Generalized Whitney formulas for broken circuits in ambigraphs and matroids*

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Abstract. We explore several generalizations of Whitney's theorem – a classical formula for the chromatic polynomial of a graph. Following Stanley, we replace the chromatic polynomial by the chromatic symmetric function. Following Dohmen and Trinks, we exclude not all but only an (arbitrarily selected) set of broken circuits, or even weigh these broken circuits with weight monomials instead of excluding them. Following Crew and Spirkl, we put weights on the vertices of the graph. Following Gebhard and Sagan, we lift the chromatic symmetric function to noncommuting variables. In addition, we replace the graph by an "ambigraph", an apparently new concept that includes both hypergraphs and multigraphs as particular cases.

We show that Whitney's formula endures all these generalizations, and a fairly simple sign-reversing involution can be used to prove it in each setting. Furthermore, if we restrict ourselves to the chromatic polynomial, then the graph can be replaced by a matroid.

We discuss an application to transitive digraphs (i.e., posets), and reprove an alternating-sum identity by Dahlberg and van Willigenburg.

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The purpose of this paper is to demonstrate several generalizations of Whitney's Broken-Circuit theorem [Whitne32, §7] – a classical formula for the chromatic polynomial of a graph (V, E) as an alternating sum over subsets of E that contain no broken circuits. We shall generalize this formula in the following directions:

- Instead of summing over the sets that contain no broken circuits, we can sum over the sets that are "$\mathfrak{K}$-free" (i.e., contain no element of $\mathfrak{K}$ as a subset), where $\mathfrak{K}$ is some fixed set of broken circuits (in particular, $\mathfrak{K}$ can be $\varnothing$, yielding another well-known formula for the chromatic polynomial). In other words, instead of excluding all broken circuits, we can choose to exclude any given set of broken circuits.

  This generalization has already been proposed by Dohmen and Trinks in [DohTri14, §3.1]; however, we give a new and self-contained proof that does not rely on Whitney's original formula.

- Even more generally, instead of summing over $\mathfrak{K}$-free subsets, we can form a weighted sum over all subsets, where the weight depends on the broken circuits contained in the subset.

- We can replace the graph by an *ambigraph*: a more general notion in which the edges are replaced by packages of edges ("edgeries"), and a proper coloring has to leave at least one edge in each such package dichromatic (i.e., color its two endpoints differently). The concept of ambigraph generalizes both multigraphs and of hypergraphs. We will discuss this concept in Sections 5 and 6.

- Analogous (and more general) results hold for Stanley's chromatic symmetric functions [Stanle95] along with two of their more recent variants: the weighted chromatic symmetric functions of Crew and Spirkl [CreSpi19] and the noncommutative chromatic symmetric functions of Gebhard and Sagan [GebSag01]. The latter variants will be studied (and generalized to ambigraphs) in Section 6.

- Analogous (and more general) results hold for matroids instead of graphs. These will be discussed in Section 8.

Note that, to my knowledge, the last two generalizations cannot be combined: Unlike graphs, matroids do not seem to have a well-defined notion of a chromatic symmetric function.

We will explore these generalizations in the work that follows. We shall also use them to prove an apparently new formula for the chromatic polynomial of a graph obtained from a transitive digraph by forgetting the orientations of the edges (Proposition 4.5). This latter formula was suggested to me as a conjecture by Alexander Postnikov, during a discussion on hyperplane arrangements on a space with a bilinear form; it is this formula which gave rise to this whole paper. The topic of hyperplane arrangements, however, will not be breached here.

As a further application, we will generalize and reprove an alternating-sum identity for chromatic polynomials found by Dahlberg and van Willigenburg (Section 7), as well as an analogous identity for characteristic polynomials of matroids (Subsection 8.4).

Acknowledgments

I thank Alexander Postnikov and Richard P. Stanley for discussions on hyperplane arrangements that led to the results described here.

1. Definitions and a main result

1.1. Graphs and colorings

We begin by recalling some basic features of finite graphs. Let us start with the definition of a graph that we shall be using:

Definition 1.1. **(a)** If V is any set, then $\mathcal{P}(V)$ will denote the powerset of V. This is the set of all subsets of V.

(b) If V is any set, then $\binom{V}{2}$ will denote the set of all 2-element subsets of V. In other words, if V is any set, then we set

$$\binom{V}{2} = \{S \in \mathcal{P}(V) \mid |S| = 2\} \\ = \{\{s,t\} \mid s \in V,\ t \in V,\ s \neq t\}.$$

(c) A *graph* means a pair (V, E), where V is a set, and where E is a subset of $\binom{V}{2}$. A graph (V, E) is said to be *finite* if the set V is finite. If $G = (V, E)$ is a graph, then the elements of V are called the *vertices* of the graph G, while the elements of E are called the *edges* of the graph G. If e is an edge of a graph G, then the two elements of e are called the *endpoints* of the edge e. If $e = \{s,t\}$ is an edge of a graph G, then we say that the edge e *connects the vertices s and t* of G.

Comparing our definition of a graph with some of the other definitions used in the literature, we thus observe that our graphs are undirected (i.e., their edges are sets, not pairs), loopless (i.e., the two endpoints of an edge must always be distinct), edge-unlabelled (i.e., their edges are just 2-element sets of vertices, rather than objects with "their own identity"), and do not have multiple edges (or, more precisely, there is no notion of several edges connecting two vertices, since the edges form a set, nor a multiset, and do not have labels). Such graphs are commonly known as *simple graphs*.

Definition 1.2. Let $G = (V, E)$ be a graph. Let X be a set.

(a) An *X-coloring* of G is defined to mean a map $V \to X$.

(b) An X-coloring f of G is said to be *proper* if every edge $\{s, t\} \in E$ satisfies $f(s) \neq f(t)$.

If f is an X-coloring of a graph $G = (V, E)$, then the value $f(v)$ for a given vertex $v \in V$ is called the *color* of this vertex v under the coloring f. We shall not use this terminology here, but we are mentioning it since it allows for a rather intuitive mental model and explains the word "coloring". An X-coloring of G is then proper if and only if each edge of G has two endpoints of different colors.

1.2. Symmetric functions

We shall now briefly introduce the notion of symmetric functions. We shall not use any nontrivial results about symmetric functions; we will merely need some notations.[1]

In the following, $\mathbb{N}$ means the set $\{0, 1, 2, \ldots\}$. Also, $\mathbb{N}_+$ shall mean the set $\{1, 2, 3, \ldots\}$.

A *partition* will mean a sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots) \in \mathbb{N}^\infty$ of nonnegative integers such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ and such that all sufficiently high integers $i \geq 1$ satisfy $\lambda_i = 0$. If $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is a partition, and if a positive integer n is such that all integers $i \geq n$ satisfy $\lambda_i = 0$, then we shall identify the partition λ with the finite sequence $(\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$. Thus, for example, the sequences $(3, 1)$ and $(3, 1, 0)$ and the partition $(3, 1, 0, 0, 0, \ldots)$ are all identified. Every weakly decreasing finite list of positive integers thus is identified with a unique partition.

Let $\mathbf{k}$ be a commutative ring with unity. We shall keep $\mathbf{k}$ fixed throughout the paper. The reader will not be missing out on anything if she assumes that $\mathbf{k} = \mathbb{Z}$.

We consider the $\mathbf{k}$-algebra $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of (commutative) power series in countably many distinct indeterminates $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. It is a topological $\mathbf{k}$-algebra[2]. A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *bounded-degree* if

[1] For an introduction to symmetric functions, see any of [Stanle99, Chapter 7], [Martin22, Chapter 9] and [GriRei14, Chapter 2] (and a variety of other texts).

[2] See [GriRei14, Section 2.6] or [Grinbe16, §2] for the definition of its topology. This topology makes sure that a sequence $(P_n)_{n \in \mathbb{N}}$ of power series converges to some power series P if and only if, for every monomial $\mathfrak{m}$, all sufficiently high $n \in \mathbb{N}$ satisfy

$$(\text{the } \mathfrak{m}\text{-coefficient of } P_n) = (\text{the } \mathfrak{m}\text{-coefficient of } P)$$

(where the meaning of "sufficiently high" can depend on the $\mathfrak{m}$).

there exists an $N \in \mathbb{N}$ such that every monomial of degree $> N$ appears with coefficient 0 in P. A power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is said to be *symmetric* if and only if P is invariant under any permutation of the indeterminates. We let Λ be the subset of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ consisting of all symmetric bounded-degree power series $P \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$. This subset Λ is a $\mathbf{k}$-subalgebra of $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$, and is called the $\mathbf{k}$-*algebra of symmetric functions* over $\mathbf{k}$.

We shall now define the few families of symmetric functions that we will be concerned with in this work. The first are the *power-sum symmetric functions*:

Definition 1.3. Let n be a positive integer. We define a power series $p_n \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$p_n = x_1^n + x_2^n + x_3^n + \cdots = \sum_{j \geq 1} x_j^n. \tag{1}$$

This power series p_n lies in Λ, and is called the *n-th power-sum symmetric function*.

We also set $p_0 = 1 \in \Lambda$. Thus, p_n is defined not only for all positive integers n, but also for all $n \in \mathbb{N}$.

Definition 1.4. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ be a partition. We define a power series $p_\lambda \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$p_\lambda = \prod_{i \geq 1} p_{\lambda_i}.$$

This is well-defined, because the infinite product $\prod_{i \geq 1} p_{\lambda_i}$ converges (indeed, all but finitely many of its factors are 1 (because every sufficiently high integer i satisfies $\lambda_i = 0$ and thus $p_{\lambda_i} = p_0 = 1$)).

We notice that every partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ (written as a finite list of nonnegative integers) satisfies

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}. \tag{2}$$

1.3. Chromatic symmetric functions

The next symmetric functions we introduce are the actual subject of this work; they are the *chromatic symmetric functions* and have been introduced by Stanley in [Stanle95, Definition 2.1]:

Definition 1.5. Let $G = (V, E)$ be a finite graph.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{x}_f$ denote the monomial $\prod_{v \in V} x_{f(v)}$ in the indeterminates $x_1, x_2, x_3, \ldots$.

(b) We define a power series $X_G \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$X_G = \sum_{\substack{f:V \to \mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f.$$

This power series X_G is called the *chromatic symmetric function* of G.

We have $X_G \in \Lambda$ for every finite graph $G = (V, E)$; this will follow from Theorem 1.8 further below (but is also rather obvious).

We remark that X_G is denoted by $\Psi[G]$ in [GriRei14, §7.3.3].

1.4. Connected components

We shall now briefly recall the notion of connected components of a graph.

Definition 1.6. Let $G = (V, E)$ be a finite graph. Let u and v be two elements of V (that is, two vertices of G). A *walk* from u to v in G will mean a sequence $(w_0, w_1, \ldots, w_k)$ of elements of V such that $w_0 = u$ and $w_k = v$ and

$$(\{w_i, w_{i+1}\} \in E \qquad \text{for every } i \in \{0, 1, \ldots, k-1\}).$$

We say that u and v are *connected (in G)* if there exists a walk from u to v in G.

Definition 1.7. Let $G = (V, E)$ be a graph.

(a) We define a binary relation $\sim_G$ (written infix) on the set V as follows: Given $u \in V$ and $v \in V$, we set $u \sim_G v$ if and only if u and v are connected (in G). It is well-known that this relation $\sim_G$ is an equivalence relation. The $\sim_G$-equivalence classes are called the *connected components* of G.

(b) Assume that the graph G is finite. We let $\lambda(G)$ denote the list of the sizes of all connected components of G, in weakly decreasing order. (Each connected component should contribute only one entry to the list.) We view $\lambda(G)$ as a partition (since $\lambda(G)$ is a weakly decreasing finite list of positive integers).

Now, we can state a formula for chromatic symmetric functions:

Theorem 1.8. Let $G = (V, E)$ be a finite graph. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda(V, F)$ is understood according to Definition 1.7 **(b)**.)

This theorem is not new; it appears, e.g., in [Stanle95, Theorem 2.5]. We shall show a far-reaching generalization of it (Theorem 1.12) soon.

1.5. Circuits and broken circuits

Let us now define the notions of cycles and circuits of a graph:

Definition 1.9. Let $G = (V, E)$ be a graph. A *cycle* of G denotes a list $(v_1, v_2, \ldots, v_{m+1})$ of elements of V with the following properties:

- We have $m > 2$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, m\}$.

If $(v_1, v_2, \ldots, v_{m+1})$ is a cycle of G, then the set $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$ is called a *circuit* of G.

For instance, if $(1, 3, 5, 7, 1)$ is a cycle of a graph G, then the corresponding circuit is $\{\{1,3\}, \{3,5\}, \{5,7\}, \{7,1\}\}$.

Definition 1.10. Let $G = (V, E)$ be a graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. We shall refer to ℓ as the *labeling function*. For every edge e of G, we shall refer to $\ell(e)$ as the *label* of e.

A *broken circuit* of G means a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C). Of course, the notion of a broken circuit of G depends on the function ℓ; however, we suppress the mention of ℓ in our notation, since we will not consider situations where two different ℓ's coexist.

Thus, if G is a graph with a labeling function ℓ, then any circuit C of G gives rise to a broken circuit provided that among the edges in C, only one attains the maximum label. (If more than one of the edges of C attains the maximum label, then C does not give rise to a broken circuit.) Notice that two different circuits may give rise to one and the same broken circuit.

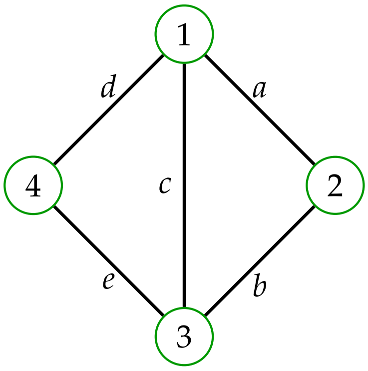
For instance, if $\{a, b, c, d\}$ is a circuit of a graph G such that $\ell(a) \leq \ell(b) \leq \ell(c) < \ell(d)$, then it gives rise to the broken circuit $\{a, b, c\}$, since its unique edge having maximum label is d. On the other hand, a circuit of the form $\{a, b, c, d\}$ with $\ell(a) \leq \ell(b) \leq \ell(c) = \ell(d)$ (and $c \neq d$) does not give rise to any broken circuit, since its edge with maximum label is not unique.

The notion of a broken circuit always depends on a labeling function $\ell : E \to X$. Any time we speak about broken circuits, we shall tacitly understand that the function $\ell : E \to X$ is used as the labeling function.

Example 1.11. Let G be the graph (V, E), where $V = \{1, 2, 3, 4\}$ and $E = \{a, b, c, d, e\}$ with

$$a = \{1, 2\}, \qquad b = \{2, 3\}, \qquad c = \{1, 3\}, \qquad d = \{1, 4\}, \qquad e = \{3, 4\}.$$

According to the standard conventions of graph theory, this graph G can be drawn as follows:



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Let $\ell : E \to X$ be a labeling function satisfying $\ell(a) < \ell(b) < \ell(c) < \ell(d) < \ell(e)$. Then, the circuits of G are

$$\{a, b, c\}, \qquad \{a, b, d, e\}, \qquad \{c, d, e\}.$$

The broken circuits of G are therefore

$$\{a, b\}, \qquad \{a, b, d\}, \qquad \{c, d\}.$$

1.6. The main results

We are now ready to state one of our main results:

Theorem 1.12. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq F}} a_K \right) p_{\lambda(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda(V, F)$ is understood according to Definition 1.7 **(b)**.)

Before we come to the proof of this result, let us explore some of its particular cases. First, a definition is in order:

Definition 1.13. Let E be a set. Let $\mathfrak{K}$ be a subset of the powerset of E (that is, a set of subsets of E). A subset F of E is said to be *$\mathfrak{K}$-free* if F contains no $K \in \mathfrak{K}$ as a subset. (For instance, if $\mathfrak{K} = \varnothing$, then every subset F of E is $\mathfrak{K}$-free.)

Here is a slightly more substantial example: If $E = \{1,2,3,4\}$ and $\mathfrak{K} = \{\{1,2\}, \{2,3\}\}$, then the subset $\{1,3\}$ of E is $\mathfrak{K}$-free whereas the subset $\{2,3,4\}$ is not (since it contains $\{2,3\} \in \mathfrak{K}$ as a subset).

Corollary 1.14. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \, p_{\lambda(V,F)}.$$

Corollary 1.15. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} \, p_{\lambda(V,F)}.$$

Corollary 1.15 appears in [Stanle95, Theorem 2.9], at least in the particular case in which ℓ is supposed to be injective.

Example 1.16. Let $G = (V, E)$ be the graph from Example 1.11, and let $\ell : E \to X$ be a labeling function as in Example 1.11. Then, the subsets of E that contain no broken circuits of G as subsets are the 18 sets

$$\begin{gathered} \varnothing, \quad \{a\}, \quad \{b\}, \quad \{c\}, \quad \{d\}, \quad \{e\}, \quad \{a,c\}, \\ \{a,d\}, \quad \{a,e\}, \quad \{b,c\}, \quad \{b,d\}, \quad \{b,e\}, \quad \{c,e\}, \\ \{d,e\}, \quad \{a,c,e\}, \quad \{a,d,e\}, \quad \{b,c,e\}, \quad \{b,d,e\}. \end{gathered}$$

Thus, the sum on the right hand side of Corollary 1.15 has 18 addends. In contrast, the sums on the right hand sides of Theorem 1.12 and of Theorem 1.8 have 32 addends. The number of addends in the sum on the right hand side of Corollary 1.14 depends on the choice of $\mathfrak{K}$.

Let us now see how Theorem 1.8, Corollary 1.14 and Corollary 1.15 can be derived from Theorem 1.12:

Proof of Corollary 1.14 using Theorem 1.12. For every subset F of E, we have

$$\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} 0 = \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free}; \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \tag{3}$$

(because if F is $\mathfrak{K}$-free, then the product $\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0$ is empty and thus equals 1; otherwise, the product $\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0$ contains at least one factor and thus equals 0). Now, Theorem 1.12 (applied to 0 instead of a_K) yields

$$\begin{aligned} X_G &= \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{\substack{=\begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} \\ \text{(by (3))}}} p_{\lambda(V,F)} \\ &= \sum_{F\subseteq E} (-1)^{|F|} \begin{cases} 1, & \text{if } F \text{ is } \mathfrak{K}\text{-free;} \\ 0, & \text{if } F \text{ is not } \mathfrak{K}\text{-free} \end{cases} p_{\lambda(V,F)} = \sum_{\substack{F\subseteq E;\\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V,F)}. \end{aligned}$$

This proves Corollary 1.14. $\square$

Proof of Corollary 1.15 using Corollary 1.14. Corollary 1.15 follows from Corollary 1.14 when $\mathfrak{K}$ is set to be the set of **all** broken circuits of G. $\square$

Proof of Theorem 1.8 using Theorem 1.12. Let X be the totally ordered set $\{1\}$, and let $\ell : E \to X$ be the only possible map. Let $\mathfrak{K}$ be the empty set. Clearly, $\mathfrak{K}$ is a set of broken circuits of G. For every $F \subseteq E$, the product $\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0$ is empty (since $\mathfrak{K}$ is the empty set), and thus equals 1. Now, Theorem 1.12 (applied to 0 instead of a_K) yields

$$X_G = \sum_{F\subseteq E} (-1)^{|F|} \underbrace{\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} 0\right)}_{=1} p_{\lambda(V,F)} = \sum_{F\subseteq E} (-1)^{|F|} p_{\lambda(V,F)}.$$

This proves Theorem 1.8. $\square$

2. Proof of Theorem 1.12

We shall now prepare for the proof of Theorem 1.12 with some notations and some lemmas. Our proof will imitate [BlaSag86, proof of Whitney's theorem]. We note that Theorem 1.12 can also be easily obtained as a consequence of [DohTri14, §2 and §3.1] using Theorem 1.8, but our proof has the advantage of not relying on Theorem 1.8 (so that it leads to a new proof of Theorem 1.8).

2.1. $\operatorname{Eqs} f$ and basic lemmas

We introduce a simple notion that measures "how non-proper" a given coloring of a graph is:[3]

Definition 2.1. Let V and X be two sets. Let $f : V \to X$ be a map. We let $\operatorname{Eqs} f$ denote the subset

$$\left\{ \{s,t\} \mid (s,t) \in V^2,\ s \neq t \text{ and } f(s) = f(t) \right\}$$

of $\dbinom{V}{2}$. (This is well-defined, because any two elements s and t of V satisfying $s \neq t$ clearly satisfy $\{s,t\} \in \dbinom{V}{2}$.)

Example 2.2. Let $V = \{1,2,3,4,5\}$ and $X = \{1,2,3\}$, and let $f : V \to X$ be the map that sends the three numbers $1,2,3$ to 1 and the remaining two numbers $4,5$ to 2. Then,

$$\operatorname{Eqs} f = \{\{1,2\},\ \{1,3\},\ \{2,3\},\ \{4,5\}\}.$$

We shall now state some first properties of this notion:

Lemma 2.3. Let $G = (V,E)$ be a graph. Let X be a set. Let $f : V \to X$ be a map. Then, the X-coloring f of G is proper if and only if $E \cap \operatorname{Eqs} f = \varnothing$.

Proof of Lemma 2.3. The set $E \cap \operatorname{Eqs} f$ is precisely the set of edges $\{s,t\}$ of G satisfying $f(s) = f(t)$; meanwhile, the X-coloring f is called proper if and only if no such edges exist. Thus, Lemma 2.3 becomes obvious. $\square$

Lemma 2.4. Let $G = (V,E)$ be a graph. Let X be a set. Let $f : V \to X$ be a map. Let C be a circuit of G. Let $e \in C$ be such that $C \setminus \{e\} \subseteq \operatorname{Eqs} f$. Then, $e \in E \cap \operatorname{Eqs} f$.

Proof of Lemma 2.4. The set C is a circuit of G. Hence, we can write C in the form

$$C = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$$

for some cycle $(v_1, v_2, \ldots, v_{m+1})$ of G. Consider this cycle $(v_1, v_2, \ldots, v_{m+1})$. According to the definition of a "cycle", the cycle $(v_1, v_2, \ldots, v_{m+1})$ is a list of elements of V having the following properties:

- We have $m > 2$.

[3] If V is a set, then V^2 denotes the Cartesian product $V \times V$, that is, the set of all ordered pairs of elements of V.

- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in E$ for every $i \in \{1, 2, \ldots, m\}$.

From the first three of these properties, we can easily conclude that the m sets $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}$ are distinct.

Recall that $e \in C$. Therefore, $e = \{v_i, v_{i+1}\}$ for some $i \in \{1, 2, \ldots, m\}$. We can thus WLOG assume that $e = \{v_m, v_{m+1}\}$ (since otherwise, we can simply relabel the vertices along the cycle $(v_1, v_2, \ldots, v_{m+1})$). Assume this. Since $C = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}\}$ and $e = \{v_m, v_{m+1}\}$, we have

$$C \setminus \{e\} = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\}\}$$

(since the m sets $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_{m+1}\}$ are distinct). For every $i \in \{1, 2, \ldots, m-1\}$, we have $f(v_i) = f(v_{i+1})$ (since

$$\{v_i, v_{i+1}\} \subseteq \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{m-1}, v_m\}\} = C \setminus \{e\} \subseteq \operatorname{Eqs} f$$

). Hence, $f(v_1) = f(v_2) = \cdots = f(v_m)$, so that $f(v_m) = f(v_1) = f(v_{m+1})$ (because $v_1 = v_{m+1}$). Thus, $\{v_m, v_{m+1}\} \in \operatorname{Eqs} f$. Thus, $e = \{v_m, v_{m+1}\} \in \operatorname{Eqs} f$. Combined with $e \in E$, this yields $e \in E \cap \operatorname{Eqs} f$. This proves Lemma 2.4. $\square$

Lemma 2.5. Let (V, B) be a finite graph. Then,

$$\sum_{\substack{f: V \to \mathbb{N}_+; \\ B \subseteq \operatorname{Eqs} f}} \mathbf{x}_f = p_{\lambda(V,B)}.$$

(Here, $\mathbf{x}_f$ is defined as in Definition 1.5 **(a)**, and the expression $\lambda(V, B)$ is understood according to Definition 1.7 **(b)**.)

Proof of Lemma 2.5. Let $(C_1, C_2, \ldots, C_k)$ be a list of all connected components of (V, B), ordered such that $|C_1| \geq |C_2| \geq \cdots \geq |C_k|$. [4] Then, $\lambda(V, B) = (|C_1|, |C_2|, \ldots, |C_k|)$ (by the definition of $\lambda(V, B)$). Hence, (2) (applied to $\lambda(V, B)$ and $|C_i|$ instead of λ and λ_i) shows that

$$p_{\lambda(V,B)} = p_{|C_1|} p_{|C_2|} \cdots p_{|C_k|} = \prod_{i=1}^{k} p_{|C_i|}. \tag{4}$$

However, for every $i \in \{1, 2, \ldots, k\}$, we have $p_{|C_i|} = \sum_{s \in \mathbb{N}_+} x_s^{|C_i|}$ (by the definition

[4] Every connected component of (V, B) should appear exactly once in this list.

of $p_{|C_i|}$). Hence, (4) becomes

$$p_{\lambda(V,B)} = \prod_{i=1}^{k} \underbrace{p_{|C_i|}}_{= \sum_{s\in\mathbb{N}_+} x_s^{|C_i|}} = \prod_{i=1}^{k} \sum_{s\in\mathbb{N}_+} x_s^{|C_i|}$$
$$= \sum_{(s_1,s_2,\ldots,s_k)\in(\mathbb{N}_+)^k} \prod_{i=1}^{k} x_{s_i}^{|C_i|} \tag{5}$$

(by the product rule).

The list $(C_1, C_2, \ldots, C_k)$ contains all connected components of (V, B), each exactly once. Thus, $V = \bigsqcup_{i=1}^{k} C_i$.

We now define a map

$$\Phi : (\mathbb{N}_+)^k \to \{f : V \to \mathbb{N}_+ \mid B \subseteq \operatorname{Eqs} f\}$$

as follows: Given any $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$, we let $\Phi(s_1, s_2, \ldots, s_k)$ be the map $V \to \mathbb{N}_+$ which sends every $v \in V$ to s_i, where $i \in \{1, 2, \ldots, k\}$ is such that $v \in C_i$. (This is well-defined, because for every $v \in V$, there exists a unique $i \in \{1, 2, \ldots, k\}$ such that $v \in C_i$; this follows from $V = \bigsqcup_{i=1}^{k} C_i$.) This map Φ is well-defined, because for every $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$, the map $\Phi(s_1, s_2, \ldots, s_k)$ actually belongs to $\{f : V \to \mathbb{N}_+ \mid B \subseteq \operatorname{Eqs} f\}$ [5].

A moment's thought reveals that the map Φ is injective[6]. Let us now show that the map Φ is surjective.

In order to show this, we must prove that every map $f : V \to \mathbb{N}_+$ satisfying $B \subseteq \operatorname{Eqs} f$ has the form $\Phi(s_1, s_2, \ldots, s_k)$ for some $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$. So let us fix a map $f : V \to \mathbb{N}_+$ satisfying $B \subseteq \operatorname{Eqs} f$. We must find some $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$ such that $f = \Phi(s_1, s_2, \ldots, s_k)$.

We have $B \subseteq \operatorname{Eqs} f$. Thus, for every $\{s, t\} \in B$, we have $\{s, t\} \in B \subseteq \operatorname{Eqs} f$ and thus

$$f(s) = f(t). \tag{6}$$

Now, if x and y are two elements of V lying in the same connected component of (V, B), then

$$f(x) = f(y) \tag{7}$$

[5] *Proof.* We just need to check that $B \subseteq \operatorname{Eqs}(\Phi(s_1, s_2, \ldots, s_k))$. But this is easy: For every $\{u, v\} \in B$, the vertices u and v of (V, B) lie in one and the same connected component C_i of the graph (V, B), and thus (by the definition of $\Phi(s_1, s_2, \ldots, s_k)$) the map $\Phi(s_1, s_2, \ldots, s_k)$ sends both of them to s_i; but this shows that $\{u, v\} \in \operatorname{Eqs}(\Phi(s_1, s_2, \ldots, s_k))$.

[6] In fact, we can reconstruct $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$ from its image $\Phi(s_1, s_2, \ldots, s_k)$, because each s_i is the image of any element of C_i under $\Phi(s_1, s_2, \ldots, s_k)$ (and this allows us to compute s_i, since C_i is nonempty).

[7]. In other words, the map f is constant on each connected component of (V, B). Thus, the map f is constant on C_i for each $i \in \{1, 2, \ldots, k\}$ (since C_i is a connected component of (V, B)). Hence, for each $i \in \{1, 2, \ldots, k\}$, we can define a positive integer $s_i \in \mathbb{N}_+$ to be the image of any element of C_i under f (this is well-defined, because f is constant on C_i and thus the choice of the element does not matter). Define $s_i \in \mathbb{N}_+$ for each $i \in \{1, 2, \ldots, k\}$ this way. Thus, we have defined a k-tuple $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$. Now, $f = \Phi(s_1, s_2, \ldots, s_k)$ (this follows immediately by recalling the definitions of Φ and s_i).

Let us now forget that we fixed f. We thus have shown that for every map $f : V \to \mathbb{N}_+$ satisfying $B \subseteq \operatorname{Eqs} f$, there exists some $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$ such that $f = \Phi(s_1, s_2, \ldots, s_k)$. In other words, the map Φ is surjective. Since Φ is both injective and surjective, we conclude that Φ is a bijection.

Moreover, it is straightforward to see that every k-tuple $(s_1, s_2, \ldots, s_k) \in (\mathbb{N}_+)^k$ satisfies

$$\mathbf{x}_{\Phi(s_1, s_2, \ldots, s_k)} = \prod_{i=1}^{k} x_{s_i}^{|C_i|} \tag{8}$$

(by the definitions of $\mathbf{x}_{\Phi(s_1, s_2, \ldots, s_k)}$ and of Φ). Now,

$$\begin{aligned}
&\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f \\
&= \sum_{(s_1,s_2,\ldots,s_k)\in(\mathbb{N}_+)^k} \underbrace{\mathbf{x}_{\Phi(s_1,s_2,\ldots,s_k)}}_{\substack{=\prod_{i=1}^{k} x_{s_i}^{|C_i|}\\ \text{(by (8))}}} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \Phi(s_1, s_2, \ldots, s_k) \text{ for } f \text{ in the sum,}\\ \text{since the map } \Phi : (\mathbb{N}_+)^k \to \{f : V \to \mathbb{N}_+ \mid B \subseteq \operatorname{Eqs} f\}\\ \text{is a bijection}\end{array}\right) \\
&= \sum_{(s_1,s_2,\ldots,s_k)\in(\mathbb{N}_+)^k} \prod_{i=1}^{k} x_{s_i}^{|C_i|} = p_{\lambda(V,B)} \qquad \text{(by (5))}.
\end{aligned}$$

This proves Lemma 2.5. □

[7]*Proof of (7):* Let x and y be two elements of V lying in the same connected component of (V, B). Then, the vertices x and y are connected by a walk in the graph (V, B) (by the definition of a "connected component"). Let $(v_0, v_1, \ldots, v_j)$ be this walk (regarded as a sequence of vertices); thus, $v_0 = x$ and $v_j = y$. For every $i \in \{0, 1, \ldots, j-1\}$, we have $\{v_i, v_{i+1}\} \in B$ (since $(v_0, v_1, \ldots, v_j)$ is a walk in the graph (V, B)) and thus $f(v_i) = f(v_{i+1})$ (by (6), applied to $(s, t) = (v_i, v_{i+1})$). In other words, $f(v_0) = f(v_1) = \cdots = f(v_j)$. Hence, $f(v_0) = f(v_j)$, so that $f\left(\underbrace{x}_{=v_0}\right) = f(v_0) = f\left(\underbrace{v_j}_{=y}\right) = f(y)$, qed.

Lemma 2.6. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let K be a broken circuit of G. Then, $K \neq \varnothing$.

Proof of Lemma 2.6. The set K is a broken circuit of G, and thus is a circuit of G with an edge removed (by the definition of a broken circuit). Thus, the set K contains at least 1 edge (since every circuit of G contains at least 2 edges). This proves Lemma 2.6. □

2.2. Alternating sums

We shall now come to less simple lemmas.

Definition 2.7. We shall use the so-called *Iverson bracket notation*: If $\mathcal{S}$ is any logical statement, then $[\mathcal{S}]$ shall mean the integer $\begin{cases} 1, & \text{if } \mathcal{S} \text{ is true;} \\ 0, & \text{if } \mathcal{S} \text{ is false} \end{cases}$.

The following lemma is probably the most crucial one in this paper:

Lemma 2.8. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$.

Let Y be any set. Let $f : V \to Y$ be any map. Then,

$$\sum_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K = [E \cap \operatorname{Eqs} f = \varnothing].$$

Proof of Lemma 2.8. We WLOG assume that $E \cap \operatorname{Eqs} f \neq \varnothing$ (since otherwise, the claim is obvious[8]). Thus, $[E \cap \operatorname{Eqs} f = \varnothing] = 0$.

[8]In (slightly) more detail: If $E \cap \operatorname{Eqs} f = \varnothing$, then the sum $\sum_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K$ has only one addend (namely, the addend for $B = \varnothing$), and thus simplifies to

$$\underbrace{(-1)^{|\varnothing|}}_{=(-1)^0=1} \underbrace{\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \varnothing}}}_{=\prod_{\substack{K \in \mathfrak{K};\\ K = \varnothing}}} a_K = \prod_{\substack{K \in \mathfrak{K};\\ K = \varnothing}} a_K = (\text{empty product}) \qquad \left(\begin{array}{c}\text{since no } K \in \mathfrak{K} \text{ satisfies } K = \varnothing \\ \text{(by Lemma 2.6)}\end{array}\right)$$

$$= 1 = [E \cap \operatorname{Eqs} f = \varnothing].$$

Pick any $d \in E \cap \operatorname{Eqs} f$ with maximum $\ell(d)$ (among all $d \in E \cap \operatorname{Eqs} f$). (This is clearly possible, since $E \cap \operatorname{Eqs} f \neq \varnothing$.) Define two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{P}(E \cap \operatorname{Eqs} f)$ as follows:

$$\mathcal{U} = \{F \in \mathcal{P}(E \cap \operatorname{Eqs} f) \mid d \notin F\};$$
$$\mathcal{V} = \{F \in \mathcal{P}(E \cap \operatorname{Eqs} f) \mid d \in F\}.$$

Thus, we have $\mathcal{P}(E \cap \operatorname{Eqs} f) = \mathcal{U} \cup \mathcal{V}$, and the sets $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Now, we define a map $\Phi : \mathcal{U} \to \mathcal{V}$ by

$$(\Phi(B) = B \cup \{d\} \qquad \text{for every } B \in \mathcal{U}).$$

This map Φ is well-defined (because for every $B \in \mathcal{U}$, we have $B \cup \{d\} \in \mathcal{V}$ [9]) and a bijection[10]. Moreover, every $B \in \mathcal{U}$ satisfies

$$(-1)^{|\Phi(B)|} = -(-1)^{|B|} \tag{9}$$

[11].

Now, we claim that, for every $B \in \mathcal{U}$ and every $K \in \mathfrak{K}$, we have the following logical equivalence:

$$(K \subseteq B) \iff (K \subseteq \Phi(B)). \tag{10}$$

Proof of (10): Let $B \in \mathcal{U}$ and $K \in \mathfrak{K}$. We must prove the equivalence (10). The definition of Φ yields $\Phi(B) = B \cup \{d\} \supseteq B$, so that $B \subseteq \Phi(B)$. Hence, if $K \subseteq B$, then $K \subseteq B \subseteq \Phi(B)$. Therefore, the forward implication of the equivalence (10) is proven. It thus remains to prove the backward implication of this equivalence. In other words, it remains to prove that if $K \subseteq \Phi(B)$, then $K \subseteq B$. So let us assume that $K \subseteq \Phi(B)$.

We want to prove that $K \subseteq B$. Assume the contrary. Thus, $K \not\subseteq B$. We have $K \in \mathfrak{K}$. Thus, K is a broken circuit of G (since $\mathfrak{K}$ is a set of broken circuits of G). In other words, K is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique edge in C having maximum label (among the edges in C) (because this is how a broken circuit is defined). Consider these C and e. Thus, $K = C \setminus \{e\}$.

The element e is the unique edge in C having maximum label (among the edges in C). Thus, if e' is any edge in C satisfying $\ell(e') \geq \ell(e)$, then

$$e' = e. \tag{11}$$

But $\underbrace{K}_{\subseteq \Phi(B) = B \cup \{d\}} \setminus \{d\} \subseteq (B \cup \{d\}) \setminus \{d\} \subseteq B$.

[9] This follows from the fact that $d \in E \cap \operatorname{Eqs} f$.

[10] Its inverse is the map $\Psi : \mathcal{V} \to \mathcal{U}$ defined by $(\Psi(B) = B \setminus \{d\} \qquad \text{for every } B \in \mathcal{V})$.

[11] *Proof.* Let $B \in \mathcal{U}$. Thus, $d \notin B$ (by the definition of $\mathcal{U}$). Now, $\left|\underbrace{\Phi(B)}_{= B \cup \{d\}}\right| = |B \cup \{d\}| = |B| + 1$ (since $d \notin B$), so that $(-1)^{|\Phi(B)|} = -(-1)^{|B|}$, qed.

If we had $d \notin K$, then we would have $K \setminus \{d\} = K$ and therefore $K = K \setminus \{d\} \subseteq B$; this would contradict $K \not\subseteq B$. Hence, we cannot have $d \notin K$. We thus must have $d \in K$. Hence, $d \in K = C \setminus \{e\}$. Hence, $d \in C$ and $d \neq e$.

But $C \setminus \{e\} = K \subseteq \Phi(B) \subseteq E \cap \operatorname{Eqs} f$ (since $\Phi(B) \in \mathcal{P}(E \cap \operatorname{Eqs} f)$), so that $C \setminus \{e\} \subseteq E \cap \operatorname{Eqs} f \subseteq \operatorname{Eqs} f$. Hence, Lemma 2.4 (applied to Y instead of X) shows that $e \in E \cap \operatorname{Eqs} f$. Thus, $\ell(d) \geq \ell(e)$ (since d was defined to be an element of $E \cap \operatorname{Eqs} f$ with maximum $\ell(d)$ among all $d \in E \cap \operatorname{Eqs} f$).

Also, $d \in C$. Since $\ell(d) \geq \ell(e)$, we can therefore apply (11) to $e' = d$. We thus obtain $d = e$. This contradicts $d \neq e$. This contradiction proves that our assumption was wrong. Hence, $K \subseteq B$ is proven. Thus, we have proven the backward implication of the equivalence (10); this completes the proof of (10).

Now, recall that we have $\mathcal{P}(E \cap \operatorname{Eqs} f) = \mathcal{U} \cup \mathcal{V}$, and the sets $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Hence, the sum $\sum\limits_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod\limits_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K$ can be split into two sums as follows:

$$\begin{aligned}
&\sum_{B \subseteq E \cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \\
&= \sum_{B \in \mathcal{U}} \underbrace{(-1)^{|B|}}_{\substack{= -(-1)^{|\Phi(B)|} \\ \text{(by (9))}}} \underbrace{\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}}}_{\substack{= \prod\limits_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} \\ \text{(because of the equivalence (10))}}} a_K + \underbrace{\sum_{B \in \mathcal{V}} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K}_{\substack{= \sum\limits_{B \in \mathcal{U}} (-1)^{|\Phi(B)|} \prod\limits_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K \\ \text{(here, we have substituted } \Phi(B) \text{ for } B \text{ in the sum,} \\ \text{since the map } \Phi: \mathcal{U} \to \mathcal{V} \text{ is a bijection)}}} \\
&= \sum_{B \in \mathcal{U}} \left(-(-1)^{|\Phi(B)|} \right) \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K + \sum_{B \in \mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K \\
&= - \sum_{B \in \mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K + \sum_{B \in \mathcal{U}} (-1)^{|\Phi(B)|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \Phi(B)}} a_K \\
&= 0 = [E \cap \operatorname{Eqs} f = \varnothing] \qquad \left(\text{since } [E \cap \operatorname{Eqs} f = \varnothing] = 0\right). \qquad (12)
\end{aligned}$$

This proves Lemma 2.8. □

We now finally proceed to the proof of Theorem 1.12:

Proof of Theorem 1.12. The definition of X_G shows that

$$\begin{aligned}
X_G &= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \left[\underbrace{f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G}_{\substack{\Longleftrightarrow \text{ (the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper)} \\ \Longleftrightarrow\ (E\cap \operatorname{Eqs} f=\varnothing) \\ \text{(by Lemma 2.3, applied to } \mathbb{N}_+ \text{ instead of } X\text{)}}} \right] \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \underbrace{[E\cap \operatorname{Eqs} f = \varnothing]}_{\substack{=\sum_{B\subseteq E\cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{K\in\mathfrak{K};\ K\subseteq B} a_K \\ \text{(by Lemma 2.8, applied to } Y=\mathbb{N}_+\text{)}}} \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \underbrace{\sum_{B\subseteq E\cap \operatorname{Eqs} f}}_{=\sum_{B\subseteq E;\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f \\
&= \underbrace{\sum_{f:V\to\mathbb{N}_+} \sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}}}_{=\sum_{B\subseteq E} \sum_{f:V\to\mathbb{N}_+;\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f \\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \mathbf{x}_f = \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f}_{\substack{=p_{\lambda(V,B)} \\ \text{(by Lemma 2.5)}}} \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right) p_{\lambda(V,B)} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K \right) p_{\lambda(V,F)}
\end{aligned}$$

(here, we have renamed the summation index B as F). This proves Theorem 1.12. ☐

Thus, Theorem 1.12 is proven; as we know, this entails the correctness of Theorem 1.8, Corollary 1.14 and Corollary 1.15.

3. The chromatic polynomial

3.1. Definition

We have so far studied the chromatic symmetric function. We shall now apply the above results to the chromatic polynomial. The definition of the chromatic polynomial rests upon the following fact:

Theorem 3.1. Let $G = (V, E)$ be a finite graph. Then, there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

$$P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G).$$

Definition 3.2. Let $G = (V, E)$ be a finite graph. Theorem 3.1 shows that there exists a polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies $P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G)$. This polynomial P is called the *chromatic polynomial* of G, and will be denoted by χ_G.

We shall later prove Theorem 3.1 (as a consequence of something stronger that we show). First, we shall state some formulas for the chromatic polynomial which are analogues of results proven before for the chromatic symmetric function.

3.2. Formulas for χ_G

Before we state several formulas for χ_G, we need to introduce one more notation:

Definition 3.3. Let G be a finite graph. We let $\operatorname{conn} G$ denote the number of connected components of G.

The following results are analogues of Theorem 1.8, Theorem 1.12, Corollary 1.14 and Corollary 1.15, respectively:

Theorem 3.4. Let $G = (V, E)$ be a finite graph. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 3.3.)

Theorem 3.5. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of

G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{\operatorname{conn}(V,F)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 3.3.)

Corollary 3.6. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

Corollary 3.7. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{\operatorname{conn}(V,F)}.$$

Except for Theorem 3.5, these results are not new; in fact, Corollary 3.6 is a particular case of [DohTri14, (12)], and of course we can obtain Corollary 3.7 and Theorem 3.4 as particular cases of Corollary 3.6.

3.3. Proofs

Nevertheless, for the sake of completeness, we shall give proofs of all the five results above (Theorem 3.1, Theorem 3.4, Theorem 3.5, Corollary 3.6 and Corollary 3.7).

There are two approaches to these results (except for Theorem 3.1): One is to prove them similarly to how we proved the analogous results about X_G; the other is to derive them from the latter. We shall take the first approach, since it yields a proof of the classical Theorem 3.1 "for free". We begin with an analogue of Lemma 2.5:

Lemma 3.8. Let (V, B) be a finite graph. Let $q \in \mathbb{N}$. Then,

$$\sum_{\substack{f : V \to \{1,2,\ldots,q\}; \\ B \subseteq \operatorname{Eqs} f}} 1 = q^{\operatorname{conn}(V,B)}.$$

(Here, the expression $\operatorname{conn}(V,B)$ is understood according to Definition 1.7 **(b)**.)

One way to prove Lemma 3.8 is to evaluate the equality given by Lemma 2.5 at $x_k = \begin{cases} 1, & \text{if } k \leq q; \\ 0, & \text{if } k > q \end{cases}$. Another proof can be obtained by mimicking our proof of Lemma 2.5:

Proof of Lemma 3.8. Define $(C_1, C_2, \ldots, C_k)$ as in the proof of Lemma 2.5. Thus, $\operatorname{conn}(V,B) = k$. Define a map Φ as in the proof of Lemma 2.5, but with $\mathbb{N}_+$ replaced by $\{1,2,\ldots,q\}$. Then,

$$\Phi : \{1,2,\ldots,q\}^k \to \{f : V \to \{1,2,\ldots,q\} \mid B \subseteq \operatorname{Eqs} f\}$$

is a bijection[12]. Now,

$$\begin{aligned}
&\sum_{\substack{f:V\to\{1,2,\ldots,q\};\\ B\subseteq \operatorname{Eqs} f}} 1 \\
&= \sum_{(s_1,s_2,\ldots,s_k)\in\{1,2,\ldots,q\}^k} 1 \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } \Phi(s_1,s_2,\ldots,s_k) \text{ for } f \text{ in the sum, since}\\ \text{the map } \Phi : \{1,2,\ldots,q\}^k \to \{f : V \to \{1,2,\ldots,q\} \mid B \subseteq \operatorname{Eqs} f\}\\ \text{is a bijection}\end{array}\right) \\
&= \left(\text{the number of all } (s_1,s_2,\ldots,s_k) \in \{1,2,\ldots,q\}^k\right) \\
&= q^k = q^{\operatorname{conn}(V,B)} \qquad (\text{since } k = \operatorname{conn}(V,B)).
\end{aligned}$$

This proves Lemma 3.8. □

We shall now show a weaker version of Theorem 3.5 (as a stepping stone to the actual theorem):

Lemma 3.9. Let $G = (V,E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Let $q \in \mathbb{N}$. Then,

$$\begin{aligned}
&(\text{the number of all proper } \{1,2,\ldots,q\}\text{-colorings of } G) \\
&= \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K\right) q^{\operatorname{conn}(V,F)}.
\end{aligned}$$

[12]This can be shown in the same way as for the map Φ in the proof of Lemma 2.5; we just have to replace every $\mathbb{N}_+$ by $\{1,2,\ldots,q\}$.

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 3.3.)

Proof of Lemma 3.9. We have[13]

$$\begin{aligned}
&(\text{the number of all proper } \{1,2,\ldots,q\}\text{-colorings of } G) \\
&= \sum_{f:V\to\{1,2,\ldots,q\}} \underbrace{\left[f \text{ is a proper } \{1,2,\ldots,q\}\text{-coloring of } G\right]}_{\substack{\iff (\text{the } \{1,2,\ldots,q\}\text{-coloring } f \text{ of } G \text{ is proper}) \\ \iff (E\cap \operatorname{Eqs} f=\varnothing) \\ (\text{by Lemma 2.3, applied to } \{1,2,\ldots,q\} \text{ instead of } X)}} \\
&= \sum_{f:V\to\{1,2,\ldots,q\}} \underbrace{[E\cap \operatorname{Eqs} f = \varnothing]}_{\substack{=\sum_{B\subseteq E\cap \operatorname{Eqs} f} (-1)^{|B|} \prod_{K\in\mathfrak{K};\ K\subseteq B} a_K \\ (\text{by Lemma 2.8, applied to } Y=\mathbb{N}_+)}} \\
&= \sum_{f:V\to\{1,2,\ldots,q\}} \underbrace{\sum_{B\subseteq E\cap \operatorname{Eqs} f}}_{=\sum_{B\subseteq E;\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) \\
&= \underbrace{\sum_{f:V\to\{1,2,\ldots,q\}} \sum_{\substack{B\subseteq E;\\ B\subseteq \operatorname{Eqs} f}}}_{=\sum_{B\subseteq E} \sum_{f:V\to\mathbb{N}_+;\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) \\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to\{1,2,\ldots,q\};\\ B\subseteq \operatorname{Eqs} f}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) = \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) \underbrace{\sum_{\substack{f:V\to\{1,2,\ldots,q\};\\ B\subseteq \operatorname{Eqs} f}} 1}_{\substack{=q^{\operatorname{conn}(V,B)} \\ (\text{by Lemma 3.8})}} \\
&= \sum_{B\subseteq E} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) q^{\operatorname{conn}(V,B)} = \sum_{F\subseteq E} (-1)^{|F|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}} a_K\right) q^{\operatorname{conn}(V,F)}
\end{aligned}$$

(here, we have renamed the summation index B as F). This proves Lemma 3.9. $\square$

[13]We are again using the Iverson bracket notation, as defined in Definition 2.7.

From Lemma 3.9, we obtain the following consequence:

Lemma 3.10. Let $G = (V, E)$ be a finite graph. Let $q \in \mathbb{N}$. Then,

$$\begin{aligned}&(\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G) \\ &= \sum_{F \subseteq E} (-1)^{|F|} q^{\operatorname{conn}(V,F)}.\end{aligned}$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\operatorname{conn}(V, F)$ is understood according to Definition 3.3.)

Proof of Lemma 3.10. This is derived from Lemma 3.9 in the same way as Theorem 1.8 was derived from Theorem 3.5. □

Next, we recall a classical fact about polynomials over fields: Namely, if a polynomial (in one variable) over a field has infinitely many roots, then this polynomial is 0. Let us state this more formally:

Proposition 3.11. Let K be a field. Let $P \in K[x]$ be a polynomial over K. Assume that there are infinitely many $\lambda \in K$ satisfying $P(\lambda) = 0$. Then, $P = 0$.

We shall use the following two consequences of this proposition:

Corollary 3.12. Let R be an integral domain. Assume that the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring R is injective. Let $P \in R[x]$ be a polynomial over R. Assume that $P(q \cdot 1_R) = 0$ for every $q \in \mathbb{N}$ (where 1_R denotes the unity of R). Then, $P = 0$.

Proof of Corollary 3.12. Let K denote the fraction field of the integral domain R. We regard R and $R[x]$ as subrings of K and $K[x]$, respectively. By assumption, we have $P(q \cdot 1_R) = 0$ for every $q \in \mathbb{N}$. But the elements $q \cdot 1_R$ of R for $q \in \mathbb{N}$ are pairwise distinct (since the canonical ring homomorphism from the ring $\mathbb{Z}$ to the ring R is injective). Hence, there are infinitely many $\lambda \in K$ satisfying $P(\lambda) = 0$ (namely, $\lambda = q \cdot 1_R$ for all $q \in \mathbb{N}$). Thus, Proposition 3.11 shows that $P = 0$. This proves Corollary 3.12. □

Corollary 3.13. Let R be an integral domain such that $\mathbb{Z}$ is a subring of R. Let $P_1 \in R[x]$ and $P_2 \in R[x]$ be two polynomials over R. Assume that every $q \in \mathbb{N}$ satisfies

$$P_1(q) = P_2(q). \qquad (13)$$

Then, $P_1 = P_2$.

Proof of Corollary 3.13. For every $q \in \mathbb{N}$, we have $(P_1 - P_2)(q) = P_1(q) - P_2(q) = 0$ (by (13)). Hence, Corollary 3.12 (applied to $P = P_1 - P_2$) yields that $P_1 - P_2 = 0$. In other words, $P_1 = P_2$. This proves Corollary 3.13. □

We can now prove the classical Theorem 3.1:

Proof of Theorem 3.1. We need to show that there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

$$P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\} \text{-colorings of } G).$$

To see that such a polynomial exists, we notice that $P = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V,F)}$ is such a polynomial (by Lemma 3.10). It remains to prove that such a polynomial is unique. But this follows directly from Corollary 3.13 (applied to $R = \mathbb{Z}$). Theorem 3.1 is therefore proven. □

Next, it is the turn of Theorem 3.5:

Proof of Theorem 3.5. Let R be the polynomial ring $\mathbb{Z}[y_K \mid K \in \mathfrak{K}]$, where y_K is a new indeterminate for each $K \in \mathfrak{K}$.

The claim of Theorem 3.5 is a polynomial identity in the elements a_K of $\mathbf{k}$. Hence, we can WLOG assume that $\mathbf{k} = R$ and $a_K = y_K$ for each $K \in \mathfrak{K}$. Assume this. Thus, $\mathbf{k}$ is an integral domain, and the ring $\mathbb{Z}$ is a subring of $\mathbf{k}$.

For every $q \in \mathbb{N}$, we have

$$\begin{aligned} \chi_G(q) &= (\text{the number of all proper } \{1, 2, \ldots, q\} \text{-colorings of } G) \\ &\qquad\qquad (\text{by the definition of the chromatic polynomial } \chi_G) \\ &= \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) q^{\operatorname{conn}(V,F)} \qquad (14) \end{aligned}$$

(by Lemma 3.9). Define a polynomial $P \in \mathbf{k}[x]$ by

$$P = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{\operatorname{conn}(V,F)}. \qquad (15)$$

Then, for every $q \in \mathbb{N}$, we have

$$P(q) = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) q^{\operatorname{conn}(V,F)} = \chi_G(q) \qquad (\text{by (14)}).$$

Thus, Corollary 3.13 (applied to $R = \mathbf{k}$ and $P_1 = P$ and $P_2 = \chi_G$) shows that $P = \chi_G$. Comparing this with (15), we obtain

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{\operatorname{conn}(V,F)}.$$

This proves Theorem 3.5. □

Now that Theorem 3.5 is proven, we could derive Theorem 3.4, Corollary 3.6 and Corollary 3.7 from it in the same way as we have derived Theorem 1.8, Corollary 1.14 and Corollary 1.15 from Theorem 1.12. We leave the details to the reader.

3.4. Special case: Whitney's Broken-Circuit Theorem

Corollary 3.7 is commonly stated in the following simplified (if less general) form:

Corollary 3.14. Let $G = (V, E)$ be a finite graph. Let X be a totally ordered set. Let $\ell : E \to X$ be an injective labeling function. Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{|V|-|F|}.$$

Corollary 3.14 is known as *Whitney's Broken-Circuit theorem* (see, e.g., [BlaSag86]). In his original 1932 paper [Whitne32, §7], Whitney stated its claim as "the $x^{|V|-i}$-coefficient of χ_G is $(-1)^i$ times the number of i-element subsets of E that contain no broken circuit as a subset", which is easily seen to be equivalent to our formulation.

Notice that ℓ is required to be injective in Corollary 3.14; the purpose of this requirement is to ensure that every circuit of G has a unique edge e with maximum $\ell(e)$, and thus induces a broken circuit of G. The proof of Corollary 3.14 relies on the following standard result:

Lemma 3.15. Let (V, F) be a finite graph. Assume that (V, F) has no circuits. Then, $\operatorname{conn}(V, F) = |V| - |F|$.

(A graph that has no circuits is commonly known as a *forest*.)

Lemma 3.15 is both extremely elementary and well-known; for example, it appears in [Bona17, Proposition 10.6], in [Bollob79, §I.2, Corollary 6] and in [Grinbe21, Theorem 6.3.15 **(e)**]. Let us now see how it entails Corollary 3.14:

Proof of Corollary 3.14. Corollary 3.14 follows from Corollary 3.7. Indeed, the injectivity of ℓ shows that every circuit of G has a unique edge e with maximum $\ell(e)$, and thus contains a broken circuit of G as a subset. Therefore, if a subset F of E contains no broken circuit of G as a subset, then F contains no circuit of G either, and therefore the graph (V, F) has no circuits; but this entails that $\operatorname{conn}(V, F) = |V| - |F|$ (by Lemma 3.15). Hence, Corollary 3.7 immediately yields Corollary 3.14. $\square$

4. Application: Transitive directed graphs

We shall now see an application of Corollary 3.6 to graphs which are obtained from certain directed graphs by "forgetting the directions of the edges". Let us first introduce the notations involved:

Definition 4.1. **(a)** A *digraph* means a pair (V, A), where V is a set, and where A is a subset of $V^2 = V \times V$. Digraphs are also called *directed graphs*. A digraph (V, A) is said to be *finite* if the set V is finite. If $D = (V, A)$ is a digraph, then the elements of V are called the *vertices* of the digraph D, while the elements of A are called the *arcs* (or the *directed edges*) of the digraph D. If $a = (v, w)$ is an arc of a digraph D, then v is called the *source* of a, whereas w is called the *target* of a.

(b) A digraph (V, A) is said to be *loopless* if every $v \in V$ satisfies $(v, v) \notin A$. (In other words, a digraph is loopless if and only if it has no arc whose source and target are identical.)

(c) A digraph (V, A) is said to be *transitive* if it has the following property: For any $u \in V$, $v \in V$ and $w \in V$ satisfying $(u, v) \in A$ and $(v, w) \in A$, we have $(u, w) \in A$.

(d) A digraph (V, A) is said to be 2-*step-free* if there exist no three elements u, v and w of V satisfying $(u, v) \in A$ and $(v, w) \in A$.

(e) Let $D = (V, A)$ be a loopless digraph. Define a map $\operatorname{set} : A \to \binom{V}{2}$ by setting

$$(\operatorname{set}(v, w) = \{v, w\} \qquad \text{for every } (v, w) \in A).$$

(It is easy to see that set is well-defined, because (V, A) is loopless.) The graph $(V, \operatorname{set} A)$ will be denoted by $\underline{D}$. (Here, $\operatorname{set} A$ means the subset $\{\operatorname{set} a \mid a \in A\}$ of $\binom{V}{2}$.)

Example 4.2. **(a)** The digraph $D = (V, A)$ with $V = \{1, 2, 3\}$ and $A = \{(1, 2), (2, 1), (2, 3), (3, 3)\}$ is not loopless (since the vertex $v = 3$ does not satisfy $(v, v) \notin A$).

(b) The digraph $D = (V, A)$ with $V = \{1, 2, 3\}$ and $A = \{(1, 2), (2, 1), (2, 3)\}$ is loopless. The corresponding (undirected) graph $\underline{D}$ is $\underline{D} = (V, \operatorname{set} A)$ with $\operatorname{set} A = \{\{1, 2\}, \{2, 3\}\}$. (Note that the two distinct arcs $(1, 2)$ and $(2, 1)$ of D yield the same edge $\{1, 2\}$ of $\underline{D}$.) Note that this digraph D is not transitive, because the three vertices $u = 1$, $v = 2$ and $w = 1$ satisfy $(u, v) \in A$ and $(v, w) \in A$ but don't satisfy $(u, w) \in A$.

(c) The digraph $D = (V, A)$ with $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (2, 3), (1, 3), (3, 4)\}$ is not transitive, since the three vertices $u = 2$, $v = 3$ and $w = 4$ satisfy $(u, v) \in A$ and $(v, w) \in A$ but don't satisfy $(u, w) \in A$.

(d) The digraph $D = (V, A)$ with $V = \{1, 2, 3, 4\}$ and $A = \{(1, 2), (2, 3), (1, 3), (4, 2), (4, 3)\}$ is loopless and transitive. It is not 2-step-free, since the three elements $u = 1$, $v = 2$ and $w = 3$ satisfy $(u, v) \in A$ and $(v, w) \in A$.

(e) The digraph $D = (V, A)$ with $V = \{1, 2, 3, 4\}$ and $A = \{(1, 3), (2, 3), (1, 4), (2, 4)\}$ is loopless, transitive and 2-step-free. (Actually, any 2-step-free digraph is transitive, for vacuous reasons.)

Remark 4.3. A transitive loopless digraph cannot have any (directed) cycles. We omit the easy proof of this fact, as we will not use it in what follows, but it illuminates some of the arguments below.

Remark 4.4. A transitive loopless digraph is more or less the same as a poset (i.e., partially ordered set). Indeed:

- If (V, A) is a transitive loopless digraph, then we can equip the set V with a (strict) partial order $<$ defined by
$$(u < v) \iff ((u, v) \in A),$$
which turns V into a poset.
- Conversely, if V is a poset, then we obtain a transitive loopless digraph (V, A) by setting $A := \left\{(u, v) \in V^2 \mid u < v\right\}$.

We find the language of digraphs to be more convenient, but the reader should be aware of the possibility of restating everything in terms of posets.

We can now state our application of Corollary 3.6, answering a question suggested by Alexander Postnikov:

Proposition 4.5. Let $D = (V, A)$ be a finite transitive loopless digraph. Then,
$$\chi_{\underline{D}} = \sum_{\substack{F \subseteq A; \\ \text{the digraph } (V,F) \text{ is 2-step-free}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{set} F)}.$$
(Here, $\operatorname{set} F$ means the subset $\{\operatorname{set} f \mid f \in F\}$ of $\dbinom{V}{2}$.)

Note that the graph $(V, \operatorname{set} F)$ in Proposition 4.5 can also be rewritten as $\underline{(V, F)}$.

Proof of Proposition 4.5. Let $E = \operatorname{set} A$. Then, the definition of $\underline{D}$ yields $\underline{D} = \left(V, \underbrace{\operatorname{set} A}_{=E}\right) = (V, E)$.

The map $\operatorname{set} : A \to \dbinom{V}{2}$ (which sends every arc $(v, w) \in A$ to $\{v, w\} \in \dbinom{V}{2}$) restricts to a surjection $A \to E$ (since $E = \operatorname{set} A$). Let us denote this surjection by

π. Thus, π is a map from A to E sending each arc $(v, w) \in A$ to $\{v, w\} \in E$. We shall soon see that π is a bijection.

We define a partial order on the set V as follows: For $i \in V$ and $j \in V$, we set $i < j$ if and only if $(i, j) \in A$ (that is, if and only if there is an arc from i to j in D). This is a well-defined strict partial order[14]. Thus, V becomes a poset. For every $i \in V$ and $j \in V$ satisfying $i \leq j$, we let $[i, j]$ denote the interval $\{k \in V \mid i \leq k \leq j\}$ of the poset V.

There exist no $i, j \in V$ such that both (i, j) and (j, i) belong to A (because if such i and j would exist, then they would satisfy $i < j$ and $j < i$, but this would contradict the fact that V is a poset). Hence, the projection $\pi : A \to E$ is injective, and thus bijective (since we already know that π is surjective). Hence, its inverse map $\pi^{-1} : E \to A$ is well-defined. For every subset F of E, we have

$$\begin{aligned} F &= \pi\left(\pi^{-1}(F)\right) \qquad \text{(since } \pi \text{ is bijective)} \\ &= \operatorname{set}\left(\pi^{-1}(F)\right) \end{aligned} \tag{16}$$

(since π is a restriction of the map set).

For any $(u, v) \in A$ and any subset F of E, we have the following logical equivalence:

$$(\{u, v\} \in F) \iff \left((u, v) \in \pi^{-1}(F)\right) \tag{17}$$

[15].

Define a function $\ell' : A \to \mathbb{N}$ by

$$\ell'(i, j) = |[i, j]| \qquad \text{for all } (i, j) \in A.$$

Define a labeling function $\ell : E \to \mathbb{N}$ by $\ell = \ell' \circ \pi^{-1}$. Thus, $\ell \circ \pi = \ell'$. Therefore,

$$\ell\left(\underbrace{\{i, j\}}_{=\pi(i,j)}\right) = \underbrace{(\ell \circ \pi)}_{=\ell'}(i, j) = \ell'(i, j) = |[i, j]| \tag{18}$$

for all $(i, j) \in A$.

[14] Indeed, the relation $<$ that we have just defined is transitive (since the digraph (V, A) is transitive) and irreflexive (since the digraph (V, A) is loopless). But any such relation is a strict partial order.

[15] *Proof of (17):* Let $(u, v) \in A$, and let F be a subset of E. We need to prove the equivalence (17).

From $(u, v) \in A$, we see that $\pi(u, v)$ is well-defined. The definition of π shows that $\pi(u, v) = \{u, v\}$. Hence, we have the following chain of equivalences:

$$\left(\underbrace{\{u, v\}}_{=\pi(u,v)} \in F\right) \iff (\pi(u, v) \in F) \iff \left((u, v) \in \pi^{-1}(F)\right).$$

This proves (17).

Let $\mathfrak{K}$ be the set

$$\{\{\{i,k\},\{k,j\}\} \mid (i,k) \in A \text{ and } (k,j) \in A\}.$$

Each $K \in \mathfrak{K}$ is a broken circuit of $\underline{D}$ [16]. Thus, $\mathfrak{K}$ is a set of broken circuits of $\underline{D}$.

A subset F of E is $\mathfrak{K}$-free if and only if the digraph $\left(V, \pi^{-1}(F)\right)$ is 2-step-free[17].

---

[16] *Proof.* Let $K \in \mathfrak{K}$. Then, $K = \{\{i,k\},\{k,j\}\}$ for some $(i,k) \in A$ and $(k,j) \in A$ (by the definition of $\mathfrak{K}$). Consider these (i,k) and (k,j). Since (V, A) is transitive, we have $(i,j) \in A$. Thus, $\{i,k\}$, $\{k,j\}$ and $\{i,j\}$ are edges of $\underline{D}$. These edges form a circuit of $\underline{D}$. In particular, i, j and k are pairwise distinct.

Applications of (18) yield $\ell(\{i,j\}) = |[i,j]|$, $\ell(\{i,k\}) = |[i,k]|$ and $\ell(\{k,j\}) = |[k,j]|$.

But we have $i < k$ (since $(i,k) \in A$) and $k < j$ (since $(k,j) \in A$). Hence, $[i,k]$ is a proper subset of $[i,j]$. (It is proper because it does not contain j, whereas $[i,j]$ does.) Hence, $|[i,k]| < |[i,j]|$. Thus, $\ell(\{i,j\}) = |[i,j]| > |[i,k]| = \ell(\{i,k\})$. Similarly, $\ell(\{i,j\}) > \ell(\{k,j\})$. The last two inequalities show that $\{i,j\}$ is the unique edge of the circuit $\{\{i,k\},\{k,j\},\{i,j\}\}$ having maximum label. Hence, $\{\{i,k\},\{k,j\},\{i,j\}\} \setminus \{\{i,j\}\}$ is a broken circuit of $\underline{D}$. Since

$$\begin{aligned}\{\{i,k\},\{k,j\},\{i,j\}\} \setminus \{\{i,j\}\} &= \{\{i,k\},\{k,j\}\} && \text{(since } i, j \text{ and } k \text{ are pairwise distinct)} \\ &= K,\end{aligned}$$

this shows that K is a broken circuit of $\underline{D}$, qed.

[17] *Proof.* Let F be a subset of E. Then, we have the following equivalence of statements:

$$\begin{aligned}
&(F \text{ is } \mathfrak{K}\text{-free}) \\
\iff &(\{\{i,k\},\{k,j\}\} \not\subseteq F \text{ whenever } (i,k) \in A \text{ and } (k,j) \in A) \\
&\quad \text{(by the definition of } \mathfrak{K}) \\
\iff &(\text{no } (i,k) \in A \text{ and } (k,j) \in A \text{ satisfy } \{\{i,k\},\{k,j\}\} \subseteq F) \\
\iff &(\text{no } (i,k) \in A \text{ and } (k,j) \in A \text{ satisfy } \{i,k\} \in F \text{ and } \{k,j\} \in F) \\
\iff &\left(\text{no } (i,k) \in A \text{ and } (k,j) \in A \text{ satisfy } (i,k) \in \pi^{-1}(F) \text{ and } \{k,j\} \in F\right) \\
&\left(\begin{array}{c}\text{because for } (i,k) \in A\text{, we have } \{i,k\} \in F \text{ if and only if } (i,k) \in \pi^{-1}(F) \\ \text{(by (17), applied to } u = i \text{ and } v = k)\end{array}\right) \\
\iff &\left(\text{no } (i,k) \in A \text{ and } (k,j) \in A \text{ satisfy } (i,k) \in \pi^{-1}(F) \text{ and } (k,j) \in \pi^{-1}(F)\right) \\
&\left(\begin{array}{c}\text{because for } (k,j) \in A\text{, we have } \{k,j\} \in F \text{ if and only if } (k,j) \in \pi^{-1}(F) \\ \text{(by (17), applied to } u = k \text{ and } v = j)\end{array}\right) \\
\iff &\left(\text{the digraph } \left(V, \pi^{-1}(F)\right) \text{ is 2-step-free}\right) \qquad \text{(by the definition of “2-step-free”)},
\end{aligned}$$

qed.

Now, Corollary 3.6 (applied to $X = \mathbb{N}$ and $G = \underline{D}$) shows that

$$\chi_{\underline{D}} = \underbrace{\sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}}}_{\substack{= \sum_{\substack{F \subseteq E; \\ \text{the digraph } (V, \pi^{-1}(F)) \text{ is 2-step-free}}} \\ \text{(since we have just shown that} \\ \text{a subset } F \text{ of } E \text{ is } \mathfrak{K}\text{-free if and only if} \\ \text{the digraph } (V, \pi^{-1}(F)) \text{ is 2-step-free)}}} \underbrace{(-1)^{|F|}}_{\substack{= (-1)^{\left|\pi^{-1}(F)\right|} \\ \text{(since } \pi \text{ is bijective)}}} \underbrace{x^{\operatorname{conn}(V,F)}}_{\substack{= x^{\operatorname{conn}\left(V, \operatorname{set}\left(\pi^{-1}(F)\right)\right)} \\ \text{(by (16))}}}$$

$$= \sum_{\substack{F \subseteq E; \\ \text{the digraph } (V, \pi^{-1}(F)) \text{ is 2-step-free}}} (-1)^{\left|\pi^{-1}(F)\right|} x^{\operatorname{conn}\left(V, \operatorname{set}\left(\pi^{-1}(F)\right)\right)}$$

$$= \sum_{\substack{B \subseteq A; \\ \text{the digraph } (V, B) \text{ is 2-step-free}}} (-1)^{|B|} x^{\operatorname{conn}(V, \operatorname{set} B)}$$

$$\left(\begin{array}{c} \text{here, we have substituted } B \text{ for } \pi^{-1}(F) \text{ in the sum,} \\ \text{since the map } \pi : A \to E \text{ is bijective and thus induces} \\ \text{a bijection from the subsets of } E \text{ to the subsets of } A \\ \text{sending each } F \subseteq E \text{ to } \pi^{-1}(F) \end{array} \right)$$

$$= \sum_{\substack{F \subseteq A; \\ \text{the digraph } (V, F) \text{ is 2-step-free}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{set} F)}$$

(here, we have renamed the summation index B as F). This proves Proposition 4.5. □

5. Ambigraphs

5.1. Definitions of ambigraphs and proper colorings

We now move on to study various generalizations of the chromatic symmetric function.

The first generalization replaces the finite graph G by what we call an *ambigraph* (short for "ambiguous graph"). To our knowledge, this is a new notion, but it serves to unify two rather well-known concepts:

- that of a *multigraph* (see [Grinbe21, Definition 6.1.1]), which is like a graph but allows for multiple parallel edges[18];
- that of a *hypergraph* (see [Berge73, Chapter 17]), which is like a graph but allows its "edges" to have any number of endpoints instead of two.

[18] More precisely, our notion of an ambigraph generalizes *loopless* multigraphs, i.e., multigraphs with no loops. Loops would be a trivial but technically awkward distraction in the study of chromatic polynomials, so we prefer to leave them out of our notions of graphs.

In both of these settings, chromatic polynomials have been defined long ago (for multigraphs perhaps since the introduction of the concept[19]; for hypergraphs since Dohmen's [Dohmen95]), and it is fairly straightforward to define chromatic symmetric functions at the same levels of generality. However, we shall instead define them for *ambigraphs*, a concept which we now introduce:

Definition 5.1. **(a)** An *ambigraph* shall mean a triple (V, E, φ), where V and E are two sets, and where $\varphi : E \to \mathcal{P}\left(\dbinom{V}{2}\right)$ is a map. (Thus, the map φ sends each $e \in E$ to a set of 2-element subsets of V.)

(b) An ambigraph (V, E, φ) is said to be *finite* if V and E are finite.

(c) Let $G = (V, E, \varphi)$ be an ambigraph. Then, the elements of V are called the *vertices* of G, whereas the elements of E are called the *edgeries* of G. If $e \in E$ is any edgery, then the elements of $\varphi(e)$ are called the *edges* of e. Note that these edges are 2-element subsets of V.

(d) Let $G = (V, E, \varphi)$ be an ambigraph. An edgery $e \in E$ is said to be *singleton* if it has exactly one edge (i.e., if $|\varphi(e)| = 1$).

We view an ambigraph (V, E, φ) as something akin to a graph, except that instead of having edges, it has edgeries – i.e., packages of edges. (This can be equivalently viewed as an edge-colored graph, but we eschew such an interpretation as we shall be using colors for other purposes.)

Example 5.2. Let V be the set $\{1, 2, 3, 4, 5\}$. Let E be the 6-element set $\{e_1, e_2, e_3, e_4, e_5, e_6\}$. Let $\varphi : E \to \mathcal{P}\left(\dbinom{V}{2}\right)$ be the map defined as follows:

$$\begin{aligned}
\varphi(e_1) &= \{\{1,3\}, \{2,5\}\}, \\
\varphi(e_2) &= \{\{1,2\}, \{2,3\}, \{3,4\}\}, \\
\varphi(e_3) &= \{\{2,5\}\}, \\
\varphi(e_4) &= \{\{1,3\}, \{2,5\}\}, \\
\varphi(e_5) &= \{\} = \varnothing, \\
\varphi(e_6) &= \{\{2,3\}, \{3,4\}\}.
\end{aligned}$$

Let G be the triple (V, E, φ). Then, G is an ambigraph. Its edgeries are $e_1, e_2, e_3, e_4, e_5, e_6$. The edgery e_3 is singleton, while the other edgeries are not. The edgeries e_1 and e_4 contain the same edges, namely $\{1, 3\}$ and $\{2, 5\}$.

Both multigraphs and hypergraphs can now be encoded as ambigraphs:

- A multigraph can be viewed as an ambigraph whose all edgeries are singleton[20].

[19] Authors often leave it vague whether their graphs are simple graphs or multigraphs.

[20] To be more precise, this is true for *loopless* multigraphs (i.e., multigraphs that have no loops). Loops can be encoded as edgeries that have no edges.

- A hypergraph can be encoded as an ambigraph by replacing each edge $\{v_1, v_2, \ldots, v_k\}$ with an edgery consisting of all edges $\{v_i, v_j\}$ with $i < j$. (Note that this encoding turns 1-element edges into empty edgeries[21]. Empty edgeries trivialize most of our results, but do not invalidate any of our proofs, so we have no reason to exclude them.)

We can now define X-colorings and proper X-colorings for ambigraphs:

Definition 5.3. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set.

(a) An *X-coloring* of G is defined to mean a map $V \to X$.

(b) If $f : V \to X$ is an X-coloring of G, and if $\{s, t\}$ is a 2-element subset of V, then this subset $\{s, t\}$ is said to be *f-dichromatic* if $f(s) \neq f(t)$.

(c) An X-coloring f of G is said to be *proper* if each edgery $e \in E$ has at least one f-dichromatic edge (i.e., for each edgery $e \in E$, there exists at least one edge $\{s, t\} \in \varphi(e)$ satisfying $f(s) \neq f(t)$).

Example 5.4. Let $G = (V, E, \varphi)$ be the ambigraph from Example 5.2. Then, G has no proper X-coloring for any X, since the edgery e_5 will never have an f-dichromatic edge, no matter what f is (because e_5 has no edge to begin with).

However, let us now modify φ by replacing $\varphi(e_5)$ by the set $\{\{1,4\}, \{2,4\}, \{3,4\}\}$. Then, for example, the X-coloring $f : V \to \{1,2,3,4\}$ given by

$$f(1) = 1, \qquad f(2) = 2, \qquad f(3) = 1,$$
$$f(4) = 1, \qquad f(5) = 3, \qquad f(6) = 1$$

is proper. For instance, the edgery e_1 has the f-dichromatic edge $\{2,5\}$, whereas the edgery e_2 has the two f-dichromatic edges $\{1,2\}$ and $\{2,3\}$. On the other hand, the X-coloring $f : V \to \{1,2,3,4\}$ given by

$$f(1) = 1, \qquad f(2) = 2, \qquad f(3) = 2,$$
$$f(4) = 2, \qquad f(5) = 3, \qquad f(6) = 1$$

is not proper, since the edgery e_6 has no f-dichromatic edge.

Example 5.5. Let $G = (V, E, \varphi)$ be the ambigraph with $V = \{1,2,3,4\}$ and $E = \{a, b\}$ and

$$\varphi(a) = \{\{2,3\}\} \qquad \text{and} \qquad \varphi(b) = \{\{1,2\}, \{3,4\}\}.$$

Let X be a set. Then, a map $f : V \to X$ is a proper X-coloring of G if and only if it satisfies

$$f(2) \neq f(3) \qquad \text{and} \qquad (f(1) \neq f(2) \text{ or } f(3) \neq f(4)).$$

[21] i.e., edgeries that have no edges

Indeed, the statement "$f(2) \neq f(3)$" is saying that the edgery a has an f-dichromatic edge, whereas the statement "$f(1) \neq f(2)$ or $f(3) \neq f(4)$" is saying that the edgery b has an f-dichromatic edge.

As Example 5.5 illustrates, the condition on an X-coloring of G to be proper is a conjunction of disjunctions of inequalities of the form $f(v) \neq f(w)$ for $(v, w) \in V^2$.

Remark 5.6. Any graph $G = (V, E)$ can be viewed as an ambigraph (V, E, φ) in a fairly obvious way: viz., by setting $\varphi(e) = \{e\}$ for each edge $e \in E$. We shall denote the latter ambigraph by G^{amb}. The proper X-colorings of this ambigraph G^{amb} are precisely the proper X-colorings of the original graph G.

Remark 5.7. Let $G = (V, E, \varphi)$ be an ambigraph, and let X be a set. If there exists an edgery $e \in E$ satisfying $\varphi(e) = \varnothing$, then there exists no proper X-coloring f of G (since the edgery e will never have an f-dichromatic edge).

5.2. The chromatic symmetric function of an ambigraph

We can now define the chromatic symmetric function of an ambigraph, by imitating Definition 1.5:

Definition 5.8. Let $G = (V, E, \varphi)$ be a finite ambigraph.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{x}_f$ denote the monomial $\prod_{v \in V} x_{f(v)}$ in the indeterminates $x_1, x_2, x_3, \ldots$.

(b) We define a power series $X_G \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$X_G = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f.$$

This power series X_G is called the *chromatic symmetric function* of G.

Example 5.9. Let $G = (V, E, \varphi)$ be the ambigraph from Example 5.5. Then,

$$\begin{aligned} X_G &= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{\mathbf{x}_f}_{=x_{f(1)}x_{f(2)}x_{f(3)}x_{f(4)}} \\ &= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} x_{f(1)}x_{f(2)}x_{f(3)}x_{f(4)} \\ &= \sum_{\substack{f:\{1,2,3,4\}\to\mathbb{N}_+; \\ f(2)\neq f(3) \text{ and } (f(1)\neq f(2) \text{ or } f(3)\neq f(4))}} x_{f(1)}x_{f(2)}x_{f(3)}x_{f(4)} \end{aligned}$$

(since a map $f : V \to \mathbb{N}_+$ is a proper $\mathbb{N}_+$-coloring of G if and only if it satisfies $f(2) \neq f(3)$ and $(f(1) \neq f(2)$ or $f(3) \neq f(4))$). If we re-encode each map $f : \{1,2,3,4\} \to \mathbb{N}_+$ as the 4-tuple $(i,j,k,\ell) = (f(1), f(2), f(3), f(4))$ of its values, then we can rewrite this equality as

$$X_G = \sum_{\substack{(i,j,k,\ell) \in (\mathbb{N}_+)^4; \\ j \neq k \text{ and } (i \neq j \text{ or } k \neq \ell)}} x_i x_j x_k x_\ell .$$

Remark 5.10. Let $G = (V, E, \varphi)$ be an ambigraph that has an edgery $e \in E$ satisfying $\varphi(e) = \varnothing$. Then, there exists no proper $\mathbb{N}_+$-coloring f of G (by Remark 5.7), and thus we have $X_G = 0$.

5.3. The union of a set of edgeries

An ambigraph (V, E, φ) can be transformed into a simple graph (V, E') by taking the union of some of its edgeries – i.e., by setting $E' := \bigcup_{e \in F} \varphi(e)$ for some subset F of E. Let us give this construction a name:

Definition 5.11. Let $G = (V, E, \varphi)$ be an ambigraph. Let F be a subset of E. Then, $\operatorname{union} F$ shall denote the subset $\bigcup_{e \in F} \varphi(e)$ of $\dbinom{V}{2}$. Thus, we obtain a graph $(V, \operatorname{union} F)$.

Example 5.12. Let $G = (V, E, \varphi)$ be the ambigraph from Example 5.2. Then,

$$\operatorname{union}\{e_2, e_3\} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{2,5\}\}$$

and

$$\operatorname{union}\{e_1, e_2, e_4\} = \{\{1,3\}, \{2,5\}, \{1,2\}, \{2,3\}, \{3,4\}\}$$

and $\operatorname{union}\{\} = \varnothing$.

We can use this notion to state our first result about ambigraphs – an analogue to Theorem 1.8. We shall prove this result at the end of the next subsection.

Theorem 5.13. Let $G = (V, E, \varphi)$ be a finite ambigraph. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda(V, \operatorname{union} F)$ is understood according to Definition 1.7 **(b)**.)

5.4. Circuits and broken circuits

Let us now define the notions of cycles, circuits and broken circuits of an ambigraph.

Definition 5.14. Let $G = (V, E, \varphi)$ be an ambigraph. A *cycle* of G denotes a list

$$(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$$

with the following properties:

- The entries $v_1, v_2, \ldots, v_{m+1}$ at the odd positions of this list belong to V, whereas the entries $e_1, e_2, \ldots, e_m$ at its even positions belong to E.
- We have $m \geq 1$.
- We have $v_{m+1} = v_1$.
- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- The edgeries $e_1, e_2, \ldots, e_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in \varphi(e_i)$ for every $i \in \{1, 2, \ldots, m\}$.

If $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$ is a cycle of G, then the set $\{e_1, e_2, \ldots, e_m\}$ is called a *circuit* of G.

Example 5.15. Let $G = (V, E, \varphi)$ be the ambigraph from Example 5.2. Then, the tuple

$$(1, e_2, 2, e_6, 3, e_4, 1)$$

is a cycle of G (chiefly because $\{1, 2\} \in \varphi(e_2)$ and $\{2, 3\} \in \varphi(e_6)$ and $\{3, 1\} \in \varphi(e_4)$). The circuit corresponding to this cycle is $\{e_2, e_6, e_4\}$.

The tuple $(1, e_2, 2, e_6, 3, e_1, 1)$ is a cycle of G as well, and leads to the circuit $\{e_2, e_6, e_1\}$.

For comparison, the similar-looking tuple $(1, e_2, 2, e_2, 3, e_4, 1)$ is not a cycle, since its edgeries e_2, e_2, e_4 are not distinct.

Definition 5.16. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. We shall refer to ℓ as the *labeling function*. For every edgery e of G, we shall refer to $\ell(e)$ as the *label* of e.

A *broken circuit* of G means a subset of E having the form $C \setminus \{e\}$, where C is a circuit of G, and where e is the unique singleton edgery in C having maximum label (among the singleton edgeries in C). Of course, the notion of a broken circuit of G depends on the function ℓ; however, we suppress the mention of ℓ in our notation, since we will not consider situations where two different ℓ's coexist.

Thus, if G is an ambigraph with a labeling function ℓ, then any circuit C of G gives rise to a broken circuit provided that

- at least one edgery in C is singleton, and
- among the singleton edgeries in C, only one attains the maximum label.

In all other cases, C does not give rise to a broken circuit. Notice that two different circuits may give rise to one and the same broken circuit.

Example 5.17. **(a)** Let $G = (V, E, \varphi)$ be the ambigraph from Example 5.2. Let X and $\ell : E \to X$ be arbitrary. Then, the circuit $\{e_2, e_6, e_4\}$ we found in Example 5.15 does not give rise to a broken circuit, since it contains no singleton edgery. However, the circuit $\{e_3, e_1\}$ (coming from the cycle $(2, e_3, 5, e_1, 2)$ does give rise to a broken circuit (namely, $\{e_1\}$), since its unique singleton edgery is e_3.

(b) For better examples, we can try an ambigraph having more singleton edgeries. For instance, we can choose some graph G and consider the corresponding ambigraph G^{amb} as defined in Remark 5.6. Then, the broken circuits of G^{amb} are precisely the broken circuits of G.

(c) Here is another example: Let $G = (V, E, \varphi)$ be the ambigraph with $V = \{1, 2, 3, 4\}$, $E = \{e_1, e_2, e_3\}$ and

$$\varphi(e_1) = \{\{1,2\}\}, \qquad \varphi(e_2) = \{\{2,3\}\}, \qquad \varphi(e_3) = \{\{3,4\}, \{1,3\}\}.$$

Let X and $\ell : E \to X$ be arbitrary. Then, the cycle $(1, e_1, 2, e_2, 3, e_3, 1)$ of G gives rise to the circuit $\{e_1, e_2, e_3\}$. This circuit gives rise to

- the broken circuit $\{e_2, e_3\}$ if $\ell(e_1) > \ell(e_2)$;
- the broken circuit $\{e_1, e_3\}$ if $\ell(e_1) < \ell(e_2)$;
- no broken circuit if $\ell(e_1) = \ell(e_2)$.

Note that $\ell(e_3)$ does not matter, since the edgery e_3 is not singleton.

The notion of a broken circuit always depends on a labeling function $\ell : E \to X$. Any time we speak about broken circuits, we shall tacitly understand that the function $\ell : E \to X$ is used as the labeling function.

5.5. The main results for ambigraphs

We can now generalize Theorem 1.12 to ambigraphs:

Theorem 5.18. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$X_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) p_{\lambda(V, \operatorname{union} F)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda(V, \operatorname{union} F)$ is understood according to Definition 1.7 **(b)**.)

This theorem generalizes Theorem 1.12 (in fact, the latter is easily obtained by applying the former to G^{amb} instead of G). Before we prove it, let us first explore some particular cases. Using Definition 1.13, we can obtain the following consequences of Theorem 1.12:

Corollary 5.19. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

Corollary 5.20. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$X_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda(V, \operatorname{union} F)}.$$

5.6. Proofs

Our proof of Theorem 5.18 is mostly similar to our above proof of Theorem 1.12, but there are some complications due to the possibility of non-singleton edgeries.

We shall use the Iverson bracket notation (Definition 2.7). We begin with a basic cancellation lemma (see, e.g., [Grinbe20, Proposition 7.8.10]):

Lemma 5.21. Let S be a finite set. Then, $\sum_{I \subseteq S} (-1)^{|I|} = [S = \varnothing]$.

In Definition 2.1, we defined a set $\operatorname{Eqs} f$ for any map $f : V \to X$. This set helped us find the edges of a graph whose endpoints received the same color under a coloring f. We shall now introduce a similar notion for ambigraphs:

Definition 5.22. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. We let $\operatorname{EQS}(G, f)$ denote the subset

$$\{e \in E \mid \varphi(e) \subseteq \operatorname{Eqs} f\}$$

of E.

Example 5.23. Let $G = (V, E, \varphi)$ be the ambigraph with $V = \{1, 2, 3, 4, 5, 6\}$ and $E = \{a, b, c\}$ and

$$\begin{aligned}
\varphi(a) &= \{\{1,3\}, \{2,4\}, \{3,6\}\}, \\
\varphi(b) &= \{\{1,3\}, \{2,6\}\}, \\
\varphi(c) &= \varnothing.
\end{aligned}$$

Let $X = \mathbb{N}$, and let $f : V \to X$ be the map that sends $1, 2, 3, 4, 5, 6$ to $1, 2, 1, 2, 1, 2$, respectively. Then,

$$\operatorname{Eqs} f = \{\{1,3\}, \{1,5\}, \{3,5\}, \{2,4\}, \{2,6\}, \{4,6\}\}$$

and $\operatorname{EQS}(G, f) = \{b, c\}$. Indeed, we have $b \in \operatorname{EQS}(G, f)$ since $\varphi(b) = \{\{1,3\}, \{2,6\}\} \subseteq \operatorname{Eqs} f$, and we have $c \in \operatorname{EQS}(G, f)$ since $\varphi(c) = \varnothing \subseteq \operatorname{Eqs} f$. On the other hand, $a \notin \operatorname{EQS}(G, f)$ since $\varphi(a) \not\subseteq \operatorname{Eqs} f$ (because $\{3, 6\}$ belongs to $\varphi(a)$ but not to $\operatorname{Eqs} f$).

Remark 5.24. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. The definition of $\operatorname{EQS}(G, f)$ yields

$$\begin{aligned}
\operatorname{EQS}(G, f) &= \{e \in E \mid \varphi(e) \subseteq \operatorname{Eqs} f\} && (19) \\
&= \{d \in E \mid \varphi(d) \subseteq \operatorname{Eqs} f\} && (20) \\
&\qquad \text{(here, we have renamed the index } e \text{ as } d\text{)} \\
&= \{d \in E \mid \text{no edge of } d \text{ is } f\text{-dichromatic}\}. && (21)
\end{aligned}$$

(The last equality is easy to check from the definitions.)

In analogy to Lemma 2.3, we can use $\operatorname{EQS}(G, f)$ to characterize when an X-coloring f is proper:

Lemma 5.25. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. Then, the X-coloring f of G is proper if and only if $\operatorname{EQS}(G, f) = \varnothing$.

Proof of Lemma 5.25. An exercise in unfolding definitions and applying de Morgan's laws. $\square$

The following simple lemma connects $\operatorname{EQS}(G, f)$ with the union F construction from Definition 5.11:

Lemma 5.26. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. Let B be a subset of E. Then, $B \subseteq \operatorname{EQS}(G, f)$ holds if and only if $\operatorname{union} B \subseteq \operatorname{Eqs} f$.

Proof of Lemma 5.26. The definition of $\operatorname{union} B$ yields $\operatorname{union} B = \bigcup_{e \in B} \varphi(e)$. Hence, we have the following chain of logical equivalences:

$$\begin{aligned} & (\operatorname{union} B \subseteq \operatorname{Eqs} f) \\ \iff & \left(\bigcup_{e \in B} \varphi(e) \subseteq \operatorname{Eqs} f \right) \\ \iff & (\varphi(e) \subseteq \operatorname{Eqs} f \text{ for each } e \in B). \end{aligned} \tag{22}$$

However, for an edgery $e \in E$, the condition $\varphi(e) \subseteq \operatorname{Eqs} f$ is equivalent to $e \in \operatorname{EQS}(G, f)$ (by the definition of $\operatorname{EQS}(G, f)$). Hence, we can rewrite the equivalence (22) as follows:

$$\begin{aligned} (\operatorname{union} B \subseteq \operatorname{Eqs} f) & \iff (e \in \operatorname{EQS}(G, f) \text{ for each } e \in B) \\ & \iff (B \subseteq \operatorname{EQS}(G, f)). \end{aligned}$$

In other words, $B \subseteq \operatorname{EQS}(G, f)$ holds if and only if $\operatorname{union} B \subseteq \operatorname{Eqs} f$. This proves Lemma 5.26. □

Next, let us show an analogue of Lemma 2.4:

Lemma 5.27. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a set. Let $f : V \to X$ be a map. Let C be a circuit of G. Let $e \in C$ be a singleton edgery such that $C \setminus \{e\} \subseteq \operatorname{EQS}(G, f)$. Then, $e \in \operatorname{EQS}(G, f)$.

Proof of Lemma 5.27. We have assumed that e is singleton. In other words, e has exactly one edge. In other words, $\varphi(e)$ is a 1-element set.

The set C is a circuit of G. Hence, we can write C in the form

$$C = \{e_1, e_2, \ldots, e_m\}$$

for some cycle $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$ of G. Consider this cycle $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$. According to the definition of a "cycle", the cycle $(v_1, e_1, v_2, e_2, \ldots, v_m, e_m, v_{m+1})$ is a list having the following properties:

- The entries $v_1, v_2, \ldots, v_{m+1}$ at the odd positions of this list belong to V, whereas the entries $e_1, e_2, \ldots, e_m$ at its even positions belong to E.
- We have $m \geq 1$.
- We have $v_{m+1} = v_1$.

- The vertices $v_1, v_2, \ldots, v_m$ are pairwise distinct.
- The edgeries $e_1, e_2, \ldots, e_m$ are pairwise distinct.
- We have $\{v_i, v_{i+1}\} \in \varphi(e_i)$ for every $i \in \{1, 2, \ldots, m\}$.

Recall that $e \in C = \{e_1, e_2, \ldots, e_m\}$. We can thus WLOG assume that $e = e_m$ (since otherwise, we can simply cyclically relabel the vertices and the edgeries along our cycle). Assume this. Since $C = \{e_1, e_2, \ldots, e_m\}$ and $e = e_m$, we have

$$C \setminus \{e\} = \{e_1, e_2, \ldots, e_{m-1}\}$$

(since the m edgeries $e_1, e_2, \ldots, e_m$ are distinct). For every $i \in \{1, 2, \ldots, m-1\}$, we thus have $e_i \in C \setminus \{e\} \subseteq \operatorname{EQS}(G, f)$ and therefore $\varphi(e_i) \subseteq \operatorname{Eqs} f$ (by the definition of $\operatorname{EQS}(G, f)$), so that $\{v_i, v_{i+1}\} \in \varphi(e_i) \subseteq \operatorname{Eqs} f$ and therefore $f(v_i) = f(v_{i+1})$ (by the definition of $\operatorname{Eqs} f$). Hence, $f(v_1) = f(v_2) = \cdots = f(v_m)$, so that $f(v_m) = f(v_1)$. Thus, $\{v_m, v_1\} \in \operatorname{Eqs} f$.

However, recall again that $\{v_i, v_{i+1}\} \in \varphi(e_i)$ for every $i \in \{1, 2, \ldots, m\}$. Applying this to $i = m$, we obtain $\{v_m, v_{m+1}\} \in \varphi(e_m)$. Since $v_{m+1} = v_1$ and $e_m = e$, we can rewrite this as $\{v_m, v_1\} \in \varphi(e)$. Since $\varphi(e)$ is a 1-element set, this entails that $\varphi(e) = \{\{v_m, v_1\}\} \subseteq \operatorname{Eqs} f$ (since $\{v_m, v_1\} \in \operatorname{Eqs} f$). In other words, $e \in \operatorname{EQS}(G, f)$ (by the definition of $\operatorname{EQS}(G, f)$). This proves Lemma 5.27. $\square$

Our next lemma will play a role in our proof of Theorem 5.18 that is similar to the role of Lemma 2.6 in the proof of Theorem 1.12 (although it is different in its claim).

Lemma 5.28. Let $G = (V, E, \varphi)$ be an ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function.

Let Y be any set. Let $f : V \to Y$ be any map. Assume that the set $\operatorname{EQS}(G, f)$ contains no singleton edgery. Then, there exists no broken circuit K of G satisfying $K \subseteq \operatorname{EQS}(G, f)$.

Proof of Lemma 5.28. Assume the contrary. Thus, there exists some broken circuit K of G satisfying $K \subseteq \operatorname{EQS}(G, f)$.

The set K is a broken circuit of G. According to the definition of a broken circuit, this means that K can be written as $K = C \setminus \{e\}$, where C is a circuit of G, and where e is the unique singleton edgery in C having maximum label (among the singleton edgeries in C). Consider these C and e.

We have $C \setminus \{e\} = K \subseteq \operatorname{EQS}(G, f)$. Thus, Lemma 5.27 (applied to Y instead of X) yields that $e \in \operatorname{EQS}(G, f)$. Thus, the set $\operatorname{EQS}(G, f)$ contains a singleton edgery (namely, e). But this contradicts the fact that the set $\operatorname{EQS}(G, f)$ contains no singleton edgery.

This contradiction shows that our assumption was false. Hence, Lemma 5.28 is proven. $\square$

We are now ready to prove the keystone lemma, which of course is an analogue of Lemma 2.8:

Lemma 5.29. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$.

Let Y be any set. Let $f : V \to Y$ be any map. Then,

$$\sum_{B \subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = [\operatorname{EQS}(G, f) = \varnothing].$$

Proof of Lemma 5.29. We are in one of the following two cases:

Case 1: The set $\operatorname{EQS}(G, f)$ contains no singleton edgery.

Case 2: The set $\operatorname{EQS}(G, f)$ contains at least one singleton edgery.

Let us first consider Case 1. In this case, the set $\operatorname{EQS}(G, f)$ contains no singleton edgery. Hence, using Lemma 5.28, it is easy to see that every subset B of $\operatorname{EQS}(G, f)$ satisfies

$$\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = 1. \tag{23}$$

Proof of (23): Let B be a subset of $\operatorname{EQS}(G, f)$. Then, Lemma 5.28 yields that there exists no broken circuit K of G satisfying $K \subseteq \operatorname{EQS}(G, f)$. Hence, there exists no $K \in \mathfrak{K}$ satisfying $K \subseteq \operatorname{EQS}(G, f)$ (since each $K \in \mathfrak{K}$ is a broken circuit of G). Therefore, there exists no $K \in \mathfrak{K}$ satisfying $K \subseteq B$ either (since $K \subseteq B$ would entail $K \subseteq B \subseteq \operatorname{EQS}(G, f)$, which would contradict the previous sentence). Hence, the product $\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K$ is empty, and thus equals 1 by definition. This proves (23).

Now,

$$\sum_{B \subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \underbrace{\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K}_{\substack{=1 \\ \text{(by (23))}}} = \sum_{B \subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} = \sum_{I \subseteq \operatorname{EQS}(G,f)} (-1)^{|I|}$$
$$= [\operatorname{EQS}(G, f) = \varnothing]$$

(by Lemma 5.21, applied to $S = \operatorname{EQS}(G, f)$). Thus, Lemma 5.29 is proved in Case 1.

Let us now consider Case 2. In this case, the set $\operatorname{EQS}(G, f)$ contains at least one singleton edgery. Pick any such singleton edgery $d \in \operatorname{EQS}(G, f)$ with maximum $d \in \operatorname{EQS}(G, f)$ (among all such singleton edgeries).

Define two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{P}(\operatorname{EQS}(G,f))$ as follows:

$$\mathcal{U} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \notin F\};$$
$$\mathcal{V} = \{F \in \mathcal{P}(\operatorname{EQS}(G,f)) \mid d \in F\}.$$

Thus, we have $\mathcal{P}(\operatorname{EQS}(G,f)) = \mathcal{U} \cup \mathcal{V}$, and the sets $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Now, we define a map $\Phi : \mathcal{U} \to \mathcal{V}$ by

$$(\Phi(B) = B \cup \{d\} \qquad \text{for every } B \in \mathcal{U}).$$

As in our above proof of Lemma 2.8, we can see that this map Φ is well-defined and a bijection, and that every $B \in \mathcal{U}$ satisfies

$$(-1)^{|\Phi(B)|} = -(-1)^{|B|}. \tag{24}$$

Furthermore, we claim that, for every $B \in \mathcal{U}$ and every $K \in \mathfrak{K}$, we have the following logical equivalence:

$$(K \subseteq B) \iff (K \subseteq \Phi(B)). \tag{25}$$

Indeed, our above proof of (10) can be easily transformed into a proof of (25) by some simple changes[22]. Thus, (25) holds.

Now, just as in the proof of Lemma 2.8, we can conclude that

$$\sum_{B \subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = 0. \tag{26}$$

However, the set $\operatorname{EQS}(G,f)$ contains at least one singleton edgery, and thus is nonempty. Hence, $[\operatorname{EQS}(G,f) = \varnothing] = 0$. Comparing this with (26), we obtain $\sum\limits_{B \subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \prod\limits_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = [\operatorname{EQS}(G,f) = \varnothing]$. Thus, Lemma 5.29 is proved in Case 2.

We have now proved Lemma 5.29 in both Cases 1 and 2. Since these two cases cover all possibilities, we thus have proved Lemma 5.29. $\square$

We are now ready to prove Theorem 5.18 and Corollaries 5.19 and 5.20 as well as Theorem 5.13:

[22]Here are the changes that we need to make to the above proof: We need to replace "edge" by "singleton edgery"; replace both sets $E \cap \operatorname{Eqs} f$ and $\operatorname{Eqs} f$ by $\operatorname{EQS}(G,f)$; and replace the reference to Lemma 2.4 by a reference to Lemma 5.27.

Proof of Theorem 5.18. The definition of X_G shows that

$$\begin{aligned}
X_G &= \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \left[\underbrace{f \text{ is a proper } \mathbb{N}_+\text{-coloring of } G}_{\substack{\iff (\text{the } \mathbb{N}_+\text{-coloring } f \text{ of } G \text{ is proper}) \\ \iff (\operatorname{EQS}(G,f)=\varnothing) \\ (\text{by Lemma 5.25, applied to } \mathbb{N}_+ \text{ instead of } X)}} \right] \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \underbrace{[\operatorname{EQS}(G,f)=\varnothing]}_{\substack{=\sum\limits_{B\subseteq \operatorname{EQS}(G,f)} (-1)^{|B|} \prod\limits_{\substack{K\in\mathfrak{K}; \\ K\subseteq B}} a_K \\ (\text{by Lemma 5.29, applied to } Y=\mathbb{N}_+)}} \mathbf{x}_f \\
&= \sum_{f:V\to\mathbb{N}_+} \underbrace{\sum_{B\subseteq \operatorname{EQS}(G,f)}}_{\substack{=\sum\limits_{\substack{B\subseteq E; \\ B\subseteq \operatorname{EQS}(G,f)}} \\ (\text{since } \operatorname{EQS}(G,f) \text{ is a subset of } E)}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K}; \\ K\subseteq B}} a_K \right) \mathbf{x}_f \\
&= \underbrace{\sum_{f:V\to\mathbb{N}_+} \sum_{\substack{B\subseteq E; \\ B\subseteq \operatorname{EQS}(G,f)}}}_{=\sum\limits_{B\subseteq E} \sum\limits_{\substack{f:V\to\mathbb{N}_+; \\ B\subseteq \operatorname{EQS}(G,f)}}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K}; \\ K\subseteq B}} a_K \right) \mathbf{x}_f \\
&= \sum_{B\subseteq E} \sum_{\substack{f:V\to\mathbb{N}_+; \\ B\subseteq \operatorname{EQS}(G,f)}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K}; \\ K\subseteq B}} a_K \right) \mathbf{x}_f
\end{aligned}$$

$$= \sum_{B \subseteq E} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{EQS}(G,f)}} \mathbf{x}_f}_{\substack{=\sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} \\ \text{(since Lemma 5.26} \\ \text{(applied to } X=\mathbb{N}_+\text{) yields that} \\ \text{the condition } B\subseteq \operatorname{EQS}(G,f) \text{ (on} \\ \text{a map } f:V\to\mathbb{N}_+\text{) is} \\ \text{equivalent to } \operatorname{union} B\subseteq \operatorname{Eqs} f\text{)}}}$$

$$= \sum_{B \subseteq E} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \right) \underbrace{\sum_{\substack{f:V\to\mathbb{N}_+;\\ \operatorname{union} B\subseteq \operatorname{Eqs} f}} \mathbf{x}_f}_{\substack{=p_{\lambda(V,\operatorname{union} B)} \\ \text{(by Lemma 2.5} \\ \text{(applied to } \operatorname{union} B \text{ instead of } B\text{))}}}$$

$$= \sum_{B \subseteq E} (-1)^{|B|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K \right) p_{\lambda(V,\operatorname{union} B)} = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq F}} a_K \right) p_{\lambda(V,\operatorname{union} F)}$$

(here, we have renamed the summation index B as F). This proves Theorem 5.18. ☐

Proof of Corollary 5.19. Analogous to the proof of Corollary 1.14. ☐

Proof of Corollary 5.20. Corollary 5.20 follows from Corollary 5.19 when $\mathfrak{K}$ is set to be the set of **all** broken circuits of G. ☐

Proof of Theorem 5.13. This follows from Theorem 5.18 in the same way as Theorem 1.8 follows from Theorem 1.12. ☐

5.7. The chromatic polynomial

We have thus proved analogues of Theorems 1.8 and 1.12 and Corollaries 1.14 and 1.15 for ambigraphs. We can just as easily prove analogues of Theorem 3.1, Definition 3.2, Theorems 3.4 and 3.5 and Corollaries 3.6 and 3.7. Here they are, in the order in which we have just mentioned them:

Theorem 5.30. Let $G = (V, E, \varphi)$ be a finite ambigraph. Then, there exists a unique polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies

$$P(q) = (\text{the number of all proper } \{1, 2, \ldots, q\}\text{-colorings of } G).$$

Definition 5.31. Let $G = (V, E, \varphi)$ be a finite ambigraph. Theorem 5.30 shows that there exists a polynomial $P \in \mathbb{Z}[x]$ such that every $q \in \mathbb{N}$ satisfies $P(q) =$ (the number of all proper $\{1, 2, \ldots, q\}$-colorings of G). This polynomial P is called the *chromatic polynomial* of G, and will be denoted by χ_G.

Theorem 5.32. Let $G = (V, E, \varphi)$ be a finite ambigraph. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{union} F)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\operatorname{conn}(V, \operatorname{union} F)$ is understood according to Definition 3.3.)

Theorem 5.33. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$\chi_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{\operatorname{conn}(V, \operatorname{union} F)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\operatorname{conn}(V, \operatorname{union} F)$ is understood according to Definition 3.3.)

Corollary 5.34. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{union} F)}.$$

Corollary 5.35. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$\chi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} x^{\operatorname{conn}(V, \operatorname{union} F)}.$$

The proofs of all these results are analogous to the proofs of the corresponding results from Section 3, so we leave them all to the reader.

One may reasonably wonder whether Corollary 3.14 has an analogue for ambigraphs as well, i.e., whether one can replace the exponent $\operatorname{conn}(V, \operatorname{union} F)$ in Corollary 5.35 by something simpler when ℓ is injective. In the case of a hypergraph, Dohmen has obtained such a result ([Dohmen95, Theorem 2.1]) under the additional condition that each cycle of G have at least one singleton edgery. Unfortunately, for ambigraphs, such a simplification does not appear possible (even under a condition like Dohmen's).

6. Weighted and noncommutative versions

In the recent decades, the chromatic symmetric function of a graph has been generalized in several directions. Two of them are the *chromatic symmetric function of a weighted graph* as defined by Crew and Spirkl ([CreSpi19, §3]), and the *noncommutative chromatic symmetric function* of Gebhard and Sagan ([GebSag01, §3]). In this section, we will recall the definitions of both of these generalizations, and extend our results to them. (The extensions will be fairly mechanical, as all the hard work has already been done.)

6.1. Weighted graphs and their chromatic symmetric functions

For us, a *weighted graph* will just mean a pair consisting of a graph $G = (V, E)$ and a weight function on V. Weight functions are defined as follows:

Definition 6.1. Let V be a set. A *weight function* on V means a function $w : V \to \mathbb{N}_+$. If $w : V \to \mathbb{N}_+$ is a weight function on V, then the *weight* of an element $v \in V$ is defined to be the positive integer $w(v) \in \mathbb{N}_+$.

Thus, a weight function on a set V just assigns a "weight" (a positive integer) to each element of V. Given such a weight function for a graph $G = (V, E)$, we can define a "weighted chromatic symmetric function":

Definition 6.2. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{x}_{f,w}$ denote the monomial $\prod_{v \in V} x_{f(v)}^{w(v)}$ in the indeterminates $x_1, x_2, x_3, \ldots$.

(b) We define a power series $X_{G,w} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$X_{G,w} = \sum_{\substack{f : V \to \mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,w}.$$

This power series $X_{G,w}$ is called the *chromatic symmetric function* of (G, w).

This chromatic symmetric function $X_{G,w}$ was introduced by Crew and Spirkl in [CreSpi19, (1)] (where it was denoted $X_{(G,w)}$). It generalizes the original chromatic symmetric function X_G, which is obtained when all the weights $w(v)$ are 1:

Example 6.3. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be the weight function that sends each $v \in V$ to 1. Then, for every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we have

$$\mathbf{x}_{f,w} = \prod_{v \in V} \underbrace{x_{f(v)}^{w(v)}}_{\substack{=x_{f(v)} \\ \text{(since } w(v)=1 \\ \text{(by the definition of } w))}} \qquad \text{(by Definition 6.2 (a))}$$
$$= \prod_{v \in V} x_{f(v)} = \mathbf{x}_f \tag{27}$$

(see Definition 1.5 **(a)** for the definition of $\mathbf{x}_f$). Thus, Definition 6.2 **(b)** yields

$$X_{G,w} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \underbrace{\mathbf{x}_{f,w}}_{\substack{=\mathbf{x}_f \\ \text{(by (27))}}} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_f = X_G$$

(by Definition 1.5 **(b)**).

6.2. The weight of a subset and the partition $\lambda(G, w)$

To state Whitney-like formulas for $X_{G,w}$, we need to adapt the partition $\lambda(G)$ defined in Definition 1.7 **(b)** to the case of a weighted graph. This adaptation consists in replacing the size of each connected component by its *weight*. Here, the weight of a subset of V is defined as follows:

Definition 6.4. Let V be a finite set. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let S be a subset of V. Then, the *weight* of S means the nonnegative integer $\sum_{v \in S} w(v)$. This weight is denoted by $w(S)$.

Note that if the subset S in this definition is nonempty, then its weight $w(S)$ is a positive integer, since it is defined as the nonempty sum $\sum_{v \in S} w(v)$ of the positive weights $w(v)$.

For example, $w(\{2, 5, 6\}) = w(2) + w(5) + w(6)$ (if $\{2, 5, 6\}$ is a subset of V).

Now, we can define the analogue of the partition $\lambda(G)$ for a weighted graph:

Definition 6.5. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. We let $\lambda(G, w)$ denote the list of the weights of all connected components of G, in weakly decreasing order. (Each connected component should contribute only one entry to the list.) We view $\lambda(G, w)$ as a partition (since $\lambda(G, w)$ is a weakly decreasing finite list of positive integers).

Example 6.6. Let $G = (V, E)$ be the finite graph with $V = \{1, 2, 3, 4, 5\}$ and $E = \{\{1, 3\}, \{2, 4\}, \{2, 5\}\}$. Then, the connected components of G are $A = \{1, 3\}$ and $B = \{2, 4, 5\}$.

Let $w : V \to \mathbb{N}_+$ be the weight function given by $w(1) = 6$ and $w(2) = 9$ and $w(3) = 6$ and $w(4) = 1$ and $w(5) = 3$. Then, the weights of the connected components A and B are

$$w(A) = w(\{1, 3\}) = w(1) + w(3) = 6 + 6 = 12 \qquad \text{and}$$
$$w(B) = w(\{2, 4, 5\}) = w(2) + w(4) + w(5) = 9 + 1 + 3 = 13.$$

Hence, the partition $\lambda(G, w)$ is the list of these two weights 12 and 13, in weakly decreasing order. In other words, $\lambda(G, w) = (13, 12)$.

6.3. Formulas for $X_{G,w}$

We are now ready to state analogues of Theorems 1.8 and 1.12 and Corollaries 1.14 and 1.15 for weighted graphs:

Theorem 6.7. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Then,

$$X_{G,w} = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda((V,F),w)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda((V, F), w)$ is understood according to Definition 6.5.)

Theorem 6.8. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$X_{G,w} = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) p_{\lambda((V,F),w)}.$$

(Here, of course, the pair (V, F) is regarded as a graph, and the expression $\lambda((V, F), w)$ is understood according to Definition 6.5.)

Corollary 6.9. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_{G,w} = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda((V,F),w)}.$$

Corollary 6.10. Let $G = (V, E)$ be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$X_{G,w} = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda((V,F),w)}.$$

Note that Theorem 6.7 is a result by Crew and Spirkl (namely, [CreSpi19, Lemma 3]).

6.4. Proofs

In this section, we shall prove Theorem 6.8, Corollary 6.9, Corollary 6.10 and Theorem 6.7. This will be fairly easy, as many of our above lemmas can be reused without any change. However, we need the following weighted version of Lemma 2.5:

Lemma 6.11. Let (V, B) be a finite graph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Then,

$$\sum_{\substack{f : V \to \mathbb{N}_+; \\ B \subseteq \operatorname{Eqs} f}} \mathbf{x}_{f,w} = p_{\lambda((V,B),w)}.$$

(Here, $\mathbf{x}_{f,w}$ is defined as in Definition 6.2 **(a)**, and the expression $\lambda((V, B), w)$ is understood according to Definition 6.5.)

Proof of Lemma 6.11. This is almost completely analogous to the proof of Lemma 2.5 that we gave long ago. (Replace each size $|C_i|$ by the weight $w(C_i)$; replace all monomials $\mathbf{x}_g$ by their weighted analogues $\mathbf{x}_{g,w}$; and of course, replace $\lambda(V, B)$ by $\lambda((V, B), w)$.) $\square$

It is now straightforward to adapt our above proofs of Theorem 1.12, Corollary 1.14, Corollary 1.15 and Theorem 1.8 to obtain proofs of Theorem 6.8, Corollary 6.9, Corollary 6.10 and Theorem 6.7. Of course, Lemma 6.11 needs to be used instead of Lemma 2.5, but everything else stays almost completely unchanged. We leave the details to the reader.

6.5. Ambigraphs redux

Just as we have imposed weights on the vertices of a graph, we can do the same to the vertices of an ambigraph. This leads to a generalization of the chromatic symmetric function X_G we introduced in Definition 5.8:

Definition 6.12. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{x}_{f,w}$ denote the monomial $\prod_{v \in V} x_{f(v)}^{w(v)}$ in the indeterminates $x_1, x_2, x_3, \ldots$.

(b) We define a power series $X_{G,w} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ by

$$X_{G,w} = \sum_{\substack{f: V \to \mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{x}_{f,w}.$$

This power series $X_{G,w}$ is called the *chromatic symmetric function* of (G, w).

We can now state generalizations of Theorems 5.13 and 5.18 and Corollaries 5.19 and 5.20:

Theorem 6.13. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Then,

$$X_{G,w} = \sum_{F \subseteq E} (-1)^{|F|} p_{\lambda((V, \operatorname{union} F), w)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda((V, \operatorname{union} F), w)$ is understood according to Definition 6.5.)

Theorem 6.14. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$X_{G,w} = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) p_{\lambda((V, \operatorname{union} F), w)}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda((V, \operatorname{union} F), w)$ is understood according to Definition 6.5.)

Corollary 6.15. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$X_{G,w} = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} p_{\lambda((V, \operatorname{union} F), w)}.$$

Corollary 6.16. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $w : V \to \mathbb{N}_+$ be a weight function on V. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$X_{G,w} = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} p_{\lambda((V, \operatorname{union} F), w)}.$$

The proofs of these four results proceed precisely like their non-weighted counterparts, again using Lemma 6.11 instead of Lemma 2.5.

6.6. Noncommutative chromatic symmetric functions

The *noncommutative chromatic symmetric function* Y_G of a graph G has been introduced by Gebhard and Sagan in [GebSag01, §3] as a lift of the chromatic symmetric function X_G to a noncommutative polynomial ring. In order to define it, we need to lift the monomials $\mathbf{x}_f$ to noncommutative monomials, which requires fixing a list of the vertices of G (since a noncommutative product is only defined if its factors appear in a chosen order). Unlike [GebSag01], we shall not require this list to contain each vertex of G exactly once; thus, we obtain a more general notion that refines not only Stanley's original X_G but also its weighted version $X_{G,w}$ discussed above.

We consider the $\mathbf{k}$-algebra $\mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$ of noncommutative power series in countably many distinct indeterminates $X_1, X_2, X_3, \ldots$ over $\mathbf{k}$. It is a topological $\mathbf{k}$-algebra[23]. A noncommutative power series $P \in \mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$ is said to be *bounded-degree* if there exists an $N \in \mathbb{N}$ such that every noncommutative monomial of degree $> N$ appears with coefficient 0 in P. A noncommutative power series $P \in \mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$ is said to be *symmetric* if and only if P is invariant under any permutation of the indeterminates. We let Λ_{NC} be the subset of $\mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$ consisting of all symmetric bounded-degree power series $P \in \mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$. This subset Λ_{NC} is a $\mathbf{k}$-subalgebra of $\mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$, and is called the $\mathbf{k}$*-algebra of symmetric functions in noncommutative indeterminates* over $\mathbf{k}$.

[23] Its topology is defined in the same way as the topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (but of course the monomials are now noncommutative monomials).

This $\mathbf{k}$-algebra Λ_{NC} is called $\Pi(\mathbf{x})$ in [RosSag04], and should not be mistaken for the algebra NSym of noncommutative symmetric functions (which is studied, e.g., in [GriRei14, §5.4]).[24]

We can now define noncommutative chromatic symmetric functions of ambigraphs:

Definition 6.17. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once.

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we let $\mathbf{X}_{f,\mathbf{t}}$ denote the noncommutative monomial $X_{f(t_1)} X_{f(t_2)} \cdots X_{f(t_N)}$ in the indeterminates $X_1, X_2, X_3, \ldots$.

(b) We define a noncommutative power series $Y_{G,\mathbf{t}} \in \mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$ by

$$Y_{G,\mathbf{t}} = \sum_{\substack{f:V\to\mathbb{N}_+ \text{ is a} \\ \text{proper } \mathbb{N}_+\text{-coloring of } G}} \mathbf{X}_{f,\mathbf{t}}. \tag{28}$$

This power series $Y_{G,\mathbf{t}}$ is called the *noncommutative chromatic symmetric function* of $(G, \mathbf{t})$.

Remark 6.18. Why did we require the list $\mathbf{t}$ to contain each element of V at least once in Definition 6.17?

Otherwise, there could be a vertex $v \in V$ that does not appear in $\mathbf{t}$. In that case, the monomials $\mathbf{X}_{f,\mathbf{t}}$ would be independent of the color $f(v)$ of this vertex, and thus we would obtain the same monomial $\mathbf{X}_{f,\mathbf{t}}$ for infinitely many different proper $\mathbb{N}_+$-colorings f (since we could arbitrarily change the color $f(v)$ without affecting the monomial $\mathbf{X}_{f,\mathbf{t}}$). Hence, the sum on the right hand side of (28) would contain equally many identical monomials, and this would render $Y_{G,\mathbf{t}}$ undefined.

The noncommutative chromatic symmetric function $Y_{G,\mathbf{t}}$ is a lift of the weighted chromatic symmetric function $X_{G,w}$ introduced in Definition 6.12 **(b)**. This can be made precise as follows:

Remark 6.19. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Let $w : V \to \mathbb{N}_+$ be the weight function on V that is defined by

$$\begin{aligned} w(v) &= (\text{number of times that } v \text{ appears in the list } \mathbf{t}) \\ &= (\text{number of all } i \in \{1, 2, \ldots, N\} \text{ such that } t_i = v) \qquad \text{for each } v \in V. \end{aligned}$$

Let $\pi : \mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle \to \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ be the topological $\mathbf{k}$-algebra homomorphism that sends the noncommuting indeterminates $X_1, X_2, X_3, \ldots$ to the respective commuting indeterminates $x_1, x_2, x_3, \ldots$. Then:

[24] The latter algebra NSym can too be viewed as a subalgebra of $\mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$, but it does not consist of symmetric power series (despite its name).

(a) For every $\mathbb{N}_+$-coloring $f : V \to \mathbb{N}_+$ of G, we have $\pi\left(\mathbf{X}_{f,\mathbf{t}}\right) = \mathbf{x}_{f,w}$. (See Definition 6.12 **(a)** and Definition 6.17 **(a)** for the meanings of $\mathbf{x}_{f,w}$ and $\mathbf{X}_{f,t}$.)

(b) We have $\pi\left(Y_{G,\mathbf{t}}\right) = X_{G,w}$.

We omit the simple proofs of these claims.

Our goal is now to state noncommutative analogues of our main theorems for ambigraphs (specifically, Theorems 6.13 and 6.14 and Corollaries 6.15 and 6.16). To do so, we need a noncommutative analogue of the power-sum symmetric functions p_λ. In the commutative case, we defined p_λ as the product $\prod\limits_{i\geq 1} p_{\lambda_i}$ (see Definition 1.4). The noncommutative case, however, will not be such a product, so we need to define it differently. This will require some preparations.

We begin by recalling the notion of a set partition:

Definition 6.20. Let X be a set.

(a) A *set partition* of X means a set $\mathbf{P}$ of disjoint nonempty subsets of X such that $\bigcup\limits_{S\in\mathbf{P}} S = X$.

(b) If $\mathbf{P}$ is a set partition of X, then the sets $S \in \mathbf{P}$ are called the *blocks* of $\mathbf{P}$.

For example:

- The set $\{\{1,3\}, \{2,4,5\}\}$ is a set partition of the set $\{1,2,3,4,5\}$, since $\{1,3\}$ and $\{2,4,5\}$ are two disjoint nonempty subsets of $\{1,2,3,4,5\}$ whose union is $\{1,3\} \cup \{2,4,5\} = \{1,2,3,4,5\}$.
- The set $\{\{1,4\}, \{2,5\}, \{3,6\}\}$ is a set partition of the set $\{1,2,3,4,5,6\}$. The blocks of this set partition are $\{1,4\}$ and $\{2,5\}$ and $\{3,6\}$.
- The set $\{\{1,2,3,4,5\}\}$ is a set partition of the set $\{1,2,3,4,5\}$. It has only one block, namely $\{1,2,3,4,5\}$.
- The set $\{\{1\}, \{2\}, \{3\}, \{4\}, \{5\}\}$ is a set partition of the set $\{1,2,3,4,5\}$. It has five blocks, namely $\{1\}, \{2\}, \{3\}, \{4\}, \{5\}$.

There is a well-known relation (actually a one-to-one correspondence) between the set partitions of a given set X and the equivalence relations on X. It can be summarized in the following theorem:

Theorem 6.21. Let X be a set.

(a) If $\sim$ is an equivalence relation on X, then the set

$$X/(\sim) := \{\text{all } \sim \text{-equivalence classes}\}$$

is a set partition of X.

(b) If $\mathbf{P}$ is a set partition of X, then we can define an equivalence relation $\sim$ on the set X as follows: For any two elements a and b of X, we shall have

$a \sim b$ if and only if the elements a and b belong to the same block of $\mathbf{P}$. This relation $\sim$ will be called $\sim_{\mathbf{P}}$.

(c) The maps

$$\begin{aligned} \{\text{equivalence relations on } X\} &\to \{\text{set partitions of } X\}, \\ (\sim) &\mapsto X/(\sim) \end{aligned}$$

and

$$\begin{aligned} \{\text{set partitions of } X\} &\to \{\text{equivalence relations on } X\}, \\ \mathbf{P} &\mapsto (\sim_{\mathbf{P}}) \end{aligned}$$

are mutually inverse bijections.

Example 6.22. Let X be the set $\{1,2,3,4,5,6\}$.

(a) If $\sim$ is the equivalence relation on X given by

$$(a \sim b) \iff (a \equiv b \bmod 2),$$

then the corresponding set partition $X/(\sim)$ of X is $\{\{1,3,5\}, \{2,4,6\}\}$.

(b) If $\mathbf{P}$ is the set partition $\{\{1,2\}, \{3,4,6\}, \{5\}\}$ of X, then the corresponding equivalence relation $\sim_{\mathbf{P}}$ on X is given by

$$1 \sim_{\mathbf{P}} 2, \qquad 3 \sim_{\mathbf{P}} 4 \sim_{\mathbf{P}} 6$$

and no further relations (except, of course, for the ones that follow from the relations just given by reflexivity, symmetry and transitivity).

We can now define the noncommutative analogues of the power-sum symmetric functions p_λ. These are indexed not by integer partitions λ but by set partitions $\mathbf{P}$:

Definition 6.23. Let $N \in \mathbb{N}$. Let $\mathbf{P}$ be a set partition of the set $\{1,2,\ldots,N\}$. Recall the relation $\sim_{\mathbf{P}}$ defined in Theorem 6.21 **(b)**.

Then, we define a noncommutative power series $P_{\mathbf{P}} \in \mathbf{k}\langle\langle X_1, X_2, X_3, \ldots\rangle\rangle$ by

$$P_{\mathbf{P}} = \sum_{\substack{(i_1,i_2,\ldots,i_N)\in(\mathbb{N}_+)^N; \\ i_a=i_b \text{ whenever } a\sim_{\mathbf{P}} b}} X_{i_1} X_{i_2} \cdots X_{i_N}.$$

Here, the condition "$i_a = i_b$ whenever $a \sim_{\mathbf{P}} b$" under the summation sign is shorthand for "$i_a = i_b$ for any two elements $a, b \in \{1,2,\ldots,N\}$ that satisfy $a \sim_{\mathbf{P}} b$".

This power series $P_{\mathbf{P}}$ is called the *power-sum symmetric function in noncommutative variables* corresponding to the set partition $\mathbf{P}$. It is not hard to see that it belongs to Λ_{NC}.

Note that $P_{\mathbf{P}}$ is called $p_{\mathbf{P}}$ in [GebSag01, §2] and in [RosSag04].

Example 6.24. (a) If $\mathbf{P}$ is the set partition $\{\{1,3\}, \{2,4,5\}\}$ of $\{1,2,3,4,5\}$, then

$$\begin{aligned} P_{\mathbf{P}} &= \sum_{\substack{(i_1,i_2,i_3,i_4,i_5)\in(\mathbb{N}_+)^5;\\ i_a=i_b \text{ whenever } a\sim_{\mathbf{P}} b}} X_{i_1}X_{i_2}X_{i_3}X_{i_4}X_{i_5} \\ &= \sum_{\substack{(i_1,i_2,i_3,i_4,i_5)\in(\mathbb{N}_+)^5;\\ i_1=i_3 \text{ and } i_2=i_4=i_5}} X_{i_1}X_{i_2}X_{i_3}X_{i_4}X_{i_5} \\ &= \sum_{(u,v)\in(\mathbb{N}_+)^2} X_uX_vX_uX_vX_v \end{aligned}$$

(here, we have substituted (u,v,u,v,v) for the summation index (i_1,i_2,i_3,i_4,i_5), since the condition "$i_1 = i_3$ and $i_2 = i_4 = i_5$" is saying precisely that the 5-tuple (i_1,i_2,i_3,i_4,i_5) can be written in the form (u,v,u,v,v)). This is a noncommutative power series that contains terms such as $X_4X_7X_4X_7X_7$ or $X_5X_5X_5X_5X_5$ (we are allowed to have $u = v$ in the above sum), but not terms such as $X_1X_2X_2X_1X_1$ (since the indeterminates don't commute).

(b) If $\mathbf{P}$ is the set partition $\{\{1,4\}, \{2,5\}, \{3,6\}\}$ of $\{1,2,3,4,5,6\}$, then

$$\begin{aligned} P_{\mathbf{P}} &= \sum_{\substack{(i_1,i_2,i_3,i_4,i_5,i_6)\in(\mathbb{N}_+)^6;\\ i_a=i_b \text{ whenever } a\sim_{\mathbf{P}} b}} X_{i_1}X_{i_2}X_{i_3}X_{i_4}X_{i_5}X_{i_6} \\ &= \sum_{\substack{(i_1,i_2,i_3,i_4,i_5,i_6)\in(\mathbb{N}_+)^6;\\ i_1=i_4 \text{ and } i_2=i_5 \text{ and } i_3=i_6}} X_{i_1}X_{i_2}X_{i_3}X_{i_4}X_{i_5}X_{i_6} \\ &= \sum_{(u,v,w)\in(\mathbb{N}_+)^3} \underbrace{X_uX_vX_wX_uX_vX_w}_{=(X_uX_vX_w)^2} = \sum_{(u,v,w)\in(\mathbb{N}_+)^3} (X_uX_vX_w)^2 . \end{aligned}$$

Note that this cannot be simplified to $\left(\sum_{u\in\mathbb{N}_+} X_u^2\right)^3$, since the indeterminates don't commute.

(c) If $\mathbf{P}$ is the set partition $\{\{1,2,3,4\}\}$ of $\{1,2,3,4\}$, then

$$\begin{aligned} P_{\mathbf{P}} &= \sum_{\substack{(i_1,i_2,i_3,i_4)\in(\mathbb{N}_+)^4;\\ i_a=i_b \text{ whenever } a\sim_{\mathbf{P}} b}} X_{i_1}X_{i_2}X_{i_3}X_{i_4} \\ &= \sum_{\substack{(i_1,i_2,i_3,i_4)\in(\mathbb{N}_+)^4;\\ i_1=i_2=i_3=i_4}} X_{i_1}X_{i_2}X_{i_3}X_{i_4} = \sum_{u\in\mathbb{N}_+} X_uX_uX_uX_u = \sum_{u\in\mathbb{N}_+} X_u^4. \end{aligned}$$

(d) If $\mathbf{P}$ is the set partition $\varnothing$ of $\{\}$ (so we have $N = 0$), then

$$\begin{aligned} P_{\mathbf{P}} &= \sum_{\substack{() \in (\mathbb{N}_+)^0; \\ i_a = i_b \text{ whenever } a \sim_{\mathbf{P}} b}} (\text{empty product}) \\ &= (\text{empty product}) \qquad \text{(since there is only one 0-tuple)} \\ &= 1. \end{aligned}$$

We shall now assign an equivalence relation to any finite graph (V, E) and any list $\mathbf{t}$ of its vertices:

Proposition 6.25. Let (V, B) be a finite graph. Then, according to Definition 1.7 **(a)**, an equivalence relation $\sim_{(V,B)}$ is defined on the set V.

Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V. Let $\approx$ be the relation on the set $\{1, 2, \ldots, N\}$ defined as follows: Two elements i and j of $\{1, 2, \ldots, N\}$ shall satisfy $i \approx j$ if and only if $t_i \sim_{(V,B)} t_j$.

Then, this relation $\approx$ is an equivalence relation.

Proof. Straightforward and easy. $\square$

As we know from Theorem 6.21, an equivalence relation is "essentially the same as" a set partition. Thus, in particular, we can turn the equivalence relation defined in Proposition 6.25 into a set partition:

Definition 6.26. Let (V, B) be a finite graph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V.

(a) Let $\approx_{(V,B,\mathbf{t})}$ be the relation $\approx$ on the set $\{1, 2, \ldots, N\}$ defined in Proposition 6.25. As we know from Proposition 6.25, this relation $\approx$ is an equivalence relation. In other words, the relation $\approx_{(V,B,\mathbf{t})}$ is an equivalence relation.

(b) Therefore, Theorem 6.21 **(a)** (applied to $X = \{1, 2, \ldots, N\}$ and $(\sim) = \left(\approx_{(V,B,\mathbf{t})}\right)$) shows that the set

$$\{1, 2, \ldots, N\} / \left(\approx_{(V,B,\mathbf{t})}\right) = \left\{\text{all } \approx_{(V,B,\mathbf{t})} \text{-equivalence classes}\right\}$$

is a set partition of $\{1, 2, \ldots, N\}$. We shall denote this set partition by $\mathbf{P}(V, B, \mathbf{t})$.

Example 6.27. Let (V, B) be the finite graph with $V = \{u, v, w, x, y\}$ and $B = \{\{u, v\}, \{v, w\}, \{x, y\}\}$. Then, the equivalence relation $\sim_{(V,B)}$ from Definition 1.7 **(a)** satisfies $u \sim_{(V,B)} v \sim_{(V,B)} w$ and $x \sim_{(V,B)} y$.

Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be the list (u, v, x, y, x, u) of elements of V (so that $N = 6$ and $t_1 = u$ and $t_2 = v$ and $t_3 = x$ and $t_4 = y$ and $t_5 = x$ and $t_6 = u$).

Then, the equivalence relation $\approx_{(V,B,\mathbf{t})}$ from Definition 6.26 **(a)** satisfies

$$1 \approx_{(V,B,\mathbf{t})} 2 \approx_{(V,B,\mathbf{t})} 6 \qquad \left(\text{since } t_1 \sim_{(V,B)} t_2 \sim_{(V,B)} t_6\right) \qquad \text{and}$$
$$3 \approx_{(V,B,\mathbf{t})} 4 \approx_{(V,B,\mathbf{t})} 5 \qquad \left(\text{since } t_3 \sim_{(V,B)} t_4 \sim_{(V,B)} t_5\right)$$

and no further relations (except for the ones that follow from the relations just given using reflexivity, symmetry and transitivity). Thus, the set partition $\mathbf{P}(V, B, \mathbf{t})$ from Definition 6.26 **(b)** is

$$\{\{1,2,6\}, \{3,4,5\}\}.$$

We now have all notations in place to state noncommutative analogues of Theorems 6.13 and 6.14 and Corollaries 6.15 and 6.16:

Theorem 6.28. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Then,

$$Y_{G,\mathbf{t}} = \sum_{F \subseteq E} (-1)^{|F|} P_{\mathbf{P}(V, \operatorname{union} F, \mathbf{t})}.$$

Theorem 6.29. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$Y_{G,\mathbf{t}} = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) P_{\mathbf{P}(V, \operatorname{union} F, \mathbf{t})}.$$

(Here, of course, the pair $(V, \operatorname{union} F)$ is regarded as a graph, and the expression $\lambda((V, \operatorname{union} F), w)$ is understood according to Definition 6.5.)

Corollary 6.30. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then,

$$Y_{G,\mathbf{t}} = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} P_{\mathbf{P}(V, \operatorname{union} F, \mathbf{t})}.$$

Corollary 6.31. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$Y_{G,\mathbf{t}} = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} P_{\mathbf{P}(V, \text{union } F, \mathbf{t})}.$$

Since any graph or loopless multigraph can be viewed as an ambigraph, it is easy to see that Theorem 6.28 and Corollary 6.31 generalize [GebSag01, Theorem 3.6] and [GebSag01, Theorem 3.8], respectively.

In order to prove these four results, we proceed similarly to the commutative case, which we have studied to exhaustion. Instead of Lemma 6.11, we need the following noncommutative analogue:

Lemma 6.32. Let (V, B) be a finite graph. Let $\mathbf{t} = (t_1, t_2, \ldots, t_N)$ be a finite list of elements of V that contains each element of V at least once. Then,

$$\sum_{\substack{f : V \to \mathbb{N}_+; \\ B \subseteq \operatorname{Eqs} f}} \mathbf{X}_{f,\mathbf{t}} = P_{\mathbf{P}(V,B,\mathbf{t})}.$$

Proof of Lemma 6.32. The definition of $P_{\mathbf{P}(V,B,\mathbf{t})}$ yields

$$P_{\mathbf{P}(V,B,\mathbf{t})} = \sum_{\substack{(i_1, i_2, \ldots, i_N) \in (\mathbb{N}_+)^N; \\ i_a = i_b \text{ whenever } a \sim_{\mathbf{P}(V,B,\mathbf{t})} b}} X_{i_1} X_{i_2} \cdots X_{i_N} \tag{29}$$

(where the condition "$i_a = i_b$ whenever $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$" is shorthand for "$i_a = i_b$ for any two elements $a, b \in \{1, 2, \ldots, N\}$ that satisfy $a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$").

However, we know from Theorem 6.21 that every equivalence relation $\sim$ on a set X can be canonically transformed into a set partition of this set (namely, the set partition $X/(\sim)$, which consists of the $\sim$-equivalence classes), and conversely, every set partition $\mathbf{P}$ of a set X can be canonically transformed into an equivalence relation $\sim_{\mathbf{P}}$ on this set. These two transformations are mutually inverse; in particular, if an equivalence relation $\sim$ gives rise to a set partition $X/(\sim)$, then the equivalence relation $\sim_{X/(\sim)}$ constructed from the latter set partition is again the original relation $\sim$.

Thus, it follows that the relation $\sim_{\mathbf{P}(V,B,\mathbf{t})}$ is precisely the relation $\approx_{(V,B,\mathbf{t})}$ (because the relation $\sim_{\mathbf{P}(V,B,\mathbf{t})}$ is constructed from the set partition $\mathbf{P}(V, B, \mathbf{t})$, but the latter set partition $\mathbf{P}(V, B, \mathbf{t})$ is in turn constructed from the equivalence relation $\approx_{(V,B,\mathbf{t})}$).

On the other hand, the relation $\approx_{(V,B,\mathbf{t})}$ is defined as the relation $\approx$ on the set $\{1, 2, \ldots, N\}$ for which two elements satisfy $i \approx j$ if and only if $t_i \sim_{(V,B)} t_j$.

Hence, for any two elements $a, b \in \{1, 2, \ldots, N\}$, we have the equivalence

$$\left(a \approx_{(V,B,\mathbf{t})} b\right) \iff \left(t_a \sim_{(V,B)} t_b\right).$$

Since the relation $\sim_{\mathbf{P}(V,B,\mathbf{t})}$ is precisely the relation $\approx_{(V,B,\mathbf{t})}$, we can rewrite this as follows: For any two elements $a, b \in \{1, 2, \ldots, N\}$, we have the equivalence

$$\left(a \sim_{\mathbf{P}(V,B,\mathbf{t})} b\right) \iff \left(t_a \sim_{(V,B)} t_b\right).$$

Therefore, we can replace the condition "$a \sim_{\mathbf{P}(V,B,\mathbf{t})} b$" under the summation sign in (29) by "$t_a \sim_{(V,B)} t_b$". As a result, (29) rewrites as follows:

$$P_{\mathbf{P}(V,B,\mathbf{t})} = \sum_{\substack{(i_1,i_2,\ldots,i_N)\in(\mathbb{N}_+)^N; \\ i_a=i_b \text{ whenever } t_a\sim_{(V,B)}t_b}} X_{i_1} X_{i_2} \cdots X_{i_N}. \tag{30}$$

Now, we define two sets

$$\mathcal{F} = \{g : V \to \mathbb{N}_+ \text{ is a map} \mid B \subseteq \operatorname{Eqs} g\}$$

and

$$\mathcal{I} = \left\{(i_1, i_2, \ldots, i_N) \in (\mathbb{N}_+)^N \mid i_a = i_b \text{ whenever } t_a \sim_{(V,B)} t_b\right\}.$$

We claim the following:

Claim 1: For any $f \in \mathcal{F}$, we have $(f(t_1), f(t_2), \ldots, f(t_N)) \in \mathcal{I}$.

[*Proof of Claim 1:* Let $f \in \mathcal{F}$. Thus, by the definition of $\mathcal{F}$, we conclude that $f : V \to \mathbb{N}_+$ is a map satisfying $B \subseteq \operatorname{Eqs} f$.

We need to show that $(f(t_1), f(t_2), \ldots, f(t_N)) \in \mathcal{I}$. By the definition of $\mathcal{I}$, this requires us to show that $f(t_a) = f(t_b)$ whenever $t_a \sim_{(V,B)} t_b$ (that is, whenever $a, b \in \{1, 2, \ldots, N\}$ are two elements satisfying $t_a \sim_{(V,B)} t_b$). So let us show this.

Let $a, b \in \{1, 2, \ldots, N\}$ be two elements satisfying $t_a \sim_{(V,B)} t_b$. We must show that $f(t_a) = f(t_b)$.

We have $B \subseteq \operatorname{Eqs} f$. Hence, if x and y are two elements of V lying in the same connected component of (V, B), then

$$f(x) = f(y). \tag{31}$$

(Indeed, this can be shown in the same way as we established (7) during our proof of Lemma 2.5.)

However, we have $t_a \sim_{(V,B)} t_b$. In other words, the elements t_a and t_b lie in the same connected component of (V, B) (since the connected components of (V, B)

are the $\sim_{(V,B)}$-equivalence classes). Thus, (31) (applied to $x = t_a$ and $y = t_b$) yields that $f(t_a) = f(t_b)$. As we explained, this completes the proof of Claim 1.]

Thanks to Claim 1, we can define a map

$$\Psi : \mathcal{F} \to \mathcal{I},$$
$$f \mapsto (f(t_1), f(t_2), \ldots, f(t_N)).$$

Consider this map Ψ. We claim the following:

Claim 2: The map Ψ is injective.

[*Proof of Claim 2:* The list $(t_1, t_2, \ldots, t_N)$ contains each element of V at least once (according to the hypotheses of Lemma 6.32). Thus, if $f \in \mathcal{F}$ is arbitrary, then the list $(f(t_1), f(t_2), \ldots, f(t_N))$ contains each value of f at least once. Therefore, any $f \in \mathcal{F}$ can be uniquely reconstructed from this list $(f(t_1), f(t_2), \ldots, f(t_N))$. In other words, any $f \in \mathcal{F}$ can be uniquely reconstructed from $\Psi(f)$ (since the definition of Ψ yields $\Psi(f) = (f(t_1), f(t_2), \ldots, f(t_N))$). In other words, the map Ψ is injective. This proves Claim 2.]

Claim 3: The map Ψ is surjective.

[*Proof of Claim 3:* Let $\mathbf{i} \in \mathcal{I}$. We shall construct an $f \in \mathcal{F}$ satisfying $\Psi(f) = \mathbf{i}$.

Indeed, $\mathbf{i} \in \mathcal{I}$. By the definition of $\mathcal{I}$, this means that $\mathbf{i}$ has the form $\mathbf{i} = (i_1, i_2, \ldots, i_N)$ for some N-tuple $(i_1, i_2, \ldots, i_N) \in (\mathbb{N}_+)^N$ that satisfies

$$i_a = i_b \text{ whenever } t_a \sim_{(V,B)} t_b. \tag{32}$$

Consider this N-tuple $(i_1, i_2, \ldots, i_N)$.

We shall now define a map $f : V \to \mathbb{N}_+$ as follows:

Let $v \in V$. The list $(t_1, t_2, \ldots, t_N)$ contains each element of V at least once (according to the hypotheses of Lemma 6.32). In particular, this list contains v at least once (since v is an element of V). In other words, there exists a $k \in \{1, 2, \ldots, N\}$ such that $t_k = v$. Pick the smallest such k, and set $f(v) := i_k$.

Thus, we have defined a positive integer $f(v) \in \mathbb{N}_+$ for each $v \in V$. In other words, we have defined a map $f : V \to \mathbb{N}_+$.

We shall now show that $B \subseteq \operatorname{Eqs} f$.

Indeed, let $e \in B$ be arbitrary. Then, $e \in B \subseteq \binom{V}{2}$, so that $e = \{x, y\}$ for two distinct vertices $x, y \in V$. Consider these x, y. Thus, (x, e, y) is a walk from x to y in the graph (V, B). Therefore, x is connected to y in this graph. In other words, $x \sim_{(V,B)} y$.

The definition of f shows that $f(x) = i_a$, where a is the smallest element of $\{1, 2, \ldots, N\}$ such that $t_a = x$. Similarly, $f(y) = i_b$, where b is the smallest element of $\{1, 2, \ldots, N\}$ such that $t_b = y$. Consider these a and b.

However, we have $x \sim_{(V,B)} y$. In other words, $t_a \sim_{(V,B)} t_b$ (since $t_a = x$ and $t_b = y$). Hence, from (32), we obtain $i_a = i_b$. In other words, $f(x) = f(y)$ (since

$f(x) = i_a$ and $f(y) = i_b$). In other words, $\{x, y\} \in \operatorname{Eqs} f$ (by the definition of $\operatorname{Eqs} f$). Hence, $e = \{x, y\} \in \operatorname{Eqs} f$. Now, forget that we fixed e. We thus have shown that $e \in \operatorname{Eqs} f$ for each $e \in B$. In other words, $B \subseteq \operatorname{Eqs} f$.

Thus, we know that f is a map $V \to \mathbb{N}_+$ and satisfies $B \subseteq \operatorname{Eqs} f$. In other words, $f \in \mathcal{F}$ (by the definition of $\mathcal{F}$).

We shall now show that $\Psi(f) = \mathbf{i}$.

Indeed, the definition of Ψ yields $\Psi(f) = (f(t_1), f(t_2), \ldots, f(t_N))$.

Now, let $j \in \{1, 2, \ldots, N\}$. We shall show that $f(t_j) = i_j$.

Indeed, the definition of f shows that $f(t_j) = i_k$, where k is the smallest element of $\{1, 2, \ldots, N\}$ such that $t_k = t_j$. Consider this k. From $t_k = t_j$, we obtain $t_k \sim_{(V,B)} t_j$ (since the relation $\sim_{(V,B)}$ is an equivalence relation). Hence, (32) (applied to $a = k$ and $b = j$) yields $i_k = i_j$. Thus, $f(t_j) = i_k = i_j$.

Forget that we fixed j. We thus have shown that $f(t_j) = i_j$ for each $j \in \{1, 2, \ldots, N\}$. In other words,

$$(f(t_1), f(t_2), \ldots, f(t_N)) = (i_1, i_2, \ldots, i_N).$$

In view of $\Psi(f) = (f(t_1), f(t_2), \ldots, f(t_N))$ and $\mathbf{i} = (i_1, i_2, \ldots, i_N)$, we can rewrite this as $\Psi(f) = \mathbf{i}$. Hence,

$$\mathbf{i} = \Psi\left(\underbrace{f}_{\in \mathcal{F}}\right) \in \Psi(\mathcal{F}).$$

Forget that we fixed $\mathbf{i}$. We thus have shown that $\mathbf{i} \in \Psi(\mathcal{F})$ for each $\mathbf{i} \in \mathcal{I}$. In other words, $\mathcal{I} \subseteq \Psi(\mathcal{F})$. In other words, the map Ψ is surjective. This proves Claim 3.]

We now know that the map Ψ is injective (by Claim 2) and surjective (by Claim 3). In other words, this map Ψ is bijective, i.e., is a bijection.

In other words, the map

$$\begin{aligned} &\mathcal{F} \to \mathcal{I}, \\ &f \mapsto (f(t_1), f(t_2), \ldots, f(t_N)) \end{aligned}$$

is a bijection (since the map is Ψ). Now, (30) becomes

$$\begin{aligned}
P_{\mathbf{P}(V,B,\mathbf{t})} &= \underbrace{\sum_{\substack{(i_1,i_2,\ldots,i_N)\in(\mathbb{N}_+)^N;\\ i_a=i_b \text{ whenever } t_a\sim_{(V,B)}t_b}}}_{\substack{=\sum_{(i_1,i_2,\ldots,i_N)\in\mathcal{I}}\\ \text{(by the definition of } \mathcal{I})}} X_{i_1}X_{i_2}\cdots X_{i_N} \\
&= \sum_{(i_1,i_2,\ldots,i_N)\in\mathcal{I}} X_{i_1}X_{i_2}\cdots X_{i_N} \\
&= \underbrace{\sum_{f\in\mathcal{F}}}_{\substack{=\sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}}\\ \text{(by the definition of } \mathcal{F})}} X_{f(t_1)}X_{f(t_2)}\cdots X_{f(t_N)} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } (f(t_1), f(t_2), \ldots, f(t_N)) \\ \text{for } (i_1, i_2, \ldots, i_N) \text{ in the sum, since the} \\ \text{map } \mathcal{F}\to\mathcal{I},\ f\mapsto (f(t_1), f(t_2), \ldots, f(t_N)) \\ \text{is a bijection}\end{array}\right) \\
&= \sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \underbrace{X_{f(t_1)}X_{f(t_2)}\cdots X_{f(t_N)}}_{\substack{=\mathbf{X}_{f,\mathbf{t}}\\ \text{(since } \mathbf{X}_{f,\mathbf{t}} \text{ was defined to be } X_{f(t_1)}X_{f(t_2)}\cdots X_{f(t_N)} \\ \text{in Definition 6.17 } \textbf{(a)})}} \\
&= \sum_{\substack{f:V\to\mathbb{N}_+;\\ B\subseteq \operatorname{Eqs} f}} \mathbf{X}_{f,\mathbf{t}}.
\end{aligned}$$

This proves Lemma 6.32. $\square$

We can now prove Theorems 6.29 and 6.28 and Corollaries 6.30 and 6.31 by making straightforward changes to the above proofs of Theorems 6.14 and 6.13 and Corollaries 6.15 and 6.16 (replacing, in particular, the use of Lemma 6.11 by a use of Lemma 6.32). We leave the details to the reader.

6.7. An abstract setting

The reader will by now have realized that we have been making the same arguments in a series of slightly different settings. In particular, the chromatic symmetric function X_G, its weighted version $X_{G,w}$ and its noncommutative version $Y_{G,\mathbf{t}}$ are all defined as sums over proper $\mathbb{N}_+$-colorings of G; they differ only in the addends being summed. We can generalize them all by allowing these addends to be arbitrary, i.e., replacing them by arbitrary elements α_f of a $\mathbb{Z}$-module M, provided that the resulting (potentially infinite) sums are still well-defined. While at that, we can also replace $\mathbb{N}_+$-colorings by Y-colorings for an arbitrary set Y. Thus, we are led to the following general setting:

Definition 6.33. Let $G = (V, E, \varphi)$ be a finite ambigraph. Let Y be any set.

Let M be a topological $\mathbb{Z}$-module. Let $\alpha_f \in M$ be an element for each Y-coloring $f : V \to Y$. Assume that the family $(\alpha_f)_{f:V\to Y}$ of these elements is summable (so that the sum $\sum\limits_{f:V\to Y} \alpha_f$ and any of its subsums is well-defined).

Then:

(a) We define an element

$$\Xi_G := \sum_{\substack{f:V\to Y \text{ is a} \\ \text{proper } Y\text{-coloring of } G}} \alpha_f \in M.$$

(b) Furthermore, if B is a subset of $\binom{V}{2}$, then we set

$$\pi_B := \sum_{\substack{f:V\to Y; \\ B\subseteq \operatorname{Eqs} f}} \alpha_f \in M.$$

Through appropriate choices of α_f, we recover the previously defined power series X_G, $X_{G,w}$ and $Y_{G,\mathbf{t}}$:

- If $Y = \mathbb{N}_+$ and $\alpha_f = \mathbf{x}_f$, then $\Xi_G = X_G$ and $\pi_B = p_{\lambda(V, \operatorname{union} B)}$.
- If $Y = \mathbb{N}_+$ and $\alpha_f = \mathbf{x}_{f,w}$ (for a given weight function $w : V \to \mathbb{N}_+$), then $\Xi_G = X_{G,w}$ and $\pi_B = p_{\lambda((V, \operatorname{union} B),w)}$.
- If $Y = \mathbb{N}_+$ and $\alpha_f = \mathbf{X}_{f,\mathbf{t}}$ (for a given list $\mathbf{t}$ of elements of V that contains each element at least once), then $\Xi_G = Y_{G,\mathbf{t}}$ and $\pi_B = P_{\mathbf{P}(V, \operatorname{union} B, \mathbf{t})}$.

We can now state analogues of Theorems 6.28 and 6.29 and Corollaries 6.30 and 6.31 in this general context:

Theorem 6.34. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Then, using the notations of Definition 6.33, we have

$$\Xi_G = \sum_{F\subseteq E} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

Theorem 6.35. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then, using the notations of Definition 6.33, we have

$$\Xi_G = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) \pi_{\operatorname{union} F}.$$

Corollary 6.36. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of G (not necessarily containing all of them). Then, using the notations of Definition 6.33, we have

$$\Xi_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

Corollary 6.37. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then, using the notations of Definition 6.33, we have

$$\Xi_G = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } G \text{ as a subset}}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

The reader will have no difficulty proving these four results by following the same well-trodden way that led us to their particular cases.

Corollary 6.37 can be used to prove certain results about list colorings (i.e., colorings of a graph or ambigraph that are not allowed to use certain colors for certain vertices); in particular, [Erey19, Lemma 3.2] follows easily from Corollary 6.37 (just turn the graph G into an ambigraph, and set α_f to be the Iverson bracket $[f(v) \neq r(v) \text{ for each } v \in V]$).

7. Application: A vanishing alternating sum

Chromatic symmetric functions of different graphs are far from being linearly independent; they satisfy several linear relations. One such relation was observed by Dahlberg and van Willigenburg in 2018 [DahWil18, Proposition 5]:

Theorem 7.1. Let $G = (V, E)$ be a graph. Let C be a circuit of G, and let $e \in C$ be arbitrary. Then,

$$\sum_{F \subseteq C \setminus \{e\}} (-1)^{|F|} X_{G \setminus F} = 0.$$

Here, whenever F is a subset of E, the notation $G \setminus F$ denotes the graph $(V, E \setminus F)$ (that is, the graph obtained from G by removing the edgeries in F).

This was extended to noncommutative chromatic symmetric functions $Y_{G,\mathbf{t}}$ by Dahlberg and van Willigenburg in [DahWil19, Proposition 3.6], and to weighted chromatic symmetric functions $X_{G,w}$ by Crew and Spirkl in [CreSpi19, Theorem 6]. Again, we shall now one-up these results by generalizing them from graphs to ambigraphs and by moving to the abstract setting of Definition 6.33. Thus, we claim the following:

Theorem 7.2. Let G, V, E, φ, Y, M and α_f be as in Definition 6.33. Let C be a circuit of G, and let $e \in C$ be a singleton edgery. Then, using the notations of Definition 6.33 **(a)**, we have

$$\sum_{F \subseteq C \setminus \{e\}} (-1)^{|F|} \Xi_{G \setminus F} = 0.$$

Here, whenever F is a subset of E, the notation $G \setminus F$ denotes the ambigraph $\left(V, E \setminus F, \varphi \mid_{E \setminus F}\right)$ (that is, the ambigraph obtained from G by removing the edges in F).

Applying this to a graph instead of an ambigraph, and setting $Y = \mathbb{N}_+$ and $\alpha_f = \mathbf{x}_f$, we recover Theorem 7.1.

Theorem 7.2 is quite easy to prove despite its generality; in fact, the beautiful sign-reversing involution argument from [DahWil19, proof of Proposition 3.6] still does the trick. However, by way of illustration, we shall now demonstrate how Theorem 7.2 can be derived from Theorem 6.34 and Corollary 6.36.

Proof of Theorem 7.2. Let us set $B := C \setminus \{e\}$. Thus, $B = C \setminus \{e\} \subseteq C \subseteq E$.

Now, we shall show the following (using the notations of Definition 6.33 **(b)**):

Claim 1: Let J be a subset of B. Then,

$$\Xi_{G \setminus J} = \sum_{\substack{F \subseteq E; \\ J \subseteq E \setminus F}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

[*Proof of Claim 1:* We know that $G \setminus J = \left(V, E \setminus J, \varphi \mid_{E \setminus J}\right)$ is an ambigraph. Thus, we can apply Theorem 6.34 to $G \setminus J$, $E \setminus J$ and $\varphi \mid_{E \setminus J}$ instead of G, E and φ. As a result, we obtain

$$\Xi_{G \setminus J} = \sum_{F \subseteq E \setminus J} (-1)^{|F|} \pi_{\operatorname{union} F} \tag{33}$$

(since the meaning of union F is the same whether we consider F as a set of edgeries of $G \setminus J$ or as a set of edgeries of G). However, a subset F of $E \setminus J$ is the same thing as a subset F of E that is disjoint from J, and this is in turn the same as a subset F of E that satisfies $J \subseteq E \setminus F$ (since $J \subseteq B \subseteq E$). Hence, we can replace the summation sign "$\sum\limits_{F \subseteq E \setminus J}$" in (33) by "$\sum\limits_{\substack{F \subseteq E; \\ J \subseteq E \setminus F}}$". As a result, (33) becomes

$$\Xi_{G \setminus J} = \sum_{\substack{F \subseteq E; \\ J \subseteq E \setminus F}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

Thus, Claim 1 is proved.]

Claim 2: We have

$$\sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} = 0. \tag{34}$$

[*Proof of Claim 2:* Theorem 6.34 yields

$$\Xi_G = \sum_{F \subseteq E} (-1)^{|F|} \pi_{\operatorname{union} F}. \tag{35}$$

On the other hand, let us define a labeling function $\ell : E \to \mathbb{N}$ by setting $\ell(e) = 1$ and setting $\ell(f) = 0$ for all $f \in E \setminus \{e\}$. Then, the edgery e is the unique singleton edgery in C having maximum label. Hence, $C \setminus \{e\}$ is a broken circuit of G. In other words, B is a broken circuit of G (since $B = C \setminus \{e\}$). Hence, $\{B\}$ is a set of broken circuits of G. Therefore, Corollary 6.36 (applied to $\mathfrak{K} = \{B\}$) yields

$$\Xi_G = \sum_{\substack{F \subseteq E; \\ F \text{ is } \{B\}\text{-free}}} (-1)^{|F|} \pi_{\operatorname{union} F} = \sum_{\substack{F \subseteq E; \\ B \not\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F}$$

(since the condition "F is $\{B\}$-free" is easily seen to be equivalent to "$B \not\subseteq F$"). Subtracting this equality from (35), we obtain

$$0 = \sum_{F \subseteq E} (-1)^{|F|} \pi_{\operatorname{union} F} - \sum_{\substack{F \subseteq E; \\ B \not\subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} = \sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F}.$$

This proves Claim 2.]

However, from $C \setminus \{e\} = B$, we obtain

$$\sum_{F \subseteq C \setminus \{e\}} (-1)^{|F|} \Xi_{G \setminus F} = \sum_{F \subseteq B} (-1)^{|F|} \Xi_{G \setminus F} = \sum_{J \subseteq B} (-1)^{|J|} \underbrace{\Xi_{G \setminus J}}_{\substack{= \sum_{\substack{F \subseteq E; \\ J \subseteq E \setminus F}} (-1)^{|F|} \pi_{\operatorname{union} F} \\ \text{(by Claim 1)}}}$$

$$\left(\begin{array}{c}\text{here, we have renamed the} \\ \text{summation index } F \text{ as } J\end{array}\right)$$

$$= \sum_{J \subseteq B} (-1)^{|J|} \sum_{\substack{F \subseteq E; \\ J \subseteq E \setminus F}} (-1)^{|F|} \pi_{\operatorname{union} F}$$

$$= \underbrace{\sum_{J \subseteq B} \sum_{\substack{F \subseteq E; \\ J \subseteq E \setminus F}}}_{= \sum_{F \subseteq E} \sum_{\substack{J \subseteq B; \\ J \subseteq E \setminus F}}} (-1)^{|J|} (-1)^{|F|} \pi_{\operatorname{union} F}$$

$$= \sum_{F \subseteq E} \underbrace{\sum_{\substack{J \subseteq B; \\ J \subseteq E \setminus F}}}_{\substack{= \sum_{J \subseteq B \cap (E \setminus F)} \\ = \sum_{J \subseteq B \setminus F} \\ \text{(since } B \subseteq E \text{ entails} \\ B \cap (E \setminus F) = B \setminus F)}} (-1)^{|J|} (-1)^{|F|} \pi_{\operatorname{union} F}$$

$$= \sum_{F \subseteq E} \underbrace{\sum_{J \subseteq B \setminus F} (-1)^{|J|}}_{\substack{= \sum_{I \subseteq B \setminus F} (-1)^{|I|} \\ = [B \setminus F = \varnothing] \\ \text{(by Lemma 5.21)}}} (-1)^{|F|} \pi_{\operatorname{union} F}$$

$$= \sum_{F \subseteq E} \underbrace{[B \setminus F = \varnothing]}_{= [B \subseteq F]} (-1)^{|F|} \pi_{\operatorname{union} F}$$

$$= \sum_{F \subseteq E} [B \subseteq F] (-1)^{|F|} \pi_{\operatorname{union} F}. \tag{36}$$

In the sum on the right hand side, we can clearly remove all addends that don't satisfy $B \subseteq F$, since the presence of the $[B \subseteq F] = 0$ factor renders all these addends equal to 0. Thus, we are left with only the addends that do satisfy

$B \subseteq F$. Hence, (36) rewrites as

$$\sum_{F \subseteq C \setminus \{e\}} (-1)^{|F|} \Xi_{G \setminus F} = \sum_{\substack{F \subseteq E; \\ B \subseteq F}} \underbrace{[B \subseteq F]}_{\substack{=1 \\ (\text{since } B \subseteq F)}} (-1)^{|F|} \pi_{\operatorname{union} F}$$
$$= \sum_{\substack{F \subseteq E; \\ B \subseteq F}} (-1)^{|F|} \pi_{\operatorname{union} F} = 0 \qquad (\text{by (34)}) .$$

This proves Theorem 7.2. ☐

8. The characteristic polynomial of a matroid

8.1. An introduction to matroids

We shall now present a result that can be considered as a generalization of Theorem 3.5 in a different direction than Theorem 1.12: namely, a formula for the characteristic polynomial of a matroid. Let us first recall the basic notions from the theory of matroids that will be needed to state it.

First, we introduce some basic poset-related terminology:

Definition 8.1. Let P be a poset.

(a) An element v of P is said to be *maximal* (with respect to P) if and only if every $w \in P$ satisfying $w \geq v$ must satisfy $w = v$.

(b) An element v of P is said to be *minimal* (with respect to P) if and only if every $w \in P$ satisfying $w \leq v$ must satisfy $w = v$.

Definition 8.2. For any set E, we shall regard the powerset $\mathcal{P}(E)$ as a poset (with respect to inclusion). Thus, any subset $\mathcal{S}$ of $\mathcal{P}(E)$ also becomes a poset, and therefore the notions of "minimal" and "maximal" elements in $\mathcal{S}$ make sense. Beware that these notions are not related to size; i.e., a maximal element of $\mathcal{S}$ might not be a maximum-size element of $\mathcal{S}$.

Now, let us define the notion of "matroid" that we will use:

Definition 8.3. **(a)** A *matroid* means a pair $(E, \mathcal{I})$ consisting of a finite set E and a set $\mathcal{I} \subseteq \mathcal{P}(E)$ satisfying the following axioms:

- *Matroid axiom 1:* We have $\varnothing \in \mathcal{I}$.
- *Matroid axiom 2:* If $Y \in \mathcal{I}$ and $Z \in \mathcal{P}(E)$ are such that $Z \subseteq Y$, then $Z \in \mathcal{I}$.
- *Matroid axiom 3:* If $Y \in \mathcal{I}$ and $Z \in \mathcal{I}$ are such that $|Y| < |Z|$, then there exists some $x \in Z \setminus Y$ such that $Y \cup \{x\} \in \mathcal{I}$.

(b) Let $(E, \mathcal{I})$ be a matroid. A subset S of E is said to be *independent* (for this matroid) if and only if $S \in \mathcal{I}$. The set E is called the *ground set* of the matroid $(E, \mathcal{I})$.

There are different definitions of a matroid in the literature; these definitions are (mostly) equivalent, but not always in the obvious way[25]. Definition 8.3 is how a matroid is defined in [Schrij13, §10.1] and in [Martin22, Definition 3.4.1] (where it is called a "(matroid) independence system"). The definition of a matroid given in Stanley's [Stanle06, Definition 3.8] is directly equivalent to Definition 8.3, with the only differences that

- Stanley replaces Matroid axiom 1 by the requirement that $\mathcal{I} \neq \varnothing$ (which is, of course, equivalent to Matroid axiom 1 as long as Matroid axiom 2 is assumed), and
- Stanley replaces Matroid axiom 3 by the requirement that for every $T \in \mathcal{P}(E)$, all maximal elements of $\mathcal{I} \cap \mathcal{P}(T)$ have the same cardinality[26] (this requirement is equivalent to Matroid axiom 3 as long as Matroid axiom 2 is assumed).

We now introduce some terminology related to matroids:

Definition 8.4. Let $M = (E, \mathcal{I})$ be a matroid.

(a) We define a function $r_M : \mathcal{P}(E) \to \mathbb{N}$ by setting

$$r_M(S) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq S\} \qquad \text{for every } S \subseteq E. \tag{37}$$

(Note that the right hand side of (37) is well-defined, because there exists at least one $Z \in \mathcal{I}$ satisfying $Z \subseteq S$ (namely, $Z = \varnothing$).) If S is a subset of E, then the nonnegative integer $r_M(S)$ is called the *rank* of S (with respect to M). It is clear that r_M is a weakly increasing function from the poset $\mathcal{P}(E)$ to $\mathbb{N}$.

(b) If $k \in \mathbb{N}$, then a *k-flat* of M means a subset of E that has rank k and is maximal among all such subsets (i.e., it is not a proper subset of any other subset having rank k). (Beware: Not all k-flats have the same size.) A *flat* of M is a subset of E which is a k-flat for some $k \in \mathbb{N}$. We let $\operatorname{Flats} M$ denote the set of all flats of M; thus, $\operatorname{Flats} M$ is a subposet of $\mathcal{P}(E)$.

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[25] Indeed, most of these definitions define a matroid as a pair (E, U) consisting of a finite set E and a subset $U \subseteq \mathcal{P}(E)$ satisfying a certain set of axioms, but these sets of axioms are not always equivalent, so they define different classes of pairs (E, U). Thus, a matroid in the sense of one definition is not necessarily a matroid in the sense of another definition. However, there are canonical bijections between one type of matroids and another (see, e.g., [Schrij13, §10.2]); these are commonly known as "cryptomorphisms".

[26] Here, we regard $\mathcal{I} \cap \mathcal{P}(T)$ as a poset with respect to inclusion (as explained in Definition 8.2). Thus, an element Y of this poset is maximal if and only if there exists no $Z \in \mathcal{I} \cap \mathcal{P}(T)$ such that Y is a proper subset of Z.

(c) A *circuit* of M means a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$. (That is, a circuit of M means a subset of E which is not independent (for M) and which is minimal among such subsets.)

(d) An element e of E is said to be a *loop* (of M) if $\{e\} \notin \mathcal{I}$. The matroid M is said to be *loopless* if no loops (of M) exist.

Notice that the function that we called r_M in Definition 8.4 **(a)** is called the *rank function* of M, and is denoted by rk in Stanley's [Stanle06, Lecture 3].

One of the most classical examples of a matroid is the *graphical matroid* of a graph:

Example 8.5. Let $G = (V, E)$ be a finite graph. Define a subset $\mathcal{I}$ of $\mathcal{P}(E)$ by

$$\mathcal{I} = \{T \in \mathcal{P}(E) \mid T \text{ contains no circuit of } G \text{ as a subset}\}.$$

Then, $(E, \mathcal{I})$ is a matroid; it is called the *graphical matroid* (or the *cycle matroid*) of G. It has the following properties:

- The matroid $(E, \mathcal{I})$ is loopless.
- For each $T \in \mathcal{P}(E)$, we have

  $$r_{(E,\mathcal{I})}(T) = |V| - \operatorname{conn}(V, T)$$

  (where $\operatorname{conn}(V, T)$ is defined as in Definition 3.3).
- The circuits of the matroid $(E, \mathcal{I})$ are precisely the circuits of the graph G.
- The flats of the matroid $(E, \mathcal{I})$ are related to colorings of G. More precisely: For each set X and each X-coloring f of G, the set $E \cap \operatorname{Eqs} f$ is a flat of $(E, \mathcal{I})$. Every flat of $(E, \mathcal{I})$ can be obtained in this way when X is chosen large enough; but often, several distinct X-colorings f lead to one and the same flat $E \cap \operatorname{Eqs} f$.

We recall three basic facts that are used countless times in arguing about matroids:

Lemma 8.6. Let $M = (E, \mathcal{I})$ be a matroid. Let $T \in \mathcal{I}$. Then, $r_M(T) = |T|$.

Proof of Lemma 8.6. We have $T \in \mathcal{I}$ and $T \subseteq T$. Thus, T is a $Z \in \mathcal{I}$ satisfying $Z \subseteq T$. Therefore, $|T| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$, so that

$$|T| \leq \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \tag{38}$$

(since any element of a set of integers is smaller or equal to the maximum of this set).

On the other hand, the definition of r_M yields

$$r_M(T) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}.$$

Hence, (38) rewrites as follows:

$$|T| \leq r_M(T).$$

Also,

$$\begin{aligned} r_M(T) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} && \text{(by the definition of } r_M\text{)} \\ &\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \end{aligned}$$

(since the maximum of any set belongs to this set). Thus, there exists a $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $r_M(T) = |Z|$. Consider this Z. From $Z \subseteq T$, we obtain $|Z| \leq |T|$, so that $r_M(T) = |Z| \leq |T|$. Combining this with $|T| \leq r_M(T)$, we obtain $r_M(T) = |T|$. This proves Lemma 8.6. □

Lemma 8.7. Let $M = (E, \mathcal{I})$ be a matroid. Let $Q \in \mathcal{P}(E) \setminus \mathcal{I}$. Then, there exists a circuit C of M such that $C \subseteq Q$.

Proof of Lemma 8.7. We have $Q \in \mathcal{P}(E) \setminus \mathcal{I}$. Thus, there exists at least one $C \in \mathcal{P}(E) \setminus \mathcal{I}$ such that $C \subseteq Q$ (namely, $C = Q$). Thus, there also exists a **minimal** such C. Consider this minimal C. We know that C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ such that $C \subseteq Q$. In other words, C is an element of $\mathcal{P}(E) \setminus \mathcal{I}$ satisfying $C \subseteq Q$, and moreover,

$$\text{every } D \in \mathcal{P}(E) \setminus \mathcal{I} \text{ satisfying } D \subseteq Q \text{ and } D \subseteq C \text{ must satisfy } D = C. \tag{39}$$

Thus, C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ [27]. In other words, C is a circuit of M (by the definition of a "circuit"). This circuit C satisfies $C \subseteq Q$. Thus, we have constructed a circuit C of M satisfying $C \subseteq Q$. Lemma 8.7 is thus proven. □

Lemma 8.8. Let $M = (E, \mathcal{I})$ be a matroid. Let T be a subset of E. Let $S \in \mathcal{I}$ be such that $S \subseteq T$. Then, there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ and $|S'| = r_M(T)$.

Proof of Lemma 8.8. Clearly, there exists at least one $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ (namely, $S' = S$). Hence, there exists a **maximal** such S'. Let Q be such a maximal S'. Thus, Q is an element of $\mathcal{I}$ satisfying $S \subseteq Q \subseteq T$.

Recall that

$$\begin{aligned} r_M(T) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} && \text{(by the definition of } r_M\text{)} \\ &\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \end{aligned}$$

[27] *Proof.* We need to show that every $D \in \mathcal{P}(E) \setminus \mathcal{I}$ satisfying $D \subseteq C$ must satisfy $D = C$ (since we already know that $C \in \mathcal{P}(E) \setminus \mathcal{I}$).

So let $D \in \mathcal{P}(E) \setminus \mathcal{I}$ be such that $D \subseteq C$. Then, $D \subseteq C \subseteq Q$. Hence, (39) shows that $D = C$. This completes our proof.

(since the maximum of any set must belong to this set). Hence, there exists some $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $r_M(T) = |Z|$. Denote such a Z by W. Thus, W is an element of $\mathcal{I}$ satisfying $W \subseteq T$ and $r_M(T) = |W|$.

We have $|Q| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$ (since $Q \in \mathcal{I}$ and $Q \subseteq T$). Since any element of a set is smaller or equal to the maximum of this set, this entails that $|Q| \leq \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} = r_M(T) = |W|$.

Now, assume (for the sake of contradiction) that $|Q| \neq |W|$. Thus, $|Q| < |W|$ (since $|Q| \leq |W|$). Hence, Matroid axiom 3 (applied to $Y = Q$ and $Z = W$) shows that there exists some $x \in W \setminus Q$ such that $Q \cup \{x\} \in \mathcal{I}$. Consider this x. We have $x \in W \setminus Q \subseteq W \subseteq T$, so that $Q \cup \{x\} \subseteq T$ (since $Q \subseteq T$). Also, $x \notin Q$ (since $x \in W \setminus Q$).

Recall that Q is a **maximal** $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$. Thus, if some $S' \in \mathcal{I}$ satisfies $S \subseteq S' \subseteq T$ and $S' \supseteq Q$, then $S' = Q$. Applying this to $S' = Q \cup \{x\}$, we obtain $Q \cup \{x\} = Q$ (since $S \subseteq Q \subseteq Q \cup \{x\} \subseteq T$ and $Q \cup \{x\} \supseteq Q$). Thus, $x \in Q$. But this contradicts $x \notin Q$. This contradiction shows that our assumption (that $|Q| \neq |W|$) was wrong. Hence, $|Q| = |W| = r_M(T)$. Thus, there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T$ and $|S'| = r_M(T)$ (namely, $S' = Q$). This proves Lemma 8.8. $\square$

8.2. The lattice of flats

We shall now show a lemma that can be regarded as an alternative criterion for a subset of E to be a flat:

Lemma 8.9. Let $M = (E, \mathcal{I})$ be a matroid. Let T be a subset of E. Then, the following statements are equivalent:

Statement $\mathfrak{F}_1$: The set T is a flat of M.

Statement $\mathfrak{F}_2$: If C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq T$, then $C \subseteq T$.

Proof of Lemma 8.9. Proof of the implication $\mathfrak{F}_1 \Longrightarrow \mathfrak{F}_2$: Assume that Statement $\mathfrak{F}_1$ holds. We must prove that Statement $\mathfrak{F}_2$ holds.

Let C be a circuit of M. Let $e \in C$ be such that $C \setminus \{e\} \subseteq T$. We must prove that $C \subseteq T$.

Assume the contrary. Thus, $C \not\subseteq T$. Combining this with $C \setminus \{e\} \subseteq T$, we obtain $e \notin T$. Hence, T is a proper subset of $T \cup \{e\}$.

We have assumed that Statement $\mathfrak{F}_1$ holds. In other words, the set T is a flat of M. In other words, there exists some $k \in \mathbb{N}$ such that T is a k-flat of M. Consider this k.

The set T is a k-flat of M, thus a subset of E that has rank k and is maximal among all such subsets. In other words, $r_M(T) = k$, but every subset S of E for which T is a proper subset of S must satisfy

$$r_M(S) \neq k. \tag{40}$$

Applying (40) to $S = T \cup \{e\}$, we obtain $r_M(T \cup \{e\}) \neq k$. Since $T \cup \{e\} \supseteq T$ (and since the function $r_M : \mathcal{P}(E) \to \mathbb{N}$ is weakly increasing), we have $r_M(T \cup \{e\}) \geq r_M(T) = k$. Combined with $r_M(T \cup \{e\}) \neq k$, this yields $r_M(T \cup \{e\}) > k = r_M(T)$.

Notice that $C \setminus \{e\}$ is a proper subset of C (since $e \in C$). The set C is a circuit of M, thus a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ (by the definition of a "circuit"). Hence, no proper subset of C belongs to $\mathcal{P}(E) \setminus \mathcal{I}$ (because C is minimal). In other words, every proper subset of C belongs to $\mathcal{I}$. Applying this to the proper subset $C \setminus \{e\}$ of C, we conclude that $C \setminus \{e\}$ belongs to $\mathcal{I}$. Hence, Lemma 8.8 (applied to $S = C \setminus \{e\}$) shows that there exists an $S' \in \mathcal{I}$ satisfying $C \setminus \{e\} \subseteq S' \subseteq T$ and $|S'| = r_M(T)$. Denote this S' by S. Thus, S is an element of $\mathcal{I}$ satisfying $C \setminus \{e\} \subseteq S \subseteq T$ and $|S| = r_M(T)$.

Furthermore, $S \subseteq T \subseteq T \cup \{e\}$. Thus, Lemma 8.8 (applied to $T \cup \{e\}$ instead of T) shows that there exists an $S' \in \mathcal{I}$ satisfying $S \subseteq S' \subseteq T \cup \{e\}$ and $|S'| = r_M(T \cup \{e\})$. Consider this S'.

We have $|S'| = r_M(T \cup \{e\}) > r_M(T)$. Hence, $S' \not\subseteq T$ [28]. Combining this with $S' \subseteq T \cup \{e\}$, we obtain $e \in S'$. Combining this with $C \setminus \{e\} \subseteq S \subseteq S'$, we find that $(C \setminus \{e\}) \cup \{e\} \subseteq S'$. Thus, $C = (C \setminus \{e\}) \cup \{e\} \subseteq S'$. Since $S' \in \mathcal{I}$, this entails that $C \in \mathcal{I}$ (by Matroid axiom 2). But $C \in \mathcal{P}(E) \setminus \mathcal{I}$ (since C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$), so that $C \notin \mathcal{I}$. This contradicts $C \in \mathcal{I}$. This contradiction shows that our assumption was wrong. Hence, $C \subseteq T$ is proven. Therefore, Statement $\mathfrak{F}_2$ holds. Thus, the implication $\mathfrak{F}_1 \Longrightarrow \mathfrak{F}_2$ is proven.

Proof of the implication $\mathfrak{F}_2 \Longrightarrow \mathfrak{F}_1$: Assume that Statement $\mathfrak{F}_2$ holds. We must prove that Statement $\mathfrak{F}_1$ holds.

Let $k = r_M(T)$. We shall show that T is a k-flat of M.

Let W be a subset of E that has rank k and satisfies $T \subseteq W$. We shall show that $T = W$.

Indeed, assume the contrary. Thus, $T \neq W$. Combined with $T \subseteq W$, this shows that T is a proper subset of W. Thus, there exists an $e \in W \setminus T$. Consider this e. We have $e \notin T$ (since $e \in W \setminus T$).

We have

$$\begin{aligned} k = r_M(T) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \qquad \text{(by the definition of } r_M) \\ &\in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \end{aligned}$$

(since the maximum of a set must belong to that set). Hence, there exists some $Z \in \mathcal{I}$ satisfying $Z \subseteq T$ and $k = |Z|$. Denote this Z by K. Thus, K is an element

[28] *Proof.* Assume the contrary. Thus, $S' \subseteq T$. Hence, S' is an element of $\mathcal{I}$ and satisfies $S' \subseteq T$. Thus, $|S'| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$.

Now, the definition of r_M yields

$$r_M(T) = \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\} \geq |S'|$$

(since $|S'| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq T\}$). This contradicts $|S'| > r_M(T)$. This contradiction proves that our assumption was wrong, qed.

of $\mathcal{I}$ and satisfies $K \subseteq T$ and $k = |K|$. Notice that $e \notin T$, so that $e \notin K$ (since $K \subseteq T$).

We have $r_M(W) = k$ (since W has rank k). Hence, $K \cup \{e\} \notin \mathcal{I}$ [29]. In other words, $K \cup \{e\} \in \mathcal{P}(E) \setminus \mathcal{I}$. Hence, Lemma 8.7 (applied to $Q = K \cup \{e\}$) shows that there exists a circuit C of M such that $C \subseteq K \cup \{e\}$. Consider this C. From $C \subseteq K \cup \{e\}$, we obtain $C \setminus \{e\} \subseteq K \subseteq T$.

From $C \setminus \{e\} \subseteq K$, we conclude (using Matroid axiom 2) that $C \setminus \{e\} \in \mathcal{I}$ (since $K \in \mathcal{I}$). On the other hand, C is a circuit of M. In other words, C is a minimal element of $\mathcal{P}(E) \setminus \mathcal{I}$ (by the definition of a "circuit"). Hence, $C \in \mathcal{P}(E) \setminus \mathcal{I}$, so that $C \notin \mathcal{I}$. Hence, $e \in C$ (since otherwise, we would have $C \setminus \{e\} = C \notin \mathcal{I}$, which would contradict $C \setminus \{e\} \in \mathcal{I}$). Now, Statement $\mathfrak{F}_2$ shows that $C \subseteq T$. Hence, $e \in C \subseteq T$, which contradicts $e \notin T$.

This contradiction shows that our assumption was wrong. Hence, $T = W$ is proven.

Now, forget that we fixed W. Thus, we have shown that if W is a subset of E that has rank k and satisfies $T \subseteq W$, then $T = W$. In other words, T is a subset of E that has rank k and is maximal among all such subsets (because we already know that T has rank $r_M(T) = k$). In other words, T is a k-flat of M (by the definition of a "k-flat"). Thus, T is a flat of M. In other words, Statement $\mathfrak{F}_1$ holds. This proves the implication $\mathfrak{F}_2 \Longrightarrow \mathfrak{F}_1$.

We have now proven the implications $\mathfrak{F}_1 \Longrightarrow \mathfrak{F}_2$ and $\mathfrak{F}_2 \Longrightarrow \mathfrak{F}_1$. Together, these implications show that Statements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent. This proves Lemma 8.9. □

Corollary 8.10. Let $M = (E, \mathcal{I})$ be a matroid. Let $F_1, F_2, \ldots, F_k$ be flats of M. Then, $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of M. (Notice that k is allowed to be 0 here; in this case, the empty intersection $F_1 \cap F_2 \cap \cdots \cap F_k$ is to be interpreted as E.)

Proof of Corollary 8.10. Lemma 8.9 gives a necessary and sufficient criterion for a subset T of E to be a flat of M. It is easy to see that if this criterion is satisfied for $T = F_1$, for $T = F_2$, etc., and for $T = F_k$, then it is satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$. In other words, if $F_1, F_2, \ldots, F_k$ are flats of M, then $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of M. [30] This proves Corollary 8.10. □

[29] *Proof.* Assume the contrary. Thus, $K \cup \{e\} \in \mathcal{I}$. Thus, $r_M(K \cup \{e\}) = |K \cup \{e\}|$ (by Lemma 8.6). Thus, $r_M(K \cup \{e\}) = |K \cup \{e\}| > |K|$ (since $e \notin K$).

But $K \cup \{e\} \subseteq W$ (since $K \subseteq T \subseteq W$ and $e \in W \setminus T \subseteq W$). Since the function r_M is weakly increasing, this yields $r_M(K \cup \{e\}) \leq r_M(W) = k = |K|$. This contradicts $r_M(K \cup \{e\}) > |K|$. This contradiction proves that our assumption was wrong, qed.

[30] Here is this argument in slightly more detail:

For every $i \in \{1, 2, \ldots, k\}$, the following statement holds: If C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_i$, then

$$C \subseteq F_i. \tag{41}$$

Proof of (41): Let $i \in \{1, 2, \ldots, k\}$. Then, the set F_i is a flat of M. In other words, Statement $\mathfrak{F}_1$ of Lemma 8.9 is satisfied for $T = F_i$. Therefore, Statement $\mathfrak{F}_2$ of Lemma 8.9 must also be satisfied for $T = F_i$ (since Lemma 8.9 shows that the Statements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent). In

Corollary 8.10 (a well-known fact, which is left to the reader to prove in [Stanle06, §3.1]) allows us to define the *closure* of a set in a matroid:

Definition 8.11. Let $M = (E, \mathcal{I})$ be a matroid. Let T be a subset of E. The *closure* of T is defined to be the intersection of all flats of M which contain T as a subset. In other words, the closure of T is defined to be $\bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$. The closure of T is denoted by $\overline{T}$.

The following proposition gathers some simple properties of closures in matroids:

Proposition 8.12. Let $M = (E, \mathcal{I})$ be a matroid.

(a) If T is a subset of E, then $\overline{T}$ is a flat of M satisfying $T \subseteq \overline{T}$.

(b) If G is a flat of M, then $\overline{G} = G$.

(c) If T is a subset of E and if G is a flat of M satisfying $T \subseteq G$, then $\overline{T} \subseteq G$.

(d) If S and T are two subsets of E satisfying $S \subseteq T$, then $\overline{S} \subseteq \overline{T}$.

(e) If the matroid M is loopless, then $\overline{\varnothing} = \varnothing$.

(f) Every subset T of E satisfies $r_M(T) = r_M\left(\overline{T}\right)$.

(g) If T is a subset of E and if G is a flat of M, then the conditions $\left(\overline{T} \subseteq G\right)$ and $(T \subseteq G)$ are equivalent.

Proof of Proposition 8.12. **(a)** The set $\operatorname{Flats} M$ is a subset of the finite set $\mathcal{P}(E)$, and thus itself finite.

Let T be a subset of E. The closure $\overline{T}$ of T is defined as $\bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$. Now, Corollary 8.10 shows that any intersection of finitely many flats of M is a flat of M. Hence, $\bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$ (being an intersection of finitely many flats of M [31]) is a flat of M. In other words, $\overline{T}$ is a flat of M (since $\overline{T} = \bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$).

Also, $T \subseteq F$ for every $F \in \operatorname{Flats} M$ satisfying $T \subseteq F$. Hence, $T \subseteq \bigcap\limits_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F = \overline{T}$. This completes the proof of Proposition 8.12 **(a)**.

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other words, if C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_i$, then $C \subseteq F_i$. This proves (41).

Now, let C be a circuit of M, and let $e \in C$ be such that $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$. For every $i \in \{1, 2, \ldots, k\}$, we have $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k \subseteq F_i$, and therefore $C \subseteq F_i$ (by (41)). So we have shown the inclusion $C \subseteq F_i$ for each $i \in \{1, 2, \ldots, k\}$. Combining these k inclusions, we obtain $C \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$.

Now, forget that we fixed C. We thus have shown that if C is a circuit of M, and if $e \in C$ is such that $C \setminus \{e\} \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$, then $C \subseteq F_1 \cap F_2 \cap \cdots \cap F_k$. In other words, Statement $\mathfrak{F}_2$ of Lemma 8.9 is satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$. Therefore, Statement $\mathfrak{F}_1$ of Lemma 8.9 must also be satisfied for $T = F_1 \cap F_2 \cap \cdots \cap F_k$ (since Lemma 8.9 shows that the Statements $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are equivalent). In other words, the set $F_1 \cap F_2 \cap \cdots \cap F_k$ is a flat of M. Qed.

[31] "Finitely many" since the set $\operatorname{Flats} M$ is finite.

(c) Let T be a subset of E, and let G be a flat of M satisfying $T \subseteq G$. Then, G is an element of $\operatorname{Flats} M$ satisfying $T \subseteq G$. Hence, G is one term in the intersection $\bigcap_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F$. Thus, $\bigcap_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F \subseteq G$. But the definition of $\overline{T}$ yields $\overline{T} = \bigcap_{\substack{F \in \operatorname{Flats} M; \\ T \subseteq F}} F \subseteq G$. This proves Proposition 8.12 **(c)**.

(b) Let G be a flat of M. Proposition 8.12 **(b)** (applied to $T = G$) yields $\overline{G} \subseteq G$. But Proposition 8.12 **(a)** (applied to $T = G$) shows that $\overline{G}$ is a flat of M satisfying $G \subseteq \overline{G}$. Combining $G \subseteq \overline{G}$ with $\overline{G} \subseteq G$, we obtain $\overline{G} = G$. This proves Proposition 8.12 **(b)**.

(d) Let S and T be two subsets of E satisfying $S \subseteq T$. Proposition 8.12 **(a)** shows that $\overline{T}$ is a flat of M satisfying $T \subseteq \overline{T}$. Now, $S \subseteq T \subseteq \overline{T}$. Hence, Proposition 8.12 **(b)** (applied to S and $\overline{T}$ instead of T and G) shows $\overline{S} \subseteq \overline{T}$. This proves Proposition 8.12 **(d)**.

(e) Assume that the matroid M is loopless. In other words, no loops (of M) exist.

The definition of r_M quickly yields $r_M(\varnothing) = 0$. In other words, the set $\varnothing$ has rank 0. We shall now show that $\varnothing$ is a 0-flat of M.

Indeed, let W be a subset of E that has rank 0 and satisfies $\varnothing \subseteq W$. We shall show that $\varnothing = W$.

Assume the contrary. Thus, $\varnothing \neq W$. Hence, W has an element w. Consider this w. The element w of E is not a loop (since no loops exist). In other words, we cannot have $\{w\} \notin \mathcal{I}$ (since w is a loop if and only if $\{w\} \notin \mathcal{I}$ (by the definition of a loop)). In other words, we must have $\{w\} \in \mathcal{I}$. Clearly, $\{w\} \subseteq W$ (since $w \in W$). Thus, $\{w\}$ is a $Z \in \mathcal{I}$ satisfying $Z \subseteq W$. Thus, $|\{w\}| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\}$.

But W has rank 0. In other words,

$$\begin{aligned}
0 = r_M(W) &= \max\{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\} \qquad (\text{by the definition of } r_M) \\
&\geq |\{w\}| \qquad (\text{since } |\{w\}| \in \{|Z| \mid Z \in \mathcal{I} \text{ and } Z \subseteq W\}) \\
&= 1,
\end{aligned}$$

which is absurd. This contradiction shows that our assumption was wrong. Hence, $\varnothing = W$ is proven.

Let us now forget that we fixed W. We thus have proven that if W is any subset of E that has rank 0 and satisfies $\varnothing \subseteq W$, then $\varnothing = W$. Thus, $\varnothing$ is a subset of E that has rank 0 and is maximal among all such subsets (because we already know that $\varnothing$ has rank 0). In other words, $\varnothing$ is a 0-flat of M (by the definition of a "0-flat"). Thus, $\varnothing$ is a flat of M. Thus, Proposition 8.12 **(b)** (applied to $G = \varnothing$) yields $\overline{\varnothing} = \varnothing$. This proves Proposition 8.12 **(e)**.

(f) Let T be a subset of E. We have $T \subseteq \overline{T}$ (by Proposition 8.12 **(a)**), and thus $r_M(T) \leq r_M(\overline{T})$ (since the function r_M is weakly increasing).

Let $k = r_M(T)$. Thus, there exists a $Q \in \mathcal{P}(E)$ satisfying $T \subseteq Q$ and $k = r_M(Q)$ (namely, $Q = T$). Hence, there exists a **maximal** such Q. Denote this Q

by R. Thus, R is a maximal $Q \in \mathcal{P}(E)$ satisfying $T \subseteq Q$ and $k = r_M(Q)$. In particular, R is an element of $\mathcal{P}(E)$ and satisfies $T \subseteq R$ and $k = r_M(R)$.

Now, R is a subset of E (since $R \in \mathcal{P}(E)$) and has rank $r_M(R) = k$. Thus, R is a subset of E that has rank k. Furthermore, R is maximal among all such subsets[32]. Thus, R is a k-flat of M (by the definition of a "k-flat"), and therefore a flat of M. Now, Proposition 8.12 **(c)** (applied to $G = R$) shows that $\overline{T} \subseteq R$. Since the function r_M is weakly increasing, this yields $r_M\left(\overline{T}\right) \leq r_M(R) = k$. Combining this with $k = r_M(T) \leq r_M\left(\overline{T}\right)$, we obtain $r_M\left(\overline{T}\right) = k = r_M(T)$. This proves Proposition 8.12 **(f)**.

(g) Let T be a subset of E. Let G be a flat of M. Proposition 8.12 **(a)** shows that $T \subseteq \overline{T}$. Hence, if $\overline{T} \subseteq G$, then $T \subseteq \overline{T} \subseteq G$. Thus, we have proven the implication $\left(\overline{T} \subseteq G\right) \Longrightarrow (T \subseteq G)$. The reverse implication (i.e., the implication $(T \subseteq G) \Longrightarrow \left(\overline{T} \subseteq G\right)$) follows from Proposition 8.12 **(c)**. Combining these two implications, we obtain the equivalence $\left(\overline{T} \subseteq G\right) \Longleftrightarrow (T \subseteq G)$. This proves Proposition 8.12 **(g)**. $\square$

We shall now recall a few more classical notions related to posets:

Definition 8.13. Let P be a poset.

(a) An element $p \in P$ is said to be a *global minimum* of P if every $q \in P$ satisfies $p \leq q$. Clearly, a global minimum of P is unique if it exists.

(b) An element $p \in P$ is said to be a *global maximum* of P if every $q \in P$ satisfies $p \geq q$. Clearly, a global maximum of P is unique if it exists.

(c) Let x and y be two elements of P. An *upper bound* of x and y (in P) means an element $z \in P$ satisfying $z \geq x$ and $z \geq y$. A *join* (or *least upper bound*) of x and y (in P) means an upper bound z of x and y such that every upper bound z' of x and y satisfies $z' \geq z$. In other words, a join of x and y is a global minimum of the subposet $\{w \in P \mid w \geq x \text{ and } w \geq y\}$ of P. Thus, a join of x and y is unique if it exists.

(d) Let x and y be two elements of P. A *lower bound* of x and y (in P) means an element $z \in P$ satisfying $z \leq x$ and $z \leq y$. A *meet* (or *greatest lower bound*) of x and y (in P) means a lower bound z of x and y such that every lower bound z' of x and y satisfies $z' \leq z$. In other words, a meet of x and y is a global maximum of the subposet $\{w \in P \mid w \leq x \text{ and } w \leq y\}$ of P. Thus, a meet of x and y is unique if it exists.

(e) The poset P is said to be a *lattice* if and only if it has a global minimum and a global maximum, and every two elements of P have a meet and a join.

[32] *Proof.* Let W be any subset of E that has rank k and satisfies $W \supseteq R$. We must prove that $W = R$.

We have $W \in \mathcal{P}(E)$, $T \subseteq R \subseteq W$ and $k = r_M(W)$ (since W has rank k). Thus, W is a $Q \in \mathcal{P}(E)$ satisfying $T \subseteq Q$ and $k = r_M(Q)$. But recall that R is a **maximal** such Q. Hence, if $W \supseteq R$, then $W = R$. Therefore, $W = R$ (since we know that $W \supseteq R$). Qed.

Proposition 8.14. Let $M = (E, \mathcal{I})$ be a matroid. The subposet $\operatorname{Flats} M$ of the poset $\mathcal{P}(E)$ is a lattice.

Proof of Proposition 8.14. By the definition of a lattice, it suffices to check the following four claims:

Claim 1: The poset $\operatorname{Flats} M$ has a global minimum.

Claim 2: The poset $\operatorname{Flats} M$ has a global maximum.

Claim 3: Every two elements of $\operatorname{Flats} M$ have a meet (in $\operatorname{Flats} M$).

Claim 4: Every two elements of $\operatorname{Flats} M$ have a join (in $\operatorname{Flats} M$).

Proof of Claim 1: Applying Proposition 8.12 **(a)** to $T = \varnothing$, we see that $\overline{\varnothing}$ is a flat of M satisfying $\varnothing \subseteq \overline{\varnothing}$. In particular, $\overline{\varnothing}$ is a flat of M, so that $\overline{\varnothing} \in \operatorname{Flats} M$. If G is a flat of M, then $\overline{\varnothing} \subseteq G$ (by Proposition 8.12 **(c)**, applied to $T = \varnothing$). Hence, $\overline{\varnothing}$ is a global minimum of the poset $\operatorname{Flats} M$. Thus, the poset $\operatorname{Flats} M$ has a global minimum. This proves Claim 1.

Proof of Claim 2: Applying Proposition 8.12 **(a)** to $T = E$, we see that $\overline{E}$ is a flat of M satisfying $E \subseteq \overline{E}$. From $E \subseteq \overline{E}$, we conclude that $\overline{E} = E$. Thus, E is a flat of M (since $\overline{E}$ is a flat of M). In other words, $E \in \operatorname{Flats} M$. If G is a flat of M, then $E \supseteq G$ (obviously). Hence, E is a global maximum of the poset $\operatorname{Flats} M$. Thus, the poset $\operatorname{Flats} M$ has a global maximum. This proves Claim 2.

Proof of Claim 3: Let F and G be two elements of $\operatorname{Flats} M$. We have to prove that F and G have a meet.

We know that F and G are elements of $\operatorname{Flats} M$, thus flats of M. Hence, Corollary 8.10 shows that $F \cap G$ is a flat of M. In other words, $F \cap G \in \operatorname{Flats} M$. Clearly, $F \cap G \subseteq F$ and $F \cap G \subseteq G$; thus, $F \cap G$ is a lower bound of F and G in $\operatorname{Flats} M$. Also, every lower bound H of F and G in $\operatorname{Flats} M$ satisfies $H \subseteq F \cap G$ [33]. Hence, $F \cap G$ is a meet of F and G. Thus, F and G have a meet. This proves Claim 3.

Proof of Claim 4: Let F and G be two elements of $\operatorname{Flats} M$. We have to prove that F and G have a join.

We know that F and G are elements of $\operatorname{Flats} M$, thus flats of M. Proposition 8.12 **(a)** (applied to $T = F \cup G$) shows that $\overline{F \cup G}$ is a flat of M satisfying $F \cup G \subseteq \overline{F \cup G}$. Now, $\overline{F \cup G} \in \operatorname{Flats} M$ (since $\overline{F \cup G}$ is a flat of M). Clearly, $F \subseteq F \cup G \subseteq \overline{F \cup G}$ and $G \subseteq F \cup G \subseteq \overline{F \cup G}$; thus, $\overline{F \cup G}$ is an upper bound of F and G in $\operatorname{Flats} M$. Also, every upper bound H of F and G in $\operatorname{Flats} M$ satisfies $H \supseteq \overline{F \cup G}$ [34]. Hence, $\overline{F \cup G}$ is a join of F and G. Thus, F and G have a join. This proves Claim 4.

We have now proven all four Claims 1, 2, 3, and 4. Thus, Proposition 8.14 is proven. $\square$

[33] *Proof.* Let H be a lower bound of F and G in $\operatorname{Flats} M$. Thus, $H \subseteq F$ and $H \subseteq G$. Combining these two inclusions, we obtain $H \subseteq F \cap G$, qed.

[34] *Proof.* Let H be an upper bound of F and G in $\operatorname{Flats} M$. Thus, $H \supseteq F$ and $H \supseteq G$. Combining these two inclusions, we obtain $H \supseteq F \cup G$. But $H \in \operatorname{Flats} M$; thus, H is a flat of M. Since H satisfies $F \cup G \subseteq H$, we therefore obtain $\overline{F \cup G} \subseteq \overline{H}$ (by Proposition 8.12 **(c)**, applied to $F \cup G$ and H instead of T and G). In other words, $H \supseteq \overline{F \cup G}$, qed.

Definition 8.15. Let $M = (E, \mathcal{I})$ be a matroid. Proposition 8.14 shows that the subposet Flats M of the poset $\mathcal{P}(E)$ is a lattice. This subposet Flats M is called the *lattice of flats* of M. (Beware: It is a subposet, but not a sublattice of $\mathcal{P}(E)$, since its join is not a restriction of the join of $\mathcal{P}(E)$.)

The lattice of flats Flats M of a matroid M is denoted by $L(M)$ in [Stanle06, §3.2].

Next, we recall the definition of the Möbius function of a poset (see, e.g., [Stanle06, Definition 1.2] or [Martin22, §2.2]):

Definition 8.16. Let P be a poset.

(a) If x and y are two elements of P satisfying $x \leq y$, then the set $\{z \in P \mid x \leq z \leq y\}$ is denoted by $[x, y]$.

(b) A subset of P is called a *closed interval* of P if it has the form $[x, y]$ for two elements x and y of P satisfying $x \leq y$.

(c) We denote by Int P the set of all closed intervals of P.

(d) If $f : \operatorname{Int} P \to \mathbb{Z}$ is any map, then the image $f([x, y])$ of a closed interval $[x, y] \in \operatorname{Int} P$ under f will be abbreviated by $f(x, y)$.

(e) Assume that every closed interval of P is finite. The *Möbius function* of the poset P is defined to be the unique function $\mu : \operatorname{Int} P \to \mathbb{Z}$ having the following two properties:

- We have
$$\mu(x, x) = 1 \qquad \text{for every } x \in P. \tag{42}$$

- We have
$$\mu(x, y) = -\sum_{\substack{z \in P; \\ x \leq z < y}} \mu(x, z) \tag{43}$$
for all $x, y \in P$ satisfying $x < y$.

(It is easy to see that these two properties indeed determine μ uniquely.) This Möbius function is denoted by μ.

We can now define the characteristic polynomial of a matroid M, following [Stanle06, (22)][35]:

Definition 8.17. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. The *characteristic polynomial* χ_M of the matroid M is defined to be the polynomial

$$\sum_{F \in \operatorname{Flats} M} \mu\left(\overline{\varnothing}, F\right) x^{m - r_M(F)} \in \mathbb{Z}[x]$$

[35]Our notation slightly differs from that in [Stanle06, (22)]. Namely, we use x as the indeterminate, while Stanley instead uses t. Stanley also denotes the global minimum $\overline{\varnothing}$ of Flats M by $\widehat{0}$.

(where μ is the Möbius function of the lattice Flats M). We further define a polynomial $\widetilde{\chi}_M \in \mathbb{Z}[x]$ by $\widetilde{\chi}_M = [\overline{\varnothing} = \varnothing] \chi_M$. Here, we are using the Iverson bracket notation (as in Definition 2.7). If the matroid M is loopless, then

$$\widetilde{\chi}_M = \underbrace{[\overline{\varnothing} = \varnothing]}_{\substack{=1 \\ \text{(by Proposition 8.12 (e))}}} \chi_M = \chi_M.$$

Example 8.18. Let $G = (V, E)$ be a finite graph. Consider the graphical matroid $(E, \mathcal{I})$ defined as in Example 8.5. Then, the characteristic polynomial $\chi_{(E,\mathcal{I})}$ of this matroid is connected to the chromatic polynomial χ_G of the graph G as follows:

$$x^{\operatorname{conn} G} \cdot \chi_{(E,\mathcal{I})}(x) = \chi_G(x).$$

This equality is a classical result (see, e.g., [Zaslav87, Proposition 7.5.1]), but can also be derived from our results below (specifically, by comparing Theorem 8.21 with Theorem 3.4).

Note that Zaslavsky, in [Zaslav87, §7.2], defines the "characteristic polynomial" of a matroid M to be our $\widetilde{\chi}_M$ instead of our χ_M; but this makes no difference when M is the graphical matroid from Example 8.5, since such a matroid M is always loopless.

8.3. Generalized Whitney formulas

Let us next define broken circuits of a matroid $M = (E, \mathcal{I})$. Stanley, in [Stanle06, §4.1], defines them in terms of a total ordering $\mathcal{O}$ on the set E, whereas we shall use a "labeling function" $\ell : E \to X$ instead (as in the case of graphs); our setting is slightly more general than Stanley's.

Definition 8.19. Let $M = (E, \mathcal{I})$ be a matroid. Let X be a totally ordered set. Let $\ell : E \to X$ be a function. We shall refer to ℓ as the *labeling function*. For every $e \in E$, we shall refer to $\ell(e)$ as the *label* of e.

A *broken circuit* of M means a subset of E having the form $C \setminus \{e\}$, where C is a circuit of M, and where e is the unique element of C having maximum label (among the elements of C). Of course, the notion of a broken circuit of M depends on the function ℓ; however, we suppress the mention of ℓ in our notation, since we will not consider situations where two different ℓ's coexist.

We shall now state analogues (and, in light of Example 8.18, generalizations, although we shall not elaborate on the few minor technicalities of seeing them as such) of Theorem 3.5, Theorem 3.4, Corollary 3.6, Corollary 3.7 and Corollary 3.14:

Theorem 8.20. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$. Then,

$$\widetilde{\chi}_M = \sum_{F \subseteq E} (-1)^{|F|} \left(\prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq F}} a_K \right) x^{m - r_M(F)}.$$

Theorem 8.21. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Then,

$$\widetilde{\chi}_M = \sum_{F \subseteq E} (-1)^{|F|} x^{m - r_M(F)}.$$

Corollary 8.22. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ is } \mathfrak{K}\text{-free}}} (-1)^{|F|} x^{m - r_M(F)}.$$

Corollary 8.23. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} x^{m - r_M(F)}.$$

Corollary 8.24. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let X be a totally ordered set. Let $\ell : E \to X$ be an injective labeling function. Then,

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} x^{m - |F|}.$$

We notice that Corollary 8.24 is equivalent to [Stanle06, Theorem 4.12] (at least when M is loopless).

Before we prove these results, let us state a lemma which will serve as an analogue of Lemma 2.8:

Lemma 8.25. Let $M = (E, \mathcal{I})$ be a matroid. Let X be a totally ordered set. Let $\ell : E \to X$ be a labeling function. Let $\mathfrak{K}$ be some set of broken circuits of M (not necessarily containing all of them). Let a_K be an element of $\mathbf{k}$ for every $K \in \mathfrak{K}$.

Let F be any flat of M. Then,

$$\sum_{B \subseteq F} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K = [F = \varnothing]. \tag{44}$$

(Again, we are using the Iverson bracket notation as in Definition 2.7.)

Proof of Lemma 8.25. Our proof will imitate the proof of Lemma 2.8 much of the time (with $E \cap \operatorname{Eqs} f$ replaced by F); thus, we will allow ourselves some more brevity.

We WLOG assume that $F \neq \varnothing$ (since otherwise, the claim is obvious[36]). Thus, $[F = \varnothing] = 0$.

Pick any $d \in F$ with maximum $\ell(d)$ (among all $d \in F$). (This is clearly possible, since $F \neq \varnothing$.) Define two subsets $\mathcal{U}$ and $\mathcal{V}$ of $\mathcal{P}(F)$ as follows:

$$\mathcal{U} = \{T \in \mathcal{P}(F) \mid d \notin T\};$$
$$\mathcal{V} = \{T \in \mathcal{P}(F) \mid d \in T\}.$$

[36] *Proof.* Assume that $F = \varnothing$. We must show that the claim is obvious.

Let us first show that no $K \in \mathfrak{K}$ satisfies $K = \varnothing$. Indeed, assume the contrary. Thus, there exists a $K \in \mathfrak{K}$ satisfying $K = \varnothing$. In other words, $\varnothing \in \mathfrak{K}$. Thus, $\varnothing$ is a broken circuit of M (since $\mathfrak{K}$ is a set of broken circuits of M). Therefore, $\varnothing$ is obtained from a circuit of M by removing one element (by the definition of a broken circuit). This latter circuit must therefore be a one-element set, i.e., it has the form $\{e\}$ for some $e \in E$. Consider this e. Thus, $\{e\}$ is a circuit of M.

But F is a flat of M. In other words, Statement $\mathfrak{F}_1$ (of Lemma 8.9) holds for $T = F$. Hence, Statement $\mathfrak{F}_2$ (of Lemma 8.9) also holds for $T = F$ (since Lemma 8.9 shows that these two statements are equivalent). Applying Statement $\mathfrak{F}_2$ to $T = F$ and $C = \{e\}$, we thus obtain $\{e\} \subseteq F$ (because $\{e\} \setminus \{e\} = \varnothing \subseteq F$). Thus, $e \in \{e\} \subseteq F = \varnothing$, which is absurd. This contradiction proves that our assumption was wrong.

Hence, we have shown that no $K \in \mathfrak{K}$ satisfies $K = \varnothing$. But from $F = \varnothing$, we see that the sum $\sum\limits_{B \subseteq F} (-1)^{|B|} \prod\limits_{\substack{K \in \mathfrak{K};\\ K \subseteq B}} a_K$ has only one addend (namely, the addend for $B = \varnothing$), and thus simplifies to

$$\underbrace{(-1)^{|\varnothing|}}_{=(-1)^0=1} \underbrace{\prod_{\substack{K \in \mathfrak{K};\\ K \subseteq \varnothing}}}_{= \prod_{\substack{K \in \mathfrak{K};\\ K = \varnothing}}} a_K = \prod_{\substack{K \in \mathfrak{K};\\ K = \varnothing}} a_K = (\text{empty product}) \qquad (\text{since no } K \in \mathfrak{K} \text{ satisfies } K = \varnothing)$$
$$= 1 = [F = \varnothing] \qquad (\text{since } F = \varnothing).$$

Thus, Lemma 8.25 is proven.

Thus, we have $\mathcal{P}(F) = \mathcal{U} \cup \mathcal{V}$, and the sets $\mathcal{U}$ and $\mathcal{V}$ are disjoint. Now, we define a map $\Phi : \mathcal{U} \to \mathcal{V}$ by

$$(\Phi(B) = B \cup \{d\} \qquad \text{for every } B \in \mathcal{U}).$$

This map Φ is well-defined (because for every $B \in \mathcal{U}$, we have $B \cup \{d\} \in \mathcal{V}$ [37]) and a bijection[38]. Moreover, every $B \in \mathcal{U}$ satisfies

$$(-1)^{|\Phi(B)|} = -(-1)^{|B|} \tag{45}$$

[39].

Now, we claim that, for every $B \in \mathcal{U}$ and every $K \in \mathfrak{K}$, we have the following logical equivalence:

$$(K \subseteq B) \iff (K \subseteq \Phi(B)). \tag{46}$$

Proof of (46): Let $B \in \mathcal{U}$ and $K \in \mathfrak{K}$. We must prove the equivalence (46). The definition of Φ yields $\Phi(B) = B \cup \{d\} \supseteq B$, so that $B \subseteq \Phi(B)$. Hence, if $K \subseteq B$, then $K \subseteq B \subseteq \Phi(B)$. Therefore, the forward implication of the equivalence (46) is proven. It thus remains to prove the backward implication of this equivalence. In other words, it remains to prove that if $K \subseteq \Phi(B)$, then $K \subseteq B$. So let us assume that $K \subseteq \Phi(B)$.

We want to prove that $K \subseteq B$. Assume the contrary. Thus, $K \not\subseteq B$. We have $K \in \mathfrak{K}$. Thus, K is a broken circuit of M (since $\mathfrak{K}$ is a set of broken circuits of M). In other words, K is a subset of E having the form $C \setminus \{e\}$, where C is a circuit of M, and where e is the unique element of C having maximum label (among the elements of C) (because this is how a broken circuit is defined). Consider these C and e. Thus, $K = C \setminus \{e\}$.

The element e is the unique element of C having maximum label (among the elements of C). Thus, if e' is any element of C satisfying $\ell(e') \geq \ell(e)$, then

$$e' = e. \tag{47}$$

But $\underbrace{K}_{\subseteq \Phi(B) = B \cup \{d\}} \setminus \{d\} \subseteq (B \cup \{d\}) \setminus \{d\} \subseteq B$.

If we had $d \notin K$, then we would have $K \setminus \{d\} = K$ and therefore $K = K \setminus \{d\} \subseteq B$; this would contradict $K \not\subseteq B$. Hence, we cannot have $d \notin K$. We thus must have $d \in K$. Hence, $d \in K = C \setminus \{e\}$. Hence, $d \in C$ and $d \neq e$.

But $C \setminus \{e\} = K \subseteq \Phi(B) \subseteq F$ (since $\Phi(B) \in \mathcal{P}(F)$). On the other hand, Statement $\mathfrak{F}_1$ (of Lemma 8.9) holds for $T = F$ (since F is a flat of M). Hence, Statement $\mathfrak{F}_2$ (of Lemma 8.9) also holds for $T = F$ (since Lemma 8.9 shows that these two statements are equivalent). Thus, from $C \setminus \{e\} \subseteq F$, we obtain $C \subseteq F$. Thus, $e \in C \subseteq F$. Consequently, $\ell(d) \geq \ell(e)$ (since d was defined to be an element of F with maximum $\ell(d)$ among all $d \in F$).

[37] This follows from the fact that $d \in F$.

[38] Its inverse is the map $\Psi : \mathcal{V} \to \mathcal{U}$ defined by $(\Psi(B) = B \setminus \{d\} \qquad \text{for every } B \in \mathcal{V})$.

[39] *Proof.* This is proven exactly like we proved (9).

Also, $d \in C$. Since $\ell(d) \geq \ell(e)$, we can therefore apply (47) to $e' = d$. We thus obtain $d = e$. This contradicts $d \neq e$. This contradiction proves that our assumption was wrong. Hence, $K \subseteq B$ is proven. Thus, we have proven the backward implication of the equivalence (46); this completes the proof of (46).

Now, proceeding as in the proof of (12), we can show that

$$\sum_{B \subseteq F} (-1)^{|B|} \prod_{\substack{K \in \mathfrak{K}; \\ K \subseteq B}} a_K = [F = \varnothing].$$

This proves Lemma 8.25. $\square$

We shall furthermore use a classical and fundamental result on the Möbius function of any finite poset:

Proposition 8.26. Let P be a finite poset. Let μ denote the Möbius function of P.

(a) For any $x \in P$ and $y \in P$, we have

$$\sum_{\substack{z \in P; \\ x \leq z \leq y}} \mu(x, z) = [x = y]. \tag{48}$$

(b) For any $x \in P$ and $y \in P$, we have

$$\sum_{\substack{z \in P; \\ x \leq z \leq y}} \mu(z, y) = [x = y]. \tag{49}$$

(c) Let $\mathbf{k}$ be a $\mathbb{Z}$-module. Let $(\beta_x)_{x \in P}$ be a family of elements of $\mathbf{k}$. Then, every $z \in P$ satisfies

$$\beta_z = \sum_{\substack{y \in P; \\ y \leq z}} \mu(y, z) \sum_{\substack{x \in P; \\ x \leq y}} \beta_x.$$

For the sake of completeness, let us give a self-contained proof of this proposition (slicker arguments appear in the literature[40]):

Proof of Proposition 8.26. **(a)** Let $x \in P$ and $y \in P$. We must prove the equality (48). We are in one of the following three cases:

Case 1: We have $x = y$.

Case 2: We have $x < y$.

Case 3: We have neither $x = y$ nor $x < y$.

[40]For example, Proposition 8.26 **(c)** is equivalent to the $\Longrightarrow$ implication of [Martin22, (2.3a)].

Let us first consider Case 1. In this case, we have $x = y$. Hence, the sum $\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x,z)$ contains only one addend – namely, the addend for $z = x$. Thus,

$$\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x,z) = \mu(x,x) = 1 \qquad \text{(by the definition of the Möbius function)}$$
$$= [x = y] \qquad (\text{since } x = y).$$

Thus, (48) is proven in Case 1.

Let us now consider Case 2. In this case, we have $x < y$. Hence, $x \neq y$, so that $[x = y] = 0$. Now, y is an element of P satisfying $x \leq y \leq y$. Thus, the sum $\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x,z)$ contains an addend for $z = y$. Splitting off this addend, we obtain

$$\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x,z) = \underbrace{\sum_{\substack{z \in P;\\ x \leq z \leq y;\ z \neq y}}}_{= \sum_{\substack{z \in P;\\ x \leq z < y}}} \mu(x,z) + \underbrace{\mu(x,y)}_{\substack{= -\sum_{\substack{z \in P;\\ x \leq z < y}} \mu(x,z) \\ \text{(by (43))}}}$$
$$= \sum_{\substack{z \in P;\\ x \leq z < y}} \mu(x,z) + \left(- \sum_{\substack{z \in P;\\ x \leq z < y}} \mu(x,z) \right) = 0 = [x = y].$$

Hence, (48) is proven in Case 2.

Finally, let us consider Case 3. In this case, we have neither $x = y$ nor $x < y$. Thus, we do not have $x \leq y$. Hence, there exists no $z \in P$ satisfying $x \leq z \leq y$. Thus,

$$\sum_{\substack{z \in P;\\ x \leq z \leq y}} \mu(x,z) = (\text{empty sum}) = 0 = [x = y]$$

(since we do not have $x = y$). Thus, (48) is proven in Case 3.

Hence, (48) is proven in all three cases. This proves Proposition 8.26 **(a)**.

(b) For any two elements u and v of P, we define a subset $[u,v]$ of P by

$$[u,v] = \{w \in P \mid u \leq w \leq v\}.$$

This subset $[u,v]$ is finite (since P is finite), and thus its size $|[u,v]|$ is a nonnegative integer.

We shall now prove Proposition 8.26 **(b)** by strong induction on $|[x,y]|$:

Induction step: Let $N \in \mathbb{N}$. Assume that Proposition 8.26 **(b)** holds whenever $|[x,y]| < N$. We must now prove that Proposition 8.26 **(b)** holds whenever $|[x,y]| = N$.

We have assumed that Proposition 8.26 **(b)** holds whenever $|[x,y]| < N$. In other words, we have assumed the following claim:

Claim 1: For any $x \in P$ and $y \in P$ satisfying $|[x,y]| < N$, we have

$$\sum_{\substack{z \in P; \\ x \leq z \leq y}} \mu(z,y) = [x = y].$$

Now, let x and y be two elements of P satisfying $|[x,y]| = N$. We are going to prove that

$$\sum_{\substack{z \in P; \\ x \leq z \leq y}} \mu(z,y) = [x = y]. \tag{50}$$

We are in one of the following three cases:

Case 1: We have $x = y$.

Case 2: We have $x < y$.

Case 3: We have neither $x = y$ nor $x < y$.

In Case 1 and in Case 3, we can prove (50) in exactly the same way as (in our above proof of Proposition 8.26 **(a)**) we have proven (48). Thus, it remains only to prove (50) in Case 2. In other words, we can WLOG assume that we are in Case 2.

Assume this. Hence, $x < y$, so that $[x = y] = 0$.

For every $t \in P$ satisfying $x \leq t < y$, we have

$$|[x,t]| < N \tag{51}$$

[41]. Therefore, for every $t \in P$ satisfying $x \leq t < y$, we have

$$\sum_{\substack{z \in P; \\ x \leq z \leq t}} \mu(z,t) = [x = t] \tag{52}$$

(by Claim 1, applied to t instead of y). Also, for every $u \in P$ and $v \in P$, we have

$$\sum_{\substack{t \in P; \\ u \leq t \leq v}} \mu(u,t) = [u = v] \tag{53}$$

[41]*Proof of (51):* Let $t \in P$ be such that $x \leq t < y$. We shall proceed in several steps:

- We have

$$\begin{aligned} [x,t] &= \{w \in P \mid x \leq w \leq t\} && (\text{by the definition of } [x,t]) \\ &\subseteq \{w \in P \mid x \leq w \leq y\} && \left(\begin{array}{c}\text{because every } w \in P \text{ satisfying } w \leq t \\ \text{must also satisfy } w \leq y \text{ (since } t < y)\end{array}\right) \\ &= [x,y] && (\text{by the definition of } [x,y]). \end{aligned}$$

- We have $t < y$. Thus, we do not have $y \leq t$. Hence, we do not have $x \leq y \leq t$. Hence, $y \notin [x,t]$. But $y \in [x,y]$ (since $x \leq y \leq y$). Hence, the sets $[x,t]$ and $[x,y]$ are distinct (since the latter contains y but the former does not). Combining this with $[x,t] \subseteq [x,y]$, we conclude that $[x,t]$ is a proper subset of $[x,y]$. Hence, $|[x,t]| < |[x,y]| = N$. This proves (51).

[42].

Now,

$$\underbrace{\sum_{\substack{(z,t)\in P^2;\\ x\leq z\leq t\leq y}}}_{=\sum\limits_{\substack{z\in P;\\ x\leq z\leq y}}\sum\limits_{\substack{t\in P;\\ z\leq t\leq y}}} \mu\left(z,t\right)$$

$$= \sum_{\substack{z\in P;\\ x\leq z\leq y}} \underbrace{\sum_{\substack{t\in P;\\ z\leq t\leq y}} \mu\left(z,t\right)}_{\substack{=[z=y]\\ \text{(by (53)}\\ \text{(applied to } u=z \text{ and } v=y))}} = \sum_{\substack{z\in P;\\ x\leq z\leq y}} [z=y]$$

$$= \sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z=y}} \underbrace{[z=y]}_{\substack{=1\\ \text{(since } z=y)}} + \sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}} \underbrace{[z=y]}_{\substack{=0\\ \text{(since } z\neq y)}}$$

$$\left(\text{since every } z\in P \text{ satisfies either } z=y \text{ or } z\neq y \text{ (but not both)}\right)$$

$$= \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z=y}}}_{=\sum\limits_{z\in\{w\in P\ \mid\ x\leq w\leq y \text{ and } w=y\}}} 1 + \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq y \text{ and } z\neq y}} 0}_{=0} = \sum_{z\in\{w\in P\ \mid\ x\leq w\leq y \text{ and } w=y\}} 1$$

$$= \left|\underbrace{\{w\in P\ \mid\ x\leq w\leq y \text{ and } w=y\}}_{=\{y\}}\right| = \left|\{y\}\right| = 1.$$

[42] *Proof of (53):* Let $u\in P$ and $v\in P$. Proposition 8.26 **(a)** (applied to $x=u$ and $y=v$) shows that $\sum\limits_{\substack{z\in P;\\ u\leq z\leq v}} \mu\left(u,z\right) = [u=v]$. Now,

$$\sum_{\substack{t\in P;\\ u\leq t\leq v}} \mu\left(u,t\right) = \sum_{\substack{z\in P;\\ u\leq z\leq v}} \mu\left(u,z\right) \qquad \text{(here, we have substituted } z \text{ for } t \text{ in the sum)}$$
$$= [u=v].$$

This proves (53).

Hence,

$$\begin{aligned}
1 &= \underbrace{\sum_{\substack{(z,t)\in P^2;\\ x\leq z\leq t\leq y}}}_{=\sum_{\substack{t\in P;\\ x\leq t\leq y}} \sum_{\substack{z\in P;\\ x\leq z\leq t}}} \mu(z,t) = \sum_{\substack{t\in P;\\ x\leq t\leq y}} \sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu(z,t) \\
&= \underbrace{\sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t=y}}}_{\substack{=\sum_{t\in\{w\in P\ \mid\ x\leq w\leq y \text{ and } w=y\}} = \sum_{t\in\{y\}} \\ (\text{since } \{w\in P\ \mid\ x\leq w\leq y \text{ and } w=y\}=\{y\})}} \sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu(z,t) \\
&\qquad + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu(z,t)}_{\substack{=[x=t] \\ (\text{by } (52)) \\ (\text{since } t<y \text{ (because } t\leq y \text{ and } t\neq y) \text{ and } x\leq t))}} \\
&\qquad\qquad (\text{since every } t\in P \text{ satisfies either } t=y \text{ or } t\neq y \text{ (but not both)}) \\
&= \underbrace{\sum_{t\in\{y\}} \sum_{\substack{z\in P;\\ x\leq z\leq t}} \mu(z,t)}_{=\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)} + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} [x=t] \\
&= \sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} [x=t].
\end{aligned}$$

Subtracting $\sum\limits_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)$ from both sides of this equality, we obtain

$$\begin{aligned}
&1 - \sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) \\
&= \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq y}} [x=t] \\
&= \underbrace{\sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t=x \text{ and } t\neq y}}}_{\substack{=\sum\limits_{t\in\{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\}} = \sum\limits_{t\in\{x\}} \\ (\text{since } \{z\in P \mid x\leq z\leq y \text{ and } z=x \text{ and } z\neq y\}=\{x\})}} \underbrace{[x=t]}_{\substack{=1\\ (\text{since } x=t)}} \\
&\qquad + \sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq x \text{ and } t\neq y}} \underbrace{[x=t]}_{\substack{=0\\ (\text{since } x\neq t)}} \\
&\qquad\qquad (\text{since every } t\in P \text{ satisfies either } t=x \text{ or } t\neq x \text{ (but not both)}) \\
&= \sum_{t\in\{x\}} 1 + \underbrace{\sum_{\substack{t\in P;\\ x\leq t\leq y \text{ and } t\neq x \text{ and } t\neq y}} 0}_{=0} = \sum_{t\in\{x\}} 1 = 1.
\end{aligned}$$

Solving this equality for $\sum\limits_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y)$, we obtain

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) = 1-1 = 0 = [x=y]$$

(since $x<y$). Thus, (50) is proven.

Let us now forget that we fixed x and y. We thus have proven that for any $x\in P$ and $y\in P$ satisfying $|[x,y]| = N$, we have

$$\sum_{\substack{z\in P;\\ x\leq z\leq y}} \mu(z,y) = [x=y].$$

In other words, Proposition 8.26 **(b)** holds whenever $|[x,y]| = N$. This completes the induction step. Thus, Proposition 8.26 **(b)** is proven by induction.

(c) For every $v \in P$, we have

$$\begin{aligned}
&\sum_{\substack{y\in P;\\ y\leq v}} \mu(y,v) \sum_{\substack{x\in P;\\ x\leq y}} \beta_x \\
&= \sum_{\substack{z\in P;\\ z\leq v}} \mu(z,v) \sum_{\substack{x\in P;\\ x\leq z}} \beta_x \qquad \left(\begin{array}{c}\text{here, we have renamed the summation}\\ \text{index } y \text{ as } z \text{ in the outer sum}\end{array}\right)\\
&= \underbrace{\sum_{\substack{z\in P;\\ z\leq v}} \sum_{\substack{x\in P;\\ x\leq z}}}_{=\sum_{x\in P} \sum_{\substack{z\in P;\\ x\leq z\leq v}}} \mu(z,v)\,\beta_x = \sum_{x\in P} \underbrace{\sum_{\substack{z\in P;\\ x\leq z\leq v}} \mu(z,v)}_{\substack{=[x=v]\\ \text{(by Proposition 8.26 \textbf{(b)}}\\ \text{(applied to } y=v\text{))}}} \beta_x \\
&= \sum_{x\in P} [x=v]\,\beta_x = \sum_{\substack{x\in P;\\ x=v}} \underbrace{[x=v]}_{\substack{=1\\ \text{(since } x=v)}} \beta_x + \sum_{\substack{x\in P;\\ x\neq v}} \underbrace{[x=v]}_{\substack{=0\\ \text{(since } x\neq v)}} \beta_x \\
&\qquad\qquad (\text{since every } x\in P \text{ satisfies either } x=v \text{ or } x\neq v \text{ (but not both)})\\
&= \sum_{\substack{x\in P;\\ x=v}} \beta_x + \underbrace{\sum_{\substack{x\in P;\\ x\neq v}} 0\beta_x}_{=0} = \sum_{\substack{x\in P;\\ x=v}} \beta_x = \beta_v \qquad (\text{since } v\in P).
\end{aligned}$$

Renaming v as z in this result, we obtain precisely Proposition 8.26 **(c)**. □

Proof of Theorem 8.20. If T is a subset of E, then $\overline{T}$ is a flat of M (by Proposition 8.12 **(a)**). In other words, if T is a subset of E, then $\overline{T} \in \operatorname{Flats} M$. Renaming T as B in this statement, we conclude that if B is a subset of E, then $\overline{B} \in \operatorname{Flats} M$.

For every $F \in \operatorname{Flats} M$, define an element $\beta_F \in \mathbf{k}$ by

$$\beta_F = \sum_{\substack{B\subseteq E;\\ \overline{B}=F}} (-1)^{|B|} \left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K \right).$$

Now, using Lemma 8.25, we can easily see that

$$\sum_{\substack{G\in \operatorname{Flats} M;\\ G\subseteq F}} \beta_G = [F=\varnothing] \qquad \text{for every } F\in \operatorname{Flats} M \tag{54}$$

[43].

[43] *Proof of (54):* Let $F \in \operatorname{Flats} M$. Thus, F is a flat of M.

If B is a subset of E, then the statements $\left(\overline{B} \subseteq F\right)$ and $(B \subseteq F)$ are equivalent. (This follows from Proposition 8.12 **(g)**, applied to $T = B$ and $G = F$.)

Let μ be the Möbius function of the lattice Flats M. The element $\overline{\varnothing}$ is the global minimum of the poset Flats M. [44] In particular, $\overline{\varnothing} \in$ Flats M and $\overline{\varnothing} \subseteq F$. Hence, $\mu\left(\overline{\varnothing}, F\right)$ is well-defined.

Now, fix $F \in$ Flats M. Proposition 8.26 **(c)** (applied to $P =$ Flats M and $z = F$)

Now,

$$\begin{aligned}
&\sum_{\substack{G\in \operatorname{Flats} M;\\ G\subseteq F}} \underbrace{\beta_G}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ \overline{B}=G}} (-1)^{|B|}\left(\prod\limits_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) \\ \text{(by the definition of } \beta_G)}} \\
&= \underbrace{\sum_{\substack{G\in \operatorname{Flats} M;\\ G\subseteq F}} \sum_{\substack{B\subseteq E;\\ \overline{B}=G}}}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ \overline{B}\subseteq F}} \\ \text{(because if } B \text{ is a subset of } E, \\ \text{then } \overline{B}\in \operatorname{Flats} M)}} (-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) \\
&= \underbrace{\sum_{\substack{B\subseteq E;\\ \overline{B}\subseteq F}}}_{\substack{=\sum\limits_{\substack{B\subseteq E;\\ B\subseteq F}} \\ \text{(because if } B \text{ is a subset of } E, \text{ then} \\ \text{the statements } \left(\overline{B}\subseteq F\right) \text{ and } (B\subseteq F) \text{ are} \\ \text{equivalent)}}} (-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) = \underbrace{\sum_{\substack{B\subseteq E;\\ B\subseteq F}}}_{=\sum\limits_{B\subseteq F}} (-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) \\
&= \sum_{B\subseteq F} (-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}} a_K\right) = [F = \varnothing] \qquad \text{(by (44))}.
\end{aligned}$$

This proves (54).

[44]This was proven during our proof of Proposition 8.14.

shows that

$$\begin{aligned}
\beta_F &= \sum_{\substack{y \in \operatorname{Flats} M;\\ y \subseteq F}} \mu(y,F) \sum_{\substack{x \in \operatorname{Flats} M;\\ x \subseteq y}} \beta_x \\
&\qquad \left(\text{since the relation } \leq \text{ of the poset } \operatorname{Flats} M \text{ is } \subseteq\right) \\
&= \sum_{\substack{H \in \operatorname{Flats} M;\\ H \subseteq F}} \mu(H,F) \underbrace{\sum_{\substack{G \in \operatorname{Flats} M;\\ G \subseteq H}} \beta_G}_{\substack{=[H=\varnothing] \\ \text{(by (54), applied to} \\ H \text{ instead of } F)}} \\
&\qquad \left(\text{here, we renamed the summation indices } y \text{ and } x \text{ as } H \text{ and } G\right) \\
&= \sum_{\substack{H \in \operatorname{Flats} M;\\ H \subseteq F}} \mu(H,F) [H=\varnothing] \\
&= \sum_{\substack{H \in \operatorname{Flats} M;\\ H \subseteq F;\\ H=\varnothing}} \mu(H,F) \underbrace{[H=\varnothing]}_{\substack{=1 \\ \text{(since } H=\varnothing)}} + \sum_{\substack{H \in \operatorname{Flats} M;\\ H \subseteq F;\\ H\neq\varnothing}} \mu(H,F) \underbrace{[H=\varnothing]}_{\substack{=0 \\ \text{(since } H\neq\varnothing)}} \\
&= \underbrace{\sum_{\substack{H \in \operatorname{Flats} M;\\ H \subseteq F;\\ H=\varnothing}}}_{\substack{=\sum\limits_{\substack{H \in \operatorname{Flats} M;\\ H=\varnothing}} \\ \text{(since the condition } H\subseteq F \\ \text{is automatically implied by} \\ \text{the condition } H=\varnothing)}} \mu(H,F) \\
&= \sum_{\substack{H \in \operatorname{Flats} M;\\ H=\varnothing}} \mu(H,F). \qquad (55)
\end{aligned}$$

Now, we shall prove that

$$\beta_F = \left[\overline{\varnothing} = \varnothing\right] \mu\left(\overline{\varnothing}, F\right). \qquad (56)$$

Proof of (56): We are in one of the following two cases:

Case 1: We have $\overline{\varnothing} = \varnothing$.

Case 2: We have $\overline{\varnothing} \neq \varnothing$.

Let us consider Case 1 first. In this case, we have $\overline{\varnothing} = \varnothing$. Hence, $\varnothing = \overline{\varnothing} \in \operatorname{Flats} M$. Thus, the sum $\sum\limits_{\substack{H \in \operatorname{Flats} M;\\ H=\varnothing}} \mu(H,F)$ has exactly one addend: namely, the addend for $H = \varnothing$. Thus, $\sum\limits_{\substack{H \in \operatorname{Flats} M;\\ H=\varnothing}} \mu(H,F) = \mu\left(\underbrace{\varnothing}_{=\overline{\varnothing}}, F\right) = \mu\left(\overline{\varnothing}, F\right)$. Thus, (55) becomes $\beta_F = \sum\limits_{\substack{H \in \operatorname{Flats} M;\\ H=\varnothing}} \mu(H,F) = \mu\left(\overline{\varnothing}, F\right)$. Comparing this with

$\underbrace{[\overline{\varnothing} = \varnothing]}_{\substack{=1 \\ (\text{since } \overline{\varnothing} = \varnothing)}} \mu(\overline{\varnothing}, F) = \mu(\overline{\varnothing}, F)$, we obtain $\beta_F = [\overline{\varnothing} = \varnothing] \mu(\overline{\varnothing}, F)$. Thus, (56) is proven in Case 1.

Let us now consider Case 2. In this case, we have $\overline{\varnothing} \neq \varnothing$. Thus, there exists no $H \in \operatorname{Flats} M$ such that $H = \overline{\varnothing}$ [45]. Hence, the sum $\sum_{\substack{H \in \operatorname{Flats} M; \\ H = \overline{\varnothing}}} \mu(H, F)$ is empty. Thus, $\sum_{\substack{H \in \operatorname{Flats} M; \\ H = \overline{\varnothing}}} \mu(H, F) = (\text{empty sum}) = 0$, so that (55) becomes $\beta_F = \sum_{\substack{H \in \operatorname{Flats} M; \\ H = \overline{\varnothing}}} \mu(H, F) = 0$. Comparing this with $\underbrace{[\overline{\varnothing} = \varnothing]}_{\substack{=0 \\ (\text{since } \overline{\varnothing} \neq \varnothing)}} \mu(\overline{\varnothing}, F) = 0$, we obtain $\beta_F = [\overline{\varnothing} = \varnothing] \mu(\overline{\varnothing}, F)$. Thus, (56) is proven in Case 2.

Now, we have proven (56) in both possible Cases 1 and 2. Thus, (56) always holds.

Now, let us forget that we fixed F. We thus have proven (56) for each $F \in \operatorname{Flats} M$.

[45]*Proof.* Assume the contrary. Thus, there exists some $H \in \operatorname{Flats} M$ such that $H = \overline{\varnothing}$. In other words, $\overline{\varnothing} \in \operatorname{Flats} M$. Hence, $\overline{\varnothing}$ is a flat of M. Proposition 8.12 **(b)** (applied to $G = \varnothing$) thus shows that $\overline{\varnothing} = \varnothing$. This contradicts $\overline{\varnothing} \neq \varnothing$. This contradiction proves that our assumption was wrong, qed.

Now,

$$\begin{aligned}
&\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}a_K\right)x^{m-r_M(F)}\\
&=\underbrace{\sum_{B\subseteq E}}_{\substack{=\sum\limits_{F\in\operatorname{Flats}M}\ \sum\limits_{\substack{B\subseteq E;\\ \overline{B}=F}}\\ \text{(because if } B \text{ is a subset of } E\text{,}\\ \text{then } \overline{B}\in\operatorname{Flats}M)}}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)\underbrace{x^{m-r_M(B)}}_{\substack{=x^{m-r_M(\overline{B})}\\ \text{(since Proposition 8.12 \textbf{(f)} (applied to } T=B\text{)}\\ \text{shows that } r_M(B)=r_M(\overline{B})\text{)}}}\\
&\qquad\qquad\left(\text{here, we have renamed the summation index } F \text{ as } B\right)\\
&=\sum_{F\in\operatorname{Flats}M}\ \sum_{\substack{B\subseteq E;\\ \overline{B}=F}}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)\underbrace{x^{m-r_M(\overline{B})}}_{\substack{=x^{m-r_M(F)}\\ \text{(since } \overline{B}=F\text{)}}}\\
&=\sum_{F\in\operatorname{Flats}M}\underbrace{\sum_{\substack{B\subseteq E;\\ \overline{B}=F}}(-1)^{|B|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq B}}a_K\right)}_{\substack{=\beta_F=[\overline{\varnothing}=\varnothing]\mu(\overline{\varnothing},F)\\ \text{(by (56))}}}x^{m-r_M(F)}\\
&=\sum_{F\in\operatorname{Flats}M}[\overline{\varnothing}=\varnothing]\,\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}\\
&=[\overline{\varnothing}=\varnothing]\sum_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}.
\end{aligned}\tag{57}$$

But the definition of χ_M yields $\chi_M=\sum\limits_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}$. The definition of $\widetilde{\chi}_M$ yields

$$\begin{aligned}
\widetilde{\chi}_M&=[\overline{\varnothing}=\varnothing]\underbrace{\chi_M}_{=\sum\limits_{F\in\operatorname{Flats}M}\mu(\overline{\varnothing},F)x^{m-r_M(F)}}=[\overline{\varnothing}=\varnothing]\sum_{F\in\operatorname{Flats}M}\mu\left(\overline{\varnothing},F\right)x^{m-r_M(F)}\\
&=\sum_{F\subseteq E}(-1)^{|F|}\left(\prod_{\substack{K\in\mathfrak{K};\\ K\subseteq F}}a_K\right)x^{m-r_M(F)}\qquad\text{(by (57))}.
\end{aligned}$$

This proves Theorem 8.20. □

Proof of Corollary 8.22. Corollary 8.22 can be derived from Theorem 8.20 in the same way as Corollary 1.14 was derived from Theorem 1.12. □

Proof of Theorem 8.21. Theorem 8.21 can be derived from Theorem 8.20 in the same way as Theorem 1.8 was derived from Theorem 1.12. □

Proof of Corollary 8.23. Corollary 8.23 follows from Corollary 8.22 when $\mathfrak{K}$ is set to be the set of **all** broken circuits of M. □

Proof of Corollary 8.24. If F is a subset of E such that F contains no broken circuit of M as a subset, then

$$r_M(F) = |F| \tag{58}$$

[46]. Now, Corollary 8.23 yields

$$\widetilde{\chi}_M = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} \underbrace{x^{m-r_M(F)}}_{\substack{=x^{m-|F|} \\ \text{(by (58))}}} = \sum_{\substack{F \subseteq E; \\ F \text{ contains no broken} \\ \text{circuit of } M \text{ as a subset}}} (-1)^{|F|} x^{m-|F|}.$$

This proves Corollary 8.24. □

8.4. A vanishing alternating sum for matroids

As an application of the above, we can prove an analogue of the alternating sum identity of Dahlberg and van Willigenburg (Theorem 7.1 above) for the characteristic polynomials of matroids:

Theorem 8.27. Let $M = (E, \mathcal{I})$ be a matroid. Let $m = r_M(E)$. Let C be a circuit of M, and let $e \in C$ be arbitrary. Then,

$$\sum_{F \subseteq C \setminus \{e\}} (-1)^{|F|} x^{m-r_M(E \setminus F)} \cdot \widetilde{\chi}_{M \setminus F} = 0.$$

Here, whenever F is a subset of E, the notation $M \setminus F$ denotes the matroid $(E \setminus F, \mathcal{I} \cap \mathcal{P}(E \setminus F))$ (that is, the matroid whose ground set is $E \setminus F$ and whose independent sets are those subsets of $E \setminus F$ that are independent in M).

Proof of Theorem 8.27. Quite similar to our above proof of Theorem 7.2, but using Theorem 8.21 and Corollary 8.22 instead of Theorem 6.34 and Corollary 6.36. (We also need to observe that the rank function $r_{M \setminus F}$ is a restriction of r_M whenever F is a subset of E.) We leave all details to the reader. □

[46] *Proof of (58):* Let F be a subset of E such that F contains no broken circuit of M as a subset.

We shall show that $F \in \mathcal{I}$. Indeed, assume the contrary. Thus, $F \notin \mathcal{I}$, so that $F \in \mathcal{P}(E) \setminus \mathcal{I}$. Hence, there exists a circuit C of M such that $C \subseteq F$ (according to Lemma 8.7, applied to $Q = F$). Consider this C. The set C is a circuit, and thus nonempty (because the empty set is in $\mathcal{I}$). Let e be the unique element of C having maximum label. (This is clearly well-defined, since the labeling function ℓ is injective.) Then, $C \setminus \{e\}$ is a broken circuit of M (by the definition of a broken circuit). Thus, F contains a broken circuit of M as a subset (since $C \setminus \{e\} \subseteq C \subseteq F$). This contradicts the fact that F contains no broken circuit of M as a subset. This contradiction shows that our assumption was wrong. Hence, $F \in \mathcal{I}$ is proven.

Thus, Lemma 8.6 (applied to $T = F$) shows that $r_M(F) = |F|$, qed.

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