

# Double posets and the antipode of QSym

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## Abstract

A quasisymmetric function is assigned to every double poset (that is, every finite set endowed with two partial orders) and any weight function on its ground set. This generalizes well-known objects such as monomial and fundamental quasisymmetric functions, (skew) Schur functions, dual immaculate functions, and quasisymmetric  $(P, \omega)$ -partition enumerators. We prove a formula for the antipode of this function that holds under certain conditions (which are satisfied when the second order of the double poset is total, but also in some other cases); this restates (in a way that to us seems more natural) a result by Malvenuto and Reutenauer, but our proof is new and self-contained. We generalize it further to an even more comprehensive setting, where a group acts on the double poset by automorphisms.

**Keywords:** antipodes, double posets, Hopf algebras, posets,  $P$ -partitions, quasisymmetric functions.

**MSC2010 Mathematics Subject Classifications:** 05E05, 05E18.

## 1. Introduction

Double posets and  $\mathbf{E}$ -partitions (for  $\mathbf{E}$  a double poset) have been introduced by Claudia Malvenuto and Christophe Reutenauer [MalReu09]; their goal was to construct a combinatorial Hopf algebra which harbors a noticeable amount of structure, including an analogue of the Littlewood-Richardson rule and a lift of the internal product operation of the Malvenuto-Reutenauer Hopf algebra of permutations. In this note, we shall employ these same notions to restate in a simpler form, and reprove in a more elementary fashion, a formula for the antipode in the Hopf algebra QSym of quasisymmetric functions due to (the same) Malvenuto and Reutenauer [MalReu98, Theorem 3.1]. We then further generalize this formula to

a setting in which a group acts on the double poset (a generalization inspired by Katharina Jochemko's [Joch13]).

In the present version of the paper, some (classical and/or straightforward) proofs are missing or sketched. A more detailed version exists, in which at least a few of these proofs are elaborated on more<sup>1</sup>.

A short summary of this paper has been submitted to the FPSAC conference [Grin16b].

## Acknowledgments

Katharina Jochemko's work [Joch13] provoked this research. I learnt a lot about QSym from Victor Reiner. The SageMath computer algebra system [Sage16] was used for some computations that suggested one of the proofs.

## Note on the published version of this paper

The document you are reading is the preprint of a paper (of the same title) that was accepted for publication in the Electronic Journal of Combinatorics in 2017. The published version differs from this preprint insubstantially<sup>2</sup>.

## 2. Quasisymmetric functions

Let us first briefly introduce the notations that will be used in the following.

We set  $\mathbb{N} = \{0, 1, 2, \dots\}$ . A *composition* means a finite sequence of positive integers. We let  $\text{Comp}$  be the set of all compositions. For  $n \in \mathbb{N}$ , a *composition of  $n$*  means a composition whose entries sum to  $n$  (that is, a composition  $(\alpha_1, \alpha_2, \dots, \alpha_k)$  satisfying  $\alpha_1 + \alpha_2 + \dots + \alpha_k = n$ ).

Let  $\mathbf{k}$  be an arbitrary commutative ring. We shall keep  $\mathbf{k}$  fixed throughout this paper. We consider the  $\mathbf{k}$ -algebra  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  of formal power series in infinitely many (commuting) indeterminates  $x_1, x_2, x_3, \dots$  over  $\mathbf{k}$ . A *monomial* shall always mean a monomial (without coefficients) in the variables  $x_1, x_2, x_3, \dots$ <sup>3</sup>

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<sup>1</sup>It can be downloaded from

<http://www.cip.ifi.lmu.de/~grinberg/algebra/dp-abstr-long.pdf> . It is also archived as an ancillary file on <http://arxiv.org/abs/1509.08355v3>, although the former website is more likely to be updated.

<sup>2</sup>The main difference is that in the published version, the long footnote in Section 2 has been relegated into a separate subsection (§2.2), whereas the remainder of Section 2 has become §2.1. Other than this, the two versions differ in formatting and editorialization.

<sup>3</sup>For the sake of completeness, let us give a detailed definition of monomials and of the topology on  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ . (This definition has been copied from [Grin14, §2], essentially unchanged.)

Let  $x_1, x_2, x_3, \dots$  be countably many distinct symbols. We let  $\text{Mon}$  be the free abelian monoid on the set  $\{x_1, x_2, x_3, \dots\}$  (written multiplicatively); it consists of elements of the form  $x_1^{a_1} x_2^{a_2} x_3^{a_3} \dots$  for finitely supported  $(a_1, a_2, a_3, \dots) \in \mathbb{N}^\infty$  (where "finitely supported" means that all but finitely many positive integers  $i$  satisfy  $a_i = 0$ ). A *monomial* will mean an element of  $\text{Mon}$ .

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Inside the  $\mathbf{k}$ -algebra  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  is a subalgebra  $\mathbf{k}[[x_1, x_2, x_3, \dots]]_{\text{bdd}}$  consisting of the *bounded-degree* formal power series; these are the power series  $f$  for which there exists a  $d \in \mathbb{N}$  such that no monomial of degree  $> d$  appears in  $f$ <sup>4</sup>. This  $\mathbf{k}$ -subalgebra  $\mathbf{k}[[x_1, x_2, x_3, \dots]]_{\text{bdd}}$  becomes a topological  $\mathbf{k}$ -algebra, by inheriting the topology from  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ .

Two monomials  $m$  and  $n$  are said to be *pack-equivalent*<sup>5</sup> if they have the forms  $x_{i_1}^{a_1} x_{i_2}^{a_2} \dots x_{i_\ell}^{a_\ell}$  and  $x_{j_1}^{a_1} x_{j_2}^{a_2} \dots x_{j_\ell}^{a_\ell}$  for two strictly increasing sequences  $(i_1 < i_2 < \dots < i_\ell)$  and  $(j_1 < j_2 < \dots < j_\ell)$  of positive integers and one (common) sequence  $(a_1, a_2, \dots, a_\ell)$  of positive integers.<sup>6</sup> A power series  $f \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$  is said to be *quasisymmetric* if it satisfies the following condition: If  $m$  and  $n$  are two pack-equivalent monomials, then the coefficient of  $m$  in  $f$  equals the coefficient of  $n$  in  $f$ .

It is easy to see that the quasisymmetric power series form a  $\mathbf{k}$ -subalgebra of

Thus, a monomial is a combinatorial object, independent of  $\mathbf{k}$ ; it does not carry a coefficient.

We consider the  $\mathbf{k}$ -algebra  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  of (commutative) power series in countably many distinct indeterminates  $x_1, x_2, x_3, \dots$  over  $\mathbf{k}$ . By abuse of notation, we shall identify every monomial  $x_1^{a_1} x_2^{a_2} x_3^{a_3} \dots \in \text{Mon}$  with the corresponding element  $x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} \dots$  of  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  when necessary (e.g., when we speak of the sum of two monomials or when we multiply a monomial with an element of  $\mathbf{k}$ ). (To be very pedantic, this identification is slightly dangerous, because it can happen that two distinct monomials in  $\text{Mon}$  get identified with two identical elements of  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ . However, this can only happen when the ring  $\mathbf{k}$  is trivial, and even then it is not a real problem unless we infer the equality of monomials from the equality of their counterparts in  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ , which we are not going to do.)

We furthermore endow the ring  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  with the following topology (as in [GriRei14, Section 2.6]):

We endow the ring  $\mathbf{k}$  with the discrete topology. To define a topology on the  $\mathbf{k}$ -algebra  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ , we (temporarily) regard every power series in  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  as the family of its coefficients (indexed by the set  $\text{Mon}$ ). More precisely, we have a  $\mathbf{k}$ -module isomorphism

$$\prod_{m \in \text{Mon}} \mathbf{k} \rightarrow \mathbf{k}[[x_1, x_2, x_3, \dots]], \quad (\lambda_m)_{m \in \text{Mon}} \mapsto \sum_{m \in \text{Mon}} \lambda_m m.$$

We use this isomorphism to transport the product topology on  $\prod_{m \in \text{Mon}} \mathbf{k}$  to  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ .

The resulting topology on  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  turns  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  into a topological  $\mathbf{k}$ -algebra; this is the topology that we will be using whenever we make statements about convergence in  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  or write down infinite sums of power series. A sequence  $(a_n)_{n \in \mathbb{N}}$  of power series converges to a power series  $a$  with respect to this topology if and only if for every monomial  $m$ , all sufficiently high  $n \in \mathbb{N}$  satisfy

$$(\text{the coefficient of } m \text{ in } a_n) = (\text{the coefficient of } m \text{ in } a).$$

Note that this topological  $\mathbf{k}$ -algebra  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  is **not** the completion of the polynomial ring  $\mathbf{k}[x_1, x_2, x_3, \dots]$  with respect to the standard grading (in which all  $x_i$  have degree 1). (They are distinct even as sets.)

<sup>4</sup>The *degree* of a monomial  $x_1^{a_1} x_2^{a_2} x_3^{a_3} \dots$  is defined to be the nonnegative integer  $a_1 + a_2 + a_3 + \dots$ . A monomial  $m$  is said to *appear* in a power series  $f \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$  if and only if the coefficient of  $m$  in  $f$  is nonzero.

<sup>5</sup>Pack-equivalence and the related notions of packed combinatorial objects that we will encounter below originate in work of Hivert, Novelli and Thibon [NovThi05]. Simple as they are, they are of great help in dealing with quasisymmetric functions.

<sup>6</sup>For instance,  $x_2^2 x_3 x_4^2$  is pack-equivalent to  $x_1^2 x_4 x_8^2$  but not to  $x_2 x_3^2 x_4^2$ .

$\mathbf{k}[[x_1, x_2, x_3, \dots]]$ . But usually one is interested in a subset of this  $\mathbf{k}$ -subalgebra: namely, the set of quasisymmetric bounded-degree power series in  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ . This latter set is a  $\mathbf{k}$ -subalgebra of  $\mathbf{k}[[x_1, x_2, x_3, \dots]]_{\text{bdd}}$ , and is known as the  *$\mathbf{k}$ -algebra of quasisymmetric functions over  $\mathbf{k}$* . It is denoted by QSym.

The symmetric functions (in the usual sense of this word in combinatorics – so, really, symmetric bounded-degree power series in  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$ ) form a  $\mathbf{k}$ -subalgebra of QSym. The quasisymmetric functions have a rich theory which is related to, and often sheds new light on, the classical theory of symmetric functions; expositions can be found in [Stan99, §§7.19, 7.23] and [GriRei14, §§5-6] and other sources.<sup>7</sup>

As a  $\mathbf{k}$ -module, QSym has a basis  $(M_\alpha)_{\alpha \in \text{Comp}}$  indexed by all compositions, where the quasisymmetric function  $M_\alpha$  for a given composition  $\alpha$  is defined as follows: Writing  $\alpha$  as  $(\alpha_1, \alpha_2, \dots, \alpha_\ell)$ , we set

$$M_\alpha = \sum_{i_1 < i_2 < \dots < i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_\ell}^{\alpha_\ell} = \sum_{\substack{\mathbf{m} \\ \text{m is a monomial pack-equivalent} \\ \text{to } x_1^{\alpha_1} x_2^{\alpha_2} \dots x_\ell^{\alpha_\ell}}} \mathbf{m}$$

(where the  $i_k$  in the first sum are positive integers). This basis  $(M_\alpha)_{\alpha \in \text{Comp}}$  is known as the *monomial basis* of QSym, and is the simplest to define among many. (We shall briefly encounter another basis in Example 3.6.)

The  $\mathbf{k}$ -algebra QSym can be endowed with a structure of a  $\mathbf{k}$ -coalgebra which, combined with its  $\mathbf{k}$ -algebra structure, turns it into a Hopf algebra. We refer to the literature both for the theory of coalgebras and Hopf algebras (see [Montg93], [GriRei14, §1], [Manchon04, §1-§2], [Abe77], [Sweed69], [DNR01] or [Fresse14, Chapter 7]) and for a deeper study of the Hopf algebra QSym (see [Malve93], [HaGuKi10, Chapter 6] or [GriRei14, §5]); in this note we shall need but the very basics of this structure, and so it is only them that we introduce.

In the following, all tensor products are over  $\mathbf{k}$  by default (i.e., the sign  $\otimes$  stands for  $\otimes_{\mathbf{k}}$  unless it comes with a subscript).

Now, we define two  $\mathbf{k}$ -linear maps  $\Delta$  and  $\varepsilon$  as follows<sup>8</sup>:

- We define a  $\mathbf{k}$ -linear map  $\Delta : \text{QSym} \rightarrow \text{QSym} \otimes \text{QSym}$  by requiring that

$$\Delta \left( M_{(\alpha_1, \alpha_2, \dots, \alpha_\ell)} \right) = \sum_{k=0}^{\ell} M_{(\alpha_1, \alpha_2, \dots, \alpha_k)} \otimes M_{(\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_\ell)} \tag{1}$$

for every  $(\alpha_1, \alpha_2, \dots, \alpha_\ell) \in \text{Comp}$ .

- We define a  $\mathbf{k}$ -linear map  $\varepsilon : \text{QSym} \rightarrow \mathbf{k}$  by requiring that

$$\varepsilon \left( M_{(\alpha_1, \alpha_2, \dots, \alpha_\ell)} \right) = \delta_{\ell,0} \quad \text{for every } (\alpha_1, \alpha_2, \dots, \alpha_\ell) \in \text{Comp}.$$

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<sup>7</sup>The notion of quasisymmetric functions goes back to Gessel in 1984 [Gessel84]; they have been studied by many authors, most significantly Malvenuto and Reutenauer [MalReu95].

<sup>8</sup>Both of their definitions rely on the fact that  $\left( M_{(\alpha_1, \alpha_2, \dots, \alpha_\ell)} \right)_{(\alpha_1, \alpha_2, \dots, \alpha_\ell) \in \text{Comp}} = (M_\alpha)_{\alpha \in \text{Comp}}$  is a basis of the  $\mathbf{k}$ -module QSym.

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(Here,  $\delta_{u,v}$  is defined to be  $\begin{cases} 1, & \text{if } u = v; \\ 0, & \text{if } u \neq v \end{cases}$  whenever  $u$  and  $v$  are two objects.)

The map  $\varepsilon$  can also be defined in a simpler (equivalent) way: Namely,  $\varepsilon$  sends every power series  $f \in \text{QSym}$  to the result  $f(0,0,0,\dots)$  of substituting zeroes for the variables  $x_1, x_2, x_3, \dots$  in  $f$ . The map  $\Delta$  can also be described in such terms, but with greater difficulty<sup>9</sup>.

It is well-known that these maps  $\Delta$  and  $\varepsilon$  make the three diagrams

$$\begin{array}{ccc}
 \text{QSym} & \xrightarrow{\Delta} & \text{QSym} \otimes \text{QSym} \\
 \Delta \downarrow & & \downarrow \Delta \otimes \text{id} \\
 \text{QSym} \otimes \text{QSym} & \xrightarrow{\text{id} \otimes \Delta} & \text{QSym} \otimes \text{QSym} \otimes \text{QSym} \\
 \text{QSym} \xrightarrow{\Delta} \text{QSym} \otimes \text{QSym} & , & \text{QSym} \xrightarrow{\Delta} \text{QSym} \otimes \text{QSym} \\
 \cong \searrow & \downarrow \varepsilon \otimes \text{id} & \cong \searrow \\
 \mathbf{k} \otimes \text{QSym} & & \text{QSym} \otimes \mathbf{k}
 \end{array}$$

(where the  $\cong$  arrows are the canonical isomorphisms) commutative, and so  $(\text{QSym}, \Delta, \varepsilon)$  is what is commonly called a  $\mathbf{k}$ -coalgebra. Furthermore,  $\Delta$  and  $\varepsilon$  are  $\mathbf{k}$ -algebra homomorphisms, which is what makes this  $\mathbf{k}$ -coalgebra  $\text{QSym}$  into a  $\mathbf{k}$ -bialgebra. Finally, let  $m : \text{QSym} \otimes \text{QSym} \rightarrow \text{QSym}$  be the  $\mathbf{k}$ -linear map sending every pure tensor  $a \otimes b$  to  $ab$ , and let  $u : \mathbf{k} \rightarrow \text{QSym}$  be the  $\mathbf{k}$ -linear map sending  $1 \in \mathbf{k}$  to  $1 \in \text{QSym}$ . Then, there exists a unique  $\mathbf{k}$ -linear map  $S : \text{QSym} \rightarrow \text{QSym}$  making the diagram

$$\begin{array}{ccccc}
 & & \text{QSym} \otimes \text{QSym} & \xrightarrow{S \otimes \text{id}} & \text{QSym} \otimes \text{QSym} & & (2) \\
 & \nearrow \Delta & & & & \searrow m & \\
 \text{QSym} & \xrightarrow{\varepsilon} & \mathbf{k} & \xrightarrow{u} & \text{QSym} & & \\
 & \searrow \Delta & & & & \nearrow m & \\
 & & \text{QSym} \otimes \text{QSym} & \xrightarrow{\text{id} \otimes S} & \text{QSym} \otimes \text{QSym} & & 
 \end{array}$$

commutative. This map  $S$  is known as the *antipode* of  $\text{QSym}$ . It is known to be an involution and an algebra automorphism of  $\text{QSym}$ , and its action on the various quasisymmetric functions defined combinatorially is the main topic of this note. The existence of the antipode  $S$  makes  $\text{QSym}$  into a *Hopf algebra*.

### 3. Double posets

Next, we shall introduce the notion of a double poset, following Malvenuto and Reutenauer [MalReu09].

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<sup>9</sup>See [GriRei14, (5.3)] for the details.

- Definition 3.1.** (a) We shall encode posets as pairs  $(E, <)$ , where  $E$  is a set and  $<$  is a strict partial order (i.e., an irreflexive, transitive and antisymmetric binary relation) on the set  $E$ ; this relation  $<$  will be regarded as the smaller relation of the poset. All binary relations will be written in infix notation: i.e., we write “ $a < b$ ” for “ $a$  is related to  $b$  by the relation  $<$ ”. (If you define binary relations as sets of pairs, then “ $a$  is related to  $b$  by the relation  $<$ ” means that  $(a, b)$  is an element of the set  $<$ .)
- (b) If  $<$  is a strict partial order on a set  $E$ , and if  $a$  and  $b$  are two elements of  $E$ , then we say that  $a$  and  $b$  are  $<$ -comparable if we have either  $a < b$  or  $a = b$  or  $b < a$ . A strict partial order  $<$  on a set  $E$  is said to be a *total order* if and only if every two elements of  $E$  are  $<$ -comparable.
- (c) If  $<$  is a strict partial order on a set  $E$ , and if  $a$  and  $b$  are two elements of  $E$ , then we say that  $a$  is  $<$ -covered by  $b$  if we have  $a < b$  and there exists no  $c \in E$  satisfying  $a < c < b$ . (For instance, if  $<$  is the standard smaller relation on  $\mathbb{Z}$ , then each  $i \in \mathbb{Z}$  is  $<$ -covered by  $i + 1$ .)
- (d) A *double poset* is defined as a triple  $(E, <_1, <_2)$  where  $E$  is a finite set and  $<_1$  and  $<_2$  are two strict partial orders on  $E$ .
- (e) A double poset  $(E, <_1, <_2)$  is said to be *special* if the relation  $<_2$  is a total order.
- (f) A double poset  $(E, <_1, <_2)$  is said to be *semispecial* if every two  $<_1$ -comparable elements of  $E$  are  $<_2$ -comparable.
- (g) A double poset  $(E, <_1, <_2)$  is said to be *tertispecial* if it satisfies the following condition: If  $a$  and  $b$  are two elements of  $E$  such that  $a$  is  $<_1$ -covered by  $b$ , then  $a$  and  $b$  are  $<_2$ -comparable.
- (h) If  $<$  is a binary relation on a set  $E$ , then the *opposite relation* of  $<$  is defined to be the binary relation  $>$  on the set  $E$  that is defined as follows: For any  $e \in E$  and  $f \in E$ , we have  $e > f$  if and only if  $f < e$ . Notice that if  $<$  is a strict partial order, then so is the opposite relation  $>$  of  $<$ .

Clearly, every special double poset is semispecial, and every semispecial double poset is tertispecial.<sup>10</sup>

<sup>10</sup>The notions of a double poset and of a special double poset come from [MalReu09]. See [Foissy13] for further results on special double posets. The notion of a “tertispecial double poset” (Dog Latin for “slightly less special than semispecial”; in hindsight, “locally special” would have been better terminology) appears to be new and arguably sounds artificial, but is the most suitable setting for some of the results below (see, e.g., Remark 4.9 below); moreover, it appears in nature, beyond the particular case of special double posets (see Example 3.3). We shall not use semispecial double posets in the following; they were only introduced as a middle-ground notion between special and tertispecial double posets having a less daunting definition.

**Definition 3.2.** If  $\mathbf{E} = (E, <_1, <_2)$  is a double poset, then an  $\mathbf{E}$ -partition shall mean a map  $\phi : E \rightarrow \{1, 2, 3, \dots\}$  such that:

- every  $e \in E$  and  $f \in E$  satisfying  $e <_1 f$  satisfy  $\phi(e) \leq \phi(f)$ ;
- every  $e \in E$  and  $f \in E$  satisfying  $e <_1 f$  and  $f <_2 e$  satisfy  $\phi(e) < \phi(f)$ .

**Example 3.3.** The notion of an  $\mathbf{E}$ -partition (which was inspired by the earlier notions of  $P$ -partitions and  $(P, \omega)$ -partitions as studied by Gessel and Stanley<sup>11</sup>) generalizes various well-known combinatorial concepts. For example:

- If  $<_2$  is the same order as  $<_1$  (or any extension of this order), then the  $\mathbf{E}$ -partitions are the weakly increasing maps from the poset  $(E, <_1)$  to the totally ordered set  $\{1, 2, 3, \dots\}$ .
- If  $<_2$  is the opposite relation of  $<_1$  (or any extension of this opposite relation), then the  $\mathbf{E}$ -partitions are the strictly increasing maps from the poset  $(E, <_1)$  to the totally ordered set  $\{1, 2, 3, \dots\}$ .

For a more interesting example, let  $\mu = (\mu_1, \mu_2, \mu_3, \dots)$  and  $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$  be two partitions such that  $\mu \subseteq \lambda$ . (See [GriRei14, §2] for the notations we are using here.) The skew Young diagram  $Y(\lambda/\mu)$  is then defined as the set of all  $(i, j) \in \{1, 2, 3, \dots\}^2$  satisfying  $\mu_i < j \leq \lambda_i$ . On this set  $Y(\lambda/\mu)$ , we define two strict partial orders  $<_1$  and  $<_2$  by

$$(i, j) <_1 (i', j') \iff (i \leq i' \text{ and } j \leq j' \text{ and } (i, j) \neq (i', j'))$$

and

$$(i, j) <_2 (i', j') \iff (i \geq i' \text{ and } j \leq j' \text{ and } (i, j) \neq (i', j')).$$

The resulting double poset  $\mathbf{Y}(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_2)$  has the property that the  $\mathbf{Y}(\lambda/\mu)$ -partitions are precisely the semistandard tableaux of shape  $\lambda/\mu$ . (Again, see [GriRei14, §2] for the meaning of these words.)

This double poset  $\mathbf{Y}(\lambda/\mu)$  is not special (in general), but it is tertispecial. (Indeed, if  $a$  and  $b$  are two elements of  $Y(\lambda/\mu)$  such that  $a$  is  $<_1$ -covered by  $b$ , then  $a$  is either the left neighbor of  $b$  or the top neighbor of  $b$ , and thus we have either  $a <_2 b$  (in the former case) or  $b <_2 a$  (in the latter case).) Some authors prefer to use a special double poset instead, which is defined as follows: We define a total order  $<_h$  on  $Y(\lambda/\mu)$  by

$$(i, j) <_h (i', j') \iff (i > i' \text{ or } (i = i' \text{ and } j < j')).$$

Then,  $\mathbf{Y}_h(\lambda/\mu) = (Y(\lambda/\mu), <_1, <_h)$  is a special double poset, and the  $\mathbf{Y}_h(\lambda/\mu)$ -partitions are precisely the semistandard tableaux of shape  $\lambda/\mu$ .

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<sup>11</sup>See [Gessel15] for the history of these notions, and see [Gessel84], [Stan71], [Stan11, §3.15] and

We now assign a certain formal power series to every double poset:

**Definition 3.4.** If  $\mathbf{E} = (E, <_1, <_2)$  is a double poset, and  $w : E \rightarrow \{1, 2, 3, \dots\}$  is a map, then we define a power series  $\Gamma(\mathbf{E}, w) \in \mathbf{k}[[x_1, x_2, x_3, \dots]]$  by

$$\Gamma(\mathbf{E}, w) = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi, w}, \quad \text{where } \mathbf{x}_{\pi, w} = \prod_{e \in E} x_{\pi(e)}^{w(e)}.$$

The following fact is easy to see (but will be reproven below):

**Proposition 3.5.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a double poset, and  $w : E \rightarrow \{1, 2, 3, \dots\}$  be a map. Then,  $\Gamma(\mathbf{E}, w) \in \text{QSym}$ .

**Example 3.6.** The power series  $\Gamma(\mathbf{E}, w)$  generalize various well-known quasisymmetric functions.

(a) If  $\mathbf{E} = (E, <_1, <_2)$  is a double poset, and  $w : E \rightarrow \{1, 2, 3, \dots\}$  is the constant function sending everything to 1, then  $\Gamma(\mathbf{E}, w) = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi}$ , where  $\mathbf{x}_{\pi} = \prod_{e \in E} x_{\pi(e)}$ . We shall denote this power series  $\Gamma(\mathbf{E}, w)$  by  $\Gamma(\mathbf{E})$ ; it is exactly what has been called  $\Gamma(\mathbf{E})$  in [MalReu09, §2.2]. All results proven below for  $\Gamma(\mathbf{E}, w)$  can be applied to  $\Gamma(\mathbf{E})$ , yielding simpler (but less general) statements.

(b) If  $E = \{1, 2, \dots, \ell\}$  for some  $\ell \in \mathbb{N}$ , if  $<_1$  is the usual total order inherited from  $\mathbb{Z}$ , and if  $<_2$  is the opposite relation of  $<_1$ , then the special double poset  $\mathbf{E} = (E, <_1, <_2)$  satisfies  $\Gamma(\mathbf{E}, w) = M_{\alpha}$ , where  $\alpha$  is the composition  $(w(1), w(2), \dots, w(\ell))$ .

Note that every  $M_{\alpha}$  can be obtained this way (by choosing  $\ell$  and  $w$  appropriately). Thus, the elements of the monomial basis  $(M_{\alpha})_{\alpha \in \text{Comp}}$  are special cases of the functions  $\Gamma(\mathbf{E}, w)$ . This shows that the  $\Gamma(\mathbf{E}, w)$  for varying  $\mathbf{E}$  and  $w$  span the  $\mathbf{k}$ -module QSym.

(c) Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_{\ell})$  be a composition of a nonnegative integer  $n$ . Let  $D(\alpha)$  be the set  $\{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + \alpha_2 + \alpha_3, \dots, \alpha_1 + \alpha_2 + \dots + \alpha_{\ell-1}\}$ . Let  $E$  be the set  $\{1, 2, \dots, n\}$ , and let  $<_1$  be the total order inherited on  $E$  from  $\mathbb{Z}$ . Let  $<_2$  be some partial order on  $E$  with the property that

$$i + 1 <_2 i \quad \text{for every } i \in D(\alpha)$$

and

$$i <_2 i + 1 \quad \text{for every } i \in \{1, 2, \dots, n - 1\} \setminus D(\alpha).$$

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[Stan99, §7.19] for some of their theory. Mind that these sources use different and sometimes incompatible notations – e.g., the  $P$ -partitions of [Stan11, §3.15] and [Gessel15] differ from those of [Gessel84] by a sign reversal.



(There are several choices for such an order; in particular, we can find one which is a total order.) Then,

$$\begin{aligned} \Gamma((E, <_1, <_2)) &= \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n; \\ i_j < i_{j+1} \text{ whenever } j \in D(\alpha)}} x_{i_1} x_{i_2} \cdots x_{i_n} \\ &= \sum_{\beta \text{ is a composition of } n; D(\beta) \supseteq D(\alpha)} M_\beta. \end{aligned}$$

This power series is known as the  $\alpha$ -th *fundamental quasisymmetric function*, usually called  $F_\alpha$  (in [Gessel84], [MalReu95, §2], [BBSSZ13, §2.4] and [Grin14, §2]) or  $L_\alpha$  (in [Stan99, §7.19] or [GriRei14, Definition 5.15]).

- (d) Let  $\mathbf{E}$  be one of the two double posets  $\mathbf{Y}(\lambda/\mu)$  and  $\mathbf{Y}_h(\lambda/\mu)$  defined as in Example 3.3 for two partitions  $\mu$  and  $\lambda$ . Then,  $\Gamma(\mathbf{E})$  is the skew Schur function  $s_{\lambda/\mu}$ .
- (e) Similarly, *dual immaculate functions* as defined in [BBSSZ13, §3.7] can be realized as  $\Gamma(\mathbf{E})$  for conveniently chosen  $\mathbf{E}$  (see [Grin14, Proposition 4.4]), which helped the author to prove one of their properties [Grin14]. (The  $\mathbf{E}$ -partitions here are the so-called *immaculate tableaux*.)
- (f) When the relation  $<_2$  of a double poset  $\mathbf{E} = (E, <_1, <_2)$  is a total order (i.e., when the double poset  $\mathbf{E}$  is special), the  $\mathbf{E}$ -partitions are precisely the reverse  $(P, \omega)$ -partitions (for  $P = (E, <_1)$  and  $\omega$  being the unique bijection  $E \rightarrow \{1, 2, \dots, |E|\}$  satisfying  $\omega^{-1}(1) <_2 \omega^{-1}(2) <_2 \dots <_2 \omega^{-1}(|E|)$ ) in the terminology of [Stan99, §7.19], and the power series  $\Gamma(\mathbf{E})$  is the  $K_{P, \omega}$  of [Stan99, §7.19]. This can also be rephrased using the notations of [GriRei14, §5.2]: When the relation  $<_2$  of a double poset  $\mathbf{E} = (E, <_1, <_2)$  is a total order, we can relabel the elements of  $E$  by the integers  $1, 2, \dots, n$  (where  $n = |E|$ ) in such a way that  $1 <_2 2 <_2 \dots <_2 n$ ; then, the  $\mathbf{E}$ -partitions are the  $P$ -partitions in the terminology of [GriRei14, Definition 5.12], where  $P$  is the labelled poset  $(E, <_1)$ ; and furthermore, our  $\Gamma(\mathbf{E})$  is the  $F_P(\mathbf{x})$  of [GriRei14, Definition 5.12]. Conversely, if  $P$  is a labelled poset, then the  $F_P(\mathbf{x})$  of [GriRei14, Definition 5.12] is our  $\Gamma((P, <_P, <_Z))$ .

## 4. The antipode theorem

We now come to the main results of this note. We first state a theorem and a corollary which are not new, but will be reproven in a more self-contained way which allows them to take their (well-deserved) place as fundamental results rather than afterthoughts in the theory of QSym.

**Definition 4.1.** We let  $S$  denote the antipode of QSym.

**Theorem 4.2.** Let  $(E, <_1, <_2)$  be a tertispecial double poset. Let  $w : E \rightarrow \{1, 2, 3, \dots\}$ . Then,  $S(\Gamma((E, <_1, <_2), w)) = (-1)^{|E|} \Gamma((E, >_1, <_2), w)$ , where  $>_1$  denotes the opposite relation of  $<_1$ .

**Corollary 4.3.** Let  $(E, <_1, <_2)$  be a tertispecial double poset. Then,  $S(\Gamma((E, <_1, <_2))) = (-1)^{|E|} \Gamma((E, >_1, <_2))$ , where  $>_1$  denotes the opposite relation of  $<_1$ .

We shall give examples for consequences of these facts shortly (Example 4.8), but let us first explain where they have already appeared. Corollary 4.3 is equivalent to [GriRei14, Corollary 5.27]<sup>12</sup> (a result found by Malvenuto and Reutenauer [MalReu98, Lemma 3.2]). Theorem 4.2 is equivalent to Malvenuto’s and Reutenauer’s [MalReu98, Theorem 3.1]<sup>13</sup>. We nevertheless believe that our versions of these facts are slicker and simpler than the ones appearing in existing literature<sup>14</sup>, and if not, then at least our proofs below are more natural.

<sup>12</sup>It is easiest to derive [GriRei14, Corollary 5.27] from our Corollary 4.3, as this only requires setting  $\mathbf{E} = (P, <_P, <_Z)$  (this is a special double poset, thus in particular a tertispecial one) and noticing that  $\Gamma((P, <_P, <_Z)) = F_P(\mathbf{x})$  and  $\Gamma((P, >_P, <_Z)) = F_{P^{\text{opp}}}(\mathbf{x})$ , where all unexplained notations are defined in [GriRei14, Chapter 5]. But one can also proceed in the opposite direction (hint: replace the partial order  $<_2$  by a linear extension, thus turning the tertispecial double poset  $(E, <_1, <_2)$  into a special one; argue that this does not change  $\Gamma((E, <_1, <_2))$  and  $\Gamma((E, >_1, <_2))$ ).

<sup>13</sup>This equivalence requires some work to set up. First of all, Malvenuto and Reutenauer, in [MalReu98], do not work with the antipode  $S$  of QSym, but instead study a certain automorphism of QSym called  $\omega$ . However, this automorphism is closely related to  $S$  (namely, for each  $n \in \mathbb{N}$  and each homogeneous element  $f \in \text{QSym}$  of degree  $n$ , we have  $\omega(f) = (-1)^n S(f)$ ); therefore, any statements about  $\omega$  can be translated into statements about  $S$  and vice versa.

Let me sketch how to derive [MalReu98, Theorem 3.1] from our Theorem 4.2. Indeed, contract all undirected edges in  $G$  and  $G'$ , denoting the (common) vertex set of the new graphs by  $E$ . Then, define two strict partial orders  $<_1$  and  $<_2$  on  $E$  by

$$(a <_1 b) \iff (a \neq b, \text{ and there exists a path from } a \text{ to } b \text{ in } G)$$

and

$$(a <_2 b) \iff (a \neq b, \text{ and there exists a path from } a \text{ to } b \text{ in } G').$$

The map  $w$  sends every  $e \in E$  to the number of vertices of  $G$  that became  $e$  when the edges were contracted. To show that the resulting double poset  $(E, <_1, <_2)$  is tertispecial, we must notice that if  $a$  is  $<_1$ -covered by  $b$ , then  $G$  had an edge from one of the vertices that became  $a$  to one of the vertices that became  $b$ . The “ $x_i$ ’s in  $X$  satisfying a set of conditions” (in the language of [MalReu98, Section 3]) are in 1-to-1 correspondence with  $(E, <_1, <_2)$ -partitions (at least when  $X = \{1, 2, 3, \dots\}$ ); this is not immediately obvious but not hard to check either (the acyclicity of  $G$  and  $G'$  is used in the proof). As a result, [MalReu98, Theorem 3.1] follows from Theorem 4.2 above. With some harder work, one can conversely derive our Theorem 4.2 from [MalReu98, Theorem 3.1].

<sup>14</sup>That said, we would not be surprised if Malvenuto and Reutenauer are aware of them; after all, they have discovered both the original version of Theorem 4.2 in [MalReu98] and the notion of double posets in [MalReu09].

To these known results, we add another, which seems to be unknown so far (probably because it is far harder to state in the terminologies of  $(P, \omega)$ -partitions or equality-and-inequality conditions appearing in literature). First, we need to introduce some notation:

**Definition 4.4.** Let  $G$  be a group, and let  $E$  be a  $G$ -set.

- (a) Let  $<$  be a strict partial order on  $E$ . We say that  $G$  *preserves the relation*  $<$  if the following holds: For every  $g \in G$ ,  $a \in E$  and  $b \in E$  satisfying  $a < b$ , we have  $ga < gb$ .
- (b) Let  $w : E \rightarrow \{1, 2, 3, \dots\}$ . We say that  $G$  *preserves*  $w$  if every  $g \in G$  and  $e \in E$  satisfy  $w(ge) = w(e)$ .
- (c) Let  $g \in G$ . Assume that the set  $E$  is finite. We say that  $g$  is  *$E$ -even* if the action of  $g$  on  $E$  (that is, the permutation of  $E$  that sends every  $e \in E$  to  $ge$ ) is an even permutation of  $E$ .
- (d) If  $X$  is any set, then the set  $X^E$  of all maps  $E \rightarrow X$  becomes a  $G$ -set in the following way: For any  $\pi \in X^E$  and  $g \in G$ , we define the element  $g\pi \in X^E$  to be the map sending each  $e \in E$  to  $\pi(g^{-1}e)$ .
- (e) Let  $F$  be a further  $G$ -set. Assume that the set  $E$  is finite. An element  $\pi \in F$  is said to be  *$E$ -coeven* if every  $g \in G$  satisfying  $g\pi = \pi$  is  $E$ -even. A  $G$ -orbit  $O$  on  $F$  is said to be  *$E$ -coeven* if all elements of  $O$  are  $E$ -coeven.

Before we come to the promised result, let us state two simple facts:

**Lemma 4.5.** Let  $G$  be a group. Let  $F$  and  $E$  be  $G$ -sets such that  $E$  is finite. Let  $O$  be a  $G$ -orbit on  $F$ . Then,  $O$  is  $E$ -coeven if and only if at least one element of  $O$  is  $E$ -coeven.

**Proposition 4.6.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a double poset. Let  $\text{Par } \mathbf{E}$  denote the set of all  $\mathbf{E}$ -partitions. Let  $G$  be a finite group which acts on  $E$ . Assume that  $G$  preserves both relations  $<_1$  and  $<_2$ .

- (a) Then,  $\text{Par } \mathbf{E}$  is a  $G$ -subset of the  $G$ -set  $\{1, 2, 3, \dots\}^E$  (see Definition 4.4 (d) for the definition of the latter).
- (b) Let  $w : E \rightarrow \{1, 2, 3, \dots\}$ . Assume that  $G$  preserves  $w$ . Let  $O$  be a  $G$ -orbit on  $\text{Par } \mathbf{E}$ . Then, the values of  $\mathbf{x}_{\pi, w}$  for all  $\pi \in O$  are equal.

**Theorem 4.7.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a tertispecial double poset. Let  $\text{Par } \mathbf{E}$  denote the set of all  $\mathbf{E}$ -partitions. Let  $w : E \rightarrow \{1, 2, 3, \dots\}$ . Let  $G$  be a finite group which acts on  $E$ . Assume that  $G$  preserves both relations  $<_1$  and  $<_2$ , and also preserves

$w$ . Then,  $G$  acts also on the set  $\text{Par } \mathbf{E}$  of all  $\mathbf{E}$ -partitions; namely,  $\text{Par } \mathbf{E}$  is a  $G$ -subset of the  $G$ -set  $\{1, 2, 3, \dots\}^E$  (according to Proposition 4.6 (a)). For any  $G$ -orbit  $O$  on  $\text{Par } \mathbf{E}$ , we define a monomial  $\mathbf{x}_{O,w}$  by

$$\mathbf{x}_{O,w} = \mathbf{x}_{\pi,w} \quad \text{for some element } \pi \text{ of } O.$$

(This is well-defined, since Proposition 4.6 (b) shows that  $\mathbf{x}_{\pi,w}$  does not depend on the choice of  $\pi \in O$ .)

Let

$$\Gamma(\mathbf{E}, w, G) = \sum_{O \text{ is a } G\text{-orbit on } \text{Par } \mathbf{E}} \mathbf{x}_{O,w}$$

and

$$\Gamma^+(\mathbf{E}, w, G) = \sum_{O \text{ is an } E\text{-coeven } G\text{-orbit on } \text{Par } \mathbf{E}} \mathbf{x}_{O,w}.$$

Then,  $\Gamma(\mathbf{E}, w, G)$  and  $\Gamma^+(\mathbf{E}, w, G)$  belong to QSym and satisfy

$$S(\Gamma(\mathbf{E}, w, G)) = (-1)^{|E|} \Gamma^+((E, >_1, <_2), w, G).$$

Here,  $>_1$  denotes the opposite relation of  $<_1$ .

This theorem, which combines Theorem 4.2 with the ideas of Pólya enumeration, is inspired by Jochemko’s reciprocity result for order polynomials [Joch13, Theorem 2.8], which can be obtained from it by specializations (see Section 8 for the details of how Jochemko’s result follows from ours).

We shall now briefly review a number of particular cases of Theorem 4.2.

**Example 4.8.** (a) Corollary 4.3 follows from Theorem 4.2 by letting  $w$  be the function which is constantly 1.

(b) Let  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$  be a composition of a nonnegative integer  $n$ , and let  $\mathbf{E} = (E, <_1, <_2)$  be the double poset defined in Example 3.6 (b). Let  $w : \{1, 2, \dots, \ell\} \rightarrow \{1, 2, 3, \dots\}$  be the map sending every  $i$  to  $\alpha_i$ . As Example 3.6 (b) shows, we have  $\Gamma(\mathbf{E}, w) = M_\alpha$ . Thus, applying Theorem 4.2 to these  $\mathbf{E}$  and  $w$  yields

$$\begin{aligned} S(M_\alpha) &= (-1)^\ell \Gamma((E, >_1, <_2), w) = (-1)^\ell \sum_{i_1 \geq i_2 \geq \dots \geq i_\ell} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_\ell}^{\alpha_\ell} \\ &= (-1)^\ell \sum_{i_1 \leq i_2 \leq \dots \leq i_\ell} x_{i_1}^{\alpha_\ell} x_{i_2}^{\alpha_{\ell-1}} \dots x_{i_\ell}^{\alpha_1} = (-1)^\ell \sum_{\substack{\gamma \text{ is a composition of } n; \\ D(\gamma) \subseteq D((\alpha_\ell, \alpha_{\ell-1}, \dots, \alpha_1))}} M_\gamma. \end{aligned}$$

This is the formula for  $S(M_\alpha)$  given in [Ehrenb96, Proposition 3.4], in [Malve93, (4.26)], in [GriRei14, Theorem 5.11], and in [BenSag14, Theorem 4.1] (originally due to Ehrenborg and to Malvenuto and Reutenauer).

- (c) Applying Corollary 4.3 to the double poset of Example 3.6 (c) (where the relation  $<_2$  is chosen to be a total order) yields the formula for the antipode of a fundamental quasisymmetric function ([Malve93, (4.27)], [GriRei14, (5.9)], [BenSag14, Theorem 5.1]).
- (d) Let us use the notations of Example 3.3. For any partition  $\lambda$ , let  $\lambda^t$  denote the conjugate partition of  $\lambda$ . Let  $\mu$  and  $\lambda$  be two partitions satisfying  $\mu \subseteq \lambda$ . Let  $>_1$  and  $>_2$  be the opposite relations of  $<_1$  and  $<_2$ . Then, there is a bijection  $\tau : Y(\lambda/\mu) \rightarrow Y(\lambda^t/\mu^t)$  sending each  $(i, j) \in Y(\lambda/\mu)$  to  $(j, i)$ . This bijection is an isomorphism of double posets from  $(Y(\lambda/\mu), >_1, <_2)$  to  $(Y(\lambda^t/\mu^t), >_1, >_2)$  (where the notion of an “isomorphism of double posets” is defined in the natural way – i.e., an isomorphism of double posets is a bijection  $\phi$  between their ground sets such that each of the two maps  $\phi$  and  $\phi^{-1}$  preserves each of the two orders). Hence,

$$\Gamma((Y(\lambda/\mu), >_1, <_2)) = \Gamma((Y(\lambda^t/\mu^t), >_1, >_2)). \quad (3)$$

But applying Corollary 4.3 to the tertispecial double poset  $\mathbf{Y}(\lambda/\mu)$ , we obtain

$$\begin{aligned} S(\Gamma(\mathbf{Y}(\lambda/\mu))) &= (-1)^{|\lambda/\mu|} \Gamma((Y(\lambda/\mu), >_1, <_2)) \\ &= (-1)^{|\lambda/\mu|} \Gamma((Y(\lambda^t/\mu^t), >_1, >_2)) \end{aligned} \quad (4)$$

(by (3)). But from Example 3.6 (d), we know that  $\Gamma(\mathbf{Y}(\lambda/\mu)) = s_{\lambda/\mu}$ . Moreover, a similar argument using [GriRei14, Remark 2.12] shows that  $\Gamma((Y(\lambda/\mu), >_1, >_2)) = s_{\lambda/\mu}$ . Applying this to  $\lambda^t$  and  $\mu^t$  instead of  $\lambda$  and  $\mu$ , we obtain  $\Gamma((Y(\lambda^t/\mu^t), >_1, >_2)) = s_{\lambda^t/\mu^t}$ . Now, (4) rewrites as

$$S(s_{\lambda/\mu}) = (-1)^{|\lambda/\mu|} s_{\lambda^t/\mu^t} \quad (5)$$

(since  $\Gamma(\mathbf{Y}(\lambda/\mu)) = s_{\lambda/\mu}$  and  $\Gamma((Y(\lambda^t/\mu^t), >_1, >_2)) = s_{\lambda^t/\mu^t}$ ). This is a well-known formula, and is usually stated for  $S$  being the antipode of the Hopf algebra of symmetric (rather than quasisymmetric) functions; but this is an equivalent statement, since the latter antipode is a restriction of the antipode of QSym.

It is also possible (but more difficult) to derive (5) by using the double poset  $\mathbf{Y}_h(\lambda/\mu)$  instead of  $\mathbf{Y}(\lambda/\mu)$ . (This boils down to what was done in [GriRei14, proof of Corollary 5.29].)

- (e) A result of Benedetti and Sagan [BenSag14, Theorem 8.2] on the antipodes of immaculate functions can be obtained from Corollary 4.3 using dualization.

**Remark 4.9.** Corollary 4.3 has a sort of converse. Namely, let us assume that  $\mathbf{k} = \mathbb{Z}$ . If  $(E, <_1, <_2)$  is a double poset satisfying  $S(\Gamma((E, <_1, <_2))) = (-1)^{|E|} \Gamma((E, >_1, <_2))$ , then  $(E, <_1, <_2)$  is tertispecial.

More precisely, the following holds: Define the *length*  $\ell(\alpha)$  of a composition  $\alpha$  to be the number of entries of  $\alpha$ . Define the *size*  $|\alpha|$  of a composition  $\alpha$  to be the sum of the entries of  $\alpha$ . Let  $\eta : \text{QSym} \rightarrow \text{QSym}$  be the  $\mathbf{k}$ -linear map defined by

$$\eta(M_\alpha) = \begin{cases} M_\alpha, & \text{if } \ell(\alpha) \geq |\alpha| - 1; \\ 0, & \text{if } \ell(\alpha) < |\alpha| - 1 \end{cases} \quad \text{for every } \alpha \in \text{Comp}.$$

Thus,  $\eta$  transforms a quasisymmetric function by removing all monomials  $m$  for which the number of indeterminates appearing in  $m$  is  $< \deg m - 1$ . We partially order the ring  $\mathbf{k}[[x_1, x_2, x_3, \dots]]$  by a coefficientwise order (i.e., two power series  $a$  and  $b$  satisfy  $a \leq b$  if and only if each coefficient of  $a$  is  $\leq$  to the corresponding coefficient of  $b$ ). Now, every double poset  $(E, <_1, <_2)$  satisfies

$$\eta\left((-1)^{|E|} S(\Gamma((E, <_1, <_2)))\right) \leq \eta(\Gamma((E, >_1, <_2))), \tag{6}$$

and equality holds if and only if the double poset  $(E, <_1, <_2)$  is tertispecial. (If we omit  $\eta$ , then the inequality fails in general.)

The proof of (6) is somewhat technical, but not too hard. A rough outline is given in the detailed version of this paper.

## 5. Lemmas: packed E-partitions and comultiplications

We shall now prepare for the proofs of our results. To this end, we introduce the notion of a *packed map*.

**Definition 5.1.** (a) An *initial interval* will mean a set of the form  $\{1, 2, \dots, \ell\}$  for some  $\ell \in \mathbb{N}$ .

(b) If  $E$  is a set and  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  is a map, then  $\pi$  is said to be *packed* if  $\pi(E)$  is an initial interval. Clearly, this initial interval must be  $\{1, 2, \dots, |\pi(E)|\}$ .

**Proposition 5.2.** Let  $E$  be a set. Let  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  be a packed map. Let  $\ell = |\pi(E)|$ .

(a) We have  $\pi(E) = \{1, 2, \dots, \ell\}$ .

(b) Let  $w : E \rightarrow \{1, 2, 3, \dots\}$  be a map. For each  $i \in \{1, 2, \dots, \ell\}$ , define an integer  $\alpha_i$  by  $\alpha_i = \sum_{e \in \pi^{-1}(i)} w(e)$ . Then,  $(\alpha_1, \alpha_2, \dots, \alpha_\ell)$  is a composition.

*Proof of Proposition 5.2.* This follows from the assumption that  $\pi$  be packed. (Details are left to the reader.)  $\square$

**Definition 5.3.** Let  $E$  be a set. Let  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  be a packed map. Let  $w : E \rightarrow \{1, 2, 3, \dots\}$  be a map. Then, the composition  $(\alpha_1, \alpha_2, \dots, \alpha_\ell)$  defined in Proposition 5.2 (b) will be denoted by  $\text{ev}_w \pi$ .

**Proposition 5.4.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a double poset. Let  $w : E \rightarrow \{1, 2, 3, \dots\}$  be a map. Then,

$$\Gamma(\mathbf{E}, w) = \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} M_{\text{ev}_w \varphi}. \quad (7)$$

*Proof of Proposition 5.4.* For every finite subset  $T$  of  $\{1, 2, 3, \dots\}$ , there exists a unique strictly increasing bijection  $\{1, 2, \dots, |T|\} \rightarrow T$ . We shall denote this bijection by  $r_T$ . For every map  $\pi : E \rightarrow \{1, 2, 3, \dots\}$ , we define the *packing* of  $\pi$  as the map  $r_{\pi(E)}^{-1} \circ \pi : E \rightarrow \{1, 2, 3, \dots\}$ ; this is a packed map (indeed, its image is  $\{1, 2, \dots, |\pi(E)|\}$ ), and will be denoted by  $\text{pack } \pi$ . This map  $\text{pack } \pi$  is an  $\mathbf{E}$ -partition if and only if  $\pi$  is an  $\mathbf{E}$ -partition<sup>15</sup>. Hence,  $\text{pack } \pi$  is a packed  $\mathbf{E}$ -partition for every  $\mathbf{E}$ -partition  $\pi$ .

We shall show that for every packed  $\mathbf{E}$ -partition  $\varphi$ , we have

$$\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}; \text{ pack } \pi = \varphi} \mathbf{x}_{\pi, w} = M_{\text{ev}_w \varphi}. \quad (8)$$

Once this is proven, it will follow that

$$\begin{aligned} \Gamma(\mathbf{E}, w) &= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \mathbf{x}_{\pi, w} = \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}; \text{ pack } \pi = \varphi} \mathbf{x}_{\pi, w}}_{= M_{\text{ev}_w \varphi} \text{ (by (8))}} \\ &\quad \text{(since } \text{pack } \pi \text{ is a packed } \mathbf{E}\text{-partition for every } \mathbf{E}\text{-partition } \pi) \\ &= \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} M_{\text{ev}_w \varphi} \end{aligned}$$

and Proposition 5.4 will be proven.

So it remains to prove (8). Let  $\varphi$  be a packed  $\mathbf{E}$ -partition. Let  $\ell = |\varphi(E)|$ ; thus  $\varphi(E) = \{1, 2, \dots, \ell\}$  (since  $\varphi$  is packed). Let  $\alpha_i = \sum_{e \in \varphi^{-1}(i)} w(e)$  for every

<sup>15</sup>Indeed,  $\text{pack } \pi = r_{\pi(E)}^{-1} \circ \pi$ . Since  $r_{\pi(E)}$  is strictly increasing, we thus see that, for any given  $e \in E$  and  $f \in E$ , the equivalences

$$((\text{pack } \pi)(e) \leq (\text{pack } \pi)(f)) \iff (\pi(e) \leq \pi(f))$$

and

$$((\text{pack } \pi)(e) < (\text{pack } \pi)(f)) \iff (\pi(e) < \pi(f))$$

hold. Hence,  $\text{pack } \pi$  is an  $\mathbf{E}$ -partition if and only if  $\pi$  is an  $\mathbf{E}$ -partition.

$i \in \{1, 2, \dots, \ell\}$ ; thus,  $\text{ev}_w \varphi = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$  (by the definition of  $\text{ev}_w \varphi$ ). Hence, the definition of  $M_{\text{ev}_w \varphi}$  yields

$$\begin{aligned}
 M_{\text{ev}_w \varphi} &= \sum_{i_1 < i_2 < \dots < i_\ell} \underbrace{x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \dots x_{i_\ell}^{\alpha_\ell}}_{= \prod_{k=1}^{\ell} x_{i_k}^{\alpha_k}} = \sum_{i_1 < i_2 < \dots < i_\ell} \prod_{k=1}^{\ell} \underbrace{x_{i_k}^{\alpha_k}}_{\substack{= x_{i_k}^{\sum_{e \in \varphi^{-1}(k)} w(e)} \\ \text{(since } \alpha_k = \sum_{e \in \varphi^{-1}(k)} w(e))}} \\
 &= \sum_{i_1 < i_2 < \dots < i_\ell} \prod_{k=1}^{\ell} \underbrace{x_{i_k}^{\sum_{e \in \varphi^{-1}(k)} w(e)}}_{= \prod_{e \in \varphi^{-1}(k)} x_{i_k}^{w(e)}} = \sum_{i_1 < i_2 < \dots < i_\ell} \prod_{k=1}^{\ell} \prod_{e \in E; \varphi(e)=k} \underbrace{x_{i_k}^{w(e)}}_{= x_{i_{\varphi(e)}}^{w(e)} \text{ (since } k = \varphi(e))} \\
 &= \sum_{i_1 < i_2 < \dots < i_\ell} \underbrace{\prod_{k=1}^{\ell} \prod_{e \in E; \varphi(e)=k} x_{i_{\varphi(e)}}^{w(e)}}_{= \prod_{e \in E} x_{i_{\varphi(e)}}^{w(e)}} = \sum_{i_1 < i_2 < \dots < i_\ell} \prod_{e \in E} x_{i_{\varphi(e)}}^{w(e)} \\
 &= \sum_{T \subseteq \{1, 2, 3, \dots\}; |T| = \ell} \prod_{e \in E} x_{r_T(\varphi(e))}^{w(e)}
 \end{aligned}$$

16. Hence,

$$\begin{aligned}
 M_{\text{ev}_w \varphi} &= \sum_{T \subseteq \{1, 2, 3, \dots\}; |T| = \ell} \underbrace{\prod_{e \in E} x_{r_T(\varphi(e))}^{w(e)}}_{= \prod_{e \in E} x_{(r_{T \circ \varphi})(e)}^{w(e)} = \mathbf{x}_{r_{T \circ \varphi}, w} \text{ (by the definition of } \mathbf{x}_{r_{T \circ \varphi}, w})} = \sum_{T \subseteq \{1, 2, 3, \dots\}; |T| = \ell} \mathbf{x}_{r_{T \circ \varphi}, w}. \quad (9)
 \end{aligned}$$

On the other hand, recall that  $\varphi$  is an  $\mathbf{E}$ -partition. Hence, every map  $\pi$  satisfying  $\text{pack } \pi = \varphi$  is an  $\mathbf{E}$ -partition (because, as we know,  $\text{pack } \pi$  is an  $\mathbf{E}$ -partition if and only if  $\pi$  is an  $\mathbf{E}$ -partition). Thus, the  $\mathbf{E}$ -partitions  $\pi$  satisfying  $\text{pack } \pi = \varphi$  are precisely the maps  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  satisfying  $\text{pack } \pi = \varphi$ . Hence,

$$\begin{aligned}
 \sum_{\pi \text{ is an } \mathbf{E}\text{-partition; } \text{pack } \pi = \varphi} \mathbf{x}_{\pi, w} &= \sum_{\pi : E \rightarrow \{1, 2, 3, \dots\}; \text{pack } \pi = \varphi} \mathbf{x}_{\pi, w} \\
 &= \sum_{T \subseteq \{1, 2, 3, \dots\}; |T| = \ell} \sum_{\pi : E \rightarrow \{1, 2, 3, \dots\}; \text{pack } \pi = \varphi; \pi(E) = T} \mathbf{x}_{\pi, w}
 \end{aligned}$$

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<sup>16</sup>In the last equality, we have used the fact that the strictly increasing sequences  $(i_1 < i_2 < \dots < i_\ell)$  of positive integers are in bijection with the subsets  $T \subseteq \{1, 2, 3, \dots\}$  such that  $|T| = \ell$ . The bijection sends a sequence  $(i_1 < i_2 < \dots < i_\ell)$  to the set of its entries; its inverse map sends every  $T$  to the sequence  $(r_T(1), r_T(2), \dots, r_T(|T|))$ .



(because if  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  is a map satisfying  $\text{pack } \pi = \varphi$ , then  $|\pi(E)| = \ell$ <sup>17</sup>). But for every  $\ell$ -element subset  $T$  of  $\{1, 2, 3, \dots\}$ , there exists exactly one  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  satisfying  $\text{pack } \pi = \varphi$  and  $\pi(E) = T$ : namely,  $\pi = r_T \circ \varphi$ <sup>18</sup>. Therefore, for every  $\ell$ -element subset  $T$  of  $\{1, 2, 3, \dots\}$ , we have

$$\sum_{\pi: E \rightarrow \{1, 2, 3, \dots\}; \text{pack } \pi = \varphi; \pi(E) = T} \mathbf{x}_{\pi, w} = \mathbf{x}_{r_T \circ \varphi, w}.$$

Hence,

$$\begin{aligned} \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}; \text{pack } \pi = \varphi} \mathbf{x}_{\pi, w} &= \sum_{T \subseteq \{1, 2, 3, \dots\}; |T| = \ell} \underbrace{\sum_{\pi: E \rightarrow \{1, 2, 3, \dots\}; \text{pack } \pi = \varphi; \pi(E) = T} \mathbf{x}_{\pi, w}}_{= \mathbf{x}_{r_T \circ \varphi, w}} \\ &= \sum_{T \subseteq \{1, 2, 3, \dots\}; |T| = \ell} \mathbf{x}_{r_T \circ \varphi, w} = M_{\text{ev } w} \varphi \end{aligned}$$

(by (9)). Thus, (8) is proven, and with it Proposition 5.4.  $\square$

*Proof of Proposition 3.5.* Proposition 3.5 follows immediately from Proposition 5.4 (since  $M_\alpha \in \mathcal{QSym}$  for every composition  $\alpha$ ).  $\square$

We shall now describe the coproduct of  $\Gamma(\mathbf{E}, w)$ , essentially giving the proof that is left to the reader in [MalReu09, Theorem 2.2].

<sup>17</sup>*Proof.* Let  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  be a map satisfying  $\text{pack } \pi = \varphi$ . The definition of  $\text{pack } \pi$  yields  $\text{pack } \pi = r_{\pi(E)}^{-1} \circ \pi$ . Hence,  $|(\text{pack } \pi)(E)| = |(r_{\pi(E)}^{-1} \circ \pi)(E)| = |r_{\pi(E)}^{-1}(\pi(E))| = |\pi(E)|$  (since  $r_{\pi(E)}^{-1}$  is a bijection). Since  $\text{pack } \pi = \varphi$ , this rewrites as  $|\varphi(E)| = |\pi(E)|$ . Hence,  $|\pi(E)| = |\varphi(E)| = \ell$ , qed.

<sup>18</sup>*Proof.* Let  $T$  be an  $\ell$ -element subset of  $\{1, 2, 3, \dots\}$ . We need to show that there exists exactly one  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  satisfying  $\text{pack } \pi = \varphi$  and  $\pi(E) = T$ : namely,  $\pi = r_T \circ \varphi$ . In other words, we need to prove the following two claims:

*Claim 1:* The map  $r_T \circ \varphi$  is a map  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  satisfying  $\text{pack } \pi = \varphi$  and  $\pi(E) = T$ .

*Claim 2:* If  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  is a map satisfying  $\text{pack } \pi = \varphi$  and  $\pi(E) = T$ , then  $\pi = r_T \circ \varphi$ .

*Proof of Claim 1.* We have  $|T| = \ell$  (since the set  $T$  is  $\ell$ -element), thus  $\ell = |T|$ . We have

$$(r_T \circ \varphi)(E) = r_T \left( \underbrace{\varphi(E)}_{=\{1, 2, \dots, \ell\}} \right) = r_T \left( \left\{ 1, 2, \dots, \underbrace{\ell}_{=|T|} \right\} \right) = r_T(\{1, 2, \dots, |T|\}) = T \text{ (by the definition of } r_T \text{).}$$

Now, the definition of  $\text{pack}(r_T \circ \varphi)$  shows that

$$\begin{aligned} \text{pack}(r_T \circ \varphi) &= r_{(r_T \circ \varphi)(E)}^{-1} \circ (r_T \circ \varphi) = r_T^{-1} \circ (r_T \circ \varphi) \quad (\text{since } (r_T \circ \varphi)(E) = T) \\ &= \varphi. \end{aligned}$$

Thus, the map  $r_T \circ \varphi : E \rightarrow \{1, 2, 3, \dots\}$  satisfies  $\text{pack}(r_T \circ \varphi) = \varphi$  and  $(r_T \circ \varphi)(E) = T$ . In other words,  $r_T \circ \varphi$  is a map  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  satisfying  $\text{pack } \pi = \varphi$  and  $\pi(E) = T$ . This proves Claim 1.

*Proof of Claim 2.* Let  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  be a map satisfying  $\text{pack } \pi = \varphi$  and  $\pi(E) = T$ . The definition of  $\text{pack } \pi$  shows that  $\text{pack } \pi = r_{\pi(E)}^{-1} \circ \pi = r_T^{-1} \circ \pi$  (since  $\pi(E) = T$ ). Hence,  $r_T^{-1} \circ \pi = \text{pack } \pi = \varphi$ , so that  $\pi = r_T \circ \varphi$ . This proves Claim 2.

Now, both Claims 1 and 2 are proven; hence, our proof is complete.

**Definition 5.5.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a double poset.

- (a) Then,  $\text{Adm } \mathbf{E}$  will mean the set of all pairs  $(P, Q)$ , where  $P$  and  $Q$  are subsets of  $E$  satisfying  $P \cap Q = \emptyset$  and  $P \cup Q = E$  and having the property that no  $p \in P$  and  $q \in Q$  satisfy  $q <_1 p$ . These pairs  $(P, Q)$  are called the *admissible partitions* of  $\mathbf{E}$ . (In the terminology of [MalReu09], they are the *decompositions* of  $(E, <_1)$ .)
- (b) For any subset  $T$  of  $E$ , we let  $\mathbf{E} \upharpoonright_T$  denote the double poset  $(T, <_1, <_2)$ , where  $<_1$  and  $<_2$  (by abuse of notation) denote the restrictions of the relations  $<_1$  and  $<_2$  to  $T$ .

**Proposition 5.6.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a double poset. Let  $w : E \rightarrow \{1, 2, 3, \dots\}$  be a map. Then,

$$\Delta(\Gamma(\mathbf{E}, w)) = \sum_{(P, Q) \in \text{Adm } \mathbf{E}} \Gamma(\mathbf{E} \upharpoonright_P, w \upharpoonright_P) \otimes \Gamma(\mathbf{E} \upharpoonright_Q, w \upharpoonright_Q). \quad (10)$$

A particular case of Proposition 5.6 (namely, the case when  $w(e) = 1$  for each  $e \in E$ ) appears in [Malve93, Théorème 4.16].

The proof of Proposition 5.6 relies on a simple bijection that an experienced combinatorialist will have no trouble finding (and proving even less); let us just give a brief outline of the argument<sup>19</sup>:

*Proof of Proposition 5.6.* Whenever  $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_\ell)$  is a composition and  $k \in \{0, 1, \dots, \ell\}$ , we introduce the notation  $\alpha[:k]$  for the composition  $(\alpha_1, \alpha_2, \dots, \alpha_k)$ , and the notation  $\alpha[k:]$  for the composition  $(\alpha_{k+1}, \alpha_{k+2}, \dots, \alpha_\ell)$ . Now, the formula (1) can be rewritten as follows:

$$\Delta(M_\alpha) = \sum_{k=0}^{\ell} M_{\alpha[:k]} \otimes M_{\alpha[k:]} \quad (11)$$

for every  $\ell \in \mathbb{N}$  and every composition  $\alpha$  with  $\ell$  entries.

Now, applying  $\Delta$  to the equality (7) yields

$$\begin{aligned} \Delta(\Gamma(\mathbf{E}, w)) &= \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} \underbrace{\Delta(M_{\text{ev}_w \varphi})}_{\substack{= \sum_{k=0}^{|\varphi(E)|} M_{(\text{ev}_w \varphi)[:k]} \otimes M_{(\text{ev}_w \varphi)[k:]} \\ \text{(by (11))}}} \\ &= \sum_{\varphi \text{ is a packed } \mathbf{E}\text{-partition}} \sum_{k=0}^{|\varphi(E)|} M_{(\text{ev}_w \varphi)[:k]} \otimes M_{(\text{ev}_w \varphi)[k:]}. \end{aligned} \quad (12)$$

<sup>19</sup>See the detailed version of this note for an (almost) completely written-out proof; I am afraid that the additional level of detail is of no help to the understanding.

On the other hand, rewriting each of the tensorands on the right hand side of (10) using (7), we obtain

$$\begin{aligned}
 & \sum_{(P,Q) \in \text{Adm } \mathbf{E}} \Gamma(\mathbf{E} \mid_P, w \mid_P) \otimes \Gamma(\mathbf{E} \mid_Q, w \mid_Q) \\
 &= \sum_{(P,Q) \in \text{Adm } \mathbf{E}} \left( \sum_{\varphi \text{ is a packed } \mathbf{E} \mid_P\text{-partition}} M_{\text{ev}_{w \mid_P} \varphi} \right) \otimes \left( \sum_{\varphi \text{ is a packed } \mathbf{E} \mid_Q\text{-partition}} M_{\text{ev}_{w \mid_Q} \varphi} \right) \\
 &= \sum_{(P,Q) \in \text{Adm } \mathbf{E}} \left( \sum_{\sigma \text{ is a packed } \mathbf{E} \mid_P\text{-partition}} M_{\text{ev}_{w \mid_P} \sigma} \right) \otimes \left( \sum_{\tau \text{ is a packed } \mathbf{E} \mid_Q\text{-partition}} M_{\text{ev}_{w \mid_Q} \tau} \right) \\
 &= \sum_{(P,Q) \in \text{Adm } \mathbf{E}} \sum_{\sigma \text{ is a packed } \mathbf{E} \mid_P\text{-partition}} \sum_{\tau \text{ is a packed } \mathbf{E} \mid_Q\text{-partition}} M_{\text{ev}_{w \mid_P} \sigma} \otimes M_{\text{ev}_{w \mid_Q} \tau}.
 \end{aligned}$$

We need to prove that the right hand sides of this equality and of (12) are equal (because then, it will follow that so are the left hand sides, and thus Proposition 5.6 will be proven). For this, it is clearly enough to exhibit a bijection between

- the pairs  $(\varphi, k)$  consisting of a packed  $\mathbf{E}$ -partition  $\varphi$  and a  $k \in \{0, 1, \dots, |\varphi(E)|\}$

and

- the triples  $((P, Q), \sigma, \tau)$  consisting of a  $(P, Q) \in \text{Adm } \mathbf{E}$ , a packed  $\mathbf{E} \mid_P$ -partition  $\sigma$  and a packed  $\mathbf{E} \mid_Q$ -partition  $\tau$

which bijection has the property that whenever it maps  $(\varphi, k)$  to  $((P, Q), \sigma, \tau)$ , we have the equalities  $(\text{ev}_w \varphi) [ : k ] = \text{ev}_{w \mid_P} \sigma$  and  $(\text{ev}_w \varphi) [ k : ] = \text{ev}_{w \mid_Q} \tau$ . Such a bijection is easy to construct: Given  $(\varphi, k)$ , it sets  $P = \varphi^{-1}(\{1, 2, \dots, k\})$ ,  $Q = \varphi^{-1}(\{k + 1, k + 2, \dots, |\varphi(E)|\})$ ,  $\sigma = \varphi \mid_P$  and  $\tau = \text{pack}(\varphi \mid_Q)$ <sup>20</sup>. Conversely, given  $((P, Q), \sigma, \tau)$ , the inverse bijection sets  $k = |\sigma(P)|$  and constructs  $\varphi$  as the

map  $E \rightarrow \{1, 2, 3, \dots\}$  which sends every  $e \in E$  to  $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}$ . Proving

that this alleged bijection and its alleged inverse bijection are well-defined and actually mutually inverse is straightforward and left to the reader<sup>21</sup>.  $\square$

<sup>20</sup>We notice that these  $P, Q, \sigma$  and  $\tau$  satisfy  $\sigma(e) = \varphi(e)$  for every  $e \in P$ , and  $\tau(e) = \varphi(e) - k$  for every  $e \in Q$ .

<sup>21</sup>The only part of the argument that is a bit trickier is proving the well-definedness of the inverse bijection: We need to show that if  $((P, Q), \sigma, \tau)$  is a triple consisting of a  $(P, Q) \in \text{Adm } \mathbf{E}$ , a packed  $\mathbf{E} \mid_P$ -partition  $\sigma$  and a packed  $\mathbf{E} \mid_Q$ -partition  $\tau$ , and if we set  $k = |\sigma(P)|$ , then the map

$\varphi : E \rightarrow \{1, 2, 3, \dots\}$  which sends every  $e \in E$  to  $\begin{cases} \sigma(e), & \text{if } e \in P; \\ \tau(e) + k, & \text{if } e \in Q \end{cases}$  is actually a packed

$\mathbf{E}$ -partition.

Indeed, it is clear that this map  $\varphi$  is packed. It remains to show that it is an  $\mathbf{E}$ -partition. To do so, we must prove the following two claims:

*Claim 1:* Every  $e \in E$  and  $f \in E$  satisfying  $e <_1 f$  satisfy  $\varphi(e) \leq \varphi(f)$ .

We note in passing that there is also a rule for multiplying quasisymmetric functions of the form  $\Gamma(\mathbf{E}, w)$ . Namely, if  $\mathbf{E}$  and  $\mathbf{F}$  are two double posets and  $u$  and  $v$  are corresponding maps, then  $\Gamma(\mathbf{E}, u) \Gamma(\mathbf{F}, v) = \Gamma(\mathbf{EF}, w)$  for a map  $w$  which is defined to be  $u$  on the subset  $\mathbf{E}$  of  $\mathbf{EF}$ , and  $v$  on the subset  $\mathbf{F}$  of  $\mathbf{EF}$ . Here,  $\mathbf{EF}$  is a double poset defined as in [MalReu09, §2.1]. Combined with Proposition 3.5, this fact gives a combinatorial proof for the fact that QSym is a  $\mathbf{k}$ -algebra, as well as for some standard formulas for multiplications of quasisymmetric functions; similarly, Proposition 5.6 can be used to derive the well-known formulas for  $\Delta M_\alpha, \Delta L_\alpha, \Delta s_{\lambda/\mu}$  etc. (although, of course, we have already used the formula for  $\Delta M_\alpha$  in our proof of Proposition 5.6).

## 6. Proof of Theorem 4.2

Before we come to the proof of Theorem 4.2, let us state five lemmas:

**Lemma 6.1.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a double poset. Let  $P$  and  $Q$  be subsets of  $E$  such that  $P \cap Q = \emptyset$  and  $P \cup Q = E$ . Assume that there exist no  $p \in P$  and  $q \in Q$  such that  $q$  is  $<_1$ -covered by  $p$ . Then,  $(P, Q) \in \text{Adm } \mathbf{E}$ .

*Proof of Lemma 6.1.* For any  $a \in E$  and  $b \in E$ , we let  $[a, b]$  denote the subset  $\{e \in E \mid a <_1 e <_1 b\}$  of  $E$ . It is easy to see that if  $a, b$  and  $c$  are three elements of  $E$  satisfying  $a <_1 c <_1 b$ , then both  $[a, c]$  and  $[c, b]$  are proper subsets of  $[a, b]$ , and therefore

$$\text{both numbers } |[a, c]| \text{ and } |[c, b]| \text{ are smaller than } |[a, b]|. \tag{13}$$

A pair  $(p, q) \in P \times Q$  is said to be a *malposition* if it satisfies  $q <_1 p$ . Now, let us assume (for the sake of contradiction) that there exists a malposition. Fix a malposition  $(u, v)$  for which the value  $|[v, u]|$  is minimum. Thus,  $(u, v) \in P \times Q$  and  $v <_1 u$ . From  $(u, v) \in P \times Q$ , we obtain  $u \in P$  and  $v \in Q$ . Hence,  $v$  is not  $<_1$ -covered by  $u$  (since there exist no  $p \in P$  and  $q \in Q$  such that  $q$  is  $<_1$ -covered

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*Claim 2:* Every  $e \in E$  and  $f \in E$  satisfying  $e <_1 f$  and  $f <_2 e$  satisfy  $\varphi(e) < \varphi(f)$ .

We shall only prove Claim 1 (as the proof of Claim 2 is similar). So let  $e \in E$  and  $f \in E$  be such that  $e <_1 f$ . We need to show that  $\varphi(e) \leq \varphi(f)$ . We are in one of the following four cases:

*Case 1:* We have  $e \in P$  and  $f \in P$ .

*Case 2:* We have  $e \in P$  and  $f \in Q$ .

*Case 3:* We have  $e \in Q$  and  $f \in P$ .

*Case 4:* We have  $e \in Q$  and  $f \in Q$ .

In Case 1, our claim  $\varphi(e) \leq \varphi(f)$  follows from the assumption that  $\sigma$  is an  $\mathbf{E} \mid_P$ -partition (because in Case 1, we have  $\varphi(e) = \sigma(e)$  and  $\varphi(f) = \sigma(f)$ ). In Case 4, it follows from the assumption that  $\tau$  is an  $\mathbf{E} \mid_Q$ -partition (since in Case 4, we have  $\varphi(e) = \tau(e) + k$  and  $\varphi(f) = \tau(f) + k$ ). In Case 2, it clearly holds (indeed, if  $e \in P$ , then the definition of  $\varphi$  yields  $\varphi(e) = \sigma(e) \leq k$ , and if  $f \in Q$ , then the definition of  $\varphi$  yields  $\varphi(f) = \tau(f) + k > k$ ; therefore, in Case 2, we have  $\varphi(e) \leq k < \varphi(f)$ ). Finally, Case 3 is impossible (because having  $e \in Q$  and  $f \in P$  and  $e <_1 f$  would contradict  $(P, Q) \in \text{Adm } \mathbf{E}$ ). Thus, we have proven the claim in each of the four cases, and consequently Claim 1 is proven. As we have said above, Claim 2 is proven similarly.

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by  $p$ ). Hence, there exists a  $w \in E$  such that  $v <_1 w <_1 u$  (since  $v <_1 u$ ). Consider this  $w$ . Applying (13) to  $a = v, c = w$  and  $b = u$ , we see that both numbers  $|[v, w]|$  and  $|[w, u]|$  are smaller than  $|[v, u]|$ , and therefore neither  $(w, v)$  nor  $(u, w)$  is a malposition (since we picked  $(u, v)$  to be a malposition with minimum  $|[v, u]|$ ). But  $w \in E = P \cup Q$ , so that either  $w \in P$  or  $w \in Q$ . If  $w \in P$ , then  $(w, v)$  is a malposition; if  $w \in Q$ , then  $(u, w)$  is a malposition. In either case, we obtain a contradiction to the fact that neither  $(w, v)$  nor  $(u, w)$  is a malposition. This contradiction shows that our assumption was wrong. Hence, there exists no malposition. In other words, there exists no  $(p, q) \in P \times Q$  satisfying  $q <_1 p$  (since this is what “malposition” means). In other words, no  $p \in P$  and  $q \in Q$  satisfy  $q <_1 p$ . Consequently,  $(P, Q) \in \text{Adm } \mathbf{E}$ . This proves Lemma 6.1.  $\square$

**Lemma 6.2.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a tertispecial double poset. Let  $(P, Q) \in \text{Adm } \mathbf{E}$ . Then,  $\mathbf{E} \upharpoonright_P$  is a tertispecial double poset.

*Proof of Lemma 6.2.* Recall that we are using the symbol  $<_1$  to denote two different relations: a strict partial order on  $E$ , and its restriction to  $P$ . This abuse of notation is usually harmless, but in the current proof it is dangerous, because it causes the statement “ $a$  is  $<_1$ -covered by  $b$ ” (for two elements  $a$  and  $b$  of  $P$ ) to carry two meanings (depending on whether the symbol  $<_1$  is interpreted as the strict partial order on  $E$ , or as its restriction to  $P$ ). (These two meanings are actually equivalent, but their equivalence is not immediately obvious.)

Thus, for the duration of this proof, we shall revert to a less ambiguous notation. Namely, the notation  $<_1$  shall only be used for the strict partial order on  $E$  which constitutes part of the double poset  $\mathbf{E}$ . The restriction of this partial order  $<_1$  to the subset  $P$  will be denoted by  $<_{1,P}$  (not by  $<_1$ ). Similarly, the restriction of the partial order  $<_2$  to the subset  $P$  will be denoted by  $<_{2,P}$  (not by  $<_2$ ). Thus, the double poset  $\mathbf{E} \upharpoonright_P$  is defined as  $\mathbf{E} \upharpoonright_P = (P, <_{1,P}, <_{2,P})$ .

We need to show that the double poset  $\mathbf{E} \upharpoonright_P = (P, <_{1,P}, <_{2,P})$  is tertispecial. In other words, we need to show that if  $a$  and  $b$  are two elements of  $P$  such that  $a$  is  $<_{1,P}$ -covered by  $b$ , then  $a$  and  $b$  are  $<_{2,P}$ -comparable.

Let  $a$  and  $b$  be two elements of  $P$  such that  $a$  is  $<_{1,P}$ -covered by  $b$ . Thus,  $a <_{1,P} b$ , and

$$\text{there exists no } c \in P \text{ satisfying } a <_{1,P} c <_{1,P} b. \tag{14}$$

We have  $a <_{1,P} b$ . In other words,  $a <_1 b$  (since  $<_{1,P}$  is the restriction of the relation  $<_1$  to  $P$ ).

Now, if  $c \in E$  is such that  $a <_1 c <_1 b$ , then  $c$  must belong to  $P$ <sup>22</sup>, and therefore

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<sup>22</sup>*Proof.* Assume the contrary. Thus,  $c \notin P$ . But  $(P, Q) \in \text{Adm } \mathbf{E}$ . Thus,  $P \cap Q = \emptyset, P \cup Q = E$ , and

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p. \tag{15}$$

From  $c \in E$  and  $c \notin P$ , we obtain  $c \in E \setminus P \subseteq Q$  (since  $P \cup Q = E$ ). Applying (15) to  $p = b$  and  $q = c$ , we thus conclude that we cannot have  $c <_1 b$ . This contradicts  $c <_1 b$ . This contradiction shows that our assumption was false, *qed*.

satisfy  $a <_{1,P} c <_{1,P} b$ <sup>23</sup>, which entails a contradiction to (14). Thus, there is no  $c \in E$  satisfying  $a <_1 c <_1 b$ . Therefore (and because we have  $a <_1 b$ ), we see that  $a$  is  $<_1$ -covered by  $b$ . Since  $\mathbf{E}$  is tertispecial, this yields that  $a$  and  $b$  are  $<_2$ -comparable. In other words, either  $a <_2 b$  or  $a = b$  or  $b <_2 a$ . Since  $a$  and  $b$  both belong to  $P$ , we can rewrite this by replacing the relation  $<_2$  by its restriction  $<_{2,P}$ . We thus conclude that either  $a <_{2,P} b$  or  $a = b$  or  $b <_{2,P} a$ . In other words,  $a$  and  $b$  are  $<_{2,P}$ -comparable.

Now, forget that we fixed  $a$  and  $b$ . Thus, we have shown that if  $a$  and  $b$  are two elements of  $P$  such that  $a$  is  $<_{1,P}$ -covered by  $b$ , then  $a$  and  $b$  are  $<_{2,P}$ -comparable. This completes the proof of Lemma 6.2.

(We could similarly show that  $\mathbf{E} \upharpoonright_Q$  is a tertispecial double poset; but we will not use this.) □

**Lemma 6.3.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a double poset. Let  $w : E \rightarrow \{1, 2, 3, \dots\}$  be a map.

- (a) If  $E = \emptyset$ , then  $\Gamma(\mathbf{E}, w) = 1$ .
- (b) If  $E \neq \emptyset$ , then  $\varepsilon(\Gamma(\mathbf{E}, w)) = 0$ .

*Proof of Lemma 6.3.* (a) Part (a) is obvious (since there is only one  $\mathbf{E}$ -partition  $\pi$  when  $E = \emptyset$ , and since this  $\mathbf{E}$ -partition  $\pi$  satisfies  $x_{\pi,w} = 1$ ).

(b) Observe that  $\Gamma(\mathbf{E}, w)$  is a homogeneous power series of degree  $\sum_{e \in E} w(e)$ .

When  $E \neq \emptyset$ , this degree is  $> 0$  (since it is then a nonempty sum of positive integers), and thus the power series  $\Gamma(\mathbf{E}, w)$  is annihilated by  $\varepsilon$  (since  $\varepsilon$  annihilates any homogeneous power series in QSym whose degree is  $> 0$ ). □

**Lemma 6.4.** Let  $(E, <_1, <_2)$  be a double poset. Let  $>_1$  be the opposite relation of  $<_1$ . Let  $P$  and  $Q$  be two subsets of  $E$  satisfying  $P \cup Q = E$ . Let  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  be a map such that  $\pi \upharpoonright_P$  is a  $(P, >_1, <_2)$ -partition. Let  $f \in P$ . Assume that

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p. \tag{16}$$

Also, assume that

$$\pi(f) \leq \pi(h) \quad \text{for every } h \in E. \tag{17}$$

Furthermore, assume that

$$\pi(f) < \pi(h) \quad \text{for every } h \in E \text{ satisfying } h <_2 f. \tag{18}$$

- (a) If  $p \in P \setminus \{f\}$  and  $q \in Q \cup \{f\}$  are such that  $q <_1 p$ , then we have neither  $q <_2 p$  nor  $p <_2 q$ .

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<sup>23</sup>*Proof.* Let  $c \in E$  be such that  $a <_1 c <_1 b$ . Then,  $c$  must belong to  $P$  (as we have just proven). Now,  $a <_1 c$ . In light of  $a \in P$  and  $c \in P$ , this rewrites as  $a <_{1,P} c$  (since  $<_{1,P}$  is the restriction of the relation  $<_1$  to  $P$ ). Similarly,  $c <_1 b$  rewrites as  $c <_{1,P} b$ . Thus,  $a <_{1,P} c <_{1,P} b$ , qed.

(b) If  $\pi|_Q$  is a  $(Q, <_1, <_2)$ -partition, then  $\pi|_{Q \cup \{f\}}$  is a  $(Q \cup \{f\}, <_1, <_2)$ -partition.

*Proof of Lemma 6.4.* From  $P \cup Q = E$ , we obtain  $\underbrace{E}_{=P \cup Q} \setminus P = (P \cup Q) \setminus P \subseteq Q$ .

(a) Let  $p \in P \setminus \{f\}$  and  $q \in Q \cup \{f\}$  be such that  $q <_1 p$ . We must show that we have neither  $q <_2 p$  nor  $p <_2 q$ .

Indeed, assume the contrary. Thus, we have either  $q <_2 p$  or  $p <_2 q$ .

We have  $q <_1 p$  and  $p \in P \setminus \{f\} \subseteq P$ . Hence, if we had  $q \in Q$ , then we would obtain a contradiction to (16). Hence, we cannot have  $q \in Q$ . Therefore,  $q = f$  (since  $q \in Q \cup \{f\}$  but not  $q \in Q$ ). Hence,  $f = q <_1 p$ , so that  $p >_1 f$ . Therefore,  $\pi(p) \leq \pi(f)$  (since  $\pi|_P$  is a  $(P, >_1, <_2)$ -partition, and since both  $f$  and  $p$  belong to  $P$ ).

Now, recall that we have either  $q <_2 p$  or  $p <_2 q$ . Since  $q = f$ , we can rewrite this as follows: We have either  $f <_2 p$  or  $p <_2 f$ . But  $p <_2 f$  cannot hold (because if we had  $p <_2 f$ , then (18) (applied to  $h = p$ ) would lead to  $\pi(f) < \pi(p)$ , which would contradict  $\pi(p) \leq \pi(f)$ ). Thus, we must have  $f <_2 p$ .

But  $\pi|_P$  is a  $(P, >_1, <_2)$ -partition. Hence,  $\pi(p) < \pi(f)$  (since  $p >_1 f$  and  $f <_2 p$ , and since  $p$  and  $f$  both lie in  $P$ ). But (17) (applied to  $h = p$ ) shows that  $\pi(f) \leq \pi(p)$ . Hence,  $\pi(p) < \pi(f) \leq \pi(p)$ , a contradiction. Thus, our assumption was wrong. This completes the proof of Lemma 6.4 (a).

(b) Assume that  $\pi|_Q$  is a  $(Q, <_1, <_2)$ -partition. We need to show that  $\pi|_{Q \cup \{f\}}$  is a  $(Q \cup \{f\}, <_1, <_2)$ -partition. In order to prove this, we need to verify the following two claims:

*Claim 1:* Every  $a \in Q \cup \{f\}$  and  $b \in Q \cup \{f\}$  satisfying  $a <_1 b$  satisfy  $\pi(a) \leq \pi(b)$ .

*Claim 2:* Every  $a \in Q \cup \{f\}$  and  $b \in Q \cup \{f\}$  satisfying  $a <_1 b$  and  $b <_2 a$  satisfy  $\pi(a) < \pi(b)$ .

*Proof of Claim 1:* Let  $a \in Q \cup \{f\}$  and  $b \in Q \cup \{f\}$  be such that  $a <_1 b$ . We need to prove that  $\pi(a) \leq \pi(b)$ . If  $a = f$ , then this follows immediately from (17) (applied to  $h = b$ ). Hence, we WLOG assume that  $a \neq f$ . Thus,  $a \in Q$  (since  $a \in Q \cup \{f\}$ ). Now, if  $b \in P$ , then  $a <_1 b$  contradicts (16) (applied to  $p = b$  and  $q = a$ ). Hence, we cannot have  $b \in P$ . Therefore,  $b \in E \setminus P \subseteq Q$ . Thus,  $\pi(a) \leq \pi(b)$  follows immediately from the fact that  $\pi|_Q$  is a  $(Q, <_1, <_2)$ -partition (since  $a \in Q$  and  $b \in Q$  and  $a <_1 b$ ). This proves Claim 1.

*Proof of Claim 2:* Let  $a \in Q \cup \{f\}$  and  $b \in Q \cup \{f\}$  be such that  $a <_1 b$  and  $b <_2 a$ . We need to prove that  $\pi(a) < \pi(b)$ . If  $a = f$ , then this follows immediately from (18) (applied to  $h = b$ ) (because if  $a = f$ , then  $b <_2 a = f$ ). Hence, we WLOG assume that  $a \neq f$ . Thus,  $a \in Q$  (since  $a \in Q \cup \{f\}$ ). Now, if  $b \in P$ , then  $a <_1 b$  contradicts (16) (applied to  $p = b$  and  $q = a$ ). Hence, we cannot have  $b \in P$ . Therefore,  $b \in E \setminus P \subseteq Q$ . Thus,  $\pi(a) < \pi(b)$  follows immediately from the fact that  $\pi|_Q$  is a  $(Q, <_1, <_2)$ -partition (since  $a \in Q$  and  $b \in Q$  and  $a <_1 b$  and  $b <_2 a$ ). This proves Claim 2.

Now, both Claim 1 and Claim 2 are proven. As already said, this completes the proof of Lemma 6.4 (b).  $\square$

**Lemma 6.5.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a tertispecial double poset satisfying  $|E| > 0$ . Let  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  be a map. Let  $>_1$  denote the opposite relation of  $<_1$ . Then,

$$\sum_{\substack{(P,Q) \in \text{Adm } \mathbf{E}; \\ \pi|_P \text{ is a } (P, >_1, <_2)\text{-partition}; \\ \pi|_Q \text{ is a } (Q, <_1, <_2)\text{-partition}}} (-1)^{|P|} = 0. \tag{19}$$

*Proof of Lemma 6.5.* Our goal is to prove (19). To do so, we denote by  $Z$  the set of all  $(P, Q) \in \text{Adm } \mathbf{E}$  such that  $\pi|_P$  is a  $(P, >_1, <_2)$ -partition and  $\pi|_Q$  is a  $(Q, <_1, <_2)$ -partition. We are going to define an involution  $T : Z \rightarrow Z$  of the set  $Z$  having the following property:

*Property P:* Let  $(P, Q) \in Z$ . If we write  $T((P, Q))$  in the form  $(P', Q')$ , then  $(-1)^{|P'|} = -(-1)^{|P|}$ .

Once such an involution  $T$  is found, it will be clear that it matches the addends on the left hand side of (19) into pairs of mutually cancelling addends<sup>24</sup>, and so (19) will follow and we will be done. It thus remains to find  $T$ .

The definition of the map  $T : Z \rightarrow Z$  is simple (although it will take us a while to prove that it is well-defined): Let  $F$  be the subset of  $E$  consisting of those  $e \in E$  for which the value  $\pi(e)$  is minimum. Then,  $F$  is a nonempty subposet<sup>25</sup> of the poset  $(E, <_2)$ , and hence has a minimal element<sup>26</sup>  $f$  (that is, an element  $f$  such that no  $g \in F$  satisfies  $g <_2 f$ ). Fix such an  $f$ . Now, the map  $T$  sends a  $(P, Q) \in Z$  to

$$\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P. \end{cases}$$

In order to prove that the map  $T$  is well-defined, we need to prove that its output values all belong to  $Z$ . In other words, we need to prove that

$$\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} \in Z \tag{20}$$

for every  $(P, Q) \in Z$ .

*Proof of (20):* Fix  $(P, Q) \in Z$ . Thus,  $(P, Q)$  is an element of  $\text{Adm } \mathbf{E}$  with the property that  $\pi|_P$  is a  $(P, >_1, <_2)$ -partition and  $\pi|_Q$  is a  $(Q, <_1, <_2)$ -partition (by the definition of  $Z$ ).

<sup>24</sup>In fact, Property P entails that  $T$  has no fixed points. Therefore, to each addend on the left hand side of (19) corresponds an addend with opposite sign, which cancels it: Namely, for each  $(A, B) \in Z$ , the addend for  $(P, Q) = (A, B)$  is cancelled by the addend for  $(P, Q) = T((A, B))$ .

<sup>25</sup>The nonemptiness of  $F$  follows from the nonemptiness of  $E$  (which, in turn, follows from  $|E| > 0$ ).

<sup>26</sup>A *minimal element* of a poset  $(P, <)$  is an element  $p \in P$  such that no  $g \in P$  satisfies  $g < p$ . It is well-known that every nonempty finite poset has at least one minimal element. We are using this fact here.



From  $(P, Q) \in \text{Adm } \mathbf{E}$ , we see that  $P \cap Q = \emptyset$  and  $P \cup Q = E$ , and furthermore that

$$\text{no } p \in P \text{ and } q \in Q \text{ satisfy } q <_1 p. \quad (21)$$

We know that  $f$  belongs to the set  $F$ , which is the subset of  $E$  consisting of those  $e \in E$  for which the value  $\pi(e)$  is minimum. Thus,

$$\pi(f) \leq \pi(h) \quad \text{for every } h \in E. \quad (22)$$

Moreover,

$$\pi(f) < \pi(h) \quad \text{for every } h \in E \text{ satisfying } h <_2 f \quad (23)$$

<sup>27</sup>.

We need to prove (20). We are in one of the following two cases:

Case 1: We have  $f \in P$ .

Case 2: We have  $f \notin P$ .

Let us first consider Case 1. In this case, we have  $f \in P$ .

Recall that  $P \cap Q = \emptyset$  and  $P \cup Q = E$ . From this, we easily obtain  $(P \setminus \{f\}) \cap (Q \cup \{f\}) = \emptyset$  and  $(P \setminus \{f\}) \cup (Q \cup \{f\}) = E$ .

Furthermore, there exist no  $p \in P \setminus \{f\}$  and  $q \in Q \cup \{f\}$  such that  $q$  is  $<_1$ -covered by  $p$  <sup>28</sup>. Hence, Lemma 6.1 (applied to  $P \setminus \{f\}$  and  $Q \cup \{f\}$  instead of  $P$  and  $Q$ ) shows that  $(P \setminus \{f\}, Q \cup \{f\}) \in \text{Adm } \mathbf{E}$ .

Furthermore,  $\pi|_P$  is a  $(P, >_1, <_2)$ -partition. Hence,  $\pi|_{P \setminus \{f\}}$  is a  $(P \setminus \{f\}, >_1, <_2)$ -partition (since  $P \setminus \{f\} \subseteq P$ ).

Furthermore,  $\pi|_{Q \cup \{f\}}$  is a  $(Q \cup \{f\}, <_1, <_2)$ -partition<sup>29</sup>.

Altogether, we now know that  $(P \setminus \{f\}, Q \cup \{f\}) \in \text{Adm } \mathbf{E}$ , that  $\pi|_{P \setminus \{f\}}$  is a  $(P \setminus \{f\}, >_1, <_2)$ -partition, and that  $\pi|_{Q \cup \{f\}}$  is a  $(Q \cup \{f\}, <_1, <_2)$ -partition. In other words,  $(P \setminus \{f\}, Q \cup \{f\}) \in Z$  (by the definition of  $Z$ ). Thus,

$$\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} = (P \setminus \{f\}, Q \cup \{f\}) \quad (\text{since } f \in P)$$

$$\in Z.$$

<sup>27</sup>*Proof of (23):* Let  $h \in E$  be such that  $h <_2 f$ . We must prove (23). Indeed, assume the contrary. Thus,  $\pi(f) \geq \pi(h)$ . But every  $g \in E$  satisfies  $\pi(f) \leq \pi(g)$  (by (22), applied to  $g$  instead of  $h$ ). Hence, every  $g \in E$  satisfies  $\pi(g) \geq \pi(f) \geq \pi(h)$ . In other words,  $h$  is one of those  $e \in E$  for which the value  $\pi(e)$  is minimum.

But recall that  $F$  is the subset of  $E$  consisting of those  $e \in E$  for which the value  $\pi(e)$  is minimum. Since  $h$  is one of these  $e \in E$ , we thus conclude that  $h \in F$ . But  $f$  is a minimal element of the subposet  $F$  of  $(E, <_2)$ . In other words, no  $g \in F$  satisfies  $g <_2 f$ . This contradicts the fact that  $h \in F$  satisfies  $h <_2 f$ . This contradiction proves that our assumption was wrong, qed.

<sup>28</sup>*Proof.* Assume the contrary. Thus, there exist  $p \in P \setminus \{f\}$  and  $q \in Q \cup \{f\}$  such that  $q$  is  $<_1$ -covered by  $p$ . Consider such  $p$  and  $q$ .

We know that  $q$  is  $<_1$ -covered by  $p$ , and thus we have  $q <_1 p$ . Hence, Lemma 6.4 (a) shows that we have neither  $q <_2 p$  nor  $p <_2 q$ . On the other hand,  $q$  is  $<_1$ -covered by  $p$ . Hence,  $q$  and  $p$  are  $<_2$ -comparable (since  $\mathbf{E}$  is tertispecial). In other words, we have either  $q <_2 p$  or  $q = p$  or  $p <_2 q$ . Hence, we must have  $q = p$  (since we have neither  $q <_2 p$  nor  $p <_2 q$ ). But this contradicts  $q <_1 p$ . This contradiction shows that our assumption was wrong, qed.

<sup>29</sup>This follows from Lemma 6.4 (b) (since  $\pi|_Q$  is a  $(Q, <_1, <_2)$ -partition).

Hence, (20) is proven in Case 1.

Let us next consider Case 2. In this case, we have  $f \notin P$ . Hence,  $f \in E \setminus P = Q$  (since  $P \cap Q = \emptyset$  and  $P \cup Q = E$ ).

Recall that  $P \cap Q = \emptyset$  and  $P \cup Q = E$ . From this, we easily obtain  $(P \cup \{f\}) \cap (Q \setminus \{f\}) = \emptyset$  and  $(P \cup \{f\}) \cup (Q \setminus \{f\}) = E$ .

We have  $f \in Q$  and  $Q \cup P = P \cup Q = E$ . Furthermore,  $>_1$  is the opposite relation of  $<_1$ , and thus is a strict partial order (since  $<_1$  is a strict partial order). Hence,  $(E, >_1, <_2)$  is a double poset. Furthermore, the relation  $<_1$  is the opposite relation of  $>_1$  (since  $>_1$  is the opposite relation of  $<_1$ ). The map  $\pi|_Q$  is a  $(Q, <_1, <_2)$ -partition. Moreover,

$$\text{no } p \in Q \text{ and } q \in P \text{ satisfy } q >_1 p \quad (24)$$

<sup>30</sup>. Hence, we can apply Lemma 6.4 to  $(E, >_1, <_2)$ ,  $<_1$ ,  $Q$  and  $P$  instead of  $(E, <_1, <_2)$ ,  $>_1$ ,  $P$  and  $Q$ .

There exist no  $p \in P \cup \{f\}$  and  $q \in Q \setminus \{f\}$  such that  $q$  is  $<_1$ -covered by  $p$  <sup>31</sup>. Hence, Lemma 6.1 (applied to  $P \cup \{f\}$  and  $Q \setminus \{f\}$  instead of  $P$  and  $Q$ ) shows that  $(P \cup \{f\}, Q \setminus \{f\}) \in \text{Adm } \mathbf{E}$ .

Furthermore,  $\pi|_Q$  is a  $(Q, <_1, <_2)$ -partition. Hence,  $\pi|_{Q \setminus \{f\}}$  is a  $(Q \setminus \{f\}, <_1, <_2)$ -partition (since  $Q \setminus \{f\} \subseteq Q$ ).

Furthermore,  $\pi|_{P \cup \{f\}}$  is a  $(P \cup \{f\}, >_1, <_2)$ -partition<sup>32</sup>.

Altogether, we now know that  $(P \cup \{f\}, Q \setminus \{f\}) \in \text{Adm } \mathbf{E}$ , that  $\pi|_{P \cup \{f\}}$  is a  $(P \cup \{f\}, >_1, <_2)$ -partition, and that  $\pi|_{Q \setminus \{f\}}$  is a  $(Q \setminus \{f\}, <_1, <_2)$ -partition. In other words,  $(P \cup \{f\}, Q \setminus \{f\}) \in Z$  (by the definition of  $Z$ ). Thus,

$$\begin{cases} (P \cup \{f\}, Q \setminus \{f\}), & \text{if } f \notin P; \\ (P \setminus \{f\}, Q \cup \{f\}), & \text{if } f \in P \end{cases} = (P \cup \{f\}, Q \setminus \{f\}) \quad (\text{since } f \notin P)$$

$$\in Z.$$

Hence, (20) is proven in Case 2.

<sup>30</sup>*Proof.* Let  $a \in Q$  and  $b \in P$  be such that  $b >_1 a$ . We shall derive a contradiction.

We have  $b >_1 a$ . In other words,  $a <_1 b$ . Thus,  $b \in P$  and  $a \in Q$  satisfy  $a <_1 b$ . This contradicts (21) (applied to  $p = b$  and  $q = a$ ).

Now, forget that we fixed  $a$  and  $b$ . We thus have found a contradiction for every  $a \in Q$  and  $b \in P$  satisfying  $b >_1 a$ . Hence, no  $a \in Q$  and  $b \in P$  satisfy  $b >_1 a$ . Renaming  $a$  and  $b$  as  $p$  and  $q$  in this statement, we obtain the following: No  $p \in Q$  and  $q \in P$  satisfy  $q >_1 p$ . This proves (24).

<sup>31</sup>*Proof.* Assume the contrary. Thus, there exist  $p \in P \cup \{f\}$  and  $q \in Q \setminus \{f\}$  such that  $q$  is  $<_1$ -covered by  $p$ . Consider such  $p$  and  $q$ .

We know that  $q$  is  $<_1$ -covered by  $p$ , and thus we have  $q <_1 p$ . In other words,  $p >_1 q$ . Thus, Lemma 6.4 (a) (applied to  $(E, >_1, <_2)$ ,  $<_1$ ,  $Q$ ,  $P$ ,  $q$  and  $p$  instead of  $(E, <_1, <_2)$ ,  $>_1$ ,  $P$ ,  $Q$ ,  $p$  and  $q$ ) yields that we have neither  $p <_2 q$  nor  $q <_2 p$ . On the other hand,  $q$  is  $<_1$ -covered by  $p$ . Hence,  $q$  and  $p$  are  $<_2$ -comparable (since  $\mathbf{E}$  is tertispecial). In other words, we have either  $q <_2 p$  or  $q = p$  or  $p <_2 q$ . Hence, we must have  $q = p$  (since we have neither  $p <_2 q$  nor  $q <_2 p$ ). But this contradicts  $q <_1 p$ . This contradiction shows that our assumption was wrong, qed.

<sup>32</sup>This follows from Lemma 6.4 (b) (applied to  $(E, >_1, <_2)$ ,  $<_1$ ,  $Q$  and  $P$  instead of  $(E, <_1, <_2)$ ,  $>_1$ ,  $P$  and  $Q$ ), since  $\pi|_P$  is a  $(P, >_1, <_2)$ -partition.

We have now proven (20) in both Cases 1 and 2. Thus, (20) always holds. In other words, the map  $T$  is well-defined.

What the map  $T$  does to a pair  $(P, Q) \in Z$  can be described as moving the element  $f$  from the set where it resides (either  $P$  or  $Q$ ) to the other set. Clearly, doing this twice gives us the original pair back. Hence, the map  $T$  is an involution. Furthermore, for any  $(P, Q) \in Z$ , if we write  $T((P, Q))$  in the form  $(P', Q')$ , then

$$(-1)^{|P'|} = -(-1)^{|P|} \text{ (because } P' = \begin{cases} P \cup \{f\}, & \text{if } f \notin P; \\ P \setminus \{f\}, & \text{if } f \in P \end{cases} \text{ and thus } |P'| = |P| \pm 1).$$

In other words, the involution  $T$  satisfies Property P. As we have already explained, this proves (19). Hence, Lemma 6.5 is proven.  $\square$

*Proof of Theorem 4.2.* We shall prove Theorem 4.2 by strong induction over  $|E|$ . The induction step proceeds as follows: Consider a tertispecial double poset  $\mathbf{E} = (E, <_1, <_2)$  and a map  $w : E \rightarrow \{1, 2, 3, \dots\}$ , and assume (as the induction hypothesis) that Theorem 4.2 is proven for all tertispecial double posets of smaller size<sup>33</sup>. Our goal is to show that

$$S(\Gamma((E, <_1, <_2), w)) = (-1)^{|E|} \Gamma((E, >_1, <_2), w). \text{ Here, as usual, } >_1 \text{ denotes the opposite relation of } <_1.$$

If  $E = \emptyset$ , then this is easy<sup>34</sup>. Thus, we WLOG assume that  $E \neq \emptyset$ . Hence,  $|E| > 0$ . Moreover, Lemma 6.3 (b) shows that  $\varepsilon(\Gamma(\mathbf{E}, w)) = 0$ . Thus,  $(u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) =$

$$u \left( \underbrace{\varepsilon(\Gamma(\mathbf{E}, w))}_{=0} \right) = u(0) = 0.$$

The upper commutative pentagon of (2) shows that  $u \circ \varepsilon = m \circ (S \otimes \text{id}) \circ \Delta$ . Applying both sides of this equality to  $\Gamma(\mathbf{E}, w)$ , we obtain  $(u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) = (m \circ (S \otimes \text{id}) \circ \Delta)(\Gamma(\mathbf{E}, w))$ . Since  $(u \circ \varepsilon)(\Gamma(\mathbf{E}, w)) = 0$ , this becomes

$$\begin{aligned} 0 &= (m \circ (S \otimes \text{id}) \circ \Delta)(\Gamma(\mathbf{E}, w)) = m((S \otimes \text{id})(\Delta(\Gamma(\mathbf{E}, w)))) \\ &= m \left( (S \otimes \text{id}) \left( \sum_{(P, Q) \in \text{Adm } \mathbf{E}} \Gamma(\mathbf{E} |_{P, w} |_{P}) \otimes \Gamma(\mathbf{E} |_{Q, w} |_{Q}) \right) \right) \quad (\text{by (10)}) \\ &= m \left( \sum_{(P, Q) \in \text{Adm } \mathbf{E}} S(\Gamma(\mathbf{E} |_{P, w} |_{P})) \otimes \Gamma(\mathbf{E} |_{Q, w} |_{Q}) \right) \\ &= \sum_{(P, Q) \in \text{Adm } \mathbf{E}} S(\Gamma(\mathbf{E} |_{P, w} |_{P})) \Gamma(\mathbf{E} |_{Q, w} |_{Q}) \\ &= S(\Gamma(\mathbf{E} |_{E, w} |_{E})) \Gamma(\mathbf{E} |_{\emptyset, w} |_{\emptyset}) + \sum_{\substack{(P, Q) \in \text{Adm } \mathbf{E}; \\ |P| < |E|}} S(\Gamma(\mathbf{E} |_{P, w} |_{P})) \Gamma(\mathbf{E} |_{Q, w} |_{Q}) \end{aligned} \tag{25}$$

<sup>33</sup>The size of a double poset  $(P, <_1, <_2)$  means the nonnegative integer  $|P|$ .

<sup>34</sup>Hint: If  $E = \emptyset$ , then both  $\Gamma((E, <_1, <_2), w)$  and  $\Gamma((E, >_1, <_2), w)$  are equal to 1 (by Lemma 6.3 (a)), but the antipode  $S$  satisfies  $S(1) = 1$  and  $(-1)^{|\emptyset|} = 1$ .

(since the only pair  $(P, Q) \in \text{Adm } \mathbf{E}$  satisfying  $|P| \geq |E|$  is  $(E, \emptyset)$ , whereas all other pairs  $(P, Q) \in \text{Adm } \mathbf{E}$  satisfy  $|P| < |E|$ ).

But whenever  $(P, Q) \in \text{Adm } \mathbf{E}$  is such that  $|P| < |E|$ , the double poset  $\mathbf{E} \upharpoonright_P = (P, <_1, <_2)$  is tertispecial (by Lemma 6.2), and therefore we have  $S(\Gamma(\mathbf{E} \upharpoonright_P, w \upharpoonright_P)) = S(\Gamma((P, <_1, <_2), w \upharpoonright_P)) = (-1)^{|P|} \Gamma((P, >_1, <_2), w \upharpoonright_P)$  (by the induction hypothesis). Hence, (25) becomes

$$\begin{aligned} 0 &= S \left( \Gamma \left( \underbrace{\mathbf{E} \upharpoonright_E}_{=\mathbf{E}}, \underbrace{w \upharpoonright_E}_{=w} \right) \right) \underbrace{\Gamma(\mathbf{E} \upharpoonright_{\emptyset}, w \upharpoonright_{\emptyset})}_{=\Gamma((\emptyset, <_1, <_2), w \upharpoonright_{\emptyset})=1} \\ &\quad \underbrace{= \Gamma((\emptyset, <_1, <_2), w \upharpoonright_{\emptyset})=1}_{\text{(by Lemma 6.3 (a))}} \\ &\quad + \sum_{\substack{(P,Q) \in \text{Adm } \mathbf{E}; \\ |P| < |E|}} \underbrace{S(\Gamma(\mathbf{E} \upharpoonright_P, w \upharpoonright_P))}_{=(-1)^{|P|} \Gamma((P, >_1, <_2), w \upharpoonright_P)} \Gamma(\mathbf{E} \upharpoonright_Q, w \upharpoonright_Q) \\ &= S(\Gamma(\mathbf{E}, w)) + \sum_{\substack{(P,Q) \in \text{Adm } \mathbf{E}; \\ |P| < |E|}} (-1)^{|P|} \Gamma((P, >_1, <_2), w \upharpoonright_P) \Gamma(\mathbf{E} \upharpoonright_Q, w \upharpoonright_Q). \end{aligned}$$

Thus,

$$S(\Gamma(\mathbf{E}, w)) = - \sum_{\substack{(P,Q) \in \text{Adm } \mathbf{E}; \\ |P| < |E|}} (-1)^{|P|} \Gamma((P, >_1, <_2), w \upharpoonright_P) \Gamma(\mathbf{E} \upharpoonright_Q, w \upharpoonright_Q). \quad (26)$$

For every subset  $P$  of  $E$ , we have

$$\begin{aligned} \Gamma((P, >_1, <_2), w \upharpoonright_P) &= \sum_{\pi \text{ is a } (P, >_1, <_2)\text{-partition}} \mathbf{x}_{\pi, w \upharpoonright_P} \\ &\quad \text{(by the definition of } \Gamma((P, >_1, <_2), w \upharpoonright_P)) \\ &= \sum_{\sigma \text{ is a } (P, >_1, <_2)\text{-partition}} \mathbf{x}_{\sigma, w \upharpoonright_P} \end{aligned} \quad (27)$$

(here, we have renamed the summation index  $\pi$  as  $\sigma$ ).

For every subset  $Q$  of  $E$ , we have

$$\begin{aligned} \Gamma \left( \underbrace{\mathbf{E} \upharpoonright_Q}_{=(Q, <_1, <_2)}, w \upharpoonright_Q \right) &= \Gamma((Q, <_1, <_2), w \upharpoonright_Q) = \sum_{\pi \text{ is a } (Q, <_1, <_2)\text{-partition}} \mathbf{x}_{\pi, w \upharpoonright_Q} \\ &\quad \text{(by the definition of } \Gamma((Q, <_1, <_2), w \upharpoonright_Q)) \\ &= \sum_{\tau \text{ is a } (Q, <_1, <_2)\text{-partition}} \mathbf{x}_{\tau, w \upharpoonright_Q} \end{aligned} \quad (28)$$

(here, we have renamed the summation index  $\pi$  as  $\tau$ ).

Now,

$$\begin{aligned}
 & \sum_{(P,Q) \in \text{Adm } \mathbf{E}} (-1)^{|P|} \underbrace{\Gamma((P, >_1, <_2), w|_P)}_{\substack{= \sum_{\sigma \text{ is a } (P, >_1, <_2)\text{-partition} \\ \text{(by (27))}} \mathbf{x}_{\sigma, w|_P} \\ = \sum_{\tau \text{ is a } (Q, <_1, <_2)\text{-partition} \\ \text{(by (28))}} \mathbf{x}_{\tau, w|_Q}} \Gamma(\mathbf{E} |_Q, w|_Q) \\
 &= \sum_{(P,Q) \in \text{Adm } \mathbf{E}} (-1)^{|P|} \left( \sum_{\sigma \text{ is a } (P, >_1, <_2)\text{-partition}} \mathbf{x}_{\sigma, w|_P} \right) \left( \sum_{\tau \text{ is a } (Q, <_1, <_2)\text{-partition}} \mathbf{x}_{\tau, w|_Q} \right) \\
 &= \sum_{(P,Q) \in \text{Adm } \mathbf{E}} (-1)^{|P|} \sum_{\sigma \text{ is a } (P, >_1, <_2)\text{-partition}} \sum_{\tau \text{ is a } (Q, <_1, <_2)\text{-partition}} \mathbf{x}_{\sigma, w|_P} \mathbf{x}_{\tau, w|_Q} \\
 &= \sum_{(P,Q) \in \text{Adm } \mathbf{E}} (-1)^{|P|} \sum_{\substack{(\sigma, \tau); \\ \sigma: P \rightarrow \{1, 2, 3, \dots\}; \\ \tau: Q \rightarrow \{1, 2, 3, \dots\}; \\ \sigma \text{ is a } (P, >_1, <_2)\text{-partition}; \\ \tau \text{ is a } (Q, <_1, <_2)\text{-partition}}} \mathbf{x}_{\sigma, w|_P} \mathbf{x}_{\tau, w|_Q} \\
 &= \sum_{(P,Q) \in \text{Adm } \mathbf{E}} (-1)^{|P|} \sum_{\substack{\pi: E \rightarrow \{1, 2, 3, \dots\}; \\ \pi|_P \text{ is a } (P, >_1, <_2)\text{-partition}; \\ \pi|_Q \text{ is a } (Q, <_1, <_2)\text{-partition}}} \underbrace{\mathbf{x}_{\pi|_P, w|_P} \mathbf{x}_{\pi|_Q, w|_Q}}_{= \mathbf{x}_{\pi, w}} \\
 & \left( \begin{array}{l} \text{here, we have substituted } (\pi|_P, \pi|_Q) \text{ for } (\sigma, \tau) \text{ in the inner sum,} \\ \text{since every pair } (\sigma, \tau) \text{ consisting of a map } \sigma: P \rightarrow \{1, 2, 3, \dots\} \\ \text{and a map } \tau: Q \rightarrow \{1, 2, 3, \dots\} \\ \text{can be written as } (\pi|_P, \pi|_Q) \text{ for a unique } \pi: E \rightarrow \{1, 2, 3, \dots\} \\ \text{(namely, for the } \pi: E \rightarrow \{1, 2, 3, \dots\} \text{ that is defined to send every} \\ e \in P \text{ to } \sigma(e) \text{ and to send every } e \in Q \text{ to } \tau(e)) \end{array} \right) \\
 &= \sum_{(P,Q) \in \text{Adm } \mathbf{E}} (-1)^{|P|} \sum_{\substack{\pi: E \rightarrow \{1, 2, 3, \dots\}; \\ \pi|_P \text{ is a } (P, >_1, <_2)\text{-partition}; \\ \pi|_Q \text{ is a } (Q, <_1, <_2)\text{-partition}}} \mathbf{x}_{\pi, w} \\
 &= \sum_{\pi: E \rightarrow \{1, 2, 3, \dots\}} \underbrace{\sum_{\substack{(P,Q) \in \text{Adm } \mathbf{E}; \\ \pi|_P \text{ is a } (P, >_1, <_2)\text{-partition}; \\ \pi|_Q \text{ is a } (Q, <_1, <_2)\text{-partition}}} (-1)^{|P|} \mathbf{x}_{\pi, w}}_{\substack{= 0 \\ \text{(by (19))}}} = \sum_{\pi: E \rightarrow \{1, 2, 3, \dots\}} 0 \mathbf{x}_{\pi, w} = 0.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 0 &= \sum_{(P,Q) \in \text{Adm } \mathbf{E}} (-1)^{|P|} \Gamma((P, >_1, <_2), w|_P) \Gamma(\mathbf{E} |_Q, w|_Q) \\
 &= (-1)^{|E|} \Gamma\left(\underbrace{(E, >_1, <_2)}_{=w}, \underbrace{w|_E}_{=\Gamma((\emptyset, <_1, <_2), w|_\emptyset)=1}\right) \\
 &\quad + \sum_{\substack{(P,Q) \in \text{Adm } \mathbf{E}; \\ |P| < |E|}} (-1)^{|P|} \Gamma((P, >_1, <_2), w|_P) \Gamma(\mathbf{E} |_Q, w|_Q) \\
 &\quad \left( \begin{array}{l} \text{because the only pair } (P, Q) \in \text{Adm } \mathbf{E} \text{ satisfying } |P| \geq |E| \\ \text{is } (P, Q) = (E, \emptyset), \\ \text{whereas all other pairs } (P, Q) \in \text{Adm } \mathbf{E} \text{ satisfy } |P| < |E| \end{array} \right) \\
 &= (-1)^{|E|} \Gamma((E, >_1, <_2), w) \\
 &\quad + \sum_{\substack{(P,Q) \in \text{Adm } \mathbf{E}; \\ |P| < |E|}} (-1)^{|P|} \Gamma((P, >_1, <_2), w|_P) \Gamma(\mathbf{E} |_Q, w|_Q),
 \end{aligned}$$

so that

$$\begin{aligned}
 (-1)^{|E|} \Gamma((E, >_1, <_2), w) &= - \sum_{\substack{(P,Q) \in \text{Adm } \mathbf{E}; \\ |P| < |E|}} (-1)^{|P|} \Gamma((P, >_1, <_2), w|_P) \Gamma(\mathbf{E} |_Q, w|_Q) \\
 &= S \left( \Gamma \left( \underbrace{\mathbf{E}}_{=(E, <_1, <_2)}, w \right) \right) \quad (\text{by (26)}) \\
 &= S(\Gamma((E, <_1, <_2), w)),
 \end{aligned}$$

and thus  $S(\Gamma((E, <_1, <_2), w)) = (-1)^{|E|} \Gamma((E, >_1, <_2), w)$ . This completes the induction step and thus the proof of Theorem 4.2.  $\square$

## 7. Proof of Theorem 4.7

Before we begin proving Theorem 4.7, we state a criterion for  $\mathbf{E}$ -partitions that is less wasteful (in the sense that it requires fewer verifications) than the definition:

**Lemma 7.1.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a tertispecial double poset. Let  $\phi : E \rightarrow \{1, 2, 3, \dots\}$  be a map. Assume that the following two conditions hold:

- *Condition 1:* If  $e \in E$  and  $f \in E$  are such that  $e$  is  $<_1$ -covered by  $f$ , and if we have  $e <_2 f$ , then  $\phi(e) \leq \phi(f)$ .
- *Condition 2:* If  $e \in E$  and  $f \in E$  are such that  $e$  is  $<_1$ -covered by  $f$ , and if we have  $f <_2 e$ , then  $\phi(e) < \phi(f)$ .

Then,  $\phi$  is an  $\mathbf{E}$ -partition.

*Proof of Lemma 7.1.* For any  $a \in E$  and  $b \in E$ , we define a subset  $[a, b]$  of  $E$  as in the proof of Lemma 6.1.

We need to show that  $\phi$  is an  $\mathbf{E}$ -partition. In other words, we need to prove the following two claims:

*Claim 1:* Every  $e \in E$  and  $f \in E$  satisfying  $e <_1 f$  satisfy  $\phi(e) \leq \phi(f)$ .

*Claim 2:* Every  $e \in E$  and  $f \in E$  satisfying  $e <_1 f$  and  $f <_2 e$  satisfy  $\phi(e) < \phi(f)$ .

*Proof of Claim 1:* Assume the contrary. Thus, there exists a pair  $(e, f) \in E \times E$  satisfying  $e <_1 f$  but not  $\phi(e) \leq \phi(f)$ . Such a pair will be called a *malrelation*. Fix a malrelation  $(u, v)$  for which the value  $|[u, v]|$  is minimum (such a  $(u, v)$  exists, since there exists a malrelation). Thus,  $u \in E$  and  $v \in E$  and  $u <_1 v$  but not  $\phi(u) \leq \phi(v)$ .

If  $u$  was  $<_1$ -covered by  $v$ , then we would obtain  $\phi(u) \leq \phi(v)$ <sup>35</sup>, which would contradict the fact that we do not have  $\phi(u) \leq \phi(v)$ . Hence,  $u$  is not  $<_1$ -covered by  $v$ . Consequently, there exists a  $w \in E$  such that  $u <_1 w <_1 v$  (since  $u <_1 v$ ). Consider this  $w$ . Applying (13) to  $a = u$ ,  $c = w$  and  $b = v$ , we see that both numbers  $|[u, w]|$  and  $|[w, v]|$  are smaller than  $|[u, v]|$ . Hence, neither  $(u, w)$  nor  $(w, v)$  is a malrelation (since we picked  $(u, v)$  to be a malrelation with minimum  $|[u, v]|$ ). Therefore, we have  $\phi(u) \leq \phi(w)$  (since  $u <_1 w$ , but  $(u, w)$  is not a malrelation) and  $\phi(w) \leq \phi(v)$  (since  $w <_1 v$ , but  $(w, v)$  is not a malrelation). Combining these two inequalities, we obtain  $\phi(u) \leq \phi(w) \leq \phi(v)$ . This contradicts the fact that we do not have  $\phi(u) \leq \phi(v)$ . This contradiction concludes the proof of Claim 1.

Instead of Claim 2, we shall prove the following stronger claim:

*Claim 3:* Every  $e \in E$  and  $f \in E$  satisfying  $e <_1 f$  and not  $e <_2 f$  satisfy  $\phi(e) < \phi(f)$ .

*Proof of Claim 3:* Assume the contrary. Thus, there exists a pair  $(e, f) \in E \times E$  satisfying  $e <_1 f$  and not  $e <_2 f$  but not  $\phi(e) < \phi(f)$ . Such a pair will be called a *malrelation*. Fix a malrelation  $(u, v)$  for which the value  $|[u, v]|$  is minimum (such a  $(u, v)$  exists, since there exists a malrelation). Thus,  $u \in E$  and  $v \in E$  and  $u <_1 v$  and not  $u <_2 v$  but not  $\phi(u) < \phi(v)$ .

If  $u$  was  $<_1$ -covered by  $v$ , then we would obtain  $\phi(u) < \phi(v)$  easily<sup>36</sup>, which would contradict the fact that we do not have  $\phi(u) < \phi(v)$ . Hence,  $u$  is not  $<_1$ -covered by  $v$ . Consequently, there exists a  $w \in E$  such that  $u <_1 w <_1 v$  (since  $u <_1 v$ ). Consider this  $w$ . Applying (13) to  $a = u$ ,  $c = w$  and  $b = v$ , we see that

<sup>35</sup>*Proof.* Assume that  $u$  is  $<_1$ -covered by  $v$ . Thus,  $u$  and  $v$  are  $<_2$ -comparable (since the double poset  $\mathbf{E}$  is tertispecial). In other words, we have either  $u <_2 v$  or  $u = v$  or  $v <_2 u$ . In the first of these three cases, we obtain  $\phi(u) \leq \phi(v)$  by applying Condition 1 to  $e = u$  and  $f = v$ . In the third of these cases, we obtain  $\phi(u) < \phi(v)$  (and thus  $\phi(u) \leq \phi(v)$ ) by applying Condition 2 to  $e = u$  and  $f = v$ . The second of these cases cannot happen because  $u <_1 v$ . Thus, we always have  $\phi(u) \leq \phi(v)$ , qed.

<sup>36</sup>*Proof.* Assume that  $u$  is  $<_1$ -covered by  $v$ . Thus,  $u$  and  $v$  are  $<_2$ -comparable (since the double poset  $\mathbf{E}$  is tertispecial). In other words, we have either  $u <_2 v$  or  $u = v$  or  $v <_2 u$ . Since neither  $u <_2 v$  nor  $u = v$  can hold (indeed,  $u <_2 v$  is ruled out by assumption, whereas  $u = v$  is ruled out by  $u <_1 v$ ), we thus have  $v <_2 u$ . Therefore,  $\phi(u) < \phi(v)$  by Condition 2 (applied to  $e = u$  and  $f = v$ ), qed.

both numbers  $|[u, w]|$  and  $|[w, v]|$  are smaller than  $|[u, v]|$ . Hence, neither  $(u, w)$  nor  $(w, v)$  is a malrelation (since we picked  $(u, v)$  to be a malrelation with minimum  $|[u, v]|$ ).

But  $\phi(v) \leq \phi(u)$  (since we do not have  $\phi(u) < \phi(v)$ ). On the other hand,  $u <_1 w$  and therefore  $\phi(u) \leq \phi(w)$  (by Claim 1, applied to  $e = u$  and  $f = w$ ). Furthermore,  $w <_1 v$  and thus  $\phi(w) \leq \phi(v)$  (by Claim 1, applied to  $e = w$  and  $f = v$ ). The chain of inequalities  $\phi(v) \leq \phi(u) \leq \phi(w) \leq \phi(v)$  ends with the same term that it begins with; therefore, it must be a chain of equalities. In other words, we have  $\phi(v) = \phi(u) = \phi(w) = \phi(v)$ .

Now, using  $\phi(w) = \phi(v)$ , we can see that  $w <_2 v$ <sup>37</sup>. The same argument (applied to  $u$  and  $w$  instead of  $w$  and  $v$ ) shows that  $u <_2 w$ . Thus,  $u <_2 w <_2 v$ , which contradicts the fact that we do not have  $u <_2 v$ . This contradiction proves Claim 3.

*Proof of Claim 2:* The condition " $f <_2 e$ " is stronger than "not  $e <_2 f$ ". Thus, Claim 2 follows from Claim 3.

Claims 1 and 2 are now both proven, and so Lemma 7.1 follows.  $\square$

*Proof of Lemma 4.5.* Consider the following three logical statements:

*Statement 1:* The  $G$ -orbit  $O$  is  $E$ -coeven.

*Statement 2:* All elements of  $O$  are  $E$ -coeven.

*Statement 3:* At least one element of  $O$  is  $E$ -coeven.

Statements 1 and 2 are equivalent (according to the definition of when a  $G$ -orbit is  $E$ -coeven). Our goal is to prove that Statements 1 and 3 are equivalent (because this is precisely what Lemma 4.5 says). Thus, it suffices to show that Statements 2 and 3 are equivalent (because we already know that Statements 1 and 2 are equivalent). Since Statement 2 obviously implies Statement 3 (in fact, the  $G$ -orbit  $O$  contains at least one element), we therefore only need to show that Statement 3 implies Statement 2. Thus, assume that Statement 3 holds. We need to prove that Statement 2 holds.

There exists at least one  $E$ -coeven  $\phi \in O$  (because we assumed that Statement 3 holds). Consider this  $\phi$ . Now, let  $\pi \in O$  be arbitrary. We shall show that  $\pi$  is  $E$ -coeven.

We know that  $\phi$  is  $E$ -coeven. In other words,

$$\text{every } g \in G \text{ satisfying } g\phi = \phi \text{ is } E\text{-even.} \quad (29)$$

Now, let  $g \in G$  be such that  $g\pi = \pi$ . Since  $\phi$  belongs to the  $G$ -orbit  $O$ , we have  $O = G\phi$ . Now,  $\pi \in O = G\phi$ . In other words, there exists some  $h \in G$  such that  $\pi = h\phi$ . Consider this  $h$ . We have  $g\pi = \pi$ . Since  $\pi = h\phi$ , this rewrites as  $gh\phi = h\phi$ . In other words,  $h^{-1}gh\phi = \phi$ . Thus, (29) (applied to  $h^{-1}gh$  instead of  $g$ ) shows that

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<sup>37</sup>*Proof.* Assume the contrary. Thus, we do not have  $w <_2 v$ . But  $\phi(w) = \phi(v)$  shows that we do not have  $\phi(w) < \phi(v)$ . Hence,  $(w, v)$  is a malrelation (since  $w <_1 v$  and not  $w <_2 v$  but not  $\phi(w) < \phi(v)$ ). This contradicts the fact that  $(w, v)$  is not a malrelation. This contradiction completes the proof.

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$h^{-1}gh$  is  $E$ -even. In other words,

$$\text{the action of } h^{-1}gh \text{ on } E \text{ is an even permutation of } E. \quad (30)$$

Now, let  $\varepsilon$  be the group homomorphism from  $G$  to  $\text{Aut } E$ <sup>38</sup> which describes the  $G$ -action on  $E$ . Then,  $\varepsilon(h^{-1}gh)$  is the action of  $h^{-1}gh$  on  $E$ , and thus is an even permutation of  $E$  (by (30)).

But since  $\varepsilon$  is a group homomorphism, we have  $\varepsilon(h^{-1}gh) = \varepsilon(h)^{-1} \varepsilon(g) \varepsilon(h)$ . Thus, the permutations  $\varepsilon(h^{-1}gh)$  and  $\varepsilon(g)$  of  $E$  are conjugate. Since the permutation  $\varepsilon(h^{-1}gh)$  is even, this shows that the permutation  $\varepsilon(g)$  is even. In other words, the action of  $g$  on  $E$  is an even permutation of  $E$ . In other words,  $g$  is  $E$ -even.

Now, let us forget that we fixed  $g$ . We thus have shown that every  $g \in G$  satisfying  $g\pi = \pi$  is  $E$ -even. In other words,  $\pi$  is  $E$ -coeven.

Let us now forget that we fixed  $\pi$ . Thus, we have proven that every  $\pi \in O$  is  $E$ -coeven. In other words, Statement 2 holds. We have thus shown that Statement 3 implies Statement 2. Consequently, Statements 2 and 3 are equivalent, and so the proof of Lemma 4.5 is complete.  $\square$

We leave the fairly straightforward proof of Proposition 4.6 to the reader.

Next, we will show three simple properties of posets on which groups act. First, we introduce a notation:

**Definition 7.2.** Let  $G$  be a group. Let  $g \in G$ . Let  $E$  be a  $G$ -set. Then, the subgroup  $\langle g \rangle$  of  $G$  (this is the subgroup of  $G$  generated by  $g$ ) also acts on  $E$ . The  $\langle g \rangle$ -orbits on  $E$  will be called the  $g$ -orbits on  $E$ . When  $E$  is clear from the context, we shall simply call them the  $g$ -orbits.

We can also describe these  $g$ -orbits explicitly: For any given  $e \in E$ , the  $g$ -orbit of  $e$  (that is, the unique  $g$ -orbit that contains  $e$ ) is  $\langle g \rangle e = \{g^k e \mid k \in \mathbb{Z}\}$ .

Equivalently, the  $g$ -orbits on  $E$  can be characterized as follows: The action of  $g$  on  $E$  is a permutation of  $E$ . The cycles of this permutation are the  $g$ -orbits on  $E$  (at least when  $E$  is finite).

**Proposition 7.3.** Let  $E$  be a set. Let  $<_1$  be a strict partial order on  $E$ . Let  $G$  be a finite group which acts on  $E$ . Assume that  $G$  preserves the relation  $<_1$ .

Let  $g \in G$ . Let  $E^g$  be the set of all  $g$ -orbits on  $E$ . Define a binary relation  $<_1^g$  on  $E^g$  by

$$(u <_1^g v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Then,  $<_1^g$  is a strict partial order.

Proposition 7.3 is precisely [Joch13, Lemma 2.4], but let us outline the proof for the sake of completeness:

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<sup>38</sup>We use the notation  $\text{Aut } E$  for the group of all permutations of the set  $E$ .

*Proof of Proposition 7.3.* Let us first show that the relation  $<_1^g$  is irreflexive. Indeed, assume the contrary. Thus, there exists a  $u \in E^g$  such that  $u <_1^g u$ . Consider this  $u$ .

We have  $u \in E^g$ . In other words,  $u$  is a  $g$ -orbit on  $E$ .

Since  $u <_1^g u$ , there exist  $a \in u$  and  $b \in u$  with  $a <_1 b$  (by the definition of the relation  $<_1^g$ ). Consider these  $a$  and  $b$ . There exists a  $k \in \mathbb{Z}$  such that  $b = g^k a$  (since  $a$  and  $b$  both lie in one and the same  $g$ -orbit  $u$ ). Consider this  $k$ .

Each element of  $G$  has finite order (since  $G$  is a finite group). In particular, the element  $g$  of  $G$  has finite order. In other words, there exists a positive integer  $n$  such that  $g^n = 1_G$ . Consider this  $n$ . Every  $p \in \mathbb{Z}$  satisfies  $g^{np} = (g^n)^p = 1_G$  (since  $g^n = 1_G$ ). Applying this to  $p = k$ , we obtain  $g^{nk} = 1_G$ .

Now,  $a <_1 b = g^k a$ . Since  $G$  preserves the relation  $<_1$ , this shows that  $ha <_1 hg^k a$  for every  $h \in G$ . Thus,  $g^{\ell k} a <_1 g^{\ell k} g^k a$  for every  $\ell \in \mathbb{N}$ . Hence,  $g^{\ell k} a <_1 g^{\ell k} g^k a = g^{\ell k + k} a = g^{(\ell+1)k} a$  for every  $\ell \in \mathbb{N}$ . Consequently,  $g^{0k} a <_1 g^{1k} a <_1 g^{2k} a <_1 \dots <_1 g^{nk} a$ . Thus,  $g^{0k} a <_1 \underbrace{g^{nk} a}_{=1_G} = a$ , which contradicts  $\underbrace{g^{0k} a}_{=g^0=1_G} = 1_G a = a$ . This

contradiction proves that our assumption was wrong. Hence, the relation  $<_1^g$  is irreflexive.

Let us next show that the relation  $<_1^g$  is transitive. Indeed, let  $u, v$  and  $w$  be three elements of  $E^g$  such that  $u <_1^g v$  and  $v <_1^g w$ . We must prove that  $u <_1^g w$ .

There exist  $a \in u$  and  $b \in v$  with  $a <_1 b$  (since  $u <_1^g v$ ). Consider these  $a$  and  $b$ .

There exist  $a' \in v$  and  $b' \in w$  with  $a' <_1 b'$  (since  $v <_1^g w$ ). Consider these  $a'$  and  $b'$ .

The set  $v$  is a  $g$ -orbit (since  $v \in E^g$ ). The elements  $b$  and  $a'$  lie in one and the same  $g$ -orbit (namely, in  $v$ ). Hence, there exists some  $k \in \mathbb{Z}$  such that  $a' = g^k b$ . Consider this  $k$ . We have  $a <_1 b$  and thus  $g^k a <_1 g^k b$  (since  $G$  preserves the relation  $<_1$ ). Hence,  $g^k a <_1 g^k b = a' <_1 b'$ . Since  $g^k a \in u$  (because  $a \in u$ , and because  $u$  is a  $g$ -orbit) and  $b' \in w$ , this shows that  $u <_1^g w$  (by the definition of the relation  $<_1^g$ ). We thus have proven that the relation  $<_1^g$  is transitive.

Now, we know that the relation  $<_1^g$  is irreflexive and transitive, and thus also antisymmetric (since every irreflexive and transitive binary relation is antisymmetric). In other words,  $<_1^g$  is a strict partial order. This proves Proposition 7.3.  $\square$

**Remark 7.4.** Proposition 7.3 can be generalized: Let  $E$  be a set. Let  $<_1$  be a strict partial order on  $E$ . Let  $G$  be a finite group which acts on  $E$ . Assume that  $G$  preserves the relation  $<_1$ . Let  $H$  be a subgroup of  $G$ . Let  $E^H$  be the set of all  $H$ -orbits on  $E$ . Define a binary relation  $<_1^H$  on  $E^H$  by

$$(u <_1^H v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Then,  $<_1^H$  is a strict partial order.

This result (whose proof is quite similar to that of Proposition 7.3) implicitly appears in [Stan84, p. 30].

**Proposition 7.5.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a tertispecial double poset. Let  $G$  be a finite group which acts on  $E$ . Assume that  $G$  preserves both relations  $<_1$  and  $<_2$ .

Let  $g \in G$ . Let  $E^g$  be the set of all  $g$ -orbits on  $E$ . Define a binary relation  $<_1^g$  on  $E^g$  by

$$(u <_1^g v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_1 b).$$

Define a binary relation  $<_2^g$  on  $E^g$  by

$$(u <_2^g v) \iff (\text{there exist } a \in u \text{ and } b \in v \text{ with } a <_2 b).$$

Let  $\mathbf{E}^g$  be the triple  $(E^g, <_1^g, <_2^g)$ . Then,  $\mathbf{E}^g$  is a tertispecial double poset.

*Proof of Proposition 7.5.* Proposition 7.3 shows that  $<_1^g$  is a strict partial order. Similarly,  $<_2^g$  is a strict partial order. Thus,  $\mathbf{E}^g = (E^g, <_1^g, <_2^g)$  is a double poset. It remains to show that this double poset  $\mathbf{E}^g$  is tertispecial.

Let  $u$  and  $v$  be two elements of  $E^g$  such that  $u$  is  $<_1^g$ -covered by  $v$ . We shall prove that  $u$  and  $v$  are  $<_2^g$ -comparable.

We have  $u <_1^g v$  (since  $u$  is  $<_1^g$ -covered by  $v$ ). In other words, there exist  $a \in u$  and  $b \in v$  with  $a <_1 b$  (by the definition of the relation  $<_1^g$ ). Consider these  $a$  and  $b$ .

If there was a  $c \in E$  satisfying  $a <_1 c <_1 b$ , then we would have  $u <_1^g w <_1^g v$  with  $w$  being the  $g$ -orbit of  $c$ , and this would contradict the condition that  $u$  is  $<_1^g$ -covered by  $v$ . Hence, no such  $c$  can exist. In other words,  $a$  is  $<_1$ -covered by  $b$  (since we know that  $a <_1 b$ ). Thus,  $a$  and  $b$  are  $<_2$ -comparable (since the double poset  $\mathbf{E}$  is tertispecial). Consequently,  $u$  and  $v$  are  $<_2^g$ -comparable.

Now, let us forget that we fixed  $u$  and  $v$ . We thus have shown that if  $u$  and  $v$  are two elements of  $E^g$  such that  $u$  is  $<_1^g$ -covered by  $v$ , then  $u$  and  $v$  are  $<_2^g$ -comparable. In other words, the double poset  $\mathbf{E}^g = (E^g, <_1^g, <_2^g)$  is tertispecial. This proves Proposition 7.5.  $\square$

**Proposition 7.6.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a tertispecial double poset. Let  $G$  be a finite group which acts on  $E$ . Assume that  $G$  preserves both relations  $<_1$  and  $<_2$ .

Let  $g \in G$ . Define the set  $E^g$ , the relations  $<_1^g$  and  $<_2^g$  and the triple  $\mathbf{E}^g$  as in Proposition 7.5. Thus,  $\mathbf{E}^g$  is a tertispecial double poset (by Proposition 7.5).

There is a bijection  $\Phi$  between

- the maps  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  satisfying  $g\pi = \pi$

and

- the maps  $\bar{\pi} : E^g \rightarrow \{1, 2, 3, \dots\}$ .

Namely, this bijection  $\Phi$  sends any map  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  satisfying  $g\pi = \pi$  to the map  $\bar{\pi} : E^g \rightarrow \{1, 2, 3, \dots\}$  defined by

$$\bar{\pi}(u) = \pi(a) \quad \text{for every } u \in E^g \text{ and } a \in u.$$

(The well-definedness of this map  $\bar{\pi}$  is easy to see: Indeed, from  $g\pi = \pi$ , we can conclude that any two elements  $a_1$  and  $a_2$  of a given  $g$ -orbit  $u$  satisfy  $\pi(a_1) = \pi(a_2)$ .)

Consider this bijection  $\Phi$ . Let  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  be a map satisfying  $g\pi = \pi$ .

- (a) If  $\pi$  is an  $\mathbf{E}$ -partition, then  $\Phi(\pi)$  is an  $\mathbf{E}^g$ -partition.
- (b) If  $\Phi(\pi)$  is an  $\mathbf{E}^g$ -partition, then  $\pi$  is an  $\mathbf{E}$ -partition.
- (c) Let  $w : E \rightarrow \{1, 2, 3, \dots\}$  be map. Define a map  $w^g : E^g \rightarrow \{1, 2, 3, \dots\}$  by

$$w^g(u) = \sum_{a \in u} w(a) \quad \text{for every } u \in E^g.$$

Then,  $\mathbf{x}_{\Phi(\pi), w^g} = \mathbf{x}_{\pi, w}$ .

*Proof of Proposition 7.6 (sketched).* The definition of  $\Phi$  shows that

$$(\Phi(\pi))(u) = \pi(a) \quad \text{for every } u \in E^g \text{ and } a \in u. \tag{31}$$

(a) Assume that  $\pi$  is an  $\mathbf{E}$ -partition. We want to show that  $\Phi(\pi)$  is an  $\mathbf{E}^g$ -partition. In order to do so, we can use Lemma 7.1 (applied to  $\mathbf{E}^g$ ,  $(E^g, <_1^g, <_2^g)$  and  $\Phi(\pi)$  instead of  $\mathbf{E}$ ,  $(E, <_1, <_2)$  and  $\phi$ ); we only need to check the following two conditions:

*Condition 1:* If  $e \in E^g$  and  $f \in E^g$  are such that  $e$  is  $<_1^g$ -covered by  $f$ , and if we have  $e <_2^g f$ , then  $(\Phi(\pi))(e) \leq (\Phi(\pi))(f)$ .

*Condition 2:* If  $e \in E^g$  and  $f \in E^g$  are such that  $e$  is  $<_1^g$ -covered by  $f$ , and if we have  $f <_2^g e$ , then  $(\Phi(\pi))(e) < (\Phi(\pi))(f)$ .

*Proof of Condition 1:* Let  $e \in E^g$  and  $f \in E^g$  be such that  $e$  is  $<_1^g$ -covered by  $f$ . Assume that we have  $e <_2^g f$ .

We have  $e <_1^g f$  (because  $e$  is  $<_1^g$ -covered by  $f$ ). In other words, there exist  $a \in e$  and  $b \in f$  satisfying  $a <_1 b$ . Consider these  $a$  and  $b$ . Since  $\pi$  is an  $\mathbf{E}$ -partition, we have  $\pi(a) \leq \pi(b)$  (since  $a <_1 b$ ). But the definition of  $\Phi(\pi)$  shows that  $(\Phi(\pi))(e) = \pi(a)$  (since  $a \in e$ ) and  $(\Phi(\pi))(f) = \pi(b)$  (since  $b \in f$ ). Thus,  $(\Phi(\pi))(e) = \pi(a) \leq \pi(b) = (\Phi(\pi))(f)$ . Hence, Condition 1 is proven.

*Proof of Condition 2:* Let  $e \in E^g$  and  $f \in E^g$  be such that  $e$  is  $<_1^g$ -covered by  $f$ . Assume that we have  $f <_2^g e$ .

We have  $e <_1^g f$  (because  $e$  is  $<_1^g$ -covered by  $f$ ). In other words, there exist  $a \in e$  and  $b \in f$  satisfying  $a <_1 b$ . Consider these  $a$  and  $b$ .

If there was a  $c \in E$  satisfying  $a <_1 c <_1 b$ , then the  $g$ -orbit  $w$  of this  $c$  would satisfy  $e <_1^g w <_1^g f$ , which would contradict the fact that  $e$  is  $<_1^g$ -covered by  $f$ . Hence, there exists no such  $c$ . In other words,  $a$  is  $<_1$ -covered by  $b$  (since  $a <_1 b$ ). Therefore,  $a$  and  $b$  are  $<_2$ -comparable (since  $\mathbf{E}$  is tertispecial). In other words, we have either  $a <_2 b$  or  $a = b$  or  $b <_2 a$ . Since  $a <_2 b$  is impossible (because if we had  $a <_2 b$ , then we would have  $e <_2^g f$  (since  $a \in e$  and  $b \in f$ ), which would contradict  $f <_2^g e$  (since  $<_2^g$  is a strict partial order)), and since  $a = b$  is impossible (because  $a <_1 b$ ), we therefore must have  $b <_2 a$ . But since  $\pi$  is an  $\mathbf{E}$ -partition, we have  $\pi(a) < \pi(b)$  (since  $a <_1 b$  and  $b <_2 a$ ). But the definition of  $\Phi(\pi)$  shows that  $(\Phi(\pi))(e) = \pi(a)$  (since  $a \in e$ ) and  $(\Phi(\pi))(f) = \pi(b)$  (since  $b \in f$ ). Thus,  $(\Phi(\pi))(e) = \pi(a) < \pi(b) = (\Phi(\pi))(f)$ . Hence, Condition 2 is proven.

Thus, Condition 1 and Condition 2 are proven. Hence, Proposition 7.6 (a) is proven.

(b) Assume that  $\Phi(\pi)$  is an  $\mathbf{E}^g$ -partition. We want to show that  $\pi$  is an  $\mathbf{E}$ -partition. In order to do so, we can use Lemma 7.1 (applied to  $\phi = \pi$ ); we only need to check the following two conditions:

*Condition 1:* If  $e \in E$  and  $f \in E$  are such that  $e$  is  $<_1$ -covered by  $f$ , and if we have  $e <_2 f$ , then  $\pi(e) \leq \pi(f)$ .

*Condition 2:* If  $e \in E$  and  $f \in E$  are such that  $e$  is  $<_1$ -covered by  $f$ , and if we have  $f <_2 e$ , then  $\pi(e) < \pi(f)$ .

*Proof of Condition 1:* Let  $e \in E$  and  $f \in E$  be such that  $e$  is  $<_1$ -covered by  $f$ . Assume that we have  $e <_2 f$ .

We have  $e <_1 f$  (since  $e$  is  $<_1$ -covered by  $f$ ). Let  $u$  and  $v$  be the  $g$ -orbits of  $e$  and  $f$ , respectively. Thus,  $u$  and  $v$  belong to  $E^g$ , and satisfy  $e \in u$  and  $f \in v$ . Hence,  $u <_1^g v$  (since  $e <_1 f$ ). Hence,  $(\Phi(\pi))(u) \leq (\Phi(\pi))(v)$  (since  $\Phi(\pi)$  is an  $\mathbf{E}^g$ -partition). But the definition of  $\Phi(\pi)$  shows that  $(\Phi(\pi))(u) = \pi(e)$  (since  $e \in u$ ) and  $(\Phi(\pi))(v) = \pi(f)$  (since  $f \in v$ ). Thus,  $\pi(e) = (\Phi(\pi))(u) \leq (\Phi(\pi))(v) = \pi(f)$ . Hence, Condition 1 is proven.

*Proof of Condition 2:* Let  $e \in E$  and  $f \in E$  be such that  $e$  is  $<_1$ -covered by  $f$ . Assume that we have  $f <_2 e$ .

We have  $e <_1 f$  (since  $e$  is  $<_1$ -covered by  $f$ ). Let  $u$  and  $v$  be the  $g$ -orbits of  $e$  and  $f$ , respectively. Thus,  $u$  and  $v$  belong to  $E^g$ , and satisfy  $e \in u$  and  $f \in v$ . Hence,  $u <_1^g v$  (since  $e <_1 f$ ) and  $v <_2^g u$  (since  $f <_2 e$ ). Hence,  $(\Phi(\pi))(u) < (\Phi(\pi))(v)$  (since  $\Phi(\pi)$  is an  $\mathbf{E}^g$ -partition). But the definition of  $\Phi(\pi)$  shows that  $(\Phi(\pi))(u) = \pi(e)$  (since  $e \in u$ ) and  $(\Phi(\pi))(v) = \pi(f)$  (since  $f \in v$ ). Thus,  $\pi(e) = (\Phi(\pi))(u) < (\Phi(\pi))(v) = \pi(f)$ . Hence, Condition 2 is proven.

Thus, Condition 1 and Condition 2 are proven. Hence, Proposition 7.6 (b) is proven.

(c) The definition of  $\mathbf{x}_{\Phi(\pi),w^s}$  shows that

$$\begin{aligned} \mathbf{x}_{\Phi(\pi),w^s} &= \prod_{e \in E^s} x_{(\Phi(\pi))(e)}^{w^s(e)} = \prod_{u \in E^s} \underbrace{x_{(\Phi(\pi))(u)}^{w^s(u)}}_{= \prod_{a \in u} x_{(\Phi(\pi))(u)}^{w(a)} \text{ (since } w^s(u) = \sum_{a \in u} w(a)\text{)}} = \prod_{u \in E^s} \prod_{a \in u} \underbrace{x_{(\Phi(\pi))(u)}^{w(a)}}_{= x_{\pi(a)}^{w(a)} \text{ (by (31))}} \\ &= \underbrace{\prod_{u \in E^s} \prod_{a \in u} x_{\pi(a)}^{w(a)}}_{= \prod_{a \in E}} = \prod_{a \in E} x_{\pi(a)}^{w(a)} = \prod_{e \in E} x_{\pi(e)}^{w(e)} = \mathbf{x}_{\pi,w} \end{aligned}$$

(by the definition of  $\mathbf{x}_{\pi,w}$ ). This proves Proposition 7.6 (c). □

Our next lemma is a standard argument in Pólya enumeration theory (compare it with the proof of Burnside’s lemma):

**Lemma 7.7.** Let  $G$  be a finite group. Let  $F$  be a  $G$ -set. Let  $O$  be a  $G$ -orbit on  $F$ , and let  $\pi \in O$ .

(a) We have

$$\frac{1}{|O|} = \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} 1. \tag{32}$$

(b) Let  $E$  be a finite  $G$ -set. For every  $g \in G$ , let  $\text{sign}_E g$  denote the sign of the permutation of  $E$  that sends every  $e \in E$  to  $ge$ . (Thus,  $g \in G$  is  $E$ -even if and only if  $\text{sign}_E g = 1$ .) Then,

$$\begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} = \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \text{sign}_E g. \tag{33}$$

*Proof of Lemma 7.7.* Let  $\text{Stab}_G \pi$  denote the stabilizer of  $\pi$ ; this is the subgroup  $\{g \in G \mid g\pi = \pi\}$  of  $G$ . (This subgroup is also known as the *stabilizer subgroup* or the *isotropy group* of  $\pi$ .) The  $G$ -orbit of  $\pi$  is  $O$  (since  $O$  is a  $G$ -orbit on  $F$ , and since  $\pi \in O$ ). In other words,  $O = G\pi$ . Therefore,  $|O| = |G\pi| = |G| / |\text{Stab}_G \pi|$  (by the orbit-stabilizer theorem). Hence,

$$\frac{1}{|O|} = \frac{1}{|G| / |\text{Stab}_G \pi|} = \frac{|\text{Stab}_G \pi|}{|G|}. \tag{34}$$

(a) We have

$$\sum_{\substack{g \in G; \\ g\pi = \pi}} 1 = \left| \underbrace{\{g \in G \mid g\pi = \pi\}}_{= \text{Stab}_G \pi} \right| = |\text{Stab}_G \pi|.$$

Hence,

$$\frac{1}{|G|} \underbrace{\sum_{\substack{g \in G; \\ g\pi = \pi}} 1}_{=|\text{Stab}_G \pi|} = \frac{1}{|G|} |\text{Stab}_G \pi| = \frac{|\text{Stab}_G \pi|}{|G|} = \frac{1}{|O|}$$

(by (34)). This proves Lemma 7.7 (a).

(b) We need to prove (33). Assume first that  $O$  is  $E$ -coeven. Thus, all elements of  $O$  are  $E$ -coeven (by the definition of what it means for  $O$  to be  $E$ -coeven). Hence,  $\pi$  is  $E$ -coeven (since  $\pi \in O$ ). This means that every  $g \in G$  satisfying  $g\pi = \pi$  is  $E$ -even. Hence, every  $g \in G$  satisfying  $g\pi = \pi$  satisfies  $\text{sign}_E g = 1$  (since  $g$  is  $E$ -even if and only if  $\text{sign}_E g = 1$ ). Thus,

$$\begin{aligned} \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \underbrace{\text{sign}_E g}_{=1} &= \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} 1 = \frac{1}{|O|} && \text{(by (32))} \\ &= \begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} && \text{(since } O \text{ is } E\text{-coeven).} \end{aligned}$$

Thus, we have proven (33) under the assumption that  $O$  is  $E$ -coeven. We can therefore WLOG assume the opposite now. Thus, assume WLOG that  $O$  is not  $E$ -coeven. Hence, no element of  $O$  is  $E$ -coeven (due to the contrapositive of Lemma 4.5). In particular,  $\pi$  is not  $E$ -coeven (since  $\pi \in O$ ). In other words, not every  $g \in G$  satisfying  $g\pi = \pi$  is  $E$ -even. In other words, not every  $g \in \text{Stab}_G \pi$  is  $E$ -even (since the elements  $g \in G$  satisfying  $g\pi = \pi$  are exactly the elements  $g \in \text{Stab}_G \pi$ ). In other words, not every  $g \in \text{Stab}_G \pi$  satisfies  $\text{sign}_E g = 1$  (since  $g$  is  $E$ -even if and only if  $\text{sign}_E g = 1$ ).

Now, the map

$$\text{Stab}_G \pi \rightarrow \{1, -1\}, \quad g \mapsto \text{sign}_E g$$

is a group homomorphism (since the action of  $G$  on  $E$  is a group homomorphism  $G \rightarrow \text{Aut } E$ , and since the sign of a permutation is multiplicative) and is not the trivial homomorphism (since not every  $g \in \text{Stab}_G \pi$  satisfies  $\text{sign}_E g = 1$ ). Hence, it must send exactly half the elements of  $\text{Stab}_G \pi$  to 1 and the other half to  $-1$ . Therefore, the addends in the sum  $\sum_{g \in \text{Stab}_G \pi} \text{sign}_E g$  cancel each other out (one half of them are 1, and the others are  $-1$ ). Therefore,  $\sum_{g \in \text{Stab}_G \pi} \text{sign}_E g = 0$ . Now,

$$\frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \text{sign}_E g = \frac{1}{|G|} \underbrace{\sum_{g \in \text{Stab}_G \pi} \text{sign}_E g}_{=0} = \begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases}$$

(since  $O$  is not  $E$ -coeven). This proves (33). Lemma 7.7 (b) is thus proven.  $\square$

*Proof of Theorem 4.7 (sketched).* Let  $g \in G$ . Define the set  $E^g$ , the relations  $<_1^g$  and  $<_2^g$  and the triple  $\mathbf{E}^g$  as in Proposition 7.5. Thus,  $\mathbf{E}^g$  is a tertispecial double poset (by Proposition 7.5). In other words,  $(E^g, <_1^g, <_2^g)$  is a tertispecial double poset (since  $\mathbf{E}^g = (E^g, <_1^g, <_2^g)$ ).

Now, forget that we fixed  $g$ . We thus have constructed a tertispecial double poset  $\mathbf{E}^g = (E^g, <_1^g, <_2^g)$  for every  $g \in G$ .

Moreover, for every  $g \in G$ , let us define  $>_1^g$  to be the opposite relation of  $<_1^g$ .

Furthermore, for every  $g \in G$ , define a map  $w^g : E^g \rightarrow \{1, 2, 3, \dots\}$  by  $w^g(u) = \sum_{a \in u} w(a)$ . (Since  $G$  preserves  $w$ , the numbers  $w(a)$  for all  $a \in u$  are equal (for given  $u$ ), and thus  $\sum_{a \in u} w(a)$  can be rewritten as  $|u| \cdot w(b)$  for any particular  $b \in u$ . But we shall not use this observation.) Now, every  $g \in G$  satisfies

$$S(\Gamma((E^g, <_1^g, <_2^g), w^g)) = (-1)^{|E^g|} \Gamma((E^g, >_1^g, <_2^g), w^g). \tag{35}$$

(Indeed, this follows from Theorem 4.2 (applied to  $(E^g, <_1^g, <_2^g)$  and  $w^g$  instead of  $(E, <_1, <_2)$  and  $w$ ) since the double poset  $(E^g, <_1^g, <_2^g)$  is tertispecial.)

For every  $g \in G$ , we have

$$\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi, w} = \Gamma(\mathbf{E}^g, w^g) \tag{36}$$



39.

It is clearly sufficient to prove Theorem 4.7 for  $\mathbf{k} = \mathbb{Z}$  (since all the power series that we are discussing are defined functorially in  $\mathbf{k}$  (and so are the Hopf algebra QSym and its antipode  $S$ ), and thus any identity between these series that holds over  $\mathbb{Z}$  must hold over any  $\mathbf{k}$ ). Therefore, it is sufficient to prove Theorem 4.7 for  $\mathbf{k} = \mathbb{Q}$  (since  $\mathbb{Z}[[x_1, x_2, x_3, \dots]]$  embeds into  $\mathbb{Q}[[x_1, x_2, x_3, \dots]]$ , and using this embedding we have  $\text{QSym}_{\mathbb{Z}} = \text{QSym}_{\mathbb{Q}} \cap \mathbb{Z}[[x_1, x_2, x_3, \dots]]$ <sup>40</sup>). Thus, we WLOG assume that  $\mathbf{k} = \mathbb{Q}$ . This will allow us to divide by positive integers.

Every  $G$ -orbit  $O$  on  $\text{Par } \mathbf{E}$  satisfies

$$\frac{1}{|O|} \sum_{\pi \in O} \underbrace{\mathbf{x}_{\pi,w}}_{= \mathbf{x}_{O,w}} = \frac{1}{|O|} \sum_{\pi \in O} \underbrace{\mathbf{x}_{O,w}}_{= |O| \mathbf{x}_{O,w}} = \frac{1}{|O|} |O| \mathbf{x}_{O,w} = \mathbf{x}_{O,w}. \quad (38)$$

(since  $\mathbf{x}_{O,w}$  is defined to be  $\mathbf{x}_{\pi,w}$ )

<sup>39</sup>Proof of (36): Let  $g \in G$ . The definition of  $\Gamma(\mathbf{E}^g, w^g)$  yields

$$\Gamma(\mathbf{E}^g, w^g) = \sum_{\pi \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\pi, w^g} = \sum_{\bar{\pi} \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\bar{\pi}, w^g} \quad (37)$$

(here, we have renamed the summation index  $\pi$  as  $\bar{\pi}$ ).

In Proposition 7.6, we have introduced a bijection  $\Phi$  between

- the maps  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  satisfying  $g\pi = \pi$
- and
- the maps  $\bar{\pi} : E^g \rightarrow \{1, 2, 3, \dots\}$ .

Parts (a) and (b) of Proposition 7.6 show that this bijection  $\Phi$  restricts to a bijection between

- the  $\mathbf{E}$ -partitions  $\pi : E \rightarrow \{1, 2, 3, \dots\}$  satisfying  $g\pi = \pi$
- and
- the  $\mathbf{E}^g$ -partitions  $\bar{\pi} : E^g \rightarrow \{1, 2, 3, \dots\}$ .

Hence, we can substitute  $\Phi(\pi)$  for  $\bar{\pi}$  in the sum  $\sum_{\bar{\pi} \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\bar{\pi}, w^g}$ . We thus obtain

$$\sum_{\bar{\pi} \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\bar{\pi}, w^g} = \sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \underbrace{\mathbf{x}_{\Phi(\pi), w^g}}_{= \mathbf{x}_{\pi, w}} = \sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi, w},$$

(by Proposition 7.6 (c))

whence  $\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition;} \\ g\pi = \pi}} \mathbf{x}_{\pi, w} = \sum_{\bar{\pi} \text{ is an } \mathbf{E}^g\text{-partition}} \mathbf{x}_{\bar{\pi}, w^g} = \Gamma(\mathbf{E}^g, w^g)$  (by (37)). This proves (36).

<sup>40</sup>Here, we are using the notation  $\text{QSym}_{\mathbf{k}}$  for the Hopf algebra QSym defined over a base ring  $\mathbf{k}$ .

Now,

$$\begin{aligned}
 \Gamma(\mathbf{E}, w, G) &= \sum_{O \text{ is a } G\text{-orbit on } \text{Par } \mathbf{E}} \underbrace{\mathbf{x}_{O,w}}_{\substack{= \frac{1}{|O|} \sum_{\pi \in O} \mathbf{x}_{\pi,w} \\ \text{(by (38))}}} = \sum_{O \text{ is a } G\text{-orbit on } \text{Par } \mathbf{E}} \frac{1}{|O|} \sum_{\pi \in O} \mathbf{x}_{\pi,w} \\
 &= \sum_{O \text{ is a } G\text{-orbit on } \text{Par } \mathbf{E}} \sum_{\pi \in O} \underbrace{\frac{1}{|O|}}_{\substack{= \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} 1 \\ \text{(by (32), applied to } F = \text{Par } \mathbf{E})}} \mathbf{x}_{\pi,w} \\
 &= \underbrace{\sum_{O \text{ is a } G\text{-orbit on } \text{Par } \mathbf{E}} \sum_{\pi \in O}}_{\substack{= \sum_{\pi \in \text{Par } \mathbf{E}} = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}}} \left( \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} 1 \right) \mathbf{x}_{\pi,w} \\
 &= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \left( \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} 1 \right) \mathbf{x}_{\pi,w} = \frac{1}{|G|} \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \sum_{\substack{g \in G; \\ g\pi = \pi}} \mathbf{x}_{\pi,w}}_{\substack{= \sum_{g \in G} \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}; \\ g\pi = \pi}} \mathbf{x}_{\pi,w}} \\
 &= \frac{1}{|G|} \sum_{g \in G} \underbrace{\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition}; \\ g\pi = \pi}} \mathbf{x}_{\pi,w}}_{\substack{= \Gamma(\mathbf{E}^g, w^g) \\ \text{(by (36))}}} \\
 &= \frac{1}{|G|} \sum_{g \in G} \Gamma \left( \underbrace{\mathbf{E}^g}_{=(E^g, <_1^g, <_2^g)}, w^g \right) = \frac{1}{|G|} \sum_{g \in G} \Gamma \left( (E^g, <_1^g, <_2^g), w^g \right). \quad (39)
 \end{aligned}$$

Hence,  $\Gamma(\mathbf{E}, w, G) \in \text{QSym}$  (by Proposition 3.5).

Applying the map  $S$  to both sides of the equality (39), we obtain

$$\begin{aligned}
 S(\Gamma(\mathbf{E}, w, G)) &= \frac{1}{|G|} \sum_{g \in G} \underbrace{S(\Gamma((E^g, <_1^g, <_2^g), w^g))}_{\substack{= (-1)^{|E^g|} \Gamma((E^g, >_1^g, <_2^g), w^g) \\ \text{(by (35))}}} \\
 &= \frac{1}{|G|} \sum_{g \in G} (-1)^{|E^g|} \Gamma((E^g, >_1^g, <_2^g), w^g). \quad (40)
 \end{aligned}$$

On the other hand, for every  $g \in G$ , let  $\text{sign}_E g$  denote the sign of the permutation of  $E$  that sends every  $e \in E$  to  $ge$ . Thus,  $g \in G$  is  $E$ -even if and only if  $\text{sign}_E g = 1$ . Now, every  $G$ -orbit  $O$  on  $\text{Par } \mathbf{E}$  and every  $\pi \in O$  satisfy

$$\begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} = \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \text{sign}_E g \quad (41)$$

(by (33), applied to  $F = \text{Par } \mathbf{E}$ ). Furthermore,

$$\text{sign}_E g = (-1)^{|E| - |E^g|} \quad (42)$$

for every  $g \in G$  <sup>41</sup>.

---

<sup>41</sup>*Proof of (42):* Let  $g \in G$ . Recall that  $\text{sign}_E g$  is the sign of the permutation of  $E$  that sends every  $e \in E$  to  $ge$ . Denote this permutation by  $\zeta$ . Thus,  $\text{sign}_E g$  is the sign of  $\zeta$ .

The permutation  $\zeta$  is the permutation of  $E$  that sends every  $e \in E$  to  $ge$ . In other words,  $\zeta$  is the action of  $g$  on  $E$ . Hence, the cycles of  $\zeta$  are the  $g$ -orbits on  $E$ . Thus, the set of all cycles of  $\zeta$  is the set of all  $g$ -orbits on  $E$ ; this latter set is  $E^g$ . Hence,  $E^g$  is the set of all cycles of  $\zeta$ .

But if  $\sigma$  is a permutation of a finite set  $X$ , then the sign of  $\sigma$  is  $(-1)^{|X| - |X^\sigma|}$ , where  $X^\sigma$  is the set of all cycles of  $\sigma$ . Applying this to  $X = E$ ,  $\sigma = \zeta$  and  $X^\sigma = E^g$ , we see that the sign of  $\zeta$  is  $(-1)^{|E| - |E^g|}$  (because  $E^g$  is the set of all cycles of  $\zeta$ ). In other words,  $\text{sign}_E g = (-1)^{|E| - |E^g|}$  (since  $\text{sign}_E g$  is the sign of  $\zeta$ ), qed.

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Now,

$$\begin{aligned}
 & \Gamma^+(\mathbf{E}, w, G) \\
 &= \sum_{\substack{O \text{ is an } E\text{-coeven} \\ G\text{-orbit on } \text{Par } \mathbf{E}}} \underbrace{\mathbf{x}_{O,w}}_{\substack{= \frac{1}{|O|} \sum_{\pi \in O} \mathbf{x}_{\pi,w} \\ \text{(by (38))}}} = \sum_{\substack{O \text{ is an } E\text{-coeven} \\ G\text{-orbit on } \text{Par } \mathbf{E}}} \frac{1}{|O|} \sum_{\pi \in O} \mathbf{x}_{\pi,w} \\
 &= \sum_{\substack{O \text{ is a } G\text{-orbit on } \text{Par } \mathbf{E}}} \begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases} \sum_{\pi \in O} \mathbf{x}_{\pi,w} \\
 & \quad \left( \begin{array}{l} \text{here, we have extended the sum to all } G\text{-orbits on } \text{Par } \mathbf{E} \\ \text{(not just the } E\text{-coeven ones); but all new addends are } 0 \\ \text{and therefore do not influence the value of the sum} \end{array} \right) \\
 &= \sum_{\substack{O \text{ is a } G\text{-orbit on } \text{Par } \mathbf{E}}} \sum_{\pi \in O} \underbrace{\begin{cases} \frac{1}{|O|}, & \text{if } O \text{ is } E\text{-coeven;} \\ 0, & \text{if } O \text{ is not } E\text{-coeven} \end{cases}}_{\substack{= \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \text{sign}_E g \\ \text{(by (41))}}} \mathbf{x}_{\pi,w} \\
 &= \underbrace{\sum_{\substack{O \text{ is a } G\text{-orbit on } \text{Par } \mathbf{E}}} \sum_{\pi \in O} \left( \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \text{sign}_E g \right)}_{\substack{= \sum_{\pi \in \text{Par } \mathbf{E}} = \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}}} \mathbf{x}_{\pi,w} \\
 &= \sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \left( \frac{1}{|G|} \sum_{\substack{g \in G; \\ g\pi = \pi}} \text{sign}_E g \right) \mathbf{x}_{\pi,w} = \frac{1}{|G|} \underbrace{\sum_{\pi \text{ is an } \mathbf{E}\text{-partition}} \sum_{\substack{g \in G; \\ g\pi = \pi}} (\text{sign}_E g) \mathbf{x}_{\pi,w}}_{\substack{= \sum_{g \in G} \sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition}; \\ g\pi = \pi}}}} \\
 &= \frac{1}{|G|} \sum_{g \in G} \underbrace{\text{sign}_E g}_{\substack{= (-1)^{|E| - |E^g|} \\ \text{(by (42))}}} \underbrace{\sum_{\substack{\pi \text{ is an } \mathbf{E}\text{-partition}; \\ g\pi = \pi}} \mathbf{x}_{\pi,w}}_{\substack{= \Gamma(\mathbf{E}^g, w^g) \\ \text{(by (36))}}} = \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma \left( \underbrace{\mathbf{E}^g}_{\substack{= (E^g, \langle \cdot \rangle_1^g, \langle \cdot \rangle_2^g)}}, w^g \right) \\
 &= \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma \left( (E^g, \langle \cdot \rangle_1^g, \langle \cdot \rangle_2^g), w^g \right). \tag{43}
 \end{aligned}$$

Hence,  $\Gamma^+(\mathbf{E}, w, G) \in \text{QSym}$  (by Proposition 3.5).

The group  $G$  preserves the relation  $>_1$  (since it preserves the relation  $<_1$ ). Furthermore, the double poset  $(E, >_1, <_2)$  is tertispecial<sup>42</sup>. Hence, we can apply (43) to  $(E, >_1, <_2)$ ,  $>_1$  and  $>_1^g$  instead of  $\mathbf{E}$ ,  $<_1$  and  $<_1^g$ . As a result, we obtain

$$\Gamma^+((E, >_1, <_2), w, G) = \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma((E^g, >_1^g, <_2^g), w^g).$$

Multiplying both sides of this equality by  $(-1)^{|E|}$ , we transform it into

$$\begin{aligned} (-1)^{|E|} \Gamma^+((E, >_1, <_2), w, G) &= (-1)^{|E|} \frac{1}{|G|} \sum_{g \in G} (-1)^{|E| - |E^g|} \Gamma((E^g, >_1^g, <_2^g), w^g) \\ &= \frac{1}{|G|} \sum_{g \in G} \underbrace{(-1)^{|E|} (-1)^{|E| - |E^g|}}_{=(-1)^{|E^g|}} \Gamma((E^g, >_1^g, <_2^g), w^g) \\ &= \frac{1}{|G|} \sum_{g \in G} (-1)^{|E^g|} \Gamma((E^g, >_1^g, <_2^g), w^g) \\ &= S(\Gamma(\mathbf{E}, w, G)) \quad (\text{by (40)}). \end{aligned}$$

This completes the proof of Theorem 4.7. □

## 8. Application: Jochemko's theorem

We shall now demonstrate an application of Theorem 4.7: namely, we will use it to provide an alternative proof of [Joch13, Theorem 2.13]. The way we derive [Joch13, Theorem 2.13] from Theorem 4.7 is classical, and in fact was what originally motivated the discovery of Theorem 4.7 (although, of course, it cannot be conversely derived from [Joch13, Theorem 2.13], so it is an actual generalization).

An intermediate step between [Joch13, Theorem 2.13] and Theorem 4.7 will be the following fact:

**Corollary 8.1.** Let  $\mathbf{E} = (E, <_1, <_2)$  be a tertispecial double poset. Let  $w : E \rightarrow \{1, 2, 3, \dots\}$ . Let  $G$  be a finite group which acts on  $E$ . Assume that  $G$  preserves both relations  $<_1$  and  $<_2$ , and also preserves  $w$ . For every  $q \in \mathbb{N}$ , let  $\text{Par}_q \mathbf{E}$  denote the set of all  $\mathbf{E}$ -partitions whose image is contained in  $\{1, 2, \dots, q\}$ . Then, the group  $G$  also acts on  $\text{Par}_q \mathbf{E}$ ; namely,  $\text{Par}_q \mathbf{E}$  is a  $G$ -subset of the  $G$ -set  $\{1, 2, \dots, q\}^E$  (see Definition 4.4 (d) for the definition of the latter).

- (a) There exists a unique polynomial  $\Omega_{\mathbf{E}, G} \in \mathbb{Q}[X]$  such that every  $q \in \mathbb{N}$  satisfies

$$\Omega_{\mathbf{E}, G}(q) = (\text{the number of all } G\text{-orbits on } \text{Par}_q \mathbf{E}). \quad (44)$$

<sup>42</sup>This can be easily derived from the fact that  $(E, <_1, <_2)$  is tertispecial. (Observe that an  $a \in E$  is  $>_1$ -covered by a  $b \in E$  if and only if  $b$  is  $<_1$ -covered by  $a$ .)

(b) This polynomial satisfies

$$\begin{aligned} \Omega_{\mathbf{E},G}(-q) &= (-1)^{|\mathbf{E}|} (\text{the number of all even } G\text{-orbits on } \text{Par}_q(\mathbf{E}, >_1, <_2)) \\ &= (-1)^{|\mathbf{E}|} (\text{the number of all even } G\text{-orbits on } \text{Par}_q(\mathbf{E}, <_1, >_2)) \end{aligned} \quad (45)$$

for all  $q \in \mathbb{N}$ .

*Proof of Corollary 8.1 (sketched).* Set  $\mathbf{k} = \mathbb{Q}$ . For any  $f \in \text{QSym}$  and any  $q \in \mathbb{N}$ , we define an element  $\text{ps}^1(f)(q) \in \mathbb{Q}$  by

$$\text{ps}^1(f)(q) = f \left( \underbrace{1, 1, \dots, 1}_{q \text{ times}}, 0, 0, 0, \dots \right)$$

(that is,  $\text{ps}^1(f)(q)$  is the result of substituting 1 for each of the variables  $x_1, x_2, \dots, x_q$  and 0 for each of the variables  $x_{q+1}, x_{q+2}, x_{q+3}, \dots$  in the power series  $f$ ).

(a) Consider the elements  $\Gamma(\mathbf{E}, w, G)$  and  $\Gamma^+(\mathbf{E}, w, G)$  of QSym defined in Theorem 4.7. Observe that  $\text{Par}_q \mathbf{E}$  is a  $G$ -subset of  $\text{Par} \mathbf{E}$ .

Now, [GriRei14, Proposition 7.7 (i)] shows that, for any given  $f \in \text{QSym}$ , there exists a unique polynomial in  $\mathbb{Q}[X]$  whose value on each  $q \in \mathbb{N}$  equals  $\text{ps}^1(f)(q)$ . Applying this to  $f = \Gamma(\mathbf{E}, w, G)$ , we conclude that there exists a unique polynomial in  $\mathbb{Q}[X]$  whose value on each  $q \in \mathbb{N}$  equals  $\text{ps}^1(\Gamma(\mathbf{E}, w, G))(q)$ . But since every  $q \in \mathbb{N}$  satisfies

$$\begin{aligned} \text{ps}^1(\Gamma(\mathbf{E}, w, G))(q) &= \underbrace{(\Gamma(\mathbf{E}, w, G))}_{\sum_{O \text{ is a } G\text{-orbit on } \text{Par} \mathbf{E}} \mathbf{x}_{O,w}} \left( \underbrace{1, 1, \dots, 1}_{q \text{ times}}, 0, 0, 0, \dots \right) \\ &= \sum_{O \text{ is a } G\text{-orbit on } \text{Par} \mathbf{E}} \underbrace{\mathbf{x}_{O,w} \left( \underbrace{1, 1, \dots, 1}_{q \text{ times}}, 0, 0, 0, \dots \right)}_{= \begin{cases} 1, & \text{if } O \subseteq \text{Par}_q \mathbf{E}; \\ 0, & \text{if } O \not\subseteq \text{Par}_q \mathbf{E} \end{cases}} \\ &= \sum_{O \text{ is a } G\text{-orbit on } \text{Par} \mathbf{E}} \begin{cases} 1, & \text{if } O \subseteq \text{Par}_q \mathbf{E}; \\ 0, & \text{if } O \not\subseteq \text{Par}_q \mathbf{E} \end{cases} \\ &= \sum_{O \text{ is a } G\text{-orbit on } \text{Par}_q \mathbf{E}} 1 = (\text{the number of all } G\text{-orbits on } \text{Par}_q \mathbf{E}), \end{aligned} \quad (46)$$

this rewrites as follows: There exists a unique polynomial in  $\mathbb{Q}[X]$  whose value on each  $q \in \mathbb{N}$  equals (the number of all  $G$ -orbits on  $\text{Par}_q \mathbf{E}$ ). This proves Corollary 8.1 (a).

(b) [GriRei14, Proposition 7.7 (i)] shows that, for any given  $f \in \text{QSym}$ , there exists a unique polynomial in  $\mathbb{Q}[X]$  whose value on each  $q \in \mathbb{N}$  equals  $\text{ps}^1(f)(q)$ . This polynomial is denoted by  $\text{ps}^1(f)$  in [GriRei14, Proposition 7.7]. From our above proof of Corollary 8.1 (a), we see that

$$\Omega_{\mathbf{E},G} = \text{ps}^1(\Gamma(\mathbf{E}, w, G)).$$

But [GriRei14, Proposition 7.7 (iii)] shows that, for any  $f \in \text{QSym}$  and  $m \in \mathbb{N}$ , we have  $\text{ps}^1(S(f))(m) = \text{ps}^1(f)(-m)$ . Applying this to  $f = \Gamma(\mathbf{E}, w, G)$ , we obtain

$$\text{ps}^1(S(\Gamma(\mathbf{E}, w, G)))(m) = \underbrace{\text{ps}^1(\Gamma(\mathbf{E}, w, G))}_{=\Omega_{\mathbf{E},G}}(-m) = \Omega_{\mathbf{E},G}(-m)$$

for any  $m \in \mathbb{N}$ . Thus, any  $m \in \mathbb{N}$  satisfies

$$\begin{aligned} \Omega_{\mathbf{E},G}(-m) &= \text{ps}^1 \left( \begin{array}{c} S(\Gamma(\mathbf{E}, w, G)) \\ =(-1)^{|\mathbf{E}|} \Gamma^+((E, >_1, <_2), w, G) \\ \text{(by Theorem 4.7)} \end{array} \right) (m) \\ &= \text{ps}^1 \left( (-1)^{|\mathbf{E}|} \Gamma^+((E, >_1, <_2), w, G) \right) (m) \\ &= (-1)^{|\mathbf{E}|} \text{ps}^1(\Gamma^+((E, >_1, <_2), w, G))(m). \end{aligned}$$

Renaming  $m$  as  $q$  in this equality, we see that every  $q \in \mathbb{N}$  satisfies

$$\Omega_{\mathbf{E},G}(-q) = (-1)^{|\mathbf{E}|} \text{ps}^1(\Gamma^+((E, >_1, <_2), w, G))(q). \quad (47)$$

But just as we proved (46), we can show that every  $q \in \mathbb{N}$  satisfies

$$\text{ps}^1(\Gamma^+(\mathbf{E}, w, G))(q) = (\text{the number of all even } G\text{-orbits on } \text{Par}_q \mathbf{E}).$$

Applying this to  $(E, >_1, <_2)$  instead of  $\mathbf{E}$ , we obtain

$$\begin{aligned} &\text{ps}^1(\Gamma^+((E, >_1, <_2), w, G))(q) \\ &= (\text{the number of all even } G\text{-orbits on } \text{Par}_q(E, >_1, <_2)). \end{aligned}$$

Now, (47) becomes

$$\begin{aligned} \Omega_{\mathbf{E},G}(-q) &= (-1)^{|\mathbf{E}|} \underbrace{\text{ps}^1(\Gamma^+((E, >_1, <_2), w, G))(q)}_{=(\text{the number of all even } G\text{-orbits on } \text{Par}_q(E, >_1, <_2))} \\ &= (-1)^{|\mathbf{E}|} (\text{the number of all even } G\text{-orbits on } \text{Par}_q(E, >_1, <_2)). \end{aligned}$$

In order to prove Corollary 8.1 (b), it thus remains to show that

$$\begin{aligned} & (\text{the number of all even } G\text{-orbits on } \text{Par}_q(E, >_1, <_2)) \\ &= (\text{the number of all even } G\text{-orbits on } \text{Par}_q(E, <_1, >_2)) \end{aligned} \quad (48)$$

for every  $q \in \mathbb{N}$ .

*Proof of (48):* Let  $q \in \mathbb{N}$ . Let  $w_0 : \{1, 2, \dots, q\} \rightarrow \{1, 2, \dots, q\}$  be the map sending each  $i \in \{1, 2, \dots, q\}$  to  $q + 1 - i$ . Then, the map

$$\text{Par}_q(E, >_1, <_2) \rightarrow \text{Par}_q(E, <_1, >_2), \quad \pi \mapsto w_0 \circ \pi$$

is an isomorphism of  $G$ -sets (this is easy to check). Thus,  $\text{Par}_q(E, >_1, <_2) \cong \text{Par}_q(E, <_1, >_2)$  as  $G$ -sets. From this, (48) follows (by functoriality, if one wishes).

The proof of Corollary 8.1 (b) is now complete.  $\square$

Now, the second formula of [Joch13, Theorem 2.13] follows from our (45), applied to  $\mathbf{E} = (P, <, <_\omega)$  (where  $<_\omega$  is the partial order on  $P$  given by  $(p <_\omega q) \iff (\omega(p) < \omega(q))$ ). The first formula of [Joch13, Theorem 2.13] can also be derived from our above arguments. We leave the details to the reader.

## 9. A final question

With the results proven above (specifically, Theorems 4.2 and 4.7), we have obtained formulas for a large class of quasisymmetric generating functions for maps from a double poset to  $\{1, 2, 3, \dots\}$ . At least one question arises:

**Question 9.1.** In [Grin16a], I have studied generalizations of Whitney's famous non-broken-circuit theorem for graphs and matroids. One of the cornerstones of that study is the bijection  $\Phi$  in [Grin16a, proofs of Lemma 2.7 and Lemma 5.25], which is uncannily reminiscent of the involution  $T$  in the proof of Theorem 4.2. (Actually, this bijection  $\Phi$  can be extended to an involution, thus making the analogy even more palpable.) Both  $\Phi$  and  $T$  are defined by toggling a certain element in or out of a subset; and this element is chosen as the argmin or argmax of a function defined on the ground set. Is there a connection between the two results, or even a common generalization?

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