Natural endomorphisms of connected graded bialgebras

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Contents

- This is an attempt at a systematic study of identities that hold for connected graded bialgebras.
- We start by recounting the definitions, then give some new examples of such identities, then develop the early sprouts of a theory.
- Lots of questions here. Some might have already been solved. The literature is fragmented (topologists have been around for the longest but I don't quite speak their language), so surprises are possible.
- **Preprints:**

  `https://www.cip.ifi.lmu.de/~grinberg/algebra/aphae-proj.pdf` (new material, still a very rough sketch);
  `https://www.cip.ifi.lmu.de/~grinberg/algebra/bernsteinproof.pdf` (old treatment of the commutative case using base change; in a way obsolete, but interesting for the methods).

1. Bialgebras

1.1. General conventions

- We fix a commutative ring $\mathbf{k}$. (No assumptions on characteristic!)
- $\otimes$ always means $\otimes_{\mathbf{k}}$ by default.
- Read "$\mathbf{k}$-" in front of each of the nouns "module", "algebra", "coalgebra", "bialgebra" or "linear" by default.

1.2. Algebras and coalgebras

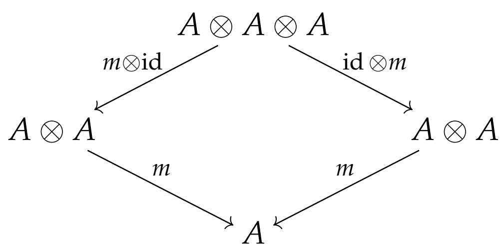
- **Definition.** An **algebra** means a module A equipped with a multiplication map

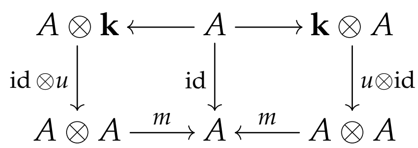
  $$m : A \otimes A \to A \qquad (a \otimes b \mapsto ab)$$

  and a unity map

  $$u : \mathbf{k} \to A \qquad (1_{\mathbf{k}} \mapsto 1_A)$$

  (both linear) such that the diagrams





commute.

One usually writes m_A and u_A for m and u (and similarly elsewhere).

- Dualizing this definition, one gets "coalgebras":

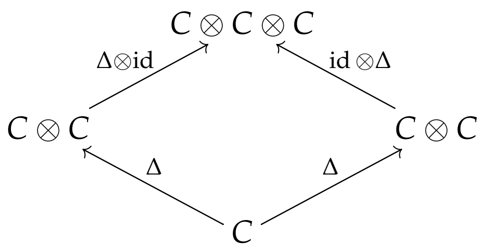
  Definition. A **coalgebra** means a module C equipped with a comultiplication map

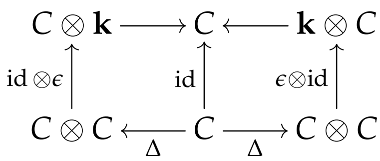
  $$\Delta : C \to C \otimes C \qquad \left(c \mapsto \sum_i c_{1,i} \otimes c_{2,i} \right)$$

  and a counit map

  $$\epsilon : C \to \mathbf{k}$$

  (both linear) such that the diagrams



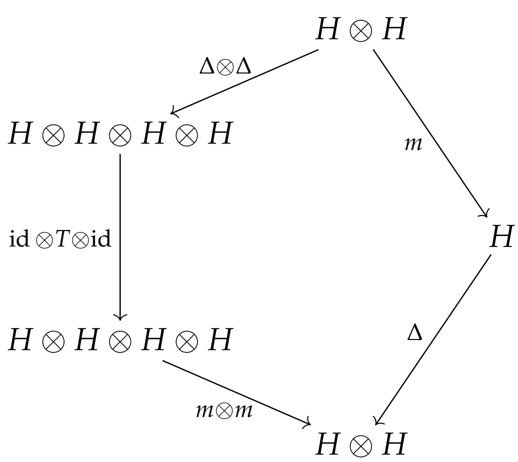


  commute.

- **Definition.** **Algebra morphisms** and **coalgebra morphisms** are defined in the least surprising way (i.e., as linear maps that commute with m and u resp. Δ and ϵ in the obvious ways).

1.3. Bialgebras

- **Definition.** A **bialgebra** is a module H that is both an algebra and a coalgebra, and that satisfies the further commutative diagrams



$$\begin{array}{ccc} H \otimes H & \xrightarrow{\epsilon \otimes \epsilon} & \mathbf{k} \otimes \mathbf{k} \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle m} \\ H & \xrightarrow[\epsilon]{} & \mathbf{k} \end{array} \qquad \begin{array}{ccc} \mathbf{k} & \xrightarrow{u} & H \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta} \\ \mathbf{k} \otimes \mathbf{k} & \xrightarrow[u \otimes u]{} & H \otimes H \end{array}$$

$$\begin{array}{ccc} \mathbf{k} & \xrightarrow{\mathrm{id}} & \mathbf{k} \\ & {\scriptstyle u}\searrow \quad \nearrow{\scriptstyle \epsilon} & \\ & H & \end{array}$$

where $T : H \otimes H \to H \otimes H$ is the **twist map** $a \otimes b \mapsto b \otimes a$.

- **Examples:**
  - $\mathbf{k}$ itself is a bialgebra (with all maps being $\mathrm{id} : \mathbf{k} \to \mathbf{k}$).
  - If M is a monoid (e.g., group), then the monoid algebra $\mathbf{k}[M]$ is a bialgebra, with

$$\begin{aligned} \Delta(g) &= g \otimes g && \text{for all } g \in M; \\ \epsilon(g) &= 1 && \text{for all } g \in M; \\ m(g \otimes h) &= gh && \text{for all } g, h \in M; \\ u(1_{\mathbf{k}}) &= e_M && \text{(that is, the unity is } e_M). \end{aligned}$$

  - If V is a $\mathbf{k}$-module, then the tensor algebra $T(V)$ is a bialgebra, with

$$\Delta\left(\underbrace{a_1 a_2 \cdots a_n}_{\text{short for } a_1 \otimes a_2 \otimes \cdots \otimes a_n}\right) = \sum_{I \subseteq \{1,2,\ldots,n\}} a_I \otimes a_{\{1,2,\ldots,n\} \setminus I}$$

    for any $a_1, a_2, \ldots, a_n \in V$. Here, a_I is the product of all a_i with $i \in I$ in increasing order.
  - There is also a shuffle algebra $\mathrm{Sh}(V)$, which is in some way dual to $T(V)$.
  - The symmetric algebra $\mathrm{Sym}\, V$ (defined as a quotient of $T(V)$) is also a bialgebra.
  - The ring Λ of symmetric functions over $\mathbf{k}$ is a bialgebra.
  - The ring QSym of quasisymmetric functions over $\mathbf{k}$ is a bialgebra.
  - Various other combinatorial bialgebras such as NSym, FQSym (= Malvenuto–Reutenauer), posets, double posets, graphs, hypergraphs, ...

1.4. Graded, connected, commutative, cocommutative

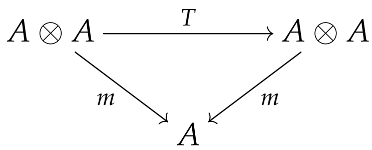
- **Definition.** A **graded** (co,bi)algebra is a (co,bi)algebra H that is graded (= $\mathbb{N}$-graded) as a module, and whose operations (m, u, Δ, ϵ, whichever apply) respect the grading. This means

$$H_a H_b \subseteq H_{a+b} \qquad \text{for all } a, b \geqslant 0;$$
$$1_H \in H_0;$$
$$\Delta(H_n) \subseteq \bigoplus_{k=0}^{n} H_k \otimes H_{n-k} \qquad \text{for all } n \geqslant 0;$$
$$\epsilon(H_n) = 0 \qquad \text{for all } n > 0.$$

- We do **not** use topologists' sign conventions.
- **Definition.** A graded (co,bi)algebra H is **connected** if and only if $H_0 \cong \mathbf{k}$ as $\mathbf{k}$-modules. (For an algebra, this automatically entails $H_0 = \mathbf{k} \cdot 1_H$.)
- For example, the tensor algebra $T(V)$ is connected graded, with

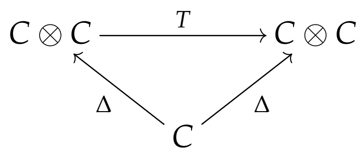
$$(T(V))_n = V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}.$$

- **Definition.** An algebra A is **commutative** if the diagram



commutes. (Again, T is the twist map.)

- **Definition.** Dually, a coalgebra C is **cocommutative** if the diagram



commutes.

- **Example:** Monoid algebras and tensor algebras are cocommutative.

  Shuffle algebras and QSym are commutative.

  Symmetric algebras and Λ are both.

1.5. Convolution

- **Definition.** If C is a coalgebra and A is an algebra, then the module

$$\operatorname{Hom}(C, A) := \{\text{all } \mathbf{k}\text{-linear maps } f : C \to A\}$$

becomes an algebra itself, equipped with the **convolution product** $\star$ defined by

$$f \star g = m_A \circ (f \otimes g) \circ \Delta_C.$$

The unity of $\operatorname{Hom}(C, A)$ is $u_A \circ \epsilon_C$.

- In particular, if C is a coalgebra, then the dual module $C^* = \operatorname{Hom}(C, \mathbf{k})$ is an algebra.

- In **nice** situations, the dual statement holds: If A is an algebra that is finite free as a module, then A^* is a coalgebra.

  Something similar holds for graded duals in the graded-finite case (= graded, and each degree is finite free).

- **Duality** is a permanent theme in bialgebra theory: You can dualize every statement, but it is not a-priori clear that the dual always holds. Still, it is typically true and often can be derived from the primal using some tricks.

- **Question:** Are there general meta-theorems that guarantee this?

- Of course, a proof that uses only element-free diagram chasing guarantees dualizability, but not every proof is so.

1.6. Hopf algebras

- If H is a bialgebra, then $\operatorname{Hom}(H, H)$ is an algebra (via convolution $\star$, as explained above).

- The identity map id_H belongs to this algebra.

- **Definition.** We call H a **Hopf algebra** if id_H has an inverse in this algebra.

  In this case, the inverse of id_H is called the **antipode** of H.

- **Theorem (Takeuchi).** If H is a connected graded bialgebra, then H is automatically a Hopf algebra.

1.7. Iterated multiplications and comultiplications

- Actually, the antipode can be computed explicitly.

- **Definition.** Let A be an algebra. For any integer $k \geqslant 0$, define the linear map
$$m^{[k]} : A^{\otimes k} \to A$$
recursively by
$$m^{[0]} = u \qquad \text{and} \qquad m^{[k]} = m \circ \left(m^{[k-1]} \otimes \mathrm{id} \right).$$
In the language of elements:
$$m^{[k]} \left(a_1 \otimes a_2 \otimes \cdots \otimes a_k \right) = a_1 a_2 \cdots a_k.$$
This map $m^{[k]}$ is called an **iterated multiplication** map. (It is commonly called $m^{(k-1)}$, but my indexing is better :)

- Dually:

  Definition. Let C be a coalgebra. For any integer $k \geqslant 0$, define the linear map
$$\Delta^{[k]} : C \to C^{\otimes k}$$
recursively by
$$\Delta^{[0]} = \epsilon \qquad \text{and} \qquad \Delta^{[k]} = \left(\Delta^{[k-1]} \otimes \mathrm{id} \right) \circ \Delta.$$
This map $\Delta^{[k]}$ is called an **iterated comultiplication** map.

- **Proposition.** Let A be an algebra, and C a coalgebra. Then, any k elements $f_1, f_2, \ldots, f_k$ of the convolution algebra $\operatorname{Hom}(C, A)$ satisfy
$$f_1 \star f_2 \star \cdots \star f_k = m^{[k]} \circ \left(f_1 \otimes f_2 \otimes \cdots \otimes f_k \right) \circ \Delta^{[k]}.$$

- **Theorem (Takeuchi's formula for the antipode).** Let H be a connected graded bialgebra. Let
$$\overline{\mathrm{id}} = \mathrm{id}_H - u \circ \epsilon = \mathrm{id} - \underbrace{p_0}_{\substack{\text{projection} \\ H \to H_0}}$$
$$= \left(\text{projection from } H = \bigoplus_{i=0}^{\infty} H_i \text{ onto } \bigoplus_{i=1}^{\infty} H_i \right).$$

Then, the antipode S of H is given by

$$S = \sum_{k=0}^{\infty} (-1)^k \underbrace{\overline{\mathrm{id}}^{\star k}}_{\substack{=\overline{\mathrm{id}}\star\overline{\mathrm{id}}\star\cdots\star\overline{\mathrm{id}} \\ =m^{[k]}\circ\overline{\mathrm{id}}^{\otimes k}\circ\Delta^{[k]}}} .$$

The sum here converges pointwise: In fact, if $x \in H_n$, then $\overline{\mathrm{id}}^{\star k}(x) = 0$ for all $k > n$.

- **Proof.** Actually quite easy!

$$\mathrm{id}_H = \underbrace{u \circ \epsilon}_{\substack{\text{unity of the} \\ \text{convolution algebra}}} + \overline{\mathrm{id}},$$

and $\overline{\mathrm{id}}$ is locally nilpotent; thus, the inverse of id_H can be found using $(1+q)^{-1} = 1 - q + q^2 - q^3 \pm \cdots$.

2. Some identities

2.1. On the order of S^2

- The antipode of a Hopf algebra is always called S.
- **Theorem (Sweedler?).** If a Hopf algebra H is commutative or cocommutative, then its antipode is an involution: that is, $S^2 = \mathrm{id}$. (Here and in the following, $S^2 = S \circ S$, not $S \star S$.)
- Not true for general H. (In general, S may even be non-invertible.)
- However:
- **Theorem (Aguiar and Lauve 2014).** If H is a connected graded bialgebra, then

$$\left(\mathrm{id} - S^2\right)^n (H_n) = 0 \qquad \text{for each } n \geqslant 0.$$

(Thus, S^2 is id "up to" a locally nilpotent "error term". In other words, S^2 is locally unipotent.)

- **Theorem (Aguiar 2017).** Even better: In the same setup,

$$\left((\mathrm{id} + S) \circ \left(\mathrm{id} - S^2\right)^{n-1}\right)(H_n) = 0 \qquad \text{for each } n > 0.$$

- For some H (for example, Malvenuto–Reutenauer), we even have

$$\left(\mathrm{id} - S^2\right)^{n-1} (H_n) = 0 \qquad \text{for each } n > 1.$$

- I generalize these in arXiv:2109.02101.

2.2. The random-to-top operator

- Here is another series of recent results (mostly unpublished – see `https://www.cip.ifi.lmu.de/~grinberg/algebra/aphae-proj.pdf` for outlined proofs –, but related work was done by Amy Pang in arXiv:1609.04312, arXiv:2108.09097).

- **Definition.** If H is any graded module (e.g., bialgebra), and if $n \geqslant 0$, then p_n shall denote the canonical projection $H \to H_n$ (regarded as a map $H \to H$).

  Note that $p_0 = u \circ \epsilon$ when H is connected.

- **Definition.** If H is a graded bialgebra, and if $n \geqslant 0$, then we set

$$\rho_n := p_n \star \mathrm{id} \in \operatorname{Hom}(H, H).$$

  In particular, $\rho_1 = p_1 \star \mathrm{id}$ is called **random-to-top operator**, since it acts on a tensor algebra $H = T(V)$ as follows:

$$\rho_1 \Bigg(\underbrace{a_1 a_2 \cdots a_n}_{\text{short for } a_1 \otimes a_2 \otimes \cdots \otimes a_n} \Bigg) = \sum_{k=1}^{n} \underbrace{a_k \cdot a_1 a_2 \cdots \widehat{a_k} \cdots a_n}_{\substack{\text{this is our input tensor,} \\ \text{with the } k\text{-th factor moved to front}}} .$$

- **Theorem.** Let H be a connected graded bialgebra.

  (a) We have $\rho_1 = 0$ on H_0, and $\rho_1 = \mathrm{id}$ on H_1.

  (b) For each $n \geqslant 2$, we have

$$(\rho_1 - n) \circ (\rho_1 - (n-2))^2 \circ \prod_{i=0}^{n-3} (\rho_1 - i)^{n-1-i} = 0 \qquad \text{on } H_n.$$

  (Note: Here and below, $\prod$ is product with respect to $\circ$, not to $\star$. Same applies to powers.)

  For example,

$$\begin{aligned} (\rho_1 - 2) \circ \rho_1^2 &= 0 && \text{on } H_2, && \text{and} \\ (\rho_1 - 3) \circ (\rho_1 - 1)^2 \circ \rho_1^2 &= 0 && \text{on } H_3, && \text{and} \\ (\rho_1 - 4) \circ (\rho_1 - 2)^2 \circ (\rho_1 - 1)^2 \circ \rho_1^3 &= 0 && \text{on } H_4. \end{aligned}$$

- It seems that this polynomial is minimal (in general). However:

- **Theorem.** If we assume further that H is commutative, or (even weaker) that $ab = ba$ for all $a, b \in H_1$, then

$$(\rho_1 - n) \circ \prod_{i=0}^{n-2} (\rho_1 - i) = 0 \qquad \text{on } H_n$$

  for any $n \geqslant 0$.

- More generally:

  Theorem. Let k be a positive integer. Assume that every two elements of $H_1 + H_2 + \cdots + H_k$ commute. Let n be a positive integer. Then,

  $$\prod_{i \in F(n,k)} (\rho_k - i) = 0 \qquad \text{on } H_n,$$

  where $F(n,k)$ is a somewhat intricate finite set of integers.

- **Question:** Does an unconditional result hold for ρ_k, similar to our first theorem for ρ_1?

2.3. But what else can we say?

- These are instances of identities that hold in every connected graded bialgebra and involve only m, u, Δ, ϵ and projections on homogeneous components. (Recall: $\overline{\operatorname{id}} = \operatorname{id} - p_0$, so that Takeuchi's formula writes S in these terms.)

- **Question:** Is there a mechanical way to prove such identities? (For a fixed n, say.)

  (We will partly answer this below.)

3. Natural transformations on a graded Hopf algebra

3.1. What is our calculus?

- The operations we are working with are defined for any graded bialgebra. They are thus **natural operations** on a graded bialgebra, i.e., natural transformations from one of the four forgetful functors

  $$\begin{aligned} \{\text{graded bialgebras}\} &\to \{\text{modules}\}, \\ \{\text{graded bialgebras}\} &\to \{\text{graded modules}\}, \\ \{\text{connected graded bialgebras}\} &\to \{\text{modules}\}, \\ \{\text{connected graded bialgebras}\} &\to \{\text{graded modules}\} \end{aligned}$$

  to itself. (These are four different but related settings.)

3.2. Descent operators

- How does a typical such operation look like?

- **Definition.** A **weak composition** means a tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of nonnegative integers.

  Example: $(3, 0, 0, 5, 1, 0)$.

- **Definition.** A **composition** means a tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of positive integers.

  Example: $(3, 5, 1)$.

- **Definition.** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a weak composition, and let $\sigma \in \mathfrak{S}_k$ be a permutation of $\{1, 2, \ldots, k\}$ (for the same k). Then, for any graded bialgebra H, we define a linear map

$$p_{\alpha,\sigma} = m^{[k]} \circ (p_{\alpha_1} \otimes p_{\alpha_2} \otimes \cdots \otimes p_{\alpha_k}) \circ T_\sigma \circ \Delta^{[k]} \\ \in \operatorname{Hom}(H, H).$$

  Here, T_σ is the σ-twist map

$$H^{\otimes k} \to H^{\otimes k}, \\ h_1 \otimes h_2 \otimes \cdots \otimes h_k \mapsto h_{\sigma(1)} \otimes h_{\sigma(2)} \otimes \cdots \otimes h_{\sigma(k)}.$$

- **Example.** For $\alpha = (3, 5)$ and $\sigma = t_{1,2}$ (the transposition swapping 1 with 2), we have

$$p_{\alpha,\sigma} = m \circ (p_3 \otimes p_5) \circ \underbrace{T}_{\text{twist}} \circ \Delta.$$

  Thus, in Sweedler notation,

$$p_{\alpha,\sigma}(x) = \sum_{(x)} p_3\left(x_{(2)}\right) p_5\left(x_{(1)}\right).$$

- We call $p_{\alpha,\sigma}$ a **descent operator** or a **BPPC operator** (short for "break, permute, project and combine"). Note that the $p_{\alpha_1} \otimes p_{\alpha_2} \otimes \cdots \otimes p_{\alpha_k}$ and T_σ parts can be (quasi)commuted:

$$(p_{\alpha_1} \otimes p_{\alpha_2} \otimes \cdots \otimes p_{\alpha_k}) \circ T_\sigma = T_\sigma \circ \left(p_{\beta_1} \otimes p_{\beta_2} \otimes \cdots \otimes p_{\beta_k}\right)$$

  for $(\beta_1, \beta_2, \ldots, \beta_k) = \left(\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \ldots, \alpha_{\sigma^{-1}(k)}\right)$.

- **Simple observation.** Any such map $p_{\alpha,\sigma}$ vanishes on H_n unless n is the sum of the entries of α.

- **Definition.** Given any weak composition α, we set

$$p_\alpha := p_{\alpha,\mathrm{id}}$$

  (where id is the identity permutation).

- **Simple observation.**

  (a) We have
  $$p_{\alpha,\sigma} = p_{\sigma\cdot\alpha} \qquad \text{if } H \text{ is commutative.}$$

  (b) We have
  $$p_{\alpha,\sigma} = p_{\alpha} \qquad \text{if } H \text{ is cocommutative.}$$

- Operators of the form p_α were studied by Patras and Reutenauer for commutative or cocommutative H. They showed that the span of such p_α operators is closed under both $\circ$ and $\star$. But this is not true for general H. Instead, we need all $p_{\alpha,\sigma}$.

- **Simple observation.** Let α be a weak composition of length k, and let α^{red} be the result of removing all zero entries from α. Let $\sigma \in \mathfrak{S}_k$ be any permutation. If H is connected, then
  $$p_{\alpha,\sigma} = p_{\alpha^{\text{red}},\tau}$$
  for an appropriate permutation τ. (To get τ, find all i such that $\alpha_i = 0$, and remove the respective $\sigma(i)$ from σ; then standardize.)

- Thus, if H is connected, all descent operators can be written as $p_{\alpha,\sigma}$ for (non-weak) compositions α.

- **Question:** Is it true that any reasonable natural transformation from the forgetful functor
  $$\{\text{connected graded bialgebras}\} \to \{\text{graded modules}\}$$
  to itself is an infinite linear combination of $p_{\alpha,\sigma}$'s?

- **Remark.** The "infinite" part is technical; we can always restrict to a given H_n, and then the combination will be finite.

- **Remark.** "Graded" is important: Otherwise, for $\mathbf{k} = \mathbb{F}_p$, the Frobenius $x \mapsto x^p$ would enter the stage.

- **Question:** But perhaps we can still characterize these natural transformations without gradedness if $\mathbf{k}$ is a field?

- In practice, all the identities we know can be stated in terms of $p_{\alpha,\sigma}$ and $\circ$ and $\star$, unless they are conditional.

3.3. Formulas for general descent operators

- **Theorem (product formulas).** Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a weak composition, and let $\sigma \in \mathfrak{S}_k$ be a permutation.

  Let $\beta = (\beta_1, \beta_2, \ldots, \beta_\ell)$ be a weak composition, and let $\tau \in \mathfrak{S}_\ell$ be a permutation. Then:

  (a) We have
  $$p_{\alpha,\sigma} \star p_{\beta,\tau} = p_{\alpha\beta,\sigma\oplus\tau},$$
  where $\alpha\beta$ is the concatenation of α and β (that is, the weak composition $(\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_\ell)$), whereas $\sigma \oplus \tau$ is the image of (σ, τ) under the obvious map $\mathfrak{S}_k \times \mathfrak{S}_\ell \to \mathfrak{S}_{k+\ell}$.

  (b) We have
  $$p_{\alpha,\sigma} \circ p_{\beta,\tau} = \sum_{\substack{\gamma_{i,j}\in\mathbb{N} \text{ for all } i\in[k] \text{ and } j\in[\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i\in[k]; \\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j\in[\ell]}} p_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}),\tau[\sigma]},$$
  where $\tau[\sigma] \in \mathfrak{S}_{k\ell}$ is the permutation that sends each $\ell(i-1)+j$ (with $i \in [k]$ and $j \in [\ell]$) to $k(\tau(j)-1)+\sigma(i)$.

  Example: Using one-line notation for permutations,
  $$p_{(a,b),[2,1]} \circ p_{(c,d),[2,1]} = \sum_{\substack{c_1+d_1=a; \\ c_2+d_2=b; \\ c_1+c_2=c; \\ d_1+d_2=d}} p_{(c_1,d_1,c_2,d_2),[4,2,3,1]}.$$

- **Proof:** easy computation for **(a)**; multi-page computation using several lemmas for **(b)**.

  (Featuring the Zolotarev shuffle, known from quadratic reciprocity.)

- Particular cases of these formulas were found by Patras in 1993 for commutative H and for cocommutative H. A Hopf monoid variant was found by Aguiar and Mahajan (Chapters 10–11 in *Bimonoids for Hyperplane Arrangements*, 2020).

3.4. The $p_{\alpha,\sigma}$ are linearly independent

- **Theorem.** Generically, the $p_{\alpha,\sigma}$ are linearly independent. That is: There is a connected graded Hopf algebra H such that the family
  $$(p_{\alpha,\sigma})_{\substack{k\in\mathbb{N}; \\ \alpha \text{ is a composition of length } k; \\ \sigma\in\mathfrak{S}_k}}$$
  (of endomorphisms of H) is $\mathbf{k}$-linearly independent.

- This H is the free **k**-algebra with generators

$$x_{i,j} \qquad \text{for all } i, j \in \mathbb{Z} \text{ satisfying } 1 \leqslant i < j,$$

which are understood to be homogeneous of degree $j - i$. The comultiplication $\Delta : H \to H \otimes H$ is given by

$$\Delta\left(x_{i,j}\right) = \sum_{k=i}^{j} x_{i,k} \otimes x_{k,j},$$

where $x_{k,k} := 1$.

(**Remark:** This is a noncommutative version of a unipotent Schur algebra.)

3.5. Universal calculus of $p_{\alpha,\sigma}$ maps

- The above theorems allow for mechanical verification of identities for connected graded bialgebras: Expand in terms of $p_{\alpha,\sigma}$'s (using the product formulas), and compare coefficients. Of course, this gets more laborious the higher n is.

- **Question.** What about non-connected graded bialgebras?

  (This includes bialgebras in general, as those are trivially graded with $H_0 = H$.)

- **Note.** Such questions would be easy if the respective categories had free objects. Do they? I don't think so (but the real question is "how close can we get")...

4. The combinatorial Hopf algebra behind this

4.1. NSym

- Patras's formulas for $p_\alpha \circ p_\beta$ when H is commutative or cocommutative can be restated in terms of a combinatorial Hopf algebra called NSym.

- **Definition.** Let NSym be the free algebra with generators $H_1, H_2, H_3, \ldots$ (that is, the tensor algebra of the free **k**-module with basis $(H_1, H_2, H_3, \ldots)$).

  (Sorry – this is standard notation, unrelated to our old H_i for the i-th degree component of H.)

  We make NSym into a graded algebra by setting each H_i homogeneous of degree i.

We make NSym into a connected graded bialgebra by setting

$$\Delta(H_n) = \sum_{i=0}^{n} H_i \otimes H_{n-i} \qquad \text{and}$$
$$\epsilon(H_n) = 0 \qquad \text{for each } n \geqslant 1.$$

Here, $H_0 := 1$.

- This connected graded bialgebra (thus Hopf algebra) NSym is called the **Hopf algebra of noncommutative symmetric functions**, since its abelianization $\mathrm{NSym}^{\mathrm{ab}}$ is the Hopf algebra $\Lambda = \mathrm{Sym}$ of symmetric functions.

  (NSym is also called the **Leibniz–Hopf algebra** by Hazewinkel, and is denoted *NCSF* by the French school.)

- There is a second multiplication defined on NSym, called the **internal product** or **Kronecker product**. Its definition needs a notation:

- **Definition.** We set

  $$H_\alpha := H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_k} \qquad \text{for any composition } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k),$$

  so that $(H_\alpha)_{\alpha \text{ is a composition}}$ is a basis of the module NSym.

- **Definition.** We define a bilinear operation $*$ on NSym, called **internal product**, by setting

  $$H_\beta * H_\gamma = \sum_{\substack{A \in \mathbb{N}^{k \times \ell};\\ \text{row } A = \beta;\\ \text{column } A = \gamma}} H_{(\text{read } A)^{\text{red}}}$$
  $$\text{for all compositions } \beta = (\beta_1, \beta_2, \ldots, \beta_k) \text{ and } \gamma = (\gamma_1, \gamma_2, \ldots, \gamma_\ell).$$

  Here, the sum ranges over all $k \times \ell$-matrices A with nonnegative integer entries such that the row sums of A are $\beta_1, \beta_2, \ldots, \beta_k$ and the column sums of A are $\gamma_1, \gamma_2, \ldots, \gamma_\ell$. The notation read A denotes the weak composition obtained by concatenating the rows of A from top to bottom.

- This internal product $*$ is called "internal" since it is not graded but rather stays inside a given degree: i.e.,

  $$\mathrm{NSym}_n * \mathrm{NSym}_m = 0 \qquad \text{for } n \neq m, \qquad \text{and}$$
  $$\mathrm{NSym}_n * \mathrm{NSym}_n \subseteq \mathrm{NSym}_n.$$

- As a consequence, $*$ has no unity in NSym, but one in each component NSym_n and one in the completion $\widehat{\mathrm{NSym}}$ (namely, $H_0 + H_1 + H_2 + \cdots$).

- We let $\operatorname{NSym}^{(2)}$ denote the non-unital algebra NSym with product $*$.
- **Theorem (Patras 1993).** Let H be a cocommutative graded bialgebra (sorry – no relation to the $H_i \in \operatorname{NSym}$). Then, H becomes a left $\operatorname{NSym}^{(2)}$-module, by having $H_\alpha \in \operatorname{NSym}^{(2)}$ act as p_α for every composition α.
- The same applies to commutative H instead of cocommutative H; just replace "left" by "right".
- This can also be reinterpreted in terms of $\operatorname{QSym}^{(2)}$-comodules (this is called the "Bernstein homomorphism" in Hazewinkel's terms; see `https://www.cip.ifi.lmu.de/~grinberg/algebra/bernsteinproof.pdf`).

4.2. The general case

- What if H is neither commutative nor cocommutative?
- **Definition.** A **mopiscotion** (please find a better name for this!) is a pair (α, σ), where α is a composition of length k (for some $k \in \mathbb{N}$) and σ is a permutation in $\mathfrak{S}_k$.

  Let PNSym be the free $\mathbf{k}$-module with basis $(F_{\alpha,\sigma})_{(\alpha,\sigma) \text{ is a mopiscotion}}$.

  If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a weak composition and $\sigma \in \mathfrak{S}_k$, then we set

  $$F_{\alpha,\sigma} := F_{\alpha^{\text{red}},\tau},$$

  where τ is obtained from σ by removing all $\sigma(i)$ for which $\alpha_i = 0$ (and standardizing).

  Define two multiplications on PNSym: one "external multiplication" (which mirrors convolution of $p_{\alpha,\sigma}$'s) given by

  $$F_{\alpha,\sigma} \cdot F_{\beta,\tau} = F_{\alpha\beta,\sigma\oplus\tau};$$

  and another "internal multiplication" (which mirrors composition of $p_{\alpha,\sigma}$'s) given by

  $$F_{\alpha,\sigma} * F_{\beta,\tau} = \sum_{\substack{\gamma_{i,j} \in \mathbb{N} \text{ for all } i \in [k] \text{ and } j \in [\ell]; \\ \gamma_{i,1}+\gamma_{i,2}+\cdots+\gamma_{i,\ell}=\alpha_i \text{ for all } i \in [k]; \\ \gamma_{1,j}+\gamma_{2,j}+\cdots+\gamma_{k,j}=\beta_j \text{ for all } j \in [\ell]}} F_{(\gamma_{1,1},\gamma_{1,2},\ldots,\gamma_{k,\ell}),\tau[\sigma]}.$$

  Also, define a comultiplication Δ on PNSym by

  $$\Delta(F_{\alpha,\sigma}) = \sum_{\substack{\beta,\gamma \text{ weak compositions;} \\ \text{entrywise sum } \beta+\gamma=\alpha}} F_{\beta,\sigma} \otimes F_{\gamma,\sigma},$$

mirroring the formula

$$(p_{\alpha,\sigma} \text{ for } H \otimes G) = \sum_{\substack{\beta,\gamma \text{ weak compositions;} \\ \text{entrywise sum } \beta+\gamma=\alpha}} (p_{\beta,\sigma} \text{ for } H) \otimes (p_{\gamma,\sigma} \text{ for } G)$$

that holds for any two graded bialgebras H and G.

- If I have not made any mistakes, then:

  Theorem. PNSym becomes a connected graded Hopf algebra when equipped with the external multiplication $\cdot$, and a (non-graded) bialgebra when equipped with the internal multiplication $*$.

- **Theorem.** Let $\text{PNSym}^{(2)}$ be the nonunital algebra PNSym with multiplication $*$. Then, every connected graded bialgebra H becomes a $\text{PNSym}^{(2)}$-module, with $F_{\alpha,\sigma}$ acting as $p_{\alpha,\sigma}$.

- **Question:** Check this all.

- **Question:** What is the combinatorial meaning of PNSym ?

- **Question:** Is there a cancellation-free formula for the antipode of PNSym ?

- **Question:** Should we expect any identities that connect the internal multiplication with the external multiplication and the coproduct? Some kind of "splitting formula"?

- **Question:** Does PNSym embed into noncommutative formal power series?

- **Remark:** An analogue of $\text{PNSym}^{(2)}$ for Hopf monoids is the **Janus algebra** of Marcelo Aguiar. Is there a way to translate results between Hopf monoids and Hopf algebras?

5. Thanks

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- ... to **you** for your patience.

- Extra kudos if you can make progress on some of the questions!