Refined dual stable Grothendieck polynomials

Darij Grinberg (MIT)

joint work with Gaku Liu (MIT) and Pavel Galashin (MIT)

23 October 2015

University of Minnesota Combinatorics Seminar

(extended version of a talk held at 3 October 2015 at the AMS Sectional Meetings, in the Special Session on Hopf Algebraic Combinatorics)

slides:
http://mit.edu/~darij/www/algebra/chicago2015.pdf

paper: http://mit.edu/~darij/www/algebra/groth1.pdf or arXiv:1509.03803

We need rather little about symmetric functions for this talk here.

- Let $\mathbf{k}$ be a commutative ring. (Often, this is $\mathbb{Z}$ or $\mathbb{Q}$.)
- We consider power series in countably many commutative variables: $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.
- Let Λ be the ring of all symmetric bounded-degree power series in these variables. It is a $\mathbf{k}$-algebra.

Skew diagrams

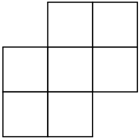
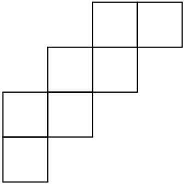
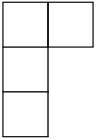
- A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of nonnegative integers ending with zeroes. We often leave out the zeroes when writing down a partition.

Skew diagrams

- A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of nonnegative integers ending with zeroes. We often leave out the zeroes when writing down a partition.
- A *skew partition* is a pair (λ, μ) of partitions such that $\mu_i \leq \lambda_i$ for each i. It is commonly written λ/μ.

- A *partition* is a weakly decreasing sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ of nonnegative integers ending with zeroes. We often leave out the zeroes when writing down a partition.
- A *skew partition* is a pair (λ, μ) of partitions such that $\mu_i \leq \lambda_i$ for each i. It is commonly written λ / μ.
- The *(skew) diagram* of a skew partition λ / μ is the subset $\left\{(i,j) \in \{1,2,3,\ldots\}^2 \mid \mu_i < j \leq \lambda_i\right\}$ of $\{1,2,3,\ldots\}^2$.

Examples of skew diagrams:

$\lambda =$	$(3,3,2)$	$(4,3,2,1)$	$(2,1,1)$
$\mu =$	(1)	$(2,1)$	$()$
diagram:			

- As usual in the theory of Young diagrams, we draw the elements of a diagram as cells, similar to the cells of a matrix (but here the matrix is no longer rectangular). More precisely, the element (i, j) is the j-th cell of the i-th row. This is the "English notation".

- As usual in the theory of Young diagrams, we draw the elements of a diagram as cells, similar to the cells of a matrix (but here the matrix is no longer rectangular). More precisely, the element (i, j) is the j-th cell of the i-th row. This is the "English notation".
- Given a skew partition λ / μ, a *semistandard tableau (SST) of shape* λ / μ is a map T from the diagram of λ / μ to $\{1, 2, 3, \ldots\}$ such that the entries of T increase weakly along rows and increase strictly down columns.
  Here, the entry of T in cell (i, j) means the image $T(i, j)$.

Examples of SSTs:

$$\begin{array}{cccc} & & 1 & 1 \\ & 1 & 3 & \\ 1 & 4 & & \\ 2 & & & \end{array} \qquad \Bigg| \qquad \begin{array}{cccc} & 1 & 2 & 2 \\ & 3 & 3 & \\ 2 & 4 & & \\ 5 & 5 & & \end{array} \qquad \Bigg| \qquad \textbf{not} \qquad \begin{array}{cccc} & 1 & {\color{red}2} & 2 \\ & 2 & {\color{red}2} & \\ 2 & 3 & & \end{array}$$

- Given a skew partition λ/μ, we define the *skew Schur function* $s_{\lambda/\mu}$ to be the formal power series

$$\sum_{T \text{ is an SST of shape } \lambda/\mu} \mathbf{x}^{\operatorname{cont} T} \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right],$$

  where

$$\mathbf{x}^{\operatorname{cont} T} = \prod_{i \geq 1} x_i^{\text{number of cells of } T \text{ having entry } i} = \prod_{c} x_{T(c)},$$

  where the second product runs over all cells c of T.
  This is a formal power series in $x_1, x_2, x_3, \ldots$ (despite the name "function").
- The usual Schur functions (non-skew) are the particular case when $\mu = ()$.
- **Theorem (classical):** $s_{\lambda/\mu} \in \Lambda$, that is, $s_{\lambda/\mu}$ is symmetric.

- **Theorem (classical):** We have $s_{\lambda/\mu} \in \Lambda$, that is, $s_{\lambda/\mu}$ is symmetric.

- **Theorem (classical):** We have $s_{\lambda/\mu} \in \Lambda$, that is, $s_{\lambda/\mu}$ is symmetric.

Idea of proof (Bender and Knuth):

- Enough to show that, for each $i \geq 1$, the power series $s_{\lambda/\mu}$ is preserved when x_i is switched with x_{i+1}.

Skew Schurs are symmetric: proof, part 1

- **Theorem (classical):** We have $s_{\lambda/\mu} \in \Lambda$, that is, $s_{\lambda/\mu}$ is symmetric.

Idea of proof (Bender and Knuth):

- Enough to show that, for each $i \geq 1$, the power series $s_{\lambda/\mu}$ is preserved when x_i is switched with x_{i+1}.
- This will be done if we can find an involution on the set of all SSTs of shape λ/μ, which involution switches the number of entries $= i$ with the number of entries $= i + 1$ (but leaves all other entries unchanged).

- **Theorem (classical):** We have $s_{\lambda/\mu} \in \Lambda$, that is, $s_{\lambda/\mu}$ is symmetric.

Idea of proof (Bender and Knuth):

- Enough to show that, for each $i \geq 1$, the power series $s_{\lambda/\mu}$ is preserved when x_i is switched with x_{i+1}.
- This will be done if we can find an involution on the set of all SSTs of shape λ/μ, which involution switches the number of entries $= i$ with the number of entries $= i+1$ (but leaves all other entries unchanged).
- We can pretend that $i = 1$, and that all entries of the SST are 1's and 2's. Indeed, we can achieve this by forgetting about all entries other than i and $i+1$ (we don't want to change them anyway), and renaming the i's and $i+1$'s as 1's and 2's.

Idea of proof (Bender and Knuth), continued:

- Fix a skew partition λ/μ (not necessarily the one we started with). Let $\mathbf{S}$ be the set of all SSTs of shape λ/μ with entries in $\{1, 2\}$. We need to find an involution of $\mathbf{S}$ which switches the number of entries $= 1$ with the number of entries $= 2$.

Idea of proof (Bender and Knuth), continued:

- Fix a skew partition λ/μ (not necessarily the one we started with). Let **S** be the set of all SSTs of shape λ/μ with entries in $\{1, 2\}$. We need to find an involution of **S** which switches the number of entries $= 1$ with the number of entries $= 2$.
- Just turning 1's into 2's and backwards doesn't work – we break increasingness:

$$\begin{ytableau} \none & \none & \none & \none & \none & 1 & 1 & 1 \\ \none & 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & 2 \end{ytableau} \mapsto \begin{ytableau} \none & \none & \none & \none & \none & 2 & 2 & 2 \\ \none & 2 & 2 & 2 & 1 & 1 & 1 \\ 2 & 1 \end{ytableau}$$

which is not an SST.

Skew Schurs are symmetric: proof, part 3

Idea of proof (Bender and Knuth), continued:

- Instead, we color all 1's that have a 2 beneath them blue, and we color all 2's that have a 1 above them blue:

					1	1	1
	1	1	1	2	2	2	
1	2						

The other 1's and 2's stay black.

Idea of proof (Bender and Knuth), continued:

- Instead, we color all 1's that have a 2 beneath them blue, and we color all 2's that have a 1 above them blue:

$$\begin{array}{cccccccc} & & & & & {\color{blue}1} & {\color{blue}1} & 1 \\ & {\color{blue}1} & 1 & 1 & 2 & {\color{blue}2} & {\color{blue}2} & \\ 1 & {\color{blue}2} & & & & & & \end{array}$$

  The other 1's and 2's stay black.
- Then, for every row, if the row has a black 1's and b black 2's, we replace them by b black 1's and a black 2's. So the above SST becomes

$$\begin{array}{cccccccc} & & & & & {\color{blue}1} & {\color{blue}1} & 2 \\ & {\color{blue}1} & 1 & 2 & 2 & {\color{blue}2} & {\color{blue}2} & \\ 2 & {\color{blue}2} & & & & & & \end{array}$$

  and that's it!

Skew Schurs are symmetric: proof, part 3

Idea of proof (Bender and Knuth), continued:

- Instead, we color all 1's that have a 2 beneath them blue, and we color all 2's that have a 1 above them blue:

					1	1	1
	1	1	1	2	2	2	
1	2						

  The other 1's and 2's stay black.

- Alternative way to describe this step: We replace all black 1's by 2's and vice versa, and then restore the "weakly increasing" condition on the rows by sorting them. So the above SST becomes

					1	1	2
	1	1	2	2	2	2	
2	2						

RPPs

- A *reverse plane partition (RPP)* is defined like an SST, but now entries increase weakly both along rows and down columns. For example,
$$\begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 1 & 2 & 2 \\ \cline{2-4} \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{1}{c}{} \\ \cline{1-3} 2 & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$$
is an RPP.
- This definition is thus more symmetric than that of an SST. Ironically, however, the obvious analogue of $s_{\lambda/\mu}$ is no longer symmetric! (Unless λ/μ particularly simple.)

- However, we **can** squeeze a symmetric function out of RPPs.
- Let $t_1, t_2, t_3, \ldots \in \mathbf{k}$. (Most commonly, $\mathbf{k}$ is taken to be $\mathbb{Z}[t_1, t_2, t_3, \ldots]$.)

- Given a skew partition λ/μ, we define the *refined stable dual Grothendieck polynomial* $\widetilde{g}_{\lambda/\mu}$ to be the formal power series

$$\sum_{T \text{ is an RPP of shape } \lambda/\mu} \mathbf{x}^{\operatorname{ircont} T} \mathbf{t}^{\operatorname{ceq} T} \in \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right],$$

where

$$\mathbf{x}^{\operatorname{ircont} T} = \prod_{i \geq 1} x_i^{\text{number of columns of } T \text{ containing entry } i}$$

and

$$\mathbf{t}^{\operatorname{ceq} T} = \prod_{i \geq 1} t_i^{\text{number of } j \text{ such that } T(i,j) = T(i+1,j)}$$

(where $T(i,j) = T(i+1,j)$ implies, in particular, that both (i,j) and $(i+1,j)$ are cells of T).
This is a formal power series in $x_1, x_2, x_3, \ldots$ (despite the name "polynomial").

Examples on $\mathbf{x}^{\operatorname{ircont} T}$:

- We have

$$\mathbf{x}^{\operatorname{ircont} T} = \prod_{i \geq 1} x_i^{\text{number of columns of } T \text{ containing entry } i}.$$

- If $T = \begin{array}{|c|c|c|c|} \cline{2-4} \multicolumn{1}{c|}{} & 1 & 2 & 2 \\ \cline{2-4} \multicolumn{1}{c|}{} & 2 & 2 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\ \cline{1-2} \end{array}$, then $\mathbf{x}^{\operatorname{ircont} T} = x_1 x_2^4 x_3$. The x_2 has exponent 4, not 5, because the two 2's in column 3 count only once.

Examples on $\mathbf{x}^{\operatorname{ircont} T}$:

- We have

$$\mathbf{x}^{\operatorname{ircont} T} = \prod_{i \geq 1} x_i^{\text{number of columns of } T \text{ containing entry } i}.$$

- If $T = \begin{array}{ccc} & \boxed{1} & \boxed{2} & \boxed{2} \\ & \boxed{2} & \boxed{2} & \\ \boxed{2} & \boxed{3} & & \end{array}$, then $\mathbf{x}^{\operatorname{ircont} T} = x_1 x_2^4 x_3$. The x_2 has exponent 4, not 5, because the two 2's in column 3 count only once.

- If T is an SST, then $\mathbf{x}^{\operatorname{ircont} T} = \mathbf{x}^{\operatorname{cont} T}$.

Examples on $\mathbf{t}^{\operatorname{ceq} T}$:

- We have

$$\mathbf{t}^{\operatorname{ceq} T} = \prod_{i \geq 1} t_i^{\text{number of } j \text{ such that } T(i,j)=T(i+1,j)}$$

- If $T = \begin{array}{|c|c|c|c|} \hline & 1 & 2 & 2 \\ \hline & 2 & 2 & \\ \hline 2 & 3 & & \\ \hline \end{array}$, then $\mathbf{t}^{\operatorname{ceq} T} = t_1$, due to $T(1,3) = T(2,3)$.

Examples on $\mathbf{t}^{\operatorname{ceq} T}$:

- We have

$$\mathbf{t}^{\operatorname{ceq} T} = \prod_{i \geq 1} t_i^{\text{number of } j \text{ such that } T(i,j) = T(i+1,j)}$$

- If $T = \begin{array}{ccc} & \boxed{1} & \boxed{2} & \boxed{2} \\ & \boxed{2} & \boxed{2} & \\ \boxed{2} & \boxed{3} & & \end{array}$, then $\mathbf{t}^{\operatorname{ceq} T} = t_1$, due to $T(1,3) = T(2,3)$.
- If T is an SST, then $\mathbf{t}^{\operatorname{ceq} T} = 1$.
- In general, $\mathbf{t}^{\operatorname{ceq} T}$ measures "how often" T breaks the SST condition.

Refined stable dual Grothendieck polynomials, part 5

- If we set $t_1 = t_2 = t_3 = \cdots = 0$, then $\widetilde{g}_{\lambda/\mu} = s_{\lambda/\mu}$.

- If we set $t_1 = t_2 = t_3 = \cdots = 0$, then $\widetilde{g}_{\lambda/\mu} = s_{\lambda/\mu}$.
- If we set $t_1 = t_2 = t_3 = \cdots = 1$, then $\widetilde{g}_{\lambda/\mu} = g_{\lambda/\mu}$, the *stable dual Grothendieck polynomial* of Lam and Pylyavskyy (arXiv:0705.2189).
- The general case, to our knowledge, is new.

- If we set $t_1 = t_2 = t_3 = \cdots = 0$, then $\widetilde{g}_{\lambda/\mu} = s_{\lambda/\mu}$.
- If we set $t_1 = t_2 = t_3 = \cdots = 1$, then $\widetilde{g}_{\lambda/\mu} = g_{\lambda/\mu}$, the *stable dual Grothendieck polynomial* of Lam and Pylyavskyy (`arXiv:0705.2189`).
- The general case, to our knowledge, is new.
- **Theorem (Galashin, G., Liu):** We have $\widetilde{g}_{\lambda/\mu} \in \Lambda$, that is, $\widetilde{g}_{\lambda/\mu}$ is symmetric in the x_i (not in the t_i).

- If we set $t_1 = t_2 = t_3 = \cdots = 0$, then $\widetilde{g}_{\lambda/\mu} = s_{\lambda/\mu}$.
- If we set $t_1 = t_2 = t_3 = \cdots = 1$, then $\widetilde{g}_{\lambda/\mu} = g_{\lambda/\mu}$, the *stable dual Grothendieck polynomial* of Lam and Pylyavskyy (`arXiv:0705.2189`).
- The general case, to our knowledge, is new.
- **Theorem (Galashin, G., Liu):** We have $\widetilde{g}_{\lambda/\mu} \in \Lambda$, that is, $\widetilde{g}_{\lambda/\mu}$ is symmetric in the x_i (not in the t_i).
- **Example 1:** If $\lambda = (n)$ and $\mu = ()$, then $\widetilde{g}_{\lambda/\mu} = h_n$, the n-th complete homogeneous symmetric function.

- If we set $t_1 = t_2 = t_3 = \cdots = 0$, then $\widetilde{g}_{\lambda/\mu} = s_{\lambda/\mu}$.
- If we set $t_1 = t_2 = t_3 = \cdots = 1$, then $\widetilde{g}_{\lambda/\mu} = g_{\lambda/\mu}$, the *stable dual Grothendieck polynomial* of Lam and Pylyavskyy (arXiv:0705.2189).
- The general case, to our knowledge, is new.
- **Theorem (Galashin, G., Liu):** We have $\widetilde{g}_{\lambda/\mu} \in \Lambda$, that is, $\widetilde{g}_{\lambda/\mu}$ is symmetric in the x_i (not in the t_i).
- **Example 1:** If $\lambda = (n)$ and $\mu = ()$, then $\widetilde{g}_{\lambda/\mu} = h_n$, the n-th complete homogeneous symmetric function.
- **Example 2:** If $\lambda = \left(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}\right)$ and $\mu = ()$, then $\widetilde{g}_{\lambda/\mu} = e_n(t_1, t_2, \ldots, t_{n-1}, x_1, x_2, x_3, \ldots)$, where e_n is the n-th elementary symmetric function.

- If we set $t_1 = t_2 = t_3 = \cdots = 0$, then $\widetilde{g}_{\lambda/\mu} = s_{\lambda/\mu}$.
- If we set $t_1 = t_2 = t_3 = \cdots = 1$, then $\widetilde{g}_{\lambda/\mu} = g_{\lambda/\mu}$, the *stable dual Grothendieck polynomial* of Lam and Pylyavskyy (arXiv:0705.2189).
- The general case, to our knowledge, is new.
- **Theorem (Galashin, G., Liu):** We have $\widetilde{g}_{\lambda/\mu} \in \Lambda$, that is, $\widetilde{g}_{\lambda/\mu}$ is symmetric in the x_i (not in the t_i).
- **Example 1:** If $\lambda = (n)$ and $\mu = ()$, then $\widetilde{g}_{\lambda/\mu} = h_n$, the n-th complete homogeneous symmetric function.
- **Example 2:** If $\lambda = \left(\underbrace{1, 1, \ldots, 1}_{n \text{ ones}}\right)$ and $\mu = ()$, then $\widetilde{g}_{\lambda/\mu} = e_n\left(t_1, t_2, \ldots, t_{n-1}, x_1, x_2, x_3, \ldots\right)$, where e_n is the n-th elementary symmetric function.
- **Example 3:** If $\lambda = (2, 1)$ and $\mu = ()$, then $\widetilde{g}_{\lambda/\mu} = \sum_{a \leq b;\ a < c} x_a x_b x_c + t_1 \sum_{a \leq b} x_a x_b$.

- **Theorem (Galashin, G., Liu):** We have $\widetilde{g}_{\lambda/\mu} \in \Lambda$, that is, $\widetilde{g}_{\lambda/\mu}$ is symmetric in the x_i (not in the t_i).

Idea of first proof:

- Again, as for the skew Schur functions, it is enough to only consider entries in $\{1, 2\}$.
- Fix a skew partition λ/μ. Let **RPP** be the set of all RPPs of shape λ/μ with entries in $\{1, 2\}$. We need to find an involution of **RPP** which switches the number of columns containing 1 with the number of columns containing 2, but at the same time preserves $\mathbf{t}^{\mathrm{ceq}\, T}$.

Idea of first proof:

- SSTs with entries in $\{1, 2\}$ are simple. RPPs with entries in $\{1, 2\}$ can be messy:

			1	1	2	2	2
	1	1	1	2	2		
1	1	1	2	2	2		
1	2						

Idea of first proof:

- We can still try to play the old game: Color the 1's blue that have a 2 somewhere below them, and the 2's that have a 1 somewhere above them. Thus obtain

			1	1	2	2	2
	1	1	1	2	2		
1	1	1	2	2	2		
1	2						

  And we can try to "flip" all black 1's and 2's (i.e., $1 \to 2$ and $2 \to 1$). But now, how do we turn the rest back into an RPP?
- We need something more systematic.

Idea of first proof:

- We fix λ/μ. Let us introduce some notations:
- A *12-table* is a map T from the diagram of λ/μ to $\{1, 2\}$ such that the entries of T weakly increase down columns.
- A column of a 12-table is:
  - *1-pure* if it contains 1's but no 2's;
  - *2-pure* if it contains 2's but no 1's;
  - *mixed* if it both 1's and 2's;
  - *empty* otherwise.

  (The mixed columns are the ones we colored blue.)

Idea of first proof:

- If T is a 12-table, and if $j \geq 1$ is such that the j-th column of T is mixed, then set

$$\operatorname{sep}_j T = \min \{ i \mid T(i,j) = 2 \} .$$

(That is, $\operatorname{sep}_j T$ marks the row where the 2's in the j-th column of T begin.)

- **Example:** If

$$T = \begin{array}{|c|c|c|c|} \hline \multicolumn{3}{c|}{} & 1 \\ \cline{3-4} \multicolumn{2}{c|}{} & 1 & 1 \\ \hline 1 & 1 & 2 & 1 \\ \hline 2 & 1 & 2 & 2 \\ \hline 2 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array}$$

then $\operatorname{sep}_1 T = 4$, $\operatorname{sep}_3 T = 3$ and $\operatorname{sep}_4 T = 4$.

Ref. st. dual Grothendiecks are symmetric: proof, part 6

Idea of first proof:

- If T is a 12-table, then seplist T denotes the list of all $\operatorname{sep}_j T$, where j ranges (in increasing order) over all positive integers such that the j-th column of T is mixed.
- A 12-table T is *benign* if seplist T is weakly decreasing.
- **Example:**

$$\operatorname{seplist} \begin{array}{|c|c|c|c|} \cline{4-4} \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 \\ \cline{3-4} \multicolumn{1}{c}{} & \multicolumn{1}{c|}{} & 1 & 1 \\ \hline 1 & 1 & 2 & 1 \\ \hline 2 & 1 & 2 & 2 \\ \hline 2 & 1 & \multicolumn{2}{c}{} \\ \cline{1-2} \end{array} = (4, 3, 4)$$

(the second column does not contribute here, since it is not mixed!). In particular, this 12-table is not benign.

Idea of first proof:

- **Tab** is the set of all 12-tables.
- **Ben** is the set of all benign 12-tables.
- **RPP** is the set of all RPPs of shape λ/μ with entries in $\{1, 2\}$.

Idea of first proof:

- **Tab** is the set of all 12-tables.
- **Ben** is the set of all benign 12-tables.
- **RPP** is the set of all RPPs of shape λ/μ with entries in $\{1, 2\}$.
- We have **RPP** $\subseteq$ **Ben** $\subseteq$ **Tab**.

Idea of first proof:

- **Tab** is the set of all 12-tables.
- **Ben** is the set of all benign 12-tables.
- **RPP** is the set of all RPPs of shape λ/μ with entries in $\{1, 2\}$.
- We have **RPP** $\subseteq$ **Ben** $\subseteq$ **Tab**.
- **Battle plan:**
  - We define a map flip : **RPP** $\to$ **Ben** as follows: Replace all 1's in 1-pure columns by 2's, and simultaneously replace all 2's in 2-pure columns by 1's.
  - We define a map norm : **Ben** $\to$ **RPP** which turns a benign 12-table into an RPP by iteratively "resolving" its "descents" (places where its rows decrease). But it is not as simple as sorting each row!
  - The map norm $\circ$ flip will be our involution on **RPP**.

Idea of first proof:

- Why flip : $\mathbf{RPP} \to \mathbf{Ben}$ is well-defined is clear: If $T \in \mathbf{RPP}$, then flip T is benign.

Idea of first proof:

- To define norm : **Ben** $\to$ **RPP**, we define an operation on benign 12-table which we call "resolving descents".
- A *descent* of a 12-table T is an $j \geq 1$ such that, for some $i \geq 1$, we have $T(i,j) = 2$ and $T(i,j+1) = 1$.
  Thus, a descent records a place where a row decreases. (More precisely, it records the column in which it happens.)

Idea of first proof:

- To define $\mathrm{norm} : \mathbf{Ben} \to \mathbf{RPP}$, we define an operation on benign 12-table which we call "resolving descents".
- A *descent* of a 12-table T is an $j \geq 1$ such that, for some $i \geq 1$, we have $T(i, j) = 2$ and $T(i, j+1) = 1$.
  Thus, a descent records a place where a row decreases. (More precisely, it records the column in which it happens.)
- If k is a descent of a benign 12-table T, then we say that this descent has
  - *type M1*, if the column k of T is mixed and the column $k+1$ of T is 1-pure;
  - *type 2M*, if the column k of T is 2-pure and the column $k+1$ of T is mixed;
  - *type 21*, if the column k of T is 2-pure and the column $k+1$ of T is 1-pure.

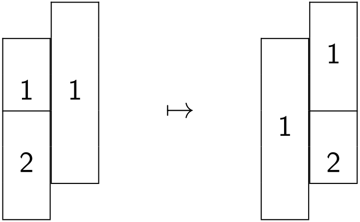
  No other types are possible; in particular, the benignness of T ensures that the columns k and $k+1$ cannot both be mixed!

Ref. st. dual Grothendiecks are symmetric: proof, part 10

Idea of first proof:

- If k is a descent of a benign 12-table T, then we define the result of *resolving the descent k in T* to be the benign 12-table T' obtained from T as follows:
    - If the descent k has type M1, then we replace all entries of column k by 1's and all entries of column $k+1$ by 1's and 2's. To decide where to use 1's and where to use 2's in column $k+1$, we require that $\operatorname{sep}_{k+1} T' = \operatorname{sep}_k T$ (that is, the 2's in column $k+1$ of T' begin on the same level as the 2's in column k of T).

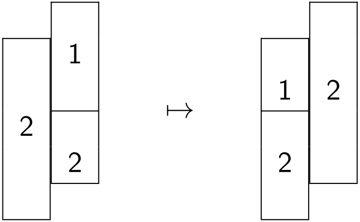
      In pictures:



Idea of first proof:

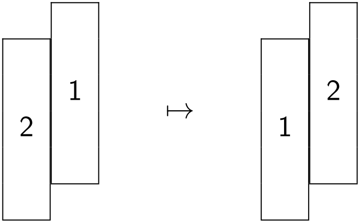
- If k is a descent of a benign 12-table T, then we define the result of *resolving the descent k in T* to be the benign 12-table T' obtained from T as follows:
  - If the descent k has type 2M, then we replace all entries of column k by 1's and 2's and all entries of column $k+1$ by 2's. To decide where to use 1's and where to use 2's in column k, we require that $\operatorname{sep}_k T' = \operatorname{sep}_{k+1} T$ (that is, the 2's in column k of T' begin on the same level as the 2's in column $k+1$ of T).

    In pictures:



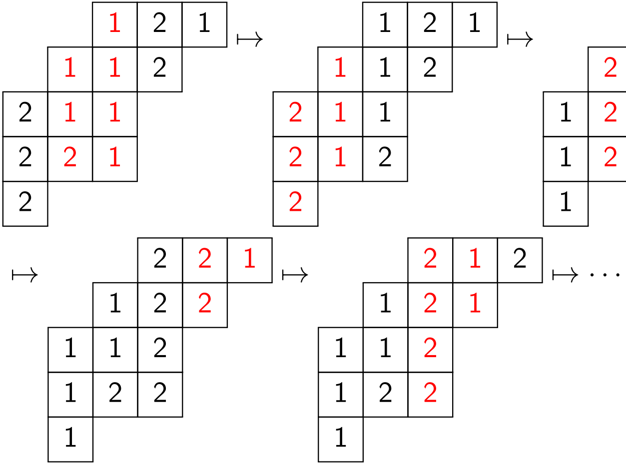
Idea of first proof:

- If k is a descent of a benign 12-table T, then we define the result of *resolving the descent k in T* to be the benign 12-table T' obtained from T as follows:
  - If the descent k has type 21, then we replace all entries of column k by 1's and all entries of column $k+1$ by 2's. In pictures:



Idea of first proof:

- Given $T \in \mathbf{Ben}$, we want to define norm T by iteratively resolving descents in T until none remain:



Idea of first proof:

- $$\cdots \mapsto \begin{array}{|c|c|c|c|c|} \hline & & \color{red}{2} & \color{red}{1} & 2 \\ \hline & 1 & \color{red}{2} & \color{red}{1} & \\ \hline 1 & 1 & \color{red}{2} & & \\ \hline 1 & 2 & \color{red}{2} & & \\ \hline 1 & & & & \\ \hline \end{array} \mapsto \begin{array}{|c|c|c|c|c|} \hline & & \color{red}{1} & 2 & 2 \\ \hline & \color{red}{1} & \color{red}{1} & 2 & \\ \hline 1 & \color{red}{1} & \color{red}{1} & & \\ \hline 1 & \color{red}{2} & \color{red}{1} & & \\ \hline 1 & & & & \\ \hline \end{array}$$
  $$\mapsto \begin{array}{|c|c|c|c|c|} \hline & & 1 & 2 & 2 \\ \hline & 1 & 1 & 2 & \\ \hline 1 & 1 & 1 & & \\ \hline 1 & 1 & 2 & & \\ \hline 1 & & & & \\ \hline \end{array}$$
  (where we color those columns in red which are going to take part in the next descent-resolution step).

Ref. st. dual Grothendiecks are symmetric: proof, part 13

Idea of first proof:

- Note that T stays benign during this procedure!
- So we want to define norm T as the result of this recursive procedure. But
  1. we first need to show that this procedure terminates, and
  2. this procedure is non-deterministic: it involves choosing a descent to resolve; we need to prove that these choices do not influence the result.

Idea of first proof:

- Note that T stays benign during this procedure!
- So we want to define norm T as the result of this recursive procedure. But
  1. we first need to show that this procedure terminates, and
  2. this procedure is non-deterministic: it involves choosing a descent to resolve; we need to prove that these choices do not influence the result.
- The first problem is easy to solve: there is a monovariant. (Roughly speaking, 1-pure columns "only move left", while 2-pure columns "only move right".)

Idea of first proof:

- Note that T stays benign during this procedure!
- So we want to define norm T as the result of this recursive procedure. But
  1. we first need to show that this procedure terminates, and
  2. this procedure is non-deterministic: it involves choosing a descent to resolve; we need to prove that these choices do not influence the result.
- The first problem is easy to solve: there is a monovariant. (Roughly speaking, 1-pure columns "only move left", while 2-pure columns "only move right".)
- The second problem is trickier. Fortunately, there is a standard tactic for such problems: the "diamond lemma" (or "Newman lemma"). (But once the dust has settled, it is easier to reword the argument as an induction on the monovariant than to state the lemma.)

Idea of first proof:

- We are defining norm T as the result of iteratively resolving descents in T until none remain.
- We need to show that this is well-defined. We shall show this by strong induction over $\ell(T)$, where

$$\ell(T) = \sum_{\substack{h \geq 1; \\ \text{the } h\text{-th column} \\ \text{of } T \text{ is mixed}}} h + \sum_{\substack{h \geq 1; \\ \text{the } h\text{-th column} \\ \text{of } T \text{ is 1-pure}}} 2h \in \mathbb{N}.$$

Idea of first proof:

- We are defining norm T as the result of iteratively resolving descents in T until none remain.
- We need to show that this is well-defined. We shall show this by strong induction over $\ell(T)$, where

$$\ell(T) = \sum_{\substack{h \geq 1; \\ \text{the } h\text{-th column} \\ \text{of } T \text{ is mixed}}} h + \sum_{\substack{h \geq 1; \\ \text{the } h\text{-th column} \\ \text{of } T \text{ is 1-pure}}} 2h \in \mathbb{N}.$$

- The nice thing about ℓ: If we resolve a descent in T, then $\ell(T)$ decreases. (There are, of course, many other functions with this property.)

Idea of first proof:

- We are defining norm T as the result of iteratively resolving descents in T until none remain.
- We need to show that this is well-defined. We shall show this by strong induction over $\ell(T)$, where

$$\ell(T) = \sum_{\substack{h \geq 1; \\ \text{the } h\text{-th column} \\ \text{of } T \text{ is mixed}}} h + \sum_{\substack{h \geq 1; \\ \text{the } h\text{-th column} \\ \text{of } T \text{ is 1-pure}}} 2h \in \mathbb{N}.$$

- The nice thing about ℓ: If we resolve a descent in T, then $\ell(T)$ decreases. (There are, of course, many other functions with this property.)
- This immediately shows that the process of iteratively resolving descents will eventually terminate. We only need to show that its result does not depend on the choices made in the process.

Idea of first proof:

- So fix $T \in \mathbf{Ben}$, and assume (as induction hypothesis) that norm S is well-defined for every $S \in \mathbf{Ben}$ with $\ell(S) < \ell(T)$. We need to prove that norm T is well-defined.

Idea of first proof:

- So fix $T \in \mathbf{Ben}$, and assume (as induction hypothesis) that norm S is well-defined for every $S \in \mathbf{Ben}$ with $\ell(S) < \ell(T)$. We need to prove that norm T is well-defined.
- If T has no descents, then there is nothing to prove. So, WLOG, assume that T has descents. Thus, the construction of norm T must start with resolving a descent of T.

Idea of first proof:

- So fix $T \in \mathbf{Ben}$, and assume (as induction hypothesis) that norm S is well-defined for every $S \in \mathbf{Ben}$ with $\ell(S) < \ell(T)$. We need to prove that norm T is well-defined.
- If T has no descents, then there is nothing to prove. So, WLOG, assume that T has descents. Thus, the construction of norm T must start with resolving a descent of T.
- Let $T_u \in \mathbf{Ben}$ be obtained from T by resolving a descent u. Let $T_v \in \mathbf{Ben}$ be obtained from T by resolving a descent v. Then, $\ell(T_u)$ and $\ell(T_v)$ are both $< \ell(T)$, and thus (by induction hypothesis) $\operatorname{norm}(T_u)$ and $\operatorname{norm}(T_v)$ are well-defined. It is enough to show that $\operatorname{norm}(T_u) = \operatorname{norm}(T_v)$.

Idea of first proof:

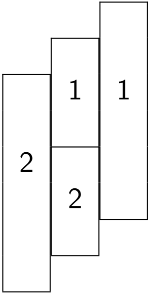
- So fix $T \in \mathbf{Ben}$, and assume (as induction hypothesis) that norm S is well-defined for every $S \in \mathbf{Ben}$ with $\ell(S) < \ell(T)$. We need to prove that norm T is well-defined.
- If T has no descents, then there is nothing to prove. So, WLOG, assume that T has descents. Thus, the construction of norm T must start with resolving a descent of T.
- Let $T_u \in \mathbf{Ben}$ be obtained from T by resolving a descent u. Let $T_v \in \mathbf{Ben}$ be obtained from T by resolving a descent v. Then, $\ell(T_u)$ and $\ell(T_v)$ are both $< \ell(T)$, and thus (by induction hypothesis) $\operatorname{norm}(T_u)$ and $\operatorname{norm}(T_v)$ are well-defined. It is enough to show that $\operatorname{norm}(T_u) = \operatorname{norm}(T_v)$.
- If $u = v$, then $T_u = T_v$ and we are done. Hence, WLOG $u \neq v$. WLOG, $u < v$.

Idea of first proof:

- Two cases: $u = v - 1$ and $u < v - 1$. We shall deal with the former case only (the latter is easier).

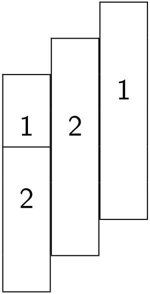
Idea of first proof:

- So $u = v - 1$. Thus, columns $v - 1, v, v + 1$ of T look as follows:



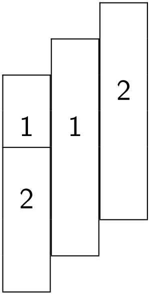
Idea of first proof:

- So $u = v - 1$. After resolving the descent $v - 1$, these columns become



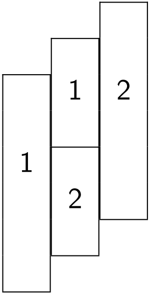
Idea of first proof:

- So $u = v - 1$. After resolving the descent $v - 1$ and then resolving the descent v, these columns become



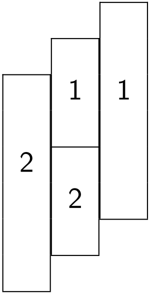
Idea of first proof:

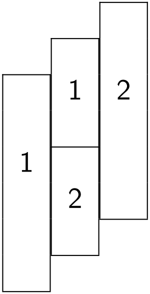
- So $u = v - 1$. After resolving the descent $v - 1$ and then resolving the descent v and then again resolving the descent $v - 1$, these columns become



Idea of first proof:

- Thus we have gone from

$T = \cdots$ 

$\cdots$ to $S := \cdots$ 

$\cdots$

  via three descent-resolution steps, passing through T_u (since we started out by resolving the descent $v - 1 = u$).

- Thus, $\operatorname{norm}(T_u) = \operatorname{norm} S$.

Idea of first proof:

- Thus, $\operatorname{norm}(T_u) = \operatorname{norm} S$.
- Similarly, $\operatorname{norm}(T_v) = \operatorname{norm} S$ (because we can just as well get from T to S by resolving the descents $v, v-1, v$ in this order, and this way we pass through T_v instead of through T_u).
- Comparing, we obtain $\operatorname{norm}(T_u) = \operatorname{norm}(T_v)$, qed. (See arXiv:1509.03803v1, §5.4 for details.)

Idea of first proof:

- Thus, $\operatorname{norm}(T_u) = \operatorname{norm} S$.
- Similarly, $\operatorname{norm}(T_v) = \operatorname{norm} S$ (because we can just as well get from T to S by resolving the descents $v, v-1, v$ in this order, and this way we pass through T_v instead of through T_u).
- Comparing, we obtain $\operatorname{norm}(T_u) = \operatorname{norm}(T_v)$, qed. (See arXiv:1509.03803v1, §5.4 for details.)
- **Note:** If you have heard of the "diamond lemma" (a.k.a. the "Newman lemma", a.k.a. the fact that a noetherian rewriting system is confluent if it is locally confluent), then you have probably realized that its proof was implicit in our argument above.
  Being explicit about it would just have made the proof longer... [See the ancillary file of arXiv:1509.03803 for a writeup of the proof that includes the diamond lemma and invokes it explicitly.]

Idea of first proof:

- So norm is defined. We are almost there.
- We need to show that $\operatorname{norm} \circ \operatorname{flip} : \mathbf{RPP} \to \mathbf{RPP}$ is an involution.

Idea of first proof:

- So norm is defined. We are almost there.
- We need to show that $\operatorname{norm} \circ \operatorname{flip} : \mathbf{RPP} \to \mathbf{RPP}$ is an involution.
- This is mostly straightforward. Main observations:
  - We can define $\operatorname{flip} T$ not only for $T \in \mathbf{RPP}$, but also for all $T \in \mathbf{Ben}$.
  - If k is a descent of a benign 12-table T, and if T' is the 12-table obtained by resolving this descent, then k is a descent of the benign 12-table $\operatorname{flip} T'$, and resolving this descent in $\operatorname{flip} T'$ gives us $\operatorname{flip} T$.

Idea of first proof:

- So norm is defined. We are almost there.
- We need to show that $\operatorname{norm} \circ \operatorname{flip} : \mathbf{RPP} \to \mathbf{RPP}$ is an involution.
- This is mostly straightforward. Main observations:
  - We can define $\operatorname{flip} T$ not only for $T \in \mathbf{RPP}$, but also for all $T \in \mathbf{Ben}$.
  - If k is a descent of a benign 12-table T, and if T' is the 12-table obtained by resolving this descent, then k is a descent of the benign 12-table $\operatorname{flip} T'$, and resolving this descent in $\operatorname{flip} T'$ gives us $\operatorname{flip} T$.
  - Both flip and norm preserve $\operatorname{seplist} T$ and $\mathbf{t}^{\operatorname{ceq} T}$ (actually, $\mathbf{t}^{\operatorname{ceq} T}$ and $\operatorname{seplist} T$ encode the same data about T).

- The involution $\mathrm{norm} \circ \mathrm{flip} : \mathbf{RPP} \to \mathbf{RPP}$ restricts to an involution on **SST** – namely, the classical (Bender-Knuth) involution on SSTs with entries 1 and 2 which we used in proving the symmetry of the $s_{\lambda/\mu}$.
- It is also, in some sense, the only "canonical" extension of this Bender-Knuth involution satisfying a certain "locality condition" (see §6 of our paper for details).
- This also gives yet another construction of our involution, therefore yet another proof of the symmetry of $\widetilde{g}_{\lambda/\mu}$.

A Littlewood-Richardson rule

- **Generalized Littlewood-Richardson rule (not yet written up):** For any partitions λ, μ and ν, we have

$$s_\nu \widetilde{g}_{\lambda/\mu} = \sum_{\substack{T \text{ is an RPP} \\ \text{of shape } \lambda/\mu \text{ such that for every} \\ j \in \{1,2,3,\ldots\}, \text{ the sequence} \\ \nu + \operatorname{ircont}\left(T|_{\operatorname{cols} \geq j}\right) \text{ is a partition}}} s_{\nu + \operatorname{ircont}(T)} \mathbf{t}^{\operatorname{ceq} T}.$$

Here, we define $T \mid_{\operatorname{cols} \geq j}$ as the result of restricting T to columns $j, j+1, j+2, \ldots$.

- **Generalized Littlewood-Richardson rule (not yet written up):** For any partitions λ, μ and ν, we have

$$s_\nu \widetilde{g}_{\lambda/\mu} = \sum_{\substack{T \text{ is an RPP} \\ \text{of shape } \lambda/\mu \text{ such that for every} \\ j \in \{1,2,3,\ldots\}, \text{ the sequence} \\ \nu + \operatorname{ircont}\left(T|_{\text{cols} \geq j}\right) \text{ is a partition}}} s_{\nu + \operatorname{ircont}(T)} \mathbf{t}^{\operatorname{ceq} T}.$$

  Here, we define $T\mid_{\text{cols}\geq j}$ as the result of restricting T to columns $j, j+1, j+2, \ldots$.

- The proof is remarkably easy: Recall Stembridge's proof of Littlewood-Richardson (Electronic Journal of Combinatorics **9** (2002), #N5), and replace the Bender-Knuth involutions by our extended involutions.
- The classical Littlewood-Richardson rule is recovered when $t_1 = t_2 = t_3 = \cdots = 0$.
- Setting $\nu = ()$ gives a Schur expansion for $\widetilde{g}_{\lambda/\mu}$.

Jacobi-Trudi identity?

- **Conjecture:** Let the conjugate partitions of λ and μ be $\lambda^t = ((\lambda^t)_1, (\lambda^t)_2, \ldots, (\lambda^t)_N)$ and $\mu^t = ((\mu^t)_1, (\mu^t)_2, \ldots, (\mu^t)_N)$. Then,

  $$\widetilde{g}_{\lambda/\mu} = \det\left(\left(e_{(\lambda^t)_i - i - (\mu^t)_j + j}\left(\mathbf{x}, \mathbf{t}\left[\left(\mu^t\right)_j + 1 : \left(\lambda^t\right)_i\right]\right)\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right).$$

  Here, $(\mathbf{x}, \mathbf{t}[k : \ell])$ denotes the alphabet $(x_1, x_2, x_3, \ldots, t_k, t_{k+1}, \ldots, t_{\ell-1})$.
  Warning: If $\ell \leq k$, then $t_k, t_{k+1}, \ldots, t_{\ell-1}$ means nothing. No "antimatter" variables!
- I have some even stronger conjectures, with less evidence...

Thank you

- **Richard Stanley** for acquainting us with the dual stable Grothendiecks.
- **Alexander Postnikov, Thomas Lam, Pavlo Pylyavskyy** for interesting discussions.
- **you** for your patience.

- Pavel Galashin, Darij Grinberg, Gaku Liu, *Refined dual stable Grothendieck polynomials and generalized Bender-Knuth involutions*, `arXiv:1509.03803`.
- Thomas Lam, Pavlo Pylyavskyy, *Combinatorial Hopf algebras and K-homology of Grassmanians*, `arXiv:0705.2189`.

Related work:

- Pavel Galashin, *A Littlewood-Richardson Rule for Dual Stable Grothendieck Polynomials*, `arXiv:1501.00051`. (This proves a weaker form of the above LR rule, but in a more interesting way.)
- Damir Yeliussizov, [work in progress]. Various results announced for dual stable Grothendiecks (but not our refined versions).