)(U \otimes V) = \alpha(U) \otimes \beta(V). \tag{35}$$

(Here, both $U \otimes V$ and $\alpha(U) \otimes \beta(V)$ have to be understood as $\mathbf{k}$-submodules of $D \otimes D$, specifically as the images of the "actual" tensor products $U \otimes V$ and $\alpha(U) \otimes \beta(V)$ under the canonical maps into $D \otimes D$.)

The proof of Lemma 3.1 is straightforward and therefore omitted.

Lemma 3.2. Let D be a $\mathbf{k}$-module. Let $\alpha, \beta, \gamma, \delta$ be four elements of $\operatorname{End} D$. Then, in $\operatorname{End}(D \otimes D)$, we have

$$(\alpha \otimes \beta) \circ (\gamma \otimes \delta) = (\alpha \circ \gamma) \otimes (\beta \circ \delta).$$

This fact can be verified easily by comparing how the left and the right hand sides transform any given pure tensor $u \otimes v \in D \otimes D$. Again, we leave the details to the reader.

Lemma 3.3. Let D be a $\mathbf{k}$-module. Let α and β be two elements of $\operatorname{End} D$. Let $n \in \mathbb{N}$. Then, in $\operatorname{End}(D \otimes D)$, we have

$$(\alpha \otimes \beta)^n = \alpha^n \otimes \beta^n.$$

Lemma 3.3 follows by induction on k using Lemma 3.2.

3.1. Proof of Theorem 2.1

Our first goal is to prove Theorem 2.1.

Proof of Theorem 2.1. We shall prove (7) and (8) by strong induction on u:

Induction step: Fix an integer $n > p$. Assume (as the induction hypothesis) that (7) and (8) hold for all $u < n$ (that is, for all integers $u > p$ satisfying $u < n$). We must prove that (7) and (8) hold for $u = n$. In other words, we must prove that

$$(e - f)^{n-p} (D_n) \subseteq \operatorname{Ker} \delta$$

and

$$(e - f)^{n-p+1} (D_n) = 0.$$

We shall focus on proving the first of these two equalities; the second will then easily follow from (1).

Consider the $\mathbf{k}$-algebras $\operatorname{End} D$ and $\operatorname{End}(D \otimes D)$. (The multiplication in each of these $\mathbf{k}$-algebras is composition of $\mathbf{k}$-linear maps.) Note that $u \otimes v \in \operatorname{End}(D \otimes D)$ for any $u, v \in \operatorname{End} D$.

We have $e, f \in \operatorname{End} D$. Let us define two elements $g \in \operatorname{End} D$ and $h \in \operatorname{End}(D \otimes D)$ by

$$g = e - f \qquad \text{and} \qquad h = e \otimes e - f \otimes f.$$

A simple computation then shows that

$$h = g \otimes f + e \otimes g$$

[4]. Moreover, from (5), we easily obtain

$$g(D_u) = 0 \qquad \text{for all } u \in \{1, 2, \ldots, p\} \tag{36}$$

[5].

[4] *Proof.* We have

$$\begin{aligned} \underbrace{g}_{=e-f} \otimes f + e \otimes \underbrace{g}_{=e-f} &= (e - f) \otimes f + e \otimes (e - f) \\ &= e \otimes f - f \otimes f + e \otimes e - e \otimes f \\ &\qquad \left(\begin{array}{c} \text{since tensoring } \mathbf{k}\text{-linear maps} \\ \text{is a } \mathbf{k}\text{-bilinear operation} \end{array} \right) \\ &= e \otimes e - f \otimes f = h. \end{aligned}$$

[5] *Proof of (36):* Let $u \in \{1, 2, \ldots, p\}$. Thus, $D_u \subseteq D_1 + D_2 + \cdots + D_p$, so that

$$\begin{aligned} g(D_u) &\subseteq g(D_1 + D_2 + \cdots + D_p) = (e - f)(D_1 + D_2 + \cdots + D_p) \qquad (\text{since } g = e - f) \\ &= 0 \qquad (\text{by (5)}). \end{aligned}$$

Thus, $g(D_u) = 0$. This proves (36).

Using (4) and $g = e - f$, we can easily see that $g \circ e = e \circ g$ [6] and $g \circ f = f \circ g$ [7]. Hence, in particular, the elements g and e of $\operatorname{End} D$ commute (since $g \circ e = e \circ g$). Hence, the $\mathbf{k}$-subalgebra of $\operatorname{End} D$ generated by g and e is commutative (because a $\mathbf{k}$-algebra generated by two commuting elements is always commutative). Therefore, for each $i \in \mathbb{N}$ and $j \in \mathbb{N}$, we have[8]

$$g^i e^j = e^j g^i \tag{37}$$

(because both g^i and e^j belong to this commutative $\mathbf{k}$-subalgebra).

Furthermore, in $\operatorname{End}(D \otimes D)$, we have the equalities

$$\begin{aligned}(g \otimes f)(e \otimes g) &= (g \otimes f) \circ (e \otimes g) = \underbrace{(g \circ e)}_{=e \circ g} \otimes (f \circ g) \qquad \text{(by Lemma 3.2)} \\ &= (e \circ g) \otimes (f \circ g)\end{aligned}$$

[6] *Proof.* This follows by comparing

$$\underbrace{g}_{=e-f} \circ e = (e - f) \circ e = e \circ e - f \circ e \qquad \left(\begin{array}{c}\text{since composition of } \mathbf{k}\text{-linear maps} \\ \text{is a } \mathbf{k}\text{-bilinear operation (and since the} \\ \text{maps } e \text{ and } f \text{ are } \mathbf{k}\text{-linear)}\end{array}\right)$$

with

$$\begin{aligned}e \circ \underbrace{g}_{=e-f} &= e \circ (e - f) = e \circ e - \underbrace{e \circ f}_{\substack{=f \circ e \\ \text{(by (4))}}} \qquad \left(\begin{array}{c}\text{since composition of } \mathbf{k}\text{-linear maps} \\ \text{is a } \mathbf{k}\text{-bilinear operation (and since the} \\ \text{maps } e \text{ and } f \text{ are } \mathbf{k}\text{-linear)}\end{array}\right) \\ &= e \circ e - f \circ e.\end{aligned}$$

[7] *Proof.* This follows by comparing

$$\underbrace{g}_{=e-f} \circ f = (e - f) \circ f = e \circ f - f \circ f \qquad \left(\begin{array}{c}\text{since composition of } \mathbf{k}\text{-linear maps} \\ \text{is a } \mathbf{k}\text{-bilinear operation (and since the} \\ \text{maps } e \text{ and } f \text{ are } \mathbf{k}\text{-linear)}\end{array}\right)$$

with

$$\begin{aligned}f \circ \underbrace{g}_{=e-f} &= f \circ (e - f) = \underbrace{f \circ e}_{\substack{=e \circ f \\ \text{(by (4))}}} - f \circ f \qquad \left(\begin{array}{c}\text{since composition of } \mathbf{k}\text{-linear maps} \\ \text{is a } \mathbf{k}\text{-bilinear operation (and since the} \\ \text{maps } e \text{ and } f \text{ are } \mathbf{k}\text{-linear)}\end{array}\right) \\ &= e \circ f - f \circ f.\end{aligned}$$

[8] We recall that the multiplication in the $\mathbf{k}$-algebra $\operatorname{End} D$ is composition of maps. Thus, $\alpha\beta = \alpha \circ \beta$ for any $\alpha, \beta \in \operatorname{End} D$. (The same holds for $\operatorname{End}(D \otimes D)$.)

and

$$\begin{aligned}(e\otimes g)(g\otimes f) &= (e\otimes g)\circ(g\otimes f) = (e\circ g)\otimes\underbrace{(g\circ f)}_{=f\circ g} \qquad \text{(by Lemma 3.2)}\\ &= (e\circ g)\otimes(f\circ g)\,.\end{aligned}$$

Comparing these two equalities, we obtain $(g\otimes f)(e\otimes g) = (e\otimes g)(g\otimes f)$. In other words, the elements $g\otimes f$ and $e\otimes g$ of $\operatorname{End}(D\otimes D)$ commute. Hence, we can apply the binomial formula[9] to $g\otimes f$ and $e\otimes g$. We thus conclude that each $k\in\mathbb{N}$ satisfies

$$\begin{aligned}(g\otimes f+e\otimes g)^k &= \sum_{r=0}^{k}\binom{k}{r}\underbrace{(g\otimes f)^r}_{\substack{=g^r\otimes f^r\\ \text{(by Lemma 3.3)}}}\underbrace{(e\otimes g)^{k-r}}_{\substack{=e^{k-r}\otimes g^{k-r}\\ \text{(by Lemma 3.3)}}} = \sum_{r=0}^{k}\binom{k}{r}\underbrace{\left(g^r\otimes f^r\right)\left(e^{k-r}\otimes g^{k-r}\right)}_{\substack{=\left(g^r\otimes f^r\right)\circ\left(e^{k-r}\otimes g^{k-r}\right)\\ =\left(g^r\circ e^{k-r}\right)\otimes\left(f^r\circ g^{k-r}\right)\\ \text{(by Lemma 3.2)}}}\\ &= \sum_{r=0}^{k}\binom{k}{r}\underbrace{\left(g^r\circ e^{k-r}\right)}_{\substack{=g^re^{k-r}\\ =e^{k-r}g^r\\ \text{(by (37))}}}\otimes\left(f^r\circ g^{k-r}\right) = \sum_{r=0}^{k}\binom{k}{r}\underbrace{\left(e^{k-r}g^r\right)}_{=e^{k-r}\circ g^r}\otimes\left(f^r\circ g^{k-r}\right)\\ &= \sum_{r=0}^{k}\binom{k}{r}\underbrace{\left(e^{k-r}\circ g^r\right)\otimes\left(f^r\circ g^{k-r}\right)}_{\substack{=\left(e^{k-r}\otimes f^r\right)\circ\left(g^r\otimes g^{k-r}\right)\\ \text{(since Lemma 3.2}\\ \text{yields } \left(e^{k-r}\otimes f^r\right)\circ\left(g^r\otimes g^{k-r}\right)=\left(e^{k-r}\circ g^r\right)\otimes\left(f^r\circ g^{k-r}\right))}}\\ &= \sum_{r=0}^{k}\binom{k}{r}\left(e^{k-r}\otimes f^r\right)\circ\left(g^r\otimes g^{k-r}\right). \qquad (38)\end{aligned}$$

For each $k\in\mathbb{N}$ and $r\in\mathbb{N}$, we define a map $h_{k,r}\in\operatorname{End}(D\otimes D)$ by

$$h_{k,r} = \binom{k}{r}\left(e^{k-r}\otimes f^r\right)\circ\left(g^r\otimes g^{k-r}\right). \qquad (39)$$

[9] We recall that the *binomial formula* says the following: If α and β are two commuting elements of a ring, then each $k\in\mathbb{N}$ satisfies $(\alpha+\beta)^k = \sum_{r=0}^{k}\binom{k}{r}\alpha^r\beta^{k-r}$.

Then, each $k \in \mathbb{N}$ satisfies

$$\begin{aligned} h^k &= (g \otimes f + e \otimes g)^k \qquad (\text{since } h = g \otimes f + e \otimes g) \\ &= \sum_{r=0}^{k} \underbrace{\binom{k}{r} \left(e^{k-r} \otimes f^r\right) \circ \left(g^r \otimes g^{k-r}\right)}_{\substack{=h_{k,r} \\ (\text{by } (39))}} \qquad (\text{by } (38)) \\ &= \sum_{r=0}^{k} h_{k,r}. \end{aligned} \tag{40}$$

Subtracting (2) from (3), we obtain

$$\begin{aligned} (e \otimes e) \circ \delta - (f \otimes f) \circ \delta &= \delta \circ e - \delta \circ f = \delta \circ \underbrace{(e - f)}_{=g} \\ &\qquad \left(\begin{array}{c} \text{since composition of } \mathbf{k}\text{-linear maps} \\ \text{is a } \mathbf{k}\text{-bilinear operation (and since} \\ \text{the maps } \delta, e \text{ and } f \text{ are } \mathbf{k}\text{-linear)} \end{array}\right) \\ &= \delta \circ g. \end{aligned}$$

Thus,

$$\begin{aligned} \delta \circ g &= (e \otimes e) \circ \delta - (f \otimes f) \circ \delta = \underbrace{(e \otimes e - f \otimes f)}_{=h} \circ \delta \\ &\qquad \left(\begin{array}{c} \text{since composition of } \mathbf{k}\text{-linear maps} \\ \text{is a } \mathbf{k}\text{-bilinear operation (and since} \\ \text{the maps } \delta, e \otimes e \text{ and } f \otimes f \text{ are } \mathbf{k}\text{-linear)} \end{array}\right) \\ &= h \circ \delta. \end{aligned} \tag{41}$$

Hence, by induction, we see that

$$\delta \circ g^k = h^k \circ \delta \qquad \text{for each } k \in \mathbb{N} \tag{42}$$

[10].

[10] *Proof of (42):* We shall prove (42) by induction on k:

Induction base: Comparing $\delta \circ \underbrace{g^0}_{=\mathrm{id}} = \delta \circ \mathrm{id} = \delta$ with $\underbrace{h^0}_{=\mathrm{id}} \circ \delta = \mathrm{id} \circ \delta = \delta$, we obtain $\delta \circ g^0 = h^0 \circ \delta$. In other words, (42) holds for $k = 0$.

Induction step: Let $\ell \in \mathbb{N}$. Assume (as the induction hypothesis) that (42) holds for $k = \ell$. We must show that (42) holds for $k = \ell + 1$.

We have assumed that (42) holds for $k = \ell$. In other words, we have $\delta \circ g^\ell = h^\ell \circ \delta$. Now,

$$\delta \circ \underbrace{g^{\ell+1}}_{=g^\ell \circ g} = \underbrace{\delta \circ g^\ell}_{=h^\ell \circ \delta} \circ g = h^\ell \circ \underbrace{\delta \circ g}_{\substack{=h \circ \delta \\ (\text{by } (41))}} = \underbrace{h^\ell \circ h}_{=h^{\ell+1}} \circ \delta = h^{\ell+1} \circ \delta.$$

In other words, (42) holds for $k = \ell + 1$. This completes the induction step. Thus, the induction proof of (42) is complete.

Our induction hypothesis says that (7) and (8) hold for all $u < n$. In particular, (8) holds for all $u < n$. In other words, for each integer $u > p$ satisfying $u < n$, we have

$$(e - f)^{u-p+1}(D_u) = 0. \tag{43}$$

Hence, it is easy to see that every positive integer $u < n$ and every positive integer $v > u - p$ satisfy

$$g^v(D_u) = 0 \tag{44}$$

[11].

Now, let $k = n - p$. Then, $k = n - p > 0$ (since $n > p$), so that $k \in \mathbb{N}$ (since k is

[11] *Proof of (44):* Let $u < n$ be a positive integer. Let $v > u - p$ be a positive integer. We must prove that $g^v(D_u) = 0$.

We are in one of the following two cases:

Case 1: We have $u \leq p$.

Case 2: We have $u > p$.

Let us first consider Case 1. In this case, we have $u \leq p$. Hence, $u \in \{1, 2, \ldots, p\}$ (since u is a positive integer) and therefore $g(D_u) = 0$ (by (36)). However, we have $v \geq 1$ (since v is a positive integer). Hence, $g^v = g^{v-1} \circ g$, so that

$$\begin{aligned} g^v(D_u) &= \left(g^{v-1} \circ g\right)(D_u) = g^{v-1}\left(\underbrace{g(D_u)}_{=0}\right) \\ &= g^{v-1}(0) = 0 \qquad \left(\text{since the map } g^{v-1} \text{ is } \mathbf{k}\text{-linear}\right). \end{aligned}$$

Hence, (44) is proved in Case 1.

Let us now consider Case 2. In this case, we have $u > p$. Therefore, (43) yields $(e - f)^{u-p+1}(D_u) = 0$. This rewrites as $g^{u-p+1}(D_u) = 0$ (since $g = e - f$). Set $u' = u - p + 1$. Thus, we have $g^{u'}(D_u) = g^{u-p+1}(D_u) = 0$.

However, from $v > u - p$, we obtain $v \geq u - p + 1$ (since v and $u - p$ are integers). In other words, $v \geq u'$ (since $u' = u - p + 1$). Thus, $g^v = g^{v-u'} \circ g^{u'}$, so that

$$\begin{aligned} g^v(D_u) &= \left(g^{v-u'} \circ g^{u'}\right)(D_u) = g^{v-u'}\left(\underbrace{g^{u'}(D_u)}_{=0}\right) \\ &= g^{v-u'}(0) = 0 \qquad \left(\text{since the map } g^{v-u'} \text{ is } \mathbf{k}\text{-linear}\right). \end{aligned}$$

Hence, (44) is proved in Case 2.

We have now proved (44) in both Cases 1 and 2. Since these two Cases cover all possibilities, we thus conclude that (44) always holds.

an integer). Furthermore, (42) yields $\delta \circ g^k = h^k \circ \delta$. Thus,

$$\left(\delta \circ g^k\right)(D_n) = \left(h^k \circ \delta\right)(D_n) = h^k \underbrace{(\delta(D_n))}_{\substack{\subseteq \sum_{i=1}^{n-1} D_i \otimes D_{n-i} \\ \text{(by (6))}}}$$
$$\subseteq h^k \left(\sum_{i=1}^{n-1} D_i \otimes D_{n-i}\right)$$
$$= \sum_{i=1}^{n-1} h^k (D_i \otimes D_{n-i}) \tag{45}$$

(since the map h^k is $\mathbf{k}$-linear).

We shall now prove that each $i \in \{1, 2, \ldots, n-1\}$ and each $r \in \{0, 1, \ldots, k\}$ satisfy

$$\left(g^r \otimes g^{k-r}\right)(D_i \otimes D_{n-i}) = 0. \tag{46}$$

[*Proof of (46):* Fix $i \in \{1, 2, \ldots, n-1\}$ and $r \in \{0, 1, \ldots, k\}$. We must prove (46).

We note that (35) (applied to $\alpha = g^r$, $\beta = g^{k-r}$, $U = D_i$ and $V = D_{n-i}$) yields

$$\left(g^r \otimes g^{k-r}\right)(D_i \otimes D_{n-i}) = g^r(D_i) \otimes g^{k-r}(D_{n-i}). \tag{47}$$

We have $i \in \{1, 2, \ldots, n-1\}$ and thus $1 \leq i \leq n-1$, so that $i \leq n-1 < n$. Thus, $n > i$, so that $n - i > 0$. Moreover, from $i \geq 1 > 0$, we obtain $n - i < n$. Furthermore, the two integers i and $n - i$ are positive (since $i > 0$ and $n - i > 0$).

We are in one of the following three cases:

Case 1: We have $r \geq k$.

Case 2: We have $r \geq i$.

Case 3: We have neither $r \geq k$ nor $r \geq i$.

Let us first consider Case 1. In this case, we have $r \geq k$. Thus, $r \geq k = \underbrace{n}_{>i} - p > i - p$. Moreover, the integer r is positive (since $r \geq k > 0$). Hence, (44) (applied to $u = i$ and $v = r$) yields $g^r(D_i) = 0$ (since $r > i - p$). Now, (47) becomes

$$\left(g^r \otimes g^{k-r}\right)(D_i \otimes D_{n-i}) = \underbrace{g^r(D_i)}_{=0} \otimes g^{k-r}(D_{n-i}) = 0 \otimes g^{k-r}(D_{n-i}) = 0.$$

Thus, (46) is proved in Case 1.

Let us next consider Case 2. In this case, we have $r \geq i$. Thus, $r \geq i > i - p$ (since $p > 0$). Moreover, the integer r is positive (since $r \geq i > 0$). Hence, (44) (applied to $u = i$ and $v = r$) yields $g^r(D_i) = 0$ (since $r > i - p$). Now, (47) becomes

$$\left(g^r \otimes g^{k-r}\right)(D_i \otimes D_{n-i}) = \underbrace{g^r(D_i)}_{=0} \otimes g^{k-r}(D_{n-i}) = 0 \otimes g^{k-r}(D_{n-i}) = 0.$$

Thus, (46) is proved in Case 2.

Finally, let us consider Case 3. In this case, we have neither $r \geq k$ nor $r \geq i$. In other words, we have $r < k$ and $r < i$. Now, the integer $k - r$ is positive (since $r < k$). Furthermore, from $k = n - p$, we obtain

$$k - r = n - p - \underbrace{r}_{<i} > n - p - i = n - i - p.$$

Hence, (44) (applied to $u = n - i$ and $v = k - r$) yields $g^{k-r}(D_{n-i}) = 0$. Now, (47) becomes

$$\left(g^r \otimes g^{k-r}\right)(D_i \otimes D_{n-i}) = g^r(D_i) \otimes \underbrace{g^{k-r}(D_{n-i})}_{=0} = g^r(D_i) \otimes 0 = 0.$$

Thus, (46) is proved in Case 3.

We have now proved (46) in all three Cases 1, 2 and 3. Thus, the proof of (46) is complete.]

Now, each $i \in \{1, 2, \ldots, n-1\}$ satisfies

$$\begin{aligned}
\underbrace{h^k}_{\substack{=\sum_{r=0}^{k} h_{k,r} \\ \text{(by (40))}}} (D_i \otimes D_{n-i}) &= \left(\sum_{r=0}^{k} h_{k,r}\right)(D_i \otimes D_{n-i}) \\
&\subseteq \sum_{r=0}^{k} \underbrace{h_{k,r}}_{\substack{=\binom{k}{r}\left(e^{k-r}\otimes f^r\right)\circ\left(g^r\otimes g^{k-r}\right) \\ \text{(by (39))}}} (D_i \otimes D_{n-i}) \\
&= \sum_{r=0}^{k} \underbrace{\left(\binom{k}{r}\left(e^{k-r}\otimes f^r\right)\circ\left(g^r\otimes g^{k-r}\right)\right)(D_i\otimes D_{n-i})}_{=\binom{k}{r}\left(e^{k-r}\otimes f^r\right)\left(\left(g^r\otimes g^{k-r}\right)(D_i\otimes D_{n-i})\right)} \\
&= \sum_{r=0}^{k}\binom{k}{r}\left(e^{k-r}\otimes f^r\right)\underbrace{\left(\left(g^r\otimes g^{k-r}\right)(D_i\otimes D_{n-i})\right)}_{\substack{=0 \\ \text{(by (46))}}} \\
&= \sum_{r=0}^{k}\binom{k}{r}\underbrace{\left(e^{k-r}\otimes f^r\right)(0)}_{\substack{=0 \\ \text{(since the map } e^{k-r}\otimes f^r \text{ is } \mathbf{k}\text{-linear)}}} = \sum_{r=0}^{k}\underbrace{\binom{k}{r}0}_{=0} \\
&= 0. \qquad (48)
\end{aligned}$$

Hence, (45) becomes

$$\left(\delta \circ g^k\right)(D_n) \subseteq \sum_{i=1}^{n-1} \underbrace{h^k\left(D_i \otimes D_{n-i}\right)}_{\substack{\subseteq 0 \\ \text{(by (48))}}} \subseteq \sum_{i=1}^{n-1} 0 = 0.$$

From this, we can easily obtain

$$g^k(D_n) \subseteq \operatorname{Ker}\delta$$

[12]. Since $g = e - f$ and $k = n - p$, we can rewrite this as follows:

$$(e-f)^{n-p}(D_n) \subseteq \operatorname{Ker}\delta. \tag{49}$$

However, we have $\operatorname{Ker}\delta \subseteq \operatorname{Ker}(e-f)$ (by (1)) and therefore $(e-f)(\operatorname{Ker}\delta) = 0$ [13]. Thus,

$$\underbrace{(e-f)^{n-p+1}}_{=(e-f)\circ(e-f)^{n-p}}(D_n) = \left((e-f)\circ(e-f)^{n-p}\right)(D_n)$$

$$= (e-f)\left(\underbrace{(e-f)^{n-p}(D_n)}_{\substack{\subseteq \operatorname{Ker}\delta \\ \text{(by (49))}}}\right) \subseteq (e-f)(\operatorname{Ker}\delta) = 0.$$

In other words,

$$(e-f)^{n-p+1}(D_n) = 0. \tag{50}$$

We have now proved the relations (49) and (50). In other words, (7) and (8) hold for $u = n$. This completes the induction step. Thus, we have proved by strong induction that (7) and (8) hold for all integers $u > p$. This proves Theorem 2.1. □

[12] *Proof.* Let $x \in g^k(D_n)$. Thus, there exists some $y \in D_n$ such that $x = g^k(y)$. Consider this y. Applying the map δ to both sides of the equality $x = g^k(y)$, we find

$$\delta(x) = \delta\left(g^k(y)\right) = \left(\delta \circ g^k\right)\left(\underbrace{y}_{\in D_n}\right) \in \left(\delta \circ g^k\right)(D_n) \subseteq 0,$$

so that $\delta(x) = 0$. Hence, $x \in \operatorname{Ker}\delta$.

Now, forget that we fixed x. We thus have shown that $x \in \operatorname{Ker}\delta$ for each $x \in g^k(D_n)$. In other words, $g^k(D_n) \subseteq \operatorname{Ker}\delta$.

[13] *Proof.* Let $y \in (e-f)(\operatorname{Ker}\delta)$. Thus, there exists some $x \in \operatorname{Ker}\delta$ such that $y = (e-f)(x)$. Consider this x.

We have $x \in \operatorname{Ker}\delta \subseteq \operatorname{Ker}(e-f)$, so that $(e-f)(x) = 0$. Hence, $y = (e-f)(x) = 0$.

Forget that we fixed y. We thus have shown that $y = 0$ for each $y \in (e-f)(\operatorname{Ker}\delta)$. In other words, $(e-f)(\operatorname{Ker}\delta) \subseteq 0$. Hence, $(e-f)(\operatorname{Ker}\delta) = 0$ (since $(e-f)(\operatorname{Ker}\delta)$ is a $\mathbf{k}$-module).

3.2. Proof of Proposition 2.4

Our next goal is to prove Proposition 2.4. We shall work towards this goal by proving some lemmas. First, we note a simple consequence of the axioms of a $\mathbf{k}$-coalgebra:

Remark 3.4. Let C be any $\mathbf{k}$-coalgebra. Let $\operatorname{can}_1 : C \otimes \mathbf{k} \to C$ be the $\mathbf{k}$-module isomorphism sending $c \otimes 1$ to c for each $c \in C$. Let $\operatorname{can}_2 : \mathbf{k} \otimes C \to C$ be the $\mathbf{k}$-module isomorphism sending $1 \otimes c$ to c for each $c \in C$.

Recall that C is a $\mathbf{k}$-coalgebra, and therefore satisfies the axioms of a $\mathbf{k}$-coalgebra. Hence, in particular, the diagram

$$\begin{tikzcd}
C \otimes \mathbf{k} \arrow[r, "\operatorname{can}_1"] & C & \mathbf{k} \otimes C \arrow[l, "\operatorname{can}_2"'] \\
C \otimes C \arrow[u, "\operatorname{id} \otimes \epsilon"] & C \arrow[l, "\Delta"] \arrow[r, "\Delta"'] \arrow[u, "\operatorname{id}"] & C \otimes C \arrow[u, "\epsilon \otimes \operatorname{id}"']
\end{tikzcd}$$

is commutative (since this is one of the axioms of a $\mathbf{k}$-coalgebra). In other words, we have

$$\operatorname{can}_1 \circ (\operatorname{id} \otimes \epsilon) \circ \Delta = \operatorname{id} \qquad \text{and} \tag{51}$$

$$\operatorname{can}_2 \circ (\epsilon \otimes \operatorname{id}) \circ \Delta = \operatorname{id}. \tag{52}$$

Next, we state a simple property of a slight generalization of primitive elements in a coalgebra:

Lemma 3.5. Let C be any $\mathbf{k}$-coalgebra. Let $a, b, d \in C$ be three elements satisfying $\epsilon(a) = 1$ and $\epsilon(b) = 1$ and $\Delta(d) = d \otimes a + b \otimes d$. Then, $\epsilon(d) = 0$.

We shall later apply Lemma 3.5 to the case when $a = b = 1_C$ (and C is either a connected filtered $\mathbf{k}$-coalgebra or a $\mathbf{k}$-bialgebra, so that 1_C does make sense); however, it is not any harder to prove it in full generality:

Proof of Lemma 3.5. Let us first observe that the map

$$\begin{aligned} C \times C &\to C \otimes C, \\ (u, v) &\mapsto u \otimes v \end{aligned} \tag{53}$$

is $\mathbf{k}$-bilinear. (This follows straight from the definition of the tensor product.)

Applying the map $\operatorname{id} \otimes \epsilon : C \otimes C \to C \otimes \mathbf{k}$ to both sides of the equality $\Delta(d) =$

$d \otimes a + b \otimes d$, we obtain

$$\begin{aligned}
(\mathrm{id} \otimes \epsilon)(\Delta(d)) &= (\mathrm{id} \otimes \epsilon)(d \otimes a + b \otimes d) \\
&= \underbrace{(\mathrm{id} \otimes \epsilon)(d \otimes a)}_{=\mathrm{id}(d) \otimes \epsilon(a)} + \underbrace{(\mathrm{id} \otimes \epsilon)(b \otimes d)}_{=\mathrm{id}(b) \otimes \epsilon(d)} \\
&\qquad \text{(since the map } \mathrm{id} \otimes \epsilon \text{ is } \mathbf{k}\text{-linear)} \\
&= \underbrace{\mathrm{id}(d)}_{=d} \otimes \underbrace{\epsilon(a)}_{=1} + \underbrace{\mathrm{id}(b)}_{=b} \otimes \underbrace{\epsilon(d)}_{=\epsilon(d)1} \\
&= d \otimes 1 + \underbrace{b \otimes \epsilon(d) 1}_{\substack{=b\epsilon(d) \otimes 1 \\ \text{(since } \epsilon(d) \text{ is a scalar in } \mathbf{k} \text{ and thus} \\ \text{can be moved past the } \otimes \text{ sign)}}} = d \otimes 1 + b\epsilon(d) \otimes 1 \\
&= (d + b\epsilon(d)) \otimes 1 \qquad (54)
\end{aligned}$$

(since the map (53) is $\mathbf{k}$-bilinear).

Define the maps can_1 and can_2 as in Remark 3.4. Applying the map can_1 to both sides of the equality (54), we obtain

$$\mathrm{can}_1((\mathrm{id} \otimes \epsilon)(\Delta(d))) = \mathrm{can}_1((d + b\epsilon(d)) \otimes 1) = d + b\epsilon(d)$$

(by the definition of can_1). Hence,

$$d + b\epsilon(d) = \mathrm{can}_1((\mathrm{id} \otimes \epsilon)(\Delta(d))) = \underbrace{(\mathrm{can}_1 \circ (\mathrm{id} \otimes \epsilon) \circ \Delta)}_{\substack{=\mathrm{id} \\ \text{(by (51))}}}(d) = \mathrm{id}(d) = d.$$

Subtracting d from both sides of this equality, we obtain $b\epsilon(d) = 0$. Applying the map ϵ to both sides of this equality, we find $\epsilon(b\epsilon(d)) = 0$. In view of

$$\begin{aligned}
\epsilon\left(\underbrace{b\epsilon(d)}_{=\epsilon(d)b}\right) &= \epsilon(\epsilon(d)b) = \epsilon(d)\underbrace{\epsilon(b)}_{=1} \qquad \text{(since the map } \epsilon \text{ is } \mathbf{k}\text{-linear)} \\
&= \epsilon(d),
\end{aligned}$$

this rewrites as $\epsilon(d) = 0$. This proves Lemma 3.5. □

Next, let us define a "reduced identity map" $\overline{\mathrm{id}}$ for any connected filtered $\mathbf{k}$-coalgebra C, and explore some of its properties:

Lemma 3.6. Let C be a connected filtered $\mathbf{k}$-coalgebra with filtration $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$. Define a $\mathbf{k}$-linear map $\overline{\mathrm{id}} : C \to C$ by setting

$$\overline{\mathrm{id}}(c) := c - \epsilon(c) 1_C \qquad \text{for each } c \in C.$$

Define a $\mathbf{k}$-linear map $\delta : C \to C \otimes C$ by setting

$$\delta(c) := \Delta(c) - c \otimes 1_C - 1_C \otimes c + \epsilon(c) 1_C \otimes 1_C \qquad \text{for each } c \in C.$$

Then:

(a) We have $\delta = \left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right) \circ \Delta$. (Here, of course, Δ denotes the comultiplication of C.)

(b) We have $\overline{\mathrm{id}}(C_{\leq n}) \subseteq C_{\leq n}$ for each $n \in \mathbb{N}$.

(c) We have $\overline{\mathrm{id}}(C_{\leq 0}) = 0$.

Proof of Lemma 3.6. Let us first observe that the map

$$\begin{aligned} C \times C &\to C \otimes C, \\ (u, v) &\mapsto u \otimes v \end{aligned} \tag{55}$$

is $\mathbf{k}$-bilinear. (This follows straight from the definition of the tensor product.)

(a) Let $c \in C$. The element $\Delta(c)$ is a tensor in $C \otimes C$, and thus can be written in the form

$$\Delta(c) = \sum_{i=1}^{m} \lambda_i c_i \otimes d_i \tag{56}$$

for some $m \in \mathbb{N}$, some $\lambda_1, \lambda_2, \ldots, \lambda_m \in \mathbf{k}$, some $c_1, c_2, \ldots, c_m \in C$ and some $d_1, d_2, \ldots, d_m \in C$. Consider this m, these $\lambda_1, \lambda_2, \ldots, \lambda_m$, these $c_1, c_2, \ldots, c_m$ and these $d_1, d_2, \ldots, d_m$.

Now, define the maps can_1 and can_2 as in Remark 3.4. Applying the map $\mathrm{id} \otimes \epsilon$ to both sides of the equality (56), we obtain

$$\begin{aligned} (\mathrm{id} \otimes \epsilon)(\Delta(c)) &= (\mathrm{id} \otimes \epsilon)\left(\sum_{i=1}^{m} \lambda_i c_i \otimes d_i\right) \\ &= \sum_{i=1}^{m} \lambda_i \underbrace{(\mathrm{id} \otimes \epsilon)(c_i \otimes d_i)}_{=\mathrm{id}(c_i) \otimes \epsilon(d_i)} \qquad \text{(since the map } \mathrm{id} \otimes \epsilon \text{ is } \mathbf{k}\text{-linear)} \\ &= \sum_{i=1}^{m} \lambda_i \underbrace{\mathrm{id}(c_i)}_{=c_i} \otimes \underbrace{\epsilon(d_i)}_{=\epsilon(d_i)1} = \sum_{i=1}^{m} \lambda_i \underbrace{c_i \otimes \epsilon(d_i) 1}_{\substack{=c_i \epsilon(d_i) \otimes 1 \\ \text{(since } \epsilon(d_i) \text{ is a scalar in } \mathbf{k} \text{ and thus} \\ \text{can be moved past the } \otimes \text{ sign)}}} \\ &= \sum_{i=1}^{m} \lambda_i c_i \epsilon(d_i) \otimes 1. \end{aligned}$$

Applying the map can_1 to both sides of this equality, we obtain

$$\begin{aligned}\mathrm{can}_1\left(\left(\mathrm{id}\otimes\epsilon\right)\left(\Delta\left(c\right)\right)\right) &= \mathrm{can}_1\left(\sum_{i=1}^{m}\lambda_i c_i \epsilon\left(d_i\right)\otimes 1\right)\\ &= \sum_{i=1}^{m}\lambda_i \underbrace{\mathrm{can}_1\left(c_i\epsilon\left(d_i\right)\otimes 1\right)}_{\substack{=c_i\epsilon(d_i)\\ \text{(by the definition of } \mathrm{can}_1\text{)}}} \qquad \text{(since the map } \mathrm{can}_1 \text{ is } \mathbf{k}\text{-linear)}\\ &= \sum_{i=1}^{m}\lambda_i c_i \epsilon\left(d_i\right).\end{aligned}$$

Comparing this with

$$\mathrm{can}_1\left(\left(\mathrm{id}\otimes\epsilon\right)\left(\Delta\left(c\right)\right)\right) = \underbrace{\left(\mathrm{can}_1\circ\left(\mathrm{id}\otimes\epsilon\right)\circ\Delta\right)}_{\substack{=\mathrm{id}\\ \text{(by (51))}}}\left(c\right) = \mathrm{id}\left(c\right) = c,$$

we obtain

$$\sum_{i=1}^{m}\lambda_i c_i \epsilon\left(d_i\right) = c. \tag{57}$$

Furthermore, applying the map $\epsilon\otimes\mathrm{id}$ to both sides of the equality (56), we obtain

$$\begin{aligned}\left(\epsilon\otimes\mathrm{id}\right)\left(\Delta\left(c\right)\right) &= \left(\epsilon\otimes\mathrm{id}\right)\left(\sum_{i=1}^{m}\lambda_i c_i\otimes d_i\right)\\ &= \sum_{i=1}^{m}\lambda_i \underbrace{\left(\epsilon\otimes\mathrm{id}\right)\left(c_i\otimes d_i\right)}_{=\epsilon(c_i)\otimes\mathrm{id}(d_i)} \qquad \text{(since the map } \epsilon\otimes\mathrm{id} \text{ is } \mathbf{k}\text{-linear)}\\ &= \sum_{i=1}^{m}\lambda_i \underbrace{\epsilon\left(c_i\right)}_{=1\epsilon(c_i)}\otimes\underbrace{\mathrm{id}\left(d_i\right)}_{=d_i} = \sum_{i=1}^{m}\lambda_i \underbrace{1\epsilon\left(c_i\right)\otimes d_i}_{\substack{=1\otimes\epsilon(c_i)d_i\\ \text{(since } \epsilon(c_i) \text{ is a scalar in } \mathbf{k} \text{ and thus}\\ \text{can be moved past the } \otimes \text{ sign)}}}\\ &= \sum_{i=1}^{m}\lambda_i 1\otimes\epsilon\left(c_i\right)d_i.\end{aligned}$$

Applying the map can_2 to both sides of this equality, we obtain

$$\begin{aligned}\mathrm{can}_2\left(\left(\epsilon\otimes\mathrm{id}\right)\left(\Delta\left(c\right)\right)\right) &= \mathrm{can}_2\left(\sum_{i=1}^{m}\lambda_i 1\otimes\epsilon\left(c_i\right)d_i\right)\\ &= \sum_{i=1}^{m}\lambda_i \underbrace{\mathrm{can}_2\left(1\otimes\epsilon\left(c_i\right)d_i\right)}_{\substack{=\epsilon(c_i)d_i\\ \text{(by the definition of } \mathrm{can}_2\text{)}}} \qquad \text{(since the map } \mathrm{can}_2 \text{ is } \mathbf{k}\text{-linear)}\\ &= \sum_{i=1}^{m}\lambda_i \epsilon\left(c_i\right)d_i.\end{aligned}$$

Comparing this with

$$\operatorname{can}_2\left(\left(\epsilon \otimes \operatorname{id}\right)\left(\Delta\left(c\right)\right)\right) = \underbrace{\left(\operatorname{can}_2 \circ \left(\epsilon \otimes \operatorname{id}\right) \circ \Delta\right)}_{\substack{=\operatorname{id} \\ \text{(by (52))}}}\left(c\right) = \operatorname{id}\left(c\right) = c,$$

we obtain

$$\sum_{i=1}^{m} \lambda_i \epsilon\left(c_i\right) d_i = c. \tag{58}$$

Applying the map $\overline{\operatorname{id}}$ to both sides of this equality, we obtain

$$\overline{\operatorname{id}}\left(\sum_{i=1}^{m} \lambda_i \epsilon\left(c_i\right) d_i\right) = \overline{\operatorname{id}}\left(c\right) = c - \epsilon\left(c\right) 1_C$$

(by the definition of $\overline{\operatorname{id}}$). Hence,

$$c - \epsilon\left(c\right) 1_C = \overline{\operatorname{id}}\left(\sum_{i=1}^{m} \lambda_i \epsilon\left(c_i\right) d_i\right) = \sum_{i=1}^{m} \lambda_i \epsilon\left(c_i\right) \overline{\operatorname{id}}\left(d_i\right) \tag{59}$$

(since the map $\overline{\operatorname{id}}$ is $\mathbf{k}$-linear).

Now, applying the map $\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}$ to both sides of the equality (56), we obtain

$$\begin{aligned}
&\left(\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}\right)\left(\Delta\left(c\right)\right) \\
&= \left(\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}\right)\left(\sum_{i=1}^{m} \lambda_i c_i \otimes d_i\right) \\
&= \sum_{i=1}^{m} \lambda_i \underbrace{\left(\overline{\operatorname{id}} \otimes \overline{\operatorname{id}}\right)\left(c_i \otimes d_i\right)}_{=\overline{\operatorname{id}}(c_i) \otimes \overline{\operatorname{id}}(d_i)} \qquad \left(\text{since the map } \overline{\operatorname{id}} \otimes \overline{\operatorname{id}} \text{ is } \mathbf{k}\text{-linear}\right) \\
&= \sum_{i=1}^{m} \lambda_i \underbrace{\overline{\operatorname{id}}\left(c_i\right)}_{\substack{=c_i - \epsilon(c_i) 1_C \\ \text{(by the definition of } \overline{\operatorname{id}}\text{)}}} \otimes \overline{\operatorname{id}}\left(d_i\right) \\
&= \sum_{i=1}^{m} \lambda_i \underbrace{\left(c_i - \epsilon\left(c_i\right) 1_C\right) \otimes \overline{\operatorname{id}}\left(d_i\right)}_{\substack{=c_i \otimes \overline{\operatorname{id}}(d_i) - \epsilon(c_i) 1_C \otimes \overline{\operatorname{id}}(d_i) \\ \text{(since the map (55) is } \mathbf{k}\text{-bilinear)}}} \\
&= \sum_{i=1}^{m} \lambda_i \left(c_i \otimes \overline{\operatorname{id}}\left(d_i\right) - \epsilon\left(c_i\right) 1_C \otimes \overline{\operatorname{id}}\left(d_i\right)\right) \\
&= \sum_{i=1}^{m} \lambda_i c_i \otimes \overline{\operatorname{id}}\left(d_i\right) - \sum_{i=1}^{m} \lambda_i \epsilon\left(c_i\right) 1_C \otimes \overline{\operatorname{id}}\left(d_i\right).
\end{aligned} \tag{60}$$

We shall now separately simplify the two sums on the right hand side of this equality.

Indeed, we have

$$\sum_{i=1}^{m} \lambda_i c_i \otimes \underbrace{\overline{\mathrm{id}}\left(d_i\right)}_{\substack{=d_i-\epsilon(d_i)1_C \\ \text{(by the definition of } \overline{\mathrm{id}}\text{)}}}$$

$$= \sum_{i=1}^{m} \underbrace{\lambda_i c_i \otimes \left(d_i - \epsilon\left(d_i\right) 1_C\right)}_{\substack{=\lambda_i c_i \otimes d_i - \lambda_i c_i \otimes \epsilon(d_i) 1_C \\ \text{(since the map (55) is } \mathbf{k}\text{-bilinear)}}}$$

$$= \sum_{i=1}^{m} \left(\lambda_i c_i \otimes d_i - \lambda_i c_i \otimes \epsilon\left(d_i\right) 1_C\right)$$

$$= \underbrace{\sum_{i=1}^{m} \lambda_i c_i \otimes d_i}_{\substack{=\Delta(c) \\ \text{(by (56))}}} - \sum_{i=1}^{m} \underbrace{\lambda_i c_i \otimes \epsilon\left(d_i\right) 1_C}_{\substack{=\lambda_i c_i \epsilon(d_i) \otimes 1_C \\ \text{(since } \epsilon(d_i) \text{ is a scalar in } \mathbf{k} \text{ and thus} \\ \text{can be moved past the } \otimes \text{ sign)}}}$$

$$= \Delta\left(c\right) - \underbrace{\sum_{i=1}^{m} \lambda_i c_i \epsilon\left(d_i\right) \otimes 1_C}_{\substack{=\left(\sum_{i=1}^{m} \lambda_i c_i \epsilon(d_i)\right) \otimes 1_C \\ \text{(since the map (55) is } \mathbf{k}\text{-bilinear)}}} = \Delta\left(c\right) - \underbrace{\left(\sum_{i=1}^{m} \lambda_i c_i \epsilon\left(d_i\right)\right)}_{\substack{=c \\ \text{(by (57))}}} \otimes 1_C$$

$$= \Delta\left(c\right) - c \otimes 1_C$$

and

$$\begin{aligned}
&\sum_{i=1}^{m} \underbrace{\lambda_i \epsilon\left(c_i\right) 1_C}_{=1_C \lambda_i \epsilon(c_i)} \otimes \overline{\mathrm{id}}\left(d_i\right) \\
&= \sum_{i=1}^{m} \underbrace{1_C \lambda_i \epsilon\left(c_i\right) \otimes \overline{\mathrm{id}}\left(d_i\right)}_{\substack{=1_C \otimes \lambda_i \epsilon(c_i)\, \overline{\mathrm{id}}(d_i) \\ \text{(since } \lambda_i \epsilon(c_i) \text{ is a scalar in } \mathbf{k} \text{ and thus} \\ \text{can be moved past the } \otimes \text{ sign)}}} \\
&= \sum_{i=1}^{m} 1_C \otimes \lambda_i \epsilon\left(c_i\right) \overline{\mathrm{id}}\left(d_i\right) = 1_C \otimes \underbrace{\left(\sum_{i=1}^{m} \lambda_i \epsilon\left(c_i\right) \overline{\mathrm{id}}\left(d_i\right)\right)}_{\substack{=c-\epsilon(c) 1_C \\ \text{(by (59))}}} \\
&\qquad \text{(since the map (55) is } \mathbf{k}\text{-bilinear)} \\
&= 1_C \otimes \left(c - \epsilon\left(c\right) 1_C\right) = 1_C \otimes c - \underbrace{1_C \otimes \epsilon\left(c\right) 1_C}_{\substack{=1_C \epsilon(c) \otimes 1_C \\ \text{(since } \epsilon(c) \text{ is a scalar in } \mathbf{k} \text{ and thus} \\ \text{can be moved past the } \otimes \text{ sign)}}} \\
&\qquad \text{(since the map (55) is } \mathbf{k}\text{-bilinear)} \\
&= 1_C \otimes c - \underbrace{1_C \epsilon\left(c\right)}_{=\epsilon(c) 1_C} \otimes 1_C = 1_C \otimes c - \epsilon\left(c\right) 1_C \otimes 1_C.
\end{aligned}$$

In light of these two equalities, we can rewrite (60) as

$$\left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right)\left(\Delta\left(c\right)\right) = \left(\Delta\left(c\right) - c \otimes 1_C\right) - \left(1_C \otimes c - \epsilon\left(c\right) 1_C \otimes 1_C\right).$$

Comparing this with

$$\begin{aligned}
\delta\left(c\right) &= \Delta\left(c\right) - c \otimes 1_C - 1_C \otimes c + \epsilon\left(c\right) 1_C \otimes 1_C \qquad \text{(by the definition of } \delta\text{)} \\
&= \left(\Delta\left(c\right) - c \otimes 1_C\right) - \left(1_C \otimes c - \epsilon\left(c\right) 1_C \otimes 1_C\right),
\end{aligned}$$

we obtain $\delta\left(c\right) = \left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right)\left(\Delta\left(c\right)\right) = \left(\left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right) \circ \Delta\right)\left(c\right)$.

Forget that we fixed c. We thus have shown that $\delta\left(c\right) = \left(\left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right) \circ \Delta\right)\left(c\right)$ for each $c \in C$. In other words, $\delta = \left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right) \circ \Delta$. This proves Lemma 3.6 **(a)**.

(b) Let $n \in \mathbb{N}$. Let $c \in C_{\leq n}$. We have $1_C \in C_{\leq 0}$ (by Definition 2.3 **(b)**). However, (9) yields $C_{\leq 0} \subseteq C_{\leq 1} \subseteq C_{\leq 2} \subseteq \cdots$ (since C is a filtered $\mathbf{k}$-coalgebra). Therefore, $C_{\leq 0} \subseteq C_{\leq n}$. Thus, $1_C \in C_{\leq 0} \subseteq C_{\leq n}$. Now, the definition of $\overline{\mathrm{id}}$ yields

$$\overline{\mathrm{id}}\left(c\right) = \underbrace{c}_{\in C_{\leq n}} - \epsilon\left(c\right) \underbrace{1_C}_{\in C_{\leq n}} \in C_{\leq n} - \epsilon\left(c\right) C_{\leq n} \subseteq C_{\leq n}$$

(since $C_{\leq n}$ is a $\mathbf{k}$-module).

Forget that we fixed c. We thus have shown that $\overline{\mathrm{id}}(c) \in C_{\leq n}$ for each $c \in C_{\leq n}$. In other words, we have $\overline{\mathrm{id}}(C_{\leq n}) \subseteq C_{\leq n}$. This proves Lemma 3.6 **(b)**.

(c) The filtered $\mathbf{k}$-coalgebra C is connected. In other words, the restriction $\epsilon|_{C_{\leq 0}}$ is a $\mathbf{k}$-module isomorphism from $C_{\leq 0}$ to $\mathbf{k}$ (by Definition 2.3 **(a)**). Hence, this restriction $\epsilon|_{C_{\leq 0}}$ is bijective, and thus injective. Also, we have $1_C \in C_{\leq 0}$ (by Definition 2.3 **(b)**).

Now, let $c \in C_{\leq 0}$. Hence, $\left(\epsilon|_{C_{\leq 0}}\right)(c)$ is well-defined. Definition 2.3 **(b)** yields $1_C = \left(\epsilon|_{C_{\leq 0}}\right)^{-1}(1_{\mathbf{k}})$. Thus, $\left(\epsilon|_{C_{\leq 0}}\right)(1_C) = 1_{\mathbf{k}}$. In other words, $\epsilon(1_C) = 1_{\mathbf{k}}$ (since $\left(\epsilon|_{C_{\leq 0}}\right)(1_C) = \epsilon(1_C)$).

Set $d = \epsilon(c)\, 1_C$. Then, $d = \epsilon(c) \underbrace{1_C}_{\in C_{\leq 0}} \in \epsilon(c)\, C_{\leq 0} \subseteq C_{\leq 0}$ (since $C_{\leq 0}$ is a $\mathbf{k}$-module). Thus, $\left(\epsilon|_{C_{\leq 0}}\right)(d)$ is well-defined.

Comparing

$$\left(\epsilon|_{C_{\leq 0}}\right)(c) = \epsilon(c)$$

with

$$\begin{aligned}\left(\epsilon|_{C_{\leq 0}}\right)(d) &= \epsilon\left(\underbrace{d}_{=\epsilon(c)1_C}\right) = \epsilon(\epsilon(c)\, 1_C) \\ &= \epsilon(c)\, \underbrace{\epsilon(1_C)}_{=1_{\mathbf{k}}} \qquad \text{(since the map } \epsilon \text{ is } \mathbf{k}\text{-linear)} \\ &= \epsilon(c),\end{aligned}$$

we obtain $\left(\epsilon|_{C_{\leq 0}}\right)(c) = \left(\epsilon|_{C_{\leq 0}}\right)(d)$.

However, the map $\epsilon|_{C_{\leq 0}}$ is injective. In other words, if u and v are two elements of $C_{\leq 0}$ satisfying $\left(\epsilon|_{C_{\leq 0}}\right)(u) = \left(\epsilon|_{C_{\leq 0}}\right)(v)$, then $u = v$. Applying this to $u = c$ and $v = d$, we obtain $c = d$ (since $\left(\epsilon|_{C_{\leq 0}}\right)(c) = \left(\epsilon|_{C_{\leq 0}}\right)(d)$). Now, the definition of $\overline{\mathrm{id}}$ yields

$$\overline{\mathrm{id}}(c) = c - \underbrace{\epsilon(c)\, 1_C}_{\substack{=d \\ \text{(by the definition of } d)}} = c - d = 0 \qquad (\text{since } c = d).$$

Forget that we have fixed c. We thus have shown that $\overline{\mathrm{id}}(c) = 0$ for each $c \in C_{\leq 0}$. In other words, $\overline{\mathrm{id}}(C_{\leq 0}) = 0$. This proves Lemma 3.6 **(c)**. □

Proof of Proposition 2.4. **(a)** Define a $\mathbf{k}$-linear map $\overline{\mathrm{id}} : C \to C$ as in Lemma 3.6. Then, Lemma 3.6 **(a)** yields $\delta = \left(\overline{\mathrm{id}} \otimes \overline{\mathrm{id}}\right) \circ \Delta$.

Now, let $n > 0$ be an integer. Then, (11) yields

$$\Delta(C_{\leq n}) \subseteq \sum_{i=0}^{n} C_{\leq i} \otimes C_{\leq n-i} \tag{61}$$

(since C is a filtered $\mathbf{k}$-coalgebra). Now,

$$\underbrace{\delta}_{=\left(\overline{\mathrm{id}}\otimes\overline{\mathrm{id}}\right)\circ\Delta}\left(C_{\leq n}\right)$$

$$= \left(\left(\overline{\mathrm{id}}\otimes\overline{\mathrm{id}}\right)\circ\Delta\right)\left(C_{\leq n}\right) = \left(\overline{\mathrm{id}}\otimes\overline{\mathrm{id}}\right)\underbrace{\left(\Delta\left(C_{\leq n}\right)\right)}_{\substack{\subseteq\sum\limits_{i=0}^{n} C_{\leq i}\otimes C_{\leq n-i} \\ \text{(by (61))}}}$$

$$\subseteq \left(\overline{\mathrm{id}}\otimes\overline{\mathrm{id}}\right)\left(\sum_{i=0}^{n} C_{\leq i}\otimes C_{\leq n-i}\right)$$

$$= \sum_{i=0}^{n} \underbrace{\left(\overline{\mathrm{id}}\otimes\overline{\mathrm{id}}\right)\left(C_{\leq i}\otimes C_{\leq n-i}\right)}_{\substack{=\overline{\mathrm{id}}(C_{\leq i})\otimes\overline{\mathrm{id}}(C_{\leq n-i}) \\ \text{(by (35), applied to } D=C,\ \alpha=\overline{\mathrm{id}},\ \beta=\overline{\mathrm{id}},\ U=C_{\leq i} \text{ and } V=C_{\leq n-i})}}$$

$$\left(\text{since the map } \overline{\mathrm{id}}\otimes\overline{\mathrm{id}} \text{ is } \mathbf{k}\text{-linear}\right)$$

$$= \sum_{i=0}^{n} \overline{\mathrm{id}}\left(C_{\leq i}\right)\otimes\overline{\mathrm{id}}\left(C_{\leq n-i}\right)$$

$$= \overline{\mathrm{id}}\left(C_{\leq 0}\right)\otimes\overline{\mathrm{id}}\left(C_{\leq n-0}\right) + \sum_{i=1}^{n-1}\overline{\mathrm{id}}\left(C_{\leq i}\right)\otimes\overline{\mathrm{id}}\left(C_{\leq n-i}\right) + \overline{\mathrm{id}}\left(C_{\leq n}\right)\otimes\underbrace{\overline{\mathrm{id}}\left(C_{\leq n-n}\right)}_{\substack{=\overline{\mathrm{id}}(C_{\leq 0}) \\ \text{(since } n-n=0)}}$$

$$\left(\begin{array}{c}\text{here, we have split off the addends for } i=0 \text{ and for } i=n \text{ from the} \\ \text{sum (and these are indeed two distinct addends, since } n>0)\end{array}\right)$$

$$= \underbrace{\overline{\mathrm{id}}\left(C_{\leq 0}\right)}_{\substack{=0 \\ \text{(by Lemma 3.6 \textbf{(c)})}}}\otimes\overline{\mathrm{id}}\left(C_{\leq n-0}\right) + \sum_{i=1}^{n-1}\overline{\mathrm{id}}\left(C_{\leq i}\right)\otimes\overline{\mathrm{id}}\left(C_{\leq n-i}\right) + \overline{\mathrm{id}}\left(C_{\leq n}\right)\otimes\underbrace{\overline{\mathrm{id}}\left(C_{\leq 0}\right)}_{\substack{=0 \\ \text{(by Lemma 3.6 \textbf{(c)})}}}$$

$$= \underbrace{0\otimes\overline{\mathrm{id}}\left(C_{\leq n-0}\right)}_{=0} + \sum_{i=1}^{n-1}\overline{\mathrm{id}}\left(C_{\leq i}\right)\otimes\overline{\mathrm{id}}\left(C_{\leq n-i}\right) + \underbrace{\overline{\mathrm{id}}\left(C_{\leq n}\right)\otimes 0}_{=0}$$

$$= \sum_{i=1}^{n-1}\underbrace{\overline{\mathrm{id}}\left(C_{\leq i}\right)}_{\substack{\subseteq C_{\leq i} \\ \text{(by Lemma 3.6 \textbf{(b)}, } \\ \text{applied to } i \text{ instead of } n)}}\otimes\underbrace{\overline{\mathrm{id}}\left(C_{\leq n-i}\right)}_{\substack{\subseteq C_{\leq n-i} \\ \text{(by Lemma 3.6 \textbf{(b)}, } \\ \text{applied to } n-i \text{ instead of } n)}}$$

$$\subseteq \sum_{i=1}^{n-1} C_{\leq i}\otimes C_{\leq n-i}.$$

This proves Proposition 2.4 **(a)**.

(b) Let $f : C \to C$ be a $\mathbf{k}$-coalgebra homomorphism satisfying $f\left(1_C\right) = 1_C$. Thus, f is a $\mathbf{k}$-coalgebra homomorphism; in other words, f is a $\mathbf{k}$-linear map satisfying

$(f \otimes f) \circ \Delta = \Delta \circ f$ and $\epsilon = \epsilon \circ f$ (by the definition of a "**k**-coalgebra homomorphism").

Let $c \in C$. Then,

$$\begin{aligned}
&((f \otimes f) \circ \delta)(c) \\
&= (f \otimes f)\left(\underbrace{\delta(c)}_{\substack{=\Delta(c)-c\otimes 1_C-1_C\otimes c+\epsilon(c)1_C\otimes 1_C \\ \text{(by the definition of } \delta)}}\right) \\
&= (f \otimes f)(\Delta(c) - c \otimes 1_C - 1_C \otimes c + \epsilon(c) 1_C \otimes 1_C) \\
&= \underbrace{(f \otimes f)(\Delta(c))}_{=((f\otimes f)\circ\Delta)(c)} - \underbrace{(f \otimes f)(c \otimes 1_C)}_{=f(c)\otimes f(1_C)} - \underbrace{(f \otimes f)(1_C \otimes c)}_{=f(1_C)\otimes f(c)} + \epsilon(c) \underbrace{(f \otimes f)(1_C \otimes 1_C)}_{=f(1_C)\otimes f(1_C)} \\
&\qquad \text{(since the map } f \otimes f \text{ is } \mathbf{k}\text{-linear)} \\
&= \underbrace{((f \otimes f) \circ \Delta)}_{=\Delta\circ f}(c) - f(c) \otimes \underbrace{f(1_C)}_{=1_C} - \underbrace{f(1_C)}_{=1_C} \otimes f(c) + \underbrace{\epsilon}_{=\epsilon\circ f}(c) \underbrace{f(1_C)}_{=1_C} \otimes \underbrace{f(1_C)}_{=1_C} \\
&= \underbrace{(\Delta \circ f)(c)}_{=\Delta(f(c))} - f(c) \otimes 1_C - 1_C \otimes f(c) + \underbrace{(\epsilon \circ f)(c)}_{=\epsilon(f(c))} 1_C \otimes 1_C \\
&= \Delta(f(c)) - f(c) \otimes 1_C - 1_C \otimes f(c) + \epsilon(f(c)) 1_C \otimes 1_C.
\end{aligned}$$

Comparing this with

$$(\delta \circ f)(c) = \delta(f(c)) = \Delta(f(c)) - f(c) \otimes 1_C - 1_C \otimes f(c) + \epsilon(f(c)) 1_C \otimes 1_C \quad \text{(by the definition of } \delta),$$

we obtain $((f \otimes f) \circ \delta)(c) = (\delta \circ f)(c)$.

Forget that we fixed c. We thus have shown that $((f \otimes f) \circ \delta)(c) = (\delta \circ f)(c)$ for each $c \in C$. In other words, $(f \otimes f) \circ \delta = \delta \circ f$. This proves Proposition 2.4 **(b)**.

(c) Definition 2.3 **(b)** yields $1_C = \left(\epsilon|_{C_{\leq 0}}\right)^{-1}(1_{\mathbf{k}})$. Thus, $\left(\epsilon|_{C_{\leq 0}}\right)(1_C) = 1_{\mathbf{k}}$. In other words, $\epsilon(1_C) = 1$ (since $\left(\epsilon|_{C_{\leq 0}}\right)(1_C) = \epsilon(1_C)$ and $1_{\mathbf{k}} = 1$).

Let $c \in (\operatorname{Ker}\delta) \cap (\operatorname{Ker}\epsilon)$. Thus, $c \in (\operatorname{Ker}\delta) \cap (\operatorname{Ker}\epsilon) \subseteq \operatorname{Ker}\delta$, so that $\delta(c) = 0$. Moreover, $c \in (\operatorname{Ker}\delta) \cap (\operatorname{Ker}\epsilon) \subseteq \operatorname{Ker}\epsilon$, so that $\epsilon(c) = 0$. However, from $\delta(c) = 0$, we obtain

$$\begin{aligned}
0 = \delta(c) &= \Delta(c) - c \otimes 1_C - 1_C \otimes c + \underbrace{\epsilon(c)}_{=0} 1_C \otimes 1_C \qquad \text{(by the definition of } \delta) \\
&= \Delta(c) - c \otimes 1_C - 1_C \otimes c + \underbrace{0 \cdot 1_C \otimes 1_C}_{=0} \\
&= \Delta(c) - c \otimes 1_C - 1_C \otimes c.
\end{aligned}$$

In other words, $\Delta(c) = c \otimes 1_C + 1_C \otimes c$. In other words, the element c of C is primitive (by the definition of "primitive"). In other words, $c \in \operatorname{Prim} C$ (since $\operatorname{Prim} C$ is defined as the set of all primitive elements of C).

Forget that we fixed c. We thus have shown that $c \in \operatorname{Prim} C$ for each $c \in (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$. In other words, $(\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) \subseteq \operatorname{Prim} C$.

Now, let $d \in \operatorname{Prim} C$. Thus, the element d of C is primitive (since $\operatorname{Prim} C$ is defined as the set of all primitive elements of C). In other words, $\Delta(d) = d \otimes 1_C + 1_C \otimes d$ (by the definition of "primitive"). Hence, Lemma 3.5 (applied to 1_C and 1_C instead of a and b) yields $\epsilon(d) = 0$ (since $\epsilon(1_C) = 1$). Hence, $d \in \operatorname{Ker} \epsilon$.

Furthermore, the definition of δ yields

$$\delta(d) = \underbrace{\Delta(d) - d \otimes 1_C - 1_C \otimes d}_{\substack{=0 \\ (\text{since } \Delta(d) = d \otimes 1_C + 1_C \otimes d)}} + \epsilon(d) 1_C \otimes 1_C = \underbrace{\epsilon(d)}_{=0} 1_C \otimes 1_C = 0.$$

Hence, $d \in \operatorname{Ker} \delta$. Combining this with $d \in \operatorname{Ker} \epsilon$, we obtain $d \in (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$.

Forget that we fixed d. We thus have shown that $d \in (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$ for each $d \in \operatorname{Prim} C$. In other words, $\operatorname{Prim} C \subseteq (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$. Combining this with $(\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) \subseteq \operatorname{Prim} C$, we obtain $\operatorname{Prim} C = (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$. This proves Proposition 2.4 **(c)**.

(d) Proposition 2.4 **(c)** yields $\operatorname{Prim} C = (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$.

The maps δ and ϵ are $\mathbf{k}$-linear. Hence, their kernels $\operatorname{Ker} \delta$ and $\operatorname{Ker} \epsilon$ are $\mathbf{k}$-submodules of C. The intersection $(\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$ of these two kernels must therefore be a $\mathbf{k}$-submodule of C as well. In other words, $\operatorname{Prim} C$ is a $\mathbf{k}$-submodule of C (since $\operatorname{Prim} C = (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon)$). This proves Proposition 2.4 **(d)**.

(e) We first observe that $1_C \in C_{\leq 0}$ (by Definition 2.3 **(b)**). However, (9) yields $C_{\leq 0} \subseteq C_{\leq 1} \subseteq C_{\leq 2} \subseteq \cdots$ (since C is a filtered $\mathbf{k}$-coalgebra). Therefore, $C_{\leq 0} \subseteq C_{\leq 1}$. Thus, $1_C \in C_{\leq 0} \subseteq C_{\leq 1}$. Hence,

$$\begin{aligned} \delta(1_C) \in \delta(C_{\leq 1}) &\subseteq \sum_{i=1}^{1-1} C_{\leq i} \otimes C_{\leq 1-i} \\ &\qquad (\text{by Proposition 2.4 } \textbf{(a)}, \text{ applied to } n = 1) \\ &= (\text{empty sum}) = 0. \end{aligned}$$

In other words, $\delta(1_C) = 0$.

Definition 2.3 **(b)** yields $1_C = \left(\epsilon|_{C_{\leq 0}}\right)^{-1}(1_{\mathbf{k}})$. Thus, $\left(\epsilon|_{C_{\leq 0}}\right)(1_C) = 1_{\mathbf{k}}$. In other words, $\epsilon(1_C) = 1$ (since $\left(\epsilon|_{C_{\leq 0}}\right)(1_C) = \epsilon(1_C)$ and $1_{\mathbf{k}} = 1$).

Let $u \in \operatorname{Ker} \delta$. Thus, $\delta(u) = 0$. Set $v = u - \epsilon(u) 1_C$. Then,

$$\begin{aligned} \delta\Big(\underbrace{v}_{=u-\epsilon(u)1_C}\Big) &= \delta(u - \epsilon(u) 1_C) = \underbrace{\delta(u)}_{=0} - \epsilon(u) \underbrace{\delta(1_C)}_{=0} \qquad (\text{since the map } \delta \text{ is } \mathbf{k}\text{-linear}) \\ &= 0 - \epsilon(u) 0 = 0, \end{aligned}$$

so that $v \in \operatorname{Ker} \delta$. Furthermore,

$$\begin{aligned} \epsilon\Big(\underbrace{v}_{=u-\epsilon(u)1_C}\Big) &= \epsilon(u - \epsilon(u) 1_C) = \epsilon(u) - \epsilon(u) \underbrace{\epsilon(1_C)}_{=1} \qquad (\text{since the map } \epsilon \text{ is } \mathbf{k}\text{-linear}) \\ &= \epsilon(u) - \epsilon(u) = 0, \end{aligned}$$

so that $v \in \operatorname{Ker} \epsilon$. Combining this with $v \in \operatorname{Ker} \delta$, we obtain $v \in (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) = \operatorname{Prim} C$ (by Proposition 2.4 **(c)**). Now, from $v = u - \epsilon(u) 1_C$, we obtain

$$u = \underbrace{\epsilon(u)}_{\in \mathbf{k}} 1_C + \underbrace{v}_{\in \operatorname{Prim} C} \in \mathbf{k} \cdot 1_C + \operatorname{Prim} C.$$

Forget that we fixed u. We thus have shown that $u \in \mathbf{k} \cdot 1_C + \operatorname{Prim} C$ for each $u \in \operatorname{Ker} \delta$. In other words,

$$\operatorname{Ker} \delta \subseteq \mathbf{k} \cdot 1_C + \operatorname{Prim} C. \tag{62}$$

On the other hand, let $w \in \mathbf{k} \cdot 1_C + \operatorname{Prim} C$. Thus, we can write w in the form $w = x + y$ for some $x \in \mathbf{k} \cdot 1_C$ and some $y \in \operatorname{Prim} C$. Consider these x and y. We have

$$\begin{aligned} y &\in \operatorname{Prim} C = (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) \qquad \text{(by Proposition 2.4 } \textbf{(c)}\text{)} \\ &\subseteq \operatorname{Ker} \delta, \end{aligned}$$

so that $\delta(y) = 0$.

We have $x \in \mathbf{k} \cdot 1_C$; in other words, $x = \lambda \cdot 1_C$ for some $\lambda \in \mathbf{k}$. Consider this λ. Now, $w = \underbrace{x}_{=\lambda \cdot 1_C} + y = \lambda \cdot 1_C + y$. Applying the map δ to both sides of this equality, we obtain

$$\begin{aligned} \delta(w) &= \delta(\lambda \cdot 1_C + y) = \lambda \cdot \underbrace{\delta(1_C)}_{=0} + \underbrace{\delta(y)}_{=0} \qquad \text{(since the map } \delta \text{ is } \mathbf{k}\text{-linear)} \\ &= \lambda \cdot 0 + 0 = 0. \end{aligned}$$

In other words, $w \in \operatorname{Ker} \delta$.

Forget that we fixed w. We thus have shown that $w \in \operatorname{Ker} \delta$ for each $w \in \mathbf{k} \cdot 1_C + \operatorname{Prim} C$. In other words,

$$\mathbf{k} \cdot 1_C + \operatorname{Prim} C \subseteq \operatorname{Ker} \delta.$$

Combining this with (62), we obtain $\operatorname{Ker} \delta = \mathbf{k} \cdot 1_C + \operatorname{Prim} C$. This proves Proposition 2.4 **(e)**. □

3.3. Proofs of the corollaries from Section 2.2

Proof of Corollary 2.5. We have $(e - f)(1_C) = \underbrace{e(1_C)}_{=1_C} - \underbrace{f(1_C)}_{=1_C} = 1_C - 1_C = 0$. Hence, $\mathbf{k} \cdot 1_C \subseteq \operatorname{Ker}(e - f)$ [14].

[14] *Proof.* Let $x \in \mathbf{k} \cdot 1_C$. Thus, $x = \lambda \cdot 1_C$ for some $\lambda \in \mathbf{k}$. Consider this λ. Now, applying the map $e - f$ to both sides of the equality $x = \lambda \cdot 1_C$, we obtain

$$\begin{aligned} (e - f)(x) &= (e - f)(\lambda \cdot 1_C) = \lambda \cdot \underbrace{(e - f)(1_C)}_{=0} \qquad \text{(since the map } e - f \text{ is } \mathbf{k}\text{-linear)} \\ &= 0. \end{aligned}$$

Clearly, $(C_{\leq 0}, C_{\leq 1}, C_{\leq 2}, \ldots)$ is a sequence of $\mathbf{k}$-submodules of C (by the definition of a filtered $\mathbf{k}$-coalgebra). Thus, $(C_{\leq 1}, C_{\leq 2}, C_{\leq 3}, \ldots)$ is a sequence of $\mathbf{k}$-submodules of C as well.

Define the $\mathbf{k}$-linear map $\delta : C \to C \otimes C$ as in Proposition 2.4. The map f is a $\mathbf{k}$-coalgebra homomorphism satisfying $f(1_C) = 1_C$. Thus, Proposition 2.4 **(b)** yields that $(f \otimes f) \circ \delta = \delta \circ f$. The same argument (applied to e instead of f) yields $(e \otimes e) \circ \delta = \delta \circ e$. Moreover, Proposition 2.4 **(e)** yields

$$\begin{aligned} \operatorname{Ker} \delta &= \underbrace{\mathbf{k} \cdot 1_C}_{\subseteq \operatorname{Ker}(e-f)} + \underbrace{\operatorname{Prim} C}_{\substack{\subseteq \operatorname{Ker}(e-f) \\ \text{(by (12))}}} \subseteq \operatorname{Ker}(e-f) + \operatorname{Ker}(e-f) \\ &\subseteq \operatorname{Ker}(e-f) \qquad (\text{since } \operatorname{Ker}(e-f) \text{ is a } \mathbf{k}\text{-module}). \end{aligned}$$

However, Proposition 2.4 **(a)** shows that

$$\delta(C_{\leq n}) \subseteq \sum_{i=1}^{n-1} C_{\leq i} \otimes C_{\leq n-i} \qquad \text{for each } n > 0.$$

Hence,

$$\delta(C_{\leq n}) \subseteq \sum_{i=1}^{n-1} C_{\leq i} \otimes C_{\leq n-i} \qquad \text{for each } n > p.$$

Moreover, the definition of a filtered $\mathbf{k}$-coalgebra yields $C_{\leq 0} \subseteq C_{\leq 1} \subseteq C_{\leq 2} \subseteq \cdots$. Hence, each $i \in \{1, 2, \ldots, p\}$ satisfies

$$C_{\leq i} \subseteq C_{\leq p} \tag{63}$$

[15]. Thus,

$$C_{\leq 1} + C_{\leq 2} + \cdots + C_{\leq p} = \sum_{i=1}^{p} \underbrace{C_{\leq i}}_{\substack{\subseteq C_{\leq p} \\ \text{(by (63))}}} \subseteq \sum_{i=1}^{p} C_{\leq p} \subseteq C_{\leq p}$$

(since $C_{\leq p}$ is a $\mathbf{k}$-module). Therefore,

$$(e-f)\left(\underbrace{C_{\leq 1} + C_{\leq 2} + \cdots + C_{\leq p}}_{\subseteq C_{\leq p}}\right) \subseteq (e-f)\left(C_{\leq p}\right) = 0$$

(by (14)), so that $(e-f)\left(C_{\leq 1} + C_{\leq 2} + \cdots + C_{\leq p}\right) = 0$.

In other words, $x \in \operatorname{Ker}(e-f)$.

Forget that we fixed x. We thus have shown that $x \in \operatorname{Ker}(e-f)$ for each $x \in \mathbf{k} \cdot 1_C$. In other words, $\mathbf{k} \cdot 1_C \subseteq \operatorname{Ker}(e-f)$.

[15] *Proof of (63):* Let $i \in \{1, 2, \ldots, p\}$. Thus, $i \in \mathbb{N}$ and $i \leq p$. However, we have $C_{\leq 0} \subseteq C_{\leq 1} \subseteq C_{\leq 2} \subseteq \cdots$. In other words, if a and b are two elements of $\mathbb{N}$ satisfying $a \leq b$, then $C_{\leq a} \subseteq C_{\leq b}$. Applying this to $a = i$ and $b = p$, we obtain $C_{\leq i} \subseteq C_{\leq p}$ (since $i \leq p$). This proves (63).

Hence, Theorem 2.1 (applied to $D = C$ and $D_i = C_{\leq i}$) shows that for any integer $u > p$, we have

$$(e - f)^{u-p} (C_{\leq u}) \subseteq \operatorname{Ker} \delta \tag{64}$$

and

$$(e - f)^{u-p+1} (C_{\leq u}) = 0. \tag{65}$$

We are now close to proving both parts of Corollary 2.5. Let us begin with part **(a)**:

(a) Let $u > p$ be an integer. Then, $u - p > 0$ (since $u > p$), so that $u - p \geq 1$ (since $u - p$ is an integer). Thus, $(e - f)^{u-p} = (e - f) \circ (e - f)^{u-p-1}$. However, f is a $\mathbf{k}$-coalgebra homomorphism, and thus satisfies $\epsilon \circ f = \epsilon$ (by the definition of a $\mathbf{k}$-coalgebra homomorphism). Similarly, $\epsilon \circ e = \epsilon$. Since composition of $\mathbf{k}$-linear maps is a $\mathbf{k}$-bilinear operation (and since the maps ϵ, e and f are $\mathbf{k}$-linear), we have

$$\epsilon \circ (e - f) = \underbrace{\epsilon \circ e}_{=\epsilon} - \underbrace{\epsilon \circ f}_{=\epsilon} = \epsilon - \epsilon = 0.$$

Thus,

$$\epsilon \circ \underbrace{(e - f)^{u-p}}_{=(e-f)\circ(e-f)^{u-p-1}} = \underbrace{\epsilon \circ (e - f)}_{=0} \circ (e - f)^{u-p-1} = 0 \circ (e - f)^{u-p-1} = 0.$$

Therefore, $(e - f)^{u-p} (C_{\leq u}) \subseteq \operatorname{Ker} \epsilon$ [16]. Combining this with (64), we obtain $(e - f)^{u-p} (C_{\leq u}) \subseteq (\operatorname{Ker} \delta) \cap (\operatorname{Ker} \epsilon) = \operatorname{Prim} C$ (by Proposition 2.4 **(c)**). This proves Corollary 2.5 **(a)**.

(b) Let $u \geq p$ be an integer. We must prove that $(e - f)^{u-p+1} (C_{\leq u}) = 0$. If $u > p$, then this follows from (65). Thus, for the rest of this proof, we WLOG assume that we don't have $u > p$. Hence, $u \leq p$. Combining this with $u \geq p$, we obtain $u = p$. Thus,

$$(e - f)^{u-p+1} (C_{\leq u}) = \underbrace{(e - f)^{p-p+1}}_{=(e-f)^1=e-f} (C_{\leq p}) = (e - f) (C_{\leq p}) = 0$$

(by (14)). This proves Corollary 2.5 **(b)**. □

Proof of Corollary 2.6. Define the $\mathbf{k}$-linear map $\delta : C \to C \otimes C$ as in Proposition 2.4. Just as we did in the proof of Corollary 2.5, we can show that $\operatorname{Ker} \delta \subseteq \operatorname{Ker} (e - f)$. However, Proposition 2.4 **(a)** (applied to $n = 1$) yields

$$\delta (C_{\leq 1}) \subseteq \sum_{i=1}^{1-1} C_{\leq i} \otimes C_{\leq 1-i} = (\text{empty sum}) = 0.$$

[16] *Proof.* Let $z \in (e - f)^{u-p} (C_{\leq u})$. Thus, z can be written in the form $z = (e - f)^{u-p} (x)$ for some $x \in C_{\leq u}$. Consider this x. From $z = (e - f)^{u-p} (x)$, we obtain $\epsilon (z) = \epsilon \left((e - f)^{u-p} (x) \right) = \underbrace{\left(\epsilon \circ (e - f)^{u-p} \right)}_{=0} (x) = 0 (x) = 0$. In other words, $z \in \operatorname{Ker} \epsilon$.

Forget that we fixed z. We thus have shown that $z \in \operatorname{Ker} \epsilon$ for each $z \in (e - f)^{u-p} (C_{\leq u})$. In other words, $(e - f)^{u-p} (C_{\leq u}) \subseteq \operatorname{Ker} \epsilon$.

Hence, it is easy to see that $C_{\leq 1} \subseteq \operatorname{Ker} \delta$ [17]. Consequently, $C_{\leq 1} \subseteq \operatorname{Ker} \delta \subseteq \operatorname{Ker}(e-f)$. Thus,

$$(e-f)(C_{\leq 1}) = 0 \tag{66}$$

[18]. Hence, we can apply Corollary 2.5 to $p = 1$.

Therefore, applying Corollary 2.5 **(a)** to $p = 1$, we conclude the following: For any integer $u > 1$, we have

$$(e-f)^{u-1}(C_{\leq u}) \subseteq \operatorname{Prim} C. \tag{67}$$

This proves Corollary 2.6 **(a)**. It remains to prove Corollary 2.6 **(b)**:

(b) Let u be a positive integer. Thus, $u \geq 1$. Hence, Corollary 2.5 **(b)** (applied to $p = 1$) shows that $(e-f)^{u-1+1}(C_{\leq u}) = 0$ (since we know that we can apply Corollary 2.5 to $p = 1$). In view of $u - 1 + 1 = u$, this rewrites as $(e-f)^{u}(C_{\leq u}) = 0$. This proves Corollary 2.6 **(b)**. □

Proof of Corollary 2.7. Clearly, $\operatorname{id} : C \to C$ is a $\mathbf{k}$-coalgebra homomorphism such that $\operatorname{id}(1_C) = 1_C$. Furthermore, $f \circ \operatorname{id} = f = \operatorname{id} \circ f$. Hence, we can apply Corollary 2.6 to $e = \operatorname{id}$. In particular, Corollary 2.6 **(a)** (applied to $e = \operatorname{id}$) yields that for any integer $u > 1$, we have

$$(\operatorname{id} - f)^{u-1}(C_{\leq u}) \subseteq \operatorname{Prim} C.$$

This proves Corollary 2.7 **(a)**. Furthermore, Corollary 2.6 **(b)** (applied to $e = \operatorname{id}$) yields that for any positive integer u, we have

$$(\operatorname{id} - f)^{u}(C_{\leq u}) = 0.$$

This proves Corollary 2.7 **(b)**. □

3.4. Proofs for Section 2.3

Before we prove the claims left unproved in Section 2.3, let us recall the defining property of the antipode of a Hopf algebra:

[17] *Proof.* Let $x \in C_{\leq 1}$. Thus, $\delta(x) \in \delta(C_{\leq 1}) \subseteq 0$, so that $\delta(x) = 0$. In other words, $x \in \operatorname{Ker} \delta$.

Forget that we fixed x. We thus have shown that $x \in \operatorname{Ker} \delta$ for each $x \in C_{\leq 1}$. In other words, $C_{\leq 1} \subseteq \operatorname{Ker} \delta$.

[18] *Proof.* For each $x \in C_{\leq 1}$, we have $(e-f)(x) = 0$ (since $x \in C_{\leq 1} \subseteq \operatorname{Ker}(e-f)$). In other words, we have $(e-f)(C_{\leq 1}) = 0$.

Remark 3.7. Let H be a $\mathbf{k}$-Hopf algebra with antipode S. Let 1_H denote the unity of the $\mathbf{k}$-algebra H. Let $m : H \otimes H \to H$ be the $\mathbf{k}$-linear map that sends each pure tensor $x \otimes y \in H \otimes H$ to the product $xy \in H$. Let $u : \mathbf{k} \to H$ be the $\mathbf{k}$-linear map that sends $1_{\mathbf{k}}$ to 1_H. Then, the diagram

$$\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes \mathrm{id}_H"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\epsilon"] \arrow[dr, "\Delta"'] & & \mathbf{k} \arrow[rr, "u"] & & H \\
 & H \otimes H \arrow[rr, "\mathrm{id}_H \otimes S"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}$$

commutes.[19] In other words, we have

$$m \circ (S \otimes \mathrm{id}_H) \circ \Delta = u \circ \epsilon \qquad \text{and} \tag{68}$$

$$m \circ (\mathrm{id}_H \otimes S) \circ \Delta = u \circ \epsilon. \tag{69}$$

Proof of Lemma 2.12. **(a)** Let $T : H \otimes H \to H \otimes H$ be the $\mathbf{k}$-linear map that sends each pure tensor $x \otimes y \in H \otimes H$ to $y \otimes x$. This map T is known as the *twist map*. It is obviously an involution, i.e., satisfies $T^2 = \mathrm{id}$. Furthermore, it is easy to see that any two $\mathbf{k}$-linear maps $\alpha, \beta \in \operatorname{End} H$ satisfy

$$(\alpha \otimes \beta) \circ T = T \circ (\beta \otimes \alpha). \tag{70}$$

Furthermore, it is well-known (see, e.g., [Abe80, Theorem 2.1.4 (iii) and (iv)] or [GriRei20, Exercise 1.4.28] or, in an equivalent form, [Radfor12, Proposition 7.1.9 (b)]) that the antipode S of H is a $\mathbf{k}$-coalgebra anti-endomorphism, i.e., that it satisfies

$$\Delta \circ S = T \circ (S \otimes S) \circ \Delta \qquad \text{and} \qquad \epsilon \circ S = \epsilon.$$

[19] Indeed, this is the diagram (1.4.3) in [GriRei20].

Now,

$$\begin{aligned}
\Delta \circ \underbrace{S^2}_{=S\circ S} &= \underbrace{\Delta \circ S}_{=T\circ(S\otimes S)\circ\Delta} \circ S = T \circ (S \otimes S) \circ \underbrace{\Delta \circ S}_{=T\circ(S\otimes S)\circ\Delta} \\
&= T \circ \underbrace{(S \otimes S) \circ T}_{\substack{=T\circ(S\otimes S) \\ \text{(by (70), applied to } \alpha=S \text{ and } \beta=S)}} \circ (S \otimes S) \circ \Delta \\
&= \underbrace{T \circ T}_{=T^2=\mathrm{id}} \circ \underbrace{(S \otimes S) \circ (S \otimes S)}_{\substack{=(S\circ S)\otimes(S\circ S) \\ \text{(by Lemma 3.2)}}} \circ \Delta = \left(\underbrace{(S \circ S)}_{=S^2} \otimes \underbrace{(S \circ S)}_{=S^2} \right) \circ \Delta \\
&= \left(S^2 \otimes S^2 \right) \circ \Delta
\end{aligned}$$

and

$$\epsilon \circ \underbrace{S^2}_{=S\circ S} = \underbrace{\epsilon \circ S}_{=\epsilon} \circ S = \epsilon \circ S = \epsilon.$$

These two equalities show that S^2 is a $\mathbf{k}$-coalgebra homomorphism (since S^2 is a $\mathbf{k}$-linear map from H to H). This proves Lemma 2.12 **(a)**.

(b) The axioms of a $\mathbf{k}$-bialgebra yield $\epsilon(1_H) = 1_{\mathbf{k}}$ (since H is a $\mathbf{k}$-bialgebra) and $\Delta(1_H) = 1_H \otimes 1_H$ (likewise).

Define the maps m and u as in Remark 3.7. Applying both sides of the equality (68) to 1_H, we obtain

$$(m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(1_H) = (u \circ \epsilon)(1_H) = u\left(\underbrace{\epsilon(1_H)}_{=1_{\mathbf{k}}} \right) = u(1_{\mathbf{k}}) = 1_H$$

(by the definition of u). Hence,

$$\begin{aligned}
1_H = (m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(1_H) &= m\left((S \otimes \mathrm{id}_H)\left(\underbrace{\Delta(1_H)}_{=1_H\otimes 1_H} \right) \right) \\
&= m\left(\underbrace{(S \otimes \mathrm{id}_H)(1_H \otimes 1_H)}_{=S(1_H)\otimes \mathrm{id}_H(1_H)} \right) = m(S(1_H) \otimes \mathrm{id}_H(1_H)) \\
&= S(1_H) \cdot \underbrace{\mathrm{id}_H(1_H)}_{=1_H} \qquad \text{(by the definition of } m) \\
&= S(1_H).
\end{aligned}$$

This proves Lemma 2.12 **(b)**.

(c) This is well-known (see, e.g., [GriRei20, Proposition 1.4.17]). For the sake of completeness, let us nevertheless give a proof:

Let x be a primitive element of H. Thus, $\Delta(x) = x \otimes 1_H + 1_H \otimes x$ (by the definition of "primitive"). Moreover, the axioms of a $\mathbf{k}$-bialgebra yield $\epsilon(1_H) = 1$ (since H is a $\mathbf{k}$-bialgebra). Hence, Lemma 3.5 (applied to $C = H$, $a = 1_H$, $b = 1_H$ and $d = x$) yields $\epsilon(x) = 0$.

Define the maps m and u as in Remark 3.7. Applying both sides of the equality (68) to x, we obtain

$$(m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(x) = (u \circ \epsilon)(x) = u\left(\underbrace{\epsilon(x)}_{=0}\right) = u(0) = 0$$

(since the map u is $\mathbf{k}$-linear). Hence,

$$\begin{aligned}
0 &= (m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(x) \\
&= m\left((S \otimes \mathrm{id}_H)\left(\underbrace{\Delta(x)}_{=x\otimes 1_H + 1_H \otimes x}\right)\right) = m\left(\underbrace{(S \otimes \mathrm{id}_H)(x \otimes 1_H + 1_H \otimes x)}_{\substack{=(S\otimes \mathrm{id}_H)(x\otimes 1_H)+(S\otimes \mathrm{id}_H)(1_H\otimes x) \\ \text{(since the map } S\otimes \mathrm{id}_H \text{ is } \mathbf{k}\text{-linear)}}}\right) \\
&= m\left(\underbrace{(S \otimes \mathrm{id}_H)(x \otimes 1_H)}_{=S(x)\otimes \mathrm{id}_H(1_H)} + \underbrace{(S \otimes \mathrm{id}_H)(1_H \otimes x)}_{=S(1_H)\otimes \mathrm{id}_H(x)}\right) \\
&= m\left(S(x) \otimes \underbrace{\mathrm{id}_H(1_H)}_{=1_H} + \underbrace{S(1_H)}_{\substack{=1_H \\ \text{(by Lemma 2.12 } \mathbf{(b)})}} \otimes \underbrace{\mathrm{id}_H(x)}_{=x}\right) = m(S(x) \otimes 1_H + 1_H \otimes x) \\
&= \underbrace{m(S(x) \otimes 1_H)}_{\substack{=S(x)\cdot 1_H \\ \text{(by the definition of } m)}} + \underbrace{m(1_H \otimes x)}_{\substack{=1_H\cdot x \\ \text{(by the definition of } m)}} \qquad \text{(since the map } m \text{ is } \mathbf{k}\text{-linear)} \\
&= S(x) \cdot 1_H + 1_H \cdot x = S(x) + x.
\end{aligned}$$

Hence, $S(x) = -x$. This proves Lemma 2.12 **(c)**.

(d) Let x be a primitive element of H. Then, Lemma 2.12 **(c)** yields $S(x) = -x$. Now,

$$\begin{aligned}
\underbrace{S^2}_{=S\circ S}(x) = (S \circ S)(x) = S\left(\underbrace{S(x)}_{=-x}\right) &= S(-x) \\
&= -\underbrace{S(x)}_{=-x} \qquad \text{(since the map } S \text{ is } \mathbf{k}\text{-linear)} \\
&= -(-x) = x.
\end{aligned}$$

This proves Lemma 2.12 **(d)**. □

Proof of Corollary 2.13. We know that H is a connected filtered $\mathbf{k}$-Hopf algebra, thus a connected filtered $\mathbf{k}$-bialgebra and therefore a connected filtered $\mathbf{k}$-coalgebra. Furthermore, Proposition 2.10 shows that the unity 1_H defined according to Definition 2.3 **(b)** equals the unity of the $\mathbf{k}$-algebra H. Thus, the notion of a "primitive element" of H does not depend on whether we regard H as a $\mathbf{k}$-bialgebra or as a connected filtered $\mathbf{k}$-coalgebra.

Lemma 2.12 **(b)** yields $S(1_H) = 1_H$. Hence,

$$\underbrace{S^2}_{=S\circ S}(1_H) = (S\circ S)(1_H) = S\left(\underbrace{S(1_H)}_{=1_H}\right) = S(1_H) = 1_H.$$

Moreover, Lemma 2.12 **(a)** yields that the map $S^2 : H \to H$ is a $\mathbf{k}$-coalgebra homomorphism. Of course, the map $\mathrm{id} : H \to H$ is a $\mathbf{k}$-coalgebra homomorphism as well, and satisfies $\mathrm{id}(1_H) = 1_H$. Furthermore, Lemma 2.12 **(d)** entails that $\operatorname{Prim} H \subseteq \operatorname{Ker}\left(\mathrm{id} - S^2\right)$ [20]. Moreover, $S^2 \circ \mathrm{id} = S^2 = \mathrm{id} \circ S^2$. Furthermore, p is a positive integer and satisfies $\left(\mathrm{id} - S^2\right)\left(H_{\leq p}\right) = 0$ (by (17)). Hence, we can apply Corollary 2.5 to $C = H$ and $C_{\leq i} = H_{\leq i}$ and $e = \mathrm{id}$ and $f = S^2$. Let us do this now.

Corollary 2.5 **(b)** (applied to $C = H$ and $C_{\leq i} = H_{\leq i}$ and $e = \mathrm{id}$ and $f = S^2$) shows that for any integer $u \geq p$, we have

$$\left(\mathrm{id} - S^2\right)^{u-p+1}\left(H_{\leq u}\right) = 0.$$

This proves Corollary 2.13 **(b)**. It remains to prove Corollary 2.13 **(a)**:

(a) Let $u > p$ be any integer. Then, Corollary 2.5 **(a)** (applied to $C = H$ and $C_{\leq i} = H_{\leq i}$ and $e = \mathrm{id}$ and $f = S^2$) shows that

$$\left(\mathrm{id} - S^2\right)^{u-p}\left(H_{\leq u}\right) \subseteq \operatorname{Prim} H.$$

This proves (18). It remains to prove (19).

First, we shall show that $(\mathrm{id} + S)(\operatorname{Prim} H) = 0$.

Indeed, let $x \in \operatorname{Prim} H$. Thus, x is a primitive element of H (since $\operatorname{Prim} H$ was defined as the set of all primitive elements of H). Thus, Lemma 2.12 **(c)** yields $S(x) = -x$. Hence, $(\mathrm{id} + S)(x) = \underbrace{\mathrm{id}(x)}_{=x} + \underbrace{S(x)}_{=-x} = x + (-x) = 0$.

Forget that we fixed x. We thus have shown that $(\mathrm{id} + S)(x) = 0$ for each $x \in \operatorname{Prim} H$. In other words, $(\mathrm{id} + S)(\operatorname{Prim} H) = 0$.

[20] *Proof.* Let $x \in \operatorname{Prim} H$. Thus, x is a primitive element of H (since $\operatorname{Prim} H$ is defined as the set of all primitive elements of H). Therefore, Lemma 2.12 **(d)** yields $S^2(x) = x$. Hence, $\left(\mathrm{id} - S^2\right)(x) = \underbrace{\mathrm{id}(x)}_{=x} - \underbrace{S^2(x)}_{=x} = x - x = 0$, so that $x \in \operatorname{Ker}\left(\mathrm{id} - S^2\right)$.

Forget that we fixed x. We thus have shown that $x \in \operatorname{Ker}\left(\mathrm{id} - S^2\right)$ for each $x \in \operatorname{Prim} H$. In other words, we have $\operatorname{Prim} H \subseteq \operatorname{Ker}\left(\mathrm{id} - S^2\right)$.

Now,

$$\left(\left(\operatorname{id}+S\right)\circ\left(\operatorname{id}-S^2\right)^{u-p}\right)\left(H_{\leq u}\right)=\left(\operatorname{id}+S\right)\left(\underbrace{\left(\operatorname{id}-S^2\right)^{u-p}\left(H_{\leq u}\right)}_{\substack{\subseteq \operatorname{Prim} H \\ \text{(by (18))}}}\right)$$
$$\subseteq\left(\operatorname{id}+S\right)\left(\operatorname{Prim} H\right)=0.$$

Therefore, $\left(\left(\operatorname{id}+S\right)\circ\left(\operatorname{id}-S^2\right)^{u-p}\right)\left(H_{\leq u}\right)=0$ (since $\left(\left(\operatorname{id}+S\right)\circ\left(\operatorname{id}-S^2\right)^{u-p}\right)\left(H_{\leq u}\right)$ is a $\mathbf{k}$-module). This proves (19). Thus, the proof of Corollary 2.13 **(a)** is complete. □

Proof of Corollary 2.14. We know that H is a connected filtered $\mathbf{k}$-Hopf algebra, thus a connected filtered $\mathbf{k}$-bialgebra and therefore a connected filtered $\mathbf{k}$-coalgebra. Furthermore, Proposition 2.10 shows that the unity 1_H defined according to Definition 2.3 **(b)** equals the unity of the $\mathbf{k}$-algebra H. Thus, the notion of a "primitive element" of H does not depend on whether we regard H as a $\mathbf{k}$-bialgebra or as a connected filtered $\mathbf{k}$-coalgebra.

In our above proof of Corollary 2.13, we have already shown that

- we have $S^2\left(1_H\right)=1_H$;
- the map $S^2:H\to H$ is a $\mathbf{k}$-coalgebra homomorphism;
- we have $\operatorname{Prim} H\subseteq\operatorname{Ker}\left(\operatorname{id}-S^2\right)$.

Hence, we can apply Corollary 2.7 to $C=H$ and $C_{\leq i}=H_{\leq i}$ and $f=S^2$. Let us do this now.

Corollary 2.7 **(b)** (applied to $C=H$ and $C_{\leq i}=H_{\leq i}$ and $f=S^2$) shows that for any positive integer u, we have

$$\left(\operatorname{id}-S^2\right)^u\left(H_{\leq u}\right)=0.$$

This proves Corollary 2.14 **(b)**. It remains to prove Corollary 2.14 **(a)**:

(a) Let $u>1$ be any integer. Then, Corollary 2.7 **(a)** (applied to $C=H$ and $C_{\leq i}=H_{\leq i}$ and $f=S^2$) shows that

$$\left(\operatorname{id}-S^2\right)^{u-1}\left(H_{\leq u}\right)\subseteq\operatorname{Prim} H.$$

This proves (21). It remains to prove (22).

We have $(\mathrm{id} + S)(\operatorname{Prim} H) = 0$ (indeed, we have already shown this in our above proof of Corollary 2.13 **(a)**). Now,

$$\left((\mathrm{id} + S) \circ \left(\mathrm{id} - S^2\right)^{u-1}\right)(H_{\leq u}) = (\mathrm{id} + S)\left(\underbrace{\left(\mathrm{id} - S^2\right)^{u-1}(H_{\leq u})}_{\substack{\subseteq \operatorname{Prim} H \\ \text{(by (21))}}}\right)$$
$$\subseteq (\mathrm{id} + S)(\operatorname{Prim} H) = 0.$$

Therefore, $\left((\mathrm{id} + S) \circ \left(\mathrm{id} - S^2\right)^{u-1}\right)(H_{\leq u}) = 0$ (since $\left((\mathrm{id} + S) \circ \left(\mathrm{id} - S^2\right)^{u-1}\right)(H_{\leq u})$ is a $\mathbf{k}$-module). This proves (22). Thus, the proof of Corollary 2.14 **(a)** is complete. □

3.5. Proofs for Section 2.4

We shall next focus on proving the claims left unproven in Section 2.4. Before we do so, let us first collect a few basic properties of connected graded Hopf algebras into a lemma for convenience:

Lemma 3.8. Let H be a connected graded $\mathbf{k}$-Hopf algebra with unity 1_H and antipode S. Then:

(a) If n is a positive integer, and if x is an element of H_n, then we have

$$\Delta(x) = 1_H \otimes x + x \otimes 1_H + w \qquad \text{for some } w \in \sum_{k=1}^{n-1} H_k \otimes H_{n-k}.$$

(b) We have $H_1 \subseteq \operatorname{Prim} H$.

(c) We have $S(ab) = ba$ for any $a, b \in H_1$.

Proof of Lemma 3.8. **(a)** This follows from [GriRei20, Exercise 1.3.20 (h)] (applied to $A = H$). (Note that what we are calling w is denoted by $\Delta_+(x)$ in [GriRei20, Exercise 1.3.20 (h)].) It also appears in [Mancho06, Proposition II.1.1] and in [Preiss16, Theorem 2.18] (using the notation $\widetilde{\Delta}(x)$ for $\Delta(x) - 1_H \otimes x - x \otimes 1_H$).

(b) Let $x \in H_1$. Then, x is an element of H_1. Hence, Lemma 3.8 **(a)** (applied to $n = 1$) yields that we have

$$\Delta(x) = 1_H \otimes x + x \otimes 1_H + w \qquad \text{for some } w \in \sum_{k=1}^{1-1} H_k \otimes H_{1-k}.$$

Consider this w. We have

$$w \in \sum_{k=1}^{1-1} H_k \otimes H_{1-k} = (\text{empty sum}) = 0,$$

so that $w = 0$. Hence, $\Delta(x) = 1_H \otimes x + x \otimes 1_H + \underbrace{w}_{=0} = 1_H \otimes x + x \otimes 1_H = x \otimes 1_H + 1_H \otimes x$. In other words, the element x of H is primitive (by the definition of a "primitive" element). In other words, $x \in \operatorname{Prim} H$ (since $\operatorname{Prim} H$ is defined as the set of all primitive elements of H).

Forget that we fixed x. We thus have shown that $x \in \operatorname{Prim} H$ for each $x \in H_1$. In other words, $H_1 \subseteq \operatorname{Prim} H$. This proves Lemma 3.8 **(b)**.

(c) Let $a, b \in H_1$. Then, $a \in H_1 \subseteq \operatorname{Prim} H$ (by Lemma 3.8 **(b)**). In other words, the element a of H is primitive (since $\operatorname{Prim} H$ is defined as the set of all primitive elements of H). Hence, $S(a) = -a$ (by Lemma 2.12 **(c)**, applied to $x = a$). Similarly, $S(b) = -b$. However, it is well-known (see, e.g., [GriRei20, Proposition 1.4.10] or [Radfor12, Proposition 7.1.9 (a)]) that the antipode S of H is a $\mathbf{k}$-algebra anti-endomorphism, i.e., that it satisfies $S(1_H) = 1_H$ and

$$S(uv) = S(v) S(u) \qquad \text{for all } u, v \in H. \tag{71}$$

Applying (71) to $u = a$ and $v = b$, we obtain $S(ab) = \underbrace{S(b)}_{=-b} \underbrace{S(a)}_{=-a} = (-b)(-a) = ba$.

This proves Lemma 3.8 **(c)**. □

Proof of Corollary 2.15. As we know, the graded $\mathbf{k}$-Hopf algebra H automatically becomes a filtered $\mathbf{k}$-Hopf algebra with filtration $(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots)$ defined by setting

$$H_{\leq n} := \bigoplus_{i=0}^{n} H_i \qquad \text{for all } n \in \mathbb{N}.$$

This filtered $\mathbf{k}$-Hopf algebra H is connected, since $H_{\leq 0} = H_0$. Thus, Corollary 2.14 can be applied.

Let u be a positive integer. Then, $H_u \subseteq H_{\leq u}$ [21].

Now, we must prove the three relations (24), (25) and (26). The third one is the easiest: We have

$$\left(\operatorname{id} - S^2\right)^u \left(\underbrace{H_u}_{\subseteq H_{\leq u}}\right) \subseteq \left(\operatorname{id} - S^2\right)^u (H_{\leq u}) = 0$$

(by Corollary 2.14 **(b)**) and therefore $\left(\operatorname{id} - S^2\right)^u (H_u) = 0$ (since $\left(\operatorname{id} - S^2\right)^u (H_u)$ is a $\mathbf{k}$-module). This proves (26).

[21] *Proof.* The definition of $H_{\leq u}$ yields $H_{\leq u} = \bigoplus_{i=0}^{u} H_i$. However, $H_u \subseteq \bigoplus_{i=0}^{u} H_i$ (since H_u is an addend of the direct sum $\bigoplus_{i=0}^{u} H_i$). In view of $H_{\leq u} = \bigoplus_{i=0}^{u} H_i$, this rewrites as $H_u \subseteq H_{\leq u}$.

We shall now focus on proving (24). Indeed, if $u > 1$, then (24) follows from

$$\left(\operatorname{id} - S^2\right)^{u-1} \left(\underbrace{H_u}_{\subseteq H_{\leq u}}\right) \subseteq \left(\operatorname{id} - S^2\right)^{u-1} \left(H_{\leq u}\right)$$
$$\subseteq \operatorname{Prim} H \qquad \text{(by (21), since } u > 1\text{)}.$$

Thus, for the rest of this proof of (24), we WLOG assume that we don't have $u > 1$. Hence, we have $u = 1$ (since u is a positive integer). Therefore, $u - 1 = 0$, so that $\left(\operatorname{id} - S^2\right)^{u-1} = \left(\operatorname{id} - S^2\right)^0 = \operatorname{id}$. Thus,

$$\underbrace{\left(\operatorname{id} - S^2\right)^{u-1}}_{=\operatorname{id}} \left(H_u\right) = \operatorname{id}\left(H_u\right) = H_u = H_1 \qquad \text{(since } u = 1\text{)}$$
$$\subseteq \operatorname{Prim} H \qquad \text{(by Lemma 3.8 } \textbf{(b)}\text{)}.$$

This completes our proof of (24).

Now, it remains to prove (25). We have $\left(\operatorname{id} + S\right)\left(\operatorname{Prim} H\right) = 0$ (indeed, we have already shown this in our above proof of Corollary 2.13 **(a)**). Now,

$$\left(\left(\operatorname{id} + S\right) \circ \left(\operatorname{id} - S^2\right)^{u-1}\right)\left(H_u\right) = \left(\operatorname{id} + S\right)\left(\underbrace{\left(\operatorname{id} - S^2\right)^{u-1}\left(H_u\right)}_{\substack{\subseteq \operatorname{Prim} H \\ \text{(by (24))}}}\right)$$
$$\subseteq \left(\operatorname{id} + S\right)\left(\operatorname{Prim} H\right) = 0.$$

Therefore, $\left(\left(\operatorname{id} + S\right) \circ \left(\operatorname{id} - S^2\right)^{u-1}\right)\left(H_u\right) = 0$ (since $\left(\left(\operatorname{id} + S\right) \circ \left(\operatorname{id} - S^2\right)^{u-1}\right)\left(H_u\right)$ is a $\mathbf{k}$-module). This proves (25).

Thus, we have shown all three relations (24), (25) and (26). This completes the proof of Corollary 2.15. □

Proof of Corollary 2.16. Let 1_H denote the unity of the $\mathbf{k}$-algebra H.

As we know, the graded $\mathbf{k}$-Hopf algebra H automatically becomes a filtered $\mathbf{k}$-Hopf algebra with filtration $\left(H_{\leq 0}, H_{\leq 1}, H_{\leq 2}, \ldots\right)$ defined by setting

$$H_{\leq n} := \bigoplus_{i=0}^{n} H_i \qquad \text{for all } n \in \mathbb{N}.$$

This filtered $\mathbf{k}$-Hopf algebra H is connected (because the graded $\mathbf{k}$-Hopf algebra H is connected, and because $H_{\leq 0} = H_0$).

We know that p is a positive integer; thus, both 0 and 1 are elements of the set $\{0, 1, \ldots, p\}$.

Now, we shall show that

$$\left(\mathrm{id} - S^2\right)\left(H_{\leq p}\right) = 0. \tag{72}$$

[*Proof of (72):* It is easy to see that $\left(\mathrm{id} - S^2\right)(H_0) = 0$ [22]. Furthermore, $\left(\mathrm{id} - S^2\right)(H_1) = 0$ [23]. Now, the definition of $H_{\leq p}$ yields $H_{\leq p} = \bigoplus_{i=0}^{p} H_i = \sum_{i=0}^{p} H_i$ (since direct sums are sums). Applying the map $\mathrm{id} - S^2$ to both sides of this equal-

[22] *Proof.* In our above proof of Corollary 2.13, we have already shown that $S^2(1_H) = 1_H$. Hence,

$$\left(\mathrm{id} - S^2\right)(1_H) = \underbrace{\mathrm{id}(1_H)}_{=1_H} - \underbrace{S^2(1_H)}_{=1_H} = 1_H - 1_H = 0.$$

Define a $\mathbf{k}$-linear map $\overline{\mathrm{id}} : H \to H$ by setting

$$\overline{\mathrm{id}}(c) := c - \epsilon(c) 1_H \qquad \text{for each } c \in H.$$

Then, Lemma 3.6 **(c)** (applied to H and $H_{\leq i}$ instead of C and $C_{\leq i}$) yields $\overline{\mathrm{id}}(H_{\leq 0}) = 0$.

The definition of $H_{\leq 0}$ yields $H_{\leq 0} = \bigoplus_{i=0}^{0} H_i = H_0$. Thus, $H_0 = H_{\leq 0}$. Applying the map $\overline{\mathrm{id}}$ to both sides of this equality, we obtain $\overline{\mathrm{id}}(H_0) = \overline{\mathrm{id}}(H_{\leq 0}) = 0$.

Now, let $c \in H_0$. Then, the definition of $\overline{\mathrm{id}}$ yields $\overline{\mathrm{id}}(c) = c - \epsilon(c) 1_H$. On the other hand, we have $\overline{\mathrm{id}}(c) = 0$ (since $\overline{\mathrm{id}}\left(\underbrace{c}_{\in H_0}\right) \in \overline{\mathrm{id}}(H_0) = 0$). Comparing these two equalities, we obtain $c - \epsilon(c) 1_H = 0$. In other words, $c = \epsilon(c) 1_H$. Now, applying the map $\mathrm{id} - S^2$ to both sides of this equality, we obtain

$$\begin{aligned}
\left(\mathrm{id} - S^2\right)(c) &= \left(\mathrm{id} - S^2\right)(\epsilon(c) 1_H) \\
&= \epsilon(c) \cdot \underbrace{\left(\mathrm{id} - S^2\right)(1_H)}_{=0} \qquad \left(\text{since the map } \mathrm{id} - S^2 \text{ is } \mathbf{k}\text{-linear}\right) \\
&= 0.
\end{aligned}$$

Forget that we fixed c. We thus have shown that $\left(\mathrm{id} - S^2\right)(c) = 0$ for each $c \in H_0$. In other words, $\left(\mathrm{id} - S^2\right)(H_0) = 0$.

[23] *Proof.* Let $x \in H_1$. Then, $x \in H_1 \subseteq \operatorname{Prim} H$ (by Lemma 3.8 **(b)**). In other words, the element x of H is primitive (since $\operatorname{Prim} H$ is defined as the set of all primitive elements of H). Hence, Lemma 2.12 **(d)** yields $S^2(x) = x$. Now,

$$\left(\mathrm{id} - S^2\right)(x) = \underbrace{\mathrm{id}(x)}_{=x} - \underbrace{S^2(x)}_{=x} = x - x = 0.$$

Forget that we fixed x. We thus have shown that $\left(\mathrm{id} - S^2\right)(x) = 0$ for each $x \in H_1$. In other words, we have $\left(\mathrm{id} - S^2\right)(H_1) = 0$.

ity, we obtain

$$\begin{aligned}
\left(\mathrm{id} - S^2\right)\left(H_{\leq p}\right) &= \left(\mathrm{id} - S^2\right)\left(\sum_{i=0}^{p} H_i\right) \\
&= \sum_{i=0}^{p} \left(\mathrm{id} - S^2\right)\left(H_i\right) \qquad \left(\text{since the map } \mathrm{id} - S^2 \text{ is } \mathbf{k}\text{-linear}\right) \\
&= \underbrace{\left(\mathrm{id} - S^2\right)\left(H_0\right)}_{=0} + \underbrace{\left(\mathrm{id} - S^2\right)\left(H_1\right)}_{=0} + \sum_{i=2}^{p} \underbrace{\left(\mathrm{id} - S^2\right)\left(H_i\right)}_{\substack{=0 \\ \text{(by (27))}}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addends for } i = 0 \\ \text{and for } i = 1 \text{ from the sum (since both 0 and 1} \\ \text{are elements of the set } \{0, 1, \ldots, p\}\text{)}\end{array}\right) \\
&= \underbrace{0 + 0}_{=0} + \underbrace{\sum_{i=2}^{p} 0}_{=0} = 0.
\end{aligned}$$

This proves (72).]

Hence, we can apply Corollary 2.13. In particular, Corollary 2.13 **(a)** shows that for any integer $u > p$, we have

$$\left(\mathrm{id} - S^2\right)^{u-p}\left(H_{\leq u}\right) \subseteq \operatorname{Prim} H$$

and

$$\left(\left(\mathrm{id} + S\right) \circ \left(\mathrm{id} - S^2\right)^{u-p}\right)\left(H_{\leq u}\right) = 0.$$

This proves Corollary 2.16 **(a)**.

Furthermore, Corollary 2.13 **(b)** shows that for any integer $u \geq p$, we have

$$\left(\mathrm{id} - S^2\right)^{u-p+1}\left(H_{\leq u}\right) = 0.$$

This proves Corollary 2.16 **(b)**. □

Proof of Corollary 2.17. **(a)** Let 1_H denote the unity of the $\mathbf{k}$-algebra H. Define the maps m and u as in Remark 3.7.

Let $x \in H_2$. Then, x is an element of H_2. Hence, Lemma 3.8 **(a)** (applied to $n = 2$) yields that we have

$$\Delta\left(x\right) = 1_H \otimes x + x \otimes 1_H + w \qquad \text{for some } w \in \sum_{k=1}^{2-1} H_k \otimes H_{2-k}.$$

Consider this w. We have

$$\begin{aligned} w &\in \sum_{k=1}^{2-1} H_k \otimes H_{2-k} = \sum_{k=1}^{1} H_k \otimes H_{2-k} \qquad (\text{since } 2-1=1) \\ &= H_1 \otimes H_{2-1} = H_1 \otimes H_1 \qquad (\text{since } 2-1=1)\,. \end{aligned}$$

Therefore, w is a tensor in $H_1 \otimes H_1$. Hence, w can be written in the form

$$w = \sum_{i=1}^{k} \lambda_i a_i \otimes b_i \tag{73}$$

for some $k \in \mathbb{N}$, some $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbf{k}$, some $a_1, a_2, \ldots, a_k \in H_1$ and some $b_1, b_2, \ldots, b_k \in H_1$. Consider this k, these $\lambda_1, \lambda_2, \ldots, \lambda_k$, these $a_1, a_2, \ldots, a_k$ and these $b_1, b_2, \ldots, b_k$.

We have $a_1, a_2, \ldots, a_k \in H_1$. Thus, for each $i \in \{1, 2, \ldots, k\}$, we have $a_i \in H_1 \subseteq \operatorname{Prim} H$ (by Lemma 3.8 **(b)**) and therefore

$$S(a_i) = -a_i \tag{74}$$

[24]. Moreover, for each $i \in \{1, 2, \ldots, k\}$, we have

$$S(a_i b_i) = a_i b_i \tag{75}$$

[25].

Applying the map $S \otimes \mathrm{id}_H : H \otimes H \to H \otimes H$ to both sides of the equality (73), we obtain

$$\begin{aligned} (S \otimes \mathrm{id}_H)(w) &= (S \otimes \mathrm{id}_H)\left(\sum_{i=1}^{k} \lambda_i a_i \otimes b_i\right) = \sum_{i=1}^{k} \lambda_i \underbrace{(S \otimes \mathrm{id}_H)(a_i \otimes b_i)}_{=S(a_i)\otimes \mathrm{id}_H(b_i)} \\ &\qquad (\text{since the map } S \otimes \mathrm{id}_H \text{ is } \mathbf{k}\text{-linear}) \\ &= \sum_{i=1}^{k} \lambda_i \underbrace{S(a_i)}_{\substack{=-a_i \\ (\text{by } (74))}} \otimes \underbrace{\mathrm{id}_H(b_i)}_{=b_i} = \sum_{i=1}^{k} \lambda_i \underbrace{(-a_i) \otimes b_i}_{\substack{=-a_i \otimes b_i \\ (\text{since the map } H\times H \to H\otimes H,\ (u,v)\mapsto u\otimes v \\ \text{is } \mathbf{k}\text{-bilinear (by the definition of the} \\ \text{tensor product))}}} \\ &= \sum_{i=1}^{k} \lambda_i (-a_i \otimes b_i) = -\underbrace{\sum_{i=1}^{k} \lambda_i a_i \otimes b_i}_{\substack{=w \\ (\text{by } (73))}} \\ &= -w. \end{aligned} \tag{76}$$

[24] *Proof of (74):* Let $i \in \{1, 2, \ldots, k\}$. Then, $a_i \in H_1 \subseteq \operatorname{Prim} H$ (by Lemma 3.8 **(b)**). In other words, the element a_i of H is primitive (since $\operatorname{Prim} H$ is defined as the set of all primitive elements of H). Therefore, Lemma 2.12 **(c)** (applied to a_i instead of x) yields $S(a_i) = -a_i$. This proves (74).

[25] *Proof of (75):* Let $i \in \{1, 2, \ldots, k\}$. Then, $a_i \in H_1$ (since $a_1, a_2, \ldots, a_k \in H_1$) and $b_i \in H_1$ (since $b_1, b_2, \ldots, b_k \in H_1$). Hence, (31) (applied to $a = a_i$ and $b = b_i$) yields $a_i b_i = b_i a_i$. On the other hand, Lemma 3.8 **(c)** (applied to $a = a_i$ and $b = b_i$) yields $S(a_i b_i) = b_i a_i$. Comparing these two equalities, we obtain $S(a_i b_i) = a_i b_i$. This proves (75).

The Hopf algebra H is graded. Hence, its counit ϵ is a graded map from H to $\mathbf{k}$ (by the definition of a graded Hopf algebra). In other words, $\epsilon(H_i) \subseteq \mathbf{k}_i$ for each $i \in \mathbb{N}$. Thus, $\epsilon(H_2) \subseteq \mathbf{k}_2 = 0$ (since the graded $\mathbf{k}$-module $\mathbf{k}$ is concentrated in degree 0). Therefore, $\epsilon\left(\underbrace{x}_{\in H_2}\right) \in \epsilon(H_2) \subseteq 0$, so that $\epsilon(x) = 0$.

Lemma 2.12 **(b)** yields $S(1_H) = 1_H$.

Applying both sides of the equality (68) to x, we obtain

$$(m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(x) = (u \circ \epsilon)(x) = u\left(\underbrace{\epsilon(x)}_{=0}\right) = u(0) = 0$$

(since the map u is $\mathbf{k}$-linear). Therefore,

$$\begin{aligned}
0 &= (m \circ (S \otimes \mathrm{id}_H) \circ \Delta)(x) \\
&= m\left((S \otimes \mathrm{id}_H)\left(\underbrace{\Delta(x)}_{=1_H \otimes x + x \otimes 1_H + w}\right)\right) \\
&= m\left(\underbrace{(S \otimes \mathrm{id}_H)(1_H \otimes x + x \otimes 1_H + w)}_{\substack{=(S\otimes \mathrm{id}_H)(1_H\otimes x)+(S\otimes \mathrm{id}_H)(x\otimes 1_H)+(S\otimes \mathrm{id}_H)(w) \\ \text{(since the map } S\otimes \mathrm{id}_H \text{ is } \mathbf{k}\text{-linear)}}}\right) \\
&= m\left(\underbrace{(S \otimes \mathrm{id}_H)(1_H \otimes x)}_{=S(1_H)\otimes \mathrm{id}_H(x)} + \underbrace{(S \otimes \mathrm{id}_H)(x \otimes 1_H)}_{=S(x)\otimes \mathrm{id}_H(1_H)} + \underbrace{(S \otimes \mathrm{id}_H)(w)}_{\substack{=-w \\ \text{(by (76))}}}\right) \\
&= m\left(\underbrace{S(1_H)}_{=1_H} \otimes \underbrace{\mathrm{id}_H(x)}_{=x} + S(x) \otimes \underbrace{\mathrm{id}_H(1_H)}_{=1_H} + (-w)\right) \\
&= m(1_H \otimes x + S(x) \otimes 1_H + (-w)) = \underbrace{m(1_H \otimes x)}_{\substack{=1_H x \\ \text{(by the definition} \\ \text{of the map } m)}} + \underbrace{m(S(x) \otimes 1_H)}_{\substack{=S(x)\cdot 1_H \\ \text{(by the definition} \\ \text{of the map } m)}} + \underbrace{m(-w)}_{\substack{=-m(w) \\ \text{(since the map } m \\ \text{is } \mathbf{k}\text{-linear)}}} \\
&\qquad \text{(since the map } m \text{ is } \mathbf{k}\text{-linear)} \\
&= \underbrace{1_H x}_{=x} + \underbrace{S(x) \cdot 1_H}_{=S(x)} + (-m(w)) = x + S(x) + (-m(w)) = x + S(x) - m(w).
\end{aligned}$$

Solving this equality for $S(x)$, we obtain

$$S(x) = m(w) - x. \tag{77}$$

Applying the map $m : H \otimes H \to H$ to both sides of the equality (73), we obtain

$$\begin{aligned} m(w) &= m\left(\sum_{i=1}^{k} \lambda_i a_i \otimes b_i\right) \\ &= \sum_{i=1}^{k} \lambda_i \underbrace{m(a_i \otimes b_i)}_{\substack{=a_i b_i \\ \text{(by the definition of the map } m)}} \qquad \text{(since the map } m \text{ is } \mathbf{k}\text{-linear)} \\ &= \sum_{i=1}^{k} \lambda_i a_i b_i. \end{aligned} \tag{78}$$

Applying the map S to both sides of this equality, we obtain

$$\begin{aligned} S(m(w)) &= S\left(\sum_{i=1}^{k} \lambda_i a_i b_i\right) = \sum_{i=1}^{k} \lambda_i \underbrace{S(a_i b_i)}_{\substack{=a_i b_i \\ \text{(by (75))}}} \qquad \text{(since the map } S \text{ is } \mathbf{k}\text{-linear)} \\ &= \sum_{i=1}^{k} \lambda_i a_i b_i = m(w) \qquad \text{(by (78))}. \end{aligned} \tag{79}$$

Now, applying the map S to both sides of the equality (77), we obtain

$$\begin{aligned} S(S(x)) &= S(m(w) - x) = \underbrace{S(m(w))}_{\substack{=m(w) \\ \text{(by (79))}}} - \underbrace{S(x)}_{\substack{=m(w)-x \\ \text{(by (77))}}} \qquad \text{(since the map } S \text{ is } \mathbf{k}\text{-linear)} \\ &= m(w) - (m(w) - x) = x. \end{aligned}$$

Now,

$$\left(\mathrm{id} - S^2\right)(x) = \underbrace{\mathrm{id}(x)}_{=x} - \underbrace{S^2}_{=S \circ S}(x) = x - \underbrace{(S \circ S)(x)}_{=S(S(x))=x} = x - x = 0.$$

Forget that we fixed x. We thus have shown that $\left(\mathrm{id} - S^2\right)(x) = 0$ for each $x \in H_2$. In other words, $\left(\mathrm{id} - S^2\right)(H_2) = 0$. This proves Corollary 2.17 **(a)**.

Now we know that $\left(\mathrm{id} - S^2\right)(H_2) = 0$ (by Corollary 2.17 **(a)**). In other words, $\left(\mathrm{id} - S^2\right)(H_i) = 0$ holds for $i = 2$. In other words, all $i \in \{2, 3, \ldots, 2\}$ satisfy $\left(\mathrm{id} - S^2\right)(H_i) = 0$ (since the only $i \in \{2, 3, \ldots, 2\}$ is 2). Hence, we can apply Corollary 2.16 to $p = 2$.

Thus, Corollary 2.16 **(a)** (applied to $p = 2$) yields that for any integer $u > 2$, we have

$$\left(\mathrm{id} - S^2\right)^{u-2}(H_{\leq u}) \subseteq \operatorname{Prim} H$$

and

$$\left((\mathrm{id} + S) \circ \left(\mathrm{id} - S^2\right)^{u-2}\right)(H_{\leq u}) = 0.$$

This proves Corollary 2.17 **(b)**.

Furthermore, Corollary 2.16 **(b)** (applied to $p = 2$) yields that for any integer $u \geq 2$, we have

$$\left(\mathrm{id} - S^2\right)^{u-2+1} (H_{\leq u}) = 0.$$

In other words, for any integer $u \geq 2$, we have

$$\left(\mathrm{id} - S^2\right)^{u-1} (H_{\leq u}) = 0$$

(since $u - 2 + 1 = u - 1$). In other words, for any integer $u > 1$, we have

$$\left(\mathrm{id} - S^2\right)^{u-1} (H_{\leq u}) = 0$$

(since "$u > 1$" is equivalent to "$u \geq 2$" when u is an integer). This proves Corollary 2.17 **(c)**. □

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