Three etudes on quasisymmetric functions

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slides:
http://mit.edu/~darij/www/algebra/brandeis06.pdf
papers:
http://mit.edu/~darij/www/algebra/dp-abstr.pdf
http://mit.edu/~darij/www/algebra/bernsteinproof.pdf
http://mit.edu/~darij/www/algebra/dimcreation.pdf

Prologue

Quasisymmetric functions and Hopf algebras

References: (e.g.)

- Darij Grinberg, Victor Reiner, *Hopf Algebras in Combinatorics*, arXiv:1409.8356.
- Michiel Hazewinkel, *Witt vectors. Part 1*, arXiv:0804.3888.
- M. Hazewinkel, N. Gubareni, V.V. Kirichenko, *Algebras, rings and modules. Lie algebras and Hopf algebras*, AMS 2010.

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Symmetric functions, part 1: definition

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- Consider the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates.
- A formal power series f is said to be *bounded-degree* if the monomials it contains are bounded (from above) in degree.
- A formal power series f is said to be *symmetric* if it is invariant under permutations of the indeterminates. Equivalently, if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1, i_2, \ldots, i_k$ are distinct and $j_1, j_2, \ldots, j_k$ are distinct.
- For example:
  - $1 + x_1 + x_2^3$ is bounded-degree but not symmetric.
  - $(1 + x_1)(1 + x_2)(1 + x_3) \cdots$ is symmetric but not bounded-degree.

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- Let Λ be the set of all symmetric bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This is a $\mathbf{k}$-subalgebra, called the *ring of symmetric functions* over $\mathbf{k}$.
  It is also known as Sym.

- The **k**-module Λ has several bases. All the important ones are indexed by partitions.
- A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of nonnegative integers such that $\lambda_i = 0$ for all sufficiently high i.
- For example, $(3, 2, 2, 0, 0, 0, \ldots)$ is a partition.

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- For example, $(3, 2, 2, 0, 0, 0, \ldots)$ is a partition.
- Always write λ_i for the i-th entry of a sequence λ.
- We identify a partition λ with the finite sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ whenever $\lambda_{k+1} = \lambda_{k+2} = \lambda_{k+3} = \cdots = 0$.
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- For example, $(3, 2, 2, 0, 0, 0) = (3, 2, 2) = (3, 2, 2, 0)$.
- Let Par be the set of all partitions.

- For every $\lambda \in \mathrm{Par}$, define

$$m_\lambda = \text{sum of all } \textbf{distinct} \text{ monomials obtained by permuting } x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3} \cdots .$$

  This is a homogeneous power series lying in Λ. Its degree is $|\lambda| := \lambda_1 + \lambda_2 + \lambda_3 + \cdots$, known as the *size* of λ.
- Examples:
  - $m_{()} = 1$.
  - $m_{(1,1)} = \sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + \cdots$.
  - $m_{(2,1)} = \sum_{i \neq j} x_i^2 x_j = x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + \cdots$.
  - $m_{(3)} = \sum_i x_i^3 = x_1^3 + x_2^3 + x_3^3 + \cdots$.

  Note: No repeated monomials even if the partition contains repeated entries!

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- In other words,

$$m_\lambda = \text{sum of all monomials whose exponents are } \lambda_1, \lambda_2, \lambda_3, \ldots \text{ in some order.}$$

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- The family $(m_\lambda)_{\lambda \in \mathrm{Par}}$ is a basis of the $\mathbf{k}$-module Λ, called the *monomial basis* (or m-basis).
- There are many more, e.g.:
  - the h-basis (complete homogeneous symmetric functions),
  - the e-basis (elementary symmetric functions),
  - the s-basis (Schur functions),
  - the p-basis (power-sum symmetric functions; these are a basis only when $\mathbb{Q} \subseteq \mathbf{k}$).

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- We shall now define the quasisymmetric functions – a bigger algebra than Λ, but still with many of its nice properties.
- A formal power series f (still in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$) is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
    - Every symmetric power series is quasisymmetric.
    - $\sum\limits_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.

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- Let QSym be the set of all quasisymmetric bounded-degree power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This is a $\mathbf{k}$-subalgebra, called the *ring of quasisymmetric functions* over $\mathbf{k}$. (Gessel, 1980s.)
- We have $\Lambda \subseteq \mathrm{QSym} \subseteq \mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

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- Let Comp be the set of all compositions.

- For every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathrm{Comp}$, define

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$
$$= \text{sum of all monomials whose nonzero exponents are } \alpha_1, \alpha_2, \ldots, \alpha_k \text{ in } \textbf{this} \text{ order.}$$

  This is a homogeneous power series lying in QSym. Its degree is $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_k$, known as the *size* of α.
- Examples:
  - $M_{()} = 1$.
  - $M_{(1,1)} = \sum_{i<j} x_i x_j = x_1x_2 + x_1x_3 + x_2x_3 + x_1x_4 + x_2x_4 + \cdots$.
  - $M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots$.
  - $M_{(3)} = \sum_{i} x_i^3 = x_1^3 + x_2^3 + x_3^3 + \cdots$.

Note: $m_{(2,1)} = M_{(2,1)} + M_{(1,2)}$.

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- There are many more, e.g.:
  - the F-basis (Gessel's fundamental basis, aka the L-basis),
  - the "quasisymmetric Schur basis",
  - the "dual immaculate basis",
  - the Ψ-basis (only when $\mathbb{Q} \subseteq \mathbf{k}$).

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  - **algebra** = **k**-module A with a multiplication map $m : A \otimes A \to A$ and a unit map $u : \mathbf{k} \to A$ satisfying some axioms;

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  - **coalgebra** $=$ $\mathbf{k}$-module C with a comultiplication map $\Delta : C \to C \otimes C$ and a counit map $\varepsilon : C \to \mathbf{k}$ satisfying the duals of these axioms;

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    (compare with: **group** = monoid M with an "inverse" map $()^{-1} : M \to M$).
- In some more detail...

- The snob's definition of an algebra (associative, unital):
  A $\mathbf{k}$-*algebra* is a $\mathbf{k}$-module A equipped with a $\mathbf{k}$-linear map $m : A \otimes A \to A$ (called "multiplication") and a $\mathbf{k}$-linear map $u : \mathbf{k} \to A$ (called "unit") such that the diagrams

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "m \otimes \mathrm{id}"] \arrow[d, "\mathrm{id} \otimes m"'] & A \otimes A \arrow[d, "m"] \\ A \otimes A \arrow[r, "m"'] & A \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbf{k} \otimes A \arrow[r, "\cong"] \arrow[dr, "u \otimes \mathrm{id}"'] & A & A \otimes \mathbf{k} \arrow[l, "\cong"'] \arrow[dl, "\mathrm{id} \otimes u"] \\ & A \otimes A \arrow[u, "m"] & \end{tikzcd}$$

commute. (All $\otimes$ signs are over $\mathbf{k}$.)

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  commute. (All $\otimes$ signs are over $\mathbf{k}$.)
- NB: The domain of m is $A \otimes A$, not $A \times A$. Thus, everything is $\mathbf{k}$-linear!

- Turning all arrows around, we can define coalgebras:
  A $\mathbf{k}$-*coalgebra* is a $\mathbf{k}$-module C equipped with a $\mathbf{k}$-linear map $\Delta : C \to C \otimes C$ (called "comultiplication") and a $\mathbf{k}$-linear map $\varepsilon : C \to \mathbf{k}$ (called "counit") such that the diagrams

$$\begin{tikzcd}
C \otimes C \otimes C & & C \otimes C \arrow[ll, "\Delta \otimes \mathrm{id}"'] \\
C \otimes C \arrow[u, "\mathrm{id} \otimes \Delta"] & & C \arrow[ll, "\Delta"] \arrow[u, "\Delta"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathbf{k} \otimes C & C \arrow[l, "\cong"'] \arrow[r, leftrightarrow, "\cong"] \arrow[d, "\Delta"'] & C \otimes \mathbf{k} \\
& C \otimes C \arrow[ul, "\varepsilon \otimes \mathrm{id}"] \arrow[ur, "\mathrm{id} \otimes \varepsilon"'] &
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  commute.
- No way to restate this without tensor products anymore!

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  commute.
- If C is a free $\mathbf{k}$-module of finite rank, then C coalgebra $\iff$ $C^* = \operatorname{Hom}(C, \mathbf{k})$ algebra. Not in general!

- A $\mathbf{k}$-*bialgebra* is a $\mathbf{k}$-module H equipped with:
  - the structure of a $\mathbf{k}$-algebra (i.e., suitable maps m and u);
  - the structure of a $\mathbf{k}$-coalgebra (i.e., suitable maps Δ and ε)

  such that the following equivalent statements hold:
  - The maps Δ and ε are $\mathbf{k}$-algebra homomorphisms (where $H \otimes H$ becomes a $\mathbf{k}$-algebra in an appropriate way).
  - The maps m and u are $\mathbf{k}$-coalgebra homomorphisms (where $H \otimes H$ becomes a $\mathbf{k}$-coalgebra in an appropriate way).
  - Some four diagrams commute.

  (We won't need these statements much.)

- A *Hopf algebra* is a $\mathbf{k}$-bialgebra H such that there exists a $\mathbf{k}$-linear map $S : H \to H$ for which the diagram

$$\begin{tikzcd}
 & H \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] \arrow[dd, "\varepsilon"] & \\
H \otimes H \arrow[dd, "\mathrm{id} \otimes S"'] & & H \otimes H \arrow[dd, "S \otimes \mathrm{id}"] \\
 & \mathbf{k} \arrow[dd, "u"] & \\
H \otimes H \arrow[dr, "m"'] & & H \otimes H \arrow[dl, "m"] \\
 & H &
\end{tikzcd}$$

  commutes.
- If S exists, then it is unique, so S can be regarded as part of the Hopf algebra structure. (And should be.) It's called the *antipode* of H.

- A *graded* $\mathbf{k}$-bialgebra is a $\mathbf{k}$-bialgebra H which is graded as $\mathbf{k}$-module, in the sense that
$$H = \bigoplus_{n \geq 0} H_n,$$
and whose structure maps m, u, Δ and ε all are graded.
- No Koszul signs here! (If you don't know what I mean, ignore this.)

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- For example, the gradedness of Δ means that

$$\Delta(H_n) \subseteq \sum_{k=0}^{n} H_k \otimes H_{n-k},$$

where $H_k \otimes H_{n-k}$ is canonically embedded into $H \otimes H$.

Hopf algebras, part 6: grading

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$$H = \bigoplus_{n \geq 0} H_n,$$
and whose structure maps m, u, Δ and ε all are graded.
- The gradedness of m is the usual condition you know from the definition of a graded algebra: $H_a H_b \subseteq H_{a+b}$.

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- A graded $\mathbf{k}$-bialgebra H is said to be *connected* if (1_H) is a basis of the $\mathbf{k}$-module H_0.
- **Takeuchi's theorem (and slightly more):** If H is a connected graded $\mathbf{k}$-bialgebra, then H is a Hopf algebra and its antipode S is graded and invertible.

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- A graded **k**-bialgebra H is said to be *connected* if (1_H) is a basis of the **k**-module H_0.
- Connected graded Hopf algebras tend to appear in combinatorics a lot. We shall now see that Λ and QSym are two such beasts.

Hopf structure on Λ: coalgebra structure

- So how does Λ become a Hopf algebra?
- The counit is easy to define:

$$\varepsilon(f) = f(0,0,0,\ldots) = \text{constant term of } f.$$

- So how does Λ become a Hopf algebra?
- The counit is easy to define:

$$\varepsilon(f) = f(0,0,0,\ldots) = \text{constant term of } f.$$

- The comultiplication, the safe way: If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \text{Par}$ with $\lambda_1, \lambda_2, \ldots, \lambda_k > 0$, then set

$$\Delta(m_\lambda) = \sum m_\mu \otimes m_\nu,$$

  where the sum is over all pairs (μ, ν) of partitions such that concatenating the lists μ and ν and then sorting the result in decreasing order gives λ.
  Don't worry, I'll make sense of this shortly.

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Don't worry, I'll make sense of this shortly.
- Example (writing $m_{\lambda_1,\lambda_2,\ldots,\lambda_k}$ for $m_{(\lambda_1,\lambda_2,\ldots,\lambda_k)}$, and writing $\varnothing$ for the empty partition $()$):

$$\Delta(m_{3,2,2}) = m_\varnothing \otimes m_{3,2,2} + m_3 \otimes m_{2,2} + m_2 \otimes m_{3,2} \\ + m_{3,2} \otimes m_2 + m_{2,2} \otimes m_3 + m_{3,2,2} \otimes m_\varnothing.$$

Hopf structure on Λ: coalgebra structure

- So how does Λ become a Hopf algebra?
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- The comultiplication, the "right" way:

$$\Delta(f) = f(x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots),$$

  where we pretend that $\mathbf{k}[[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]] \cong \mathbf{k}[[x_1, x_2, x_3, \ldots]] \otimes \mathbf{k}[[y_1, y_2, y_3, \ldots]]$ (although it is not true). Making this formal requires work, but this is the actual meaning of comultiplication: it "doubles the alphabet", i.e., splits the indeterminates into two groups. Note that symmetry of f is used here.

- Equipped with these Δ and ε, the $\mathbf{k}$-algebra Λ becomes a connected graded $\mathbf{k}$-bialgebra. Thus, it is a Hopf algebra (by Takeuchi's theorem). What is its antipode?

- Equipped with these Δ and ε, the **k**-algebra Λ becomes a connected graded **k**-bialgebra. Thus, it is a Hopf algebra (by Takeuchi's theorem). What is its antipode?
- The antipode S of Λ is given by

$$S(f) = (-1)^n \omega(f) \qquad \text{for every homogeneous } f \in \Lambda \text{ of degree } n,$$

where ω is the "omega involution".
One way to define ω: It is the **k**-algebra endomorphism of Λ sending each h_n to e_n, where

$$h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \in \Lambda \qquad \text{and}$$

$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \in \Lambda.$$

As the name says, it is an involution, and so is S.

Hopf structure on QSym: coalgebra structure

- So how does QSym become a Hopf algebra?
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$$\varepsilon(f) = f(0, 0, 0, \ldots) = \text{constant term of } f.$$

- The comultiplication, the safe way: If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \text{Comp}$, then set

$$\Delta(M_\alpha) = \sum_{i=0}^{k} M_{(\alpha_1, \alpha_2, \ldots, \alpha_i)} \otimes M_{(\alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_k)}.$$

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- Example (writing $M_{\alpha_1,\alpha_2,\ldots,\alpha_k}$ for $M_{\left(\alpha_1,\alpha_2,\ldots,\alpha_k\right)}$, and writing $\varnothing$ for the empty composition $()$):

$$\begin{aligned}\Delta\left(M_{2,3,2}\right)&=M_\varnothing\otimes M_{2,3,2}+M_2\otimes M_{3,2}\\&\quad+M_{2,3}\otimes M_2+M_{2,3,2}\otimes M_\varnothing.\end{aligned}$$

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- The comultiplication, the "right" way:

$$\Delta(f) = f(x_1 < x_2 < x_3 < \cdots < y_1 < y_2 < y_3 < \cdots).$$

This is even harder to rigorously justify than for Λ, since f is no longer symmetric. We can still apply f to a totally ordered set of indeterminates (thus the $<$ signs), but this is not a-priori clear.
The "safe" way is the better one here for most purposes.

Hopf structure on QSym: the antipode

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  for every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathrm{Comp}$.
- Alternatively, this can be written

$$S\left(M_\alpha\right) = (-1)^k \sum_{\gamma \in \mathrm{Comp};\ \gamma \preceq \operatorname{rev} \alpha} M_\gamma,$$

  where $\preceq$ is a certain partial order on Comp, and where $\operatorname{rev} \alpha = (\alpha_k, \alpha_{k-1}, \ldots, \alpha_1)$.
  (Classical result. Malvenuto? Reutenauer? Gessel?)

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  (Classical result. Malvenuto? Reutenauer? Gessel?)
- Again, the antipode is just a sign away from the "omega involution", if you know the latter.

- **Standard theorem:** If A and B are two Hopf algebras, and $f : A \to B$ is a $\mathbf{k}$-bialgebra homomorphism (i.e., a $\mathbf{k}$-linear map preserving m, u, Δ and ε), then f is a Hopf algebra homomorphism (i.e., also preserves S).

Homomorphisms preserve antipodes

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- **Corollary:** If A is a Hopf subalgebra of a Hopf algebra B, then the antipode of A is the restriction of the antipode of B.
- Thus, the antipode of Λ is the restriction of the antipode of QSym.

Chapter 1

E-partitions and the antipode

Reference:

- Darij Grinberg, *Double posets and the antipode of QSym*, arXiv:1509.08355.
  (The version on my website is newer than the arXiv one, currently at least.)

- Recall: The antipode S of QSym sends

$$M_\alpha \mapsto (-1)^k \sum_{i_1 \geq i_2 \geq \cdots \geq i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

for every $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathrm{Comp}$.

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- Thus, $<$ becomes $\geq$, and a sign appears. This is a classical phenomenon (e.g., Ehrhart reciprocity). Is this specific to the M_α's?
- **It isn't.** Numerous antipode formulas for power series in QSym and in other combinatorial Hopf algebras share the same pattern. We shall show what might be the most general such result for QSym. (See Carolina Benedetti, Bruce Sagan, *Antipodes and involutions*, arXiv:1410.5023 for other Hopf algebras.)

Double posets and E-partitions

- A *double poset* is a triple $(E, <_1, <_2)$, where E is a finite set, and where $<_1$ and $<_2$ are two strict partial orders on E. ("Strict" means "smaller", not "smaller or equal".)

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- Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. An $\mathbf{E}$-*partition* shall mean a map $\phi : E \to \{1, 2, 3, \ldots\}$ such that:
  - every $e \in E$ and $f \in E$ satisfying $e <_1 f$ satisfy $\phi(e) \leq \phi(f)$;
  - every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.

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  - every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.
- **Examples:**
  - If $<_2$ is the same as $<_1$ (or, more, generally, if $<_2$ extends $<_1$), then the **E**-partitions are the **weakly** increasing maps $(E, <_1) \to \{1, 2, 3, \ldots\}$ (also known as poset homomorphisms).
  - If $<_2$ is the same as $>_1$ (or, more, generally, if $<_2$ extends $>_1$), then the **E**-partitions are the **strictly** increasing maps $(E, <_1) \to \{1, 2, 3, \ldots\}$.

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  - every $e \in E$ and $f \in E$ satisfying $e <_1 f$ and $f <_2 e$ satisfy $\phi(e) < \phi(f)$.
- **Examples:**
  - In the general case, you get "something inbetween": weakly increasing maps $(E, <_1) \to \{1, 2, 3, \ldots\}$ satisfying some strict inequalities.
  - One specific examples: Semistandard tableaux of shape λ/μ are $\mathbf{E}$-partitions for a special choice of $\mathbf{E}$.

Double posets: $\Gamma(\mathbf{E}, w)$

- Double posets are a generous source of quasisymmetric functions. To wit:
  If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1, 2, 3, \ldots\}$ is a map, then we define a power series $\Gamma(\mathbf{E}, w) \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$\Gamma(\mathbf{E}, w) = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi,w}, \qquad \text{where } \mathbf{x}_{\pi,w} = \prod_{e \in E} x_{\pi(e)}^{w(e)}.$$

- Easy to see: $\Gamma(\mathbf{E}, w) \in \operatorname{QSym}$.

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- **Examples:**
  - Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition. Let $E = \{1, 2, \ldots, k\}$. Let $<_1$ be the usual $<$ relation, and let $<_2$ be the $>$ relation. Let $w : E \to \{1, 2, 3, \ldots\}$ send each i to α_i. Then,

$$\Gamma(\mathbf{E}, w) = M_\alpha.$$

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- Easy to see: $\Gamma(\mathbf{E}, w) \in \mathrm{QSym}$.
- **Examples:**
  - Let $n \in \mathbb{N}$ and $I \subseteq \{1, 2, \ldots, n-1\}$. Then, there exists a double poset $\mathbf{E}$ and a map w with

$$\Gamma(\mathbf{E}, w) = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in I}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

    This power series is known as the α-th *fundamental quasisymmetric function*, usually called F_α or L_α. Here, α is a composition formed by the "gaps" between the elements of $I \cup \{0, n\}$.

Double posets: $\Gamma(\mathbf{E}, w)$

- If $\mathbf{E} = (E, <_1, <_2)$ is a double poset, and $w : E \to \{1, 2, 3, \ldots\}$ is a map, then we define a power series $\Gamma(\mathbf{E}, w) \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

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- Easy to see: $\Gamma(\mathbf{E}, w) \in \operatorname{QSym}$.
- **Examples:**
  - Schur functions (including those of skew shapes and worse).
  - Dual immaculate functions.
  - P-partition enumerators (in Gessel's language), or (P, ω)-partition enumerators (in Stanley's).
  - "Weighted" versions of the above.

- **What is the antipode of $\Gamma(\mathbf{E}, w)$?**

- **What is the antipode of $\Gamma(\mathbf{E}, w)$?**
- No general answer, but one for *tertispecial* double posets.

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- Special $\Longrightarrow$ semispecial $\Longrightarrow$ tertispecial.
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- **Examples:**
    - The posets whose $\mathbf{E}$-partitions are semistandard tableaux are tertispecial.
    - The posets generating M_α and F_α are special (at least if appropriately chosen).

- **Theorem (Malvenuto, Reutenauer, 1998).** Let $(E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots\}$. Then, the antipode S of QSym satisfies

$$S\left(\Gamma\left(\left(E, <_1, <_2\right), w\right)\right) = (-1)^{|E|} \Gamma\left(\left(E, >_1, <_2\right), w\right),$$

where $>_1$ denotes the opposite relation of $<_1$.

- **Theorem (Malvenuto, Reutenauer, 1998).** Let $(E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots\}$. Then, the antipode S of QSym satisfies

$$S(\Gamma((E, <_1, <_2), w)) = (-1)^{|E|} \Gamma((E, >_1, <_2), w),$$

where $>_1$ denotes the opposite relation of $<_1$.
- **Examples:**
  - The formula for $S(M_\alpha)$ given above.

The antipode of $\Gamma(\mathbf{E}, w)$

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  where $>_1$ denotes the opposite relation of $<_1$.
- **Examples:**
  - The formula for $S(M_\alpha)$ given above.
  - A classical formula for $S(F_\alpha)$.
  - The formula $S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t}$ for skew Schur functions.
  - Benedetti's and Sagan's formulas for antipodes of dual immaculate functions (only for a few simple shapes).
  - Gessel's(?) formula for the antipode of a P-partition enumerator.

- **Theorem (G.).** Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $\operatorname{Par} \mathbf{E}$ denote the set of all $\mathbf{E}$-partitions. Let $w : E \to \{1, 2, 3, \ldots\}$. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$, and also preserves w. Then, G acts also on the set $\operatorname{Par} \mathbf{E}$.

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  For any G-orbit O on $\operatorname{Par}\mathbf{E}$, we define a monomial $\mathbf{x}_{O,w}$ by
  $$\mathbf{x}_{O,w} = \mathbf{x}_{\pi,w} \qquad \text{for some element } \pi \text{ of } O$$
  (this does not depend on the choice of π). Let
  $$\Gamma(\mathbf{E}, w, G) = \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \mathbf{x}_{O,w};$$
  $$\Gamma^{+}(\mathbf{E}, w, G) = \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \mathbf{x}_{O,w}.$$
  Here, an orbit O is said to be E-*coeven* if for every $g \in G$ and every $\pi \in O$ satisfying $g\pi = \pi$, the action of g on E is an even permutation of E.

- **Theorem (G.).** Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $\operatorname{Par} \mathbf{E}$ denote the set of all $\mathbf{E}$-partitions. Let $w : E \to \{1, 2, 3, \ldots\}$. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$, and also preserves w. Then, G acts also on the set $\operatorname{Par} \mathbf{E}$.
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  $$\Gamma^+(\mathbf{E}, w, G) = \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \operatorname{Par} \mathbf{E}} \mathbf{x}_{O,w}.$$
  Then, $\Gamma(\mathbf{E}, w, G) \in \operatorname{QSym}$, $\Gamma^+(\mathbf{E}, w, G) \in \operatorname{QSym}$, and
  $$S(\Gamma(\mathbf{E}, w, G)) = (-1)^{|E|} \Gamma^+((E, >_1, <_2), w, G).$$

- **Theorem (G.).** Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $\operatorname{Par}\mathbf{E}$ denote the set of all $\mathbf{E}$-partitions. Let $w : E \to \{1, 2, 3, \ldots\}$. Let G be a finite group which acts on E. Assume that G preserves both relations $<_1$ and $<_2$, and also preserves w. Then, G acts also on the set $\operatorname{Par}\mathbf{E}$.
For any G-orbit O on $\operatorname{Par}\mathbf{E}$, we define a monomial $\mathbf{x}_{O,w}$ by

$$\mathbf{x}_{O,w} = \mathbf{x}_{\pi,w} \qquad \text{for some element } \pi \text{ of } O$$

(this does not depend on the choice of π). Let

$$\Gamma(\mathbf{E}, w, G) = \sum_{O \text{ is a } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \mathbf{x}_{O,w};$$

$$\Gamma^+(\mathbf{E}, w, G) = \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \operatorname{Par}\mathbf{E}} \mathbf{x}_{O,w}.$$

Then, $\Gamma(\mathbf{E}, w, G) \in \operatorname{QSym}$, $\Gamma^+(\mathbf{E}, w, G) \in \operatorname{QSym}$, and

$$S(\Gamma(\mathbf{E}, w, G)) = (-1)^{|E|}\,\Gamma^+((E, >_1, <_2), w, G).$$

Inspired by Katharina Jochemko's work (arXiv:1310.0838).

Proof of Malvenuto-Reutenauer formula: preliminaries

- I am not going to focus on my generalization. Instead I shall sketch my proof of the Malvenuto-Reutenauer result, which I consider the better part of my paper.

- I am not going to focus on my generalization. Instead I shall sketch my proof of the Malvenuto-Reutenauer result, which I consider the better part of my paper.
- **More notations:** Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset.
  - Then, $\operatorname{Adm} \mathbf{E}$ will mean the set of all pairs (P, Q), where P and Q are subsets of E such that:
    - $P \cap Q = \varnothing$,
    - $P \cup Q = E$,
    - no $p \in P$ and $q \in Q$ satisfy $q <_1 p$.

    These pairs (P, Q) are called the *admissible partitions* of $\mathbf{E}$.

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    These pairs (P, Q) are called the *admissible partitions* of $\mathbf{E}$.
  - For any $T \subseteq E$, we let $\mathbf{E}\mid_T$ denote the double poset $(T, <_1, <_2)$, where $<_1$ and $<_2$ (by abuse of notation) denote the restrictions of the relations $<_1$ and $<_2$ to T.

- **Lemma.** Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then,

$$\Delta\left(\Gamma\left(\mathbf{E}, w\right)\right) = \sum_{(P,Q)\in \operatorname{Adm} \mathbf{E}} \Gamma\left(\mathbf{E}\mid_P, w\mid_P\right) \otimes \Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right).$$

- **Lemma.** Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then,

$$\Delta\left(\Gamma\left(\mathbf{E}, w\right)\right) = \sum_{(P,Q)\in \operatorname{Adm} \mathbf{E}} \Gamma\left(\mathbf{E}\mid_P, w\mid_P\right) \otimes \Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right).$$

- This is easy to prove if you believe in the formula

$$\Delta\left(f\right) = f\left(x_1 < x_2 < x_3 < \cdots < y_1 < y_2 < y_3 < \cdots\right).$$

This formula would suggest that $\Delta\left(\Gamma\left(\mathbf{E}, w\right)\right)$ is a sum over "$\mathbf{E}$-partitions into the totally ordered set $\{x_1 < x_2 < x_3 < \cdots < y_1 < y_2 < y_3 < \cdots\}$". Any such $\mathbf{E}$-partition π splits E into two subsets $P = \pi^{-1}\left(\{x_1 < x_2 < x_3 < \cdots\}\right)$ and $Q = \pi^{-1}\left(\{y_1 < y_2 < y_3 < \cdots\}\right)$, which satisfy $(P, Q) \in \operatorname{Adm} \mathbf{E}$.

- **Lemma.** Let $\mathbf{E} = (E, <_1, <_2)$ be a double poset. Let $w : E \to \{1, 2, 3, \ldots\}$ be a map. Then,

$$\Delta(\Gamma(\mathbf{E}, w)) = \sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} \Gamma(\mathbf{E} \mid_P, w \mid_P) \otimes \Gamma(\mathbf{E} \mid_Q, w \mid_Q).$$

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- There is also a pedestrian proof in my preprint, which needs no such witchcraft.

- Once again, this is what we are proving:
  Theorem (Malvenuto, Reutenauer, 1998). Let $(E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots\}$. Then, the antipode S of QSym satisfies

$$S\left(\Gamma\left(\left(E, <_1, <_2\right), w\right)\right) = (-1)^{|E|}\, \Gamma\left(\left(E, >_1, <_2\right), w\right),$$

  where $>_1$ denotes the opposite relation of $<_1$.

- Once again, this is what we are proving:
  Notation: If $\mathbf{E} = (E, <_1, <_2)$, then write $\mathbf{E}^c$ for $(E, >_1, <_2)$.
  Theorem (Malvenuto, Reutenauer, 1998). Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots\}$. Then, the antipode S of QSym satisfies
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  Theorem (Malvenuto, Reutenauer, 1998). Let $\mathbf{E} = (E, <_1, <_2)$ be a tertispecial double poset. Let $w : E \to \{1, 2, 3, \ldots\}$. Then, the antipode S of QSym satisfies
  $$S(\Gamma(\mathbf{E}, w)) = (-1)^{|E|} \Gamma(\mathbf{E}^c, w),$$
- Proceed by strong induction over $|E|$. So, WLOG, the Theorem is proven for all smaller tertispecial double posets.
- The case $|E| = 0$ is easy, so WLOG assume $|E| > 0$.

- Recall the commutative diagram for the antipode:

$$\begin{tikzcd}
 & H \arrow[dl, "\Delta"'] \arrow[dd, "\varepsilon"] \arrow[dr, "\Delta"] & \\
H \otimes H \arrow[d, "\mathrm{id} \otimes S"'] & & H \otimes H \arrow[d, "S \otimes \mathrm{id}"] \\
H \otimes H \arrow[dr, "m"'] & \mathbf{k} \arrow[d, "u"] & H \otimes H \arrow[dl, "m"] \\
 & H &
\end{tikzcd}$$

It yields $m \circ (S \otimes \mathrm{id}) \circ \Delta = u \circ \varepsilon$. Thus,

$$(m \circ (S \otimes \mathrm{id}) \circ \Delta) (\Gamma (\mathbf{E}, w)) = (u \circ \varepsilon) (\Gamma (\mathbf{E}, w)) = 0$$

(since $\Gamma (\mathbf{E}, w)$ has no constant term, and thus is annihilated by ε).

Proof of Malvenuto-Reutenauer formula: Strategy, part 3

- Thus,

$$0 = (m \circ (S \otimes \mathrm{id}) \circ \Delta)(\Gamma(\mathbf{E}, w)) = m((S \otimes \mathrm{id})(\Delta(\Gamma(\mathbf{E}, w))))$$

$$= m\left((S \otimes \mathrm{id})\left(\sum_{(P,Q)\in \mathrm{Adm}\,\mathbf{E}} \Gamma(\mathbf{E}\mid_P, w\mid_P) \otimes \Gamma(\mathbf{E}\mid_Q, w\mid_Q)\right)\right)$$

(by the lemma on $\Delta(\Gamma(\mathbf{E}, w))$)

$$= \sum_{(P,Q)\in \mathrm{Adm}\,\mathbf{E}} S(\Gamma(\mathbf{E}\mid_P, w\mid_P))\,\Gamma(\mathbf{E}\mid_Q, w\mid_Q)$$

- Thus,

$$0 = (m \circ (S \otimes \mathrm{id}) \circ \Delta)(\Gamma(\mathbf{E}, w)) = m((S \otimes \mathrm{id})(\Delta(\Gamma(\mathbf{E}, w))))$$
$$= m\left((S \otimes \mathrm{id})\left(\sum_{(P,Q)\in \operatorname{Adm} \mathbf{E}} \Gamma(\mathbf{E}\mid_P, w\mid_P) \otimes \Gamma(\mathbf{E}\mid_Q, w\mid_Q)\right)\right)$$
$$\text{(by the lemma on } \Delta(\Gamma(\mathbf{E}, w))\text{)}$$
$$= \sum_{(P,Q)\in \operatorname{Adm} \mathbf{E}} S(\Gamma(\mathbf{E}\mid_P, w\mid_P))\,\Gamma(\mathbf{E}\mid_Q, w\mid_Q)$$

- On the other hand, let's say that we can show

$$0 = \sum_{(P,Q)\in \operatorname{Adm} \mathbf{E}} (-1)^{|P|}\,\Gamma(\mathbf{E}^c\mid_P, w\mid_P)\,\Gamma(\mathbf{E}\mid_Q, w\mid_Q)$$

Then, I claim, we will be done!

- Why will we be done?
- The two equalities

$$0 = \sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} S\left(\Gamma\left(\mathbf{E}\mid_P, w\mid_P\right)\right)\Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right);$$

$$0 = \sum_{(P,Q) \in \operatorname{Adm} \mathbf{E}} (-1)^{|P|}\,\Gamma\left(\mathbf{E}^c\mid_P, w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right)$$

have the same LHS. The sums on their RHSes are equal to each other term by term **except possibly** the $(P, Q) = (E, \varnothing)$ terms.

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  have the same LHS. The sums on their RHSes are equal to each other term by term **except possibly** the $(P,Q) = (E, \varnothing)$ terms.
  - Why? We need to show that if $(P,Q) \in \operatorname{Adm} \mathbf{E}$ satisfies $(P,Q) \neq (E,\varnothing)$, then $S\left(\Gamma\left(\mathbf{E}\mid_P, w\mid_P\right)\right) = (-1)^{|P|}\,\Gamma\left(\mathbf{E}^c\mid_P, w\mid_P\right)$.
  - This is easy: Check that $\mathbf{E}\mid_P$ is tertispecial and that $\left(\mathbf{E}\mid_P\right)^c = \mathbf{E}^c\mid_P$. Now, $|P| < |E|$, so the induction hypothesis yields $S\left(\Gamma\left(\mathbf{E}\mid_P, w\mid_P\right)\right) = (-1)^{|P|}\,\Gamma\left(\left(\mathbf{E}\mid_P\right)^c, w\mid_P\right) = (-1)^{|P|}\,\Gamma\left(\mathbf{E}^c\mid_P, w\mid_P\right)$.

Proof of Malvenuto-Reutenauer formula: Strategy, part 4

- Why will we be done?
- The two equalities

$$0 = \sum_{(P,Q)\in \operatorname{Adm} \mathbf{E}} S\left(\Gamma\left(\mathbf{E}\mid_P, w\mid_P\right)\right)\Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right);$$

$$0 = \sum_{(P,Q)\in \operatorname{Adm} \mathbf{E}} (-1)^{|P|}\Gamma\left(\mathbf{E}^c\mid_P, w\mid_P\right)\Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right)$$

  have the same LHS. The sums on their RHSes are equal to each other term by term **except possibly** the $(P, Q) = (E, \varnothing)$ terms.
- So the $(P, Q) = (E, \varnothing)$ terms must also be equal. But this means $S\left(\Gamma\left(\mathbf{E}, w\right)\right) = (-1)^{|E|}\Gamma\left(\mathbf{E}^c, w\right)$.

  So, yeah, we will be done then.

The crucial identity

- So it remains to prove

$$0 = \sum_{(P,Q)\in \operatorname{Adm} \mathbf{E}} (-1)^{|P|} \Gamma\left(\mathbf{E}^c \mid_P, w \mid_P\right) \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right).$$

Proof of the crucial identity, part 1

- We have

$$\sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} (-1)^{|P|} \Gamma\left(\mathbf{E}^c\mid_P, w\mid_P\right) \Gamma\left(\mathbf{E}\mid_Q, w\mid_Q\right)$$

$$= \sum_{(P,Q)\in\operatorname{Adm}\mathbf{E}} (-1)^{|P|} \left(\sum_{\sigma \text{ is a } (P,>_1,<_2)\text{-partition}} \mathbf{x}_{\sigma,w|_P} \right) \left(\sum_{\tau \text{ is a } (Q,<_1,<_2)\text{-partition}} \mathbf{x}_{\tau,w|_Q} \right)$$

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$$= \sum_{(P,Q)\in \operatorname{Adm} \mathbf{E}} (-1)^{|P|} \sum_{\substack{(\sigma,\tau);\\ \sigma:P\to\{1,2,3,\ldots\};\\ \tau:Q\to\{1,2,3,\ldots\};\\ \sigma \text{ is a } (P,>_1,<_2)\text{-partition};\\ \tau \text{ is a } (Q,<_1,<_2)\text{-partition}}} \mathbf{x}_{\sigma,w|_P} \mathbf{x}_{\tau,w|_Q}$$

- We have

$$\sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} (-1)^{|P|} \Gamma\left(\mathbf{E}^c \mid_P, w \mid_P\right) \Gamma\left(\mathbf{E} \mid_Q, w \mid_Q\right)$$

$$= \sum_{(P,Q)\in \operatorname{Adm}\mathbf{E}} (-1)^{|P|} \sum_{\substack{\pi : E \to \{1,2,3,\ldots\};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} \underbrace{\mathbf{x}_{\pi|_P, w|_P} \mathbf{x}_{\pi|_Q, w|_Q}}_{=\mathbf{x}_{\pi,w}}$$

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$$= \sum_{\pi : E \to \{1,2,3,\ldots\}} \left(\sum_{\substack{(P,Q)\in \operatorname{Adm} \mathbf{E};\\ \pi|_P \text{ is a } (P, >_1, <_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q, <_1, <_2)\text{-partition}}} (-1)^{|P|} \right) \mathbf{x}_{\pi,w}.$$

Proof of the crucial identity, part 1

- We have

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$$= \sum_{\pi:E\to\{1,2,3,\ldots\}} \left(\sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} (-1)^{|P|} \right) \mathbf{x}_{\pi,w}.$$

- So it suffices to prove that, for every map $\pi : E \to \{1,2,3,\ldots\}$, we have

$$\sum_{\substack{(P,Q)\in\operatorname{Adm}\mathbf{E};\\ \pi|_P \text{ is a } (P,>_1,<_2)\text{-partition};\\ \pi|_Q \text{ is a } (Q,<_1,<_2)\text{-partition}}} (-1)^{|P|} = 0.$$

Proof of the crucial identity, part 2

- We must prove that, for every map $\pi : E \to \{1, 2, 3, \ldots\}$, we have
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- The strategy: Find an involution T on the set of all $(P, Q) \in \operatorname{Adm} \mathbf{E}$ satisfying the two conditions under the summation sign. Show that this T reverses the sign of $(-1)^{|P|}$.

Proof of the crucial identity, part 2

- We must prove that, for every map $\pi : E \to \{1, 2, 3, \ldots\}$, we have
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- The strategy: Find an involution T on the set of all $(P, Q) \in \operatorname{Adm} \mathbf{E}$ satisfying the two conditions under the summation sign. Show that this T reverses the sign of $(-1)^{|P|}$.
- The definition of T is simple: Let
$$F = \{e \in E \mid \pi(e) \text{ is minimum}\}.$$
Choose **minimal** element f of the poset $(F, <_2)$ (that is, no $g \in F$ satisfies $g <_2 f$). Now, the map T sends a $(P, Q) \in Z$ to $\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases}.$
- Not so simple: the proof that it works. See the preprint.

A few words on the generalization with the group action

- The generalization where G acts on E is proven by reduction to the Malvenuto-Reutenauer formula. The reduction uses a "quotient poset" construction (idea from Jochemko).

A few words on the generalization with the group action

- The generalization where G acts on E is proven by reduction to the Malvenuto-Reutenauer formula. The reduction uses a "quotient poset" construction (idea from Jochemko).
- Neat exercise (a classical lemma used in the proof):
  Let $(E, <)$ be a poset, and G be a finite group acting on E, preserving $<$. Let $g \in G$. Let E^g be the set of all orbits of g on E. Define a binary relation $<^g$ on E^g by

  $$(u <^g v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ such that } a < b).$$

  Then, $(E^g, <^g)$ is a poset.
  Actually, g can be replaced by a subgroup here.

References

This chapter was about:

- Darij Grinberg, *Double posets and the antipode of QSym*.

The work incorporated ideas from:

- Ira M. Gessel, *Multipartite P-partitions and Inner Products of Skew Schur Functions*, 1984. (I think QSym originates here. Antipodes are not studied, but lots of the groundwork is laid.)
- Claudia Malvenuto, Christophe Reutenauer, *Plethysm and conjugation of quasi-symmetric functions*, 1995. (Original statement of Malvenuto-Reutenauer antipode formula, relatively clumsy (IMHO).)
- Claudia Malvenuto, Christophe Reutenauer, *A self paired Hopf algebra on double posets and a Littlewood-Richardson rule*, 2011. (The notion of a double poset is introduced here, along with the $\Gamma(\mathbf{E}, w)$ and the lemma about $\Delta(\Gamma(\mathbf{E}, w))$. Surprisingly, no connection is made to the antipode formula!)
- Katharina Jochemko, *Order polynomials and Pólya's enumeration theorem*, 2014. (The group action comes from here, though the setting is a lot less general and the claims weaker.)

This chapter was about:

- Darij Grinberg, *Double posets and the antipode of QSym*.

Related work:

- Carolina Benedetti, Bruce Sagan, *Antipodes and involutions*, 2015. (Other Hopf algebras. For QSym, much weaker result, but the method is more combinatorial.)
- See references in my preprint for more.

Thanks to Katharina Jochemko for the original inspiration, to Victor Reiner for teaching me P-partitions, and to you for listening!

Chapter 2

The Bernstein homomorphism

Reference:

- Darij Grinberg, *The Bernstein homomorphism via Aguiar-Bergeron-Sottile universality*.

More on graded modules

- We shall need some more general Hopf algebra theory.
- If H is a graded $\mathbf{k}$-module ($H = \bigoplus_{n \in \mathbb{N}} H_n$, where $0 \in \mathbb{N}$), then
  - for any $n \in \mathbb{N}$, we let $\pi_n : H \to H$ denote the canonical projection onto H_n;
  - for any composition $\alpha = (a_1, a_2, \ldots, a_k)$, we let $\pi_\alpha : H^{\otimes k} \to H^{\otimes k}$ denote the map $\pi_{a_1} \otimes \pi_{a_2} \otimes \cdots \otimes \pi_{a_k}$.

- For any $\mathbf{k}$-algebra A and any $k \in \mathbb{N}$, we define a $\mathbf{k}$-linear $m^{(k-1)} : A^{\otimes k} \to k$ recursively as follows: We set $m^{(-1)} = u_A$ (the unity map, sending $1_{\mathbf{k}}$ to 1_A), $m^{(1)} = \mathrm{id}_A$ and

$$m^{(k)} = m \circ \left(\mathrm{id}_A \otimes m^{(k-1)} \right) \qquad \text{for every } k \geq 1.$$

  The maps $m^{(k-1)} : A^{\otimes k} \to A$ are called the *iterated multiplication maps* of A.
- Or, in simpler language: $m^{(k-1)}$ is the $\mathbf{k}$-linear map $A^{\otimes k} \to A$ which sends every $a_1 \otimes a_2 \otimes \cdots \otimes a_k \in A^{\otimes k}$ to $a_1 a_2 \cdots a_k$.

- For any $\mathbf{k}$-algebra A and any $k \in \mathbb{N}$, we define a $\mathbf{k}$-linear $m^{(k-1)} : A^{\otimes k} \to k$ recursively as follows: We set $m^{(-1)} = u_A$ (the unity map, sending $1_{\mathbf{k}}$ to 1_A), $m^{(1)} = \mathrm{id}_A$ and
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- Or, in simpler language: $m^{(k-1)}$ is the $\mathbf{k}$-linear map $A^{\otimes k} \to A$ which sends every $a_1 \otimes a_2 \otimes \cdots \otimes a_k \in A^{\otimes k}$ to $a_1 a_2 \cdots a_k$.
- But we want to dualize... and the simpler language won't help us do this.

- For any **k**-coalgebra C and any $k \in \mathbb{N}$, we define a **k**-linear map $\Delta^{(k-1)} : C \to C^{\otimes k}$ recursively as follows: We set $\Delta^{(-1)} = \varepsilon_C$ (the counit), $\Delta^{(1)} = \mathrm{id}_C$ and

$$\Delta^{(k)} = \left(\mathrm{id}_C \otimes \Delta^{(k-1)}\right) \circ \Delta \qquad \text{for every } k \geq 1.$$

  The maps $\Delta^{(k-1)} : C \to C^{\otimes k}$ are called the *iterated comultiplication maps* of C.

The second (internal) coalgebra structure on QSym, part 1

- Now to "something completely different": a second way to turn QSym into a **k**-bialgebra. (Implicit in Gessel's 1984 paper.)
- The counit, again, is easy to define:

$$\varepsilon_P(f) = f(1, 0, 0, 0, \ldots).$$

Equivalently, $\varepsilon_P(M_\alpha) = 1$ if the composition α has ≤ 1 entry, and $= 0$ otherwise.

- The comultiplication, the safe way: If α is a composition, then set
$$\Delta_P(M_\alpha) = \sum_A M_{\text{row } A} \otimes M_{\text{column } A}$$
where the sum is over all matrices A with entries in $\mathbb{N}$ such that
  - no row of A is the zero vector;
  - no column of A is the zero vector;
  - reading the entries of A (from left to right, row by row, starting with the top row) gives the composition α, possibly with some zeroes interspersed.

  Here,
  - row A denotes the entrywise sum of all columns of A;
  - column A denotes the entrywise sum of all rows of A.

- The comultiplication, the "right" way:

$$\begin{aligned}\Delta(f) = f\,(&x_1y_1 < x_1y_2 < x_1y_3 < \cdots \\ &< x_2y_1 < x_2y_2 < x_2y_3 < \cdots \\ &< x_3y_1 < x_3y_2 < x_3y_3 < \cdots)\end{aligned}$$

  (these are all products x_iy_j in lexicographic order). Again, we pretend that this is all well-defined.

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  (these are all products $x_i y_j$ in lexicographic order). Again, we pretend that this is all well-defined.

- The comultiplication Δ_P and the counit ε_P make the $\mathbf{k}$-algebra QSym into a $\mathbf{k}$-bialgebra (but not graded and not a Hopf algebra). They are called the *second*, or *internal*, comultiplication and counit.

ABS (Aguiar-Bergeron-Sottile) universal property, part 1

- Aguiar, Bergeron and Sottile, in their 2003 paper "Combinatorial Hopf algebras and generalized Dehn-Sommerville relations" (updated version: http://www.math.cornell.edu/~maguiar/CHalgebra.pdf) proved a universal property (henceforth *ABS property*) for QSym.
- The proof is surprisingly simple (the dual of QSym is NSym, which is a free algebra and thus has a universal property; there are a few more details, but that's basically it), but the property quite useful.

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- The proof is surprisingly simple (the dual of QSym is NSym, which is a free algebra and thus has a universal property; there are a few more details, but that's basically it), but the property quite useful.
- The version you'll see here is not the strongest.

ABS (Aguiar-Bergeron-Sottile) universal property, part 2

- **ABS property:**
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$$\begin{tikzcd} H \arrow[rr, "\Psi"] \arrow[dr, "\zeta"'] & & \mathrm{QSym} \arrow[dl, "\varepsilon_P"] \\ & \mathbf{k} & \end{tikzcd}$$

  is commutative.

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  (b) This Ψ is a $\mathbf{k}$-Hopf algebra homomorphism.

- **ABS property (continued):**
  (c) For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $\mathbf{k}$-linear map $\zeta_\alpha : H \to \mathbf{k}$ as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \xrightarrow{\pi_\alpha} H^{\otimes k} \xrightarrow{\zeta^{\otimes k}} \mathbf{k}^{\otimes k} \xrightarrow[\cong]{m^{(k-1)}} \mathbf{k} .$$

  (Most of these arrows have been defined a few slides before.)
  Then, the unique Ψ above is given by the formula

$$\Psi(h) = \sum_{\alpha \in \mathrm{Comp}} \zeta_\alpha(h) \cdot M_\alpha \qquad \text{for every } h \in H$$

  (in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).

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  (in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).
  (d) Assume that the $\mathbf{k}$-coalgebra H is cocommutative. Then, the unique Ψ above satisfies $\Psi(H) \subseteq \Lambda$, where Λ is the $\mathbf{k}$-algebra of symmetric functions over $\mathbf{k}$.

What is the connection?

- So the ABS property involves the second counit, ε_P. It characterizes $(\mathrm{QSym}, \varepsilon_P)$ as a terminal object in an arrow category (of connected graded Hopf algebras with a $\mathbf{k}$-algebra homomorphism to $\mathbf{k}$).
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- $$\Delta_P : \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym};$$ $$\Psi : H \to \mathrm{QSym}.$$

  These don't look like they match...
- ... until we change the base ring (a.k.a. extend scalars).
- Idea: Regard $\mathrm{QSym} \otimes \mathrm{QSym}$ as "QSym over QSym".
- For greater generality, we can replace some of the QSyms by a commutative algebra A.

Some notations

- Let A be a commutative $\mathbf{k}$-algebra. Then, A is just as good a base ring as $\mathbf{k}$. Every $\mathbf{k}$-coalgebra H gives rise to an A-coalgebra $\underline{A} \otimes H$ by extension of scalars.
  Here, $\underline{A}$ just denotes the $\mathbf{k}$-algebra A, with all its additional structure forgotten. (So far A has no additional structure, but we will later set $A = \operatorname{QSym}$, and then there will be a grading and a coalgebra structure to forget.)
- Similarly, every $\mathbf{k}$-Hopf algebra H induces an A-Hopf algebra $\underline{A} \otimes H$.
  Similarly for algebras, bialgebras, graded Hopf algebras, ...
- For example, $\underline{A} \otimes \operatorname{QSym} \cong \operatorname{QSym}_A$, where QSym_A denotes the graded Hopf algebra QSym defined over A instead of $\mathbf{k}$.

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- For example, $\underline{A} \otimes \operatorname{QSym} \cong \operatorname{QSym}_A$, where QSym_A denotes the graded Hopf algebra QSym defined over A instead of $\mathbf{k}$.
- Let H be a $\mathbf{k}$-coalgebra, and let G be an A-coalgebra. A $\mathbf{k}$-linear map $f : H \to G$ is said to be a $(\mathbf{k}, \underline{A})$*-coalgebra homomorphism* if the A-linear map

$$\underline{A} \otimes H \to G, \qquad a \otimes h \mapsto af(h)$$

  is an A-coalgebra homomorphism.

- **Consequence of ABS property:**
  Let A be a commutative $\mathbf{k}$-algebra. Let H be a connected graded Hopf algebra (over the commutative ring $\mathbf{k}$). Let $\xi : H \to A$ be a $\mathbf{k}$-algebra homomorphism.
  (a) Then, there exists a unique graded $(\mathbf{k}, A)$-coalgebra homomorphism $\Xi : H \to \underline{A} \otimes \mathrm{QSym}$ for which the diagram

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  (b) This Ξ is a $\mathbf{k}$-algebra homomorphism.

- **Consequence of ABS property (continued):**
  (c) For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $\mathbf{k}$-linear map $\xi_\alpha : H \to A$ as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \xrightarrow{\pi_\alpha} H^{\otimes k} \xrightarrow{\xi^{\otimes k}} A^{\otimes k} \xrightarrow{m^{(k-1)}} A .$$

  Then, the unique Ξ above is given by the formula

$$\Xi(h) = \sum_{\alpha \in \mathrm{Comp}} \xi_\alpha(h) \otimes M_\alpha \qquad \text{for every } h \in H$$

  (in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).
  Note: Alternative definition of ξ_α, if you know about convolution:

$$\xi_\alpha = (\xi \circ \pi_{a_1}) \bigstar (\xi \circ \pi_{a_2}) \bigstar \cdots \bigstar (\xi \circ \pi_{a_k}) .$$

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  (d) Assume that the **k**-coalgebra H is cocommutative. Then, the unique Ξ above satisfies $\Xi(H) \subseteq \underline{A} \otimes \Lambda$, where Λ is the **k**-algebra of symmetric functions over **k**.

ABS property after change of bases, part 3

- **Proof idea:** Apply the ABS property to $\underline{A}$, $\underline{A} \otimes H$ and $\xi^\sharp$ instead of $\mathbf{k}$, H and ζ, where

$$\xi^\sharp : \underline{A} \otimes H \to \underline{A}, \qquad a \otimes h \mapsto a\xi(h).$$

- There is a bijection between the $\mathbf{k}$-linear maps $H \to \underline{A} \otimes G$ and the A-linear maps $\underline{A} \otimes H \to \underline{A} \otimes G$ (where G is any A-module). (This is just the adjointness of induction and restriction.) Use it back and forth, many times.
- Some amount of work required to check that this bijection takes maps where you expect it to take them, and that all the identifications (e.g., ε_P defined over A is identified with $\mathrm{id}_A \otimes \varepsilon_P$) play well together. (See the preprint.)

- Apply the Consequence of ABS property to $A = H$ and $\xi = \mathrm{id}$. Obtain the following:
- **Bernstein homomorphism theorem:**
  Let H be a **commutative** connected graded Hopf algebra (over the commutative ring $\mathbf{k}$).
  (a) Then, there exists a unique graded $(\mathbf{k}, H)$-coalgebra homomorphism $\Xi : H \to \underline{H} \otimes \mathrm{QSym}$ for which the diagram

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  is commutative (that is, we have $\mathrm{id}_H = (\mathrm{id}_H \otimes \varepsilon_P) \circ \Xi$).
  Denote this Ξ by β_H.

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  (b) This β_H is a $\mathbf{k}$-algebra homomorphism.

- **Bernstein homomorphism theorem (continued):**
  (c) For every composition $\alpha = (a_1, a_2, \ldots, a_k)$, define a $\mathbf{k}$-linear map $\xi_\alpha : H \to \mathbf{k}$ as the composition

$$H \xrightarrow{\Delta^{(k-1)}} H^{\otimes k} \xrightarrow{\pi_\alpha} H^{\otimes k} \xrightarrow{m^{(k-1)}} A .$$

  Then, β_H ($=$ the unique Ξ above) is given by the formula

$$\beta_H(h) = \sum_{\alpha \in \operatorname{Comp}} \xi_\alpha(h) \otimes M_\alpha \qquad \text{for every } h \in H$$

  (in particular, the sum on the right hand side of this equality has only finitely many nonzero addends).
  Note: Alternative definition of ξ_α, if you know about convolution:

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  (d) Assume that the $\mathbf{k}$-coalgebra H is cocommutative. Then, $\beta_H(H) \subseteq \underline{H} \otimes \Lambda$, where Λ is the $\mathbf{k}$-algebra of symmetric functions over $\mathbf{k}$.

- **Bernstein homomorphism theorem (continued):** Additionally, it is not hard to show:
  (e) Let τ denote the twist map $\mathrm{QSym} \otimes \mathrm{QSym} \to \mathrm{QSym} \otimes \mathrm{QSym}, \ a \otimes b \mapsto b \otimes a$. Let $\Delta'_P = \tau \circ \Delta_P$ be the "twisted second comultiplication" of QSym.
  Then,
  $$\beta_{\mathrm{QSym}} = \Delta'_P.$$
  Thus, we have obtained Δ'_P as a particular case of the Bernstein homomorphism.

- **Bernstein homomorphism theorem (continued):**
- The uniqueness part of the Consequence of ABS property can be used to prove identities with β_H. For example:
  (f) The diagram

$$\begin{tikzcd} H \arrow[rr, "\beta_H"] \arrow[d, "\beta_H"'] & & \underline{H} \otimes \mathrm{QSym} \arrow[d, "\beta_H \otimes \mathrm{id}"'] \\ \underline{H} \otimes \mathrm{QSym} \arrow[rr, "\mathrm{id} \otimes \Delta'_P"'] & & \underline{H} \otimes \underline{\mathrm{QSym}} \otimes \mathrm{QSym} \end{tikzcd}$$

  is commutative.
  Thus, every commutative connected graded Hopf algebra H becomes a comodule over the "second bialgebra" QSym.
- This easily gives an alternative proof of the fact that the second bialgebra is indeed a bialgebra (coassociative, etc.).

- The (restricted) Bernstein homomorphism $\beta_H : H \to \underline{H} \otimes \Lambda$ for commutative-and-cocommutative H appeared in Hazewinkel's "Witt vectors", §18.24. It is attributed to Joseph N. Bernstein (who apparently used it to classify PSH-algebras – see Zelevinsky's LNM book).
- I think the more general "Bernstein homomorphism" $\beta_H : H \to \underline{H} \otimes \mathrm{QSym}$ constructed above (not assuming cocommutativity) is new.
- That said, the Marne-la-Vallée school has its own methods (alphabets, generalized Cauchy kernels, etc.) which probably lead to many of the same results. I don't know the details :(

- **Bernstein homomorphism theorem (continued):**
  (g) (Same notations as before.) We have
  $$\beta_H \circ S_H = (\mathrm{id}_H \otimes S_{\mathrm{QSym}}) \circ \beta_H,$$
  where S_G means the antipode of a Hopf algebra G.
- Again, this comes "for free" out of universal properties.
- Probably more can be gotten this way. Idea:
  β_H is injective (since $(\mathrm{id}_H \otimes \varepsilon_P) \circ \beta_H = \mathrm{id}_H$), and thus "embeds" H into $\underline{H} \otimes \mathrm{QSym} \cong \mathrm{QSym}_H$. Scare quotes because the "embedding" changes the base ring. But
  $\underline{H} \otimes H \to \underline{H} \otimes \mathrm{QSym} \cong \mathrm{QSym}_H$, $a \otimes b \mapsto a\beta_H(b)$ is a honest graded H-Hopf algebra homomorphism.

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$\underline{H} \otimes H \to \underline{H} \otimes \mathrm{QSym} \cong \mathrm{QSym}_H$, $a \otimes b \mapsto a\beta_H(b)$ is a honest graded H-Hopf algebra homomorphism.
$\Longrightarrow$ **Propagandistic slogan:** Every commutative connected graded Hopf algebra embeds into QSym. (Fine print: over itself. But this should still be useful!)

References

This chapter was about:

- Darij Grinberg, *The Bernstein homomorphism via Aguiar-Bergeron-Sottile universality*.

The work incorporated ideas from:

- Andrey V. Zelevinsky, *Representations of Finite Classical Groups – A Hopf Algebra Approach*, LNM #869, Springer 1981. (This is where the first "Bernstein homomorphism" appeared.)
- Michiel Hazewinkel, *Witt vectors. Part 1*, arXiv:0804.3888. (An unusual introduction to symmetric functions, emphasizing categorical methods (among other things).)
- Marcelo Aguiar, Nantel Bergeron, Frank Sottile, *Combinatorial Hopf algebras and generalized Dehn-Sommerville relations*. (The universal property of QSym with several applications.)

Thanks to Marcelo Aguiar for helpful emails!

Chapter 3

On dual immaculate functions

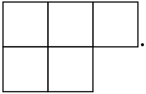
Reference:

- Darij Grinberg, *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions*.

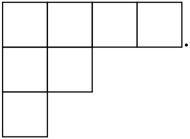
- I won't actually use Schur functions, but I'll talk about an analogue; so let me recall their definition.
- Let λ be a partition. The *Young diagram* of λ is like a matrix, but the rows have different lengths, and are left-aligned; the i-th row has λ_i cells.

  Examples:

  - The Young diagram of $(3, 2)$ has the form

  - The Young diagram of $(4, 2, 1)$ has the form

- A *semistandard tableau* of shape λ is the Young diagram of λ, filled with positive integers, such that
  - the entries in each **row** are **weakly** increasing;
  - the entries in each **column** are **strictly** increasing.

Examples:

- A semistandard tableau of shape $(3, 2)$ is

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}.$$

- A semistandard tableau of shape $(4, 2, 1)$ is

$$\begin{array}{|c|c|c|c|}\hline 2 & 2 & 3 & 4 \\ \hline 3 & 4 \\ \cline{1-2} 5 \\ \cline{1-1}\end{array}.$$

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- The semistandard tableaux of shape $(3,2)$ are the arrays of the form

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \multicolumn{1}{c}{} \\ \cline{1-2}\end{array}$$

with $a \leq b \leq c$ and $d \leq e$ and $a < d$ and $b < e$.

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a	b	c
d	e	

    with $a \leq b \leq c$ and $d \leq e$ and $a < d$ and $b < e$.
- So the semistandard tableaux of a given shape λ are the **E**-partitions, for a certain double poset **E** (which we can choose to be special).

Schur functions, part 3

- Given a partition λ, we define the *Schur function* s_λ as the power series

$$s_\lambda = \sum_{\substack{T \text{ is a semistandard} \\ \text{tableau of shape } \lambda}} \mathbf{x}_T, \qquad \text{where } \mathbf{x}_T = \prod_{p \text{ is a cell of } T} x_{T(p)}$$

(where $T(p)$ denotes the entry of T in p).

- **Example:**
  - $$s_{(3,2)} = \sum_{\substack{a \leq b \leq c,\ d \leq e, \\ a < d,\ b < e}} x_a x_b x_c x_d x_e,$$

    because the semistandard tableau

    $$T = \begin{array}{|c|c|c|} \hline a & b & c \\ \hline d & e & \\ \cline{1-2} \end{array}$$

    contributes the addend $\mathbf{x}_T = x_a x_b x_c x_d x_e$.

- **Classical theorem:** The Schur function s_λ is a symmetric function (= an element of Λ) for any partition λ.
- **Classical theorem:** The family $(s_\lambda)_{\lambda \in \mathrm{Par}}$ is a basis of the $\mathbf{k}$-module Λ.

- Dual immaculate functions are a quasisymmetric analogue (one of several) of the Schur functions. They have been introduced in 2012 by Berg, Bergeron, Saliola, Serrano and Zabrocki (arXiv:1208.5191v3).
- Their original definition is complicated (they are defined as the dual basis to a basis of NSym, which is defined using "creation operators" – thus the name). We shall give an equivalent definition.

- Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition. The *Young diagram* of α is like a matrix, but the rows have different lengths, and are left-aligned; the i-th row has α_i cells (there are k rows in total).

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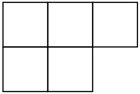
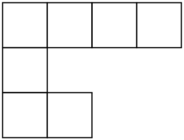
    .

  - The Young diagram of $(4, 1, 2)$ has the form

    .

Dual immaculate functions, part 3

- An *immaculate tableau* of shape α is the Young diagram of α, filled with positive integers, such that
  - the entries in each **row** are **weakly** increasing;
  - the entries in **the first column** are **strictly** increasing.

  No requirements on the second, third, etc. columns!

  Examples:
  - An immaculate tableau of shape $(3, 2)$ is

$$\begin{array}{|c|c|c|}\hline 2 & 3 & 3 \\ \hline 3 & 3 \\ \cline{1-2}\end{array}\ .$$

  - An immaculate tableau of shape $(4, 1, 2)$ is

$$\begin{array}{|c|c|c|c|}\hline 2 & 6 & 7 & 7 \\ \hline 3 \\ \cline{1-2} 5 & 5 \\ \cline{1-2}\end{array}\ .$$

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  - the entries in each **row** are **weakly** increasing;
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  No requirements on the second, third, etc. columns!

  Examples:
  - The immaculate tableaux of shape $(3, 2)$ are the arrays of the form

$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \\ \cline{1-2}\end{array}$$

    with $a \leq b \leq c$ and $d \leq e$ and $a < d$.

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  Examples:
  - The immaculate tableaux of shape $(3, 2)$ are the arrays of the form
    $$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e & \\ \cline{1-2}\end{array}$$
    with $a \leq b \leq c$ and $d \leq e$ and $a < d$.
- So the immaculate tableaux of a given shape α are the **E**-partitions, for a certain double poset **E** (which we can choose to be special).
- Immaculate tableaux can also be viewed as certain labellings of a binary tree (a special one – a "comb").

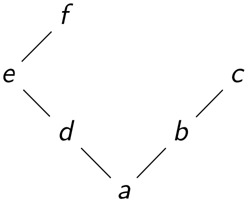
Dual immaculate functions, digression

- **Digression** (won't be used, but good to know):
  I just said that immaculate tableaux can be viewed as certain labellings of a binary tree; here is what I mean:

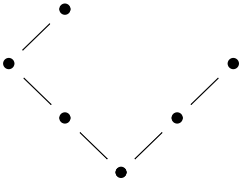
- The immaculate tableaux of shape $(3,1,2)$ are the arrays

a	b	c
d		
e	f	

satisfying $a \leq b \leq c$ and $e \leq f$ and $a < d < e$. In other words, they are labellings



of the binary tree



such that the label on each node is $<$ to the label on its left child and $\leq$ to the label on its right child.

- Given a composition α, we define the *dual immaculate function* $\mathfrak{S}_{\alpha}^{*}$ as the power series

$$\mathfrak{S}_{\alpha}^{*}=\sum_{\substack{T \text{ is an immaculate} \\ \text{tableau of shape } \alpha}} \mathbf{x}_T, \qquad \text{where } \mathbf{x}_T=\prod_{p \text{ is a cell of } T} x_{T(p)}$$

  (where $T(p)$ denotes the entry of T in p).
- **Example:**
  - $$\mathfrak{S}_{(3,2)}^{*}=\sum_{\substack{a\leq b\leq c,\ d\leq e, \\ a<d}} x_a x_b x_c x_d x_e,$$

    because the immaculate tableau

    $$T=\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & e \\ \cline{1-2}\end{array}$$

    contributes the addend $\mathbf{x}_T=x_a x_b x_c x_d x_e$.

- **Theorem:** The dual immaculate function $\mathfrak{S}^*_\alpha$ is a quasisymmetric function (= an element of QSym) for any composition α.
  (This is actually pretty obvious, since the immaculate tableaux are **E**-partitions.)
- **Theorem:** The family $(\mathfrak{S}^*_\alpha)_{\alpha \in \mathsf{Comp}}$ is a basis of the **k**-module QSym.
- It is natural to consider this basis an analogue of the Schur functions (although there are other contenders whose claims are equally valid).

- I mentioned the fundamental quasisymmetric functions F_α in Chapter 1, but now I'll need to define them in detail.

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- Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ be a composition of size $n = \alpha_1 + \alpha_2 + \cdots + \alpha_k$. Define a subset $D(\alpha)$ of $\{1, 2, \ldots, n-1\}$ by

$$\begin{aligned} D(\alpha) &= \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\} \\ &= \{\text{all nonempty partial sums of } \alpha \text{, except} \\ &\qquad\qquad \text{for the total sum}\}. \end{aligned}$$

- Define a power series F_α (also known as L_α) by

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

- I mentioned the fundamental quasisymmetric functions F_α in Chapter 1, but now I'll need to define them in detail.
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- **Theorem:** The family $(F_\alpha)_{\alpha \in \mathrm{Comp}}$ is a basis of the $\mathbf{k}$-module QSym, called *Gessel's Fundamental basis*.

- **Theorem.** For any composition α, we have
$$F_\alpha = \sum_{\substack{\beta \in \mathrm{Comp}; \\ \beta \text{ refines } \alpha}} M_\beta .$$
Here, a composition β is said to *refine* α if and only if β can be obtained by splitting each entry α_i of α into several which sum to α_i.
Example. $(2,1,3,4,2,2,1,3)$ refines $(3,3,8,4)$, because ...

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Example. $\left(\underbrace{2,1}_{\text{sum } 3}, \underbrace{3}_{\text{sum } 3}, \underbrace{4,2,2}_{\text{sum } 8}, \underbrace{1,3}_{\text{sum } 4}\right)$ refines $(3,3,8,4)$.

Fundamental quasisymmetric functions, part 2

- **Theorem.** For any composition α, we have

$$F_\alpha = \sum_{\substack{\beta \in \operatorname{Comp}; \\ \beta \text{ refines } \alpha}} M_\beta .$$

  Here, a composition β is said to *refine* α if and only if β can be obtained by splitting each entry α_i of α into several which sum to α_i.

- The antipode S of QSym satisfies

$$S\left(F_\alpha\right) = (-1)^{|\alpha|} F_{\omega(\alpha)}$$

  for each $\alpha \in \operatorname{Comp}$. Here, if α is a composition of n, then $\omega\left(\alpha\right)$ is the composition of n defined by $D\left(\omega\left(\alpha\right)\right) = \{1, 2, \ldots, n-1\} \setminus \underbrace{\left(n - D\left(\alpha\right)\right)}_{=\{n-g \mid g \in D(\alpha)\}}$.

  (This follows from Chapter 1.)

- Let β be a composition. We define the $\mathbf{k}$-linear map $R_\beta^\perp : \operatorname{QSym} \to \operatorname{QSym}$ (called *skewing by the ribbon function* R_β) by

$$R_\beta^\perp (F_\alpha) = \begin{cases} F_{(\alpha_{i+1},\alpha_{i+2},\ldots\alpha_k)}, & \text{if } \beta = (\alpha_1, \alpha_2, \ldots, \alpha_i) \text{ for} \\ & \text{some } 0 \leq i \leq k; \\ 0, & \text{otherwise} \end{cases}$$

  for every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$.
  (This works because $(F_\alpha)_{\alpha \in \operatorname{Comp}}$ is a basis of the $\mathbf{k}$-module QSym.)
- This is not the "official" definition of $R_\beta^\perp$, but is equivalent and self-contained. (The "official" definition involves the Hopf algebra NSym, which is the graded dual of QSym.)

Skewing by R_β: a quick definition

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- This is not the "official" definition of $R_\beta^\perp$, but is equivalent and self-contained. (The "official" definition involves the Hopf algebra NSym, which is the graded dual of QSym.)
- Note: $R_\beta^\perp$ lowers the degree by $|\beta|$.

Zabrocki's "dual creation operators"

- For each composition α, define a composition $\alpha \odot (m)$ as follows: If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, then $\alpha \odot (m) = (\alpha_1, \alpha_2, \ldots, \alpha_{k-1}, \alpha_k + m)$. (If α is the empty composition, then $\alpha \odot (m) = (m)$.)

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- For each positive integer m, define a **k**-linear map $\mathbf{W}_m : \mathrm{QSym} \to \mathrm{QSym}$ by

$$\mathbf{W}_m = \sum_{\alpha \in \mathrm{Comp}} (-1)^{|\alpha|} F_{\alpha \odot (m)} R^{\perp}_{\omega(\alpha)}.$$

  (The right hand side converges pointwise: Every $f \in \mathrm{QSym}$ is annihilated by all but finitely many addends.)
- This construction is similar to the creation operators defined on NSym for the (non-dual) immaculate functions, and to the Bernstein creation operators on Λ (**not** the Bernstein homomorphism of Chapter 2, but due to the same Bernstein, and also appearing in Zelevinsky's book).

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  (The right hand side converges pointwise: Every $f \in \mathrm{QSym}$ is annihilated by all but finitely many addends.)
- **Theorem (Zabrocki, G.).** For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, the dual immaculate function $\mathfrak{S}^*_\alpha$ is

$$\mathfrak{S}^*_\alpha = \left(\mathbf{W}_{\alpha_1} \circ \mathbf{W}_{\alpha_2} \circ \cdots \circ \mathbf{W}_{\alpha_k}\right)(1).$$

- My proof uses a *dendriform algebra* structure on QSym. I won't explain the proof here, but I will define this structure.
- I think the structure is foreshadowed (if not implicitly introduced) in the works of Glânffrwd P. Thomas (mainly *Frames, Young tableaux, and Baxter sequences*). But I need to think about it more carefully.
  (There are well-known adjoint functors between Rota-Baxter algebras and dendriform algebras: Kurusch Ebrahimi-Fard, Li Guo, *Rota-Baxter Algebras and Dendriform Algebras*, arXiv:math/0503647)

Dendriform structure on QSym, introduction

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- Also, does anyone have a copy of Thomas's thesis?
  (Glânffrwd P. Thomas, *Baxter Algebras and Schur Functions*, Ph.D. Thesis, University College of Swansea, September 1974.)

- For any monomial $\mathfrak{m}$, let $\operatorname{Supp}\mathfrak{m}$ denote the set $\{i \mid x_i \text{ appears in } \mathfrak{m}\}$.
- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.

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- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.
- We define a binary operation $\prec$ on the $\mathbf{k}$-module $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ as follows:
  - On monomials, it should be given by

    $$\mathfrak{m} \prec \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min\left(\operatorname{Supp}\mathfrak{m}\right) < \min\left(\operatorname{Supp}\mathfrak{n}\right); \\ 0, & \text{if } \min\left(\operatorname{Supp}\mathfrak{m}\right) \geq \min\left(\operatorname{Supp}\mathfrak{n}\right) \end{cases}$$

    for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbf{k}$-bilinear.
  - It should be continuous (i.e., its $\mathbf{k}$-bilinearity also applies to infinite $\mathbf{k}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $\left(x_2^2 x_4\right) \prec \left(x_3^2 x_5\right) = x_2^2 x_3^2 x_4 x_5$, but $\left(x_2^2 x_4\right) \prec \left(x_2^2 x_5\right) = 0$.

- For any monomial $\mathfrak{m}$, let $\operatorname{Supp}\mathfrak{m}$ denote the set $\{i \mid x_i \text{ appears in } \mathfrak{m}\}$.
- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.
- We define a binary operation $\succeq$ on the $\mathbf{k}$-module $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ as follows:
  - On monomials, it should be given by
    $$\mathfrak{m} \succeq \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min\left(\operatorname{Supp}\mathfrak{m}\right) \geq \min\left(\operatorname{Supp}\mathfrak{n}\right); \\ 0, & \text{if } \min\left(\operatorname{Supp}\mathfrak{m}\right) < \min\left(\operatorname{Supp}\mathfrak{n}\right) \end{cases}$$
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- Well-definedness is pretty clear.
- **Example.** $\left(x_2^2 x_4\right) \succeq \left(x_3^2 x_5\right) = 0$, but $\left(x_2^2 x_4\right) \succeq \left(x_2^2 x_5\right) = x_2^4 x_4 x_5$.

- We now have defined two binary operations $\prec$ and $\succeq$ on $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$. They satisfy:

$$
\begin{aligned}
a \prec b + a \succeq b &= ab; \\
(a \prec b) \prec c &= a \prec (bc); \\
(a \succeq b) \prec c &= a \succeq (b \prec c); \\
a \succeq (b \succeq c) &= (ab) \succeq c.
\end{aligned}
$$

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- This says that $(\mathbf{k}[[x_1, x_2, x_3, \ldots]], \prec, \succeq)$ is a *dendriform algebra* in the sense of Loday (see, e.g., Zinbiel, *Encyclopedia of types of algebras 2010*, arXiv:1101.0267).

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- This says that $(\mathbf{k}[[x_1, x_2, x_3, \ldots]], \prec, \succeq)$ is a *dendriform algebra* in the sense of Loday (see, e.g., Zinbiel, *Encyclopedia of types of algebras 2010*, arXiv:1101.0267).
- QSym becomes a dendriform subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. (This is somewhat tangential to the actual proof.)

- **Crucial lemma 1.** For every positive integer m and every $f \in \operatorname{QSym}$, we have

$$\mathbf{W}_m f = h_m \prec f,$$

where h_m is the m-th complete homogeneous symmetric function (that is, $\sum_{i_1 \leq i_2 \leq \cdots \leq i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$).

- **Crucial lemma 1.** For every positive integer m and every $f \in \operatorname{QSym}$, we have
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where h_m is the m-th complete homogeneous symmetric function (that is, $\sum\limits_{i_1 \leq i_2 \leq \cdots \leq i_m} x_{i_1} x_{i_2} \cdots x_{i_m}$).
- **Crucial lemma 2.** For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, the dual immaculate function $\mathfrak{S}^*_\alpha$ is
$$\mathfrak{S}^*_\alpha = h_{\alpha_1} \prec (h_{\alpha_2} \prec (\cdots \prec (h_{\alpha_k} \prec 1) \cdots)) .$$
(This actually follows easily from our definition of $\mathfrak{S}^*_\alpha$.)

- In the proof of Crucial lemma 1, another binary operation on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ appears, which I call Φ.
- We define a binary operation Φ on the $\mathbf{k}$-module $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ as follows:
  - On monomials, it should be given by
  $$\mathfrak{m} \Phi \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \leq \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) > \min(\operatorname{Supp} \mathfrak{n}) \end{cases}$$
  for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
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- **Example.** $(x_2^2 x_4) \Phi (x_4^2 x_5) = x_2^2 x_4^3 x_5$ and $(x_2^2 x_4) \Phi (x_3^2 x_5) = 0$.

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- **Example.** $\left(x_2^2 x_4\right) ⚹ \left(x_4^2 x_5\right) = 0$, but $\left(x_2^2 x_4\right) ⚹ \left(x_5^2 x_6\right) = x_2^2 x_4 x_5^2 x_6$.

Main ideas of the proof, part 2

- In the proof of Crucial lemma 1, another binary operation on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ appears, which I call ᛄ.
- Belgthor (ᛄ) and Tvimadur (ᛯ) are two calendar runes signifying two of the 19 years of the Metonic cycle. I sought two (unused) symbols that (roughly) look like "putting one thing (monomial) atop another", allowing overlap (ᛄ) and disallowing overlap (ᛯ).
  You be the judge whether I succeeded...

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  You be the judge whether I succeeded...
- My apologies to the editors of the Canadian Journal of Mathematics who'll have to put up with a LaTeX package for runes in a mathematical document.

- A lemma (for Crucial lemma 1): For any $a \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ and $b \in \operatorname{QSym}$, we have

$$\sum_{(b)} \left(S\left(b_{(1)}\right) \Phi a\right) b_{(2)} = a \prec b,$$

where we use Sweedler's notation $\sum_{(b)} b_{(1)} \otimes b_{(2)}$ for $\Delta(b)$.

A few more words about the runes

- The operations Φ and Ж restrict to QSym.
- They are associative and unital (with unity 1).
- They satisfy

$$(a \Phi b) \text{Ж} c - a \Phi (b \text{Ж} c) = \varepsilon(b)(a \text{Ж} c - a \Phi c);$$
$$(a \text{Ж} b) \Phi c - a \text{Ж} (b \Phi c) = \varepsilon(b)(a \Phi c - a \text{Ж} c),$$

where $\varepsilon : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}$ sends f to $f(0,0,0,\ldots)$ as before.

- The operations Φ and $\divideontimes$ restrict to QSym.
- They are associative and unital (with unity 1).
- They satisfy

$$(a \mathbin{\Phi} b) \divideontimes c - a \mathbin{\Phi} (b \divideontimes c) = \varepsilon(b)(a \divideontimes c - a \mathbin{\Phi} c);$$
$$(a \divideontimes b) \mathbin{\Phi} c - a \divideontimes (b \mathbin{\Phi} c) = \varepsilon(b)(a \mathbin{\Phi} c - a \divideontimes c),$$

  where $\varepsilon : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}$ sends f to $f(0, 0, 0, \ldots)$ as before.
- As a consequence,

$$(a \mathbin{\Phi} b) \divideontimes c + (a \divideontimes b) \mathbin{\Phi} c = a \mathbin{\Phi} (b \divideontimes c) + a \divideontimes (b \mathbin{\Phi} c).$$

  This says that $(\mathrm{QSym}, \Phi, \divideontimes)$ is a $As^{\langle 2 \rangle}$*-algebra* (in the sense of Loday).

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- They satisfy

$$(a \Phi b) \divideontimes c - a \Phi (b \divideontimes c) = \varepsilon(b)(a \divideontimes c - a \Phi c);$$
$$(a \divideontimes b) \Phi c - a \divideontimes (b \Phi c) = \varepsilon(b)(a \Phi c - a \divideontimes c),$$

  where $\varepsilon : \mathbf{k}[[x_1, x_2, x_3, \ldots]] \to \mathbf{k}$ sends f to $f(0,0,0,\ldots)$ as before.
- As a consequence,

$$(a \Phi b) \divideontimes c + (a \divideontimes b) \Phi c = a \Phi (b \divideontimes c) + a \divideontimes (b \Phi c).$$

  This says that $(\mathrm{QSym}, \Phi, \divideontimes)$ is a *$As^{\langle 2\rangle}$-algebra* (in the sense of Loday).
- What other identities do they satisfy? What identities do Φ, $\divideontimes$, $\prec$ and $\succeq$ satisfy together?

- $\Lambda \subseteq \mathrm{QSym}$ is not the end of the road.
- There are higher "combinatorial Hopf algebras" around, such as the noncommutative algebras FQSym and WQSym.
- All above-mentioned operations can be lifted to WQSym. Some also to FQSym. (See the preprint.)
- Actually, dendriform structures on WQSym and FQSym are known (Foissy, Novelli, Thibon).

References

This chapter was about:

- Darij Grinberg, *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions*, to appear in the Canadian Journal of Mathematics.

Related work:

- Jean-Christophe Novelli, Jean-Yves Thibon, *Construction of dendriform trialgebras*, arXiv:math/0510218.
- Glânffrwd P. Thomas, *Frames, Young tableaux, and Baxter sequences*.
- Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, *A lift of the Schur and Hall-Littlewood bases to non-commutative symmetric functions*, arXiv:1208.5191.

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