Shuffle-compatibility for the exterior peak set

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/dartmouth18.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf
project: https://github.com/darijgr/gzshuf

Section 1

Shuffle-compatibility

Reference:

- Ira M. Gessel, Yan Zhuang, *Shuffle-compatible permutation statistics*, arXiv:1706.00750, Adv. in Math. **332** (2018), pp. 85–141.

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  We say that π is *nonempty* if $n > 0$.
- If π is an n-permutation and $i \in \{1, 2, \ldots, n\}$, then π_i denotes the i-th entry of π.

Permutations & permutation statistics: Definitions 2

- Two n-permutations α and β (with the same n) are *order-equivalent* if all $i, j \in \{1, 2, \ldots, n\}$ satisfy $(\alpha_i < \alpha_j) \iff (\beta_i < \beta_j)$.
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  Intuition: A statistic computes some "fingerprint" of a permutation that only depends on the relative order of its letters.

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  Note. A statistic need not be integer-valued! It can be set-valued, or list-valued for example.

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- If π is an n-permutation, then a *descent* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$.

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  Thus, Des is a statistic.
  Example: $\operatorname{Des}(3, 1, 5, 2, 4) = \{1, 3\}$.

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- The *major index* $\operatorname{maj} \pi$ of a permutation π is the **sum** of all descents of π.
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  Example: $\operatorname{maj}(3, 1, 5, 2, 4) = 1 + 3 = 4$.

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  Thus, maj is a statistic.
  Example: $\operatorname{maj}(3, 1, 5, 2, 4) = 1 + 3 = 4$.
- The *Coxeter length* inv (i.e., *number of inversions*) and the *set of inversions* are statistics, too.

Examples of permutation statistics, 2: peaks

- If π is an n-permutation, then a *peak* of π means an $i \in \{2, 3, \ldots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$.
  (Thus, peaks can only exist if $n \geq 3$.
  The name refers to the plot of π, where peaks look like this: $/\backslash$.)

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- The *peak set* $\operatorname{Pk} \pi$ of a permutation π is the set of all peaks of π.
  Thus, Pk is a statistic.
  Examples:
  - $\operatorname{Pk}(3, 1, 5, 2, 4) = \{3\}$.
  - $\operatorname{Pk}(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
  - $\operatorname{Pk}(3, 2) = \{\}$.

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- The *peak number* $\operatorname{pk} \pi$ of a permutation π is the number of all peaks of π: that is, $\operatorname{pk} \pi = |\operatorname{Pk} \pi|$.
  Thus, pk is a statistic.
  Example: $\operatorname{pk}(3, 1, 5, 2, 4) = 1$.

Examples of permutation statistics, 3: left peaks

- If π is an n-permutation, then a *left peak* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$.
  (Thus, left peaks are the same as peaks, except that 1 counts as a left peak if $\pi_1 > \pi_2$.)
- The *left peak set* $\operatorname{Lpk} \pi$ of a permutation π is the set of all left peaks of π.
  Thus, Lpk is a statistic.
  Examples:
  - $\operatorname{Lpk}(3, 1, 5, 2, 4) = \{1, 3\}$.
  - $\operatorname{Lpk}(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
  - $\operatorname{Lpk}(3, 2) = \{1\}$.
- The *left peak number* $\operatorname{lpk} \pi$ of a permutation π is the number of all left peaks of π: that is, $\operatorname{lpk} \pi = |\operatorname{Lpk} \pi|$.
  Thus, lpk is a statistic.
  Example: $\operatorname{lpk}(3, 1, 5, 2, 4) = 2$.

Examples of permutation statistics, 4: right peaks

- If π is an n-permutation, then a *right peak* of π means an $i \in \{2, 3, \ldots, n\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_{n+1} = 0$.
  (Thus, right peaks are the same as peaks, except that n counts as a right peak if $\pi_{n-1} < \pi_n$.)
- The *right peak set* $\operatorname{Rpk} \pi$ of a permutation π is the set of all right peaks of π.
  Thus, Rpk is a statistic.
  Examples:
  - $\operatorname{Rpk}(3, 1, 5, 2, 4) = \{3, 5\}$.
  - $\operatorname{Rpk}(1, 3, 2, 5, 4, 6) = \{2, 4, 6\}$.
  - $\operatorname{Rpk}(3, 2) = \{\}$.
- The *right peak number* $\operatorname{rpk} \pi$ of a permutation π is the number of all right peaks of π: that is, $\operatorname{rpk} \pi = |\operatorname{Rpk} \pi|$.
  Thus, rpk is a statistic.
  Example: $\operatorname{rpk}(3, 1, 5, 2, 4) = 2$.

Examples of permutation statistics, 5: exterior peaks

- If π is an n-permutation, then an *exterior peak* of π means an $i \in \{1, 2, \ldots, n\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$ *and* $\pi_{n+1} = 0$.
  (Thus, exterior peaks are the same as peaks, except that 1 counts if $\pi_1 > \pi_2$, and n counts if $\pi_{n-1} < \pi_n$.)
- The *exterior peak set* $\operatorname{Epk} \pi$ of a permutation π is the set of all exterior peaks of π.
  Thus, Epk is a statistic.
  Examples:
  - $\operatorname{Epk}(3, 1, 5, 2, 4) = \{1, 3, 5\}$.
  - $\operatorname{Epk}(1, 3, 2, 5, 4, 6) = \{2, 4, 6\}$.
  - $\operatorname{Epk}(3, 2) = \{1\}$.
- Thus, $\operatorname{Epk} \pi = \operatorname{Lpk} \pi \cup \operatorname{Rpk} \pi$ if $n \geq 2$.
- The *exterior peak number* $\operatorname{epk} \pi$ of a permutation π is the number of all exterior peaks of π: that is, $\operatorname{epk} \pi = |\operatorname{Epk} \pi|$.
  Thus, epk is a statistic.
  Example: $\operatorname{epk}(3, 1, 5, 2, 4) = 3$.

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  (And thus, no other letters can appear in τ.)
- We let $S(\pi, \sigma)$ be the set of all shuffles of π and σ.
- **Example:**

$$S((4,1),(2,5)) = \{(4,1,2,5),(4,2,1,5),(4,2,5,1),\\ (2,4,1,5),(2,4,5,1),(2,5,4,1)\}.$$

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- Observe that π and σ have $\binom{m+n}{m}$ shuffles, in bijection with m-element subsets of $\{1,2,\ldots,m+n\}$.

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- A statistic st is said to be *shuffle-compatible* if for any two disjoint permutations π and σ, the multiset

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- In other words, st is shuffle-compatible if and only the distribution of st on the set $S(\pi,\sigma)$ stays unchaged if π and σ are replaced by two other disjoint permutations of the same size and same st-values.
In particular, it has to stay unchanged if π and σ are replaced by two permutations order-equivalent to them: e.g., st must have the same distribution on the three sets

$$S((4,1),(2,5)), \qquad S((2,1),(3,5)), \qquad S((9,8),(2,3)).$$

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- Statistics that are **not shuffle-compatible**: inv, $\operatorname{des} + \operatorname{maj}$, maj_2 (sending π to the sum of the squares of its descents), $(\operatorname{Pk}, \operatorname{des})$ (sending π to $(\operatorname{Pk} \pi, \operatorname{des} \pi)$), and others.

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- Their proofs use a mixture of enumerative combinatorics (including some known formulas of MacMahon, Stanley, ...), quasisymmetric functions, Hopf algebra theory, P-partitions (and variants by Stembridge and Petersen), Eulerian polynomials (based on earlier work by Zhuang, and even earlier work by Foata and Strehl).

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- **Theorem (G.).** The statistic Epk is shuffle-compatible (as conjectured in Gessel/Zhuang).

LR-shuffle-compatibility

- We further introduce a finer version of shuffle-compatibility: "LR-shuffle-compatibility".
- Given two disjoint nonempty permutations π and σ,
  - a *left shuffle* of π and σ is a shuffle of π and σ that starts with a letter of π;
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- A statistic st is said to be *LR-shuffle-compatible* if for any two disjoint nonempty permutations π and σ, the multisets

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  depend only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, $|\pi|$, $|\sigma|$ and the truth value of $\pi_1 > \sigma_1$.

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- **Theorem (G.).** Des, des, Lpk and Epk are LR-shuffle-compatible. (But not maj or Rpk or Pk.)

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- The "LR" in "LR-shuffle-compatibility" stands for "left and right". Indeed:
- A statistic st is said to be *left-shuffle-compatible* if for any two disjoint nonempty permutations π and σ such that

$$\pi_1 > \sigma_1,$$

the multiset

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depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, $|\pi|$ and $|\sigma|$.

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- **Proposition.** A permutation statistic st is LR-shuffle-compatible if and only if it is both left-shuffle-compatible and right-shuffle-compatible.

Section 2

Methods of proof

References:

- Darij Grinberg, *Shuffle-compatible permutation statistics II: the exterior peak set*.
- John R. Stembridge, *Enriched P-partitions*, Trans. Amer. Math. Soc. 349 (1997), no. 2, pp. 763–788.
- T. Kyle Petersen, *Enriched P-partitions and peak algebras*, Adv. in Math. 209 (2007), pp. 561–610.

Roadmap to Epk

- Now to the general ideas of our proof that Epk is shuffle-compatible.
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- More precisely, we define $\mathcal{Z}$**-enriched** P**-partitions**: a generalization of
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- More precisely, we define **$\mathcal{Z}$-enriched P-partitions**: a generalization of
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  which are used in the proofs for Des, Pk and Lpk, respectively.
- The idea is simple, but the proof takes work. Let me just show the highlights without using P-partition language.

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- Let $\mathcal{N}$ be the totally ordered set $\{0 < 1 < 2 < \cdots < \infty\}$.
- Let $\operatorname{Pow}\mathcal{N}$ be the ring of power series over $\mathbb{Q}$ in the indeterminates $x_0, x_1, x_2, \ldots, x_\infty$.

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- If $n \in \mathbb{N}$ and if Λ is any subset of $[n]$, then we define a power series $K_{n,\Lambda}^{\mathcal{Z}} \in \operatorname{Pow} \mathcal{N}$ by

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  - we let $k(g)$ be the number of **distinct** entries of this n-tuple g, not counting those that equal 0 or ∞.
- **Product formula.** If π is an n-permutation and σ is an m-permutation, then

$$K_{n,\operatorname{Epk}\pi}^{\mathcal{Z}} \cdot K_{m,\operatorname{Epk}\sigma}^{\mathcal{Z}} = \sum_{\tau \in S(\pi,\sigma)} K_{n+m,\operatorname{Epk}\tau}^{\mathcal{Z}}.$$

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- Proof idea: $K^{\mathcal{Z}}_{n,\operatorname{Epk}\pi}$ is the generating function of $\mathcal{Z}$-enriched P-partitions for a certain totally ordered set P.

Lacunar subsets and linear independence

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  (And conversely – although we don't need it –, any such subset has the form $\operatorname{Epk}\pi$ for some π.)
- **Lemma.** The family

$$\left(K_{n,\Lambda}^{\mathcal{Z}}\right)_{n\in\mathbb{N};\ \Lambda\subseteq[n] \text{ is lacunar and nonempty}}$$

  is $\mathbb{Q}$-linearly independent.
- These lemmas, and the above product formula, prove the shuffle-compatibility of Epk.

LR-shuffle-compatibility redux

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- Recall again the definitions:
- We let $S_{\prec}(\pi, \sigma)$ be the set of all left shuffles of π and σ (= the shuffles that start with π_1).
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- A statistic st is said to be *LR-shuffle-compatible* if for any two disjoint nonempty permutations π and σ, the multisets

  $$\{\operatorname{st}\tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multiset}} \quad \text{and} \quad \{\operatorname{st}\tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multiset}}$$

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- We claim that Des, des, Lpk and Epk are LR-shuffle-compatible.

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- A permutation statistic st is said to be *head-graft-compatible* if for any nonempty permutation π and any letter a that does not appear in π, the element $\operatorname{st}(a:\pi)$ depends only on $\operatorname{st}(\pi)$, $|\pi|$ and on the truth value of $a > \pi_1$.
  Here, $a:\pi$ is the permutation obtained from π by appending a at the front:

$$\pi = (\pi_1, \pi_2, \ldots, \pi_n) \qquad \Longrightarrow \qquad a:\pi = (a, \pi_1, \pi_2, \ldots, \pi_n) .$$

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- For example, Epk is head-graft-compatible, since

$$\operatorname{Epk}(a:\pi) = \begin{cases} \operatorname{Epk}\pi + 1, & \text{if not } a > \pi_1; \\ ((\operatorname{Epk}\pi + 1) \setminus \{2\}) \cup \{1\}, & \text{if } a > \pi_1. \end{cases}$$

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- Likewise, Des, Lpk and des are head-graft-compatible.

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- **Theorem (G.).** A statistic st is LR-shuffle-compatible **if and only if** it is shuffle-compatible and head-graft-compatible.

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- **Theorem (G.).** A statistic st is LR-shuffle-compatible **if and only if** it is shuffle-compatible and head-graft-compatible.
- Hence, Epk, Des, Lpk and des are LR-shuffle-compatible.

Proof idea for $\Longleftarrow$

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  If π and σ are two disjoint permutations, then

$$\begin{aligned} S_{\prec}(\pi, \sigma) &= S_{\succ}(\sigma, \pi); \\ S_{\prec}(\pi, \sigma) &= S_{\succ}(\pi_{\sim 1}, \pi_1 : \sigma) && \text{if } \pi \text{ is nonempty;} \\ S_{\succ}(\pi, \sigma) &= S_{\prec}(\sigma_1 : \pi, \sigma_{\sim 1}) && \text{if } \sigma \text{ is nonempty.} \end{aligned}$$

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  These allow for an inductive argument.
- Note that the concept of LR-shuffle-compatibility is not invariant under reversal: st can be LR-shuffle-compatible while $\text{st} \circ \text{rev}$ is not, where

$$\text{rev}(\pi_1, \pi_2, \ldots, \pi_n) = (\pi_n, \pi_{n-1}, \ldots, \pi_1).$$

  For example, Lpk is LR-shuffle-compatible, but Rpk is not.

Section 3

The QSym connection

References:

- Ira M. Gessel, Yan Zhuang, *Shuffle-compatible permutation statistics*, arXiv:1706.00750.
- Darij Grinberg, Victor Reiner, *Hopf Algebras in Combinatorics*, arXiv:1409.8356, and various other texts on combinatorial Hopf algebras.

Descent statistics

- Gessel and Zhuang prove **most** of their shuffle-compatibilities algebraically. Their methods involve combinatorial Hopf algebras (QSym and NSym).
- These methods work for **descent statistics** only. What is a descent statistic?

Descent statistics

- Gessel and Zhuang prove **most** of their shuffle-compatibilities algebraically. Their methods involve combinatorial Hopf algebras (QSym and NSym).
- These methods work for **descent statistics** only. What is a descent statistic?
- A *descent statistic* is a statistic st such that $\operatorname{st}\pi$ depends only on $|\pi|$ and $\operatorname{Des}\pi$ (in other words: if π and σ are two n-permutations with $\operatorname{Des}\pi = \operatorname{Des}\sigma$, then $\operatorname{st}\pi = \operatorname{st}\sigma$).
  Intuition: A descent statistic is a statistic which "factors through Des in each size".

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- For example, $(1, 3, 2)$ is a composition of 6.
- Let $n \in \mathbb{N}$, and let $[n-1] = \{1, 2, \ldots, n-1\}$.
  Then, there are mutually inverse bijections

$$\mathrm{Des} : \{\text{compositions of } n\} \to \{\text{subsets of } [n-1]\},$$
$$(i_1, i_2, \ldots, i_k) \mapsto \{i_1 + i_2 + \cdots + i_j \mid 1 \leq j \leq k-1\}$$

and

$$\mathrm{Comp} : \{\text{subsets of } [n-1]\} \to \{\text{compositions of } n\},$$
$$\{s_1 < s_2 < \cdots < s_k\} \mapsto (s_1 - s_0, s_2 - s_1, \ldots, s_{k+1} - s_k)$$

(using the notations $s_0 = 0$ and $s_{k+1} = n$).

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- Let $n \in \mathbb{N}$, and let $[n-1] = \{1, 2, \ldots, n-1\}$.
  Then, there are mutually inverse bijections Des and Comp between $\{\text{subsets of } [n-1]\}$ and $\{\text{compositions of } n\}$.
  If π is an n-permutation, then $\operatorname{Comp}(\operatorname{Des}\pi)$ is called the *descent composition* of π, and is written $\operatorname{Comp}\pi$.

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  If π is an n-permutation, then $\operatorname{Comp}(\operatorname{Des} \pi)$ is called the *descent composition* of π, and is written $\operatorname{Comp} \pi$.
- If st is a descent statistic, then we use the notation $\operatorname{st} \alpha$ (where α is a composition) for $\operatorname{st} \pi$, where π is any permutation with $\operatorname{Comp} \pi = \alpha$.
- **Warning:**

$$\operatorname{Des}((1, 5, 2) \text{ the composition}) = \{1, 6\};$$
$$\operatorname{Des}((1, 5, 2) \text{ the permutation}) = \{2\}.$$

Same for other statistics! Context must disambiguate.

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- inv is not a descent statistic: The permutations $(2, 1, 3)$ and $(3, 1, 2)$ have the same descents, but different numbers of inversions.

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- **However:** Every LR-shuffle-compatible statistic is a descent statistic.
  (Better yet, every head-graft-compatible statistic is a descent statistic.)

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- A formal power series $f \in \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
  - Every symmetric power series is quasisymmetric.
  - $\sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.

Quasisymmetric functions, part 1: definition

- Consider the ring $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates.
- A formal power series f is said to be *bounded-degree* if the monomials it contains are bounded (from above) in degree.
- A formal power series $f \in \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
  - Every symmetric power series is quasisymmetric.
  - $\sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.
- Let QSym be the set of all quasisymmetric bounded-degree power series in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. This is a $\mathbb{Q}$-subalgebra, called the *ring of quasisymmetric functions* over $\mathbb{Q}$. (Gessel, 1980s.)

Quasisymmetric functions, part 2: the monomial basis

- For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, define

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

$$= \text{sum of all monomials whose nonzero exponents are } \alpha_1, \alpha_2, \ldots, \alpha_k \text{ in } \textbf{this} \text{ order.}$$

  This is a homogeneous power series of degree $|\alpha|$ (the *size* of α, defined by $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_k$).
- Examples:
  - $M_{()} = 1$.
  - $M_{(1,1)} = \sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + \cdots$.
  - $M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots$.
  - $M_{(3)} = \sum_i x_i^3 = x_1^3 + x_2^3 + x_3^3 + \cdots$.

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$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

$=$ sum of all monomials whose nonzero exponents are $\alpha_1, \alpha_2, \ldots, \alpha_k$ in **this** order.

  This is a homogeneous power series of degree $|\alpha|$ (the *size* of α, defined by $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_k$).
- The family $(M_\alpha)_{\alpha \text{ is a composition}}$ is a basis of the $\mathbb{Q}$-vector space QSym, called the *monomial basis* (or M-basis).

- For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, define

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ for all } j \in \operatorname{Des} \alpha}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$= \sum_{\substack{\beta \text{ is a composition of } n; \\ \operatorname{Des} \beta \supseteq \operatorname{Des} \alpha}} M_\beta, \qquad \text{where } n = |\alpha|.$$

  This is a homogeneous power series of degree $|\alpha|$ again.
- Examples:
  - $F_{()} = 1$.
  - $F_{(1,1)} = \sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + \cdots$.
  - $F_{(2,1)} = \sum_{i \leq j < k} x_i x_j x_k$.
  - $F_{(3)} = \sum_{i \leq j \leq k} x_i x_j x_k$.

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  This is a homogeneous power series of degree $|\alpha|$ again.
- The family $(F_\alpha)_{\alpha \text{ is a composition}}$ is a basis of the $\mathbb{Q}$-vector space QSym, called the *fundamental basis* (or F-basis). Sometimes, F_α is also denoted L_α.

- For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, define

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ for all } j \in \operatorname{Des} \alpha}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$= \sum_{\substack{\beta \text{ is a composition of } n; \\ \operatorname{Des} \beta \supseteq \operatorname{Des} \alpha}} M_\beta, \qquad \text{where } n = |\alpha| .$$

This is a homogeneous power series of degree $|\alpha|$ again.

- What connects QSym with shuffles of permutations is the following fact:

Theorem. If π and σ are two disjoint permutations, then

$$F_{\operatorname{Comp} \pi} \cdot F_{\operatorname{Comp} \sigma} = \sum_{\tau \in S(\pi, \sigma)} F_{\operatorname{Comp} \tau} .$$

The kernel criterion for shuffle-compatibility

- If st is a descent statistic, then two compositions α and β are said to be st-*equivalent* if $|\alpha| = |\beta|$ and $\operatorname{st}\alpha = \operatorname{st}\beta$.
  (Remember: $\operatorname{st}\alpha$ means $\operatorname{st}\pi$ for any permutation π satisfying $\operatorname{Comp}\pi = \alpha$.)

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- The *kernel* $\mathcal{K}_{\mathrm{st}}$ of a descent statistic st is the $\mathbb{Q}$-vector subspace of QSym spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st-equivalent compositions:

$$\mathcal{K}_{\mathrm{st}} = \left\langle F_\alpha - F_\beta \ \mid \ |\alpha| = |\beta| \text{ and } \operatorname{st}\alpha = \operatorname{st}\beta \right\rangle_{\mathbb{Q}}.$$

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- **Theorem.** The descent statistic st is shuffle-compatible if and only if $\mathcal{K}_{\text{st}}$ is an ideal of QSym.
  (This is essentially due to Gessel & Zhuang.)

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- **Theorem.** The descent statistic st is shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is an ideal of QSym. (This is essentially due to Gessel & Zhuang.)
- Since Epk is shuffle-compatible, its kernel $\mathcal{K}_{\mathrm{Epk}}$ is an ideal of QSym. How can we describe it?
- Two ways: using the F-basis and using the M-basis.

The kernel $\mathcal{K}_{\mathrm{Epk}}$ in terms of the F-basis

- If $J = (j_1, j_2, \ldots, j_m)$ and K are two compositions, then we write $J \to K$ if there exists an $\ell \in \{2, 3, \ldots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \ldots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m)$. (In other words, we write $J \to K$ if K can be obtained from J by "splitting" some non-initial entry $j_\ell > 2$ into two consecutive entries 1 and $j_\ell - 1$.)
- **Example.** Here are all instances of the $\to$ relation on compositions of size ≤ 5:

$$(1,3) \to (1,1,2), \qquad (1,4) \to (1,1,3),$$
$$(1,3,1) \to (1,1,2,1), \qquad (1,1,3) \to (1,1,1,2),$$
$$(2,3) \to (2,1,2).$$

- **Proposition.** The ideal $\mathcal{K}_{\mathrm{Epk}}$ of QSym is spanned (as a $\mathbb{Q}$-vector space) by all differences of the form $F_J - F_K$, where J and K are two compositions satisfying $J \to K$.

The kernel $\mathcal{K}_{\mathrm{Epk}}$ in terms of the M-basis

- If $J = (j_1, j_2, \ldots, j_m)$ and K are two compositions, then we write $J \underset{M}{\to} K$ if there exists an $\ell \in \{2, 3, \ldots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \ldots, j_{\ell-1}, 2, j_\ell - 2, j_{\ell+1}, j_{\ell+2}, \ldots, j_m)$. (In other words, we write $J \underset{M}{\to} K$ if K can be obtained from J by "splitting" some non-initial entry $j_\ell > 2$ into two consecutive entries 2 and $j_\ell - 2$.)
- **Example.** Here are all instances of the $\underset{M}{\to}$ relation on compositions of size ≤ 5:

$$(1,3) \underset{M}{\to} (1,2,1), \qquad (1,4) \underset{M}{\to} (1,2,2),$$
$$(1,3,1) \underset{M}{\to} (1,2,1,1), \qquad (1,1,3) \underset{M}{\to} (1,1,2,1),$$
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- **Proposition.** The ideal $\mathcal{K}_{\mathrm{Epk}}$ of QSym is spanned (as a $\mathbb{Q}$-vector space) by all sums of the form $M_J + M_K$, where J and K are two compositions satisfying $J \underset{M}{\to} K$.

What about other statistics?

- **Question.** Do other descent statistics allow for similar descriptions of $\mathcal{K}_{\mathrm{st}}$?
(See the paper for some experimental results.)

What does LR-shuffle-compatibility mean algebraically?

- If shuffle-compatible descent statistics induce ideals of QSym, then what do LR-shuffle-compatible descent statistics induce?

$$(\text{shuffle-compatible des. statistics}) \leftrightarrow ((\text{some})\ \text{ideals of QSym})\,;$$
$$(\text{LR-shuffle-compatible des. statistics}) \leftrightarrow ??$$

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- We will answer this question using the *dendriform algebra* structure on QSym.

What does LR-shuffle-compatibility mean algebraically?

- We will answer this question using the *dendriform algebra* structure on QSym.
  This structure first appeared in:
  Darij Grinberg, *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions*, Canad. J. Math. 69 (2017), pp. 21–53.
  But the ideas go back to:
  - Glânffrwd P. Thomas, *Frames, Young tableaux, and Baxter sequences*, Advances in Mathematics, Volume 26, Issue 3, December 1977, Pages 275–289.
  - Jean-Christophe Novelli, Jean-Yves Thibon, *Construction of dendriform trialgebras*, arXiv:math/0510218.

  Something similar also appeared in: Aristophanes Dimakis, Folkert Müller-Hoissen, *Quasi-symmetric functions and the KP hierarchy*, Journal of Pure and Applied Algebra, Volume 214, Issue 4, April 2010, Pages 449–460.

Dendriform structure on QSym, part 1

- For any monomial $\mathfrak{m}$, let $\operatorname{Supp}\mathfrak{m}$ denote the set $\{i \mid x_i \text{ appears in } \mathfrak{m}\}$.
- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.

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- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.
- We define a binary operation $\prec$ on the $\mathbb{Q}$-vector space $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ as follows:
  - On monomials, it should be given by

    $$\mathfrak{m} \prec \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) < \min(\operatorname{Supp}\mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) \geq \min(\operatorname{Supp}\mathfrak{n}) \end{cases}$$

    for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbb{Q}$-bilinear.
  - It should be continuous (i.e., its $\mathbb{Q}$-bilinearity also applies to infinite $\mathbb{Q}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $\left(x_2^2 x_4\right) \prec \left(x_3^2 x_5\right) = x_2^2 x_3^2 x_4 x_5$, but $\left(x_2^2 x_4\right) \prec \left(x_2^2 x_5\right) = 0$.

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- We define a binary operation $\succeq$ on the $\mathbb{Q}$-vector space $\mathbb{Q}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ as follows:
  - On monomials, it should be given by
    $$\mathfrak{m} \succeq \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min\left(\operatorname{Supp}\mathfrak{m}\right) \geq \min\left(\operatorname{Supp}\mathfrak{n}\right); \\ 0, & \text{if } \min\left(\operatorname{Supp}\mathfrak{m}\right) < \min\left(\operatorname{Supp}\mathfrak{n}\right) \end{cases}$$
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Dendriform structure on QSym, part 2

- We now have defined two binary operations $\prec$ and $\succeq$ on $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. They satisfy:

$$a \prec b + a \succeq b = ab;$$
$$(a \prec b) \prec c = a \prec (bc);$$
$$(a \succeq b) \prec c = a \succeq (b \prec c);$$
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- This says that $(\mathbb{Q}[[x_1, x_2, x_3, \ldots]], \prec, \succeq)$ is a *dendriform algebra* in the sense of Loday (see, e.g., Zinbiel, *Encyclopedia of types of algebras 2010*, arXiv:1101.0267).

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- QSym is closed under both operations $\prec$ and $\succeq$. Thus, QSym becomes a dendriform subalgebra of $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$.

The kernel criterion for LR-shuffle-compatibility

- Recall the **Theorem:** The descent statistic st is shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is an ideal of QSym.

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- Similarly, **Theorem:** The descent statistic st is LR-shuffle-compatible if and only if

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(that is, $\mathcal{K}_{\mathrm{st}}$ is an ideal of the **dendriform** algebra QSym).

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- Thus, for example, $\mathcal{K}_{\text{Epk}}$ is an ideal of the **dendriform** algebra QSym, and the quotient $\text{QSym} / \mathcal{K}_{\text{Epk}}$ is a dendriform algebra.

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- Thus, for example, $\mathcal{K}_{\mathrm{Epk}}$ is an ideal of the **dendriform** algebra QSym, and the quotient $\mathrm{QSym}/\mathcal{K}_{\mathrm{Epk}}$ is a dendriform algebra.
- This actually inspired the (combinatorial) proof of LR-shuffle-compatibility hinted at above.

A few questions

- **Question.** What mileage do we get out of $\mathcal{Z}$-enriched (P, γ)-partitions for other choices of $\mathcal{N}$ and $\mathcal{Z}$ than the ones used in the known proofs?
- **Question.** What ring do the $K_{n,\Lambda}^{\mathcal{Z}}$ span?
- **Question.** Hsiao and Petersen have generalized enriched (P, γ)-partitions to "colored (P, γ)-partitions" (with $\{+, -\}$ replaced by an m-element set). Does this generalize our results?
- **Question.** How do the kernels $\mathcal{K}_{\mathrm{st}}$ look like for $\mathrm{st} = \mathrm{Pk}, \mathrm{Lpk}, \ldots$?
- **Question.** Are the quotients $\mathrm{QSym} / \mathcal{K}_{\mathrm{st}}$ for $\mathrm{st} = \mathrm{des}, \mathrm{Lpk}, \mathrm{Epk}$ known dendriform algebras?

Section 4

Quadri-compatibility (work in progress)

References:

- a forthcoming preprint.
- Marcelo Aguiar, Jean-Louis Loday, *Quadri-algebras*, Journal of Pure and Applied Algebra, Volume 191 (2004), Issue 3, Pages 205–221.
- Loïc Foissy, *Free quadri-algebras and dual quadri-algebras*, arXiv:1504.06056.

WIP: Quadri-compatibility, 1: definition

- We can refine LR-shuffle-compatibility even further.
- Given two disjoint nonempty permutations $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ and $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$, define sets $S_{i,j}(\pi, \sigma)$ for all $i, j \in \{1, 2\}$ as follows:

$$S_{1,1}(\pi, \sigma) = \{\tau \in S(\pi, \sigma) \mid \tau_1 = \pi_1 \text{ and } \tau_{n+m} = \pi_n\};$$
$$S_{1,2}(\pi, \sigma) = \{\tau \in S(\pi, \sigma) \mid \tau_1 = \pi_1 \text{ and } \tau_{n+m} = \sigma_m\};$$
$$S_{2,1}(\pi, \sigma) = \{\tau \in S(\pi, \sigma) \mid \tau_1 = \sigma_1 \text{ and } \tau_{n+m} = \pi_n\};$$
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$$S_{2,2}(\pi,\sigma) = \{\tau \in S(\pi,\sigma) \mid \tau_1 = \sigma_1 \text{ and } \tau_{n+m} = \sigma_m\}.$$

- A statistic st is said to be *quadri-compatible* if for any two disjoint nonempty permutations π and σ and any $i, j \in \{1, 2\}$, the multiset

$$\{\operatorname{st}\tau \mid \tau \in S_{i,j}(\pi,\sigma)\}_{\text{multiset}}$$

depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, $|\pi|$, $|\sigma|$, i, j, the truth value of $\pi_1 > \sigma_1$, and the truth value of $\pi_n > \sigma_m$.

- A permutation statistic st is said to be *tail-graft-compatible* if for any nonempty permutation $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ and any letter a that does not appear in π, the element $\operatorname{st}(\pi : a)$ depends only on $\operatorname{st}(\pi)$, $|\pi|$ and on the truth value of $a > \pi_n$. Here, $\pi : a$ is the permutation obtained from π by appending a at the end:

$$\pi = (\pi_1, \pi_2, \ldots, \pi_n) \qquad \Longrightarrow \qquad \pi : a = (a, \pi_1, \pi_2, \ldots, \pi_n, a).$$

WIP: Quadri-compatibility, 2: criterion

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$$\pi = (\pi_1, \pi_2, \ldots, \pi_n) \qquad \Longrightarrow \qquad \pi : a = (a, \pi_1, \pi_2, \ldots, \pi_n, a).$$

- **(Almost-)Theorem (G.)** A statistic st is quadri-compatible **if and only if** it is shuffle-compatible, head-graft-compatible and tail-graft-compatible.
- My proof uses both induction and QSym and still needs to be written up. (Hopefully it survives the process.)

WIP: Quadri-compatibility, 2: criterion

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$$\pi = (\pi_1, \pi_2, \ldots, \pi_n) \qquad \Longrightarrow \qquad \pi : a = (a, \pi_1, \pi_2, \ldots, \pi_n, a).$$

- **(Almost-)Theorem (G.)** A statistic st is quadri-compatible **if and only if** it is shuffle-compatible, head-graft-compatible and tail-graft-compatible.
- My proof uses both induction and QSym and still needs to be written up. (Hopefully it survives the process.)
- Hence, Des, des, and Epk are quadri-compatible.

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- The proof (so far) uses a refined version of dendriform algebras: the *quadri-algebras* of Aguiar and Loday (arXiv:math/0309171, arXiv:1504.06056).

Thanks

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Thank you for attending!

slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/dartmouth18.pdf

paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf

project: https://github.com/darijgr/gzshuf