

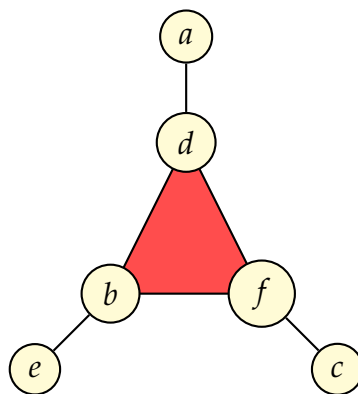
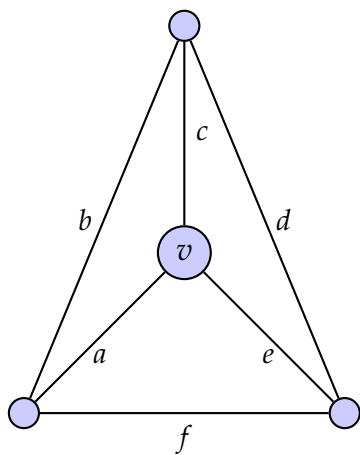
The Elser nuclei sum revisited

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Abstract. Fix a finite undirected graph Γ and a vertex v of Γ . Let E be the set of edges of Γ . We call a subset F of E *pandemic* if each edge of Γ has at least one endpoint that can be connected to v by an F -path (i.e., a path using edges from F only). In 1984, Elser showed that the sum of $(-1)^{|F|}$ over all pandemic subsets F of E is 0 if $E \neq \emptyset$. We give a simple proof of this result via a sign-reversing involution, and discuss variants, generalizations and refinements, revealing connections to abstract convexity (the notion of an antimatroid) and discrete Morse theory.

Keywords: graph theory, nuclei, simplicial complex, discrete Morse theory, alternating sum, enumerative combinatorics, inclusion/exclusion, convexity, antimatroids.

Future versions of this text will be available from the first author's website:
<http://www.cip.ifi.lmu.de/~grinberg/algebra/elsersum.pdf>
<http://www.cip.ifi.lmu.de/~grinberg/algebra/elsersum-long.pdf> (detailed version).

Typeset with L^AT_EX.

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In [Elser84], Veit Elser studied the probabilities of clusters forming when n points are sampled randomly in a d -dimensional volume. In the process, he found a purely graph-theoretical lemma [Elser84, Lemma 1], which served a crucial role in his work. For decades, the lemma stayed hidden from the eyes of combinatorialists in a physics journal, until it resurfaced in recent work [DHLetc19] by Dorpalen-Barry, Hettle, Livingston, Martin, Nasr, Vega and Whitlatch. In this note, I will show a simpler proof of the lemma using a sign-reversing involution. The proof also suggests multiple venues of generalization that I will explore in the later sections; one extends the lemma to a statement about arbitrary antimatroids (and even a wider setting). Finally, I will strengthen the lemma to a Morse-theoretical result, stating the collapsibility of a certain simplicial complex. Some open questions will be posed.

Note to the reader

The pictures on the title page illustrate the simplicial complex \mathcal{A} from Proposition 5.2 on an example. The left picture is a graph Γ (with the vertex labelled v playing the role of v), whereas the right picture shows the corresponding simplicial complex \mathcal{A} for $G = E$ (that is, the simplicial complex whose faces are the subsets of E that are not pandemic).

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Remark on alternative versions

You are reading the detailed version of this paper. For the standard version (which is shorter by virtue of omitting some details in proofs), see [Grinbe20].

1. Elser’s result

Let us first introduce our setting, which is slightly more general (and perhaps also simpler) than that used in [Elser84]. (In Section 4, we will move to a more general setup.)

We fix an arbitrary graph Γ with vertex set V and edge set E . Here, “graph” means “finite undirected multigraph” – i.e., it can have self-loops and parallel edges, but it has finitely many vertices and edges, and its edges are undirected.

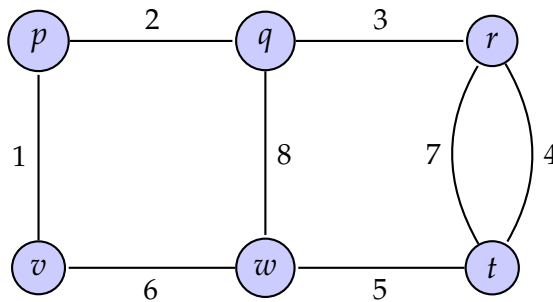
We fix a vertex $v \in V$.

If $F \subseteq E$, then an F -path shall mean a path of Γ such that all edges of the path belong to F .

If $e \in E$ is any edge and $F \subseteq E$ is any subset, then we say that F infects e if there exists an F -path from v to some endpoint of e . (The terminology is inspired by the idea of an infectious disease starting in the vertex v and being transmitted along edges.)¹

A subset $F \subseteq E$ is said to be *pandemic* if it infects each edge $e \in E$.

Example 1.1. Let Γ be the following graph:



(where the vertex v is the vertex labelled v). Then, for example, the set $\{1,2\} \subseteq E$ infects edges 1,2,3,6,8 (but none of the other edges). The set $\{1,2,5\}$ infects the same edges as $\{1,2\}$ (indeed, the additional edge 5 does not increase its infectiousness, since it is not on any $\{1,2,5\}$ -path from v). The set $\{1,2,3\}$ infects every edge other than 5. The set $\{1,2,3,4\}$ infects each edge, and thus is pandemic.

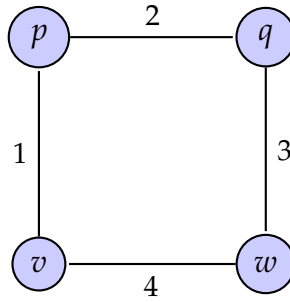
Now, we can state our version of [Elser84, Lemma 1]:

Theorem 1.2. Assume that $E \neq \emptyset$. Then,

$$\sum_{\substack{F \subseteq E \text{ is} \\ \text{pandemic}}} (-1)^{|F|} = 0. \tag{1}$$

¹Note that if an edge e contains the vertex v , then any subset F of E (even the empty one) infects e , since there is a trivial (edgeless) F -path from v to v .

Example 1.3. Let Γ be the following graph:



(where the vertex v is the vertex labelled v). Then, the pandemic subsets of E are the sets

$$\{1,2\}, \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, \{1,2,4\}, \{2,3,4\}, \{1,2,3,4\}.$$

The sizes of these subsets are 2,2,2,3,3,3,3,4, respectively. Hence, (1) says that

$$(-1)^2 + (-1)^2 + (-1)^2 + (-1)^3 + (-1)^3 + (-1)^3 + (-1)^3 + (-1)^4 = 0.$$

We note that the equality (1) can be restated as “there are equally many pandemic subsets $F \subseteq E$ of even size and pandemic subsets $F \subseteq E$ of odd size”. Thus, in particular, the number of all pandemic subsets F of E is even (when $E \neq \emptyset$).

Remark 1.4. Theorem 1.2 is a bit more general than [Elser84, Lemma 1]. To see why, we assume that the graph Γ is connected and simple (i.e., has no self-loops and parallel edges). Then, a *nucleus* is defined in [Elser84] as a subgraph N of Γ with the properties that

1. the subgraph N is connected, and
2. each edge of Γ has at least one endpoint in N .

Given a subgraph N of Γ , we let $E(N)$ denote the set of all edges of N . Now, [Elser84, Lemma 1] claims that if $E \neq \emptyset$, then

$$\sum_{\substack{N \text{ is a nucleus} \\ \text{containing } v}} (-1)^{|E(N)|} = 0.$$

But this is equivalent to (1), because there is a bijection

$$\{\text{nuclei containing } v\} \rightarrow \{\text{pandemic subsets } F \subseteq E\}, \\ N \mapsto E(N).$$

We leave it to the reader to check this in detail; what needs to be checked are the following three statements:

- If N is a nucleus containing v , then $E(N)$ is a pandemic subset of E .
- Every nucleus N containing v is uniquely determined by the set $E(N)$. (Indeed, since a nucleus has to be connected, each of its vertices must be an endpoint of one of its edges, unless its only vertex is v .)
- If F is a pandemic subset of E , then there is a nucleus N containing v such that $E(N) = F$. (Indeed, N can be defined as the subgraph of Γ whose vertices are the endpoints of all edges in F as well as the vertex v , and whose edges are the edges in F . To see that this subgraph N is connected, it suffices to argue that each of its vertices has a path to v ; but this follows from the definition of “pandemic”, since each vertex of N other than v belongs to at least one edge in F .)

Thus, Theorem 1.2 is equivalent to [Elser84, Lemma 1] in the case when Γ is connected and simple.

Remark 1.5. It might appear more natural to talk about a subset $F \subseteq E$ infecting a vertex rather than an edge. (Namely, we can say that F infects a vertex w if there is an F -path from v to w .) However, the analogue of Theorem 1.2 in which pandemicity is defined via infecting all vertices is not true. The graph of Example 1.3 provides a counterexample.

2. The proof

2.1. Shades

Our proof of Theorem 1.2 will rest on a few notions. The first is that of a *shade*:

Definition 2.1. Let F be a subset of E . Then, we define a subset $\text{Shade } F$ of E by

$$\text{Shade } F = \{e \in E \mid F \text{ infects } e\}. \quad (2)$$

We refer to $\text{Shade } F$ as the *shade* of F .

Thus, the shade of a subset $F \subseteq E$ is the set of all edges of Γ that are infected by F . (In more standard graph-theoretical lingo, this means that $\text{Shade } F$ is the set of edges that contain at least one vertex of the connected component containing v of the graph (V, F) .)

Example 2.2. In Example 1.1, we have $\text{Shade } \{1,2\} = \{1,2,3,6,8\}$ and $\text{Shade } \{1\} = \{1,2,6\}$ and $\text{Shade } \{8\} = \{1,6\}$.

The following property of shades is rather obvious:

Lemma 2.3. Let A and B be two subsets of E such that $A \subseteq B$. Then, $\text{Shade } A \subseteq \text{Shade } B$.

Proof of Lemma 2.3. Let $q \in \text{Shade } A$. Thus, $q \in \text{Shade } A = \{e \in E \mid A \text{ infects } e\}$ (by the definition of $\text{Shade } A$). In other words, q is an $e \in E$ such that A infects e . In other words, q is an element of E with the property that A infects q .

We know that A infects q . In other words, there exists a A -path from v to some endpoint of q (by the definition of “infects”). Hence, there exists a B -path from v to some endpoint of q (since any A -path is automatically a B -path²). In other words, B infects q (by the definition of “infects”). Thus, q is an $e \in E$ such that B infects e (since q is an element of E). In other words, $q \in \{e \in E \mid B \text{ infects } e\}$. In other words, $q \in \text{Shade } B$ (since the definition of $\text{Shade } B$ yields $\text{Shade } B = \{e \in E \mid B \text{ infects } e\}$).

Forget that we fixed q . We thus have proved that $q \in \text{Shade } B$ for each $q \in \text{Shade } A$. In other words, $\text{Shade } A \subseteq \text{Shade } B$. This proves Lemma 2.3. \square

The major property of shades that we will need is the following:

Lemma 2.4. Let F be a subset of E . Let $u \in E$ be such that $u \notin \text{Shade } F$. Then,

$$\text{Shade } (F \cup \{u\}) = \text{Shade } F \quad (3)$$

and

$$\text{Shade } (F \setminus \{u\}) = \text{Shade } F. \quad (4)$$

Proof of Lemma 2.4. We shall prove (3) and (4) separately:

[*Proof of (3):* Note that $F \cup \{u\}$ is a subset of E (since F is a subset of E , and since $u \in E$). Thus, $\text{Shade } (F \cup \{u\})$ is well-defined.

Let $q \in \text{Shade } (F \cup \{u\})$. We shall show that $q \in \text{Shade } F$.

We have $q \in \text{Shade } (F \cup \{u\}) = \{e \in E \mid F \cup \{u\} \text{ infects } e\}$ (by the definition of $\text{Shade } (F \cup \{u\})$). In other words, q is an $e \in E$ such that $F \cup \{u\}$ infects e . In other words, q is an element of E with the property that $F \cup \{u\}$ infects q .

We shall now show that F infects q . Indeed, assume the contrary. Thus, F does not infect q . In other words, there exists no F -path from v to any endpoint of q (by the definition of “infects”).

We know that $F \cup \{u\}$ infects q . In other words, there exists an $(F \cup \{u\})$ -path from v to some endpoint of q (by the definition of “infects”). Let π be this $(F \cup \{u\})$ -path. If this $(F \cup \{u\})$ -path π did not contain the edge u , then it

²because $A \subseteq B$

would be an F -path, which would contradict the fact that there exists no F -path from v to any endpoint of q . Hence, this $(F \cup \{u\})$ -path π must contain the edge u . By removing u , we can thus cut this path π into two segments: The first segment is a path from v to some endpoint of u , while the second segment is a path from the other endpoint of u to some endpoint of q . Both segments are F -paths (since they arise by removing u from an $(F \cup \{u\})$ -path). Thus, in particular, the first segment is an F -path from v to some endpoint of u . Hence, there exists an F -path from v to some endpoint of u . In other words, F infects u (by the definition of “infects”). Hence, u is an $e \in E$ such that F infects e (since $u \in E$). In other words, $u \in \{e \in E \mid F \text{ infects } e\}$. This can be rewritten as $u \in \text{Shade } F$ (because of (2)). This contradicts $u \notin \text{Shade } F$.

This contradiction shows that our assumption was false. Hence, we have proved that F infects q . Thus, q is an $e \in E$ such that F infects e (since $u \in E$). In other words, $q \in \{e \in E \mid F \text{ infects } e\}$. This can be rewritten as $q \in \text{Shade } F$ (because of (2)).

Forget that we fixed q . We thus have shown that $q \in \text{Shade } F$ for each $q \in \text{Shade}(F \cup \{u\})$. In other words, $\text{Shade}(F \cup \{u\}) \subseteq \text{Shade } F$. On the other hand, $F \subseteq F \cup \{u\}$; therefore, $\text{Shade } F \subseteq \text{Shade}(F \cup \{u\})$ (by Lemma 2.3, applied to $A = F$ and $B = F \cup \{u\}$). Combining this with $\text{Shade}(F \cup \{u\}) \subseteq \text{Shade } F$, we obtain $\text{Shade}(F \cup \{u\}) = \text{Shade } F$. This proves (3).]

[*Proof of (4):* Note that $F \setminus \{u\}$ is a subset of E (since $F \setminus \{u\} \subseteq F \subseteq E$). Thus, $\text{Shade}(F \setminus \{u\})$ is well-defined.

We must prove that $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$. This is obvious if $F \setminus \{u\} = F$. Thus, for the rest of this proof, we WLOG assume that $F \setminus \{u\} \neq F$. Hence, $u \in F$ (since otherwise, we would have $u \notin F$ and thus $F \setminus \{u\} = F$, which would contradict $F \setminus \{u\} \neq F$). Thus, $(F \setminus \{u\}) \cup \{u\} = F$.

We have $F \setminus \{u\} \subseteq F$ and thus $\text{Shade}(F \setminus \{u\}) \subseteq \text{Shade } F$ (by Lemma 2.3, applied to $A = F \setminus \{u\}$ and $B = F$). Hence, from $u \notin \text{Shade } F$, we obtain $u \notin \text{Shade}(F \setminus \{u\})$ (because otherwise, we would have $u \in \text{Shade}(F \setminus \{u\}) \subseteq \text{Shade } F$, which would contradict $u \notin \text{Shade } F$). Therefore, (3) (applied to $F \setminus \{u\}$ instead of F) yields $\text{Shade}((F \setminus \{u\}) \cup \{u\}) = \text{Shade}(F \setminus \{u\})$. Thus, $\text{Shade}(F \setminus \{u\}) = \text{Shade}(\underbrace{(F \setminus \{u\}) \cup \{u\}}_{=F}) = \text{Shade } F$. This proves (4).]

We have now proved both (3) and (4). Thus, Lemma 2.4 is proved. \square

2.2. A slightly more general claim

Lemma 2.4 might not look very powerful, but it contains all we need to prove Theorem 1.2. Better yet, we shall prove the following slightly more general version of Theorem 1.2:

Theorem 2.5. Let G be any subset of E . Assume that $E \neq \emptyset$. Then,

$$\sum_{\substack{F \subseteq E; \\ G \subseteq \text{Shade } F}} (-1)^{|F|} = 0.$$

We will soon prove Theorem 2.5 and explain how Theorem 1.2 follows from it. First, however, let us give an equivalent (but slightly easier to prove) version of Theorem 2.5:

Theorem 2.6. Let G be any subset of E . Then,

$$\sum_{\substack{F \subseteq E; \\ G \not\subseteq \text{Shade } F}} (-1)^{|F|} = 0.$$

Proof of Theorem 2.6. Let

$$\mathcal{A} = \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}. \quad (5)$$

Thus, \mathcal{A} is a subset of the power set of E .

We equip the finite set E with a total order (chosen arbitrarily, but fixed henceforth). If $F \in \mathcal{A}$, then there exists a unique **smallest** edge $e \in G \setminus \text{Shade } F$ ³. This unique smallest edge e will be denoted by $\varepsilon(F)$.

We note that the edge $\varepsilon(F)$ (for a set $F \in \mathcal{A}$) depends only on $\text{Shade } F$, but not on F itself (because it was defined as the smallest edge $e \in G \setminus \text{Shade } F$). Thus, if two sets $F_1 \in \mathcal{A}$ and $F_2 \in \mathcal{A}$ satisfy $\text{Shade}(F_1) = \text{Shade}(F_2)$, then

$$\varepsilon(F_1) = \varepsilon(F_2). \quad (6)$$

We also notice the following simple fact: If F and F' are two subsets of E such that $F \in \mathcal{A}$ and $\text{Shade}(F') = \text{Shade } F$, then

$$F' \in \mathcal{A} \quad (7)$$

⁴

We now define two subsets \mathcal{A}_+ and \mathcal{A}_- of \mathcal{A} by

$$\mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\} \quad \text{and} \quad \mathcal{A}_- = \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}.$$

³*Proof.* Let $F \in \mathcal{A}$. Thus, $F \in \mathcal{A} = \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}$. In other words, F is a $P \subseteq E$ satisfying $G \not\subseteq \text{Shade } P$. In other words, F is a subset of E and satisfies $G \not\subseteq \text{Shade } F$. From $G \not\subseteq \text{Shade } F$, we see that $G \setminus \text{Shade } F \neq \emptyset$. In other words, there exists at least one edge $e \in G \setminus \text{Shade } F$. Hence, there exists a unique **smallest** edge $e \in G \setminus \text{Shade } F$ (because the set $G \setminus \text{Shade } F$ is finite (being a subset of E) and totally ordered (being a subset of E)). Qed.

⁴*Proof of (7):* Let F and F' be two subsets of E such that $F \in \mathcal{A}$ and $\text{Shade}(F') = \text{Shade } F$. We have $F \in \mathcal{A} = \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}$. In other words, F is a $P \subseteq E$ satisfying $G \not\subseteq \text{Shade } P$. In other words, F is a subset of E and satisfies $G \not\subseteq \text{Shade } F$. Thus, $G \not\subseteq \text{Shade } F = \text{Shade}(F')$ (since $\text{Shade}(F') = \text{Shade } F$). Now, F' is a subset of E and satisfies $G \not\subseteq \text{Shade}(F')$. In other words, F' is a $P \subseteq E$ satisfying $G \not\subseteq \text{Shade } P$. In other words, $F' \in \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}$. This can be rewritten as $F' \in \mathcal{A}$ (because of (5)). This proves (7).

Each $F \in \mathcal{A}_+$ satisfies $F \setminus \{\varepsilon(F)\} \in \mathcal{A}_-$ ⁵. Thus, we can define a map

$$\begin{aligned} \Phi : \mathcal{A}_+ &\rightarrow \mathcal{A}_-, \\ F &\mapsto F \setminus \{\varepsilon(F)\}. \end{aligned}$$

Each $F \in \mathcal{A}_-$ satisfies $F \cup \{\varepsilon(F)\} \in \mathcal{A}_+$ ⁶. Thus, we can define a map

$$\begin{aligned} \Psi : \mathcal{A}_- &\rightarrow \mathcal{A}_+, \\ F &\mapsto F \cup \{\varepsilon(F)\}. \end{aligned}$$

⁵*Proof.* Let $F \in \mathcal{A}_+$. We must show that $F \setminus \{\varepsilon(F)\} \in \mathcal{A}_-$.

Indeed, let us set $F' = F \setminus \{\varepsilon(F)\}$. Then, $F \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\} \subseteq \mathcal{A} = \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}$. In other words, F is a $P \subseteq E$ satisfying $G \not\subseteq \text{Shade } P$. Hence, F is a subset of E . Thus, F' is a subset of E (since $F' = F \setminus \{\varepsilon(F)\} \subseteq F \subseteq E$).

Recall that $\varepsilon(F)$ is the smallest edge $e \in G \setminus \text{Shade } F$ (by the definition of $\varepsilon(F)$). Hence, $\varepsilon(F) \in G \setminus \text{Shade } F$. In other words, $\varepsilon(F) \in G$ and $\varepsilon(F) \notin \text{Shade } F$. Thus, $\varepsilon(F) \in G \subseteq E$ and $\varepsilon(F) \notin \text{Shade } F$. Therefore, (4) (applied to $u = \varepsilon(F)$) yields $\text{Shade}(F \setminus \{\varepsilon(F)\}) = \text{Shade } F$. This can be rewritten as $\text{Shade}(F') = \text{Shade } F$ (since $F' = F \setminus \{\varepsilon(F)\}$). Hence, (7) yields $F' \in \mathcal{A}$. Therefore, (6) (applied to F' and F instead of F_1 and F_2) yields $\varepsilon(F') = \varepsilon(F)$ (since $F \in \mathcal{A}$ and $F' \in \mathcal{A}$ and $\text{Shade}(F') = \text{Shade } F$). However, $\varepsilon(F) \notin F'$ (since $F' = F \setminus \{\varepsilon(F)\}$). In other words, $\varepsilon(F') \notin F'$ (since $\varepsilon(F') = \varepsilon(F)$). Hence, F' is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \notin P$ (since $F' \in \mathcal{A}$ and $\varepsilon(F') \notin F'$). In other words, $F' \in \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}$. In other words, $F' \in \mathcal{A}_-$ (since $\mathcal{A}_- = \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}$). In other words, $F \setminus \{\varepsilon(F)\} \in \mathcal{A}_-$ (since $F' = F \setminus \{\varepsilon(F)\}$). Qed.

⁶*Proof.* Let $F \in \mathcal{A}_-$. We must show that $F \cup \{\varepsilon(F)\} \in \mathcal{A}_+$.

Indeed, let us set $F' = F \cup \{\varepsilon(F)\}$. Recall that $\varepsilon(F)$ is the smallest edge $e \in G \setminus \text{Shade } F$ (by the definition of $\varepsilon(F)$). Hence, $\varepsilon(F) \in G \setminus \text{Shade } F$. In other words, $\varepsilon(F) \in G$ and $\varepsilon(F) \notin \text{Shade } F$. Thus, $\varepsilon(F) \in G \subseteq E$ and $\varepsilon(F) \notin \text{Shade } F$.

We have $F \in \mathcal{A}_- = \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\} \subseteq \mathcal{A} = \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}$. In other words, F is a $P \subseteq E$ satisfying $G \not\subseteq \text{Shade } P$. Thus, F is a subset of E . Therefore, $F \cup \{\varepsilon(F)\}$ is a subset of E as well (since $\varepsilon(F) \in E$). In other words, F' is a subset of E (since $F' = F \cup \{\varepsilon(F)\}$).

Moreover, (3) (applied to $u = \varepsilon(F)$) yields $\text{Shade}(F \cup \{\varepsilon(F)\}) = \text{Shade } F$ (since $\varepsilon(F) \notin \text{Shade } F$). This can be rewritten as $\text{Shade}(F') = \text{Shade } F$ (since $F' = F \cup \{\varepsilon(F)\}$). Hence, (7) yields $F' \in \mathcal{A}$. Also, (6) (applied to F' and F instead of F_1 and F_2) yields $\varepsilon(F') = \varepsilon(F)$ (since $F \in \mathcal{A}$ and $F' \in \mathcal{A}$ and $\text{Shade}(F') = \text{Shade } F$). Hence, $\varepsilon(F') = \varepsilon(F) \in \{\varepsilon(F)\} \subseteq F \cup \{\varepsilon(F)\} = F'$ (since $F' = F \cup \{\varepsilon(F)\}$). Therefore, F' is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \in P$ (since $F' \in \mathcal{A}$ and $\varepsilon(F') \in F'$). In other words, $F' \in \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$. In other words, $F' \in \mathcal{A}_+$ (since $\mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$). In other words, $F \cup \{\varepsilon(F)\} \in \mathcal{A}_+$ (since $F' = F \cup \{\varepsilon(F)\}$). Qed.

We have $\Phi \circ \Psi = \text{id}$ ⁷ and $\Psi \circ \Phi = \text{id}$ ⁸. Thus, the maps Φ and Ψ are mutually inverse. Hence, the map Φ is invertible, thus a bijection.

⁷*Proof.* Let $F \in \mathcal{A}_-$. Thus, $F \in \mathcal{A}_- = \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}$. In other words, F is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \notin P$. In other words, F is an element of \mathcal{A} and satisfies $\varepsilon(F) \notin F$.

We set $u = \varepsilon(F)$ and $F' = F \cup \{u\}$. Thus, $u = \varepsilon(F) \notin F$ and $F' = F \cup \{u\} = F \cup \{\varepsilon(F)\}$ (since $u = \varepsilon(F)$).

We have $F \in \mathcal{A} = \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}$. In other words, F is a $P \subseteq E$ satisfying $G \not\subseteq \text{Shade } P$. Thus, F is a subset of E .

Recall that $\varepsilon(F)$ is the smallest edge $e \in G \setminus \text{Shade } F$ (by the definition of $\varepsilon(F)$). Hence, $\varepsilon(F) \in G \setminus \text{Shade } F$. In other words, $u \in G \setminus \text{Shade } F$ (since $u = \varepsilon(F)$). In other words, $u \in G$ and $u \notin \text{Shade } F$. Thus, $u \in G \subseteq E$ and $u \notin \text{Shade } F$. Therefore, $F \cup \{u\}$ is a subset of E (since F is a subset of E , and since $u \in E$). In other words, F' is a subset of E (since $F' = F \cup \{u\}$).

Furthermore, (3) yields $\text{Shade}(F \cup \{u\}) = \text{Shade } F$ (since $u \notin \text{Shade } F$). This can be rewritten as $\text{Shade}(F') = \text{Shade } F$ (since $F' = F \cup \{u\}$). Hence, (7) yields $F' \in \mathcal{A}$. Also, (6) (applied to F' and F instead of F_1 and F_2) yields $\varepsilon(F') = \varepsilon(F)$ (since $F \in \mathcal{A}$ and $F' \in \mathcal{A}$ and $\text{Shade}(F') = \text{Shade } F$).

The definition of Ψ yields $\Psi(F) = F \cup \{\varepsilon(F)\} = F'$ (since $F' = F \cup \{\varepsilon(F)\}$), so that $F' = \Psi(F) \in \mathcal{A}_+$ (since Ψ is a map from \mathcal{A}_- to \mathcal{A}_+). Hence, the definition of Φ yields

$$\begin{aligned} \Phi(F') &= F' \setminus \{\varepsilon(F')\} = \underbrace{F'}_{=F \cup \{u\}} \setminus \{u\} && \text{(since } \varepsilon(F') = \varepsilon(F) = u) \\ &= (F \cup \{u\}) \setminus \{u\} = F && \text{(since } u \notin F). \end{aligned}$$

In view of $\Psi(F) = F'$, this can be rewritten as $\Phi(\Psi(F)) = F$. Thus, $(\Phi \circ \Psi)(F) = \Phi(\Psi(F)) = F = \text{id}(F)$.

Forget that we fixed F . We thus have shown that $(\Phi \circ \Psi)(F) = \text{id}(F)$ for each $F \in \mathcal{A}_-$. In other words, $\Phi \circ \Psi = \text{id}$.

⁸*Proof.* Let $F \in \mathcal{A}_+$. Thus, $F \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$. In other words, F is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \in P$. In other words, F is an element of \mathcal{A} and satisfies $\varepsilon(F) \in F$.

We set $u = \varepsilon(F)$ and $F' = F \setminus \{u\}$. Hence, $u = \varepsilon(F) \in F$ and $F' = F \setminus \{u\} = F \setminus \{\varepsilon(F)\}$ (since $u = \varepsilon(F)$).

We have $F \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\} \subseteq \mathcal{A} = \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}$. In other words, F is a $P \subseteq E$ satisfying $G \not\subseteq \text{Shade } P$. Thus, F is a subset of E . Therefore, $F \setminus \{u\}$ is a subset of E as well (since $F \setminus \{u\} \subseteq F \subseteq E$). In other words, F' is a subset of E (since $F' = F \setminus \{u\}$).

Recall that $\varepsilon(F)$ is the smallest edge $e \in G \setminus \text{Shade } F$ (by the definition of $\varepsilon(F)$). Hence, $\varepsilon(F) \in G \setminus \text{Shade } F$. In other words, $u \in G \setminus \text{Shade } F$ (since $u = \varepsilon(F)$). In other words, $u \in G$ and $u \notin \text{Shade } F$. Thus, $u \in G \subseteq E$ and $u \notin \text{Shade } F$. Therefore, (4) yields $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$. This can be rewritten as $\text{Shade}(F') = \text{Shade } F$ (since $F' = F \setminus \{u\}$). Hence, (7) yields $F' \in \mathcal{A}$. Also, (6) (applied to F' and F instead of F_1 and F_2) yields $\varepsilon(F') = \varepsilon(F)$ (since $F \in \mathcal{A}$ and $F' \in \mathcal{A}$ and $\text{Shade}(F') = \text{Shade } F$).

The definition of Φ yields $\Phi(F) = F \setminus \{\varepsilon(F)\} = F'$ (since $F' = F \setminus \{\varepsilon(F)\}$), so that $F' = \Phi(F) \in \mathcal{A}_-$ (since Φ is a map from \mathcal{A}_+ to \mathcal{A}_-). Hence, the definition of Ψ yields

$$\begin{aligned} \Psi(F') &= F' \cup \{\varepsilon(F')\} = \underbrace{F'}_{=F \setminus \{u\}} \cup \{u\} && \text{(since } \varepsilon(F') = \varepsilon(F) = u) \\ &= (F \setminus \{u\}) \cup \{u\} = F && \text{(since } u \in F). \end{aligned}$$

In view of $\Phi(F) = F'$, this can be rewritten as $\Psi(\Phi(F)) = F$. Thus, $(\Psi \circ \Phi)(F) =$

Moreover, each $F \in \mathcal{A}_+$ satisfies

$$(-1)^{|\Phi(F)|} = -(-1)^{|F|}. \quad (8)$$

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Now,

$$\begin{aligned}
& \sum_{\substack{F \subseteq E; \\ G \not\subseteq \text{Shade } F}} (-1)^{|F|} \\
&= \sum_{F \in \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}} \\
&= \sum_{F \in \mathcal{A}} \\
&\text{(since } \{P \subseteq E \mid G \not\subseteq \text{Shade } P\} = \mathcal{A}\text{)} \\
&= \sum_{F \in \mathcal{A}} (-1)^{|F|} = \sum_{\substack{F \in \mathcal{A}; \\ \varepsilon(F) \in F}} (-1)^{|F|} + \sum_{\substack{F \in \mathcal{A}; \\ \varepsilon(F) \notin F}} (-1)^{|F|} \\
&= \sum_{\substack{F \in \{P \in \mathcal{A} \mid \varepsilon(P) \in P\} \\ = \mathcal{A}_+}} (-1)^{|F|} + \sum_{\substack{F \in \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\} \\ = \mathcal{A}_-}} (-1)^{|F|} \\
&\text{(since } \{P \in \mathcal{A} \mid \varepsilon(P) \in P\} = \mathcal{A}_+ \text{) \quad (since } \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\} = \mathcal{A}_- \text{)} \\
&\left(\text{since each } F \in \mathcal{A} \text{ satisfies either } \varepsilon(F) \in F \text{ or } \varepsilon(F) \notin F \text{ (but not both at the same time)} \right) \\
&= \sum_{F \in \mathcal{A}_+} (-1)^{|F|} + \sum_{F \in \mathcal{A}_-} (-1)^{|F|} = \sum_{F \in \mathcal{A}_+} (-1)^{|F|} + \sum_{F \in \mathcal{A}_+} \underbrace{(-1)^{|\Phi(F)|}}_{= -(-1)^{|F|} \text{ (by (8))}} \\
&\left(\text{here, we have substituted } \Phi(F) \text{ for } F \text{ in the second sum, since the map } \Phi : \mathcal{A}_+ \rightarrow \mathcal{A}_- \text{ is a bijection} \right) \\
&= \sum_{F \in \mathcal{A}_+} (-1)^{|F|} + \underbrace{\sum_{F \in \mathcal{A}_+} (-(-1)^{|F|})}_{= - \sum_{F \in \mathcal{A}_+} (-1)^{|F|}} = \sum_{F \in \mathcal{A}_+} (-1)^{|F|} - \sum_{F \in \mathcal{A}_+} (-1)^{|F|} = 0.
\end{aligned}$$

This proves Theorem 2.6. \square

In order to derive Theorem 2.5 from Theorem 2.6, we need the following innocent lemma – which is one of the simplest facts in enumerative combinatorics:

$$\Psi(\Phi(F)) = F = \text{id}(F).$$

Forget that we fixed F . We thus have shown that $(\Psi \circ \Phi)(F) = \text{id}(F)$ for each $F \in \mathcal{A}_+$.

In other words, $\Psi \circ \Phi = \text{id}$.

⁹*Proof of (8):* Let $F \in \mathcal{A}_+$. Thus, $F \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$. In other words, F is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \in P$. In other words, F is an element of \mathcal{A} and satisfies $\varepsilon(F) \in F$.

The definition of Φ yields $\Phi(F) = F \setminus \{\varepsilon(F)\}$. Hence, $|\Phi(F)| = |F \setminus \{\varepsilon(F)\}| = |F| - 1$ (since $\varepsilon(F) \in F$). Therefore, $(-1)^{|\Phi(F)|} = (-1)^{|F|-1} = -(-1)^{|F|}$. This proves (8).

Lemma 2.7. Let U be a finite set with $U \neq \emptyset$. Then,

$$\sum_{F \subseteq U} (-1)^{|F|} = 0.$$

Lemma 2.7 can easily be derived from the fact that $\sum_{k=0}^n (-1)^k \binom{n}{k} = 0$ for any positive integer n (as follows readily from the binomial theorem). However, keeping true to the spirit of this paper, let us give a bijective proof for it:

Proof of Lemma 2.7. This is a standard argument that underlies many combinatorial proofs of alternating sum identities (see, for example, [Sagan20, proof of (2.4)] or [BenQui08, proof of (1)]). For the sake of completeness, let us nevertheless recall it.

We have $U \neq \emptyset$; hence, there exists some $u \in U$. Consider this u .

For every $F \subseteq U$ satisfying $u \in F$, we have $F \setminus \{u\} \subseteq U$ ¹⁰ and $u \notin F \setminus \{u\}$. Hence, the map

$$\begin{aligned} \Phi : \{F \subseteq U \mid u \in F\} &\rightarrow \{F \subseteq U \mid u \notin F\}, \\ F &\mapsto F \setminus \{u\} \end{aligned}$$

is well-defined. Consider this map Φ .

For every $F \subseteq U$ satisfying $u \notin F$, we have $F \cup \{u\} \subseteq U$ ¹¹ and $u \in F \cup \{u\}$. Hence, the map

$$\begin{aligned} \Psi : \{F \subseteq U \mid u \notin F\} &\rightarrow \{F \subseteq U \mid u \in F\}, \\ F &\mapsto F \cup \{u\} \end{aligned}$$

is well-defined. Consider this map Ψ .

The maps Φ and Ψ we just defined are clearly mutually inverse. Thus, they are invertible, i.e., are bijections. Hence, in particular, Φ is a bijection. Thus, we can substitute $\Phi(F)$ for F in the sum $\sum_{\substack{F \subseteq U; \\ u \notin F}} (-1)^{|F|}$. We thus obtain

$$\begin{aligned} \sum_{\substack{F \subseteq U; \\ u \notin F}} (-1)^{|F|} &= \sum_{\substack{F \subseteq U; \\ u \in F}} \underbrace{(-1)^{|\Phi(F)|}}_{= (-1)^{|F \setminus \{u\}|}} = \sum_{\substack{F \subseteq U; \\ u \in F}} \underbrace{(-1)^{|F \setminus \{u\}|}}_{= (-1)^{|F|-1}} \\ &\quad \text{(since } \Phi(F) = F \setminus \{u\} \text{ (by the definition of } \Phi)) \quad \text{(since } |F \setminus \{u\}| = |F| - 1 \text{ (because } u \in F))} \\ &= \sum_{\substack{F \subseteq U; \\ u \in F}} \underbrace{(-1)^{|F|-1}}_{= -(-1)^{|F|}} = - \sum_{\substack{F \subseteq U; \\ u \in F}} (-1)^{|F|}. \end{aligned}$$

¹⁰since $F \setminus \{u\} \subseteq F \subseteq U$

¹¹since $F \subseteq U$ and $u \in U$

However, each $F \subseteq U$ satisfies either $u \in F$ or $u \notin F$ (but not both). Hence,

$$\begin{aligned} \sum_{F \subseteq U} (-1)^{|F|} &= \sum_{\substack{F \subseteq U; \\ u \in F}} (-1)^{|F|} + \underbrace{\sum_{\substack{F \subseteq U; \\ u \notin F}} (-1)^{|F|}}_{=-\sum_{\substack{F \subseteq U; \\ u \in F}} (-1)^{|F|}} = \sum_{\substack{F \subseteq U; \\ u \in F}} (-1)^{|F|} - \sum_{\substack{F \subseteq U; \\ u \in F}} (-1)^{|F|} = 0. \end{aligned}$$

This proves Lemma 2.7. \square

We can now easily derive Theorem 2.5 from Theorem 2.6:

Proof of Theorem 2.5. Each subset F of E satisfies either $G \subseteq \text{Shade } F$ or $G \not\subseteq \text{Shade } F$ (but not both at the same time). Hence,

$$\sum_{F \subseteq E} (-1)^{|F|} = \sum_{\substack{F \subseteq E; \\ G \subseteq \text{Shade } F}} (-1)^{|F|} + \underbrace{\sum_{\substack{F \subseteq E; \\ G \not\subseteq \text{Shade } F}} (-1)^{|F|}}_{=0 \text{ (by Theorem 2.6)}} = \sum_{\substack{F \subseteq E; \\ G \subseteq \text{Shade } F}} (-1)^{|F|}.$$

Therefore,

$$\sum_{\substack{F \subseteq E; \\ G \subseteq \text{Shade } F}} (-1)^{|F|} = \sum_{F \subseteq E} (-1)^{|F|} = 0$$

(by Lemma 2.7, applied to $U = E$). This proves Theorem 2.5. \square

2.3. Proving Theorem 1.2

Theorem 1.2 is now a simple particular case of Theorem 2.5:

Proof of Theorem 1.2. Let G be the set E . Thus, $G = E$. Hence, for each subset F of E , we have the following chain of logical equivalences:

$$\begin{aligned} (G \subseteq \text{Shade } F) &\iff (E \subseteq \text{Shade } F) \\ &\iff (\text{each } u \in E \text{ satisfies } u \in \text{Shade } F) \\ &\iff (\text{each } u \in E \text{ satisfies } u \in \{e \in E \mid F \text{ infects } e\}) \\ &\quad (\text{since } \text{Shade } F = \{e \in E \mid F \text{ infects } e\}) \\ &\iff (\text{each } u \in E \text{ has the property that } F \text{ infects } u) \\ &\iff (F \text{ infects each } u \in E) \\ &\iff (F \text{ infects each } e \in E) \\ &\iff (F \text{ is pandemic}) \quad (\text{by the definition of "pandemic"}). \end{aligned}$$

Thus, the summation sign " $\sum_{\substack{F \subseteq E; \\ G \subseteq \text{Shade } F}}$ " can be rewritten as " $\sum_{\substack{F \subseteq E \text{ is} \\ \text{pandemic}}}$ ". Hence,

$$\sum_{\substack{F \subseteq E; \\ G \subseteq \text{Shade } F}} (-1)^{|F|} = \sum_{\substack{F \subseteq E \text{ is} \\ \text{pandemic}}} (-1)^{|F|}.$$

Therefore,

$$\sum_{\substack{F \subseteq E \text{ is} \\ \text{pandemic}}} (-1)^{|F|} = \sum_{\substack{F \subseteq E; \\ G \subseteq \text{Shade } F}} (-1)^{|F|} = 0$$

(by Theorem 2.5). This proves Theorem 1.2. □

3. Vertex infection and other variants

In our study of graphs so far, we have barely ever mentioned vertices (even though they are, of course, implicit in the notion of a path). It may appear somewhat strange to talk about a subset infecting an edge, when the infection is spread from vertex to vertex. One might thus wonder if there is also a vertex counterpart of Theorem 1.2. So let us define analogues of our notions for vertices:

If $F \subseteq V$, then an F -vertex-path shall mean a path of Γ such that all vertices of the path except (possibly) for its two endpoints belong to F . (Thus, if a path has only one edge or none, then it automatically is an F -vertex-path.)

If $w \in V \setminus \{v\}$ is any vertex and $F \subseteq V \setminus \{v\}$ is any subset, then we say that F vertex-infects w if there exists an F -vertex-path from v to w . (This is always true when w is v or a neighbor of v .)

A subset $F \subseteq V \setminus \{v\}$ is said to be *vertex-pandemic* if it vertex-infects each vertex $w \in V \setminus \{v\}$.

Example 3.1. Let Γ be as in Example 1.3. Then, the path $v \xrightarrow{1} p \xrightarrow{2} q$ is an F -vertex-path for any subset $F \subseteq V$ that satisfies $p \in F$. The subset $\{p\}$ of $V \setminus \{v\}$ vertex-infects each vertex (for example, $v \xrightarrow{1} p \xrightarrow{2} q$ is a $\{p\}$ -vertex-path from v to q , and $v \xrightarrow{4} w$ is a $\{p\}$ -vertex-path from v to w), and thus is vertex-pandemic. The vertex-pandemic subsets of $V \setminus \{v\}$ are the sets

$$\{p\}, \{w\}, \{p, q\}, \{p, w\}, \{q, w\}, \{p, q, w\}.$$

We now have the following analogue of Theorem 1.2:

Theorem 3.2. Assume that $V \setminus \{v\} \neq \emptyset$. Then,

$$\sum_{\substack{F \subseteq V \setminus \{v\} \text{ is} \\ \text{vertex-pandemic}}} (-1)^{|F|} = 0.$$

Proof of Theorem 3.2. With just a few easy modifications, our above proof of Theorem 1.2 can be repurposed as a proof of Theorem 3.2. Namely:

- We need to replace “edge” by “vertex” throughout the argument (including Definition 2.1, Lemma 2.3, Lemma 2.4, Theorem 2.5 and Theorem 2.6), as well as replace E by $V \setminus \{v\}$.
- The words “ F -path”, “infects” and “pandemic” have to be replaced by “ F -vertex-path”, “vertex-infects” and “vertex-pandemic”, respectively.
- In the proofs of Lemma 2.3 and Lemma 2.4, the words “an endpoint of” (as well as “any endpoint of” and “some endpoint of”) need to be removed (since the notion of “vertex-infects” is defined not in terms of paths to an endpoint of a given edge, but in terms of paths to a given vertex).
- In the proof of Lemma 2.4, specifically in the proof of (3), the path π is now cut not by removing the edge u , but by splitting the path π at the vertex u .

The reader may check that these changes result in a valid proof of Theorem 3.2. \square

Another variant of Theorem 1.2 (and Theorem 2.5 and Theorem 2.6) is obtained by replacing the undirected graph Γ with a directed graph (while, of course, replacing paths by directed paths). More generally, we can replace Γ by a “hybrid” graph with some directed and some undirected edges.¹² No changes are required to the above proofs. Yet another variation can be obtained by replacing “endpoint” by “source” (for directed edges). We cannot, however, replace “endpoint” by “target”.

4. An abstract perspective

Seeing how little graph theory we have used in proving Theorem 1.2, and how easily the same argument adapted to Theorem 3.2, we get the impression that there might be some general theory lurking behind it. What follows is an attempt at building this theory.

4.1. Shade maps

Let $\mathcal{P}(E)$ denote the power set of E . In Definition 2.1, we have encoded the “infects” relation as a map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ defined by

$$\text{Shade } F = \{e \in E \mid F \text{ infects } e\}.$$

¹²We understand that a directed edge still has two endpoints: its source and its target.

As we recall, Theorem 2.5 (a generalization of Theorem 1.2) states that

$$\sum_{\substack{F \subseteq E; \\ G \subseteq \text{Shade } F}} (-1)^{|F|} = 0 \tag{9}$$

for any $G \subseteq E$, under the assumption that $E \neq \emptyset$.

To generalize this, we forget about the graph Γ and the map Shade , and instead start with an **arbitrary** finite set E . (This set E corresponds to the set E in Theorem 1.2 and to the set $V \setminus \{v\}$ in Theorem 3.2.) Let $\mathcal{P}(E)$ be the power set of E . Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be an arbitrary map (meant to generalize the map Shade from the previous paragraph). We may now ask:

Question 4.1. What (combinatorial) properties must Shade satisfy in order for (9) to hold for any $G \subseteq E$ under the assumption that $E \neq \emptyset$?

A partial answer to this question can be given by analyzing our above proof of Theorem 2.5 and extracting what was used:

Definition 4.2. Let E be a set. A *shade map* on E shall mean a map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ that satisfies the following two axioms:

Axiom 1: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \cup \{u\}) = \text{Shade } F$.

Axiom 2: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$.

Theorem 4.3. Let E be a finite set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be a shade map on E .

Assume that $E \neq \emptyset$. Let G be any subset of E . Then,

$$\sum_{\substack{F \subseteq E; \\ G \subseteq \text{Shade } F}} (-1)^{|F|} = 0.$$

Proof sketch. Again, the proof is analogous to our above proof of Theorem 2.5. (This time, in the proof of Lemma 2.4, the equalities (3) and (4) follow directly from Axiom 1 and Axiom 2, respectively.) \square

How do shade maps relate to known concepts in the combinatorics of set families (such as topologies, clutters, matroids, or submodular functions)? Are they just one of these known concepts in disguise? We shall answer two versions of this question in the following subsections. Specifically:

- In Subsection 4.3, we will show that inclusion-reversing shade maps on E (i.e., shade maps Shade that satisfy $\text{Shade } B \subseteq \text{Shade } A$ whenever $A \subseteq B$) are in bijection with *antimatroidal quasi-closure operators* (a slight variant of antimatroids) on E .
- In Subsection 4.4, we will show that arbitrary shade maps are in bijection with *Boolean interval partitions* of $\mathcal{P}(E)$ (that is, set partitions of $\mathcal{P}(E)$ into intervals of the Boolean lattice $\mathcal{P}(E)$).

Before we come to these characterizations, we shall however make a few elementary remarks on shade maps.

First, we observe that Axioms 1 and 2 in Definition 4.2 can be weakened to the following statements:

Axiom 1': If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \cup \{u\}) \subseteq \text{Shade } F$.

Axiom 2': If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \setminus \{u\}) \subseteq \text{Shade } F$.

Axiom 1' is weaker than Axiom 1, and likewise Axiom 2' is weaker than Axiom 2. However, Axioms 1' and 2' combined are equivalent to Axioms 1 and 2 combined:

Proposition 4.4. Let E be a set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be any map. Then, Shade is a shade map on E if and only if Shade satisfies the two Axioms 1' and 2' stated above.

Proof of Proposition 4.4. \implies : Assume that Shade is a shade map on E . We must show that Shade satisfies the two Axioms 1' and 2'.

Indeed, Shade is a shade map (by assumption), and thus satisfies Axioms 1 and 2 in Definition 4.2. Now, Axiom 1' clearly follows from Axiom 1, whereas Axiom 2' clearly follows from Axiom 2. Hence, Shade satisfies the two Axioms 1' and 2'. This proves the " \implies " direction of Proposition 4.4.

\impliedby : Assume that Shade satisfies the two Axioms 1' and 2'. We must prove that Shade is a shade map on E . To this purpose, we shall show that Shade satisfies Axioms 1 and 2 in Definition 4.2.

[*Proof of Axiom 1:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must show that $\text{Shade}(F \cup \{u\}) \subseteq \text{Shade } F$.

Indeed, if $u \in F$, then this is obvious (since we have $F \cup \{u\} = F$ in this case). Thus, for the rest of this proof, we WLOG assume that $u \notin F$. Hence, $(F \cup \{u\}) \setminus \{u\} = F$.

We have assumed that Axioms 1' and 2' hold. Thus, Axiom 1' shows that $\text{Shade}(F \cup \{u\}) \subseteq \text{Shade } F$. Hence, $E \setminus \underbrace{\text{Shade}(F \cup \{u\})}_{\subseteq \text{Shade } F} \supseteq E \setminus \text{Shade } F$, so that

$E \setminus \text{Shade } F \subseteq E \setminus \text{Shade } (F \cup \{u\})$. Now, $u \in E \setminus \text{Shade } F \subseteq E \setminus \text{Shade } (F \cup \{u\})$. Hence, Axiom 2' (applied to $F \cup \{u\}$ instead of F) yields $\text{Shade } ((F \cup \{u\}) \setminus \{u\}) \subseteq \text{Shade } (F \cup \{u\})$. In view of $(F \cup \{u\}) \setminus \{u\} = F$, this rewrites as $\text{Shade } F \subseteq \text{Shade } (F \cup \{u\})$. Combining this with $\text{Shade } (F \cup \{u\}) \subseteq \text{Shade } F$, we obtain $\text{Shade } (F \cup \{u\}) = \text{Shade } F$. This proves Axiom 1.]

[*Proof of Axiom 2:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must show that $\text{Shade } (F \setminus \{u\}) = \text{Shade } F$.

Indeed, if $u \notin F$, then this is obvious (since we have $F \setminus \{u\} = F$ in this case). Thus, for the rest of this proof, we WLOG assume that $u \in F$. Hence, $(F \setminus \{u\}) \cup \{u\} = F$.

We have assumed that Axioms 1' and 2' hold. Thus, Axiom 2' shows that $\text{Shade } (F \setminus \{u\}) \subseteq \text{Shade } F$. Hence, $E \setminus \underbrace{\text{Shade } (F \setminus \{u\})}_{\subseteq \text{Shade } F} \supseteq E \setminus \text{Shade } F$, so that

$E \setminus \text{Shade } F \subseteq E \setminus \text{Shade } (F \setminus \{u\})$. Now, $u \in E \setminus \text{Shade } F \subseteq E \setminus \text{Shade } (F \setminus \{u\})$. Hence, Axiom 1' (applied to $F \setminus \{u\}$ instead of F) yields $\text{Shade } ((F \setminus \{u\}) \cup \{u\}) \subseteq \text{Shade } (F \setminus \{u\})$. In view of $(F \setminus \{u\}) \cup \{u\} = F$, this rewrites as $\text{Shade } F \subseteq \text{Shade } (F \setminus \{u\})$. Combining this with $\text{Shade } (F \setminus \{u\}) \subseteq \text{Shade } F$, we obtain $\text{Shade } (F \setminus \{u\}) = \text{Shade } F$. This proves Axiom 2.]

We have now shown that the map Shade satisfies both Axioms 1 and 2 in Definition 4.2. Hence, Shade is a shade map. This proves the " \Leftarrow " direction of Proposition 4.4.

Thus, the proof of Proposition 4.4 is complete. \square

Here is one further restatement of both Axioms 1' and 2': For any $F \subseteq E$ and any two elements $u, v \in E \setminus \text{Shade } F$, we have $v \notin \text{Shade } (F \cup \{u\})$ (Axiom 1') and $v \notin \text{Shade } (F \setminus \{u\})$ (Axiom 2').

Axioms 1 and 2 can also be combined into one common axiom:

Axiom 3: If $F \in \mathcal{P}(E)$ and $u \in E \setminus F$, then we have $\text{Shade } F = \text{Shade } (F \cup \{u\})$ or $u \in (\text{Shade } F) \cap \text{Shade } (F \cup \{u\})$.

Proposition 4.5. Let E be a set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be any map. Then, Shade is a shade map on E if and only if Shade satisfies Axiom 3.

Proof of Proposition 4.5. \implies : Assume that Shade is a shade map on E . We must show that Shade satisfies Axiom 3.

[*Proof of Axiom 3:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus F$. We must prove that we have $\text{Shade } F = \text{Shade } (F \cup \{u\})$ or $u \in (\text{Shade } F) \cap \text{Shade } (F \cup \{u\})$.

Indeed, assume the contrary. Thus, we have neither $\text{Shade } F = \text{Shade } (F \cup \{u\})$ nor $u \in (\text{Shade } F) \cap \text{Shade } (F \cup \{u\})$. In other words, we have $\text{Shade } F \neq \text{Shade } (F \cup \{u\})$ and $u \notin (\text{Shade } F) \cap \text{Shade } (F \cup \{u\})$.

We know that Shade is a shade map (by assumption), and thus satisfies Axioms 1 and 2 in Definition 4.2. Hence, if we had $u \in E \setminus \text{Shade } F$, then we would have $\text{Shade } (F \cup \{u\}) = \text{Shade } F$ (by Axiom 1), which would contradict

$\text{Shade } F \neq \text{Shade } (F \cup \{u\})$. Thus, we cannot have $u \in E \setminus \text{Shade } F$. Hence, we have $u \in E \setminus (E \setminus \text{Shade } F) = \text{Shade } F$ (since $\text{Shade } F \subseteq E$).

Therefore, if we had $u \in \text{Shade } (F \cup \{u\})$, then we would have $u \in (\text{Shade } F) \cap \text{Shade } (F \cup \{u\})$, which would contradict $u \notin (\text{Shade } F) \cap \text{Shade } (F \cup \{u\})$. Thus, we cannot have $u \in \text{Shade } (F \cup \{u\})$. Hence, we have $u \notin \text{Shade } (F \cup \{u\})$. Therefore, $u \in E \setminus \text{Shade } (F \cup \{u\})$. Hence, Axiom 2 (applied to $F \cup \{u\}$ instead of F) yields $\text{Shade } ((F \cup \{u\}) \setminus \{u\}) = \text{Shade } (F \cup \{u\})$. However, $u \notin F$ (since $u \in E \setminus F$) and thus $(F \cup \{u\}) \setminus \{u\} = F$. Hence, the equality $\text{Shade } ((F \cup \{u\}) \setminus \{u\}) = \text{Shade } (F \cup \{u\})$ rewrites as $\text{Shade } F = \text{Shade } (F \cup \{u\})$. This contradicts $\text{Shade } F \neq \text{Shade } (F \cup \{u\})$.

This contradiction shows that our assumption was wrong. Hence, we have proved that we have $\text{Shade } F = \text{Shade } (F \cup \{u\})$ or $u \in (\text{Shade } F) \cap \text{Shade } (F \cup \{u\})$. This proves Axiom 3.]

We thus have shown that Shade satisfies Axiom 3. This proves the “ \implies ” direction of Proposition 4.5.

\impliedby : Assume that Shade satisfies Axiom 3. We must prove that Shade is a shade map on E . To this purpose, we shall show that Shade satisfies Axioms 1 and 2 in Definition 4.2.

[*Proof of Axiom 1:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must show that $\text{Shade } (F \cup \{u\}) = \text{Shade } F$.

Indeed, assume the contrary. Thus, $\text{Shade } (F \cup \{u\}) \neq \text{Shade } F$, so that $F \cup \{u\} \neq F$. Consequently, $u \notin F$ (because otherwise, we would have $u \in F$ and thus $F \cup \{u\} = F$, contradicting $F \cup \{u\} \neq F$). Hence, $u \in E \setminus F$.

We assumed that Axiom 3 holds. Thus, Axiom 3 shows that we have $\text{Shade } F = \text{Shade } (F \cup \{u\})$ or $u \in (\text{Shade } F) \cap \text{Shade } (F \cup \{u\})$. Hence, $u \in (\text{Shade } F) \cap \text{Shade } (F \cup \{u\})$ (since $\text{Shade } F = \text{Shade } (F \cup \{u\})$) would contradict $\text{Shade } (F \cup \{u\}) \neq \text{Shade } F$. Thus, $u \in (\text{Shade } F) \cap \text{Shade } (F \cup \{u\}) \subseteq \text{Shade } F$. However, $u \in E \setminus \text{Shade } F$ entails $u \notin \text{Shade } F$. This contradicts $u \in \text{Shade } F$. This contradiction shows that our assumption was false. Hence, we have proved $\text{Shade } (F \cup \{u\}) = \text{Shade } F$; thus Axiom 1 is proven.]

[*Proof of Axiom 2:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must show that $\text{Shade } (F \setminus \{u\}) = \text{Shade } F$.

Indeed, assume the contrary. Thus, $\text{Shade } (F \setminus \{u\}) \neq \text{Shade } F$, so that $F \setminus \{u\} \neq F$. Consequently, $u \in F$ (because otherwise, we would have $u \notin F$ and thus $F \setminus \{u\} = F$, contradicting $F \setminus \{u\} \neq F$). Hence, $(F \setminus \{u\}) \cup \{u\} = F$.

We have $u \in E \setminus (F \setminus \{u\})$ (since $u \in E$ and $u \notin F \setminus \{u\}$). We assumed that Axiom 3 holds. Thus, Axiom 3 (applied to $F \setminus \{u\}$ instead of F) shows that we have $\text{Shade } (F \setminus \{u\}) = \text{Shade } ((F \setminus \{u\}) \cup \{u\})$ or $u \in (\text{Shade } (F \setminus \{u\})) \cap \text{Shade } ((F \setminus \{u\}) \cup \{u\})$. In view of $(F \setminus \{u\}) \cup \{u\} = F$, this rewrites as follows: We have $\text{Shade } (F \setminus \{u\}) = \text{Shade } F$ or $u \in (\text{Shade } (F \setminus \{u\})) \cap \text{Shade } F$. Hence, $u \in (\text{Shade } (F \setminus \{u\})) \cap \text{Shade } F$ (since we assumed that we don't have $\text{Shade } (F \setminus \{u\}) = \text{Shade } F$). Thus, $u \in (\text{Shade } (F \setminus \{u\})) \cap \text{Shade } F \subseteq \text{Shade } F$. However, $u \in E \setminus \text{Shade } F$ entails $u \notin \text{Shade } F$. This contradicts $u \in \text{Shade } F$. This contradiction shows that our assumption was false. Hence, we have

proved $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$; thus Axiom 2 is proven.]

We have now shown that the map Shade satisfies both Axioms 1 and 2 in Definition 4.2. Hence, Shade is a shade map. This proves the “ \Leftarrow ” direction of Proposition 4.5.

Thus, the proof of Proposition 4.5 is complete. \square

We will soon see some examples. First, let us introduce two more basic concepts that will help clarify these examples:

Definition 4.6. Let E be a set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be any map (not necessarily a shade map).

(a) We say that Shade is *inclusion-preserving* if it satisfies the following property: If A and B are two subsets of E such that $A \subseteq B$, then $\text{Shade } A \subseteq \text{Shade } B$.

(b) We say that Shade is *inclusion-reversing* if it satisfies the following property: If A and B are two subsets of E such that $A \subseteq B$, then $\text{Shade } B \subseteq \text{Shade } A$.

For instance, the map Shade from Definition 2.1 is inclusion-preserving (because of Lemma 2.3) and is a shade map (by Lemma 2.4). The same holds for the analogue of the map Shade that uses vertex-infection instead of infection. This does not mean that any shade map is inclusion-preserving. Indeed, we shall soon see some inclusion-reversing shade maps, and it is not hard to construct shade maps that are neither inclusion-preserving nor inclusion-reversing.

Let us observe that there is a simple bijection between inclusion-preserving and inclusion-reversing maps, and this bijection preserves shadeness:

Proposition 4.7. Let E be a set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be any map (not necessarily a shade map). Let $\text{Shade}' : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be the map that sends each $F \in \mathcal{P}(E)$ to $\text{Shade}(E \setminus F) \in \mathcal{P}(E)$. Then:

(a) The map Shade is inclusion-preserving if and only if the map Shade' is inclusion-reversing.

(b) The map Shade is a shade map if and only if the map Shade' is a shade map.

Proof. This is easy and left to the reader. (For part (b), observe that Axioms 1 and 2 for the map Shade translate into Axioms 2 and 1 for the map Shade' .) \square

Definition 4.8. Let E , Shade and Shade' be as in Proposition 4.7. We then say that the map Shade' is *dual* to Shade .

4.2. Some examples of shade maps

As we already mentioned, Lemma 2.4 and its analogue for vertex-infection provide two examples of inclusion-preserving shade maps Shade . An example of an inclusion-reversing shade map comes from the theory of posets:

Example 4.9. Let E be a poset. For any $F \subseteq E$, we define

$$F_{\downarrow} = \{e \in E \mid \text{there exists an } f \in F \text{ with } e < f\}$$

and

$$\text{Shade } F = E \setminus F_{\downarrow}.$$

Then, this map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is an inclusion-reversing shade map.

Proof of Example 4.9 (sketched). We shall prove the following three statements:

Statement 0: If A and B are two subsets of E such that $A \subseteq B$, then $\text{Shade } B \subseteq \text{Shade } A$.

Statement 1: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \cup \{u\}) = \text{Shade } F$.

Statement 2: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$.

[*Proof of Statement 0:* Let A and B be two subsets of E such that $A \subseteq B$. We must show that $\text{Shade } B \subseteq \text{Shade } A$.

Indeed, $A \subseteq B$ entails $A_{\downarrow} \subseteq B_{\downarrow}$ (by the definitions of A_{\downarrow} and B_{\downarrow}). Hence, $E \setminus A_{\downarrow} \supseteq E \setminus B_{\downarrow}$, so that $E \setminus B_{\downarrow} \subseteq E \setminus A_{\downarrow}$. But the definition of Shade yields $\text{Shade } A = E \setminus A_{\downarrow}$ and $\text{Shade } B = E \setminus B_{\downarrow}$. Hence, $\text{Shade } B = E \setminus B_{\downarrow} \subseteq E \setminus A_{\downarrow} = \text{Shade } A$. This proves Statement 0.]

[*Proof of Statement 1:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must prove that $\text{Shade}(F \cup \{u\}) = \text{Shade } F$.

We have $u \in E \setminus \text{Shade } F = F_{\downarrow}$ (since $\text{Shade } F = E \setminus F_{\downarrow}$). In other words, there exists a $g \in F$ with $u < g$ (by the definition of F_{\downarrow}). Consider this g .

We must prove that $\text{Shade}(F \cup \{u\}) = \text{Shade } F$. In other words, we must prove that $(F \cup \{u\})_{\downarrow} = F_{\downarrow}$ (since $\text{Shade}(F \cup \{u\}) = E \setminus (F \cup \{u\})_{\downarrow}$ and $\text{Shade } F = E \setminus F_{\downarrow}$). Since $F_{\downarrow} \subseteq (F \cup \{u\})_{\downarrow}$ is rather obvious (by the definitions of F_{\downarrow} and $(F \cup \{u\})_{\downarrow}$), we only need to prove that $(F \cup \{u\})_{\downarrow} \subseteq F_{\downarrow}$.

So let $v \in (F \cup \{u\})_{\downarrow}$. We must show that $v \in F_{\downarrow}$.

We have $v \in (F \cup \{u\})_{\downarrow}$. In other words, there exists an $f \in F \cup \{u\}$ with $v < f$ (by the definition of $(F \cup \{u\})_{\downarrow}$). Consider this f . If $f \in F$, then we immediately obtain $v \in F_{\downarrow}$ (since $v < f$ and $f \in F$), and thus our proof is complete. Hence, we WLOG assume that $f \notin F$. Hence, $f = u$ (since $f \in F \cup \{u\}$ but $f \notin F$) and thus $v < f = u < g$. This entails $v \in F_{\downarrow}$ (by the definition of F_{\downarrow} , since $g \in F$).

Forget that we fixed v . We thus have shown that $v \in F_{\downarrow}$ for each $v \in (F \cup \{u\})_{\downarrow}$. In other words, $(F \cup \{u\})_{\downarrow} \subseteq F_{\downarrow}$. As we explained above, this completes the proof of Statement 1.]

[*Proof of Statement 2:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must prove that $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$.

We have $u \in E \setminus \text{Shade } F = F_{\downarrow}$ (since $\text{Shade } F = E \setminus F_{\downarrow}$). In other words, there exists a $g \in F$ with $u < g$ (by the definition of F_{\downarrow}). Consider this g . Note that $u < g$ entails $g \neq u$, so that $g \in F \setminus \{u\}$ (since $g \in F$).

We must prove that $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$. In other words, we must prove that $(F \setminus \{u\})_{\downarrow} = F_{\downarrow}$ (since $\text{Shade}(F \setminus \{u\}) = E \setminus (F \setminus \{u\})_{\downarrow}$ and $\text{Shade } F = E \setminus F_{\downarrow}$). Since $(F \setminus \{u\})_{\downarrow} \subseteq F_{\downarrow}$ is rather obvious (by the definitions of F_{\downarrow} and $(F \setminus \{u\})_{\downarrow}$), we only need to prove that $F_{\downarrow} \subseteq (F \setminus \{u\})_{\downarrow}$.

So let $v \in F_{\downarrow}$. We must show that $v \in (F \setminus \{u\})_{\downarrow}$.

We have $v \in F_{\downarrow}$. In other words, there exists an $f \in F$ with $v < f$ (by the definition of F_{\downarrow}). Consider this f . If $f \in F \setminus \{u\}$, then we immediately obtain $v \in (F \setminus \{u\})_{\downarrow}$ (since $v < f$ and $f \in F \setminus \{u\}$), and thus our proof is complete. Hence, we WLOG assume that $f \notin F \setminus \{u\}$. Hence, $f = u$ (since $f \in F$ but $f \notin F \setminus \{u\}$) and thus $v < f = u < g$. This entails $v \in (F \setminus \{u\})_{\downarrow}$ (by the definition of $(F \setminus \{u\})_{\downarrow}$, since $g \in F \setminus \{u\}$).

Forget that we fixed v . We thus have shown that $v \in (F \setminus \{u\})_{\downarrow}$ for each $v \in F_{\downarrow}$. In other words, $F_{\downarrow} \subseteq (F \setminus \{u\})_{\downarrow}$. As we explained above, this completes the proof of Statement 2.]

Now, we have proved Statements 1 and 2. Thus, the map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ satisfies the two axioms in Definition 4.2. In other words, this map is a shade map. Moreover, this map is inclusion-reversing (by Statement 0). This proves Example 4.9. \square

Another example of a shade map comes from discrete geometry:

Example 4.10. Let A be an affine space over \mathbb{R} . If S is a finite subset of A , then a *nontrivial convex combination* of S will mean a point of the form $\sum_{s \in S} \lambda_s s \in A$,

where the coefficients λ_s are nonnegative reals smaller than 1 and satisfying $\sum_{s \in S} \lambda_s = 1$.

Fix a finite subset E of A . For any $F \subseteq E$, we define

$$\text{Shade } F = \{e \in E \mid e \text{ is \textbf{not} a nontrivial convex combination of } F\}.$$

Then, this map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is an inclusion-reversing shade map.

Proof of Example 4.10 (sketched). We shall prove the following three statements:

Statement 0: If X and Y are two subsets of E such that $X \subseteq Y$, then $\text{Shade } Y \subseteq \text{Shade } X$.

Statement 1: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \cup \{u\}) = \text{Shade } F$.

Statement 2: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$.

[*Proof of Statement 0:* Let X and Y be two subsets of E such that $X \subseteq Y$. We must prove that $\text{Shade } Y \subseteq \text{Shade } X$.

Indeed, we have $X \subseteq Y$. Thus, any nontrivial convex combination of X is a nontrivial convex combination of Y . Therefore, any $e \in E$ that is **not** a nontrivial convex combination of Y also cannot be a nontrivial convex combination of X . In other words, $\text{Shade } Y \subseteq \text{Shade } X$ (by the definition of Shade). This proves Statement 0.]

[*Proof of Statement 1:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must prove that $\text{Shade}(F \cup \{u\}) = \text{Shade } F$. If $u \in F$, then this is obvious (since $F \cup \{u\} = F$ in this case); thus, we WLOG assume that $u \notin F$.

We must prove that $\text{Shade}(F \cup \{u\}) = \text{Shade } F$. We have $\text{Shade}(F \cup \{u\}) \subseteq \text{Shade } F$ (by Statement 0, since $F \subseteq F \cup \{u\}$). Thus, it suffices to show that $\text{Shade } F \subseteq \text{Shade}(F \cup \{u\})$. So let $p \in \text{Shade } F$. We must then show that $p \in \text{Shade}(F \cup \{u\})$. In other words, we must prove that p is **not** a nontrivial convex combination of $F \cup \{u\}$.

Assume the contrary. Thus, p is a nontrivial convex combination of $F \cup \{u\}$. In other words, $p = \sum_{s \in F \cup \{u\}} \mu_s s$ for some nonnegative reals μ_s smaller than 1 and satisfying $\sum_{s \in F \cup \{u\}} \mu_s = 1$. Consider these reals μ_s .

We have $u \in E \setminus \text{Shade } F$. Hence, $u \in E$ and $u \notin \text{Shade } F$. From $u \notin \text{Shade } F$, we see that u is a nontrivial convex combination of F . In other words, $u = \sum_{f \in F} \lambda_f f$ for some nonnegative reals λ_f smaller than 1 and satisfying $\sum_{f \in F} \lambda_f = 1$. Consider these reals λ_f .

Now,

$$\begin{aligned} p &= \sum_{s \in F \cup \{u\}} \mu_s s = \underbrace{\sum_{s \in F} \mu_s s}_{= \sum_{f \in F} \mu_f f} + \mu_u \underbrace{u}_{= \sum_{f \in F} \lambda_f f} \\ &\quad \left(\begin{array}{c} \text{here, we have split off the addend for } s = u \\ \text{from the sum, since } u \notin F \end{array} \right) \\ &= \sum_{f \in F} \mu_f f + \mu_u \sum_{f \in F} \lambda_f f = \sum_{f \in F} (\mu_f + \mu_u \lambda_f) f. \end{aligned} \tag{10}$$

Clearly, the coefficients $\mu_f + \mu_u \lambda_f$ on the right hand side of this equality are nonnegative reals. Moreover, they are easily seen to be smaller than 1¹³.

¹³*Proof.* Assume the contrary. Thus, $\mu_f + \mu_u \lambda_f \geq 1$ for some $f \in F$. Consider this f . Note that $\mu_f < 1$ (since all coefficients μ_s are smaller than 1) and $\lambda_f < 1$ (for similar reasons). Now, $\mu_f + \mu_u \lambda_f \geq 1 > \mu_f$ (since $\mu_f < 1$), so that $\mu_u \lambda_f > 0$. This entails $\mu_u \lambda_f \neq 0$ and thus $\mu_u \neq 0$, so that $\mu_u > 0$ (since μ_u is nonnegative). Now, $\underbrace{\mu_u \lambda_f}_{< 1} < \mu_u$ (since $\mu_u > 0$).

However, $f \neq u$ (since $f \in F$ and $u \notin F$), so that $\mu_f + \mu_u$ is a subsum of the sum $\sum_{s \in F \cup \{u\}} \mu_s$.

Finally, they satisfy

$$\begin{aligned} \sum_{f \in F} (\mu_f + \mu_u \lambda_f) &= \sum_{f \in F} \mu_f + \mu_u \underbrace{\sum_{f \in F} \lambda_f}_{=1} = \sum_{f \in F} \mu_f + \mu_u = \sum_{f \in FU\{u\}} \mu_f \quad (\text{since } u \notin F) \\ &= \sum_{s \in FU\{u\}} \mu_s = 1. \end{aligned}$$

Hence, (10) shows that p is a nontrivial convex combination of F . In other words, $p \notin \text{Shade } F$. This contradicts $p \in \text{Shade } F$. This contradiction shows that our assumption was false. This completes the proof of Statement 1.]

[*Proof of Statement 2:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must prove that $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$. If $u \notin F$, then this is obvious (since $F \setminus \{u\} = F$ in this case); thus, we WLOG assume that $u \in F$.

We have $u \in E \setminus \text{Shade } F$. Hence, $u \in E$ and $u \notin \text{Shade } F$. From $u \notin \text{Shade } F$, we see that u is a nontrivial convex combination of F . In other words, $u = \sum_{f \in F} \lambda_f f$ for some nonnegative reals λ_f smaller than 1 and satisfying $\sum_{f \in F} \lambda_f = 1$.

Consider these reals λ_f .

$$\text{We have } \sum_{g \in F} \lambda_g = \sum_{f \in F} \lambda_f = 1.$$

We have $\lambda_u < 1$ (since all reals λ_f are smaller than 1). Hence, $1 - \lambda_u > 0$, so that $\frac{1}{1 - \lambda_u}$ is a well-defined positive real.

We have

$$u = \sum_{f \in F} \lambda_f f = \sum_{f \in F \setminus \{u\}} \lambda_f f + \lambda_u u$$

(here, we have split off the addend for $f = u$ from the sum, since $u \in F$). In other words, $u - \lambda_u u = \sum_{f \in F \setminus \{u\}} \lambda_f f$. In other words, $(1 - \lambda_u) u = \sum_{f \in F \setminus \{u\}} \lambda_f f$.

Since $1 - \lambda_u > 0$, we can divide this equality by $1 - \lambda_u$, and obtain

$$u = \frac{1}{1 - \lambda_u} \sum_{f \in F \setminus \{u\}} \lambda_f f = \sum_{f \in F \setminus \{u\}} \frac{\lambda_f}{1 - \lambda_u} f. \quad (11)$$

Clearly, the coefficients $\frac{\lambda_f}{1 - \lambda_u}$ on the right hand side of this equality are nonnegative reals (since $1 - \lambda_u > 0$). Moreover, they are easily seen to be smaller

Since all the μ_s are nonnegative, this subsum thus must be $\leq \sum_{s \in FU\{u\}} \mu_s$. In other words, $\mu_f + \mu_u \leq \sum_{s \in FU\{u\}} \mu_s$. Hence, $\mu_f + \underbrace{\mu_u \lambda_f}_{< \mu_u} < \mu_f + \mu_u \leq \sum_{s \in FU\{u\}} \mu_s = 1$. This contradicts $\mu_f + \mu_u \lambda_f \geq 1$. This contradiction shows that our assumption was wrong, qed.

than 1¹⁴. Finally, they satisfy

$$\begin{aligned} \sum_{f \in F \setminus \{u\}} \frac{\lambda_f}{1 - \lambda_u} &= \frac{1}{1 - \lambda_u} \underbrace{\sum_{f \in F \setminus \{u\}} \lambda_f}_{= \sum_{f \in F} \lambda_f - \lambda_u} = \frac{1}{1 - \lambda_u} \left(\underbrace{\sum_{f \in F} \lambda_f}_{=1} - \lambda_u \right) \\ &= \frac{1}{1 - \lambda_u} (1 - \lambda_u) = 1. \end{aligned}$$

Hence, (11) shows that u is a nontrivial convex combination of $F \setminus \{u\}$. In other words, $u \notin \text{Shade}(F \setminus \{u\})$. Thus, $u \in E \setminus \text{Shade}(F \setminus \{u\})$. Hence, Statement 1 (applied to $F \setminus \{u\}$ instead of F) yields $\text{Shade}((F \setminus \{u\}) \cup \{u\}) =$

¹⁴*Proof.* Assume the contrary. Thus, $\frac{\lambda_f}{1 - \lambda_u} \geq 1$ for some $f \in F \setminus \{u\}$. Consider this f . From

$$\frac{\lambda_f}{1 - \lambda_u} \geq 1, \text{ we obtain } \lambda_f \geq 1 - \lambda_u \text{ (since } 1 - \lambda_u > 0), \text{ thus } \lambda_f + \lambda_u \geq 1.$$

However, $f \neq u$ (since $f \in F \setminus \{u\}$), so that $\lambda_f + \lambda_u$ is a subsum of the sum $\sum_{g \in F} \lambda_g$.

Since all the λ_g are nonnegative, this subsum thus must be $\leq \sum_{g \in F} \lambda_g$. In other words,

$$\lambda_f + \lambda_u \leq \sum_{g \in F} \lambda_g. \text{ Hence, } \lambda_f + \lambda_u \leq \sum_{g \in F} \lambda_g = 1. \text{ Combining this with } \lambda_f + \lambda_u \geq 1, \text{ we}$$

obtain $\lambda_f + \lambda_u = 1$. Thus, $\lambda_f = 1 - \lambda_u$.

Now, we observe that $\lambda_f + \lambda_u \geq 1 > \lambda_u$ (since $\lambda_u < 1$), so that $\lambda_f > 0$. Moreover, if $h \in F$ satisfies $h \neq f$ and $h \neq u$, then $\lambda_f + \lambda_u + \lambda_h$ is a subsum of the sum $\sum_{g \in F} \lambda_g$

and thus satisfies $\lambda_f + \lambda_u + \lambda_h \leq \sum_{g \in F} \lambda_g$ (since all the λ_g are nonnegative) and therefore

$$\lambda_f + \lambda_u + \lambda_h \leq \sum_{g \in F} \lambda_g = 1 = \lambda_f + \lambda_u, \text{ so that } \lambda_h \leq 0 \text{ and therefore } \lambda_h = 0 \text{ (since } \lambda_h \text{ is a}$$

nonnegative real). Thus, we have shown that all elements $h \in F$ except for (possibly) f and u satisfy $\lambda_h = 0$. Thus,

$$\sum_{h \in F} \lambda_h h = \lambda_f f + \lambda_u u \quad (\text{since } f \text{ and } u \text{ are distinct elements of } F).$$

However, recall that $u = \sum_{h \in F} \lambda_h h$. (This is just the old equation $u = \sum_{f \in F} \lambda_f f$, except that we had to rename the summation index f as h since f means something else now.) Thus,

$$u = \sum_{h \in F} \lambda_h h = \lambda_f f + \lambda_u u,$$

so that $u - \lambda_u u = \lambda_f f$. In view of $u - \lambda_u u = (1 - \lambda_u)u$ and $\lambda_f = 1 - \lambda_u$, this rewrites as $(1 - \lambda_u)u = (1 - \lambda_u)f$. We can cancel $1 - \lambda_u$ from this equality (since $1 - \lambda_u > 0$), and thus obtain $u = f$. This contradicts $f \neq u$. This contradiction shows that our assumption was wrong, qed.

Shade $(F \setminus \{u\})$. Therefore,

$$\text{Shade}(F \setminus \{u\}) = \text{Shade} \left(\underbrace{(F \setminus \{u\}) \cup \{u\}}_{\substack{=F \\ (\text{since } u \in F)}} \right) = \text{Shade } F.$$

This proves Statement 2.]

Now, we have proved Statements 1 and 2. Thus, the map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ satisfies the two axioms in Definition 4.2. In other words, this map is a shade map. Moreover, this map is inclusion-reversing (by Statement 0). This proves Example 4.10. \square

As a contrast to Example 4.10, let us mention a not-quite-example (satisfying only one of the two axioms in Theorem 4.3):

Example 4.11. Let V be a vector space over \mathbb{R} . If S is a finite subset of V , then a *nontrivial conic combination* of S will mean a vector of the form $\sum_{s \in S} \lambda_s s \in V$, where the coefficients λ_s are nonnegative reals with the property that at least two elements $s \in S$ satisfy $\lambda_s > 0$.

Fix a finite subset E of V . For any $F \subseteq E$, we define

$$\text{Shade } F = \{e \in E \mid e \text{ is not a nontrivial conic combination of } F\}.$$

It can be shown that this map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ satisfies Axiom 1 in Definition 4.2. In general, it does not satisfy Axiom 2. Thus, it is not a shade map in general.

4.3. Antimatroids and inclusion-reversing shade maps

Examples 4.9 and 4.10 are instances of a general class of examples: shade maps coming from *antimatroids*. Not unlike matroids, antimatroids are a combinatorial concept with many equivalent avatars (see, e.g., [KoLoSc91, Chapter III]). Here we shall view them through one of these avatars: that of *antimatroidal quasi-closure operators* (roughly equivalent to *convex geometries*). We begin by defining the notions we need:

Definition 4.12. Let E be any set.

(a) A *quasi-closure operator* on E means a map $\tau : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ with the following properties:

1. We have $A \subseteq \tau(A)$ for any $A \subseteq E$.
2. If A and B are two subsets of E satisfying $A \subseteq B$, then $\tau(A) \subseteq \tau(B)$.

3. We have $\tau(\tau(A)) = \tau(A)$ for any $A \subseteq E$.

(b) A quasi-closure operator τ on E is said to be *antimatroidal* if it has the following additional property:

4. If X is a subset of E , and if y and z are two distinct elements of $E \setminus \tau(X)$ satisfying $z \in \tau(X \cup \{y\})$, then $y \notin \tau(X \cup \{z\})$.

(c) A *closure operator* on E means a quasi-closure operator τ on E that satisfies $\tau(\emptyset) = \emptyset$.

(d) If τ is an antimatroidal closure operator on E , then the pair (E, τ) is called a *convex geometry*.

Here are some examples of antimatroidal quasi-closure operators:

Example 4.13. Let E be a poset. For any $F \subseteq E$, we define

$$\tau(F) = \{e \in E \mid \text{there exists an } f \in F \text{ with } e \leq f\}.$$

Then, τ is an antimatroidal closure operator on E . (This example is the “downset alignment” from [EdeJam85, §3, Example II], and is equivalent to the “poset antimatroid” from [KoLoSc91, §III.2.3].)

Example 4.14. Let E be a poset. For any $F \subseteq E$, we define

$$\tau(F) = \{e \in E \mid \text{there exist } f \in F \text{ and } g \in F \text{ with } g \leq e \leq f\}.$$

Then, τ is an antimatroidal closure operator on E . (This example is the “order convex alignment” from [EdeJam85, §3, Example II], and is the “double shelling of a poset” example from [KoLoSc91, §III.2.4].)

Example 4.15. Let A be an affine space over \mathbb{R} . If S is a finite subset of A , then a *convex combination* of S will mean a point of the form $\sum_{s \in S} \lambda_s s \in A$, where the coefficients λ_s are nonnegative reals satisfying $\sum_{s \in S} \lambda_s = 1$.

Fix a finite subset E of A . For any $F \subseteq E$, we define

$$\tau(F) = \{e \in E \mid e \text{ is a convex combination of } F\}.$$

Then, τ is an antimatroidal closure operator on E . (This example is [EdeJam85, §3, Example I]; it gave the name “convex geometry” to the notion defined in Definition 4.12 **(d)**.)

Example 4.16. Let Γ be any graph with edge set E . Fix a vertex v of Γ . We say that a subset $F \subseteq E$ *blocks* an edge $e \in E$ if each path of Γ that contains v and e must contain at least one edge of F . (In particular, this is automatically the case when $e \in F$.) For each $F \subseteq E$, we define

$$\tau(F) = \{e \in E \mid F \text{ blocks } e\}.$$

Then, τ is an antimatroidal quasi-closure operator on E . (This example is the “line-search antimatroid” from [KoLoSc91, §III.2.11].)

If Γ is connected, then τ is actually a closure operator.

Further examples of antimatroidal closure operators can be found in [KoLoSc91, §III.2] and [EdeJam85, §3].

We shall be dealing with quasi-closure operators rather than closure operators most of the time. However, since the latter concept is somewhat more widespread, let us comment on the connection between the two. Roughly speaking, the relation between quasi-closure and closure operators is comparable to the relation between semigroups and monoids, or between nonunital rings and unital rings, or (perhaps the best analogue) between simplicial complexes in general and simplicial complexes without ghost vertices (i.e., simplicial complexes for which every element of the ground set is a dimension-0 face). More concretely, specifying a quasi-closure operator on a set E is tantamount to specifying a subset of E and a closure operator on this subset. To wit:

Proposition 4.17. Let E be a set. Let L be a subset of E .

(a) Then, there is a bijection from

$$\{\text{quasi-closure operators } \tau \text{ on } E \text{ satisfying } \tau(\emptyset) = L\}$$

to

$$\{\text{closure operators } \sigma \text{ on } E \setminus L\}$$

that is defined as follows: It sends each quasi-closure operator τ to the closure operator σ that sends each $F \subseteq E \setminus L$ to $\tau(F) \setminus L$.

(b) This bijection restricts to a bijection from

$$\{\text{antimatroidal quasi-closure operators } \tau \text{ on } E \text{ satisfying } \tau(\emptyset) = L\}$$

to

$$\{\text{antimatroidal closure operators } \sigma \text{ on } E \setminus L\}.$$

Proposition 4.17 will not be important to what follows, so we omit the (rather straightforward) proof.

Now, we claim the following:

Theorem 4.18. Let E be a set. Let $\tau : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be an antimatroidal quasi-closure operator on E . For any $F \subseteq E$, we define

$$\text{Shade } F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}. \quad (12)$$

Then, this map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is an inclusion-reversing shade map.

Theorem 4.18 generalizes Examples 4.9 and 4.10. Indeed, applying Theorem 4.18 to the setting of Example 4.13, we can easily recover the claim of Example 4.9. Likewise, applying Theorem 4.18 to the setting of Example 4.15, we can recover the claim of Example 4.10. Less directly, Lemma 2.4 and its vertex-infection analogue are particular cases of Theorem 4.18 as well (even though they involve shade maps that are inclusion-preserving rather than inclusion-reversing). Indeed, if we apply Theorem 4.18 to the setting of Example 4.16, then we obtain the claim of Lemma 2.4 with $\text{Shade } F$ replaced by $\text{Shade}(E \setminus F)$; this is easily seen to be equivalent to Lemma 2.4 (by the duality stated in Proposition 4.7).

We shall soon prove Theorem 4.18; first we set up two lemmas:

Lemma 4.19. Let E be a set. Let τ be a quasi-closure operator on E . Let X be a subset of E , and let $z \in \tau(X)$. Then, $\tau(X \cup \{z\}) = \tau(X)$.

Proof of Lemma 4.19. We have $z \in \tau(X)$, so that $\tau(X) \cup \{z\} = \tau(X)$. Hence, $\tau(\tau(X) \cup \{z\}) = \tau(\tau(X)) = \tau(X)$ (by Property 3 in Definition 4.12 (a)). However, $X \subseteq \tau(X)$ (by Property 1 in Definition 4.12 (a)) and thus $\underbrace{X}_{\subseteq \tau(X)} \cup \{z\} \subseteq \tau(X) \cup \{z\}$. Hence, Property 2 in Definition 4.12 (a) (applied to $A = X \cup \{z\}$ and $B = \tau(X) \cup \{z\}$) yields $\tau(X \cup \{z\}) \subseteq \tau(\tau(X) \cup \{z\}) = \tau(X)$. On the other hand, $X \subseteq X \cup \{z\}$; thus, Property 2 in Definition 4.12 (a) (applied to $A = X$ and $B = X \cup \{z\}$) yields $\tau(X) \subseteq \tau(X \cup \{z\})$. Combining this with $\tau(X \cup \{z\}) \subseteq \tau(X)$, we obtain $\tau(X \cup \{z\}) = \tau(X)$. This proves Lemma 4.19. \square

Lemma 4.20. Let E be a set. Let τ be an antimatroidal quasi-closure operator on E . Let X be a subset of E , and let y and z be two distinct elements of E satisfying $z \in \tau(X \cup \{y\})$ and $y \in \tau(X \cup \{z\})$. Then, $y \in \tau(X)$.

Proof of Lemma 4.20. Assume the contrary. Thus, $y \notin \tau(X)$, so that $y \in E \setminus \tau(X)$.

If we had $z \in \tau(X)$, then Lemma 4.19 would yield $\tau(X \cup \{z\}) = \tau(X)$ and therefore $y \in \tau(X \cup \{z\}) = \tau(X)$, which would contradict $y \notin \tau(X)$. Hence, we cannot have $z \in \tau(X)$. Thus, we have $z \notin \tau(X)$. Therefore, $z \in E \setminus \tau(X)$.

Now, we know that y and z are two distinct elements of $E \setminus \tau(X)$ satisfying $z \in \tau(X \cup \{y\})$. Hence, Property 4 in Definition 4.12 (b) shows that $y \notin \tau(X \cup \{z\})$. This contradicts $y \in \tau(X \cup \{z\})$. This contradiction shows that our assumption was false. Thus, Lemma 4.20 is proven. \square

Note that Lemma 4.20 has a converse: If τ is a quasi-closure operator on E satisfying the claim of Lemma 4.20, then τ is antimatroidal. This is easy to see but will not be used in what follows.

Proof of Theorem 4.18. We shall prove the following three statements:

Statement 0: If A and B are two subsets of E such that $A \subseteq B$, then $\text{Shade } B \subseteq \text{Shade } A$.

Statement 1: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \cup \{u\}) = \text{Shade } F$.

Statement 2: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$.

[*Proof of Statement 0:* Let A and B be two subsets of E such that $A \subseteq B$. We must prove that $\text{Shade } B \subseteq \text{Shade } A$.

Let $u \in \text{Shade } B$. Thus,

$$u \in \text{Shade } B = \{e \in E \mid e \notin \tau(B \setminus \{e\})\}$$

(by the definition of $\text{Shade } B$). In other words, $u \in E$ and $u \notin \tau(B \setminus \{u\})$.

However, $A \setminus \{u\} \subseteq B \setminus \{u\}$ (since $A \subseteq B$) and thus $\tau(A \setminus \{u\}) \subseteq \tau(B \setminus \{u\})$ (by Property 2 in Definition 4.12 (a), applied to $A \setminus \{u\}$ and $B \setminus \{u\}$ instead of A and B). Hence, from $u \notin \tau(B \setminus \{u\})$, we obtain $u \notin \tau(A \setminus \{u\})$. Therefore, $u \in \{e \in E \mid e \notin \tau(A \setminus \{e\})\} = \text{Shade } A$ (by the definition of $\text{Shade } A$).

Forget that we fixed u . We thus have shown that $u \in \text{Shade } A$ for each $u \in \text{Shade } B$. In other words, $\text{Shade } B \subseteq \text{Shade } A$. This proves Statement 0.]

[*Proof of Statement 2:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must prove that $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$.

We have $u \in E \setminus \text{Shade } F = \{e \in E \mid e \in \tau(F \setminus \{e\})\}$ (since $\text{Shade } F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}$). In other words, $u \in E$ and $u \in \tau(F \setminus \{u\})$.

We have $F \setminus \{u\} \subseteq F$ and thus $\text{Shade } F \subseteq \text{Shade}(F \setminus \{u\})$ (by Statement 0, applied to $A = F \setminus \{u\}$ and $B = F$).

Now, let $v \in \text{Shade}(F \setminus \{u\}) \setminus \text{Shade } F$. We shall derive a contradiction.

We have

$$v \in \underbrace{\text{Shade}(F \setminus \{u\})}_{\subseteq E} \setminus \text{Shade } F \subseteq E \setminus \text{Shade } F = \{e \in E \mid e \in \tau(F \setminus \{e\})\}.$$

In other words, $v \in E$ and $v \in \tau(F \setminus \{v\})$.

On the other hand,

$$\begin{aligned} v &\in \text{Shade}(F \setminus \{u\}) \setminus \text{Shade } F \\ &\subseteq \text{Shade}(F \setminus \{u\}) = \{e \in E \mid e \notin \tau((F \setminus \{u\}) \setminus \{e\})\} \end{aligned}$$

(by the definition of $\text{Shade}(F \setminus \{u\})$). In other words,

$$v \in E \quad \text{and} \quad v \notin \tau((F \setminus \{u\}) \setminus \{v\}).$$

Let $X = (F \setminus \{u\}) \setminus \{v\}$. Then, $X \cup \{v\} \supseteq F \setminus \{u\}$, so that $F \setminus \{u\} \subseteq X \cup \{v\}$ and therefore $\tau(F \setminus \{u\}) \subseteq \tau(X \cup \{v\})$ (by Property 2 in Definition 4.12 **(a)**, applied to $A = F \setminus \{u\}$ and $B = X \cup \{v\}$). Hence, $u \in \tau(F \setminus \{u\}) \subseteq \tau(X \cup \{v\})$.

Also, from $X = (F \setminus \{u\}) \setminus \{v\} = (F \setminus \{v\}) \setminus \{u\}$, we obtain $X \cup \{u\} \supseteq F \setminus \{v\}$, so that $F \setminus \{v\} \subseteq X \cup \{u\}$ and therefore $\tau(F \setminus \{v\}) \subseteq \tau(X \cup \{u\})$ (by Property 2 in Definition 4.12 **(a)**, applied to $A = F \setminus \{v\}$ and $B = X \cup \{u\}$). Hence, $v \in \tau(F \setminus \{v\}) \subseteq \tau(X \cup \{u\})$.

If we had $v = u$, then we would have $(F \setminus \{u\}) \setminus \{v\} = (F \setminus \{u\}) \setminus \{u\} = F \setminus \{u\} = F \setminus \{v\}$ (since $u = v$) and therefore $v \notin \tau\left(\underbrace{(F \setminus \{u\}) \setminus \{v\}}_{=F \setminus \{v\}}\right) = \tau(F \setminus \{v\})$, which would contradict $v \in \tau(F \setminus \{v\})$. Thus, we cannot have $v = u$. Hence, v and u are distinct.

Thus, Lemma 4.20 (applied to $y = v$ and $z = u$) yields $v \in \tau(X)$ (since $v \in \tau(X \cup \{u\})$ and $u \in \tau(X \cup \{v\})$). In other words, $v \in \tau((F \setminus \{u\}) \setminus \{v\})$ (since $X = (F \setminus \{u\}) \setminus \{v\}$). But this contradicts $v \notin \tau((F \setminus \{u\}) \setminus \{v\})$.

Forget that we fixed v . We thus have found a contradiction for each $v \in \text{Shade}(F \setminus \{u\}) \setminus \text{Shade} F$. Hence, there exists no such v . In other words, the set $\text{Shade}(F \setminus \{u\}) \setminus \text{Shade} F$ is empty. Hence, $\text{Shade}(F \setminus \{u\}) \subseteq \text{Shade} F$. Combining this with $\text{Shade} F \subseteq \text{Shade}(F \setminus \{u\})$, we obtain $\text{Shade}(F \setminus \{u\}) = \text{Shade} F$. This proves Statement 2.]

[*Proof of Statement 1:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade} F$. We must prove that $\text{Shade}(F \cup \{u\}) = \text{Shade} F$. If $u \in F$, then this is obvious (since $F \cup \{u\} = F$ in this case). Thus, we WLOG assume that $u \notin F$. Hence, $(F \cup \{u\}) \setminus \{u\} = F$.

We have $F \subseteq F \cup \{u\}$ and thus $\text{Shade}(F \cup \{u\}) \subseteq \text{Shade} F$ (by Statement 0, applied to $A = F$ and $B = F \cup \{u\}$). Hence, $E \setminus \underbrace{\text{Shade}(F \cup \{u\})}_{\subseteq \text{Shade} F} \supseteq E \setminus \text{Shade} F$,

so that $E \setminus \text{Shade} F \subseteq E \setminus \text{Shade}(F \cup \{u\})$.

Now, $u \in E \setminus \text{Shade} F \subseteq E \setminus \text{Shade}(F \cup \{u\})$. Hence, Statement 2 (applied to $F \cup \{u\}$ instead of F) yields $\text{Shade}((F \cup \{u\}) \setminus \{u\}) = \text{Shade}(F \cup \{u\})$. In view of $(F \cup \{u\}) \setminus \{u\} = F$, this rewrites as $\text{Shade} F = \text{Shade}(F \cup \{u\})$. Hence, $\text{Shade}(F \cup \{u\}) = \text{Shade} F$. This proves Statement 1.]

Now, we have proved Statements 1 and 2. Thus, the map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ satisfies the two axioms in Definition 4.2. In other words, this map is a shade map. Moreover, this map is inclusion-reversing (by Statement 0). Thus, Theorem 4.18 is proved. \square

We note that the quasi-closure operator τ in Theorem 4.18 can be reconstructed from the map Shade . This does not even require τ to be antimatroidal; the following holds for any quasi-closure operator:

Proposition 4.21. Let E be a set. Let $\tau : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be a quasi-closure operator on E . For any $F \subseteq E$, we define

$$\text{Shade } F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}.$$

Then, each $F \subseteq E$ satisfies

$$\tau(F) = F \cup (E \setminus \text{Shade } F). \quad (13)$$

Proof of Proposition 4.21. Let F be a subset of E . We must prove (13).

We shall prove the two inclusions $\tau(F) \subseteq F \cup (E \setminus \text{Shade } F)$ and $F \cup (E \setminus \text{Shade } F) \subseteq \tau(F)$ separately:

[*Proof of $\tau(F) \subseteq F \cup (E \setminus \text{Shade } F)$:* Let $u \in \tau(F)$. We shall show that $u \in F \cup (E \setminus \text{Shade } F)$. Indeed, assume the contrary. Thus, $u \notin F \cup (E \setminus \text{Shade } F)$. In other words, $u \notin F$ and $u \notin E \setminus \text{Shade } F$. From $u \in \tau(F) \subseteq E$ and $u \notin E \setminus \text{Shade } F$, we obtain

$$u \in E \setminus (E \setminus \text{Shade } F) \subseteq \text{Shade } F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}.$$

In other words, $u \in E$ and $u \notin \tau(F \setminus \{u\})$. However, $u \notin F$ shows that $F \setminus \{u\} = F$. Now, $u \notin \tau(F \setminus \{u\}) = \tau(F)$ (since $F \setminus \{u\} = F$) contradicts $u \in \tau(F)$. This contradiction shows that our assumption was false. Hence, $u \in F \cup (E \setminus \text{Shade } F)$ is proved.

Now, forget that we fixed u . We thus have proved that $u \in F \cup (E \setminus \text{Shade } F)$ for each $u \in \tau(F)$. In other words, $\tau(F) \subseteq F \cup (E \setminus \text{Shade } F)$.]

[*Proof of $F \cup (E \setminus \text{Shade } F) \subseteq \tau(F)$:* We have $F \subseteq \tau(F)$ (by Property 1 in Definition 4.12 (a)).

Now, let $v \in E \setminus \text{Shade } F$. Thus,

$$v \in E \setminus \text{Shade } F = \{e \in E \mid e \in \tau(F \setminus \{e\})\}$$

(since $\text{Shade } F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}$). In other words, $v \in E$ and $v \in \tau(F \setminus \{v\})$. However, $F \setminus \{v\} \subseteq F$ and thus $\tau(F \setminus \{v\}) \subseteq \tau(F)$ (by Property 2 in Definition 4.12 (a), applied to $A = F \setminus \{v\}$ and $B = F$). Thus, $v \in \tau(F \setminus \{v\}) \subseteq \tau(F)$.

Forget that we fixed v . We thus have shown that $v \in \tau(F)$ for each $v \in E \setminus \text{Shade } F$. In other words, $E \setminus \text{Shade } F \subseteq \tau(F)$. Combining this with $F \subseteq \tau(F)$, we obtain $F \cup (E \setminus \text{Shade } F) \subseteq \tau(F)$.]

Combining the two inclusions $\tau(F) \subseteq F \cup (E \setminus \text{Shade } F)$ and $F \cup (E \setminus \text{Shade } F) \subseteq \tau(F)$, we obtain $\tau(F) = F \cup (E \setminus \text{Shade } F)$. This proves (13). Proposition 4.21 is thus proven. \square

It turns out that if one applies the formula (13) to an inclusion-reversing shade map Shade , then the resulting map τ is an antimatroidal quasi-closure operator, at least when E is finite. In fact, we have the following:

Proposition 4.22. Let E be a finite set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be an inclusion-reversing shade map. Define a map $\tau : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ by setting

$$\tau(F) = F \cup (E \setminus \text{Shade} F) \quad (14)$$

for each $F \subseteq E$. Then, τ is an antimatroidal quasi-closure operator on E .

The proof of this proposition rests on the following lemma:

Lemma 4.23. Let E be a set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be a shade map. Let A and B be two subsets of E such that B is finite and $B \cap \text{Shade} A = \emptyset$. Then, $\text{Shade}(A \cup B) = \text{Shade} A$.

Proof of Lemma 4.23. We proceed by induction on $|B|$:

Induction base: Lemma 4.23 holds for $|B| = 0$ ¹⁵.

Induction step: Let $k \in \mathbb{N}$. Assume that Lemma 4.23 holds for $|B| = k$. We must prove that Lemma 4.23 holds for $|B| = k + 1$.

We have assumed that Lemma 4.23 holds for $|B| = k$. In other words, if A and B are two subsets of E such that B is finite and $B \cap \text{Shade} A = \emptyset$ and $|B| = k$, then

$$\text{Shade}(A \cup B) = \text{Shade} A. \quad (15)$$

Now, let A and B be two subsets of E such that B is finite and $B \cap \text{Shade} A = \emptyset$ and $|B| = k + 1$. We shall prove that $\text{Shade}(A \cup B) = \text{Shade} A$.

We assumed that Shade is a shade map. Thus, in particular, Shade satisfies Axiom 1 in Definition 4.2.

The set B is nonempty (since $|B| = k + 1 > k \geq 0$). Thus, there exists some $u \in B$. Consider this u . If we had $u \in \text{Shade} A$, then we would have $u \in B \cap \text{Shade} A$ (since $u \in B$ and $u \in \text{Shade} A$), which would entail $B \cap \text{Shade} A \neq \emptyset$; but this would contradict $B \cap \text{Shade} A = \emptyset$. Hence, we cannot have $u \in \text{Shade} A$. In other words, we have $u \notin \text{Shade} A$. Hence, $u \in E \setminus \text{Shade} A$.

From $u \in B$, we obtain $|B \setminus \{u\}| = |B| - 1 = k$ (since $|B| = k + 1$). Moreover, $(B \setminus \{u\}) \cap \text{Shade} A = \emptyset$ (since $\underbrace{(B \setminus \{u\})}_{\subseteq B} \cap \text{Shade} A \subseteq B \cap \text{Shade} A = \emptyset$).

Thus, we can apply (15) to $B \setminus \{u\}$ instead of B . As a consequence, we obtain $\text{Shade}(A \cup (B \setminus \{u\})) = \text{Shade} A$. Thus, $u \in E \setminus \text{Shade}(A \cup (B \setminus \{u\}))$ (since $u \in E \setminus \text{Shade} A$). Thus, Axiom 1 in Definition 4.2 (applied to $F = A \cup (B \setminus \{u\})$) yields

$$\text{Shade}((A \cup (B \setminus \{u\})) \cup \{u\}) = \text{Shade}(A \cup (B \setminus \{u\})) = \text{Shade} A.$$

In view of

$$(A \cup (B \setminus \{u\})) \cup \{u\} = A \cup \underbrace{(B \setminus \{u\}) \cup \{u\}}_{\substack{=B \\ (\text{since } u \in B)}} = A \cup B,$$

¹⁵*Proof.* Assume that $|B| = 0$. Thus, $B = \emptyset$, so that $A \cup B = A \cup \emptyset = A$. Hence, $\text{Shade}(A \cup B) = \text{Shade} A$. Thus, Lemma 4.23 is proven for $|B| = 0$.

this rewrites as $\text{Shade}(A \cup B) = \text{Shade } A$.

Forget that we fixed A and B . We thus have shown that if A and B are two subsets of E such that B is finite and $B \cap \text{Shade } A = \emptyset$ and $|B| = k + 1$, then $\text{Shade}(A \cup B) = \text{Shade } A$. In other words, Lemma 4.23 holds for $|B| = k + 1$. This completes the induction step. Thus, Lemma 4.23 is proven. \square

Proof of Proposition 4.22. We shall show that the four Properties 1, 2, 3 and 4 from Definition 4.12 are satisfied for our map τ .

[*Proof of Property 1:* Let A be any subset of E . We must show that $A \subseteq \tau(A)$.

The definition of τ yields $\tau(A) = A \cup (E \setminus \text{Shade } A) \supseteq A$, so that $A \subseteq \tau(A)$. Thus, Property 1 is proved.]

[*Proof of Property 2:* Let A and B be two subsets of E satisfying $A \subseteq B$. We must prove that $\tau(A) \subseteq \tau(B)$.

We have $A \subseteq B$ and therefore $\text{Shade } B \subseteq \text{Shade } A$ (since Shade is inclusion-reversing). Thus, $E \setminus \underbrace{\text{Shade } B}_{\subseteq \text{Shade } A} \supseteq E \setminus \text{Shade } A$, so that $E \setminus \text{Shade } A \subseteq E \setminus \text{Shade } B$.

However, the definition of τ yields $\tau(A) = A \cup (E \setminus \text{Shade } A)$ and $\tau(B) = B \cup (E \setminus \text{Shade } B)$. Therefore,

$$\tau(A) = \underbrace{A}_{\subseteq B} \cup \underbrace{(E \setminus \text{Shade } A)}_{\subseteq E \setminus \text{Shade } B} \subseteq B \cup (E \setminus \text{Shade } B) = \tau(B).$$

This proves Property 2.]

[*Proof of Property 3:* Let A be any subset of E . We must show that $\tau(\tau(A)) = \tau(A)$.

The set $E \setminus \text{Shade } A$ is finite (since E is finite) and satisfies $(E \setminus \text{Shade } A) \cap \text{Shade } A = \emptyset$. Thus, Lemma 4.23 (applied to $B = E \setminus \text{Shade } A$) yields $\text{Shade}(A \cup (E \setminus \text{Shade } A)) = \text{Shade } A$. However, the definition of τ yields $\tau(A) = A \cup (E \setminus \text{Shade } A)$. Thus,

$$\text{Shade}(\tau(A)) = \text{Shade}(A \cup (E \setminus \text{Shade } A)) = \text{Shade } A.$$

Now, the definition of τ yields

$$\begin{aligned} \tau(\tau(A)) &= \underbrace{\tau(A)}_{=A \cup (E \setminus \text{Shade } A)} \cup \left(E \setminus \underbrace{\text{Shade}(\tau(A))}_{=\text{Shade } A} \right) \\ &= A \cup \underbrace{(E \setminus \text{Shade } A) \cup (E \setminus \text{Shade } A)}_{=(E \setminus \text{Shade } A)} = A \cup (E \setminus \text{Shade } A) = \tau(A). \end{aligned}$$

This proves Property 3.]

[*Proof of Property 4:* Let X be a subset of E , and let y and z be two distinct elements of $E \setminus \tau(X)$ satisfying $z \in \tau(X \cup \{y\})$. We must prove that $y \notin \tau(X \cup \{z\})$.

Indeed, assume the contrary. Thus, $y \in \tau(X \cup \{z\})$.

From $y \in E \setminus \tau(X)$, we obtain $y \notin \tau(X) = X \cup (E \setminus \text{Shade } X)$ (by the definition of τ). In other words, $y \notin X$ and $y \notin E \setminus \text{Shade } X$. Combining $y \in E$ and $y \notin E \setminus \text{Shade } X$, we obtain $y \in E \setminus (E \setminus \text{Shade } X) \subseteq \text{Shade } X$.

We have thus proved that $y \notin X$ and $y \in \text{Shade } X$. The same argument (applied to z instead of y) shows that $z \notin X$ and $z \in \text{Shade } X$.

We assumed that Shade is a shade map. Thus, in particular, Shade satisfies both Axioms 1 and 2 in Definition 4.2.

We have $z \neq y$ (since y and z are distinct) and thus $z \notin \{y\}$. Combining this with $z \notin X$, we obtain $z \notin X \cup \{y\}$. However, by assumption, we have

$$z \in \tau(X \cup \{y\}) = (X \cup \{y\}) \cup (E \setminus \text{Shade}(X \cup \{y\}))$$

(by the definition of τ). In other words, we have $z \in X \cup \{y\}$ or $z \in E \setminus \text{Shade}(X \cup \{y\})$. Since $z \in X \cup \{y\}$ is impossible (because $z \notin X \cup \{y\}$), we thus obtain $z \in E \setminus \text{Shade}(X \cup \{y\})$. Hence, Axiom 1 in Definition 4.2 (applied to $F = X \cup \{y\}$ and $u = z$) yields

$$\text{Shade}((X \cup \{y\}) \cup \{z\}) = \text{Shade}(X \cup \{y\}).$$

The same argument (with the roles of y and z swapped) shows that

$$\text{Shade}((X \cup \{z\}) \cup \{y\}) = \text{Shade}(X \cup \{z\})$$

(since $y \in \tau(X \cup \{z\})$). Hence,

$$\begin{aligned} \text{Shade}(X \cup \{z\}) &= \text{Shade} \left(\underbrace{(X \cup \{z\}) \cup \{y\}}_{=(X \cup \{y\}) \cup \{z\}} \right) \\ &= \text{Shade}((X \cup \{y\}) \cup \{z\}) = \text{Shade}(X \cup \{y\}). \end{aligned}$$

However, recall that $z \in E \setminus \underbrace{\text{Shade}(X \cup \{y\})}_{=\text{Shade}(X \cup \{z\})} = E \setminus \text{Shade}(X \cup \{z\})$. Therefore, Axiom 2 in Definition 4.2 (applied to $F = X \cup \{z\}$ and $u = z$) yields

$$\text{Shade}((X \cup \{z\}) \setminus \{z\}) = \text{Shade}(X \cup \{z\}) = \text{Shade}(X \cup \{y\}).$$

Thus,

$$\text{Shade}(X \cup \{y\}) = \text{Shade} \left(\underbrace{(X \cup \{z\}) \setminus \{z\}}_{\substack{=X \\ (\text{since } z \notin X)}} \right) = \text{Shade } X.$$

Recall that $z \in E \setminus \underbrace{\text{Shade}(X \cup \{y\})}_{=\text{Shade } X} = E \setminus \text{Shade } X$. Hence, $z \notin \text{Shade } X$. This clearly contradicts $z \in \text{Shade } X$. This contradiction shows that our assumption was false. Hence, we must have $y \notin \tau(X \cup \{z\})$. This proves Property 4.]

Now, the map τ is a quasi-closure operator (since Properties 1, 2 and 3 in Definition 4.12 (a) are satisfied for it), and is antimatroidal (since Property 4 in Definition 4.12 (b) is satisfied for it). Thus, Proposition 4.22 is proved. \square

Proposition 4.21 has a (sort of) converse:

Proposition 4.24. Let E be a set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be an inclusion-reversing shade map. For any $F \subseteq E$, we define

$$\tau(F) = F \cup (E \setminus \text{Shade } F).$$

Then, each $F \subseteq E$ satisfies

$$\text{Shade } F = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}. \quad (16)$$

Proof of Proposition 4.24. We assumed that Shade is a shade map. Thus, in particular, Shade satisfies both Axioms 1 and 2 in Definition 4.2.

Let F be a subset of E . We set $X = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}$. We shall prove the inclusions $\text{Shade } F \subseteq X$ and $X \subseteq \text{Shade } F$ separately.

[*Proof of $\text{Shade } F \subseteq X$:* Let $u \in \text{Shade } F$. We shall show that $u \in X$.

Indeed, assume the contrary. Thus, $u \notin X$. Combining this with $u \in \text{Shade } F \subseteq E$, we find $u \in E \setminus X = \{e \in E \mid e \in \tau(F \setminus \{e\})\}$ (since $X = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}$). In other words, $u \in E$ and $u \in \tau(F \setminus \{u\})$. However, the definition of τ yields $\tau(F \setminus \{u\}) = (F \setminus \{u\}) \cup (E \setminus \text{Shade}(F \setminus \{u\}))$. Thus,

$$u \in \tau(F \setminus \{u\}) = (F \setminus \{u\}) \cup (E \setminus \text{Shade}(F \setminus \{u\})).$$

In other words, we have $u \in F \setminus \{u\}$ or $u \in E \setminus \text{Shade}(F \setminus \{u\})$. Since $u \in F \setminus \{u\}$ is impossible, we thus have $u \in E \setminus \text{Shade}(F \setminus \{u\})$. Hence, $u \in E$ and $u \notin \text{Shade}(F \setminus \{u\})$. However, Shade is inclusion-reversing; thus, from $F \setminus \{u\} \subseteq F$, we obtain $\text{Shade } F \subseteq \text{Shade}(F \setminus \{u\})$. Thus, $u \in \text{Shade } F \subseteq \text{Shade}(F \setminus \{u\})$. But this contradicts $u \notin \text{Shade}(F \setminus \{u\})$. This contradiction shows that our assumption was false. Hence, $u \in X$ is proved.

Now, forget that we fixed u . We thus have shown that $u \in X$ for each $u \in \text{Shade } F$. In other words, $\text{Shade } F \subseteq X$.]

[*Proof of $X \subseteq \text{Shade } F$:* Let $u \in X$. We shall show that $u \in \text{Shade } F$.

Indeed, assume the contrary. Thus, $u \notin \text{Shade } F$. This entails $u \in E \setminus \text{Shade } F$ (since $u \in X \subseteq E$). Hence, Axiom 2 in Definition 4.2 yields $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$.

However, from $u \in X = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}$, we see that $u \in E$ and $u \notin \tau(F \setminus \{u\})$. Thus,

$$u \notin \tau(F \setminus \{u\}) = (F \setminus \{u\}) \cup (E \setminus \text{Shade}(F \setminus \{u\}))$$

(by the definition of τ). Hence, in particular, $u \notin E \setminus \text{Shade}(F \setminus \{u\})$. Since $u \in E$, we thus have $u \in \text{Shade}(F \setminus \{u\}) = \text{Shade } F$. But this contradicts $u \notin \text{Shade } F$. This contradiction shows that our assumption was false. Hence, $u \in \text{Shade } F$ is proved.

Now, forget that we fixed u . We thus have shown that $u \in \text{Shade } F$ for each $u \in X$. In other words, $X \subseteq \text{Shade } F$.]

Combining $\text{Shade } F \subseteq X$ with $X \subseteq \text{Shade } F$, we obtain $\text{Shade } F = X = \{e \in E \mid e \notin \tau(F \setminus \{e\})\}$. This proves Proposition 4.24. \square

Combining many of the results in this section, we obtain the following description of inclusion-reversing shade maps:

Theorem 4.25. Let E be a finite set. Then, there is a bijection from the set

$$\{\text{inclusion-reversing shade maps } \text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)\}$$

to the set

$$\{\text{antimatroidal quasi-closure operators } \tau : \mathcal{P}(E) \rightarrow \mathcal{P}(E)\}.$$

It sends each map Shade to the map τ defined by (14). Its inverse map sends each map τ to the map Shade defined by (12).

Proof of Theorem 4.25. Let

$$\mathcal{S} = \{\text{inclusion-reversing shade maps } \text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)\}$$

and

$$\mathcal{A} = \{\text{antimatroidal quasi-closure operators } \tau : \mathcal{P}(E) \rightarrow \mathcal{P}(E)\}.$$

Then, the map $\mathcal{S} \rightarrow \mathcal{A}$ that sends each map $\text{Shade} \in \mathcal{S}$ to the map τ defined by (14) is well-defined (by Proposition 4.22). Furthermore, the map $\mathcal{A} \rightarrow \mathcal{S}$ that sends each map $\tau \in \mathcal{A}$ to the map Shade defined by (12) is well-defined (by Theorem 4.18). These two maps $\mathcal{S} \rightarrow \mathcal{A}$ and $\mathcal{A} \rightarrow \mathcal{S}$ are mutually inverse (by Proposition 4.21 and Proposition 4.24), and thus are bijections. This proves Theorem 4.25. \square

Theorem 4.25 classifies **inclusion-reversing** shade maps in terms of antimatroidal quasi-closure operators¹⁶. The latter can in turn be described in terms of antimatroidal closure operators (by Proposition 4.17), i.e., in terms of antimatroids. Thus, inclusion-reversing shade maps “boil down” to antimatroids. The same can be said of **inclusion-preserving** shade maps (because Proposition 4.7 establishes a bijection between them and the inclusion-reversing ones). In the next subsection, we shall classify **arbitrary** shade maps in terms of what we will call *Boolean interval partitions*.

4.4. Boolean interval partitions and arbitrary shade maps

Let us first define Boolean interval partitions:

¹⁶In the parlance of matroid theorists, it shows that inclusion-reversing shade maps are *cryptomorphic* to antimatroidal quasi-closure operators.

Definition 4.26. Let E be a set.

(a) If U and V are two subsets of E satisfying $U \subseteq V$, then $[U, V]$ shall denote the subset $\{I \in \mathcal{P}(E) \mid U \subseteq I \subseteq V\}$ of $\mathcal{P}(E)$. This is the set of all subsets of E that lie between U and V (meaning that they contain U as a subset, but in turn are contained in V as subsets).

(b) A *Boolean interval* of $\mathcal{P}(E)$ shall mean a subset of $\mathcal{P}(E)$ that has the form $[U, V]$ for two subsets U and V of E satisfying $U \subseteq V$. Note that each Boolean interval $[U, V]$ of $\mathcal{P}(E)$ is nonempty (as it contains U and V), and the two subsets U and V can easily be reconstructed from it (namely, U is the intersection of all $I \in [U, V]$, whereas V is the union of all $I \in [U, V]$).

(c) A *Boolean interval partition* of $\mathcal{P}(E)$ means a set of pairwise disjoint Boolean intervals of $\mathcal{P}(E)$ whose union is $\mathcal{P}(E)$.

(d) If \mathbf{P} is a Boolean interval partition of $\mathcal{P}(E)$, then the elements of \mathbf{P} (that is, the Boolean intervals that belong to \mathbf{P}) are called the *blocks* of \mathbf{P} .

Example 4.27. For this example, let $E = \{1, 2, 3\}$. We shall use the shorthand $i_1 i_2 \cdots i_k$ for a subset $\{i_1, i_2, \dots, i_k\}$ of E . (For example, 13 means the subset $\{1, 3\}$.)

(a) We have

$$[1, 123] = \{I \in \mathcal{P}(123) \mid 1 \subseteq I \subseteq 123\} = \{1, 12, 13, 123\}$$

and

$$[1, 13] = \{I \in \mathcal{P}(123) \mid 1 \subseteq I \subseteq 13\} = \{1, 13\}.$$

(b) There are $3^3 = 27$ Boolean intervals of $\mathcal{P}(E)$. (More generally, if E is an n -element set, then there are 3^n Boolean intervals of $\mathcal{P}(E)$.)

(c) Here is one of many Boolean interval partitions of $\mathcal{P}(E)$ (where E is still $\{1, 2, 3\}$):

$$\left\{ \underbrace{\{\emptyset\}}_{=[\emptyset, \emptyset]}, \underbrace{\{1, 13\}}_{=[1, 13]}, \underbrace{\{3\}}_{=[3, 3]}, \underbrace{\{2, 12, 23, 123\}}_{=[2, 123]} \right\}.$$

Here is another:

$$\left\{ \underbrace{\{\emptyset, 1\}}_{=[\emptyset, 1]}, \underbrace{\{3, 13\}}_{=[3, 13]}, \underbrace{\{2, 23\}}_{=[2, 23]}, \underbrace{\{12\}}_{=[12, 12]}, \underbrace{\{123\}}_{=[123, 123]} \right\}.$$

The former has four blocks; the latter has five.

Here are two ways to think of Boolean interval partitions of $\mathcal{P}(E)$:

- The following is just a slick restatement of Definition 4.26 (c) using standard combinatorial lingo: A Boolean interval partition of $\mathcal{P}(E)$ is a set

partition of the Boolean lattice $\mathcal{P}(E)$ into intervals.

- It is well-known that the set partitions of a given set are in a canonical bijection with the equivalence relations on this set. In light of this, the Boolean interval partitions of $\mathcal{P}(E)$ can be viewed as the equivalence relations on $\mathcal{P}(E)$ whose equivalence classes are Boolean intervals. In other words, they can be viewed as the equivalence relations \sim on $\mathcal{P}(E)$ satisfying the axiom “if $U, V, I \in \mathcal{P}(E)$ satisfy $U \sim V$ and $U \cap V \subseteq I \subseteq U \cup V$, then $U \sim I \sim V$ ”. The reader can prove this alternative characterization as an easy exercise in Boolean algebra.

We note a crucial property of Boolean interval partitions (which we shall use tacitly):

Unique block property of Boolean interval partitions: Let E be a set. Let \mathbf{P} be a Boolean interval partition of $\mathcal{P}(E)$. Let $F \in \mathcal{P}(E)$. Then, there is a **unique** block of \mathbf{P} that contains F . Moreover, this block is a Boolean interval, i.e., it has the form $[\alpha(F), \tau(F)]$ for some $\alpha(F) \in \mathcal{P}(E)$ and some $\tau(F) \in \mathcal{P}(E)$. Finally, these $\alpha(F)$ and $\tau(F)$ are uniquely determined by F and \mathbf{P} .

Boolean interval partitions have come up in combinatorics before (e.g., [BrOlNo09], [Dawson80], [DedTit20], [GorMah97]).

We shall now construct a shade map from any Boolean interval partition:

Theorem 4.28. Let E be a set. Let \mathbf{P} be a Boolean interval partition of $\mathcal{P}(E)$.

For any $F \in \mathcal{P}(E)$, let $[\alpha(F), \tau(F)]$ denote the (unique) block of \mathbf{P} that contains F .

We define a map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ by setting

$$\text{Shade } F = E \setminus (\tau(F) \setminus \alpha(F)) \quad \text{for any } F \in \mathcal{P}(E).$$

Then:

(a) The map Shade is a shade map on E .

(b) We have $\alpha(F) = F \cap \text{Shade } F$ and $\tau(F) = F \cup (E \setminus \text{Shade } F)$ for any $F \in \mathcal{P}(E)$.

(c) We have $\mathbf{P} = \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}$.

The proof of Theorem 4.28 is rather easy. We lighten our burden somewhat with a simple lemma:

Lemma 4.29. Let X, Y, Z and E be four sets such that $X \subseteq Y \subseteq Z \subseteq E$. Then:

(a) We have $Y \cup (Z \setminus X) = Z$.

(b) We have $Y \cap (E \setminus (Z \setminus X)) = X$.

Proof of Lemma 4.29. This is straightforward set theory; it is easily checked (e.g.) using Venn diagrams. \square

Proof of Theorem 4.28. **(a)** We shall prove the following two statements:

Statement 1: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \cup \{u\}) = \text{Shade } F$.

Statement 2: If $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$, then $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$.

[*Proof of Statement 1:* Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must prove that $\text{Shade}(F \cup \{u\}) = \text{Shade } F$.

The definition of $\alpha(F)$ and $\tau(F)$ reveals that $[\alpha(F), \tau(F)]$ is the (unique) block of \mathbf{P} that contains F . Thus, $[\alpha(F), \tau(F)]$ is a block of \mathbf{P} , therefore a Boolean interval of $\mathcal{P}(E)$ (since \mathbf{P} is a Boolean interval partition of $\mathcal{P}(E)$). Hence, $\alpha(F)$ and $\tau(F)$ belong to $\mathcal{P}(E)$, hence are subsets of E . Hence, $\tau(F) \setminus \alpha(F)$ is a subset of E as well.

The definition of Shade yields $\text{Shade } F = E \setminus (\tau(F) \setminus \alpha(F))$. Now,

$$u \in E \setminus \underbrace{\text{Shade } F}_{=E \setminus (\tau(F) \setminus \alpha(F))} = E \setminus (E \setminus (\tau(F) \setminus \alpha(F))) = \tau(F) \setminus \alpha(F)$$

(since $\tau(F) \setminus \alpha(F)$ is a subset of E). In other words, $u \in \tau(F)$ and $u \notin \alpha(F)$.

On the other hand, the Boolean interval $[\alpha(F), \tau(F)]$ contains F (since $[\alpha(F), \tau(F)]$ is the (unique) block of \mathbf{P} that contains F). In other words, $\alpha(F) \subseteq F \subseteq \tau(F)$.

Now, set $F' = F \cup \{u\}$. From $u \in \tau(F)$, we obtain $\{u\} \subseteq \tau(F)$. Combining $F \subseteq \tau(F)$ with $\{u\} \subseteq \tau(F)$, we obtain $F \cup \{u\} \subseteq \tau(F)$. In other words, $F' \subseteq \tau(F)$ (since $F' = F \cup \{u\}$). Combining this with $\alpha(F) \subseteq F \subseteq F \cup \{u\} = F'$, we obtain the chain of inclusions $\alpha(F) \subseteq F' \subseteq \tau(F)$. In other words, $F' \in [\alpha(F), \tau(F)]$. Hence, $[\alpha(F), \tau(F)]$ is a block of \mathbf{P} that contains F' (since we already know that $[\alpha(F), \tau(F)]$ is a block of \mathbf{P}).

However, the definition of $\alpha(F')$ and $\tau(F')$ shows that $[\alpha(F'), \tau(F')]$ is the (unique) block of \mathbf{P} that contains F' . Since we know that $[\alpha(F), \tau(F)]$ is a block of \mathbf{P} that contains F' , we therefore conclude that $[\alpha(F'), \tau(F')] = [\alpha(F), \tau(F)]$. Hence, we have

$$\alpha(F') = \alpha(F) \quad \text{and} \quad \tau(F') = \tau(F)$$

(because a Boolean interval $[U, V]$ uniquely determines both U and V). Now, the definition of Shade yields

$$\text{Shade}(F') = E \setminus \left(\underbrace{\tau(F')}_{=\tau(F)} \setminus \underbrace{\alpha(F')}_{=\alpha(F)} \right) = E \setminus (\tau(F) \setminus \alpha(F)) = \text{Shade } F.$$

In view of $F' = F \cup \{u\}$, this rewrites as $\text{Shade}(F \cup \{u\}) = \text{Shade } F$. This proves Statement 1.]

[Proof of Statement 2: Let $F \in \mathcal{P}(E)$ and $u \in E \setminus \text{Shade } F$. We must prove that $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$.

We proceed exactly as in our above proof of Statement 1 up until the point where we define F' . Instead of setting $F' = F \cup \{u\}$, we now set $F' = F \setminus \{u\}$. Now, let us call to mind the obvious fact that if two sets U and V and some object w satisfy $U \subseteq V$ and $w \notin U$, then $U \subseteq V \setminus \{w\}$. We can apply this to $U = \alpha(F)$ and $V = F$ and $w = u$ (since $\alpha(F) \subseteq F$ and $u \notin \alpha(F)$); thus we obtain $\alpha(F) \subseteq F \setminus \{u\}$. In other words, $\alpha(F) \subseteq F'$ (since $F' = F \setminus \{u\}$). Combining this with $F' = F \setminus \{u\} \subseteq F \subseteq \tau(F)$, we obtain the chain of inclusions $\alpha(F) \subseteq F' \subseteq \tau(F)$. From this chain, we can obtain $\text{Shade}(F') = \text{Shade } F$ by the same argument that we used back in the proof of Statement 1. In view of $F' = F \setminus \{u\}$, this rewrites as $\text{Shade}(F \setminus \{u\}) = \text{Shade } F$. This proves Statement 2.]

Now, we have proved Statements 1 and 2. Thus, the map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ satisfies the two axioms in Definition 4.2. In other words, this map is a shade map. This proves Theorem 4.28 (a).

(b) Let $F \in \mathcal{P}(E)$. We must prove that $\alpha(F) = F \cap \text{Shade } F$ and $\tau(F) = F \cup (E \setminus \text{Shade } F)$.

The definition of $\alpha(F)$ and $\tau(F)$ reveals that $[\alpha(F), \tau(F)]$ is the (unique) block of \mathbf{P} that contains F . Hence, $[\alpha(F), \tau(F)]$ contains F . In other words, $\alpha(F) \subseteq F \subseteq \tau(F)$. Moreover, $[\alpha(F), \tau(F)]$ is a block of \mathbf{P} (since $[\alpha(F), \tau(F)]$ is the (unique) block of \mathbf{P} that contains F), therefore a Boolean interval of $\mathcal{P}(E)$ (since \mathbf{P} is a Boolean interval partition of $\mathcal{P}(E)$). Hence, $\alpha(F)$ and $\tau(F)$ belong to $\mathcal{P}(E)$, hence are subsets of E . Hence, $\tau(F) \setminus \alpha(F)$ is a subset of E as well.

The definition of Shade yields $\text{Shade } F = E \setminus (\tau(F) \setminus \alpha(F))$. Hence,

$$\begin{aligned} E \setminus \underbrace{\text{Shade } F}_{=E \setminus (\tau(F) \setminus \alpha(F))} &= E \setminus (E \setminus (\tau(F) \setminus \alpha(F))) = \tau(F) \setminus \alpha(F) \end{aligned}$$

(since $\tau(F) \setminus \alpha(F)$ is a subset of E).

Now, $\alpha(F) \subseteq F \subseteq \tau(F) \subseteq E$ (since $\tau(F)$ is a subset of E). Hence, Lemma 4.29 (b) (applied to $X = \alpha(F)$ and $Y = F$ and $Z = \tau(F)$) yields that $F \cap (E \setminus (\tau(F) \setminus \alpha(F))) = \alpha(F)$. In view of $\text{Shade } F = E \setminus (\tau(F) \setminus \alpha(F))$, this rewrites as $F \cap \text{Shade } F = \alpha(F)$. In other words, $\alpha(F) = F \cap \text{Shade } F$.

Furthermore, Lemma 4.29 (a) (applied to $X = \alpha(F)$ and $Y = F$ and $Z = \tau(F)$) yields that $F \cup (\tau(F) \setminus \alpha(F)) = \tau(F)$. In view of $E \setminus \text{Shade } F = \tau(F) \setminus \alpha(F)$, this rewrites as $F \cup (E \setminus \text{Shade } F) = \tau(F)$. In other words, $\tau(F) = F \cup (E \setminus \text{Shade } F)$. Thus, Theorem 4.28 (b) is proven.

(c) It is clear that $\mathbf{P} \subseteq \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}$ ¹⁷ and

¹⁷Proof. Let $Q \in \mathbf{P}$. Hence, Q is a Boolean interval of $\mathcal{P}(E)$ (since \mathbf{P} is a Boolean interval partition of $\mathcal{P}(E)$). Therefore, Q is nonempty (since every Boolean interval is nonempty). In other words, there exists some $F \in Q$. Consider this F . Thus, Q contains F (since $F \in Q$). Moreover, Q is a block of \mathbf{P} (since $Q \in \mathbf{P}$), and we have $F \in Q \subseteq \mathcal{P}(E)$ (since Q is a Boolean interval of $\mathcal{P}(E)$).

The definition of $\alpha(F)$ and $\tau(F)$ reveals that $[\alpha(F), \tau(F)]$ is the (unique) block of \mathbf{P} that contains F . Thus, $[\alpha(F), \tau(F)] = Q$ (since Q is a block of \mathbf{P} that contains F). In other

$\{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\} \subseteq \mathbf{P}$ ¹⁸. Combining these two inclusions, we obtain $\mathbf{P} = \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}$. This proves Theorem 4.28 (c). \square

A converse to Theorem 4.28 is provided by the following theorem:

Theorem 4.30. Let E be a finite set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be a shade map on E . Define a map $\alpha : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ by setting

$$\alpha(F) = F \cap \text{Shade } F \quad \text{for any } F \in \mathcal{P}(E).$$

Define a map $\tau : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ by setting

$$\tau(F) = F \cup (E \setminus \text{Shade } F) \quad \text{for any } F \in \mathcal{P}(E).$$

Let

$$\mathbf{P} = \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}. \quad (17)$$

Then:

- (a) We have $F \in [\alpha(F), \tau(F)]$ for any $F \in \mathcal{P}(E)$.
- (b) If $F \in \mathcal{P}(E)$ and $G \in [\alpha(F), \tau(F)]$, then $[\alpha(F), \tau(F)] = [\alpha(G), \tau(G)]$.
- (c) The set \mathbf{P} is a Boolean interval partition of $\mathcal{P}(E)$.
- (d) We have $\text{Shade } F = E \setminus (\tau(F) \setminus \alpha(F))$ for any $F \in \mathcal{P}(E)$.

To prove this theorem, we will need the following variant of Lemma 4.23:

Lemma 4.31. Let E be a set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be a shade map. Let A and B be two subsets of E such that B is finite and $B \cap \text{Shade } A = \emptyset$. Then, $\text{Shade}(A \setminus B) = \text{Shade } A$.

Proof of Lemma 4.31. We proceed by induction on $|B|$:

Induction base: Lemma 4.31 holds for $|B| = 0$ ¹⁹.

words, $Q = [\alpha(G), \tau(G)]$.

Thus, Q has the form $[\alpha(G), \tau(G)]$ for some $G \in \mathcal{P}(E)$ (namely, for $G = F$). In other words, $Q \in \{[\alpha(G), \tau(G)] \mid G \in \mathcal{P}(E)\}$.

Forget that we fixed Q . We thus have shown that $Q \in \{[\alpha(G), \tau(G)] \mid G \in \mathcal{P}(E)\}$ for each $Q \in \mathbf{P}$. In other words, $\mathbf{P} \subseteq \{[\alpha(G), \tau(G)] \mid G \in \mathcal{P}(E)\}$. Hence,

$$\mathbf{P} \subseteq \{[\alpha(G), \tau(G)] \mid G \in \mathcal{P}(E)\} = \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}$$

(here, we have renamed the index G as F).

¹⁸*Proof.* Let $F \in \mathcal{P}(E)$. The definition of $\alpha(F)$ and $\tau(F)$ reveals that $[\alpha(F), \tau(F)]$ is the (unique) block of \mathbf{P} that contains F . Thus, $[\alpha(F), \tau(F)]$ is a block of \mathbf{P} . In other words, $[\alpha(F), \tau(F)] \in \mathbf{P}$.

Forget that we fixed F . We thus have shown that $[\alpha(F), \tau(F)] \in \mathbf{P}$ for each $F \in \mathcal{P}(E)$. In other words, $\{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\} \subseteq \mathbf{P}$.

¹⁹*Proof.* Assume that $|B| = 0$. Thus, $B = \emptyset$, so that $A \setminus B = A \setminus \emptyset = A$. Hence, $\text{Shade}(A \setminus B) = \text{Shade } A$. Thus, Lemma 4.31 is proven for $|B| = 0$.

Induction step: Let $k \in \mathbb{N}$. Assume that Lemma 4.31 holds for $|B| = k$. We must prove that Lemma 4.31 holds for $|B| = k + 1$.

We have assumed that Lemma 4.31 holds for $|B| = k$. In other words, if A and B are two subsets of E such that B is finite and $B \cap \text{Shade } A = \emptyset$ and $|B| = k$, then

$$\text{Shade}(A \setminus B) = \text{Shade } A. \quad (18)$$

Now, let A and B be two subsets of E such that B is finite and $B \cap \text{Shade } A = \emptyset$ and $|B| = k + 1$. We shall prove that $\text{Shade}(A \setminus B) = \text{Shade } A$.

We assumed that Shade is a shade map. Thus, in particular, Shade satisfies Axiom 2 in Definition 4.2.

The set B is nonempty (since $|B| = k + 1 > k \geq 0$). Thus, there exists some $u \in B$. Consider this u . If we had $u \in \text{Shade } A$, then we would have $u \in B \cap \text{Shade } A$ (since $u \in B$ and $u \in \text{Shade } A$), which would entail $B \cap \text{Shade } A \neq \emptyset$; but this would contradict $B \cap \text{Shade } A = \emptyset$. Hence, we cannot have $u \in \text{Shade } A$. In other words, we have $u \notin \text{Shade } A$. Hence, $u \in E \setminus \text{Shade } A$.

From $u \in B$, we obtain $|B \setminus \{u\}| = |B| - 1 = k$ (since $|B| = k + 1$). Moreover, $(B \setminus \{u\}) \cap \text{Shade } A = \emptyset$ (since $\underbrace{(B \setminus \{u\})}_{\subseteq B} \cap \text{Shade } A \subseteq B \cap \text{Shade } A = \emptyset$).

Thus, we can apply (18) to $B \setminus \{u\}$ instead of B . As a consequence, we obtain $\text{Shade}(A \setminus (B \setminus \{u\})) = \text{Shade } A$. Thus, $u \in E \setminus \text{Shade}(A \setminus (B \setminus \{u\}))$ (since $u \in E \setminus \text{Shade } A$). Thus, Axiom 2 in Definition 4.2 (applied to $F = A \setminus (B \setminus \{u\})$) yields

$$\text{Shade}((A \setminus (B \setminus \{u\})) \setminus \{u\}) = \text{Shade}(A \setminus (B \setminus \{u\})) = \text{Shade } A.$$

In view of

$$(A \setminus (B \setminus \{u\})) \setminus \{u\} = A \setminus \underbrace{((B \setminus \{u\}) \cup \{u\})}_{\substack{=B \\ (\text{since } u \in B)}} = A \setminus B,$$

this rewrites as $\text{Shade}(A \setminus B) = \text{Shade } A$.

Forget that we fixed A and B . We thus have shown that if A and B are two subsets of E such that B is finite and $B \cap \text{Shade } A = \emptyset$ and $|B| = k + 1$, then $\text{Shade}(A \setminus B) = \text{Shade } A$. In other words, Lemma 4.31 holds for $|B| = k + 1$. This completes the induction step. Thus, Lemma 4.31 is proven. \square

We will also use another simple set-theoretical lemma:

Lemma 4.32. Let E be a set. Let X and Y be two subsets of E . Then,

$$E \setminus Y = (X \cup (E \setminus Y)) \setminus (X \cap Y).$$

Proof of Lemma 4.32. This is straightforward set theory; it is easily checked (e.g.) using Venn diagrams. \square

Proof of Theorem 4.30. (a) Let $F \in \mathcal{P}(E)$. The definition of α yields $\alpha(F) = F \cap \text{Shade } F \subseteq F$. The definition of τ yields $\tau(F) = F \cup (E \setminus \text{Shade } F) \supseteq F$, so that $F \subseteq \tau(F)$. Thus, we have the chain of inclusions $\alpha(F) \subseteq F \subseteq \tau(F)$. In other words, $F \in [\alpha(F), \tau(F)]$. This proves Theorem 4.30 (a).

(b) Let $F \in \mathcal{P}(E)$ and $G \in [\alpha(F), \tau(F)]$. From $G \in [\alpha(F), \tau(F)]$, we obtain the chain of inclusions $\alpha(F) \subseteq G \subseteq \tau(F)$. Thus, $G \subseteq \tau(F) = F \cup (E \setminus \text{Shade } F)$ (by the definition of τ). Hence,

$$\underbrace{G}_{=F \cup (E \setminus \text{Shade } F)} \setminus F = (F \cup (E \setminus \text{Shade } F)) \setminus F \subseteq E \setminus \text{Shade } F$$

(since $(X \cup Y) \setminus X \subseteq Y$ for any two sets X and Y). Therefore,

$$\underbrace{(G \setminus F)}_{\subseteq E \setminus \text{Shade } F} \cap \text{Shade } F \subseteq (E \setminus \text{Shade } F) \cap \text{Shade } F = \emptyset$$

(since $(X \setminus Y) \cap Y = \emptyset$ for any two sets X and Y). Thus,

$$(G \setminus F) \cap \text{Shade } F = \emptyset. \quad (19)$$

Note that $G \setminus F \subseteq G \subseteq \tau(F) \subseteq E$ (since $\tau(F) \in \mathcal{P}(E)$) and $F \cap G \subseteq F \subseteq E$ (since $F \in \mathcal{P}(E)$) and $F \setminus G \subseteq F \subseteq E$.

Furthermore, we have $\alpha(F) = F \cap \text{Shade } F$ (by the definition of α). Hence, $F \cap \text{Shade } F = \alpha(F) \subseteq G$. Thus, $G \supseteq F \cap \text{Shade } F$, so that

$$F \setminus \underbrace{G}_{\supseteq F \cap \text{Shade } F} \subseteq F \setminus (F \cap \text{Shade } F) = F \setminus \text{Shade } F$$

(since $X \setminus (X \cap Y) = X \setminus Y$ for any two sets X and Y). Hence,

$$\underbrace{(F \setminus G)}_{\subseteq F \setminus \text{Shade } F} \cap \text{Shade } F \subseteq (F \setminus \text{Shade } F) \cap \text{Shade } F = \emptyset$$

(since $(X \setminus Y) \cap Y = \emptyset$ for any two sets X and Y). Thus,

$$(F \setminus G) \cap \text{Shade } F = \emptyset.$$

Hence, Lemma 4.31 (applied to $A = F$ and $B = F \setminus G$) yields $\text{Shade}(F \setminus (F \setminus G)) = \text{Shade } F$. In view of $F \setminus (F \setminus G) = F \cap G$, this rewrites as $\text{Shade}(F \cap G) = \text{Shade } F$. Hence, (19) rewrites as

$$(G \setminus F) \cap \text{Shade}(F \cap G) = \emptyset.$$

Thus, Lemma 4.23 (applied to $A = F \cap G$ and $B = G \setminus F$) yields $\text{Shade}((F \cap G) \cup (G \setminus F)) = \text{Shade}(F \cap G) = \text{Shade } F$. In view of $(F \cap G) \cup (G \setminus F) = G$, this rewrites as $\text{Shade } G = \text{Shade } F$. In other words,

$$\text{Shade } F = \text{Shade } G.$$

The definition of α yields $\alpha(F) = F \cap \text{Shade } F$ and $\alpha(G) = G \cap \text{Shade } G$. Now, $\alpha(F) = F \cap \text{Shade } F \subseteq \text{Shade } F$, so that

$$\alpha(F) = \underbrace{\alpha(F)}_{\subseteq G} \cap \underbrace{\text{Shade } F}_{=\text{Shade } G} \subseteq G \cap \text{Shade } G = \alpha(G).$$

Combining this with

$$\begin{aligned} \alpha(G) &= \underbrace{G}_{\subseteq \tau(F)} \cap \underbrace{\text{Shade } G}_{=\text{Shade } F} \subseteq (F \cup (E \setminus \text{Shade } F)) \cap \text{Shade } F \\ &= \underbrace{(F \cap \text{Shade } F)}_{=\alpha(F)} \cup \underbrace{((E \setminus \text{Shade } F) \cap \text{Shade } F)}_{=\emptyset} \\ &\quad (\text{since } (X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z) \text{ for any three sets } X, Y, Z) \\ &= \alpha(F) \cup \emptyset = \alpha(F), \end{aligned}$$

we obtain $\alpha(F) = \alpha(G)$.

The definition of τ yields $\tau(F) = F \cup (E \setminus \text{Shade } F)$ and $\tau(G) = G \cup (E \setminus \text{Shade } G)$. Now, $\tau(F) = F \cup (E \setminus \text{Shade } F) \supseteq E \setminus \text{Shade } F$, so that

$$\tau(F) = \underbrace{\tau(F)}_{\supseteq G} \cup \left(\underbrace{E \setminus \text{Shade } F}_{=\text{Shade } G} \right) \supseteq G \cup (E \setminus \text{Shade } G) = \tau(G).$$

(since $G \subseteq \tau(F)$)

Combining this with

$$\begin{aligned} \tau(F) &= \underbrace{F}_{=(F \cap \text{Shade } F) \cup (F \setminus \text{Shade } F)} \cup (E \setminus \text{Shade } F) \\ &\quad (\text{since } X = (X \cap Y) \cup (X \setminus Y) \text{ for any two sets } X \text{ and } Y) \\ &= (F \cap \text{Shade } F) \cup \left(\underbrace{F \setminus \text{Shade } F}_{\subseteq E} \right) \cup (E \setminus \text{Shade } F) \\ &\subseteq \underbrace{(F \cap \text{Shade } F)}_{=\alpha(F)} \cup \underbrace{(E \setminus \text{Shade } F) \cup (E \setminus \text{Shade } F)}_{=E \setminus \text{Shade } F} \\ &= \underbrace{\alpha(F)}_{\subseteq G} \cup \left(\underbrace{E \setminus \text{Shade } F}_{=\text{Shade } G} \right) \subseteq G \cup (E \setminus \text{Shade } G) = \tau(G), \end{aligned}$$

we obtain $\tau(F) = \tau(G)$.

$$\text{Now, } \left[\begin{array}{cc} \underbrace{\alpha(F)}_{=\alpha(G)} & \underbrace{\tau(F)}_{=\tau(G)} \end{array} \right] = [\alpha(G), \tau(G)]. \text{ Theorem 4.30 (b) is thus proven.}$$

(c) We have $\mathbf{P} = \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}$. Thus, \mathbf{P} is a set of Boolean intervals of $\mathcal{P}(E)$ (since $[\alpha(F), \tau(F)]$ is a Boolean interval of $\mathcal{P}(E)$ whenever $F \in \mathcal{P}(E)$). We shall refer to the elements of \mathbf{P} as the *blocks* of \mathbf{P} (even though we don't know yet that \mathbf{P} is a Boolean interval partition).

Next, we shall show that the blocks of \mathbf{P} are pairwise disjoint.

[*Proof.* Let Q and R be two distinct blocks of \mathbf{P} .

Let $G \in Q \cap R$. Now, we have $Q = [\alpha(F), \tau(F)]$ for some $F \in \mathcal{P}(E)$ (since $Q \in \mathbf{P} = \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}$). Consider this F . We have $G \in Q \cap R \subseteq Q = [\alpha(F), \tau(F)]$. Therefore, Theorem 4.30 (b) yields $[\alpha(F), \tau(F)] = [\alpha(G), \tau(G)]$. Hence, $Q = [\alpha(F), \tau(F)] = [\alpha(G), \tau(G)]$. The same argument (applied to R instead of Q) yields $R = [\alpha(G), \tau(G)]$. Comparing these two equalities, we obtain $Q = R$; this contradicts the assumption that Q and R are distinct.

Forget that we fixed G . We thus have obtained a contradiction for each $G \in Q \cap R$. Hence, there exists no $G \in Q \cap R$. In other words, $Q \cap R = \emptyset$. In other words, Q and R are disjoint.

Forget that we fixed Q and R . We thus have shown that any two distinct blocks Q and R of \mathbf{P} are disjoint. In other words, the blocks of \mathbf{P} are pairwise disjoint.]

Recall that \mathbf{P} is a set of Boolean intervals of $\mathcal{P}(E)$. Hence, \mathbf{P} is a set of pairwise disjoint Boolean intervals of $\mathcal{P}(E)$ (since the blocks of \mathbf{P} are pairwise disjoint).

Now, let $G \in \mathcal{P}(E)$. Then, $[\alpha(G), \tau(G)] \in \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\} = \mathbf{P}$. Thus, the set $[\alpha(G), \tau(G)]$ is one of the sets that are being united in the union $\bigcup_{R \in \mathbf{P}} R$. Therefore, $[\alpha(G), \tau(G)] \subseteq \bigcup_{R \in \mathbf{P}} R$. However, Theorem 4.30 (a) (applied to $F = G$) yields $G \in [\alpha(G), \tau(G)] \subseteq \bigcup_{R \in \mathbf{P}} R$.

Forget that we fixed G . We thus have shown that $G \in \bigcup_{R \in \mathbf{P}} R$ for each $G \in \mathcal{P}(E)$. In other words, $\mathcal{P}(E) \subseteq \bigcup_{R \in \mathbf{P}} R$. Combining this inclusion with the inclusion $\bigcup_{R \in \mathbf{P}} R \subseteq \mathcal{P}(E)$ (which is obvious, since each $R \in \mathbf{P}$ is a subset of $\mathcal{P}(E)$), we obtain $\bigcup_{R \in \mathbf{P}} R = \mathcal{P}(E)$. In other words, the union of all blocks of \mathbf{P} is $\mathcal{P}(E)$.

Now, recall that \mathbf{P} is a set of pairwise disjoint Boolean intervals of $\mathcal{P}(E)$. Hence, \mathbf{P} is a set of pairwise disjoint Boolean intervals of $\mathcal{P}(E)$ whose union is $\mathcal{P}(E)$ (since the union of all blocks of \mathbf{P} is $\mathcal{P}(E)$). In other words, \mathbf{P} is a Boolean interval partition of $\mathcal{P}(E)$. This proves Theorem 4.30 (c).

(d) Let $F \in \mathcal{P}(E)$. Then, $\text{Shade } F$ belongs to $\mathcal{P}(E)$ as well (since Shade is a map from $\mathcal{P}(E)$ to $\mathcal{P}(E)$). In other words, $\text{Shade } F$ is a subset of E .

Also, F is a subset of E (since $F \in \mathcal{P}(E)$). Hence, Lemma 4.32 (applied to

$X = F$ and $Y = \text{Shade } F$) yields

$$E \setminus \text{Shade } F = \underbrace{(F \cup (E \setminus \text{Shade } F))}_{=\tau(F)} \setminus \underbrace{(F \cap \text{Shade } F)}_{=\alpha(F)} = \tau(F) \setminus \alpha(F).$$

(since $\tau(F) = F \cup (E \setminus \text{Shade } F)$) (since $\alpha(F) = F \cap \text{Shade } F$)

However, $\text{Shade } F$ is a subset of E ; thus, $E \setminus (E \setminus \text{Shade } F) = \text{Shade } F$. Therefore,

$$\text{Shade } F = E \setminus \underbrace{(E \setminus \text{Shade } F)}_{=\tau(F) \setminus \alpha(F)} = E \setminus (\tau(F) \setminus \alpha(F)).$$

This proves Theorem 4.30 (d). □

Combining Theorem 4.28 with Theorem 4.30, we obtain the following:

Theorem 4.33. Let E be a finite set. Then, there is a bijection from the set

$$\{\text{shade maps } \text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)\}$$

to the set

$$\{\text{Boolean interval partitions of } \mathcal{P}(E)\}.$$

It sends each map Shade to the Boolean interval partition \mathbf{P} defined in Theorem 4.30 (c). Its inverse map sends each Boolean interval partition \mathbf{P} to the map Shade defined in Theorem 4.28.

Proof of Theorem 4.33. If $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is a shade map, then the \mathbf{P} defined in Theorem 4.30 (c) is a Boolean interval partition of $\mathcal{P}(E)$ (by Theorem 4.30 (c)). Hence, we can define a map

$$\mathbf{B} : \{\text{shade maps } \text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)\} \rightarrow \{\text{Boolean interval partitions of } \mathcal{P}(E)\}$$

that sends each map Shade to the Boolean interval partition \mathbf{P} defined in Theorem 4.30 (c). Consider this map \mathbf{B} .

If \mathbf{P} is a Boolean interval partition of $\mathcal{P}(E)$, then the map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ defined in Theorem 4.28 is a shade map (by Theorem 4.28 (a)). Hence, we can define a map

$$\mathbf{S} : \{\text{Boolean interval partitions of } \mathcal{P}(E)\} \rightarrow \{\text{shade maps } \text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)\}$$

that sends each Boolean interval partition \mathbf{P} to the map Shade defined in Theorem 4.28. Consider this map \mathbf{S} .

We shall now show that the maps \mathbf{B} and \mathbf{S} are mutually inverse. Indeed, we claim the following:

Claim 1: We have $\mathbf{B} \circ \mathbf{S} = \text{id}$.

[*Proof of Claim 1:* Let \mathbf{P} be a Boolean interval partition of $\mathcal{P}(E)$.

For any $F \in \mathcal{P}(E)$, let $[\alpha(F), \tau(F)]$ denote the (unique) block of \mathbf{P} that contains F . Thus, we have defined two sets $\alpha(F) \in \mathcal{P}(E)$ and $\tau(F) \in \mathcal{P}(E)$ for each $F \in \mathcal{P}(E)$. In other words, we have defined two maps $\alpha : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ and $\tau : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$.

We define a map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ by setting

$$\text{Shade } F = E \setminus (\tau(F) \setminus \alpha(F)) \quad \text{for any } F \in \mathcal{P}(E).$$

This map Shade is precisely the map Shade defined in Theorem 4.28. Hence, $\mathbf{S}(\mathbf{P}) = \text{Shade}$ (by the definition of \mathbf{S}).

Now, any $F \in \mathcal{P}(E)$ satisfies $\alpha(F) = F \cap \text{Shade } F$ (by Theorem 4.28 (b)). Thus, our map $\alpha : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is precisely the map α defined in Theorem 4.30.

Furthermore, any $F \in \mathcal{P}(E)$ satisfies $\tau(F) = F \cup (E \setminus \text{Shade } F)$ (by Theorem 4.28 (b)). Thus, our map $\tau : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ is precisely the map τ defined in Theorem 4.30.

Moreover, we have $\mathbf{P} = \{[\alpha(F), \tau(F)] \mid F \in \mathcal{P}(E)\}$ (by Theorem 4.28 (c)). Thus, our Boolean interval partition \mathbf{P} is precisely the Boolean interval partition \mathbf{P} defined in Theorem 4.30 (c) (because we already know that our maps α and τ are precisely the maps α and τ defined in Theorem 4.30). However, the definition of \mathbf{B} shows that $\mathbf{B}(\text{Shade})$ is the Boolean interval partition \mathbf{P} defined in Theorem 4.30 (c). Thus, $\mathbf{B}(\text{Shade}) = \mathbf{P}$ (since our Boolean interval partition \mathbf{P} is precisely the Boolean interval partition \mathbf{P} defined in Theorem 4.30 (c)). Now,

$$(\mathbf{B} \circ \mathbf{S})(\mathbf{P}) = \mathbf{B} \left(\underbrace{\mathbf{S}(\mathbf{P})}_{=\text{Shade}} \right) = \mathbf{B}(\text{Shade}) = \mathbf{P} = \text{id}(\mathbf{P}).$$

Now, forget that we fixed \mathbf{P} . We thus have shown that $(\mathbf{B} \circ \mathbf{S})(\mathbf{P}) = \text{id}(\mathbf{P})$ for any Boolean interval partition \mathbf{P} of $\mathcal{P}(E)$. In other words, $\mathbf{B} \circ \mathbf{S} = \text{id}$. This proves Claim 1.]

Claim 2: We have $\mathbf{S} \circ \mathbf{B} = \text{id}$.

[*Proof of Claim 2:* Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be a shade map on E . Define two maps α and τ as in Theorem 4.30. Define a Boolean interval partition \mathbf{P} as in Theorem 4.30 (c). Then, $\mathbf{B}(\text{Shade}) = \mathbf{P}$ (by the definition of \mathbf{B}).

Let $F \in \mathcal{P}(E)$. Then, $[\alpha(F), \tau(F)] \in \mathbf{P}$ (by (17)). In other words, $[\alpha(F), \tau(F)]$ is a block of the Boolean interval partition \mathbf{P} . Furthermore, this block $[\alpha(F), \tau(F)]$ contains F (since Theorem 4.30 (a) yields $F \in [\alpha(F), \tau(F)]$).

However, there is a unique block of \mathbf{P} that contains F (since \mathbf{P} is a Boolean interval partition of $\mathcal{P}(E)$). This unique block must be $[\alpha(F), \tau(F)]$ (since $[\alpha(F), \tau(F)]$ is a block of the Boolean interval partition \mathbf{P} and contains F). Thus, $[\alpha(F), \tau(F)]$ is the (unique) block of \mathbf{P} that contains F .

Forget that we fixed F . We thus have shown that if $F \in \mathcal{P}(E)$, then $[\alpha(F), \tau(F)]$ is the (unique) block of \mathbf{P} that contains F . In other words, our definition of $\alpha(F)$ and $\tau(F)$ agrees with the definition of $\alpha(F)$ and $\tau(F)$ given in Theorem 4.28 (because the latter definition defines $\alpha(F)$ and $\tau(F)$ by requiring that $[\alpha(F), \tau(F)]$ be the (unique) block of \mathbf{P} that contains F). Hence, everything said about $\alpha(F)$ and $\tau(F)$ in Theorem 4.28 can be applied to our $\alpha(F)$ and $\tau(F)$ as well.

Moreover, Theorem 4.30 (d) shows that we have $\text{Shade } F = E \setminus (\tau(F) \setminus \alpha(F))$ for any $F \in \mathcal{P}(E)$. Hence, our map Shade is precisely the map Shade defined in Theorem 4.28 (because we already know that our definition of $\alpha(F)$ and $\tau(F)$ agrees with the definition of $\alpha(F)$ and $\tau(F)$ given in Theorem 4.28). However, the definition of \mathbf{S} shows that $\mathbf{S}(\mathbf{P})$ is the map Shade defined in Theorem 4.28. Thus, $\mathbf{S}(\mathbf{P}) = \text{Shade}$ (since our map Shade is precisely the map Shade defined in Theorem 4.28). Now,

$$(\mathbf{S} \circ \mathbf{B})(\text{Shade}) = \mathbf{S} \left(\underbrace{\mathbf{B}(\text{Shade})}_{=\mathbf{P}} \right) = \mathbf{S}(\mathbf{P}) = \text{Shade} = \text{id}(\text{Shade}).$$

Now, forget that we fixed Shade . We thus have shown that $(\mathbf{S} \circ \mathbf{B})(\text{Shade}) = \text{id}(\text{Shade})$ for any shade map $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$. In other words, $\mathbf{S} \circ \mathbf{B} = \text{id}$. This proves Claim 2.]

Combining Claim 1 with Claim 2, we conclude that the maps \mathbf{B} and \mathbf{S} are mutually inverse. Hence, these two maps \mathbf{B} and \mathbf{S} are bijections. Thus, there is a bijection from the set

$$\{\text{shade maps } \text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)\}$$

to the set

$$\{\text{Boolean interval partitions of } \mathcal{P}(E)\}$$

(namely, the map \mathbf{B}), and this bijection sends each map Shade to the Boolean interval partition \mathbf{P} defined in Theorem 4.30 (c) (because this is how \mathbf{B} was defined). Moreover, its inverse map is the map \mathbf{S} (because \mathbf{B} and \mathbf{S} are mutually inverse), and therefore sends each Boolean interval partition \mathbf{P} to the map Shade defined in Theorem 4.28 (because this is how \mathbf{S} was defined). Thus, the proof of Theorem 4.33 is complete. \square

Question 4.34. According to Theorem 4.18, any antimatroid gives rise to an inclusion-reversing shade map, which in turn gives rise to a Boolean interval partition by Theorem 4.30. Is this construction equivalent to the construction of a Boolean interval partition from an antimatroid described by Gordon and McMahon in [GorMah97, Theorem 2.5]?

5. The topological viewpoint

Now we return to the setting of Section 1. We aim to reinterpret Theorem 2.6 in the terms of combinatorial topology (specifically, finite simplicial complexes) and strengthen it. We recall the definition of a *simplicial complex*:²⁰

Definition 5.1. Let E be a finite set. A *simplicial complex* on ground set E means a subset \mathcal{A} of the power set of E with the following property:

$$\text{If } P \in \mathcal{A} \text{ and } Q \subseteq P, \text{ then } Q \in \mathcal{A}.$$

Thus, in terms of posets, a simplicial complex on ground set E means a down-closed subset of the Boolean lattice on E . Note that a simplicial complex contains the empty set \emptyset unless it is empty itself.

We refer to [Kozlov20] for context and theory about simplicial complexes. We shall restrict ourselves to the few definitions relevant to what we will prove. The following is fairly simple:

Proposition 5.2. Let us use the notations from Section 1 as well as Definition 2.1. Let G be any subset of E . Let

$$\mathcal{A} = \{F \subseteq E \mid G \not\subseteq \text{Shade } F\}. \quad (20)$$

Then, \mathcal{A} is a simplicial complex on ground set E .

Proof of Proposition 5.2. Clearly, \mathcal{A} is a subset of the power set of E . Thus, we only need to verify the following claim:

Claim 1: If $P \in \mathcal{A}$ and $Q \subseteq P$, then $Q \in \mathcal{A}$.

[*Proof of Claim 1:* Let $P \in \mathcal{A}$ and let $Q \subseteq P$. We must show that $Q \in \mathcal{A}$.

We have $P \in \mathcal{A} = \{F \subseteq E \mid G \not\subseteq \text{Shade } F\}$. In other words, P is a subset F of E satisfying $G \not\subseteq \text{Shade } F$. In other words, P is a subset of E and satisfies $G \not\subseteq \text{Shade } P$. Now, Q is a subset of E (since $Q \subseteq P \subseteq E$). Also, from $Q \subseteq P$, we obtain $\text{Shade } Q \subseteq \text{Shade } P$ (by Lemma 2.3, applied to $A = Q$ and $B = P$). Hence, from $G \not\subseteq \text{Shade } P$, we obtain $G \not\subseteq \text{Shade } Q$. Thus, Q is a subset F of E satisfying $G \not\subseteq \text{Shade } F$. In other words, $Q \in \{F \subseteq E \mid G \not\subseteq \text{Shade } F\}$. This can be rewritten as $Q \in \mathcal{A}$ (by (20)). Thus, Claim 1 is proved.]

Claim 1 shows that \mathcal{A} is a simplicial complex on ground set E . This proves Proposition 5.2. \square

To state the main result of this section, we need a few more notions:

²⁰We forget all the conventions we have introduced so far. (Thus, for example, E no longer means the edge set of a graph Γ .)

Definition 5.3. Let A and B be two sets. Then, we say that $A \prec B$ if we have $B = A \cup \{b\}$ for some $b \in B \setminus A$.

Equivalently, two sets A and B satisfy $A \prec B$ if and only if $A \subseteq B$ and $|B \setminus A| = 1$.

Definition 5.4. Let E be a finite set. Let \mathcal{A} be a simplicial complex on ground set E .

(a) A *complete matching* of \mathcal{A} means a triple $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$, where \mathcal{A}_- and \mathcal{A}_+ are two disjoint subsets of \mathcal{A} satisfying $\mathcal{A}_- \cup \mathcal{A}_+ = \mathcal{A}$, and where $\Phi : \mathcal{A}_+ \rightarrow \mathcal{A}_-$ is a bijection with the property that

$$\text{each } F \in \mathcal{A}_+ \text{ satisfies } \Phi(F) \prec F. \quad (21)$$

(b) A complete matching $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ of \mathcal{A} is said to be *acyclic* if there exists no tuple (B_1, B_2, \dots, B_n) of distinct sets $B_1, B_2, \dots, B_n \in \mathcal{A}_+$ with the property that $n \geq 2$ and that

$$\Phi(B_i) \prec B_{i+1} \quad \text{for each } i \in \{1, 2, \dots, n-1\}$$

and

$$\Phi(B_n) \prec B_1.$$

(c) The simplicial complex \mathcal{A} is said to be *collapsible* if it has an acyclic complete matching.

These definitions are essentially equivalent to the definitions in [Kozlov20], although it takes a bit of work to match them up precisely. Our notion of a “complete matching” as defined in Definition 5.4 (a) is a particular case of the notion introduced in [Kozlov20, Chapter 10], as we restrict ourselves to simplicial complexes (i.e., down-closed sets of Boolean lattices) instead of arbitrary posets. To be fully precise, our complete matchings are triples $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$, whereas the complete matchings of [Kozlov20, Chapter 10] are certain fixed-point-free involutions²¹ $\mu : \mathcal{A} \rightarrow \mathcal{A}$; the equivalence between these two objects is fairly easy to see (in particular, if $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ is a complete matching in our sense, then the corresponding complete matching $\mu : \mathcal{A} \rightarrow \mathcal{A}$ in the sense of [Kozlov20, Chapter 10] is the map that sends each $B \in \mathcal{A}_+$ to $\Phi(B) \in \mathcal{A}_-$ and sends each $B \in \mathcal{A}_-$ to $\Phi^{-1}(B) \in \mathcal{A}_+$). Our notion of “collapsible” as defined in Definition 5.4 (c) is equivalent to the classical notion of “collapsible” (even though the latter is usually defined differently) because of [Kozlov20, Theorem 10.9].

We now claim:

²¹A *fixed-point-free involution* means an involution (i.e., a map that is inverse to itself) that has no fixed point.

Theorem 5.5. Let us use the notations from Section 1 as well as Definition 2.1. Let G be any subset of E . Define \mathcal{A} as in (20). Then, the simplicial complex \mathcal{A} is collapsible.

Collapsible simplicial complexes are well-behaved in various ways – in particular, they are contractible ([Kozlov20, Corollary 9.19]), and thus have trivial homotopy and homology groups (in positive degrees). Moreover, the reduced Euler characteristic of any collapsible simplicial complex is 0 (for obvious reasons: having a complete matching suffices, even if it is not acyclic); thus, Theorem 2.6 follows from Theorem 5.5.

Our proof of Theorem 5.5 will rely on the following simple lemma:

Lemma 5.6. Let X and Y be two sets. Let $u \in X \cap Y$. If $X \setminus \{u\} \prec Y$, then $X = Y$.

Proof of Lemma 5.6. Set $X' = X \setminus \{u\}$. Thus, $\underbrace{X'}_{=X \setminus \{u\}} \cup \{u\} = (X \setminus \{u\}) \cup \{u\} = X$ (since $u \in X \cap Y \subseteq X$). Also, clearly, $u \notin X \setminus \{u\}$ (since $u \notin P \setminus \{u\}$ for any set P). In other words, $u \notin X'$ (since $X' = X \setminus \{u\}$).

Now, assume that $X \setminus \{u\} \prec Y$. In other words, $X' \prec Y$ (since $X' = X \setminus \{u\}$). In other words, we have

$$Y = X' \cup \{b\} \tag{22}$$

for some $b \in Y \setminus X'$ (by Definition 5.3). Consider this b . Combining $u \in X \cap Y \subseteq Y = X' \cup \{b\}$ with $u \notin X'$, we obtain

$$u \in (X' \cup \{b\}) \setminus X' \subseteq \{b\}.$$

In other words, $u = b$. Thus, (22) can be rewritten as $Y = X' \cup \{u\}$. Hence, $Y = X' \cup \{u\} = X$, so that $X = Y$. This proves Lemma 5.6. \square

Proof of Theorem 5.5. We know from Proposition 5.2 that \mathcal{A} is a simplicial complex. It remains to show that \mathcal{A} is collapsible.

We have

$$\mathcal{A} = \{F \subseteq E \mid G \not\subseteq \text{Shade } F\} = \{P \subseteq E \mid G \not\subseteq \text{Shade } P\}$$

(here, we have renamed the index F as P). Thus, our set \mathcal{A} is precisely the set \mathcal{A} defined in the proof of Theorem 2.6 above.

We equip the finite set E with a total order (chosen arbitrarily, but fixed henceforth).

If $F \in \mathcal{A}$, then we define the edge $\varepsilon(F) \in G \setminus \text{Shade } F$ as in the proof of Theorem 2.6. That is, we define $\varepsilon(F)$ as the **smallest** edge $e \in G \setminus \text{Shade } F$.

If two sets $F_1 \in \mathcal{A}$ and $F_2 \in \mathcal{A}$ satisfy $\text{Shade}(F_1) = \text{Shade}(F_2)$, then

$$\varepsilon(F_1) = \varepsilon(F_2). \tag{23}$$

(Indeed, this is precisely the equality (6) from the above proof of Theorem 2.6.)

We define two subsets \mathcal{A}_+ and \mathcal{A}_- of \mathcal{A} as in the proof of Theorem 2.6. That is, we set

$$\mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\} \quad \text{and} \quad \mathcal{A}_- = \{P \in \mathcal{A} \mid \varepsilon(P) \notin P\}.$$

Thus, each $P \in \mathcal{A}$ satisfies either $P \in \mathcal{A}_-$ or $P \in \mathcal{A}_+$ but not both at the same time (since it satisfies either $\varepsilon(P) \notin P$ or $\varepsilon(P) \in P$ but not both at the same time). Hence, \mathcal{A}_- and \mathcal{A}_+ are two disjoint subsets of \mathcal{A} satisfying $\mathcal{A}_- \cup \mathcal{A}_+ = \mathcal{A}$.

We define a map $\Phi : \mathcal{A}_+ \rightarrow \mathcal{A}_-$ as in the proof of Theorem 2.6. That is, we set

$$\Phi(F) = F \setminus \{\varepsilon(F)\} \quad \text{for each } F \in \mathcal{A}_+.$$

We know (from the proof of Theorem 2.6) that the map Φ is a bijection. Moreover, it is clear that each $F \in \mathcal{A}_+$ satisfies $\Phi(F) \prec F$ ²². Hence, the triple $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ is a complete matching of \mathcal{A} .

We shall now prove that this complete matching $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ is acyclic. Indeed, let (B_1, B_2, \dots, B_n) be a tuple of distinct sets $B_1, B_2, \dots, B_n \in \mathcal{A}_+$ with the property that $n \geq 2$ and that

$$\Phi(B_i) \prec B_{i+1} \quad \text{for each } i \in \{1, 2, \dots, n-1\} \quad (24)$$

and

$$\Phi(B_n) \prec B_1. \quad (25)$$

We shall derive a contradiction.

Set $B_{n+1} = B_1$. Then, (25) can be rewritten as $\Phi(B_n) \prec B_{n+1}$. In other words, we have $\Phi(B_i) \prec B_{i+1}$ for $i = n$. Combining this with (24), we conclude that

$$\Phi(B_i) \prec B_{i+1} \quad \text{for each } i \in \{1, 2, \dots, n\}. \quad (26)$$

Now, set $A_i = \text{Shade}(B_i)$ for each $i \in \{1, 2, \dots, n+1\}$. Then, $A_{n+1} = A_1$ (since $B_{n+1} = B_1$).

We now claim the following:

Claim 1: We have $A_i \subseteq A_{i+1}$ for each $i \in \{1, 2, \dots, n\}$.

[*Proof of Claim 1:* Let $i \in \{1, 2, \dots, n\}$. Then, the definition of A_i yields $A_i = \text{Shade}(B_i)$. Likewise, $A_{i+1} = \text{Shade}(B_{i+1})$.

We have $B_i \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$. In other words, B_i is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \in P$. In other words, B_i is an element of \mathcal{A} and satisfies $\varepsilon(B_i) \in B_i$.

²²*Proof.* Let $F \in \mathcal{A}_+$. Thus, $F \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$. In other words, F is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \in P$. In other words, $F \in \mathcal{A}$ and $\varepsilon(F) \in F$. From $\varepsilon(F) \in F$, we obtain $F = (F \setminus \{\varepsilon(F)\}) \cup \{\varepsilon(F)\}$ and $\varepsilon(F) \in F \setminus (F \setminus \{\varepsilon(F)\})$. Hence, $F \setminus \{\varepsilon(F)\} \prec F$ (by Definition 5.3). In other words, $\Phi(F) \prec F$ (since the definition of Φ yields $\Phi(F) = F \setminus \{\varepsilon(F)\}$). Qed.

We set $u = \varepsilon(B_i)$. The definition of Φ yields $\Phi(B_i) = B_i \setminus \{\varepsilon(B_i)\} = B_i \setminus \{u\}$ (since $\varepsilon(B_i) = u$).

Recall that $\varepsilon(B_i)$ is the smallest edge $e \in G \setminus \text{Shade}(B_i)$ (by the definition of $\varepsilon(B_i)$). Hence, $\varepsilon(B_i) \in G \setminus \text{Shade}(B_i)$. In other words, $u \in G \setminus \text{Shade}(B_i)$ (since $u = \varepsilon(B_i)$). In other words, $u \in G$ and $u \notin \text{Shade}(B_i)$. Thus, $u \in G \subseteq E$ and $u \notin \text{Shade}(B_i)$. Therefore, (4) (applied to $F = B_i$) yields $\text{Shade}(B_i \setminus \{u\}) = \text{Shade}(B_i)$. This can be rewritten as $\text{Shade}(\Phi(B_i)) = A_i$ (since $\Phi(B_i) = B_i \setminus \{u\}$ and $A_i = \text{Shade}(B_i)$).

But (26) yields $\Phi(B_i) \prec B_{i+1}$, so that $\Phi(B_i) \subseteq B_{i+1}$ and thus $\text{Shade}(\Phi(B_i)) \subseteq \text{Shade}(B_{i+1})$ (by Lemma 2.3, applied to $A = \Phi(B_i)$ and $B = B_{i+1}$). In view of $\text{Shade}(\Phi(B_i)) = A_i$ and $\text{Shade}(B_{i+1}) = A_{i+1}$, this can be rewritten as $A_i \subseteq A_{i+1}$. This proves Claim 1.]

Claim 1 shows that $A_i \subseteq A_{i+1}$ for each $i \in \{1, 2, \dots, n\}$. In other words,

$$A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq A_{n+1}.$$

This is a chain of inclusions, but its last entry equals the first: indeed, $A_{n+1} = A_1$. Thus, all inclusions in this chain must be equalities. That is, we have

$$A_1 = A_2 = \dots = A_n = A_{n+1}. \tag{27}$$

Hence, in particular, $A_n = A_1$. However, $A_n = \text{Shade}(B_n)$ (by the definition of A_n) and $A_1 = \text{Shade}(B_1)$ (by the definition of A_1). Hence, $\text{Shade}(B_n) = A_n = A_1 = \text{Shade}(B_1)$. Thus, (23) (applied to $F_1 = B_n$ and $F_2 = B_1$) yields $\varepsilon(B_n) = \varepsilon(B_1)$ (since $B_n \in \mathcal{A}_+ \subseteq \mathcal{A}$ and $B_1 \in \mathcal{A}_+ \subseteq \mathcal{A}$).

Set $u = \varepsilon(B_n)$. Thus, $u = \varepsilon(B_n) = \varepsilon(B_1)$.

Recall that the sets B_1, B_2, \dots, B_n are distinct. Hence, $B_n \neq B_1$ (since $n \geq 2$).

The definition of Φ yields $\Phi(B_n) = B_n \setminus \{\varepsilon(B_n)\} = B_n \setminus \{u\}$ (since $\varepsilon(B_n) = u$). However, (25) says that $\Phi(B_n) \prec B_1$. In other words, $B_n \setminus \{u\} \prec B_1$ (since $\Phi(B_n) = B_n \setminus \{u\}$).

On the other hand, $B_1 \in \mathcal{A}_+ = \{P \in \mathcal{A} \mid \varepsilon(P) \in P\}$. In other words, B_1 is a $P \in \mathcal{A}$ satisfying $\varepsilon(P) \in P$. In other words, B_1 is an element of \mathcal{A} and satisfies $\varepsilon(B_1) \in B_1$. Now, $u = \varepsilon(B_1) \in B_1$. The same argument (applied to B_n instead of B_1) yields $u \in B_n$ (since $u = \varepsilon(B_n)$). Combining $u \in B_n$ with $u \in B_1$, we obtain $u \in B_n \cap B_1$.

Thus, Lemma 5.6 (applied to $X = B_n$ and $Y = B_1$) yields $B_n = B_1$ (since $B_n \setminus \{u\} \prec B_1$). This contradicts $B_n \neq B_1$.

Forget that we fixed (B_1, B_2, \dots, B_n) . We thus have found a contradiction whenever (B_1, B_2, \dots, B_n) is a tuple of distinct sets $B_1, B_2, \dots, B_n \in \mathcal{A}_+$ with the property that $n \geq 2$ and that (24) and (25). Hence, there exists no such tuple. In other words, the complete matching $(\mathcal{A}_-, \mathcal{A}_+, \Phi)$ is acyclic. Therefore, the simplicial complex \mathcal{A} has an acyclic complete matching, and thus is collapsible (by Definition 5.4 (c)). This finishes the proof of Theorem 5.5. \square

The analogues of Proposition 5.2 and of Theorem 5.5 for vertex-infection (instead of usual infection) also hold (with the same proofs). More generally,

Proposition 5.2 and Theorem 5.5 can be generalized to any inclusion-preserving shade map:

Theorem 5.7. Let E be any set. Let $\text{Shade} : \mathcal{P}(E) \rightarrow \mathcal{P}(E)$ be an inclusion-preserving shade map on E . Let G be any subset of E . Let

$$\mathcal{A} = \{F \subseteq E \mid G \not\subseteq \text{Shade } F\}.$$

Then:

- (a) This \mathcal{A} is a simplicial complex on ground set E .
- (b) This simplicial complex \mathcal{A} is collapsible.

Proof sketch. Part (a) is a straightforward generalization of Proposition 5.2, while part (b) is a straightforward generalization of Theorem 5.5. The proofs we gave above generalize (mutatis mutandis). \square

However, Theorem 5.7 cannot be lifted to the full generality of arbitrary shade maps, since \mathcal{A} will generally not be a simplicial complex unless the shade map is inclusion-preserving. (However, for inclusion-reversing shade maps, we can obtain a variant of Theorem 5.7 by applying Theorem 5.7 to the dual shade map Shade' from Proposition 4.7.)

6. Open questions

I shall now comment on two natural directions of research so far unexplored.

6.1. The Alexander dual

Any simplicial complex has an *Alexander dual*, which is defined as follows:

Definition 6.1. Let E be a finite set. Let \mathcal{A} be a simplicial complex on ground set E . Then, we define a new simplicial complex \mathcal{A}^\vee on ground set E by

$$\mathcal{A}^\vee = \{F \subseteq E \mid E \setminus F \notin \mathcal{A}\}.$$

(That is, \mathcal{A}^\vee consists of those subsets of E whose complements don't belong to \mathcal{A} .) This simplicial complex \mathcal{A}^\vee is called the *Alexander dual* of \mathcal{A} .

It is well-known that a simplicial complex \mathcal{A} and its Alexander dual \mathcal{A}^\vee share many properties; in particular, the reduced homology of \mathcal{A} is isomorphic to the reduced cohomology of \mathcal{A}^\vee (see, e.g., [BjoTan09, Theorem 1.1]). However, the collapsibility and the homotopy types of \mathcal{A} and \mathcal{A}^\vee are not always related. Thus, the following question is suggested but not answered by Theorem 5.5:

Question 6.2. Let us use the notations from Section 1 as well as Definition 2.1. Let G be any subset of E . Define \mathcal{A} as in (20). Is the simplicial complex

$$\mathcal{A}^\vee = \{F \subseteq E \mid G \subseteq \text{Shade}(E \setminus F)\}$$

collapsible? Is it contractible?

6.2. Several vertices v

Elser’s nuclei-based viewpoint in [Elser84] (and [DHLetc19, Conjecture 9.1]) suggests yet another question.

Our definition of $\text{Shade } F$ (Definition 2.1), and the underlying notion of “infecting” an edge, implicitly relied on the choice of vertex v . It thus is advisable to rename the set $\text{Shade } F$ as $\text{Shade}_v F$ and combine such sets for different values of v . In particular, we can define

$$\mathcal{A}_U = \{F \subseteq E \mid G \not\subseteq \text{Shade}_v F \text{ for some } v \in U\}$$

for any subset U of V . This \mathcal{A}_U is a simplicial complex (being the union of a family of simplicial complexes), and thus we can ask the same questions about it as we did about \mathcal{A} :

Question 6.3. What can we say about the homotopy and discrete Morse theory of \mathcal{A}_U ? What about its Alexander dual?

For $G = E$ and $|U| > 0$, this simplicial complex \mathcal{A}_U is the Alexander dual of the “ U -nucleus complex” Δ_U^G from [DHLetc19, Definition 3.2] (when G is connected). If [DHLetc19, Conjecture 9.1 for $|U| > 1$] is correct, then the homology of \mathcal{A}_U with real coefficients should be concentrated in a single degree; this suggests the possible existence of an acyclic partial matching with all critical faces in one degree.

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