Some simplicial complexes in combinatorics

[Corrected slides of a talk at UConn Algebra Seminar, 2021-03-11]

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Abstract. A number of combinatorial identities are concerned with certain classes of subsets of a finite set (e.g., matchings of a graph); they can be viewed as saying (roughly speaking) that equal numbers of these subsets have even size and odd size. In this talk, I will discuss a few such identities – some of them new – and their topological meaning. As a common theme, the "parity bias" (or lack thereof) is the Euler characteristic of a simplicial complex, and thus any expression for it is potentially the tip of a topological iceberg. Underneath are questions of homology, homotopy or even discrete Morse theory. Aside from the specific complexes in question, I hope to provide one more pair of "simplex glasses" through which combinatorial identities appear in a new light.

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`https://arxiv.org/abs/2009.11527`

1. Introduction

1.1. Alternating sums

- Enumerative combinatorics is full of *alternating sums*. Some examples:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \qquad \text{for integers } n > 0;$$
$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} \qquad \text{for } m \geqslant 0;$$
$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} = (1 \text{ or } 0 \text{ or } -1);$$
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{ak+b}{c} = 0 \qquad \text{for } c, n \in \mathbb{N} \text{ with } c < n;$$
$$\sum_{i=0}^{m} (-1)^i \operatorname{sur}(m,i) = (-1)^m,$$

  where $\operatorname{sur}(m,i) = (\# \text{ of surjections from } \{1,2,\ldots,m\} \text{ to } \{1,2,\ldots,i\})$.

- These alternating sums are among the most helpful tools in proving identities. (They often play a similar role as the formula $1 + \zeta + \zeta^2 + \cdots + \zeta^{n-1} = 0$ for ζ being a nontrivial n-th root of unity plays in the discrete Fourier transform.)

- An alternating sum identity generally looks like this:

$$\sum_{\text{(some finite set)}} (-1)^{\text{(something)}} \text{(something)} = \text{(something typically simpler)}.$$

- In this talk, I shall
  - present some alternating sum identities and their combinatorial proofs by "**toggling**" or "sign-reversing involutions";
  - discuss how a few of these identities can be lifted to **topological** statements about simplicial complexes,
  - and how these topological statements can be lifted to **combinatorial** statements again using discrete Morse theory.

- This is not a theory talk; you'll hear my personal favorites, not the most general or most important results.

- There will be various open questions.

2. Toggling

2.1. All subsets

- We start with the first identity listed above:

  Theorem. Let n be a positive integer. Then,

  $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

- There are many ways to prove this:

  e.g., expand $(1-1)^n$ using the binomial theorem.

- Here is a **combinatorial proof**:

  Set $[n] = \{1, 2, \ldots, n\}$. Then,

  $$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = \sum_{I \subseteq [n]} (-1)^{|I|}.$$

  Claim: In the sum on the RHS, all the addends cancel out.

  Proof. For each subset I of $[n]$, we can

  1. insert 1 into I if $1 \notin I$, or
  2. remove 1 from I if $1 \in I$.

  This gives us a new subset of $[n]$, which we denote by $I \triangle \{1\}$.

  Easy to see: The map

  $$\{\text{subsets of } [n]\} \to \{\text{subsets of } [n]\}, \\ I \mapsto I \triangle \{1\}$$

  is an involution (i.e., applying it twice gives the identity), and it flips the sign (meaning $(-1)^{|I \triangle \{1\}|} = -(-1)^{|I|}$ for any subset I of $[n]$).

  Hence, all addends in the sum $\sum_{I \subseteq [n]} (-1)^{|I|}$ cancel out (the I-addend cancelling the $I \triangle \{1\}$-addend). Thus, the sum is 0, qed.

- Our notation $I \triangle \{1\}$ is a particular case of the notation

  $$\begin{aligned} I \triangle J &= (I \cup J) \setminus (I \cap J) \\ &= (I \setminus J) \cup (J \setminus I) \\ &= \{\text{all elements that belong to } \textbf{exactly one} \text{ of } I \text{ and } J\} \end{aligned}$$

  for any two sets I and J.

- If a is any element, then the operation of replacing a set I by $I \triangle \{a\}$ (that is, inserting a into I if $a \notin I$, and removing a from I otherwise) is called *toggling* a in I. This is always an involution: $(I \triangle \{a\}) \triangle \{a\} = I$ for any I and a.

- **Remark:** It was actually sufficient for our proof that the map $I \mapsto I \triangle \{1\}$ is a bijection, not necessarily an involution. But all such maps we will encounter are involutions.

2.2. All subsets not too large

- Let us try the second identity:

  Theorem. Let n be any number (e.g., a real), and let m be a nonnegative integer. Then,
  $$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$

- **Proof.** First of all, we are proving a polynomial identity in n, so we WLOG assume that n is a positive integer (since two polynomials over a field are equal if they agree on sufficiently many points).

  We have
  $$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = \sum_{\substack{I \subseteq [n]; \\ |I| \leqslant m}} (-1)^{|I|}.$$

  Now, we try the involution from the previous proof:
  $$\{\text{subsets of } [n]\} \to \{\text{subsets of } [n]\},$$
  $$I \mapsto I \triangle \{1\}.$$

  Unfortunately, applying it to a set I might break the $|I| \leqslant m$ restriction. But it restricts to an involution
  $$\mathcal{A} \to \mathcal{A},$$
  $$I \mapsto I \triangle \{1\},$$
  where
  $$\mathcal{A} = \{\text{subsets } I \text{ of } [n] \text{ with } |I \setminus \{1\}| < m\}.$$

  Thus, all addends in the sum $\sum_{\substack{I \subseteq [n]; \\ |I| \leqslant m}} (-1)^{|I|}$ cancel except for those with $|I \setminus \{1\}| = m$. We get
  $$\sum_{\substack{I \subseteq [n]; \\ |I| \leqslant m}} (-1)^{|I|} = \sum_{\substack{I \subseteq [n]; \\ |I| \leqslant m; \\ |I \setminus \{1\}| = m}} (-1)^{|I|} = \sum_{\substack{I \subseteq [n]; \\ 1 \notin I; \\ |I| = m}} (-1)^{|I|} = (-1)^m \binom{n-1}{m},$$

since there are exactly $\binom{n-1}{m}$ many subsets I of $[n]$ satisfying $1 \notin I$ and $|I| = m$. This completes our proof.

2.3. Lacunar subsets

- Now to the third identity:

  Theorem. Let n be a nonnegative integer. Then,

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} = \begin{cases} 1, & \text{if } n\%6 \in \{0,1\}\,; \\ 0, & \text{if } n\%6 \in \{2,5\}\,; \\ -1, & \text{if } n\%6 \in \{3,4\}\,, \end{cases}$$

  where $n\%6$ means the remainder of n divided by 6.

- To prove this combinatorially, we need to find out what $\binom{n-k}{k}$ counts.

- **Convention.** We shall write $[m]$ for $\{1, 2, \ldots, m\}$ whenever $m \in \mathbb{Z}$.

- **Definition.** A set I of integers is said to be *lacunar* if it contains no two consecutive integers (i.e., there is no $i \in I$ such that $i + 1 \in I$).

- For example, $\{1, 3, 6\}$ is lacunar, but $\{1, 3, 4\}$ is not. Empty and one-element sets are always lacunar.

- Note that any lacunar subset of $[n-1]$ has size $\leqslant \lfloor n/2 \rfloor$.

- **Proposition.** For any $n \geqslant k \geqslant 0$, the number of lacunar k-element subsets of $[n-1]$ is $\binom{n-k}{k}$.

- **Proof.** Write "elt" for "element", and "subs" for "subsets".

  There is a bijection

$$\{\text{lacunar } k\text{-elt subs of } [n-1]\} \to \{k\text{-elt subs of } \{0, 1, \ldots, n-k-1\}\}\,,$$
$$\{i_1 < i_2 < \cdots < i_k\} \mapsto \{i_1 - 1 < i_2 - 2 < \cdots < i_k - k\}\,.$$

- Thus, we can start a **combinatorial proof** of our theorem as follows:

$$\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n-k}{k} = \sum_{\substack{I \subseteq [n-1]; \\ I \text{ is lacunar}}} (-1)^{|I|}\,.$$

  We want to prove that this is 1 or 0 or -1.

So let us try to construct a sign-reversing involution on the set

$$\{\text{lacunar subsets of } [n-1]\}$$

except for possibly one element.

Let I be a lacunar subset of $[n-1]$.

- We try to toggle 1 in I, but we only do this if the result is lacunar.

  If we succeed (i.e., if the result is lacunar), then we are done.

  [**Examples:**

  * If $I = \{1,3,7\}$, then we toggle 1, and obtain the set $\{3,7\}$. Thus, in this case, we succeed and have found the image of I under our involution.
  * If $I = \{3,7\}$, then we toggle 1, and obtain the set $\{1,3,7\}$. Thus, in this case, we succeed and have found the image of I under our involution.
  * If $I = \{2,7\}$, then we cannot toggle 1, since this would produce the non-lacunar set $\{1,2,7\}$. Thus, in this case, we don't succeed and move on to the next step.]

- If we have not succeeded in the previous step, then $2 \in I$ and thus $3 \notin I$.

  Thus we try to toggle 4 in I, but we only do this if the result is lacunar.

  If we succeed, then we are done.

  [**Examples:**

  * If $I = \{2,4,9\}$, then we toggle 4, and obtain the set $\{2,9\}$. Thus, in this case, we succeed and have found the image of I under our involution.
  * If $I = \{2,9\}$, then we toggle 4, and obtain the set $\{2,4,9\}$. Thus, in this case, we succeed and have found the image of I under our involution.
  * If $I = \{2,5,8\}$, then we cannot toggle 4, since this would produce the non-lacunar set $\{2,4,5,8\}$. Thus, in this case, we don't succeed and move on to the next step.
  * If $I = \{1,3,7\}$, then we do not get to this step in the first place, since the first step has already succeeded (turning I into $\{3,7\}$).]

- If we have not succeeded in the previous step, then $5 \in I$ and thus $6 \notin I$.

  Thus we try to toggle 7 in I, but we only do this if the result is lacunar.

  If we succeed, then we are done.

  - And so on.

  This operation goes on until we have run out of elements of $[n-1]$ to toggle. The only case in which we fail to toggle anything is if

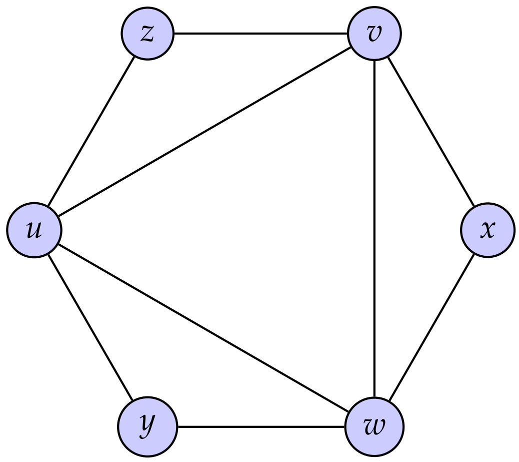
$$n \not\equiv 2 \bmod 3 \text{ and } I = \{2, 5, 8, \ldots\} \cap [n-1].$$

  Thus we have found a sign-reversing involution on the set $\{\text{lacunar subsets of } [n-1]\}$ with the exception of a single lacunar subset if $n \not\equiv 2 \bmod 3$ (and with no exceptions if $n \equiv 2 \bmod 3$). The conclusion easily follows.

- This proof is in [BenQui08] (where it is worded using domino tilings instead of lacunar subsets).

2.4. Independent sets of a graph

- Let us generalize this.
- **Definition.** Let $\Gamma = (V, E)$ be an (undirected) graph. An *independent set* of Γ means a subset I of V such that no two vertices in I are adjacent (i.e., no edge of Γ connects two vertices in I).
- **Example.** For the following graph:



  the independent sets are

$$\{x, y\}, \{y, z\}, \{z, x\}, \{u, x\}, \{v, y\}, \{w, z\}, \{x, y, z\}$$

  as well as all 1-element sets and the empty set.

- For any $m \geqslant 0$, let the *m-path* be the graph

$$1 - 2 - 3 - \cdots - m$$

  (that is, the graph with vertices $1, 2, \ldots, m$ and edges $\{i, i+1\}$ for each $0 < i < m$).

  Then, the lacunar subsets of $[m]$ are the independent sets of the m-path.

- Now we can generalize our previous theorem as follows:

  Question: For what graphs Γ do we have

$$\sum_{\substack{I \text{ is an independent} \\ \text{set of } \Gamma}} (-1)^{|I|} \in \{1, 0, -1\} \text{ ?}$$

- Certainly not for all graphs Γ (e.g., the 3-cycle is a counterexample).
- But we know it's true for path graphs. For what other graphs?
- We can try to construct a sign-reversing involution again, and see where we fail.
- What order do we try to toggle the vertices in?
- Well, we can always pick some order at random.
- Unfortunately, toggling a vertex might be blocked by several vertices.
- Trying to solve the resulting conflicts often fails (e.g., for a 4-cycle, even though the sum is -1 for a 4-cycle).
- Our above proof can be adapted when Γ is a tree.
- However, a much more general result holds:
- **Theorem (conjectured by Kalai and Meshulam, 1990s, proved by Chudnovsky, Scott, Seymour, Spirkl, 2018 ([CSSS18])):** Let Γ be a simple loopless undirected graph that has no induced cycle of length divisible by 3. Then,

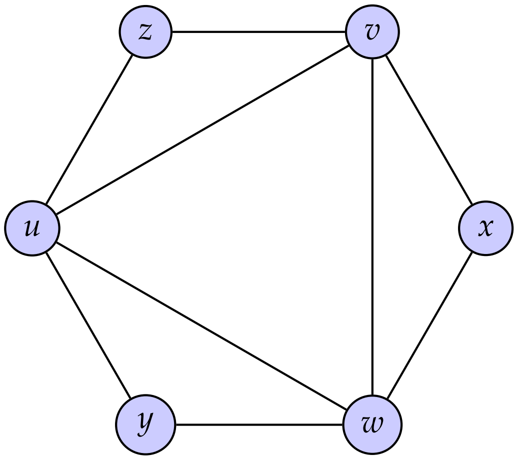
$$\sum_{\substack{I \text{ is an independent} \\ \text{set of } \Gamma}} (-1)^{|I|} \in \{1, 0, -1\} .$$

- **Question:** Is there any proof under 10 pages length?

2.5. Dominating sets of a graph

- **Definition.** Let $\Gamma = (V, E)$ be an (undirected) graph. A *dominating set* of Γ means a subset I of V such that each vertex of Γ belongs to I or has a neighbor in I.

- **Example.** For the following graph:



the **non-dominating** sets are

$$\{u, y, z\}, \quad \{v, z, x\}, \quad \{w, x, y\}$$

as well as all their subsets.

- **Theorem (e.g., Brouwer 2009 ([Brouwe09], [BrCsSc09])):** The number of dominating sets of a graph Γ is always odd.

- **Theorem (Heinrich, Tittmann, 2017 ([HeiTit17], [Grinbe17, Theorem 3.2.2])):** The number of dominating sets of a graph $\Gamma = (V, E)$ is

$$2^{|V|} - 1 + \underbrace{\sum_{\substack{\text{pairs } (A,B) \text{ of disjoint} \\ \text{nonempty subsets of } V; \\ \{a,b\} \notin E \text{ for all } a \in A \text{ and } b \in B; \\ |A| \equiv |B| \bmod 2}} (-1)^{|A|}}_{\substack{\text{This is even for symmetry reasons} \\ \text{(for any } (A,B)\text{, there is a } (B,A))}} .$$

- What about the alternating sum

$$\sum_{\substack{I \text{ is a dominating} \\ \text{set of } \Gamma}} (-1)^{|I|} \quad ?$$

Is it ± 1 ?

- No; for example:

Theorem (Alikhani, 2012 ([Alikha12, Lemma 1])): If Γ is an n-cycle (for $n > 0$), then this alternating sum is

$$\begin{cases} 3, & \text{if } n \equiv 0 \bmod 4; \\ -1, & \text{otherwise.} \end{cases}$$

Exercise: Prove this! (Is there a nice proof without too much casework?)

- **Theorem (Ehrenborg, Hetyei, 2005 ([EhrHet06, §7])):** The alternating sum is ± 1 whenever Γ is a forest.

3. Simplicial complexes

3.1. Basic definitions

- The sums we have been discussing so far didn't range over some random collections of sets. Most of them had a commonality: If a set I appeared in the sum, then so did any subset of I.

- Such collections of sets are called **simplicial complexes**.

- Formally:

  Definition. A *simplicial complex* means a pair (S, Δ), where S is a finite set and Δ is a collection (= set) of subsets of S such that

  $$\text{any } I \in \Delta \text{ and } J \subseteq I \text{ satisfy } J \in \Delta.$$

- We often just write Δ for a simplicial complex (S, Δ).

- A *face* of a simplicial complex Δ means a set $I \in \Delta$.

- Note that $\{\}$ and $\{\varnothing\}$ are two different simplicial complexes on any set S.

- **Examples** of simplicial complexes:

  - $\{$all subsets of $S\}$ for a given finite set S.
  - $\{$all lacunar subsets of $[m]\}$ for a given $m \in \mathbb{N}$.
  - $\{$all independent sets of $\Gamma\}$ for a given graph Γ.
  - **not** $\{$all dominating sets of $\Gamma\}$ for a given graph Γ.
  - $\{$all non-dominating sets of $\Gamma\}$ and $\{$all complements of dominating sets of $\Gamma\}$ for a given graph Γ.

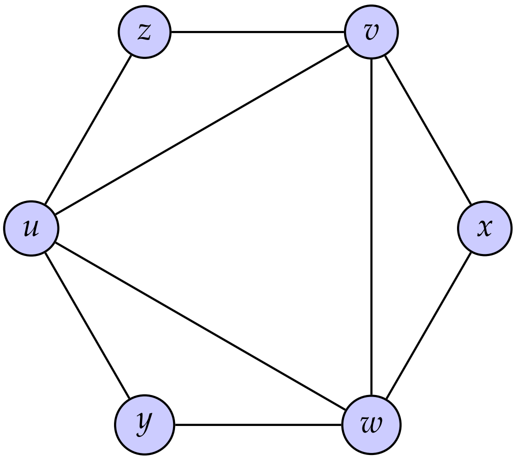
  (Here the ground set is the set of vertices of Γ.)

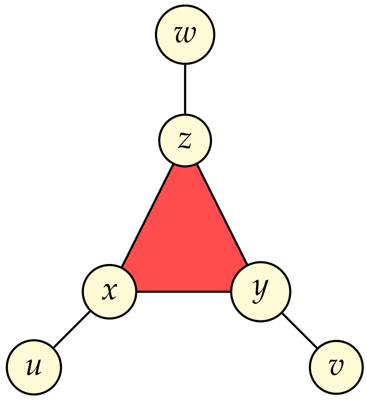
3.2. Geometric realizations

- Each simplicial complex (S, Δ) has a *geometric realization* $|\Delta|$, which is a topological space glued out of (geometric) simplices. The easiest way to define it is by assuming (WLOG) that $S = [n]$ for some $n \in \mathbb{N}$, and setting

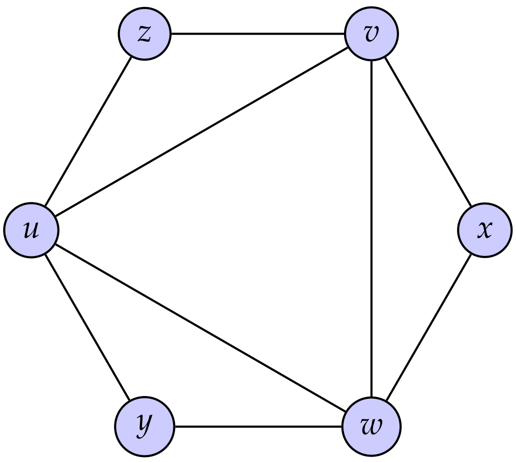
  $$|\Delta| = \left\{ (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n_{\geqslant 0} \ \mid \ t_1 + t_2 + \cdots + t_n = 1 \right.$$
  $$\left. \text{and } \{ i \ \mid \ t_i > 0 \} \in \Delta \right\}.$$

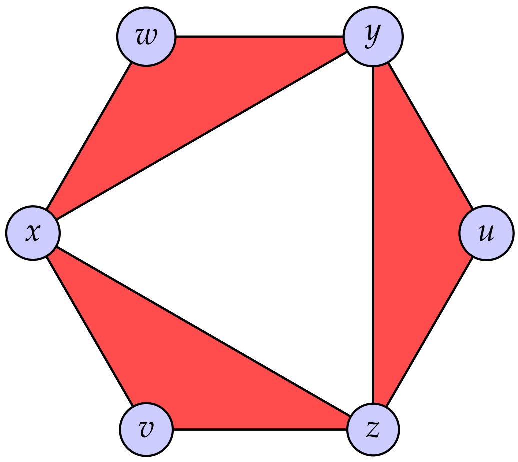
- Normally we don't draw the literal $|\Delta|$ (since $\mathbb{R}^n$ has too high dimension) but just something homeomorphic to it (usually in a smaller space).
- Some examples:
  - The complex {all independent sets of Γ} of the graph Γ on the left is the simplicial complex drawn on the right:





  - The complex {all non-dominating sets of Γ} of the graph Γ on the left is the simplicial complex drawn on the right:





3.3. Homotopy and homology

- A lot of features come for free with the geometric realization:

  The *homotopy type*, the *homology* and the *reduced Euler characteristic* of a simplicial complex Δ are defined to be the homotopy type, the homology and the reduced Euler characteristic of its geometric realization.

- Explicitly, the Euler characteristic of a complex Δ is simply

$$\sum_{I \in \Delta} (-1)^{|I|-1}.$$

(The "-1" in the exponent just negates the whole sum.)

- Thus, the alternating sums we have been computing are actually Euler characteristics in disguise.

- Homology is a stronger invariant than Euler characteristic, and homotopy type is an even stronger invariant than homology:

$$\begin{aligned} &(\text{homotopy type}) \twoheadrightarrow (\text{homology over } \mathbb{Z}) \twoheadrightarrow (\text{homology over } \mathbb{Q}) \\ &\twoheadrightarrow (\text{Euler characteristic}). \end{aligned}$$

Our results above are all about Euler characteristics; can we lift them to those stronger invariants?

- Note that homology can be easily redefined combinatorially in terms of Δ. (Homotopy type, too, but less easily; see [Kozlov20, Proposition 9.28].)

3.4. Examples of homotopy types

- Our first theorem said that the reduced Euler characteristic of the simplicial complex

$$\{\text{all subsets of } E\}$$

is 0 for any nonempty finite set E. This lifts all the way up to homotopy level:

Proposition. This simplicial complex is contractible (i.e., homotopy-equivalent to a point).

Geometrically, this is clear: Its geometric realization is a simplex, hence homeomorphic to an $(n-1)$-ball, where $n = |E|$.

- Our second theorem was

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$

This corresponds to the simplicial complex

$$\{\text{all subsets of } [n] \text{ having size } \leqslant m\}.$$

This is called the $(m-1)$-*skeleton* of the $(n-1)$-ball. By classical algebraic topology, it is homotopy-equivalent to a bouquet of $\binom{n-1}{m}$ many $(m-1)$-spheres, which again explains the Euler characteristic.

- Now, recall the independent sets of graphs.

  Theorem (Kalai, Meshulam, Engström, Chudnovsky, Scott, Seymour, Spirkl, Zhang, Wu, Kim, 2021 ([ZhaWu20], [Kim21])): Let Γ be a simple loopless undirected graph that has no induced cycle of length divisible by 3. Then, the simplicial complex

  $$\{\text{independent sets of } \Gamma\}$$

  is either contractible or homotopy-equivalent to a sphere (whence its reduced Euler characteristic is in $\{1, 0, -1\}$).

- As we recall, the dominating sets of a graph do not form a simplicial complex, but their complements do, and so do the non-dominating sets. As far as the alternating sum $\sum\limits_{I} (-1)^{|I|}$ is concerned, these are just as good (switching between dominating and non-dominating sets or between the sets and their complements changes the sum by a factor of ± 1).

  Theorem (Ehrenborg, Hetyei, 2005 ([EhrHet06, §7])): Let Γ be a forest. Then, both simplicial complexes

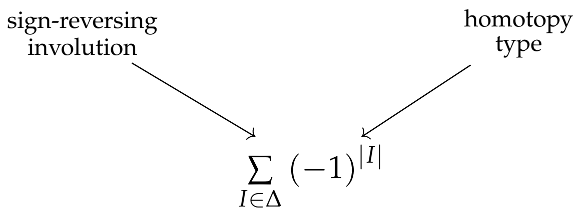
  $$\{\text{non-dominating sets of } \Gamma\} \qquad \text{and}$$
  $$\{\text{complements of dominating sets of } \Gamma\}$$

  are either contractible or homotopy-equivalent to a sphere.

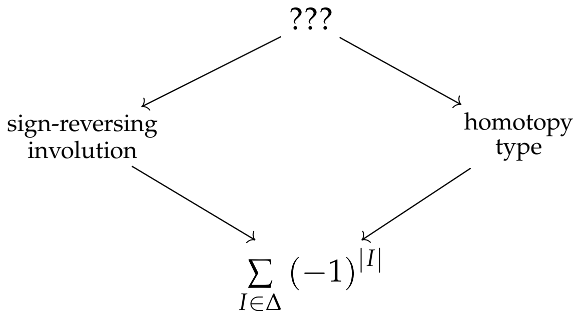
- **Question:** What can be said about the case when Γ is an n-cycle?

3.5. Discrete Morse theory

- Thus we have two approaches to proving formulas for alternating sums:



- Could these two approaches be combined? I.e., is there a technique that gets us both homotopy information and a sign-reversing involution in one (possibly harder) swoop?



- *Discrete Morse theory* is an answer. We will use some of its very basics.

- **Definition.** For two sets I and J, we write $I \prec J$ if $J = I \cup \{\text{a single element}\}$ (that is, if $I \subseteq J$ and $|J \setminus I| = 1$). Equivalently, we write $J \succ I$ for this.

- **Definition.** Let (S, Δ) be a simplicial complex. A *partial matching* on Δ shall mean an involution $\mu : \Delta \to \Delta$ such that

$$\mu(I) = I \text{ or } \mu(I) \prec I \text{ or } \mu(I) \succ I \qquad \text{for each } I \in \Delta.$$

  In other words, $\mu(I)$ is either I itself or is obtained from I by removing or inserting a single element.

- **Definition.** If μ is a partial matching on Δ, then the sets $I \in \Delta$ satisfying $\mu(I) = I$ will be called *unmatched* (by μ).

- Thus, if μ is a partial matching on Δ, then

$$\sum_{I\in\Delta}(-1)^{|I|} = \sum_{\substack{I\in\Delta \text{ is} \\ \text{unmatched}}}(-1)^{|I|}$$

  (by cancellation).

- Thus, partial matchings are just our partial sign-reversing involutions rewritten (instead of taking some sets out of our complex, we are now leaving them fixed).

- What about the homotopy information? We cannot in general "cancel" matched faces from a simplicial complex and hope that the homotopy information is preserved.

- However, we can restrict our matchings in a way that will make them homotopy-friendly! This is one of the main contributions of Forman that became discrete Morse theory ([Forman02, §3, §6], [Kozlov20]):

- **Definition.** Let (S, Δ) be a simplicial complex. A partial matching μ on Δ is said to be *acyclic* (or a *Morse matching*) if there exists no "cycle" of the form
$$I_1 \succ \mu(I_1) \prec I_2 \succ \mu(I_2) \prec I_3 \succ \cdots \prec I_n \succ \mu(I_n) \prec I_1$$
with $n \geqslant 2$ and with $I_1, I_2, \ldots, I_n$ distinct.

- **Intuition:** The easiest way to ensure this is by making sure that when μ adds an element to a face I, then it does so in an "optimal" way (i.e., among all ways to add an element to I and still obtain a face of Δ, it picks the "best" one in some sense). This way, in the above "cycle", the faces $I_1, I_2, \ldots, I_n, I_1$ become "better and better", so the cycle cannot exist. There is freedom in defining what "optimal"/"best" is (it means specifying some partial order on the faces of any given size).

  This is why Forman calls acyclic matchings "*gradient vector fields*" in [Forman02].

- **Empiric fact(?):** Sign-reversing involutions in combinatorics tend to be acyclic partial matchings.

- **Question:** Really? Check some of the more complicated ones!

- **Theorem (Forman, I believe).** Let (S, Δ) be a simplicial complex, and μ an acyclic partial matching on Δ. For each $k \in \mathbb{N}$, let c_k be the number of unmatched size-k faces of Δ.

  Then, there is a CW-complex homotopy-equivalent to Δ that has exactly c_k faces of dimension $k-1$ for each $k \in \mathbb{N}$.

- **Corollary. (a)** If a simplicial complex (S, Δ) has an acyclic partial matching that leaves no face unmatched, then it is contractible.

  (b) If a simplicial complex (S, Δ) has an acyclic partial matching that leaves exactly one face unmatched, then it is homotopy-equivalent to a sphere.

- As a consequence, having a good Morse matching gets us good (if not 100% complete) information both about the homotopy type and about the combinatorics of a simplicial complex.

- For example, all the sign-reversing involutions we used in our proofs above are Morse matchings.

4. Elser's "pandemic" complex

- A remarkable alternating sum identity appeared in a 1984 paper by Elser on mathematical physics (percolation theory) [Elser84]. I shall restate it in a slightly simpler language.

- Fix a (finite undirected multi)graph Γ with vertex set V and edge set E.

  Fix a vertex $v \in V$.

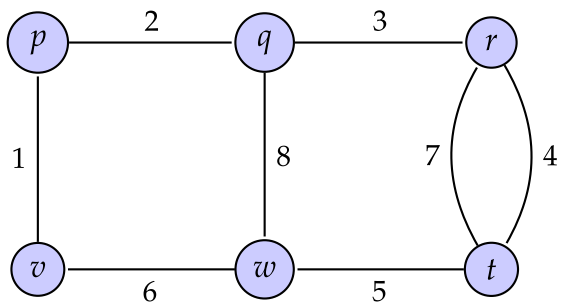
- If $F \subseteq E$, then an *F-path* shall mean a path of Γ such that all edges of the path belong to F.

- If $e \in E$ is any edge and $F \subseteq E$ is any subset, then we say that F *infects* e if there exists an F-path from v to some endpoint of e.

  (My go-to mental model: A virus starts out in v and spreads along any F-edge it can get to. Then, F infects e if the virus will eventually reach an endpoint of e. Note that F always infects any edge through v.)

- A subset $F \subseteq E$ is said to be *pandemic* if it infects each edge $e \in E$.

- **Example:** Let Γ be



  .

  Then:

  - The set $\{1, 2\} \subseteq E$ infects edges $1, 2, 3, 6, 8$ (but no others), since the virus gets to the vertices v, p, q.
  - The set $\{1, 2, 5\}$ infects the same edges.
  - The set $\{1, 2, 3\}$ infects every edge other than 5.
  - The set $\{1, 2, 3, 4\}$ infects each edge, and thus is pandemic (even though the virus never gets to vertex w).

- **Theorem (Elser, 1984 ([Elser84, Lemma 1], [Grinbe20, Theorem 1.2])):** Assume that $E \neq \varnothing$. Then,

$$\sum_{\substack{F \subseteq E \text{ is} \\ \text{pandemic}}} (-1)^{|F|} = 0.$$

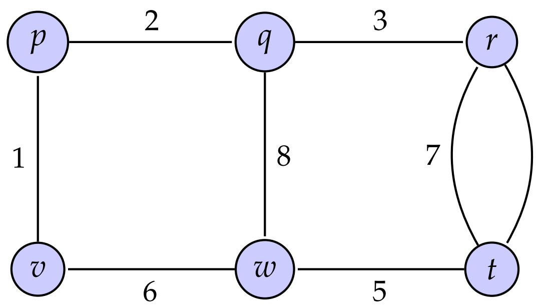
- **Remark:** A version of pandemicity in which F has to infect all vertices (rather than all edges) would fail to produce such a theorem.

4.1. More generally

- If F is a subset of E, then we define a subset $\operatorname{Shade} F$ of E by

$$\operatorname{Shade} F = \{e \in E \ \mid \ F \text{ infects } e\}.$$

- **Example:** Let Γ be



Then, $\operatorname{Shade}\{1,2\} = \{1,2,3,6,8\}$ and $\operatorname{Shade}\{1\} = \{1,2,6\}$ and $\operatorname{Shade}\{8\} = \{1,6\}$.

- **Theorem ([Grinbe20, Theorem 2.5], generalizing Elser's theorem):** Let G be any subset of E. Assume that $E \neq \varnothing$. Then,

$$\sum_{\substack{F \subseteq E; \\ G \subseteq \operatorname{Shade} F}} (-1)^{|F|} = 0.$$

- **Theorem ([Grinbe20, Theorem 2.6], equivalent restatement of previous theorem):** Let G be any subset of E. Then,

$$\sum_{\substack{F \subseteq E; \\ G \not\subseteq \operatorname{Shade} F}} (-1)^{|F|} = 0.$$

- This restatement looks useful since it gets rid of the $E \neq \varnothing$ assumption. That's a good sign!

4.2. Proof idea

- Let's prove this latter restatement. Here is it again:

  Theorem ([Grinbe20, Theorem 2.6], equivalent restatement of previous theorem): Let G be any subset of E. Then,

$$\sum_{\substack{F \subseteq E; \\ G \not\subseteq \operatorname{Shade} F}} (-1)^{|F|} = 0.$$

- **Proof.** Let
$$\mathcal{A} = \{F \subseteq E \mid G \not\subseteq \operatorname{Shade} F\}.$$
Equip the set E with a total order. If $F \in \mathcal{A}$, then let $\varepsilon(F)$ be the **smallest** edge $e \in G \setminus \operatorname{Shade} F$.

  Define a sign-reversing involution
$$\mathcal{A} \to \mathcal{A},$$
$$F \mapsto F \triangle \{\varepsilon(F)\}.$$
Check that this works! (The key observation: $\operatorname{Shade} F$ does not change when we toggle $\varepsilon(F)$ in F.)

4.3. Variants

- We cannot replace "infects all edges" by "infects all vertices" as long as we work with sets of edges.
- However, we can work with sets of vertices instead (mutatis mutandis).
- In detail:
- If $F \subseteq V$, then an *F-vertex-path* shall mean a path of Γ such that all vertices of the path except (possibly) for its two endpoints belong to F. (Thus, if a path has only one edge or none, then it automatically is an F-vertex-path.)
- If $w \in V \setminus \{v\}$ is any vertex and $F \subseteq V \setminus \{v\}$ is any subset, then we say that F *vertex-infects* w if there exists an F-vertex-path from v to w. (This is always true when w is v or a neighbor of v.)
- A subset $F \subseteq V \setminus \{v\}$ is said to be *vertex-pandemic* if it vertex-infects each vertex $w \in V \setminus \{v\}$.
- **Theorem ([Grinbe20, Theorem 3.2]).** Assume that $V \setminus \{v\} \neq \varnothing$. Then,
$$\sum_{\substack{F \subseteq V \setminus \{v\} \text{ is} \\ \text{vertex-pandemic}}} (-1)^{|F|} = 0.$$
- Generalizations similar to the one above also hold.

4.4. A hammer in search of nails

- The proofs of the original Elser's theorem and of its vertex variant are suspiciously similar.

- Even worse, they use barely any graph theory. All we needed is that E is a finite set, and that $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ (where $\mathcal{P}(E) = \{\text{all subsets of } E\}$) is a map with the property that

$$\begin{aligned} &\operatorname{Shade}(F \triangle \{u\}) = \operatorname{Shade} F \\ &\qquad \text{for any } F \subseteq E \text{ and } u \in E \setminus \operatorname{Shade} F. \end{aligned}$$

  I call such a map Shade a *shade map*. Our above argument then shows that

$$\sum_{\substack{F \subseteq E; \\ G \not\subseteq \operatorname{Shade} F}} (-1)^{|F|} = 0 \qquad \text{for any } G \subseteq E.$$

- **Question.** Have you seen other maps satisfying this property in the wild?

- **Answer 1.** Let A be an affine space over $\mathbb{R}$. Fix a finite subset E of A. For any $F \subseteq E$, we define

$$\operatorname{Shade} F = \{e \in E \mid e \text{ is } \textbf{not} \text{ a nontrivial convex combination of } F\}.$$

  (A convex combination is said to be *nontrivial* if all coefficients are < 1.)

  Then, this map $\operatorname{Shade} : \mathcal{P}(E) \to \mathcal{P}(E)$ is a shade map.

- Other answers? Can you get shade maps from matroids? spanning trees? closure operators? lattices?

4.5. The topological viewpoint

- Now let us return to the case of a graph $\Gamma = (V, E)$. Fix a subset G of E, and let

$$\begin{aligned} \mathcal{A} &= \{F \subseteq E \mid G \not\subseteq \operatorname{Shade} F\} \\ &= \{F \subseteq E \mid \text{not every edge in } G \text{ is infected by } F\} \end{aligned}$$

  as in the proof above.

- This $\mathcal{A}$ is clearly a simplicial complex on ground set E.

- **Theorem (G., 2020 ([Grinbe20, Theorem 5.5])).** This simplicial complex has a Morse matching (i.e., an acyclic partial matching) with no unmatched faces. Thus, it is contractible.

- **Proof idea.** Argue that the sign-reversing involution above is a Morse matching.

4.6. The Alexander dual

- The complex
$$\mathcal{A} = \{F \subseteq E \mid G \not\subseteq \operatorname{Shade} F\}$$
is not the only simplicial complex we can obtain from our setup. There is also
$$\mathcal{A}^* = \{F \subseteq E \mid G \subseteq \operatorname{Shade}(E \setminus F)\}.$$

- More generally, if (S, Δ) is any simplicial complex, then we can define a new simplicial complex (S, Δ^*), where
$$\Delta^* := \{I \subseteq S \mid S \setminus I \notin \Delta\}$$
$$= \{\text{the complements of the non-faces of } \Delta\}.$$
This (S, Δ^*) is called the *Alexander dual* of (S, Δ).

- The homologies of (S, Δ^*) and (S, Δ) are isomorphic (folklore – see, e.g., [BjoTan09]); thus the Euler characteristics agree up to sign.

  But the homotopy types are not in general equivalent! Nor is the existence of a Morse matching with good properties.

- Thus, for any homotopy type question we can answer, we can state an analogous one for its dual.

- **Question.** What is the homotopy type of the $\mathcal{A}^*$ above?

4.7. Multi-shades?

- I can't help spreading yet another open question that essentially comes from Dorpalen-Barry et al. [DHLetc19, Conjecture 9.1].

- Return to the setup of a graph $\Gamma = (V, E)$, but don't fix the vertex v this time.

- Rename $\operatorname{Shade} F$ as $\operatorname{Shade}_v F$ to stress its dependence on v.

- For any subset $U \subseteq V$, define the simplicial complex
$$\mathcal{A}_U := \{F \subseteq E \mid G \not\subseteq \operatorname{Shade}_v F \text{ for some } v \in U\}.$$

- **Question:** What can we say about the homotopy and discrete Morse theory of $\mathcal{A}_U$? What about its Alexander dual?

- An optimistic yet reasonable expectation would be: a Morse matching whose unmatched faces all have the same size. (Thus, $\mathcal{A}_U$ should be homotopy-equivalent to a bouquet of spheres.)

5. Bonus: Path-free and path-missing complexes

- This is **joint work with Lukas Katthän and Joel Brewster Lewis** [GrKaLe21].
- Fix a **directed** graph $G = (V, E)$ and two vertices s and t. We define the two simplicial complexes

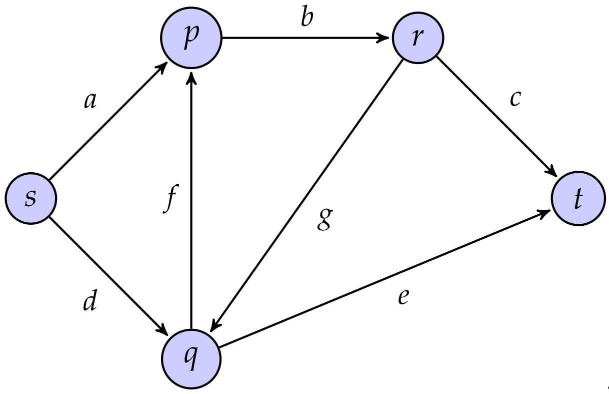
$$\mathcal{PF}(G) = \{F \subseteq E \mid \text{there is no } F\text{-path from } s \text{ to } t\}$$
$$(\text{the } \textit{"path-free" complex} \text{ of } G)$$

and

$$\mathcal{PM}(G) = \{F \subseteq E \mid \text{there is an } (E \setminus F)\text{-path from } s \text{ to } t\}$$
$$(\text{the } \textit{"path-missing" complex} \text{ of } G).$$

(These are Alexander duals of each other.)

- **Example:** Let G be the following directed graph:



.

Then:

  - The faces of the simplicial complex $\mathcal{PF}(G)$ are the sets

$$\{b, c, e, f, g\}, \{a, c, e, f, g\}, \{b, c, d, g\}, \{a, c, d, f, g\}, \{a, b, e, f\}, \{a, b, d, f, g\}$$

as well as all their subsets.

  - The faces of the simplicial complex $\mathcal{PM}(G)$ are the sets

$$\{d, e, f, g\}, \{c, d, f\}, \{a, b, c, f, g\}, \{a, e, g\}$$

as well as all their subsets.

- **Theorem (G., Katthän, Lewis, 2021 ([GrKaLe21])).** Assume that $s \neq t$ and $E \neq \varnothing$ (the other cases are trivial). Then, both complexes $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$ are contractible or homotopy-equivalent to spheres. The dimensions of the spheres can be determined explicitly. The complexes are contractible if and only if G has a useless edge (i.e., an edge that appears in no path from s to t) or a (directed) cycle.

- **Theorem (G., Katthän, Lewis, 2021+ ([GrKaLe21, future version])).** Both complexes $\mathcal{PF}(G)$ and $\mathcal{PM}(G)$ have Morse matchings with at most one unmatched face.
- The proofs use (fairly intricate) deletion/contraction arguments.
- **Question.** Is there a good combinatorial description of these Morse matchings?

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