18.747: Infinite-dimensional Lie algebras (Spring term 2012 at MIT)

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Contents

0.1. Version notes

Only Chapters 1 and 2 of these notes are currently anywhere near completion. Chapter 3 is done in parts, but some material is still sketchy and/or wrong. The beginning of Chapter 4 is done, but the rest is still an unusable mess.

These notes are mostly based on what is being said and written on the blackboard in the lectures, and less so on Pavel Etingof's handwritten notes posted on `http://www-math.mit.edu/~etingof/` . They cover less material than Etingof's handwritten notes, but are more detailed in what they do cover.

Thanks to Pavel Etingof for his patience in explaining me things until I actually understand them. Thanks to Dorin Boger for finding mistakes.

0.2. Remark on the level of detail

This is the "brief" version of the lecture notes, meaning that there is a more detailed one, which can be obtained by replacing
`\excludecomment{verlong}`
`\includecomment{vershort}`
by
`\includecomment{verlong}`
`\excludecomment{vershort}`
in the preamble of the LaTeX sourcecode and then compiling to PDF. That detailed version, however, is not recommended, since it differs from the brief one mostly in boring computations and straightforward arguments being carried out rather than sketched. The amount of detail in the brief version is usually enough for understanding (unless it is a part of the lecture I didn't understand myself and just copied from the blackboard; but in that case the detailed version is of no help either). There is currently a large number of proofs which are only sketched in either version.

0.3. Introduction

These notes follow a one-semester graduate class by Pavel Etingof at MIT in the Spring term of 2012. The class was also accompanied by the lecturer's handwritten notes, downloadable from `http://www-math.mit.edu/~etingof/` .

The goal of these lectures is to discuss the structure and the representation theory (mainly the latter) of some of the most important infinite-dimensional Lie algebras.[1] Occasionally, we are also going to show some connections of this subject to other fields of mathematics (such as conformal field theory and the theory of integrable systems).

The prerequisites for reading these notes vary from section to section. We are going to liberally use linear algebra, the basics of algebra (rings, fields, formal power series, categories, tensor products, tensor algebras, symmetric algebras, exterior algebras, etc.) and fundamental notions of Lie algebra theory. At certain points we will also use some results from the representation theory of finite-dimensional Lie algebras, as well as some properties of symmetric polynomials (Schur polynomials in particular) and representations of associative algebras. Analysis and geometry will appear very rarely, and mostly to provide intuition or alternative proofs.

The biggest difference between the theory of finite-dimensional Lie algebras and that of infinite-dimensional ones is that in the finite-dimensional case, we have a complete picture (we can classify simple Lie algebras and their finite-dimensional representations, etc.), whereas most existing results for the infinite-dimensional case are case studies. For example, there are lots and lots of simple infinite-dimensional Lie algebras and we have no real hope to classify them; what we can do is study some very specific classes and families. As far as their representations are concerned, the amount of general results is also rather scarce, and one mostly studies concrete families[2].

The main classes of Lie algebras that we will study in this course are:

1. The Heisenberg algebra (aka oscillator algebra) $\mathcal{A}$ and its Lie subalgebra $\mathcal{A}_0$.

2. The Virasoro algebra Vir.

3. The Lie algebra $\mathfrak{gl}_\infty$ and some variations on it ($\overline{\mathfrak{a}_\infty}$, $\mathfrak{a}_\infty$, $\mathfrak{u}_\infty$).

4. Kac-Moody algebras (this class contains semisimple Lie algebras and also affine Lie algebras, which are central extensions of $\mathfrak{g}\left[t, t^{-1}\right]$ where $\mathfrak{g}$ is simple finite-dimensional).

0.4. References

The standard text on infinite-dimensional Lie algebras (although we will not really follow it) is:

- V. G. Kac, A. K. Raina, *(Bombay Lectures on) Highest Weight Representations of Infinite Dimensional Lie Algebras*, World Scientific 1987.

Further recommended sources are:

[1]It should be noticed that most of the infinite-dimensional Lie algebras studied in these notes are $\mathbb{Z}$-graded and have both their positive and their negative parts infinite-dimensional. This is in contrast to many Lie algebras appearing in algebraic combinatorics (such as free Lie algebras over non-graded vector spaces, and the Lie algebras of primitive elements of many combinatorial Hopf algebras), which tend to be concentrated in nonnegative degrees. So a better title for these notes might have been "Two-sided infinite-dimensional Lie algebras".

[2]Though, to be honest, we are mostly talking about infinite-dimensional representations here, and these are not very easy to handle even for finite-dimensional Lie algebras.

- Victor G. Kac, *Infinite dimensional Lie algebras*, Third Edition, CUP 1995.

- B. L. Feigin, A. Zelevinsky, *Representations of contragredient Lie algebras and the Kac-Macdonald identities*, a paper in: Representations of Lie groups and Lie algebras (Budapest, 1971), pp. 25-77, Akad. Kiadó, Budapest, 1985.

0.5. General conventions

We will almost always work over $\mathbb{C}$ in this course. All algebras are over $\mathbb{C}$ unless specified otherwise. Characteristic p is too complicated for us, although very interesting. Sometimes we will work over $\mathbb{R}$, and occasionally even over rings (as auxiliary constructions require this).

Some remarks on notation:

- In the following, $\mathbb{N}$ will always denote the set $\{0, 1, 2, ...\}$ (and not $\{1, 2, 3, ...\}$).

- All rings are required to have a unity (but not necessarily be commutative). If R is a ring, then all R-algebras are required to have a unity and satisfy $(\lambda a) b = a (\lambda b) = \lambda (ab)$ for all $\lambda \in R$ and all a and b in the algebra. (Some people call such R-algebras *central R-algebras*, but for us this is part of the notion of an R-algebra.)

- When a Lie algebra $\mathfrak{g}$ acts on a vector space M, we will denote the image of an element $m \in M$ under the action of an element $a \in \mathfrak{g}$ by any of the three notations am, $a \cdot m$ and $a \rightharpoonup m$. (One day, I will probably come to an agreement with myself and decide which of these notations to use, but for now expect to see all of them used synonymously in this text. Some authors also use the notation $a \circ m$ for the image of m under the action of a, but we won't use this notation.)

- If V is a vector space, then the tensor algebra of V will be denoted by $T(V)$; the symmetric algebra of V will be denoted by $S(V)$; the exterior algebra of V will be denoted by $\wedge V$.

- For every $n \in \mathbb{N}$, we let S_n denote the n-th symmetric group (that is, the group of all permutations of the set $\{1, 2, \ldots, n\}$). On occasion, the notation S_n will denote some other things as well; we hope that context will suffice to keep these meanings apart.

1. The main examples

1.1. The Heisenberg algebra

We start with the definition of the Heisenberg algebra. Before we formulate it, let us introduce polynomial differential forms on $\mathbb{C}^\times$ (in the algebraic sense):

Definition 1.1.1. Recall that $\mathbb{C}[t, t^{-1}]$ denotes the $\mathbb{C}$-algebra of Laurent polynomials in the variable t over $\mathbb{C}$.

Consider the free $\mathbb{C}[t, t^{-1}]$-module on the basis (dt) (where dt is just a symbol). The elements of this module are called *polynomial differential forms on* $\mathbb{C}^\times$. Thus, polynomial differential forms on $\mathbb{C}^\times$ are just formal expressions of the form fdt where $f \in \mathbb{C}[t, t^{-1}]$.

Whenever $g \in \mathbb{C}[t, t^{-1}]$ is a Laurent polynomial, we define a polynomial differential form dg by $dg = g'dt$. This notation dg does not conflict with the previously defined notation dt (which was a symbol), because the polynomial t satisfies $t' = 1$.

Definition 1.1.2. For every polynomial differential form fdt on $\mathbb{C}^\times$ (with $f \in \mathbb{C}[t, t^{-1}]$), we define a complex number $\operatorname{Res}_{t=0}(fdt)$ to be the coefficient of the Laurent polynomial f before t^{-1}. In other words, we define $\operatorname{Res}_{t=0}(fdt)$ to be a_{-1}, where f is written as $\sum\limits_{i \in \mathbb{Z}} a_i t^i$ (with $a_i \in \mathbb{C}$ for all $i \in \mathbb{Z}$).

This number $\operatorname{Res}_{t=0}(fdt)$ is called the *residue* of the form fdt at 0.

(The same definition could have been done for Laurent series instead of Laurent polynomials, but this would require us to consider a slightly different notion of differential forms, and we do not want to do this here.)

Remark 1.1.3. **(a)** Every Laurent polynomial $f \in \mathbb{C}[t, t^{-1}]$ satisfies $\operatorname{Res}_{t=0}(df) = 0$.

(b) Every Laurent polynomial $f \in \mathbb{C}[t, t^{-1}]$ satisfies $\operatorname{Res}_{t=0}(fdf) = 0$.

Proof of Remark 1.1.3. **(a)** Write f in the form $\sum\limits_{i \in \mathbb{Z}} b_i t^i$ (with $b_i \in \mathbb{C}$ for all $i \in \mathbb{Z}$). Then, $f' = \sum\limits_{i \in \mathbb{Z}} i b_i t^{i-1} = \sum\limits_{i \in \mathbb{Z}} (i+1) b_{i+1} t^i$. Now, $df = f'dt$, so that

$$\begin{aligned} \operatorname{Res}_{t=0}(df) &= \operatorname{Res}_{t=0}(f'dt) = \left(\text{the coefficient of the Laurent polynomial } f' \text{ before } t^{-1}\right) \\ &= \underbrace{(-1+1)}_{=0} b_{-1+1} \qquad \left(\text{since } f' = \sum_{i \in \mathbb{Z}} (i+1) b_{i+1} t^i\right) \\ &= 0, \end{aligned}$$

proving Remark 1.1.3 **(a)**.

(b) *First proof of Remark 1.1.3 **(b)**:* By the Leibniz identity, $(f^2)' = ff' + f'f = 2ff'$, so that $ff' = \frac{1}{2}(f^2)'$ and thus $f \underbrace{df}_{=f'dt} = \underbrace{ff'}_{=\frac{1}{2}(f^2)'} dt = \frac{1}{2} \underbrace{(f^2)' dt}_{=d(f^2)} = \frac{1}{2} d(f^2)$. Thus,

$$\operatorname{Res}_{t=0}(fdf) = \operatorname{Res}_{t=0}\left(\frac{1}{2} d\left(f^2\right)\right) = \frac{1}{2} \underbrace{\operatorname{Res}_{t=0}\left(d\left(f^2\right)\right)}_{\substack{=0 \text{ (by Remark 1.1.3 (a),} \\ \text{applied to } f^2 \text{ instead of } f)}} = 0,$$

and Remark 1.1.3 **(b)** is proven.

*Second proof of Remark 1.1.3 **(b)**:* Write f in the form $\sum\limits_{i \in \mathbb{Z}} b_i t^i$ (with $b_i \in \mathbb{C}$ for all $i \in \mathbb{Z}$). Then, $f' = \sum\limits_{i \in \mathbb{Z}} i b_i t^{i-1} = \sum\limits_{i \in \mathbb{Z}} (i+1) b_{i+1} t^i$. Now,

$$ff' = \left(\sum_{i \in \mathbb{Z}} b_i t^i\right) \left(\sum_{i \in \mathbb{Z}} (i+1) b_{i+1} t^i\right) = \sum_{n \in \mathbb{Z}} \left(\sum_{\substack{(i,j) \in \mathbb{Z}^2; \\ i+j=n}} b_i \cdot (j+1) b_{j+1}\right) t^n$$

(by the definition of the product of Laurent polynomials). Also, $df = f'dt$, so that

$$\operatorname{Res}_{t=0}(fdf) = \operatorname{Res}_{t=0}(ff'dt) = \left(\text{the coefficient of the Laurent polynomial } ff' \text{ before } t^{-1}\right)$$

$$= \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ i+j=-1}} b_i \cdot (j+1)\, b_{j+1} \qquad \left(\text{since } ff' = \sum_{n\in\mathbb{Z}} \left(\sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ i+j=n}} b_i \cdot (j+1)\, b_{j+1} \right) t^n \right)$$

$$= \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ i+j=0}} b_i \cdot j b_j \qquad \left(\text{here, we substituted } (i,j) \text{ for } (i,j+1) \text{ in the sum}\right)$$

$$= \sum_{j\in\mathbb{Z}} b_{-j} \cdot j b_j = \underbrace{\sum_{\substack{j\in\mathbb{Z};\\ j<0}} b_{-j} \cdot j b_j}_{\substack{= \sum_{\substack{j\in\mathbb{Z};\\ j>0}} b_{-(-j)}\cdot(-j) b_{-j} \\ \text{(here, we substituted } j \text{ for } -j \text{ in the sum)}}} + \underbrace{b_{-0} \cdot 0 b_0}_{=0} + \sum_{\substack{j\in\mathbb{Z};\\ j>0}} b_{-j} \cdot j b_j$$

$$= \sum_{\substack{j\in\mathbb{Z};\\ j>0}} \underbrace{b_{-(-j)} \cdot (-j)\, b_{-j}}_{=b_j(-j)b_{-j} = -b_{-j}\cdot j b_j} + \sum_{\substack{j\in\mathbb{Z};\\ j>0}} b_{-j} \cdot j b_j = \sum_{\substack{j\in\mathbb{Z};\\ j>0}} \left(-b_{-j} \cdot j b_j\right) + \sum_{\substack{j\in\mathbb{Z};\\ j>0}} b_{-j} \cdot j b_j = 0.$$

This proves Remark 1.1.3 **(b)**.

Note that the first proof of Remark 1.1.3 **(b)** made use of the fact that 2 is invertible in $\mathbb{C}$, whereas the second proof works over any commutative ring instead of $\mathbb{C}$.

Now, finally, we define the Heisenberg algebra:

Definition 1.1.4. The *oscillator algebra* $\mathcal{A}$ is the vector space $\mathbb{C}[t, t^{-1}] \oplus \mathbb{C}$ endowed with the Lie bracket

$$[(f,\alpha), (g,\beta)] = (0, \operatorname{Res}_{t=0}(gdf)).$$

Since this Lie bracket satisfies the Jacobi identity (because the definition quickly yields that $[[x,y],z] = 0$ for all $x,y,z \in \mathcal{A}$) and is skew-symmetric (due to Remark 1.1.3 **(b)**), this $\mathcal{A}$ is a Lie algebra.

This oscillator algebra $\mathcal{A}$ is also known as the *Heisenberg algebra*.

Thus, $\mathcal{A}$ has a basis

$$\{a_n \mid n \in \mathbb{Z}\} \cup \{K\},$$

where $a_n = (t^n, 0)$ and $K = (0,1)$. The bracket is given by

$$[a_n, K] = 0 \qquad (\text{thus, } K \text{ is central});$$
$$[a_n, a_m] = n\delta_{n,-m} K$$

(in fact, $[a_n, a_{-n}] = \operatorname{Res}_{t=0}(t^{-n} dt^n) K = \operatorname{Res}_{t=0}(nt^{-1}dt) K = nK$). Thus, $\mathcal{A}$ is a 1-dimensional central extension of the abelian Lie algebra $\mathbb{C}[t,t^{-1}]$; this means that we have a short exact sequence

$$0 \longrightarrow \mathbb{C}K \longrightarrow \mathcal{A} \longrightarrow \mathbb{C}[t,t^{-1}] \longrightarrow 0,$$

where $\mathbb{C}K$ is contained in the center of $\mathcal{A}$ and where $\mathbb{C}[t,t^{-1}]$ is an abelian Lie algebra.

Note that $\mathcal{A}$ is a 2-nilpotent Lie algebra. Also note that the center of $\mathcal{A}$ is spanned by a_0 and K.

1.2. The Witt algebra

The next introductory example will be the Lie algebra of vector fields:

Definition 1.2.1. Consider the free $\mathbb{C}[t, t^{-1}]$-module on the basis (∂) (where ∂ is just a symbol). This module, regarded as a $\mathbb{C}$-vector space, will be denoted by W. Thus, the elements of W are formal expressions of the form $f\partial$ where $f \in \mathbb{C}[t, t^{-1}]$. (Thus, $W \cong \mathbb{C}[t, t^{-1}]$.)

Define a Lie bracket on the $\mathbb{C}$-vector space W by

$$[f\partial, g\partial] = (fg' - gf')\,\partial \qquad \text{for all } f \in \mathbb{C}\left[t, t^{-1}\right] \text{ and } g \in \mathbb{C}\left[t, t^{-1}\right].$$

This Lie bracket is easily seen to be skew-symmetric and satisfy the Jacobi identity. Thus, it makes W into a Lie algebra. This Lie algebra is called the *Witt algebra*.

The elements of W are called *polynomial vector fields on* $\mathbb{C}^\times$.

The symbol ∂ is often denoted by $\dfrac{d}{dt}$.

Remark 1.2.2. It is not by chance that ∂ is also known as $\dfrac{d}{dt}$. In fact, this notation allows us to view the elements of W as actual polynomial vector fields on $\mathbb{C}^\times$ in the sense of algebraic geometry over $\mathbb{C}$. The Lie bracket of the Witt algebra W is then exactly the usual Lie bracket of vector fields (because if $f \in \mathbb{C}[t, t^{-1}]$ and $g \in \mathbb{C}[t, t^{-1}]$ are two Laurent polynomials, then a simple application of the Leibniz rule shows that the commutator of the differential operators $f\dfrac{d}{dt}$ and $g\dfrac{d}{dt}$ is indeed the differential operator $(fg' - gf')\dfrac{d}{dt}$).

A basis of the Witt algebra W is $\{L_n \mid n \in \mathbb{Z}\}$, where L_n means $-t^{n+1}\dfrac{d}{dt} = -t^{n+1}\partial$. (Note that some other references like to define L_n as $t^{n+1}\partial$ instead, thus getting a different sign in many formulas.) It is easy to see that the Lie bracket of the Witt algebra is given on this basis by

$$[L_n, L_m] = (n - m)\, L_{n+m} \qquad \text{for every } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$

1.3. A digression: Lie groups (and the absence thereof)

Let us make some remarks about the relationship between Lie algebras and Lie groups. In analysis and geometry, linearizations (tangent spaces etc.) usually only give a crude approximation of non-linear things (manifolds etc.). This is what makes the theory of Lie groups special: The linearization of a finite-dimensional Lie group (i. e., its corresponding Lie algebra) carries very much information about the Lie group. The relation between finite-dimensional Lie groups and finite-dimensional Lie algebras is almost a one-to-one correspondence (at least if we restrict ourselves to simply connected Lie groups). This correspondence breaks down in the infinite-dimensional case. There are lots of important infinite-dimensional Lie groups, but their relation to Lie algebras is not as close as in the finite-dimensional case anymore. One example for this is that there is no Lie group corresponding to the Witt algebra W. There are a few things that come close to such a Lie group:

We can consider the real subalgebra $W_{\mathbb{R}}$ of W, consisting of the vector fields in W which are tangent to S^1 (the unit circle in $\mathbb{C}$). This is a real Lie algebra satisfying $W_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} \cong W$ (thus, $W_{\mathbb{R}}$ is what is called a *real form* of W). And we can say that $\widehat{W_{\mathbb{R}}} = \operatorname{Lie}(\operatorname{Diff} S^1)$ (where $\operatorname{Diff} S^1$ denotes the group of all diffeomorphisms $S^1 \to S^1$) for some kind of completion $\widehat{W_{\mathbb{R}}}$ of $W_{\mathbb{R}}$ (although $W_{\mathbb{R}}$ itself is not the Lie algebra of any Lie group).[3] Now if we take two one-parameter families

$$\begin{array}{lll} g_s \in \operatorname{Diff} S^1, & g_s \mid_{s=0} = \mathrm{id}, & g_s' \mid_{s=0} = \varphi; \\ h_u \in \operatorname{Diff} S^1, & h_u \mid_{u=0} = \mathrm{id}, & h_u' \mid_{u=0} = \psi, \end{array}$$

then

$$\begin{aligned} g_s(\theta) &= \theta + s\varphi(\theta) + O\left(s^2\right); \\ h_u(\theta) &= \theta + u\psi(\theta) + O\left(u^2\right); \\ \left(g_s \circ h_u \circ g_s^{-1} \circ h_u^{-1}\right)(\theta) &= \theta + su\left(\varphi\psi' - \psi\varphi'\right)(\theta) + (\text{cubic terms in } s \text{ and } u \text{ and higher}). \end{aligned}$$

So we get something resembling the standard Lie-group-Lie-algebra correspondence, but only for the completion of the real part. For the complex one, some people have done some work yielding something like Lie semigroups (the so-called “semigroup of annuli” of G. Segal), but no Lie groups.

Anyway, this was a digression, just to show that we don’t have Lie groups corresponding to our Lie algebras. Still, this should not keep us from heuristically thinking of Lie algebras as linearizations of Lie groups. We can even formalize this heuristic, by using the purely algebraic notion of formal groups.

1.4. The Witt algebra acts on the Heisenberg algebra by derivations

Let’s return to topic. The following proposition is a variation on a well-known theme:

Proposition 1.4.1. Let $\mathfrak{n}$ be a Lie algebra. Let $f : \mathfrak{n} \to \mathfrak{n}$ and $g : \mathfrak{n} \to \mathfrak{n}$ be two derivations of $\mathfrak{n}$. Then, $[f, g]$ is a derivation of $\mathfrak{n}$. (Here, the Lie bracket is to be understood as the Lie bracket on $\operatorname{End}\mathfrak{n}$, so that we have $[f, g] = f \circ g - g \circ f$.)

[3] Here is how this completion $\widehat{W_{\mathbb{R}}}$ is defined exactly: Notice that

$$W_{\mathbb{R}} = \left\{ \varphi(\theta) \frac{d}{d\theta} \mid \begin{array}{c} \varphi \text{ is a trigonometric polynomial, i. e.,} \\ \varphi(\theta) = a_0 + \sum\limits_{n>0} a_n \cos n\theta + \sum\limits_{n>0} b_n \sin n\theta \\ \text{where both sums are finite} \end{array} \right\},$$

where $\theta = \frac{1}{i} \ln t$ and $\frac{d}{d\theta} = it\frac{d}{dt}$. Now, define the completion $\widehat{W_{\mathbb{R}}}$ by

$$\widehat{W_{\mathbb{R}}} = \left\{ \varphi(\theta) \frac{d}{d\theta} \mid \begin{array}{c} \varphi(\theta) = a_0 + \sum\limits_{n>0} a_n \cos n\theta + \sum\limits_{n>0} b_n \sin n\theta \\ \text{where both sums are infinite sums with rapidly} \\ \text{decreasing coefficients} \end{array} \right\}.$$

Definition 1.4.2. For every Lie algebra $\mathfrak{g}$, we will denote by $\operatorname{Der}\mathfrak{g}$ the Lie subalgebra $\{f \in \operatorname{End}\mathfrak{g} \mid f \text{ is a derivation}\}$ of $\operatorname{End}\mathfrak{g}$. (This is well-defined because Proposition 1.4.1 shows that $\{f \in \operatorname{End}\mathfrak{g} \mid f \text{ is a derivation}\}$ is a Lie subalgebra of $\operatorname{End}\mathfrak{g}$.) We call $\operatorname{Der}\mathfrak{g}$ the *Lie algebra of derivations* of $\mathfrak{g}$.

Lemma 1.4.3. There is a natural homomorphism $\eta : W \to \operatorname{Der}\mathcal{A}$ of Lie algebras given by

$$(\eta(f\partial))(g,\alpha) = (fg', 0) \qquad \text{for all } f \in \mathbb{C}\left[t,t^{-1}\right],\ g \in \mathbb{C}\left[t,t^{-1}\right] \text{ and } \alpha \in \mathbb{C}.$$

First proof of Lemma 1.4.3. Lemma 1.4.3 can be proven by direct calculation:
For every $f\partial \in W$, the map

$$\mathcal{A} \to \mathcal{A}, \qquad (g,\alpha) \mapsto (fg', 0)$$

is a derivation of $\mathcal{A}$ [4], thus lies in $\operatorname{Der}\mathcal{A}$. Hence, we can define a map $\eta : W \to \operatorname{Der}\mathcal{A}$ by

$$\eta(f\partial) = (\mathcal{A} \to \mathcal{A}, \qquad (g,\alpha) \mapsto (fg', 0)) \qquad \text{for all } f \in \mathbb{C}\left[t,t^{-1}\right].$$

In other words, we can define a map $\eta : W \to \operatorname{Der}\mathcal{A}$ by

$$(\eta(f\partial))(g,\alpha) = (fg', 0) \qquad \text{for all } f \in \mathbb{C}\left[t,t^{-1}\right],\ g \in \mathbb{C}\left[t,t^{-1}\right] \text{ and } \alpha \in \mathbb{C}.$$

[4] *Proof.* Let $f\partial$ be an element of W. (In other words, let f be an element of $\mathbb{C}\left[t,t^{-1}\right]$.) Let τ denote the map

$$\mathcal{A} \to \mathcal{A}, \qquad (g,\alpha) \mapsto (fg', 0).$$

Then, we must prove that τ is a derivation of $\mathcal{A}$.

In fact, first it is clear that τ is $\mathbb{C}$-linear. Moreover, any $(u,\beta) \in \mathcal{A}$ and $(v,\gamma) \in \mathcal{A}$ satisfy

$$\begin{aligned}\tau\left(\underbrace{[(u,\beta),(v,\gamma)]}_{=(0,\operatorname{Res}_{t=0}(vdu))}\right) &= \tau(0, \operatorname{Res}_{t=0}(vdu)) = (f0, 0) \qquad \text{(by the definition of } \tau\text{)} \\ &= (0,0)\end{aligned}$$

and

$$\begin{aligned}&\left[\underbrace{\tau(u,\beta)}_{=(fu',0)}, (v,\gamma)\right] + \left[(u,\beta), \underbrace{\tau(v,\gamma)}_{=(fv',0)}\right] \\ &= \underbrace{[(fu',0),(v,\gamma)]}_{=(0,\operatorname{Res}_{t=0}(vd(fu')))} + \underbrace{[(u,\beta),(fv',0)]}_{=(0,\operatorname{Res}_{t=0}(fv'du))} \\ &= (0, \operatorname{Res}_{t=0}(vd(fu'))) + (0, \operatorname{Res}_{t=0}(fv'du)) \\ &= (0, \operatorname{Res}_{t=0}(vd(fu') + fv'du)) = (0, \operatorname{Res}_{t=0}(d(vfu'))) \\ &\quad \left(\begin{array}{c}\text{since } v \underbrace{d(fu')}_{=(fu')'dt} + fv' \underbrace{du}_{=u'dt} = v(fu')'\,dt + fv'u'dt \\ = \left(v(fu')' + fv'u'\right)dt = \underbrace{\left(v(fu')' + v'(fu')\right)}_{=(vfu')'}dt = (vfu')'\,dt = d(vfu')\end{array}\right) \\ &= (0,0) \qquad \text{(since Remark 1.1.3 **(a)** (applied to } vfu' \text{ instead of } f\text{) yields } \operatorname{Res}_{t=0}(d(vfu')) = 0),\end{aligned}$$

so that $\tau([(u,\beta),(v,\gamma)]) = [\tau(u,\beta),(v,\gamma)] + [(u,\beta),\tau(v,\gamma)]$. Thus, τ is a derivation of $\mathcal{A}$, qed.

Now, it remains to show that this map η is a homomorphism of Lie algebras.

In fact, any $f_1 \in \mathbb{C}[t,t^{-1}]$ and $f_2 \in \mathbb{C}[t,t^{-1}]$ and any $g \in \mathbb{C}[t,t^{-1}]$ and $\alpha \in \mathbb{C}$ satisfy

$$\left(\eta\left(\underbrace{[f_1\partial, f_2\partial]}_{=(f_1f_2'-f_2f_1')\partial}\right)\right)(g,\alpha) = (\eta((f_1f_2'-f_2f_1')\partial))(g,\alpha) = ((f_1f_2'-f_2f_1')g', 0)$$

and

$$\begin{aligned}
&[\eta(f_1\partial), \eta(f_2\partial)](g,\alpha) \\
&= (\eta(f_1\partial))\underbrace{((\eta(f_2\partial))(g,\alpha))}_{=(f_2g',0)} - (\eta(f_2\partial))\underbrace{((\eta(f_1\partial))(g,\alpha))}_{=(f_1g',0)} \\
&= \underbrace{(\eta(f_1\partial))(f_2g',0)}_{=(f_1(f_2g')',0)} - \underbrace{(\eta(f_2\partial))(f_1g',0)}_{=(f_2(f_1g')',0)} = \left(f_1(f_2g')',0\right) - \left(f_2(f_1g')',0\right) \\
&= \left(f_1(f_2g')' - f_2(f_1g')',0\right) = ((f_1f_2'-f_2f_1')g',0) \\
&\qquad \left(\begin{array}{c} \text{since } f_1\underbrace{(f_2g')'}_{=f_2'g'+f_2g''} - f_2\underbrace{(f_1g')'}_{=f_1'g'+f_1g''} = f_1(f_2'g'+f_2g'') - f_2(f_1'g'+f_1g'') \\ = f_1f_2'g' + f_1f_2g'' - f_2f_1'g' - f_1f_2g'' = f_1f_2'g' - f_2f_1'g' = (f_1f_2'-f_2f_1')g' \end{array}\right),
\end{aligned}$$

so that

$$(\eta([f_1\partial, f_2\partial]))(g,\alpha) = ((f_1f_2'-f_2f_1')g',0) = [\eta(f_1\partial), \eta(f_2\partial)](g,\alpha).$$

Thus, any $f_1 \in \mathbb{C}[t,t^{-1}]$ and $f_2 \in \mathbb{C}[t,t^{-1}]$ satisfy $\eta([f_1\partial, f_2\partial]) = [\eta(f_1\partial), \eta(f_2\partial)]$. This proves that η is a Lie algebra homomorphism, and thus Lemma 1.4.3 is proven.

Second proof of Lemma 1.4.3 (sketched). The following proof I don't understand, so don't expect my version of it to make any sense. See Akhil Matthew's blog post `http://amathew.wordpress.com/2012/03/01/the-heisenberg-and-witt-algebras/` for a much better writeup.

The following proof is a bit of an overkill; however, it is supposed to provide some motivation for Lemma 1.4.3. We won't be working completely formally, so the reader should expect some imprecision.

Let us really interpret the elements of W as vector fields on $\mathbb{C}^\times$. The bracket $[\cdot,\cdot]$ of the Lie algebra $\mathcal{A}$ was defined in an invariant way:

$$[f,g] = \operatorname{Res}_{t=0}(gdf) = \frac{1}{2\pi i}\oint_{|z|=1} gdf \qquad \text{(by Cauchy's residue theorem)}$$

is an integral of a 1-form, thus invariant under diffeomorphisms, thus invariant under "infinitesimal diffeomorphisms" such as the ones given by elements of W. Thus, Lemma 1.4.3 becomes obvious. [This proof needs revision.]

The first of these two proofs is obviously the more straightforward one (and generalizes better to fields other than $\mathbb{C}$), but it does not offer any explanation why Lemma 1.4.3 is more than a mere coincidence. Meanwhile, the second proof gives Lemma 1.4.3 a philosophical reason to be true.

1.5. The Virasoro algebra

In representation theory, one often doesn't encounter representations of W directly, but instead one finds representations of a 1-dimensional central extension of W called the Virasoro algebra. I will now construct this extension and show that it is the only one (up to isomorphism of extensions).

Let us recollect the theory of central extensions of Lie algebras (more precisely, the 1-dimensional ones):

Definition 1.5.1. If L is a Lie algebra, then a 1-dimensional central extension of L is a Lie algebra $\widehat{L}$ along with an exact sequence

$$0 \to \mathbb{C} \to \widehat{L} \to L \to 0, \tag{1}$$

where $\mathbb{C}$ is central in $\widehat{L}$. Since all exact sequences of vector spaces split, we can pick a splitting of this exact sequence on the level of vector spaces, and thus identify $\widehat{L}$ with $L \oplus \mathbb{C}$ as a vector space (not as a Lie algebra). Upon this identification, the Lie bracket of $\widehat{L}$ can be written as

$$[(a, \alpha), (b, \beta)] = ([a, b], \omega(a, b)) \qquad \text{for } a \in L,\ \alpha \in \mathbb{C},\ b \in L,\ \beta \in \mathbb{C}, \tag{2}$$

for some skew-symmetric bilinear form $\omega : L \times L \to \mathbb{C}$. (We can also write this skew-symmetric bilinear form $\omega : L \times L \to \mathbb{C}$ as a linear form $\wedge^2 L \to \mathbb{C}$.) But ω cannot be a completely arbitrary skew-symmetric bilinear form. It needs to satisfy the so-called 2-*cocycle condition*

$$\omega([a, b], c) + \omega([b, c], a) + \omega([c, a], b) = 0 \qquad \text{for all } a, b, c \in L. \tag{3}$$

This condition comes from the requirement that the bracket in $\widehat{L}$ have to satisfy the Jacobi identity.

In the following, a 2-*cocycle on* L will mean a skew-symmetric bilinear form $\omega : L \times L \to \mathbb{C}$ (not necessarily obtained from a central extension!) which satisfies the equation (3). (The name "2-cocycle" comes from Lie algebra cohomology, where 2-cocycles are indeed the cocycles in the 2-nd degree.) Thus, we have assigned a 2-cocycle on L to every 1-dimensional central extension of L (although the assignment depended on the splitting).

Conversely, if ω is any 2-cocycle on L, then we can define a 1-dimensional central extension $\widehat{L}_\omega$ of L such that the 2-cocycle corresponding to this extension is ω. In fact, we can construct such a central extension $\widehat{L}_\omega$ by setting $\widehat{L}_\omega = L \oplus \mathbb{C}$ as a vector space, and defining the Lie bracket on this vector space by (2). (The maps $\mathbb{C} \to \widehat{L}_\omega$ and $\widehat{L}_\omega \to L$ are the canonical ones coming from the direct sum decomposition $\widehat{L}_\omega = L \oplus \mathbb{C}$.) Thus, every 2-cocycle on L canonically determines a 1-dimensional central extension of L.

However, our assignment of the 2-cocycle ω to the central extension $\widehat{L}$ was not canonical, but depended on the splitting of the exact sequence (1). If we change the splitting by some $\xi \in L^*$, then ω is changed by $d\xi$ (this means that ω is being replaced by $\omega + d\xi$), where $d\xi$ is the 2-cocycle on L defined by

$$d\xi(a, b) = \xi([a, b]) \qquad \text{for all } a, b \in L.$$

The 2-cocycle $d\xi$ is called a 2-*coboundary*. As a conclusion, 1-dimensional central extensions of L are parametrized up to isomorphism by the vector space

$$\text{(2-cocycles)} \diagup \text{(2-coboundaries)} = H^2(L).$$

(Note that "up to isomorphism" means "up to isomorphism of extensions" here, not "up to isomorphism of Lie algebras".) The vector space $H^2(L)$ is called the 2-*nd cohomology space* (or just the 2-nd cohomology) of the Lie algebra L.

Theorem 1.5.2. The vector space $H^2(W)$ is 1-dimensional and is spanned by the residue class of the 2-cocycle ω given by

$$\omega(L_n, L_m) = \frac{n^3 - n}{6}\delta_{n,-m} \qquad \text{for all } n, m \in \mathbb{Z}.$$

Note that in this theorem, we could have replaced the factor $\frac{n^3-n}{6}$ by $n^3 - n$ (since the vector space spanned by a vector obviously doesn't change if we rescale the vector by a nonzero scalar factor), or even by n^3 (since the 2-cocycle $(L_n, L_m) \mapsto n\delta_{n,-m}$ is a coboundary, and two 2-cocycles which differ by a coboundary give the same residue class in $H^2(W)$). But we prefer $\frac{n^3-n}{6}$ since this is closer to how this class appears in representation theory (and, also, comes up in the proof below).

Proof of Theorem 1.5.2. First of all, it is easy to prove by computation that the bilinear form $\omega : W \times W \to \mathbb{C}$ given by

$$\omega(L_n, L_m) = \frac{n^3 - n}{6}\delta_{n,-m} \qquad \text{for all } n, m \in \mathbb{Z}$$

is indeed a 2-cocycle. Now, let us prove that every 2-cocycle on W is congruent to a multiple of ω modulo the 2-coboundaries.

Let β be a 2-cocycle on W. We must prove that β is congruent to a multiple of ω modulo the 2-coboundaries.

Pick $\xi \in W^*$ such that $\xi(L_n) = \frac{1}{n}\beta(L_n, L_0)$ for all $n \neq 0$ (such a ξ clearly exists, but is not unique since we have complete freedom in choosing $\xi(L_0)$). Let $\widetilde{\beta}$ be the 2-cocycle $\beta - d\xi$. Then,

$$\widetilde{\beta}(L_n, L_0) = \underbrace{\beta(L_n, L_0)}_{\substack{=n\xi(L_n) \\ \left(\text{since } \xi(L_n) = \frac{1}{n}\beta(L_n,L_0)\right)}} - \xi\left(\underbrace{[L_n, L_0]}_{=nL_n}\right) = n\xi(L_n) - \xi(nL_n) = 0$$

for every $n \neq 0$. Thus, by replacing β by $\widetilde{\beta}$, we can WLOG assume that $\beta(L_n, L_0) = 0$ for every $n \neq 0$. This clearly also holds for $n = 0$ since β is skew-symmetric. Hence, $\beta(X, L_0) = 0$ for every $X \in W$. Now, by the 2-cocycle condition, we have

$$\beta([L_0, L_m], L_n) + \beta([L_n, L_0], L_m) + \beta([L_m, L_n], L_0) = 0$$

for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. Thus,

$$\begin{aligned}
0 &= \beta\left(\underbrace{[L_0, L_m]}_{=-mL_m}, L_n\right) + \beta\left(\underbrace{[L_n, L_0]}_{=nL_n}, L_m\right) + \underbrace{\beta\left(\left[L_m, L_n\right], L_0\right)}_{=0 \text{ (since } \beta(X,L_0)=0 \text{ for every } X\in W)} \\
&= -m \underbrace{\beta\left(L_m, L_n\right)}_{\substack{=-\beta(L_n,L_m) \\ \text{(since } \beta \text{ is skew-symmetric)}}} + n\beta\left(L_n, L_m\right) = m\beta\left(L_n, L_m\right) + n\beta\left(L_n, L_m\right) \\
&= (n+m)\,\beta\left(L_n, L_m\right)
\end{aligned}$$

for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. Hence, for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ with $n + m \neq 0$, we have $\beta\left(L_n, L_m\right) = 0$. In other words, there exists some sequence $\left(b_n\right)_{n\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}}$ such that

$$\beta\left(L_n, L_m\right) = b_n \delta_{n,-m} \qquad \text{for all } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}. \tag{4}$$

This sequence satisfies

$$b_{-n} = -b_n \qquad \text{for every } n \in \mathbb{Z} \tag{5}$$

(since β is skew-symmetric and thus $\beta\left(L_n, L_{-n}\right) = -\beta\left(L_{-n}, L_n\right)$) and thus, in particular, $b_0 = 0$. We will now try to get a recursive equation for this sequence.

Let m, n and p be three integers satisfying $m + n + p = 0$. Then, the 2-cocycle condition yields

$$\beta\left(\left[L_p, L_n\right], L_m\right) + \beta\left(\left[L_m, L_p\right], L_n\right) + \beta\left(\left[L_n, L_m\right], L_p\right) = 0.$$

Due to

$$\begin{aligned}
\beta\left(\underbrace{\left[L_p, L_n\right]}_{=(p-n)L_{p+n}}, L_m\right) &= (p-n) \underbrace{\beta\left(L_{p+n}, L_m\right)}_{\substack{=-\beta(L_m,L_{p+n}) \\ \text{(since } \beta \text{ is skew-symmetric)}}} = -(p-n) \underbrace{\beta\left(L_m, L_{p+n}\right)}_{\substack{=b_m\delta_{m,-(p+n)} \\ \text{(by (4))}}} \\
&= -(p-n)\, b_m \underbrace{\delta_{m,-(p+n)}}_{\substack{=1 \\ \text{(since } m+n+p=0)}} = -(p-n)\, b_m
\end{aligned}$$

and the two cyclic permutations of this equality, this rewrites as

$$\left(-(p-n)\, b_m\right) + \left(-(m-p)\, b_n\right) + \left(-(n-m)\, b_p\right) = 0.$$

In other words,

$$(n-m)\, b_p + (m-p)\, b_n + (p-n)\, b_m = 0. \tag{6}$$

Now define a form $\xi_0 \in W^*$ by $\xi_0\left(L_0\right) = 1$ and $\xi_0\left(L_i\right) = 0$ for all $i \neq 0$.

By replacing β with $\beta - \dfrac{b_1}{2} d\xi_0$, we can assume WLOG that $b_1 = 0$.

Now let $n \in \mathbb{Z}$ be arbitrary. Setting $m = 1$ and $p = -(n+1)$ in (6) (this is allowed since $1 + n + (-(n+1)) = 0$), we get

$$(n-1)\, b_{-(n+1)} + (1 - (-(n+1)))\, b_n + (n-1)\, b_1 = 0.$$

Thus,

$$\begin{aligned}0 &= (n-1) \underbrace{b_{-(n+1)}}_{=-b_{n+1} \text{ (by (5))}} + \underbrace{(1-(-(n+1)))}_{=n+2} b_n + (n-1) \underbrace{b_1}_{=0} \\ &= -(n-1) b_{n+1} + (n+2) b_n,\end{aligned}$$

so that $(n-1) b_{n+1} = (n+2) b_n$. This recurrence equation rewrites as $b_{n+1} = \dfrac{n+2}{n-1} b_n$ for $n \geq 2$. Thus, by induction we see that every $n \geq 2$ satisfies

$$b_n = \frac{n+1}{n-2} \cdot \frac{n}{n-3} \cdot \frac{n-1}{n-4} \cdot \ldots \cdot \frac{4}{1} b_2 = \frac{(n+1) \cdot n \cdot \ldots \cdot 4}{(n-2) \cdot (n-3) \cdot \ldots \cdot 1} b_2 = \frac{(n+1)(n-1) n}{6} b_2 = \frac{n^3 - n}{6} b_2.$$

But $b_n = \dfrac{n^3-n}{6} b_2$ also holds for $n = 1$ (since $b_1 = 0$ and $\dfrac{1^3-1}{6} = 0$) and for $n = 0$ (since $b_0 = 0$ and $\dfrac{0^3-0}{6} = 0$). Hence, $b_n = \dfrac{n^3-n}{6} b_2$ holds for every $n \geq 0$. By (5), we conclude that $b_n = \dfrac{n^3-n}{6} b_2$ holds also for every $n \leq 0$. Thus, every $n \in \mathbb{Z}$ satisfies $b_n = \dfrac{n^3-n}{6} b_2$. From (4), we thus see that β is a scalar multiple of ω.

We thus have proven that every 2-cocycle β on W is congruent to a multiple of ω modulo the 2-coboundaries. This yields that the space $H^2(W)$ is *at most* 1-dimensional and is spanned by the residue class of the 2-cocycle ω. In order to complete the proof of Theorem 1.5.2, we have yet to prove that $H^2(W)$ is indeed 1-dimensional (and not 0-dimensional), i. e., that the 2-cocycle ω is *not* a 2-coboundary. But this is easy[5]. The proof of Theorem 1.5.2 is thus complete.

The 2-cocycle $\dfrac{1}{2}\omega$ (where ω is the 2-cocycle introduced in Theorem 1.5.2) gives a central extension of the Witt algebra W: the so-called Virasoro algebra. Let us recast the definition of this algebra in elementary terms:

[5] *Proof.* Assume the contrary. Then, the 2-cocycle ω is a 2-coboundary. This means that there exists a linear map $\eta : W \to \mathbb{C}$ such that $\omega = d\eta$. Pick such a η. Then,

$$\omega(L_2, L_{-2}) = (d\eta)(L_2, L_{-2}) = \eta\left(\underbrace{[L_2, L_{-2}]}_{=4L_0}\right) = 4\eta(L_0)$$

and

$$\omega(L_1, L_{-1}) = (d\eta)(L_1, L_{-1}) = \eta\left(\underbrace{[L_1, L_{-1}]}_{=2L_0}\right) = 2\eta(L_0).$$

Hence,

$$2\underbrace{\omega(L_1, L_{-1})}_{=2\eta(L_0)} = 4\eta(L_0) = \omega(L_2, L_{-2}).$$

But this contradicts with the equalities $\omega(L_1, L_{-1}) = 0$ and $\omega(L_2, L_{-2}) = 1$ (which easily follow from the definition of ω). This contradiction shows that our assumption was wrong, and thus the 2-cocycle ω is not a 2-coboundary, qed.

Definition 1.5.3. The *Virasoro algebra* Vir is defined as the vector space $W \oplus \mathbb{C}$ with Lie bracket defined by

$$\begin{aligned} [L_n, L_m] &= (n-m)\, L_{n+m} + \frac{n^3 - n}{12} \delta_{n,-m} C; \\ [L_n, C] &= 0, \end{aligned}$$

where L_n denotes $(L_n, 0)$ for every $n \in \mathbb{Z}$, and where C denotes $(0,1)$. Note that $\{L_n \mid n \in \mathbb{Z}\} \cup \{C\}$ is a basis of Vir.

If we change the denominator 12 to any other nonzero complex number, we get a Lie algebra isomorphic to Vir (it is just a rescaling of C). It is easy to show that the Virasoro algebra is not isomorphic to the Lie-algebraic direct sum $W \oplus \mathbb{C}$. Thus, Vir is the unique (up to Lie algebra isomorphism) nontrivial 1-dimensional central extension of W.

1.6. Recollection on $\mathfrak{g}$-invariant forms

Before we show the next important family of infinite-dimensional Lie algebras, let us define some standard notions. First, let us define the notion of a $\mathfrak{g}$-invariant form, in full generality (that is, for any two $\mathfrak{g}$-modules):

Definition 1.6.1. Let $\mathfrak{g}$ be a Lie algebra over a field k. Let M and N be two $\mathfrak{g}$-modules. Let $\beta : M \times N \to k$ be a k-bilinear form. Then, this form β is said to be $\mathfrak{g}$-*invariant* if and only if every $x \in \mathfrak{g}$, $a \in M$ and $b \in N$ satisfy

$$\beta\left(x \rightharpoonup a, b\right) + \beta\left(a, x \rightharpoonup b\right) = 0.$$

Instead of "$\mathfrak{g}$-invariant", one often says "invariant".

The following remark gives an alternative characterization of $\mathfrak{g}$-invariant bilinear forms (which is occasionally used as an alternative definition thereof):

Remark 1.6.2. Let $\mathfrak{g}$ be a Lie algebra over a field k. Let M and N be two $\mathfrak{g}$-modules. Consider the tensor product $M \otimes N$ of the two $\mathfrak{g}$-modules M and N; this is known to be a $\mathfrak{g}$-module again. Consider also k as a $\mathfrak{g}$-module (with the trivial $\mathfrak{g}$-module structure).

Let $\beta : M \times N \to k$ be a k-bilinear form. Let B be the linear map $M \otimes N \to k$ induced by the k-bilinear map $\beta : M \times N \to k$ using the universal property of the tensor product.

Then, β is $\mathfrak{g}$-invariant if and only if B is a $\mathfrak{g}$-module homomorphism.

We leave the proof of this remark as an instructive exercise for those who are not already aware of it.

Very often, the notion of a "$\mathfrak{g}$-invariant" bilinear form (as defined in Definition 1.6.1) is applied to forms on $\mathfrak{g}$ itself. In this case, it has to be interpreted as follows:

Convention 1.6.3. Let $\mathfrak{g}$ be a Lie algebra over a field k. Let $\beta : \mathfrak{g} \times \mathfrak{g} \to k$ be a bilinear form. When we say that β is $\mathfrak{g}$-invariant without specifying the $\mathfrak{g}$-module structure on $\mathfrak{g}$, we always tacitly understand that the $\mathfrak{g}$-module structure on $\mathfrak{g}$ is the adjoint one (i. e., the one defined by $x \rightharpoonup a = [x, a]$ for all $x \in \mathfrak{g}$ and $a \in \mathfrak{g}$).

The following remark provides two equivalent criteria for a bilinear form on the Lie algebra $\mathfrak{g}$ itself to be $\mathfrak{g}$-invariant; they will often be used tacitly:

Remark 1.6.4. Let $\mathfrak{g}$ be a Lie algebra over a field k. Let $\beta : \mathfrak{g} \times \mathfrak{g} \to k$ be a k-bilinear form.

(a) The form β is $\mathfrak{g}$-invariant if and only if every elements a, b and c of $\mathfrak{g}$ satisfy $\beta([a, b], c) + \beta(b, [a, c]) = 0$.

(b) The form β is $\mathfrak{g}$-invariant if and only if every elements a, b and c of $\mathfrak{g}$ satisfy $\beta([a, b], c) = \beta(a, [b, c])$.

The proof of this remark is, again, completely straightforward.

An example of a $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$ itself for $\mathfrak{g}$ finite-dimensional is given by the so-called Killing form:

Proposition 1.6.5. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k. Then, the form

$$\begin{aligned} &\mathfrak{g} \times \mathfrak{g} \to k, \\ &(x, y) \mapsto \operatorname{Tr}_{\mathfrak{g}}((\operatorname{ad} x) \circ (\operatorname{ad} y)) \end{aligned}$$

is a symmetric $\mathfrak{g}$-invariant bilinear form. This form is called the *Killing form* of the Lie algebra $\mathfrak{g}$.

Proposition 1.6.6. Let $\mathfrak{g}$ be a finite-dimensional semisimple Lie algebra over $\mathbb{C}$.

(a) The Killing form of $\mathfrak{g}$ is nondegenerate.

(b) Any $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$ is a scalar multiple of the Killing form of $\mathfrak{g}$. (Hence, if $\mathfrak{g} \neq 0$, then the vector space of $\mathfrak{g}$-invariant bilinear forms on $\mathfrak{g}$ is 1-dimensional and spanned by the Killing form.)

1.7. Affine Lie algebras

Now let us introduce the so-called affine Lie algebras; this is a very general construction from which a lot of infinite-dimensional Lie algebras emerge (including the Heisenberg algebra defined above).

Definition 1.7.1. Let $\mathfrak{g}$ be a Lie algebra.

(a) The $\mathbb{C}$-Lie algebra $\mathfrak{g}$ induces (by extension of scalars) a $\mathbb{C}[t, t^{-1}]$-Lie algebra

$$\mathbb{C}\left[t, t^{-1}\right] \otimes \mathfrak{g} = \left\{ \sum_{i \in \mathbb{Z}} a_i t^i \;\mid\; a_i \in \mathfrak{g};\ \text{all but finitely many } i \in \mathbb{Z} \text{ satisfy } a_i = 0 \right\}.$$

This Lie algebra $\mathbb{C}[t, t^{-1}] \otimes \mathfrak{g}$, considered as a $\mathbb{C}$-Lie algebra, will be called the *loop algebra* of $\mathfrak{g}$, and denoted by $\mathfrak{g}[t, t^{-1}]$.

(b) Let $(\cdot,\cdot)$ be a symmetric bilinear form on $\mathfrak{g}$ (that is, a symmetric bilinear map $\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$) which is $\mathfrak{g}$-invariant (this means that $([a,b],c)+(b,[a,c])=0$ for all $a,b,c\in\mathfrak{g}$).

Then, we can define a 2-cocycle ω on the loop algebra $\mathfrak{g}[t,t^{-1}]$ by

$$\omega(f,g)=\sum_{i\in\mathbb{Z}} i(f_i,g_{-i}) \qquad \text{for every } f\in\mathfrak{g}\left[t,t^{-1}\right] \text{ and } g\in\mathfrak{g}\left[t,t^{-1}\right] \tag{7}$$

(where we write f in the form $f=\sum_{i\in\mathbb{Z}} f_it^i$ with $f_i\in\mathfrak{g}$, and where we write g in the form $g=\sum_{i\in\mathbb{Z}} g_it^i$ with $g_i\in\mathfrak{g}$).

Proving that ω is a 2-cocycle is an exercise. So we can define a 1-dimensional central extension $\mathfrak{g}[t,t^{-1}]_\omega=\mathfrak{g}[t,t^{-1}]\oplus\mathbb{C}$ with bracket defined by ω.

We are going to abbreviate $\mathfrak{g}[t,t^{-1}]_\omega$ by $\widehat{\mathfrak{g}}_\omega$, or, more radically, by $\widehat{\mathfrak{g}}$.

Remark 1.7.2. The equation (7) can be rewritten in the (laconical but suggestive) form $\omega(f,g)=\operatorname{Res}_{t=0}(df,g)$. Here, (df,g) is to be understood as follows: Extend the bilinear form $(\cdot,\cdot):\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ to a bilinear form $(\cdot,\cdot):\mathfrak{g}[t,t^{-1}]\times\mathfrak{g}[t,t^{-1}]\to\mathbb{C}[t,t^{-1}]$ by setting

$$\left(at^i,bt^j\right)=(a,b)\,t^{i+j} \qquad \text{for all } a\in\mathfrak{g},\ b\in\mathfrak{g},\ i\in\mathbb{Z} \text{ and } j\in\mathbb{Z}.$$

Also, for every $f\in\mathfrak{g}[t,t^{-1}]$, define the "derivative" f' of f to be the element $\sum_{i\in\mathbb{Z}} if_it^{i-1}$ of $\mathfrak{g}[t,t^{-1}]$ (where we write f in the form $f=\sum_{i\in\mathbb{Z}} f_it^i$ with $f_i\in\mathfrak{g}$). In analogy to the notation $dg=g'dt$ which we introduced in Definition 1.1.1, set (df,g) to mean the polynomial differential form $(f',g)\,dt$ for any $f\in\mathfrak{g}[t,t^{-1}]$ and $g\in\mathfrak{g}[t,t^{-1}]$. Then, it is very easy to see that $\operatorname{Res}_{t=0}(df,g)=\sum_{i\in\mathbb{Z}} i(f_i,g_{-i})$ (where we write f in the form $f=\sum_{i\in\mathbb{Z}} f_it^i$ with $f_i\in\mathfrak{g}$, and where we write g in the form $g=\sum_{i\in\mathbb{Z}} g_it^i$ with $g_i\in\mathfrak{g}$), so that we can rewrite (7) as $\omega(f,g)=\operatorname{Res}_{t=0}(df,g)$.

We already know one example of the construction in Definition 1.7.1:

Remark 1.7.3. If $\mathfrak{g}$ is the abelian Lie algebra $\mathbb{C}$, and $(\cdot,\cdot)$ is the bilinear form $\mathbb{C}\times\mathbb{C}\to\mathbb{C}$, $(x,y)\mapsto xy$, then the 2-cocycle ω on the loop algebra $\mathbb{C}[t,t^{-1}]$ is given by

$$\omega(f,g)=\operatorname{Res}_{t=0}(gdf)=\sum_{i\in\mathbb{Z}} if_ig_{-i} \qquad \text{for every } f,g\in\mathbb{C}\left[t,t^{-1}\right]$$

(where we write f in the form $f=\sum_{i\in\mathbb{Z}} f_it^i$ with $f_i\in\mathbb{C}$, and where we write g in the form $g=\sum_{i\in\mathbb{Z}} g_it^i$ with $g_i\in\mathbb{C}$). Hence, in this case, the central extension $\mathfrak{g}[t,t^{-1}]_\omega=\widehat{\mathfrak{g}}_\omega$ is precisely the Heisenberg algebra $\mathcal{A}$ as introduced in Definition 1.1.4.

The main example that we will care about is when $\mathfrak{g}$ is a simple finite-dimensional Lie algebra and $(\cdot,\cdot)$ is the unique (up to scalar) invariant symmetric bilinear form (i.

e., a multiple of the Killing form). In this case, the Lie algebra $\widehat{\mathfrak{g}} = \widehat{\mathfrak{g}}_\omega$ is called an *affine Lie algebra*.

Theorem 1.7.4. If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, then $H^2\left(\mathfrak{g}\left[t, t^{-1}\right]\right)$ is 1-dimensional and spanned by the cocycle ω corresponding to $(\cdot, \cdot)$.

Corollary 1.7.5. If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, then the Lie algebra $\mathfrak{g}\left[t, t^{-1}\right]$ has a unique (up to isomorphism of Lie algebras, not up to isomorphism of extensions) nontrivial 1-dimensional central extension $\widehat{\mathfrak{g}}_\omega$.

Definition 1.7.6. The Lie algebra $\widehat{\mathfrak{g}}_\omega$ defined in Corollary 1.7.5 (for $(\cdot, \cdot)$ being the Killing form of $\mathfrak{g}$) is called the *affine Kac-Moody algebra* corresponding to $\mathfrak{g}$. (Or, more precisely, the *untwisted affine Kac-Moody algebra* corresponding to $\mathfrak{g}$.)

In order to prepare for the proof of Theorem 1.7.4, we recollect some facts from the cohomology of Lie algebras:

Definition 1.7.7. Let $\mathfrak{g}$ be a Lie algebra. Let M be a $\mathfrak{g}$-module. We define the *semidirect product* $\mathfrak{g} \ltimes M$ to be the Lie algebra which, as a vector space, is $\mathfrak{g} \oplus M$, but whose Lie bracket is defined by

$$[(a, \alpha), (b, \beta)] = ([a, b], a \rightharpoonup \beta - b \rightharpoonup \alpha) \qquad \text{for all } a \in \mathfrak{g},\ \alpha \in M,\ b \in \mathfrak{g} \text{ and } \beta \in M .$$

(The symbol $\rightharpoonup$ means action here; i. e., a term like $c \rightharpoonup m$ (with $c \in \mathfrak{g}$ and $m \in M$) means the action of c on m.) Thus, the canonical injection $\mathfrak{g} \to \mathfrak{g} \ltimes M$, $a \mapsto (a, 0)$ is a Lie algebra homomorphism, and so is the canonical projection $\mathfrak{g} \ltimes M \to \mathfrak{g}$, $(a, \alpha) \mapsto a$. Also, M is embedded into $\mathfrak{g} \ltimes M$ by the injection $M \to \mathfrak{g} \ltimes M$, $\alpha \mapsto (0, \alpha)$; this makes M an abelian Lie subalgebra of $\mathfrak{g} \ltimes M$.

All statements made in Definition 1.7.7 (including the tacit statement that the Lie bracket on $\mathfrak{g} \ltimes M$ defined in Definition 1.7.7 satisfies antisymmetry and the Jacobi identity) are easy to verify by computation. The semidirect product that we have just defined is not the most general notion of a semidirect product. We will later (Definition 3.2.1) define a more general one, where M itself may have a Lie algebra structure and this structure has an effect on that of $\mathfrak{g} \ltimes M$. But for now, Definition 1.7.7 suffices for us.

Definition 1.7.8. Let $\mathfrak{g}$ be a Lie algebra. Let M be a $\mathfrak{g}$-module.

(a) A 1-*cocycle of* $\mathfrak{g}$ *with coefficients in* M is a linear map $\eta : \mathfrak{g} \to M$ such that

$$\eta([a, b]) = a \rightharpoonup \eta(b) - b \rightharpoonup \eta(a) \qquad \text{for all } a \in \mathfrak{g} \text{ and } b \in \mathfrak{g}.$$

(The symbol $\rightharpoonup$ means action here; i. e., a term like $c \rightharpoonup m$ (with $c \in \mathfrak{g}$ and $m \in M$) means the action of c on m.)

It is easy to see (and known) that 1-cocycles of $\mathfrak{g}$ with coefficients in M are in bijection with Lie algebra homomorphisms $\mathfrak{g} \to \mathfrak{g} \ltimes M$. This bijection sends every 1-cocycle η to the map $\mathfrak{g} \to \mathfrak{g} \ltimes M$, $a \mapsto (a, \eta(a))$.

Notice that 1-cocycles of $\mathfrak{g}$ with coefficients in the $\mathfrak{g}$-module $\mathfrak{g}$ are exactly the same as derivations of $\mathfrak{g}$.

(b) A 1-*coboundary of* $\mathfrak{g}$ *with coefficients in* M means a linear map $\eta : \mathfrak{g} \to M$ which has the form $a \mapsto a \rightharpoonup m$ for some $m \in M$. Every 1-coboundary of $\mathfrak{g}$ with coefficients in M is a 1-cocycle.

(c) The space of 1-cocycles of $\mathfrak{g}$ with coefficients in M is denoted by $Z^1(\mathfrak{g}, M)$. The space of 1-coboundaries of $\mathfrak{g}$ with coefficients in M is denoted by $B^1(\mathfrak{g}, M)$. We have $B^1(\mathfrak{g}, M) \subseteq Z^1(\mathfrak{g}, M)$. The quotient space $Z^1(\mathfrak{g}, M) \diagup B^1(\mathfrak{g}, M)$ is denoted by $H^1(\mathfrak{g}, M)$ is called the 1-*st cohomology space* of $\mathfrak{g}$ *with coefficients in* M.

Of course, these spaces $Z^1(\mathfrak{g}, M)$, $B^1(\mathfrak{g}, M)$ and $H^1(\mathfrak{g}, M)$ are but particular cases of more general constructions $Z^i(\mathfrak{g}, M)$, $B^i(\mathfrak{g}, M)$ and $H^i(\mathfrak{g}, M)$ which are defined for every $i \in \mathbb{N}$. (In particular, $H^0(\mathfrak{g}, M)$ is the subspace $\{m \in M \mid a \rightharpoonup m = 0 \text{ for all } a \in \mathfrak{g}\}$ of M, and often denoted by $M^{\mathfrak{g}}$.) The spaces $H^i(\mathfrak{g}, M)$ (or, more precisely, the functors assigning these spaces to every $\mathfrak{g}$-module M) can be understood as the so-called derived functors of the functor $M \mapsto M^{\mathfrak{g}}$. However, we won't use $H^i(\mathfrak{g}, M)$ for any i other than 1 here.

We record a relation between $H^1(\mathfrak{g}, M)$ and the Ext bifunctor:

$$H^1(\mathfrak{g}, M) = \operatorname{Ext}^1_{\mathfrak{g}}(\mathbb{C}, M).$$

More generally, $\operatorname{Ext}^1_{\mathfrak{g}}(N, M) = H^1(\mathfrak{g}, \operatorname{Hom}_{\mathbb{C}}(N, M))$ for any two $\mathfrak{g}$-modules N and M.

Theorem 1.7.9 (Whitehead)**.** If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, and M is a finite-dimensional $\mathfrak{g}$-module, then $H^1(\mathfrak{g}, M) = 0$.

Proof of Theorem 1.7.9. Since $\mathfrak{g}$ is a simple Lie algebra, Weyl's theorem says that finite-dimensional $\mathfrak{g}$-modules are completely reducible. Hence, if N and M are finite-dimensional $\mathfrak{g}$-modules, we have $\operatorname{Ext}^1_{\mathfrak{g}}(N, M) = 0$. In particular, $\operatorname{Ext}^1_{\mathfrak{g}}(\mathbb{C}, M) = 0$. Since $H^1(\mathfrak{g}, M) = \operatorname{Ext}^1_{\mathfrak{g}}(\mathbb{C}, M)$, this yields $H^1(\mathfrak{g}, M) = 0$. Theorem 1.7.9 is thus proven.

Lemma 1.7.10. Let ω be a 2-cocycle on a Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_0 \subseteq \mathfrak{g}$ be a Lie subalgebra, and $M \subseteq \mathfrak{g}$ be a $\mathfrak{g}_0$-submodule. Then, $\omega \mid_{\mathfrak{g}_0 \times M}$, when considered as a map $\mathfrak{g}_0 \to M^*$, belongs to $Z^1(\mathfrak{g}_0, M^*)$.

The proof of Lemma 1.7.10 is a straightforward manipulation of formulas:

Proof of Lemma 1.7.10. Let η denote the 2-cocycle $\omega \mid_{\mathfrak{g}_0 \times M}$, considered as a map $\mathfrak{g}_0 \to M^*$. Thus, η is defined by

$$\eta(x) = (M \to \mathbb{C}, \qquad y \mapsto \omega(x, y)) \qquad \text{for all } x \in \mathfrak{g}_0.$$

Hence,

$$(\eta(x))(y) = \omega(x, y) \qquad \text{for all } x \in \mathfrak{g}_0 \text{ and } y \in M. \tag{8}$$

Thus, any $a \in \mathfrak{g}_0$, $b \in \mathfrak{g}_0$ and $c \in M$ satisfy $(\eta([a,b]))(c) = \omega([a,b],c)$ and

$$\begin{aligned}
&(a \rightharpoonup \eta(b) - b \rightharpoonup \eta(a))(c) \\
&= \underbrace{(a \rightharpoonup \eta(b))(c)}_{\substack{=-(\eta(b))([a,c]) \\ \text{(by the definition of the dual of a } \mathfrak{g}_0\text{-module)}}} - \underbrace{(b \rightharpoonup \eta(a))(c)}_{\substack{=-(\eta(a))([b,c]) \\ \text{(by the definition of the dual of a } \mathfrak{g}_0\text{-module)}}} \\
&= \left(-\underbrace{(\eta(b))([a,c])}_{\substack{=\omega(b,[a,c]) \\ \text{(by (8))}}}\right) - \left(-\underbrace{(\eta(a))([b,c])}_{\substack{=\omega(a,[b,c]) \\ \text{(by (8))}}}\right) = (-\omega(b,[a,c])) - (-\omega(a,[b,c])) \\
&= -\omega\left(b, \underbrace{[a,c]}_{=-[c,a]}\right) + \omega(a,[b,c]) = \underbrace{\omega(b,[c,a])}_{\substack{=-\omega([c,a],b) \\ \text{(since } \omega \text{ is antisymmetric)}}} + \underbrace{\omega(a,[b,c])}_{\substack{=-\omega([b,c],a) \\ \text{(since } \omega \text{ is antisymmetric)}}} \\
&= -\omega([c,a],b) - \omega([b,c],a) = \omega([a,b],c) \qquad \text{(by (3))},
\end{aligned}$$

so that $(\eta([a,b]))(c) = (a \rightharpoonup \eta(b) - b \rightharpoonup \eta(a))(c)$. Thus, any $a \in \mathfrak{g}_0$ and $b \in \mathfrak{g}_0$ satisfy $\eta([a,b]) = a \rightharpoonup \eta(b) - b \rightharpoonup \eta(a)$. This shows that η is a 1-cocycle, i. e., belongs to $Z^1(\mathfrak{g}_0, M^*)$. Lemma 1.7.10 is proven.

Proof of Theorem 1.7.4. First notice that any $a, b, c \in \mathfrak{g}$ satisfy

$$([a,b],c) = ([b,c],a) = ([c,a],b) \tag{9}$$

[6]. Moreover,

$$\text{there exist } a,b,c \in \mathfrak{g} \text{ such that } ([a,b],c) = ([b,c],a) = ([c,a],b) \neq 0. \tag{10}$$

[7] This will be used later in our proof; but as for now, forget about these a, b, c.

It is easy to see that the 2-cocycle ω on $\mathfrak{g}[t,t^{-1}]$ defined by (7) is not a 2-coboundary.[8]

[6] *Proof.* First of all, any $a, b, c \in \mathfrak{g}$ satisfy

$$\begin{aligned}
([a,b],c) &= (a,[b,c]) && \text{(since the form } (\cdot,\cdot) \text{ is invariant)} \\
&= ([b,c],a) && \text{(since the form } (\cdot,\cdot) \text{ is symmetric)}.
\end{aligned}$$

Applying this to b, c, a instead of a, b, c, we obtain $([b,c],a) = ([c,a],b)$. Hence, $([a,b],c) = ([b,c],a) = ([c,a],b)$, so that (9) is proven.

[7] *Proof.* Since $\mathfrak{g}$ is simple, we have $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ and thus $([\mathfrak{g},\mathfrak{g}],\mathfrak{g}) = (\mathfrak{g},\mathfrak{g}) \neq 0$ (since the form $(\cdot,\cdot)$ is nondegenerate). Hence, there exist $a, b, c \in \mathfrak{g}$ such that $([a,b],c) \neq 0$. The rest is handled by (9).

[8] *Proof.* Assume the contrary. Then, this 2-cocycle ω is a coboundary, i. e., there exists a linear map $\xi : \mathfrak{g}[t,t^{-1}] \to \mathbb{C}$ such that $\omega = d\xi$.

Now, pick some $a \in \mathfrak{g}$ and $b \in \mathfrak{g}$ such that $(a,b) \neq 0$ (this is possible since the form $(\cdot,\cdot)$ is nondegenerate). Then,

$$\underbrace{\omega}_{=d\xi}\left(at, bt^{-1}\right) = (d\xi)\left(at, bt^{-1}\right) = \xi\left(\underbrace{\left[at, bt^{-1}\right]}_{=[a,b]}\right) = \xi([a,b])$$

and

$$\underbrace{\omega}_{=d\xi}(a,b) = (d\xi)(a,b) = \xi([a,b]),$$

Now let us consider the structure of $\mathfrak{g}[t,t^{-1}]$. We have $\mathfrak{g}[t,t^{-1}] = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}t^n \supseteq \mathfrak{g}t^0 = \mathfrak{g}$. This is, actually, an inclusion of Lie algebras. So $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}[t,t^{-1}]$, and $\mathfrak{g}t^n$ is a $\mathfrak{g}$-submodule of $\mathfrak{g}[t,t^{-1}]$ isomorphic to $\mathfrak{g}$ for every $n \in \mathbb{Z}$.

Let ω be an arbitrary 2-cocycle on $\mathfrak{g}[t,t^{-1}]$ (not necessarily the one defined by (7)).

Let $n \in \mathbb{Z}$. Then, $\omega \mid_{\mathfrak{g}\times\mathfrak{g}t^n}$, when considered as a map $\mathfrak{g} \to (\mathfrak{g}t^n)^*$, belongs to $Z^1(\mathfrak{g},(\mathfrak{g}t^n)^*)$ (by Lemma 1.7.10, applied to $\mathfrak{g}$, $\mathfrak{g}t^n$ and $\mathfrak{g}[t,t^{-1}]$ instead of $\mathfrak{g}_0$, M and $\mathfrak{g}$), i. e., is a 1-cocycle. But by Theorem 1.7.9, we have $H^1(\mathfrak{g},(\mathfrak{g}t^n)^*) = 0$, so this rewrites as $\omega \mid_{\mathfrak{g}\times\mathfrak{g}t^n} \in B^1(\mathfrak{g},(\mathfrak{g}t^n)^*)$. In other words, there exists some $\xi_n \in (\mathfrak{g}t^n)^*$ such that $\omega \mid_{\mathfrak{g}\times\mathfrak{g}t^n} = d\xi_n$. Pick such a ξ_n. Thus,

$$\omega(a,bt^n) = \underbrace{(\omega \mid_{\mathfrak{g}\times\mathfrak{g}t^n})}_{=d\xi_n}(a,bt^n) = (d\xi_n)(a,bt^n) = \xi_n([a,bt^n]) \qquad \text{for all } a,b\in\mathfrak{g}.$$

Define a map $\xi : \mathfrak{g}[t,t^{-1}] \to \mathbb{C}$ by requiring that $\xi \mid_{\mathfrak{g}t^n} = \xi_n$ for every $n \in \mathbb{Z}$.

Now, let $\widetilde{\omega} = \omega - d\xi$. Then,

$$\widetilde{\omega}(x,y) = \omega(x,y) - \xi([x,y]) \qquad \text{for all } x,y \in \mathfrak{g}\left[t,t^{-1}\right].$$

Replace ω by $\widetilde{\omega}$ (this doesn't change the residue class of ω in $H^2(\mathfrak{g}[t,t^{-1}])$, since $\widetilde{\omega}$ differs from ω by a 2-coboundary). By doing this, we have reduced to a situation when

$$\omega(a,bt^n) = 0 \qquad \text{for all } a,b\in\mathfrak{g} \text{ and } n\in\mathbb{Z}.$$

[9] Since ω is antisymmetric, this yields

$$\omega(bt^n,a) = 0 \qquad \text{for all } a,b\in\mathfrak{g} \text{ and } n\in\mathbb{Z}. \tag{11}$$

Now, fix some $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. Since ω is a 2-cocycle, the 2-cocycle condition yields

$$\begin{aligned}
0 &= \omega\left(\underbrace{[a,bt^n]}_{=[a,b]t^n}, ct^m\right) + \omega\left(\underbrace{[ct^m,a]}_{\substack{=[c,a]t^m \\ =-[a,c]t^m}}, bt^n\right) + \omega\left(\underbrace{[bt^n,ct^m]}_{=[b,c]t^{n+m}}, a\right) \\
&= \omega([a,b]t^n, ct^m) + \underbrace{\omega(-[a,c]t^m, bt^n)}_{=\omega(bt^n,[a,c]t^m)} + \underbrace{\omega\left([b,c]t^{n+m}, a\right)}_{\substack{=0 \\ \text{(by (11))}}} \\
&= \omega([a,b]t^n, ct^m) + \omega(bt^n,[a,c]t^m) \qquad \text{for all } a,b,c\in\mathfrak{g}.
\end{aligned}$$

In other words, the bilinear form on $\mathfrak{g}$ given by $(b,c) \mapsto \omega(bt^n,ct^m)$ is $\mathfrak{g}$-invariant. But every $\mathfrak{g}$-invariant bilinear form on $\mathfrak{g}$ must be a multiple of our bilinear form $(\cdot,\cdot)$ (since

so that $\omega\left(at,bt^{-1}\right) = \omega(a,b)$. But by the definition of ω, we easily see that $\omega\left(at,bt^{-1}\right) = 1\underbrace{(a,b)}_{\neq 0} \neq 0$ and $\omega(a,b) = 0\,(a,b) = 0$, which yields a contradiction.

[9] But all the ξ-freedom has been used up in this reduction - i. e., if the new ω is nonzero, then the original ω was not a 2-coboundary. This gives us an alternative way of proving that the 2-cocycle ω on $\mathfrak{g}\left[t,t^{-1}\right]$ defined by (7) is not a 2-coboundary.

$\mathfrak{g}$ is simple, and thus the space of all $\mathfrak{g}$-invariant bilinear forms on $\mathfrak{g}$ is 1-dimensional[10]). Hence, there exists some constant $\gamma_{n,m} \in \mathbb{C}$ (depending on n and m) such that

$$\omega\left(bt^n, ct^m\right) = \gamma_{n,m} \cdot (b, c) \qquad \text{for all } b, c \in \mathfrak{g}. \tag{12}$$

It is easy to see that

$$\gamma_{n,m} = -\gamma_{m,n} \qquad \text{for all } n, m \in \mathbb{Z}, \tag{13}$$

since the bilinear form ω is skew-symmetric whereas the bilinear form $(\cdot, \cdot)$ is symmetric.

Now, for any $m \in \mathbb{Z}$, $n \in \mathbb{Z}$ and $p \in \mathbb{Z}$, the 2-cocycle condition yields

$$\omega\left(\left[at^n, bt^m\right], ct^p\right) + \omega\left(\left[bt^m, ct^p\right], at^n\right) + \omega\left(\left[ct^p, at^n\right], bt^m\right) = 0 \qquad \text{for all } a, b, c \in \mathfrak{g}.$$

Due to

$$\omega\left(\underbrace{\left[at^n, bt^m\right]}_{=[a,b]t^{n+m}}, ct^p\right) = \omega\left(\left[a, b\right] t^{n+m}, ct^p\right) = \gamma_{n+m,p} \cdot \left(\left[a, b\right], c\right) \qquad \text{(by (12))}$$

and the two cyclic permutations of this identity, this rewrites as

$$\gamma_{n+m,p} \cdot \left(\left[a, b\right], c\right) + \gamma_{m+p,n} \cdot \left(\left[b, c\right], a\right) + \gamma_{p+n,m} \cdot \left(\left[c, a\right], b\right) = 0.$$

Since this holds for all $a, b, c \in \mathfrak{g}$, we can use (10) to transform this into

$$\gamma_{n+m,p} + \gamma_{m+p,n} + \gamma_{p+n,m} = 0.$$

Due to (13), this rewrites as

$$\gamma_{n,m+p} + \gamma_{m,p+n} + \gamma_{p,m+n} = 0.$$

Denoting by s the sum $m + n + p$, we can rewrite this as

$$\gamma_{n,s-n} + \gamma_{m,s-m} - \gamma_{m+n,s-m-n} = 0.$$

In other words, for fixed $s \in \mathbb{Z}$, the function $\mathbb{Z} \to \mathbb{C}$, $n \mapsto \gamma_{n,s-n}$ is additive. Hence, $\gamma_{n,s-n} = n\gamma_{1,s-1}$ and $\gamma_{s-n,n} = (s - n)\gamma_{1,s-1}$ for every $n \in \mathbb{Z}$. Thus,

$$\begin{aligned}(s - n)\gamma_{1,s-1} &= \gamma_{s-n,n} = -\gamma_{n,s-n} \qquad &&\text{(by (13))}\\ &= -n\gamma_{1,s-1} &&\text{for every } n \in \mathbb{Z}\end{aligned}$$

Hence, $s\gamma_{1,s-1} = 0$. Thus, for every $s \neq 0$, we conclude that $\gamma_{1,s-1} = 0$ and hence $\gamma_{n,s-n} = n\underbrace{\gamma_{1,s-1}}_{=0} = 0$ for every $n \in \mathbb{Z}$. In other words, $\gamma_{n,m} = 0$ for every $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfying $n + m \neq 0$.

What happens for $s = 0$? For $s = 0$, the equation $\gamma_{n,s-n} = n\gamma_{1,s-1}$ becomes $\gamma_{n,-n} = n\gamma_{1,-1}$.

Thus we have proven that $\gamma_{n,m} = 0$ for every $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfying $n+m \neq 0$, and that every $n \in \mathbb{Z}$ satisfies $\gamma_{n,-n} = n\gamma_{1,-1}$.

Hence, the form ω must be a scalar multiple of the form which sends every (f, g) to $\operatorname{Res}_{t=0} \underbrace{(df, g)}_{\text{scalar-valued 1-form}} = \sum\limits_{i\in\mathbb{Z}} i\left(f_i, g_{-i}\right)$. We have thus proven that every 2-cocycle ω is a scalar multiple of the 2-cocycle ω defined by (7) modulo the 2-coboundaries. Since we also know that the 2-cocycle ω defined by (7) is not a 2-coboundary, this yields that the space $H^2\left(\mathfrak{g}\left[t, t^{-1}\right]\right)$ is 1-dimensional and spanned by the residue class of the 2-cocycle ω defined by (7). This proves Theorem 1.7.4.

[10] and spanned by the Killing form

2. Representation theory: generalities

2.1. Representation theory: general facts

The first step in the representation theory of any objects (groups, algebras, etc.) is usually proving some kind of Schur's lemma. There is one form of Schur's lemma that holds almost tautologically: This is the form that claims that every morphism between irreducible representations is either 0 or an isomorphism.[11] However, the more often used form of Schur's lemma is a bit different: It claims that, over an algebraically closed field, every endomorphism of a finite-dimensional irreducible representation is a scalar multiple of the identity map. This is usually proven using eigenvalues, and this proof depends on the fact that eigenvalues exist; this (in general) requires the irreducible representation to be *finite-dimensional*. Hence, it should not come as a surprise that this latter form of Schur's lemma does not generally hold for infinite-dimensional representations. This makes this lemma not particularly useful in the case of infinite-dimensional Lie algebras. But we still can show the following version of Schur's lemma over $\mathbb{C}$:

Lemma 2.1.1 (Dixmier's Lemma)**.** Let A be an algebra over $\mathbb{C}$, and let V be an irreducible A-module of countable dimension. Then, any A-module homomorphism $\phi : V \to V$ is a scalar multiple of the identity.

This lemma is called *Dixmier's lemma*, and its proof is similar to the famous proof of the Nullstellensatz over $\mathbb{C}$ using the uncountability of $\mathbb{C}$.

Proof of Lemma 2.1.1. Let $D = \operatorname{End}_A V$. Then, D is a division algebra (in fact, the endomorphism ring of an irreducible representation always is a division algebra).

For any nonzero $v \in V$, we have $Av = V$ (otherwise, Av would be a nonzero proper A-submodule of V, contradicting the fact that V is irreducible and thus does not have any such submodules). In other words, for any nonzero $v \in V$, every element of V can be written as av for some $a \in A$. Thus, for any nonzero $v \in V$, any element $\phi \in D$ is completely determined by $\phi(v)$ (because $\phi(av) = a\phi(v)$ for every $a \in A$, so that the value $\phi(v)$ uniquely determines the value of $\phi(av)$ for every $a \in A$, and thus (since we know that every element of V can be written as av for some $a \in A$) every value of ϕ is uniquely determined). Thus, we have an embedding of D into V. Hence, D is countably-dimensional (since V is countably-dimensional). But a countably-dimensional division algebra D over $\mathbb{C}$ must be $\mathbb{C}$ itself[12], so that $D = \mathbb{C}$, and this is exactly what we wanted to show. Lemma 2.1.1 is proven.

Note that Lemma 2.1.1 is a general fact, not particular to Lie algebras; however, it is not as general as it seems: It really makes use of the uncountability of $\mathbb{C}$, not just

[11] There are also variations on this assertion:

1) Every morphism from an irreducible representation to a representation is either 0 or injective.

2) Every morphism from a representation to an irreducible representation is either 0 or surjective.

Both of these variations follow very easily from the definition of "irreducible".

[12] *Proof.* Indeed, assume the contrary. So there exists some $\phi \in D$ not belonging to $\mathbb{C}$. Then, ϕ is transcendental over $\mathbb{C}$, so that $\mathbb{C}(\phi) \subseteq D$ is the field of rational functions in one variable ϕ over $\mathbb{C}$. Now, $\mathbb{C}(\phi)$ contains the rational function $\dfrac{1}{\phi - \lambda}$ for every $\lambda \in \mathbb{C}$, and these rational functions for varying λ are linearly independent. Since $\mathbb{C}$ is uncountable, we thus have an uncountable linearly independent set of elements of $\mathbb{C}(\phi)$, contradicting the fact that $\mathbb{C}(\phi)$ is a subspace of the countably-dimensional space D, qed.

of the fact that $\mathbb{C}$ is an algebraically closed field of characteristic 0. It would be wrong if we would replace $\mathbb{C}$ by (for instance) the algebraic closure of $\mathbb{Q}$.

Remark 2.1.2. Let A be a countably-dimensional algebra over $\mathbb{C}$, and let V be an irreducible A-module. Then, V itself is countably dimensional.

Proof of Remark 2.1.2. For any nonzero $v \in V$, we have $Av = V$ (by the same argument as in the proof of Lemma 2.1.1), and thus $\dim (Av) = \dim V$. Since $\dim (Av) \leq \dim A$, we thus have $\dim V = \dim (Av) \leq \dim A$, so that V has countable dimension (since A has countable dimension). This proves Remark 2.1.2.

Corollary 2.1.3. Let A be an algebra over $\mathbb{C}$, and let V be an irreducible A-module of countable dimension. Let C be a central element of A. Then, $C \mid_V$ is a scalar (i. e., a scalar multiple of the identity map).

Proof of Corollary 2.1.3. Since C is central, the element C commutes with any element of A. Thus, $C \mid_V$ is an A-module homomorphism, and hence (by Lemma 2.1.1, applied to $\phi = C \mid_V$) a scalar multiple of the identity. This proves Corollary 2.1.3.

2.2. Representations of the Heisenberg algebra $\mathcal{A}$

2.2.1. General remarks

Consider the oscillator algebra (aka Heisenberg algebra) $\mathcal{A} = \langle a_i \mid i \in \mathbb{Z} \rangle + \langle K \rangle$. Recall that

$$[a_i, a_j] = i\delta_{i,-j}K \qquad \text{for any } i, j \in \mathbb{Z};$$
$$[K, a_i] = 0 \qquad \text{for any } i \in \mathbb{Z}.$$

Let us try to classify the irreducible $\mathcal{A}$-modules.

Let V be an irreducible $\mathcal{A}$-module. Then, V is countably-dimensional (by Remark 2.1.2, since $U(\mathcal{A})$ is countably-dimensional), so that by Corollary 2.1.3, the endomorphism $K \mid_V$ is a scalar (because K is a central element of $\mathcal{A}$ and thus also a central element of $U(\mathcal{A})$).

If $K \mid_V = 0$, then V is a module over the Lie algebra $\mathcal{A} \diagup \mathbb{C}K = \langle a_i \mid i \in \mathbb{Z} \rangle$. But since $\langle a_i \mid i \in \mathbb{Z} \rangle$ is an abelian Lie algebra, irreducible modules over $\langle a_i \mid i \in \mathbb{Z} \rangle$ are 1-dimensional (again by Corollary 2.1.3), so that V must be 1-dimensional in this case. Thus, the case when $K \mid_V = 0$ is not an interesting case.

Now consider the case when $K \mid_V = k \neq 0$. Then, we can WLOG assume that $k = 1$, because the Lie algebra $\mathcal{A}$ has an automorphism sending K to λK for any arbitrary $\lambda \neq 0$ (this automorphism is given by $a_i \mapsto \lambda a_i$ for $i > 0$, and $a_i \mapsto a_i$ for $i \leq 0$).

We are thus interested in irreducible representations V of $\mathcal{A}$ satisfying $K \mid_V = 1$. These are in an obvious 1-to-1 correspondence with irreducible representations of $U(\mathcal{A}) \diagup (K - 1)$.

Proposition 2.2.1. We have an algebra isomorphism

$$\xi : U(\mathcal{A}) \diagup (K - 1) \to D(x_1, x_2, x_3, \ldots) \otimes \mathbb{C}[x_0],$$

where $D(x_1, x_2, x_3, ...)$ is the algebra of differential operators in the variables x_1, x_2, x_3, ... with polynomial coefficients. This isomorphism is given by

$$\xi(a_{-i}) = x_i \qquad \text{for } i \geq 1;$$
$$\xi(a_i) = i\frac{\partial}{\partial x_i} \qquad \text{for } i \geq 1;$$
$$\xi(a_0) = x_0.$$

Note that we are sloppy with notation here: Since ξ is a homomorphism from $U(\mathcal{A}) \diagup (K-1)$ (rather than $U(\mathcal{A})$), we should write $\xi(\overline{a_{-i}})$ instead of $\xi(a_{-i})$, etc.. We are using the same letters to denote elements of $U(\mathcal{A})$ and their residue classes in $U(\mathcal{A}) \diagup (K-1)$, and are relying on context to keep them apart. We hope that the reader will forgive us this abuse of notation.

Proof of Proposition 2.2.1. It is clear[13] that there exists a unique algebra homomorphism $\xi : U(\mathcal{A}) \diagup (K-1) \to D(x_1, x_2, x_3, ...)$ satisfying

$$\xi(a_{-i}) = x_i \qquad \text{for } i \geq 1;$$
$$\xi(a_i) = i\frac{\partial}{\partial x_i} \qquad \text{for } i \geq 1;$$
$$\xi(a_0) = x_0.$$

It is also clear that this ξ is surjective (since all the generators x_i, $\dfrac{\partial}{\partial x_i}$ and x_0 of the algebra $D(x_1, x_2, x_3, ...) \otimes \mathbb{C}[x_0]$ are in its image).

In the following, a map $\varphi : A \to \mathbb{N}$ (where A is some set) is said to be *finitely supported* if all but finitely many $a \in A$ satisfy $\varphi(a) = 0$. Sequences (finite, infinite, or two-sided infinite) are considered as maps (from finite sets, $\mathbb{N}$ or $\mathbb{Z}$, or occasionally other sets). Thus, a sequence is finitely supported if and only if all but finitely many of its elements are zero.

If A is a set, then $\mathbb{N}^A_{\text{fin}}$ will denote the set of all finitely supported maps $A \to \mathbb{N}$.

By the easy part of the Poincaré-Birkhoff-Witt theorem (this is the part which states that the increasing monomials *span* the universal enveloping algebra[14]), the family[15]

$$\left(\overrightarrow{\prod_{i\in\mathbb{Z}}} a_i^{n_i} \cdot K^m\right)_{(...,n_{-2},n_{-1},n_0,n_1,n_2,...)\in\mathbb{N}^{\mathbb{Z}}_{\text{fin}},\ m\in\mathbb{N}}$$

is a spanning set of the vector space $U(\mathcal{A})$. Hence, the family

$$\left(\overrightarrow{\prod_{i\in\mathbb{Z}}} a_i^{n_i}\right)_{(...,n_{-2},n_{-1},n_0,n_1,n_2,...)\in\mathbb{N}^{\mathbb{Z}}_{\text{fin}}}$$

[13] from the universal property of the universal enveloping algebra, and the universal property of the quotient algebra

[14] The hard part says that these increasing monomials are linearly independent.

[15] Here, $\overrightarrow{\prod_{i\in\mathbb{Z}}} a_i^{n_i}$ denotes the product $...a_{-2}^{n_{-2}} a_{-1}^{n_{-1}} a_0^{n_0} a_1^{n_1} a_2^{n_2}....$ (This product is infinite, but still has a value since only finitely many n_i are nonzero.)

is a spanning set of $U(\mathcal{A}) \diagup (K-1)$, and since this family maps to a linearly independent set under ξ (this is very easy to see), it follows that ξ is injective. Thus, ξ is an isomorphism, so that Proposition 2.2.1 is proven.

Definition 2.2.2. Define a vector subspace $\mathcal{A}_0$ of $\mathcal{A}$ by $\mathcal{A}_0 = \langle a_i \mid i \in \mathbb{Z} \diagdown \{0\} \rangle + \langle K \rangle$.

Proposition 2.2.3. This subspace $\mathcal{A}_0$ is a Lie subalgebra of $\mathcal{A}$, and $\mathbb{C}a_0$ is also a Lie subalgebra of $\mathcal{A}$. We have $\mathcal{A} = \mathcal{A}_0 \oplus \mathbb{C}a_0$ as Lie algebras. Hence,

$$U(\mathcal{A}) \diagup (K-1) = U(\mathcal{A}_0 \oplus \mathbb{C}a_0) \diagup (K-1) \cong \underbrace{(U(\mathcal{A}_0) \diagup (K-1))}_{\cong D(x_1,x_2,x_3,...)} \otimes \underbrace{\mathbb{C}[a_0]}_{\cong \mathbb{C}[x_0]}$$

(since $K \in \mathcal{A}_0$). Here, the isomorphism $U(\mathcal{A}_0) \diagup (K-1) \cong D(x_1, x_2, x_3, ...)$ is defined as follows: In analogy to Proposition 2.2.1, we have an algebra isomorphism

$$\widetilde{\xi} : U(\mathcal{A}_0) \diagup (K-1) \to D(x_1, x_2, x_3, ...)$$

given by

$$\widetilde{\xi}(a_{-i}) = x_i \qquad \text{for } i \geq 1;$$
$$\widetilde{\xi}(a_i) = i\frac{\partial}{\partial x_i} \qquad \text{for } i \geq 1.$$

The proof of Proposition 2.2.3 is analogous to that of Proposition 2.2.1 (where it is not completely straightforward).

2.2.2. The Fock space

From Proposition 2.2.3, we know that

$$U(\mathcal{A}_0) \diagup (K-1) \cong D(x_1, x_2, x_3, ...) \subseteq \operatorname{End}(\mathbb{C}[x_1, x_2, x_3, ...]).$$

Hence, we have a $\mathbb{C}$-algebra homomorphism $U(\mathcal{A}_0) \to \operatorname{End}(\mathbb{C}[x_1, x_2, x_3, ...])$. This makes $\mathbb{C}[x_1, x_2, x_3, ...]$ into a representation of the Lie algebra $\mathcal{A}_0$. Let us state this as a corollary:

Corollary 2.2.4. The Lie algebra $\mathcal{A}_0$ has a representation $F = \mathbb{C}[x_1, x_2, x_3, ...]$ which is given by

$$a_{-i} \mapsto x_i \qquad \text{for every } i \geq 1;$$
$$a_i \mapsto i\frac{\partial}{\partial x_i} \qquad \text{for every } i \geq 1,$$
$$K \mapsto 1$$

(where "$a_{-i} \mapsto x_i$" is just shorthand for "$a_{-i} \mapsto$ (multiplication by x_i)"). For every $\mu \in \mathbb{C}$, we can upgrade F to a representation F_μ of $\mathcal{A}$ by adding the condition that $a_0 \mid_{F_\mu} = \mu \cdot \mathrm{id}$.

Definition 2.2.5. The representation F of $\mathcal{A}_0$ introduced in Corollary 2.2.4 is called the *Fock module* or the *Fock representation*. For every $\mu \in \mathbb{C}$, the representation F_μ of $\mathcal{A}$ introduced in Corollary 2.2.4 will be called the *μ-Fock representation* of $\mathcal{A}$. The vector space F itself is called the *Fock space*.

Let us now define some gradings to make these infinite-dimensional spaces more manageable:

Definition 2.2.6. Let us grade the vector space $\mathcal{A}$ by $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}[n]$, where $\mathcal{A}[n] = \langle a_n \rangle$ for $n \neq 0$, and where $\mathcal{A}[0] = \langle a_0, K \rangle$. With this grading, we have $[\mathcal{A}[n], \mathcal{A}[m]] \subseteq \mathcal{A}[n+m]$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. (In other words, the Lie algebra $\mathcal{A}$ with the decomposition $\mathcal{A} = \bigoplus_{n\in\mathbb{Z}} \mathcal{A}[n]$ is a $\mathbb{Z}$-graded Lie algebra. The notion of a "$\mathbb{Z}$-graded Lie algebra" that we have just used is defined in Definition 2.5.1.)

Note that we are denoting the n-th homogeneous component of $\mathcal{A}$ by $\mathcal{A}[n]$ rather than $\mathcal{A}_n$, since otherwise the notation $\mathcal{A}_0$ would have two different meanings.

Definition 2.2.7. We grade the polynomial algebra F by setting $\deg(x_i) = -i$ for each i. Thus, $F = \bigoplus_{n\geq 0} F[-n]$, where $F[-n]$ is the space of polynomials of degree $-n$, where the degree is our degree defined by $\deg(x_i) = -i$ (so that, for instance, $x_1^2 + x_2$ is homogeneous of degree -2). With this grading, $\dim(F[-n])$ is the number $p(n)$ of all partitions of n. Hence,

$$\sum_{n\geq 0} \dim(F[-n])\, q^n = \sum_{n\geq 0} p(n)\, q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} = \frac{1}{\prod_{i\geq 1}(1-q^i)}$$

in the ring of power series $\mathbb{Z}[[q]]$.

We use the same grading for F_μ for every $\mu \in \mathbb{C}$. That is, we define the grading on F_μ by $F_\mu[n] = F[n]$ for every $n \in \mathbb{Z}$.

Remark 2.2.8. Some people prefer to grade F_μ somewhat differently from F: namely, they shift the grading for F_μ by $\frac{\mu^2}{2}$, so that $\deg 1 = -\frac{\mu^2}{2}$ in F_μ, and generally $F_\mu[z] = F\left[\frac{\mu^2}{2} + z\right]$ (as vector spaces) for every $z \in \mathbb{C}$. This is a grading by complex numbers rather than integers (in general). (The advantage of this grading is that we will eventually find an operator whose eigenspace to the eigenvalue n is $F_\mu[n] = F\left[\frac{\mu^2}{2} + n\right]$ for every $n \in \mathbb{C}$.)

With this grading, the equality $\sum_{n\geq 0} \dim(F[-n])\, q^n = \frac{1}{\prod_{i\geq 1}(1-q^i)}$ rewrites as $\sum_{n\in\mathbb{C}} \dim(F_\mu[-n])\, q^{n+\frac{\mu^2}{2}} = \frac{q^{\mu^2}}{\prod_{i\geq 1}(1-q^i)}$, if we allow power series with complex exponents. We define a "power series" $\operatorname{ch}(F_\mu)$ by

$$\operatorname{ch}(F_\mu) = \sum_{n\in\mathbb{C}} \dim(F_\mu[-n])\, q^{n+\dfrac{\mu^2}{2}} = \frac{q^{\mu^2}}{\prod\limits_{i\geq 1}(1-q^i)}.$$

But we will not use this grading; instead we will use the grading defined in Definition 2.2.7.

Proposition 2.2.9. The representation F is an irreducible representation of $\mathcal{A}_0$.

Lemma 2.2.10. For every $P \in F$, we have

$$P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot 1 = P \qquad \text{in } F.$$

(Here, the term $P(a_{-1}, a_{-2}, a_{-3}, ...)$ denotes the evaluation of the polynomial P at $(x_1, x_2, x_3, ...) = (a_{-1}, a_{-2}, a_{-3}, ...)$. This evaluation is a well-defined element of $U(\mathcal{A}_0)$, since the elements a_{-1}, a_{-2}, a_{-3}, ... of $U(\mathcal{A}_0)$ commute.)

Proof of Lemma 2.2.10. For every $Q \in F$, let $\operatorname{mult} Q$ denote the map $F \to F$, $R \mapsto QR$. (In Proposition 2.2.1, we abused notations and denoted this map simply by Q; but we will not do this in this proof.) Then, by the definition of ξ, we have $\xi(a_{-i}) = \operatorname{mult}(x_i)$ for every $i \geq 1$.

Since we have defined an endomorphism $\operatorname{mult} Q \in \operatorname{End} F$ for every $Q \in F$, we thus obtain a map $\operatorname{mult} : F \to \operatorname{End} F$. This map mult is an algebra homomorphism (since it describes the action of F on the F-module F).

Let $P \in F$. Since ξ is an algebra homomorphism, and thus commutes with polynomials, we have

$$\begin{aligned}
&\xi(P(a_{-1}, a_{-2}, a_{-3}, ...)) \\
&= P(\xi(a_{-1}), \xi(a_{-2}), \xi(a_{-3}), ...) = P(\operatorname{mult}(x_1), \operatorname{mult}(x_2), \operatorname{mult}(x_3), ...) \\
&\qquad\qquad (\text{since } \xi(a_{-i}) = \operatorname{mult}(x_i) \text{ for every } i \geq 1) \\
&= \operatorname{mult}\left(\underbrace{P(x_1, x_2, x_3, ...)}_{=P}\right) \qquad \left(\begin{array}{c}\text{since mult is an algebra homomorphism,} \\ \text{and thus commutes with polynomials}\end{array}\right) \\
&= \operatorname{mult} P.
\end{aligned}$$

Thus,

$$P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot 1 = (\operatorname{mult} P)(1) = P \cdot 1 = P.$$

This proves Lemma 2.2.10.

Proof of Proposition 2.2.9. **1)** The representation F is generated by 1 as a $U(\mathcal{A}_0)$-module (due to Lemma 2.2.10). In other words, $F = U(\mathcal{A}_0) \cdot 1$.

2) Let us forget about the grading on F which we defined in Definition 2.2.7, and instead, once again, define a grading on F by $\deg(x_i) = 1$ for every $i \in \{1, 2, 3, ...\}$. Thus, the degree of a polynomial $P \in F$ with respect to this grading is what is usually referred to as the degree of the polynomial P.

If $P \in F$ and if $\alpha \cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} ...$ is a monomial in P of degree $\deg P$, with $\alpha \neq 0$, then $\dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} \dfrac{\partial_{x_3}^{m_3}}{m_3!} ... P = \alpha$ [16].

Thus, for every nonzero $P \in F$, we have $1 \in U(\mathcal{A}_0) \cdot P$ [17]. Combined with **1)**, this yields that for every nonzero $P \in F$, the representation F is generated by P as a $U(\mathcal{A}_0)$-module (since $F = U(\mathcal{A}_0) \cdot \underbrace{1}_{\in U(\mathcal{A}_0) \cdot P} \subseteq U(\mathcal{A}_0) \cdot U(\mathcal{A}_0) \cdot P = U(\mathcal{A}_0) \cdot P$). Consequently, F is irreducible. Proposition 2.2.9 is proven.

Proposition 2.2.11. Let V be an irreducible $\mathcal{A}_0$-module on which K acts as 1. Assume that for any $v \in V$, the space $\mathbb{C}[a_1, a_2, a_3, ...] \cdot v$ is finite-dimensional, and the a_i with $i > 0$ act on it by nilpotent operators. Then, $V \cong F$ as $\mathcal{A}_0$-modules.

Before we prove this, a simple lemma:

[16] *Proof.* Let $P \in F$. Let $\alpha \cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} ...$ be a monomial in P of degree $\deg P$, with $\alpha \neq 0$.

WLOG, no variable other than x_1, x_2, ..., x_k appears in P, for some $k \in \mathbb{N}$. Thus, $x_1^{m_1} x_2^{m_2} x_3^{m_3} ... = x_1^{m_1} x_2^{m_2} ... x_k^{m_k}$ and $\dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} \dfrac{\partial_{x_3}^{m_3}}{m_3!} ... = \dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} ... \dfrac{\partial_{x_k}^{m_k}}{m_k!}$.

Thus, $\alpha \cdot x_1^{m_1} x_2^{m_2} ... x_k^{m_k} = \alpha \cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} ...$ is a monomial in P of degree $\deg P$.

When we apply the differential operator $\dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} ... \dfrac{\partial_{x_k}^{m_k}}{m_k!}$ to P, all monomials $\beta \cdot x_1^{n_1} x_2^{n_2} ... x_k^{n_k}$ with ($n_\ell < m_\ell$ for at least one $\ell \in \{1, 2, ..., k\}$) are annihilated (because if $n_\ell < m_\ell$ for some ℓ, then $\dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} ... \dfrac{\partial_{x_k}^{m_k}}{m_k!} (\beta \cdot x_1^{n_1} x_2^{n_2} ... x_k^{n_k}) = 0$). Hence, the only monomials in P which survive under this operator are monomials of the form $\beta \cdot x_1^{n_1} x_2^{n_2} ... x_k^{n_k}$ with each n_ℓ being $\geq$ to the corresponding m_ℓ. But since $m_1 + m_2 + ... + m_k = \deg P$ (because $\alpha \cdot x_1^{m_1} x_2^{m_2} ... x_k^{m_k}$ is a monomial of degree $\deg P$), the only such monomial in P is $\alpha \cdot x_1^{m_1} x_2^{m_2} ... x_k^{m_k}$ (because for every other monomial of the form $\beta \cdot x_1^{n_1} x_2^{n_2} ... x_k^{n_k}$ with each n_ℓ being $\geq$ to the corresponding m_ℓ, the sum $n_1 + n_2 + ... + n_k$ must be greater than $m_1 + m_2 + ... + m_k = \deg P$, and thus such a monomial cannot occur in P). Hence, the only monomial in P which survives is the monomial $\alpha \cdot x_1^{m_1} x_2^{m_2} ... x_k^{m_k}$. This monomial clearly gets mapped to α by the differential operator $\dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} ... \dfrac{\partial_{x_k}^{m_k}}{m_k!}$. Thus, $\dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} ... \dfrac{\partial_{x_k}^{m_k}}{m_k!} P = \alpha$. Since $\dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} ... \dfrac{\partial_{x_k}^{m_k}}{m_k!} = \dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} \dfrac{\partial_{x_3}^{m_3}}{m_3!} ...$, this rewrites as $\dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} \dfrac{\partial_{x_3}^{m_3}}{m_3!} ... P = \alpha$, qed.

[17] *Proof.* Let $P \in F$ be nonzero. Then, there exist a monomial $\alpha \cdot x_1^{m_1} x_2^{m_2} x_3^{m_3} ...$ in P of degree P with $\alpha \neq 0$. Consider such a monomial. As shown above, we have $\dfrac{\partial_{x_1}^{m_1}}{m_1!} \dfrac{\partial_{x_2}^{m_2}}{m_2!} \dfrac{\partial_{x_3}^{m_3}}{m_3!} ... P = \alpha$. But we know that $a_i \in \mathcal{A}_0$ acts as $i \dfrac{\partial}{\partial x_i}$ on F for every $i \geq 1$. Thus, $\dfrac{1}{i} a_i \in \mathcal{A}_0$ acts as $\dfrac{\partial}{\partial x_i} = \partial_{x_i}$ on F for every $i \geq 1$. Hence,

$$\frac{\left(\frac{1}{1} a_1\right)^{m_1}}{m_1!} \frac{\left(\frac{1}{2} a_2\right)^{m_2}}{m_2!} \frac{\left(\frac{1}{3} a_3\right)^{m_3}}{m_3!} ... P = \frac{\partial_{x_1}^{m_1}}{m_1!} \frac{\partial_{x_2}^{m_2}}{m_2!} \frac{\partial_{x_3}^{m_3}}{m_3!} ... P = \alpha.$$

Consequently,

$$\alpha = \frac{\left(\frac{1}{1} a_1\right)^{m_1}}{m_1!} \frac{\left(\frac{1}{2} a_2\right)^{m_2}}{m_2!} \frac{\left(\frac{1}{3} a_3\right)^{m_3}}{m_3!} ... P \in U(\mathcal{A}_0) \cdot P.$$

Since $\alpha \neq 0$, we can divide this relation by α, and obtain $1 \in \dfrac{1}{\alpha} \cdot U(\mathcal{A}_0) \cdot P \subseteq U(\mathcal{A}_0) \cdot P$, qed.

Lemma 2.2.12. Let V be an $\mathcal{A}_0$-module. Let $u \in V$ be such that $a_i u = 0$ for all $i > 0$, and such that $Ku = u$. Then, there exists a homomorphism $\eta : F \to V$ of $\mathcal{A}_0$-modules such that $\eta(1) = u$. (This homomorphism η is unique, although we won't need this.)

We give two proofs of this lemma. The first one is conceptual and gives us a glimpse into the more general theory (it proceeds by constructing an $\mathcal{A}_0$-module $\operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C}$, which is an example of what we will later call a Verma highest-weight module in Definition 2.5.14). The second one is down-to-earth and proceeds by direct construction and computation.

First proof of Lemma 2.2.12. Define a vector subspace $\mathcal{A}_0^+$ of $\mathcal{A}_0$ by $\mathcal{A}_0^+ = \langle a_i \mid i \text{ positive integer}\rangle$. It is clear that the internal direct sum $\mathbb{C}K \oplus \mathcal{A}_0^+$ is well-defined and an abelian Lie subalgebra of $\mathcal{A}_0$. We can make $\mathbb{C}$ into an $(\mathbb{C}K \oplus \mathcal{A}_0^+)$-module by setting

$$K\lambda = \lambda \qquad \text{for every } \lambda \in \mathbb{C};$$
$$a_i\lambda = 0 \qquad \text{for every } \lambda \in \mathbb{C} \text{ and every positive integer } i.$$

Now, consider the $\mathcal{A}_0$-module $\operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C} = U(\mathcal{A}_0) \otimes_{U(\mathbb{C}K \oplus \mathcal{A}_0^+)} \mathbb{C}$. Denote the element $1 \otimes_{U(\mathbb{C}K \oplus \mathcal{A}_0^+)} 1 \in U(\mathcal{A}_0) \otimes_{U(\mathbb{C}K \oplus \mathcal{A}_0^+)} \mathbb{C}$ of this module by 1.

We will now show the following important property of this module:

$$\left(\begin{array}{c} \text{For any } \mathcal{A}_0\text{-module } T, \text{ and any } t \in T \text{ satisfying } (a_i t = 0 \text{ for all } i > 0) \text{ and } Kt = t, \\ \text{there exists a homomorphism } \overline{\eta}_{T,t} : \operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C} \to T \text{ of } \mathcal{A}_0\text{-modules such that } \overline{\eta}_{T,t}(1) = t \end{array} \right). \tag{14}$$

Once this is proven, we will (by considering $\overline{\eta}_{F,1}$) show that $\operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}} \mathbb{C} \cong F$, so this property will translate into the assertion of Lemma 2.2.12.

Proof of (14). Let $\tau : \mathbb{C} \to T$ be the map which sends every $\lambda \in \mathbb{C}$ to $\lambda t \in T$. Then, τ is $\mathbb{C}$-linear and satisfies

$$\tau \underbrace{(K\lambda)}_{=\lambda} = \tau(\lambda) = \lambda \underbrace{t}_{=Kt} = \lambda \cdot Kt = K \cdot \underbrace{\lambda t}_{=\tau(\lambda)} = K \cdot \tau(\lambda) \qquad \text{for every } \lambda \in \mathbb{C}$$

and

$$\tau \underbrace{(a_i\lambda)}_{=0} = \tau(0) = 0 = \lambda \cdot \underbrace{0}_{=a_i t} = \lambda \cdot a_i t = a_i \cdot \underbrace{\lambda t}_{=\tau(\lambda)} = a_i \tau(\lambda)$$
$$\text{for every } \lambda \in \mathbb{C} \text{ and every positive integer } i.$$

Thus, τ is a $(\mathbb{C}K \oplus \mathcal{A}_0^+)$-module map. In other words, $\tau \in \operatorname{Hom}_{\mathbb{C}K \oplus \mathcal{A}_0^+}\left(\mathbb{C}, \operatorname{Res}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} T\right)$.

By Frobenius reciprocity, we have

$$\operatorname{Hom}_{\mathcal{A}_0}\left(\operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C}, T\right) \cong \operatorname{Hom}_{\mathbb{C}K \oplus \mathcal{A}_0^+}\left(\mathbb{C}, \operatorname{Res}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} T\right).$$

The preimage of $\tau \in \operatorname{Hom}_{\mathbb{C}K \oplus \mathcal{A}_0^+}\left(\mathbb{C}, \operatorname{Res}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} T\right)$ under this isomorphism is an $\mathcal{A}_0$-module map $\overline{\eta}_{T,t} : \operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C} \to T$ such that

$$\begin{aligned} \overline{\eta}_{T,t} \underbrace{(1)}_{=1 \otimes_{U(\mathbb{C}K \oplus \mathcal{A}_0^+)} 1} &= \overline{\eta}_{T,t}\left(1 \otimes_{U(\mathbb{C}K \oplus \mathcal{A}_0^+)} 1\right) = 1 \underbrace{\tau(1)}_{=1t=t} \qquad \text{(by the proof of Frobenius reciprocity)} \\ &= 1t = t. \end{aligned}$$

Hence, there exists a homomorphism $\overline{\eta}_{T,t} : \operatorname{Ind}_{\mathbb{C}K\oplus\mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C} \to T$ of $\mathcal{A}_0$-modules such that $\overline{\eta}_{T,t}(1) = t$. This proves (14).

It is easy to see that the element $1 \in F$ satisfies ($a_i 1 = 0$ for all $i > 0$) and $K1 = 1$. Thus, (14) (applied to $T = F$ and $t = 1$) yields that there exists a homomorphism $\overline{\eta}_{F,1} : \operatorname{Ind}_{\mathbb{C}K\oplus\mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C} \to F$ of $\mathcal{A}_0$-modules such that $\overline{\eta}_{F,1}(1) = 1$. This homomorphism $\overline{\eta}_{F,1}$ is surjective, since

$$\begin{aligned} F &= U(\mathcal{A}_0) \cdot \underbrace{1}_{=\overline{\eta}_{F,1}(1)} && \text{(as proven in the proof of Proposition 2.2.9)} \\ &= U(\mathcal{A}_0) \cdot \overline{\eta}_{F,1}(1) = \overline{\eta}_{F,1}(U(\mathcal{A}_0) \cdot 1) && \big(\text{since } \overline{\eta}_{F,1} \text{ is an } \mathcal{A}_0\text{-module map}\big) \\ &\subseteq \operatorname{Im} \overline{\eta}_{F,1}. \end{aligned}$$

Now we will prove that this homomorphism $\overline{\eta}_{F,1}$ is injective.

In the following, a map $\varphi : A \to \mathbb{N}$ (where A is any set) is said to be *finitely supported* if all but finitely many $a \in A$ satisfy $\varphi(a) = 0$. Sequences (finite, infinite, or two-sided infinite) are considered as maps (from finite sets, $\mathbb{N}$ or $\mathbb{Z}$, or occasionally other sets). Thus, a sequence is finitely supported if and only if all but finitely many of its elements are zero.

If A is a set, then $\mathbb{N}_{\text{fin}}^{A}$ will denote the set of all finitely supported maps $A \to \mathbb{N}$.

By the easy part of the Poincaré-Birkhoff-Witt theorem (this is the part which states that the increasing monomials *span* the universal enveloping algebra), the family[18]

$$\left(\overrightarrow{\prod_{i\in\mathbb{Z}\setminus\{0\}}} a_i^{n_i} \cdot K^m \right)_{(\ldots,n_{-2},n_{-1},n_1,n_2,\ldots)\in\mathbb{N}_{\text{fin}}^{\mathbb{Z}\setminus\{0\}},\ m\in\mathbb{N}}$$

is a spanning set of the vector space $U(\mathcal{A}_0)$.

Hence, the family

$$\left(\left(\overrightarrow{\prod_{i\in\mathbb{Z}\setminus\{0\}}} a_i^{n_i} \cdot K^m \right) \otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} 1 \right)_{(\ldots,n_{-2},n_{-1},n_1,n_2,\ldots)\in\mathbb{N}_{\text{fin}}^{\mathbb{Z}\setminus\{0\}},\ m\in\mathbb{N}}$$

is a spanning set of the vector space $U(\mathcal{A}_0) \otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} \mathbb{C} = \operatorname{Ind}_{\mathbb{C}K\oplus\mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C}$.

Let us first notice that this family is redundant: Each of its elements is contained in the smaller family

$$\left(\left(\overrightarrow{\prod_{i\in\mathbb{Z}\setminus\{0\}}} a_i^{n_i} \right) \otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} 1 \right)_{(\ldots,n_{-2},n_{-1},n_1,n_2,\ldots)\in\mathbb{N}_{\text{fin}}^{\mathbb{Z}\setminus\{0\}}}.$$

[18] Here, $\overrightarrow{\prod\limits_{i\in\mathbb{Z}\setminus\{0\}}} a_i^{n_i}$ denotes the product $\ldots a_{-2}^{n_{-2}} a_{-1}^{n_{-1}} a_1^{n_1} a_2^{n_2} \ldots$. (This product is infinite, but still has a value since only finitely many n_i are nonzero.)

[19] Hence, this smaller family is also a spanning set of the vector space $\operatorname{Ind}_{\mathbb{C}K\oplus\mathcal{A}_0^+}^{\mathcal{A}_0}\mathbb{C}$.

This smaller family is still redundant: Every of its elements corresponding to a sequence $(..., n_{-2}, n_{-1}, n_1, n_2, ...) \in \mathbb{N}_{\text{fin}}^{\mathbb{Z}\diagdown\{0\}}$ satisfying $n_1+n_2+n_3+... > 0$ is zero[20], and zero elements in a spanning set are automatically redundant. Hence, we can replace

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[19]This is because any sequence $(..., n_{-2}, n_{-1}, n_1, n_2, ...) \in \mathbb{N}_{\text{fin}}^{\mathbb{Z}\diagdown\{0\}}$ and any $m \in \mathbb{N}$ satisfy

$$\begin{aligned}
&\left(\overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}}} a_i^{n_i}\cdot K^m\right)\otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} 1\\
&=\left(\overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}}} a_i^{n_i}\right)\otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} \underbrace{(K^m 1)}_{\substack{=1\\ \text{(by repeated application of } K1=1)}} \qquad \left(\text{since } K^m\in U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)\right)\\
&=\left(\overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}}} a_i^{n_i}\right)\otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} 1.
\end{aligned}$$

[20]*Proof.* Let $(..., n_{-2}, n_{-1}, n_1, n_2, ...) \in \mathbb{N}_{\text{fin}}^{\mathbb{Z}\diagdown\{0\}}$ be a sequence satisfying $n_1+n_2+n_3+... > 0$. Then, the sequence $(..., n_{-2}, n_{-1}, n_1, n_2, ...)$ is finitely supported (as it is an element of $\in \mathbb{N}_{\text{fin}}^{\mathbb{Z}\diagdown\{0\}}$), so that only finitely many n_i are nonzero.

There exists some positive integer ℓ satisfying $n_\ell > 0$ (since $n_1+n_2+n_3+... > 0$). Let j be the greatest such ℓ (this is well-defined, since only finitely many n_i are nonzero).

Since j is the greatest positive integer ℓ satisfying $n_\ell > 0$, it is clear that j is the greatest integer ℓ satisfying $n_\ell > 0$. In other words, $a_j^{n_j}$ is the rightmost factor in the product $\overrightarrow{\prod_{i\in\mathbb{Z}}} a_i^{n_i}$ which is not equal to 1. Thus,

$$\overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}}} a_i^{n_i} = \overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}\diagdown\{j\}}} a_i^{n_i}\cdot \underbrace{a_j^{n_j}}_{\substack{=a_j^{n_j-1}a_j\\ \text{(since } n_j>0)}} = \overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}\diagdown\{j\}}} a_i^{n_i}\cdot a_j^{n_j-1}a_j,$$

so that

$$\begin{aligned}
\left(\overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}}} a_i^{n_i}\right)\otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} 1 &= \left(\overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}\diagdown\{j\}}} a_i^{n_i}\cdot a_j^{n_j-1}a_j\right)\otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} 1\\
&= \overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}\diagdown\{j\}}} a_i^{n_i}\cdot a_j^{n_j-1}\otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} \underbrace{a_j 1}_{\substack{=0\\ \text{(since } j>0\text{, so that}\\ a_j1=j\frac{\partial}{\partial x_j}1=0)}}\\
&\qquad \left(\text{since } a_j\in U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)\right)\\
&= 0.
\end{aligned}$$

We have thus proven that every sequence $(..., n_{-2}, n_{-1}, n_1, n_2, ...) \in \mathbb{N}_{\text{fin}}^{\mathbb{Z}\diagdown\{0\}}$ satisfying $n_1+n_2+n_3+... > 0$ satisfies $\left(\overrightarrow{\prod_{i\in\mathbb{Z}\diagdown\{0\}}} a_i^{n_i}\right)\otimes_{U(\mathbb{C}K\oplus\mathcal{A}_0^+)} 1 = 0$, qed.

this smaller family by the even smaller family

$$\left(\left(\prod_{i\in\mathbb{Z}\diagdown\{0\}}^{\rightarrow} a_i^{n_i}\right)\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)} 1\right)_{(\ldots,n_{-2},n_{-1},n_1,n_2,\ldots)\in\mathbb{N}_{\text{fin}}^{\mathbb{Z}\diagdown\{0\}};\text{ we do \textit{not} have } n_1+n_2+n_3+\ldots>0}$$
$$=\left(\left(\prod_{i\in\mathbb{Z}\diagdown\{0\}}^{\rightarrow} a_i^{n_i}\right)\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)} 1\right)_{(\ldots,n_{-2},n_{-1},n_1,n_2,\ldots)\in\mathbb{N}_{\text{fin}}^{\mathbb{Z}\diagdown\{0\}};\ n_1=n_2=n_3=\ldots=0}$$
$$\left(\begin{array}{c}\text{since the condition (we do \textit{not} have } n_1+n_2+n_3+\ldots>0\text{)}\\ \text{is equivalent to the condition } (n_1=n_2=n_3=\ldots=0)\\ \text{(because } n_i\in\mathbb{N}\text{ for all } i\in\mathbb{Z}\diagdown\{0\}\text{)}\end{array}\right),$$

and we still have a spanning set of the vector space $\operatorname{Ind}_{\mathbb{C}K\oplus\mathcal{A}_0^+}^{\mathcal{A}_0}\mathbb{C}$.

Clearly, sequences $(\ldots,n_{-2},n_{-1},n_1,n_2,\ldots)\in\mathbb{N}_{\text{fin}}^{\mathbb{Z}\diagdown\{0\}}$ satisfying $n_1=n_2=n_3=\ldots=0$ are in 1-to-1 correspondence with sequences $(\ldots,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{\ldots,-3,-2,-1\}}$. Hence, we can reindex the above family as follows:

$$\left(\left(\prod_{i\in\{\ldots,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)} 1\right)_{(\ldots,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{\ldots,-3,-2,-1\}}}.$$

So we have proven that the family

$$\left(\left(\prod_{i\in\{\ldots,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)} 1\right)_{(\ldots,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{\ldots,-3,-2,-1\}}}$$

is a spanning set of the vector space $\operatorname{Ind}_{\mathbb{C}K\oplus\mathcal{A}_0^+}^{\mathcal{A}_0}\mathbb{C}$. But the map $\overline{\eta}_{F,1}$ sends this family to

$$\left(\overline{\eta}_{F,1}\left(\left(\prod_{i\in\{\ldots,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)} 1\right)\right)_{(\ldots,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{\ldots,-3,-2,-1\}}}$$
$$=\left(\prod_{i\in\{\ldots,-3,-2,-1\}}^{\rightarrow} x_{-i}^{n_i}\right)_{(\ldots,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{\ldots,-3,-2,-1\}}}$$

[21]. Since the family $\left(\prod\limits_{i\in\{...,-3,-2,-1\}}^{\rightarrow} x_{-i}^{n_i}\right)_{(...,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{...,-3,-2,-1\}}}$ is a basis of the vector space F (in fact, this family consists of all monomials of the polynomial ring $\mathbb{C}[x_1,x_2,x_3,...]=F$), we thus conclude that $\overline{\eta}_{F,1}$ sends a spanning family of the vector space $\operatorname{Ind}_{\mathbb{C}K\oplus\mathcal{A}_0^+}^{\mathcal{A}_0}\mathbb{C}$ to a basis of the vector space F. Thus, $\overline{\eta}_{F,1}$ must be injective[22].

Altogether, we now know that $\overline{\eta}_{F,1}$ is a surjective and injective $\mathcal{A}_0$-module map. Thus, $\overline{\eta}_{F,1}$ is an isomorphism of $\mathcal{A}_0$-modules.

Now, apply (14) to $T=V$ and $t=u$. This yields that there exists a homomorphism $\overline{\eta}_{V,u}:\operatorname{Ind}_{\mathbb{C}K\oplus\mathcal{A}_0^+}^{\mathcal{A}_0}\mathbb{C}\to V$ of $\mathcal{A}_0$-modules such that $\overline{\eta}_{V,u}(1)=u$.

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[21] *Proof.* Let $(...,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{...,-3,-2,-1\}}$ be arbitrary. Then,

$$\overline{\eta}_{F,1}\left(\underbrace{\left(\prod_{i\in\{...,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)}1}_{=\left(\prod\limits_{i\in\{...,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\left(1\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)}1\right)}\right)$$
$$=\overline{\eta}_{F,1}\left(\left(\prod_{i\in\{...,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\left(1\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)}1\right)\right)$$
$$=\left(\prod_{i\in\{...,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\overline{\eta}_{F,1}\underbrace{\left(1\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)}1\right)}_{=1}\qquad\left(\text{since }\overline{\eta}_{F,1}\text{ is an }\mathcal{A}_0\text{-module map}\right)$$
$$=\left(\prod_{i\in\{...,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\underbrace{\overline{\eta}_{F,1}(1)}_{=1}=\left(\prod_{i\in\{...,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)1=\left(\prod_{i\in\{...,-3,-2,-1\}}^{\rightarrow} x_{-i}^{n_i}\right)1$$
$$\left(\text{because each }a_i\text{ with negative }i\text{ acts on }F\text{ by multiplication with }x_{-i}\right)$$
$$=\prod_{i\in\{...,-3,-2,-1\}}^{\rightarrow} x_{-i}^{n_i}=\prod_{i\in\{...,-3,-2,-1\}} x_{-i}^{n_i}\qquad\left(\text{since }F\text{ is commutative}\right).$$

Now forget that we fixed $(...,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{...,-3,-2,-1\}}$. We thus have shown that every $(...,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{...,-3,-2,-1\}}$ satisfies $\overline{\eta}_{F,1}\left(\left(\prod\limits_{i\in\{...,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)}1\right)=\prod\limits_{i\in\{...,-3,-2,-1\}} x_{-i}^{n_i}$. Thus,

$$\left(\overline{\eta}_{F,1}\left(\left(\prod_{i\in\{...,-3,-2,-1\}}^{\rightarrow} a_i^{n_i}\right)\otimes_{U\left(\mathbb{C}K\oplus\mathcal{A}_0^+\right)}1\right)\right)_{(...,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{...,-3,-2,-1\}}}$$
$$=\left(\prod_{i\in\{...,-3,-2,-1\}}^{\rightarrow} x_{-i}^{n_i}\right)_{(...,n_{-2},n_{-1})\in\mathbb{N}_{\text{fin}}^{\{...,-3,-2,-1\}}},$$

qed.

[22] Here we are using the following trivial fact from linear algebra: If a linear map $\varphi:V\to W$ sends a spanning family of the vector space V to a basis of the vector space W (as families, not just as sets), then this map φ must be injective.

Now, the composition $\overline{\eta}_{V,u} \circ \overline{\eta}_{F,1}^{-1}$ is a homomorphism $F \to V$ of $\mathcal{A}_0$-modules such that

$$\left(\overline{\eta}_{V,u} \circ \overline{\eta}_{F,1}^{-1}\right)(1) = \overline{\eta}_{V,u} \underbrace{\left(\overline{\eta}_{F,1}^{-1}(1)\right)}_{\substack{=1 \\ (\text{since } \overline{\eta}_{F,1}(1)=1)}} = \overline{\eta}_{V,u}(1) = u.$$

Thus, there exists a homomorphism $\eta : F \to V$ of $\mathcal{A}_0$-modules such that $\eta(1) = u$ (namely, $\eta = \overline{\eta}_{V,u} \circ \overline{\eta}_{F,1}^{-1}$). This proves Lemma 2.2.12.

Second proof of Lemma 2.2.12. Let η be the map $F \to V$ which sends every polynomial $P \in F = \mathbb{C}[x_1, x_2, x_3, ...]$ to $P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot u \in V$. [23] This map η is clearly $\mathbb{C}$-linear, and satisfies $\eta(F) \subseteq U(\mathcal{A}_0) \cdot u$. In order to prove that η is an $\mathcal{A}_0$-module homomorphism, we must prove that

$$\eta(a_i P) = a_i \eta(P) \qquad \text{for every } i \in \mathbb{Z} \diagdown \{0\} \text{ and } P \in F \tag{15}$$

and that

$$\eta(KP) = K\eta(P) \qquad \text{for every } P \in F. \tag{16}$$

First we show that

$$Kv = v \qquad \text{for every } v \in U(\mathcal{A}_0) \cdot u. \tag{17}$$

Proof of (17). Since K lies in the center of the Lie algebra $\mathcal{A}_0$, it is clear that K lies in the center of the universal enveloping algebra $U(\mathcal{A}_0)$. Thus, $Kx = xK$ for every $x \in U(\mathcal{A}_0)$.

Now let $v \in U(\mathcal{A}_0) \cdot u$. Then, there exists some $x \in U(\mathcal{A}_0)$ such that $v = xu$. Thus, $Kv = Kxu = x \underbrace{Ku}_{=u} = xu = v$. This proves (17).

Proof of (16). Since K acts as the identity on F, we have $KP = P$ for every $P \in F$. Thus, for every $P \in F$, we have

$$\eta(KP) = \eta(P) = K\eta(P) \qquad \left(\begin{array}{c}\text{since (17) (applied to } v = \eta(P)\text{) yields } K\eta(P) = \eta(P) \\ \text{(because } \eta(P) \in \eta(F) \subseteq U(\mathcal{A}_0) \cdot u\text{)}\end{array}\right).$$

This proves (16).

Proof of (15). Let $i \in \mathbb{Z} \diagdown \{0\}$. If $i < 0$, then (15) is pretty much obvious (because in this case, a_i acts as x_{-i} on F, so that $a_i P = x_{-i} P$ and thus

$$\eta(a_i P) = \eta(x_{-i} P) = (x_{-i} P)(a_{-1}, a_{-2}, a_{-3}, ...) \cdot u = a_i \underbrace{P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot u}_{=\eta(P)} = a_i \eta(P)$$

for every $P \in F$). Hence, from now on, we can WLOG assume that i is not < 0. Assume this. Then, $i \geq 0$, so that $i > 0$ (since $i \in \mathbb{Z} \diagdown \{0\}$).

In order to prove the equality (15) for all $P \in F$, it is enough to prove it for the case when P is a monomial of the form $x_{\ell_1} x_{\ell_2} ... x_{\ell_m}$ for some $m \in \mathbb{N}$ and some

[23] Note that the term $P(a_{-1}, a_{-2}, a_{-3}, ...)$ denotes the evaluation of the polynomial P at $(x_1, x_2, x_3, ...) = (a_{-1}, a_{-2}, a_{-3}, ...)$. This evaluation is a well-defined element of $U(\mathcal{A}_0)$, since the elements a_{-1}, a_{-2}, a_{-3}, ... of $U(\mathcal{A}_0)$ commute.

$(\ell_1, \ell_2, ..., \ell_m) \in \{1, 2, 3, ...\}^m$. [24] In other words, in order to prove the equality (15), it is enough to prove that

$$\eta\left(a_i\left(x_{\ell_1}x_{\ell_2}...x_{\ell_m}\right)\right) = a_i\eta\left(x_{\ell_1}x_{\ell_2}...x_{\ell_m}\right) \qquad \text{for every } m \in \mathbb{N} \text{ and every } (\ell_1, \ell_2, ..., \ell_m) \in \{1, 2, 3, ...\}^m . \tag{18}$$

Thus, let us now prove (18). In fact, we are going to prove (18) by induction over m. The induction base is very easy (using $a_i 1 = i\frac{\partial}{\partial x_i}1 = 0$ and $a_i u = 0$) and thus left to the reader. For the induction step, fix some positive $M \in \mathbb{N}$, and assume that (18) is already proven for $m = M - 1$. Our task is now to prove (18) for $m = M$.

So let $(\ell_1, \ell_2, ..., \ell_M) \in \{1, 2, 3, ...\}^M$ be arbitrary. Denote by Q the polynomial $x_{\ell_2}x_{\ell_3}...x_{\ell_M}$. Then, $x_{\ell_1}Q = x_{\ell_1}x_{\ell_2}x_{\ell_3}...x_{\ell_M} = x_{\ell_1}x_{\ell_2}...x_{\ell_M}$.

Since (18) is already proven for $m = M - 1$, we can apply (18) to $M - 1$ and $(\ell_2, \ell_3, ..., \ell_M)$ instead of m and $(\ell_1, \ell_2, ..., \ell_m)$. We obtain $\eta\left(a_i\left(x_{\ell_2}x_{\ell_3}...x_{\ell_M}\right)\right) = a_i\eta$ $\left(x_{\ell_2}x_{\ell_3}...x_{\ell_M}\right)$. Since $x_{\ell_2}x_{\ell_3}...x_{\ell_M} = Q$, this rewrites as $\eta\left(a_iQ\right) = a_i\eta\left(Q\right)$.

Since any $x \in \mathcal{A}_0$ and $y \in \mathcal{A}_0$ satisfy $xy = yx + [x, y]$ (by the definition of $U(\mathcal{A}_0)$), we have

$$a_i a_{-\ell_1} = a_{-\ell_1}a_i + \underbrace{[a_i, a_{-\ell_1}]}_{=i\delta_{i,-(-\ell_1)}K} = a_{-\ell_1}a_i + i\underbrace{\delta_{i,-(-\ell_1)}}_{=\delta_{i,\ell_1}}K = a_{-\ell_1}a_i + i\delta_{i,\ell_1}K.$$

On the other hand, by the definition of η, every $P \in F$ satisfies the two equalities $\eta(P) = P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot u$ and

$$\begin{aligned}\eta\left(x_{\ell_1}P\right) &= \underbrace{\left(x_{\ell_1}P\right)\left(a_{-1}, a_{-2}, a_{-3}, ...\right)}_{=a_{-\ell_1}\cdot P(a_{-1},a_{-2},a_{-3},...)}\cdot u = a_{-\ell_1}\cdot\underbrace{P\left(a_{-1}, a_{-2}, a_{-3}, ...\right)\cdot u}_{=\eta(P)} \\ &= a_{-\ell_1}\cdot\eta(P).\end{aligned} \tag{19}$$

Since a_i acts on F as $i\frac{\partial}{\partial x_i}$, we have $a_i\left(x_{\ell_1}Q\right) = i\frac{\partial}{\partial x_i}\left(x_{\ell_1}Q\right)$ and $a_iQ = i\frac{\partial}{\partial x_i}Q$. Now,

$$\begin{aligned}a_i\left(\underbrace{x_{\ell_1}x_{\ell_2}...x_{\ell_M}}_{=x_{\ell_1}Q}\right) &= a_i\left(x_{\ell_1}Q\right) = i\frac{\partial}{\partial x_i}\left(x_{\ell_1}Q\right) = i\left(\left(\frac{\partial}{\partial x_i}x_{\ell_1}\right)Q + x_{\ell_1}\left(\frac{\partial}{\partial x_i}Q\right)\right) \\ &\qquad \text{(by the Leibniz rule)} \\ &= i\underbrace{\left(\frac{\partial}{\partial x_i}x_{\ell_1}\right)}_{=\delta_{i,\ell_1}}Q + x_{\ell_1}\cdot\underbrace{i\frac{\partial}{\partial x_i}Q}_{=a_iQ} = i\delta_{i,\ell_1}Q + x_{\ell_1}\cdot a_iQ = x_{\ell_1}\cdot a_iQ + i\delta_{i,\ell_1}Q,\end{aligned}$$

so that

$$\begin{aligned}\eta\left(a_i\left(x_{\ell_1}x_{\ell_2}...x_{\ell_M}\right)\right) &= \eta\left(x_{\ell_1}\cdot a_iQ + i\delta_{i,\ell_1}Q\right) = \underbrace{\eta\left(x_{\ell_1}\cdot a_iQ\right)}_{\substack{=a_{-\ell_1}\cdot\eta(a_iQ) \\ \text{(by (19), applied to } P=a_iQ\text{)}}} + i\delta_{i,\ell_1}\eta(Q) \\ &= a_{-\ell_1}\cdot\underbrace{\eta\left(a_iQ\right)}_{=a_i\eta(Q)} + i\delta_{i,\ell_1}\eta(Q) = a_{-\ell_1}\cdot a_i\eta(Q) + i\delta_{i,\ell_1}\eta(Q).\end{aligned}$$

[24] This is because such monomials generate F as a $\mathbb{C}$-vector space, and because the equality (15) is linear in P.

Compared to

$$a_i\eta\left(\underbrace{x_{\ell_1}x_{\ell_2}...x_{\ell_M}}_{=x_{\ell_1}Q}\right) = a_i \underbrace{\eta\left(x_{\ell_1}Q\right)}_{\substack{=a_{-\ell_1}\cdot\eta(Q) \\ \text{(by (19), applied to } P=Q)}} = \underbrace{a_i a_{-\ell_1}}_{=a_{-\ell_1}a_i+i\delta_{i,\ell_1}K}\cdot\eta\left(Q\right)$$
$$= \left(a_{-\ell_1}a_i + i\delta_{i,\ell_1}K\right)\cdot\eta\left(Q\right) = a_{-\ell_1}\cdot a_i\eta\left(Q\right) + i\delta_{i,\ell_1}\underbrace{K\eta\left(Q\right)}_{\substack{=\eta(Q) \\ \text{(by (17), applied to } v=\eta(Q)) \\ \text{(since } \eta(Q)\in\eta(F)\subseteq U(\mathcal{A}_0)\cdot u))}}$$
$$= a_{-\ell_1}\cdot a_i\eta\left(Q\right) + i\delta_{i,\ell_1}\eta\left(Q\right),$$

this yields $\eta\left(a_i\left(x_{\ell_1}x_{\ell_2}...x_{\ell_M}\right)\right) = a_i\eta\left(x_{\ell_1}x_{\ell_2}...x_{\ell_M}\right)$. Since we have proven this for every $\left(\ell_1,\ell_2,...,\ell_M\right)\in\left\{1,2,3,...\right\}^M$, we have thus proven (18) for $m=M$. This completes the induction step, and thus the induction proof of (18) is complete. As we have seen above, this proves (15).

From (15) and (16), it is clear that η is $\mathcal{A}_0$-linear (since $\mathcal{A}_0$ is spanned by the a_i for $i\in\mathbb{Z}\diagdown\left\{0\right\}$ and K). Since $\eta\left(1\right)=u$ is obvious, this proves Lemma 2.2.12.

Proof of Proposition 2.2.11. Pick some nonzero vector $v\in V$. Let $W=\mathbb{C}\left[a_1,a_2,a_3,...\right]\cdot v$. Then, by the condition, we have $\dim W<\infty$, and $a_i : W\to W$ are commuting nilpotent operators[25]. Hence, $\bigcap\limits_{i\geq 1}\operatorname{Ker}a_i\neq 0$ [26]. Hence, there exists some nonzero $u\in\bigcap\limits_{i\geq 1}\operatorname{Ker}a_i$. Pick such a u. Then, $a_iu=0$ for all $i>0$, and $Ku=u$ (since K acts as 1 on V). Thus, there exists a homomorphism $\eta : F\to V$ of $\mathcal{A}_0$-modules such that $\eta\left(1\right)=u$ (by Lemma 2.2.12). Since both F and V are irreducible and $\eta\neq 0$, this yields that η is an isomorphism. This proves Proposition 2.2.11.

2.2.3. Classification of $\mathcal{A}_0$-modules with locally nilpotent action of $\mathbb{C}\left[a_1,a_2,a_3,...\right]$

Proposition 2.2.13. Let V be any $\mathcal{A}_0$-module having a locally nilpotent action of $\mathbb{C}\left[a_1,a_2,a_3,...\right]$. (Here, we say that the $\mathcal{A}_0$-module V has a *locally nilpotent action of* $\mathbb{C}\left[a_1,a_2,a_3,...\right]$ if for any $v\in V$, the space $\mathbb{C}\left[a_1,a_2,a_3,...\right]\cdot v$ is finite-dimensional, and the a_i with $i>0$ act on it by nilpotent operators.) Assume that K acts as 1 on V. Assume that for every $v\in V$, there exists some $N\in\mathbb{N}$ such that for every $n\geq N$, we have $a_nv=0$. Then, $V\cong F\otimes U$ as $\mathcal{A}_0$-modules for some vector space U. (The vector space U is not supposed to carry any $\mathcal{A}_0$-module structure.)

Remark 2.2.14. From Proposition 2.2.13, we cannot remove the condition that for every $v\in V$, there exists some $N\in\mathbb{N}$ such that for every $n\geq N$, we have $a_nv=0$. In fact, here is a counterexample of how Proposition 2.2.13 can fail without this condition:

[25] Of course, when we write $a_i : W\to W$, we don't mean the elements a_i of $\mathcal{A}_0$ themselves, but their actions on W.

[26] Here, we are using the following linear-algebraic fact:

If T is a nonzero finite-dimensional vector space over an algebraically closed field, and if b_1, b_2, b_3, ... are commuting linear maps $T\to T$, then there exists a nonzero common eigenvector of b_1, b_2, b_3, ... If b_1, b_2, b_3, ... are nilpotent, this yields $\bigcap\limits_{i\geq 1}\operatorname{Ker}b_i\neq 0$ (since any eigenvector of a nilpotent map must lie in its kernel).

Let V be the representation $\mathbb{C}[x_1, x_2, x_3, ...][y] \diagup (y^2)$ of $\mathcal{A}_0$ given by

$$a_{-i} \mapsto x_i \qquad \text{for every } i \geq 1;$$
$$a_i \mapsto i\frac{\partial}{\partial x_i} + y \qquad \text{for every } i \geq 1,$$
$$K \mapsto 1$$

(where we are being sloppy and abbreviating the residue class $\overline{y} \in \mathbb{C}[x_1, x_2, x_3, ...][y] \diagup (y^2)$ by y, and similarly all other residue classes). We have an exact sequence

$$0 \longrightarrow F \xrightarrow{i} V \xrightarrow{\pi} F \longrightarrow 0$$

of $\mathcal{A}_0$-modules, where the map $i : F \to V$ is given by

$$i(P) = yP \qquad \text{for every } p \in F = \mathbb{C}[x_1, x_2, x_3, ...],$$

and the map $\pi : V \to F$ is the canonical projection $V \to V \diagup (y) \cong F$. Thus, V is an extension of F by F. It is easily seen that V has a locally nilpotent action of $\mathbb{C}[a_1, a_2, a_3, ...]$. But V is not isomorphic to $F \otimes U$ as $\mathcal{A}_0$-modules for any vector space U, since there is a vector $v \in V$ satisfying $V = U(\mathcal{A}_0) \cdot v$ (for example, $v = 1$), whereas there is no vector $v \in F \otimes U$ satisfying $F \otimes U = U(\mathcal{A}_0) \cdot v$ if $\dim U > 1$, and the case $\dim U \leq 1$ is easily ruled out (in this case, $\dim U$ would have to be 1, so that V would be $\cong F$ and thus irreducible, and thus the homomorphisms i and π would have to be isomorphisms, which is absurd).

Before we prove Proposition 2.2.13, we need to define the notion of complete coflags:

Definition 2.2.15. Let k be a field. Let V be a k-vector space. Let W be a vector subspace of V. Assume that $\dim(V \diagup W) < \infty$. Then, a **complete coflag from V to W** will mean a sequence $(V_0, V_1, ..., V_N)$ of vector subspaces of V (with N being an integer) satisfying the following conditions:

- We have $V_0 \supseteq V_1 \supseteq ... \supseteq V_N$.
- Every $i \in \{0, 1, ..., N\}$ satisfies $\dim(V \diagup V_i) = i$.
- We have $V_0 = V$ and $V_N = W$.

(Note that the condition $V_0 = V$ is superfluous (since it follows from the condition that every $i \in \{0, 1, ..., N\}$ satisfies $\dim(V \diagup V_i) = i$), but has been given for the sake of intuition.)

We will also denote the complete coflag $(V_0, V_1, ..., V_N)$ by $V = V_0 \supseteq V_1 \supseteq ... \supseteq V_N = W$.

It is clear that if k is a field, V is a k-vector space, and W is a vector subspace of V satisfying $\dim(V \diagup W) < \infty$, then a complete coflag from V to W exists.[27]

[27] In fact, it is known that the finite-dimensional vector space $V \diagup W$ has a complete flag $(F_0, F_1, ..., F_N)$; now, if we let p be the canonical projection $V \to V \diagup W$, then $\left(p^{-1}(F_N), p^{-1}(F_{N-1}), ..., p^{-1}(F_0)\right)$ is easily seen to be a complete coflag from V to W.

Definition 2.2.16. Let k be a field. Let V be a k-algebra. Let W be a vector subspace of V. Let $\mathfrak{i}$ be an ideal of V. Then, an **$\mathfrak{i}$-coflag from V to W** means a complete coflag $(V_0, V_1, ..., V_N)$ from V to W such that

$$\text{every } i \in \{0, 1, ..., N-1\} \text{ satisfies } \mathfrak{i} \cdot V_i \subseteq V_{i+1}.$$

Lemma 2.2.17. Let k be a field. Let B be a commutative k-algebra. Let I be an ideal of B such that the k-vector space $B \diagup I$ is finite-dimensional. Let $\mathfrak{i}$ be an ideal of B. Let $M \in \mathbb{N}$. Then, there exists an $\mathfrak{i}$-coflag from B to $\mathfrak{i}^M + I$.

Proof of Lemma 2.2.17. We will prove Lemma 2.2.17 by induction over M:

Induction base: Lemma 2.2.17 is trivial in the case when $M = 0$, because $\underbrace{\mathfrak{i}^0}_{=B} + I = B + I = B$. This completes the induction base.

Induction base: Let $m \in \mathbb{N}$. Assume that Lemma 2.2.17 is proven in the case when $M = m$. We now must prove Lemma 2.2.17 in the case when $M = m + 1$.

Since Lemma 2.2.17 is proven in the case when $M = m$, there exists an $\mathfrak{i}$-coflag $(J_0, J_1, ..., J_K)$ from B to $\mathfrak{i}^m + I$. This $\mathfrak{i}$-coflag clearly is a complete coflag from B to $\mathfrak{i}^m + I$.

Since

$$\begin{aligned}
\dim\left((\mathfrak{i}^m + I) \diagup \left(\mathfrak{i}^{m+1} + I\right)\right) &\leq \dim\left(B \diagup \left(\mathfrak{i}^{m+1} + I\right)\right) \\
&\qquad \left(\text{because } (\mathfrak{i}^m + I) \diagup \left(\mathfrak{i}^{m+1} + I\right) \text{ injects into } B \diagup \left(\mathfrak{i}^{m+1} + I\right)\right) \\
&\leq \dim(B \diagup I) \qquad \left(\text{since } B \diagup \left(\mathfrak{i}^{m+1} + I\right) \text{ is a quotient of } B \diagup I\right) \\
&< \infty \qquad (\text{since } B \diagup I \text{ is finite-dimensional}),
\end{aligned}$$

there exists a complete coflag $(U_0, U_1, ..., U_P)$ from $\mathfrak{i}^m + I$ to $\mathfrak{i}^{m+1} + I$.

Since $(U_0, U_1, ..., U_P)$ is a complete coflag from $\mathfrak{i}^m + I$ to $\mathfrak{i}^{m+1} + I$, we have $U_0 = \mathfrak{i}^m + I$, and each of the vector spaces U_0, U_1, ..., U_P contains $\mathfrak{i}^{m+1} + I$ as a subspace.

Also, every $i \in \{0, 1, ..., P\}$ satisfies $U_i \subseteq \mathfrak{i}^m + I$ (again since $(U_0, U_1, ..., U_P)$ is a complete coflag from $\mathfrak{i}^m + I$ to $\mathfrak{i}^{m+1} + I$).

Since $(J_0, J_1, ..., J_K)$ is a complete coflag from B to $\mathfrak{i}^m + I$, while $(U_0, U_1, ..., U_P)$ is a complete coflag from $\mathfrak{i}^m + I$ to $\mathfrak{i}^{m+1} + I$, it is clear that

$$(J_0, J_1, ..., J_K, U_1, U_2, ..., U_P) = (J_0, J_1, ..., J_{K-1}, U_0, U_1, ..., U_P)$$

is a complete coflag from B to $\mathfrak{i}^{m+1} + I$. We now will prove that this complete coflag

$$(J_0, J_1, ..., J_K, U_1, U_2, ..., U_P) = (J_0, J_1, ..., J_{K-1}, U_0, U_1, ..., U_P)$$

actually is an $\mathfrak{i}$-coflag.

In order to prove this, we must show the following two assertions:

Assertion 1: Every $i \in \{0, 1, ..., K-1\}$ satisfies $\mathfrak{i} \cdot J_i \subseteq J_{i+1}$.

Assertion 2: Every $i \in \{0, 1, ..., P-1\}$ satisfies $\mathfrak{i} \cdot U_i \subseteq U_{i+1}$.

Assertion 1 follows directly from the fact that $(J_0, J_1, ..., J_K)$ is an $\mathfrak{i}$-coflag.

Assertion 2 follows from the fact that $\mathfrak{i} \cdot \underbrace{U_i}_{\subseteq \mathfrak{i}^m + I} \subseteq \mathfrak{i} \cdot (\mathfrak{i}^m + I) \subseteq \underbrace{\mathfrak{i} \cdot \mathfrak{i}^m}_{= \mathfrak{i}^{m+1}} + \underbrace{\mathfrak{i} \cdot I}_{\substack{\subseteq I \\ (\text{since } I \text{ is an ideal})}} \subseteq \mathfrak{i}^{m+1} + I \subseteq U_{i+1}$ (because we know that each of the vector spaces U_0, U_1, ..., U_P contains $\mathfrak{i}^{m+1} + I$ as a subspace, so that (in particular) $\mathfrak{i}^{m+1} + I \subseteq U_{i+1}$).

Hence, both Assertions 1 and 2 are proven, and we conclude that

$$(J_0, J_1, ..., J_K, U_1, U_2, ..., U_P) = (J_0, J_1, ..., J_{K-1}, U_0, U_1, ..., U_P)$$

is an $\mathfrak{i}$-coflag. This is clearly an $\mathfrak{i}$-coflag from B to $\mathfrak{i}^{m+1} + I$. Thus, there exists an $\mathfrak{i}$-coflag from B to $\mathfrak{i}^{m+1} + I$. This proves Lemma 2.2.17 in the case when $M = m + 1$. The induction step is complete, and with it the proof of Lemma 2.2.17.

Proof of Proposition 2.2.13. Let $v \in V$ be arbitrary. Let $I_v \subseteq \mathbb{C}[a_1, a_2, a_3, ...]$ be the annihilator of v. Then, the canonical $\mathbb{C}$-algebra map $\mathbb{C}[a_1, a_2, a_3, ...] \to \operatorname{End}(\mathbb{C}[a_1, a_2, a_3, ...] \cdot v)$ (this map comes from the action of the $\mathbb{C}$-algebra $\mathbb{C}[a_1, a_2, a_3, ...]$ on $\mathbb{C}[a_1, a_2, a_3, ...] \cdot v$) gives rise to an *injective* map $\mathbb{C}[a_1, a_2, a_3, ...] \diagup I_v \to \operatorname{End}(\mathbb{C}[a_1, a_2, a_3, ...] \cdot v)$. Since this map is injective, we have $\dim(\mathbb{C}[a_1, a_2, a_3, ...] \diagup I_v) \leq \dim(\operatorname{End}(\mathbb{C}[a_1, a_2, a_3, ...] \cdot v)) < \infty$ (since $\mathbb{C}[a_1, a_2, a_3, ...] \cdot v$ is finite-dimensional). In other words, the vector space $\mathbb{C}[a_1, a_2, a_3, ...] \diagup I_v$ is finite-dimensional.

Let W be the $\mathcal{A}_0$-submodule of V generated by v. In other words, let $W = U(\mathcal{A}_0) \cdot v$. Then, W is a quotient of $U(\mathcal{A}_0)$ (as an $\mathcal{A}_0$-module). Since K acts as 1 on W, it follows that W is a quotient of $U(\mathcal{A}_0) \diagup (K-1) \cong D(x_1, x_2, x_3, ...)$. Since I_v annihilates v, it follows that W is a quotient of $D(x_1, x_2, ...) \diagup (D(x_1, x_2, ...) I_v)$. Let us denote the $\mathcal{A}_0$-module $D(x_1, x_2, ...) \diagup (D(x_1, x_2, ...) I_v)$ by $\widetilde{W}$.

We now will prove that $\widetilde{W}$ is a finite-length $\mathcal{A}_0$-module with all composition factors isomorphic to F. [28]

Let $\mathfrak{i}$ be the ideal $(a_1, a_2, a_3, ...)$ of the commutative algebra $\mathbb{C}[a_1, a_2, a_3, ...]$.

Since I_v is an ideal of the commutative algebra $\mathbb{C}[a_1, a_2, a_3, ...]$, the quotient $\mathbb{C}[a_1, a_2, a_3, ...] \diagup I_v$ is an algebra. For every $q \in \mathbb{C}[a_1, a_2, a_3, ...]$, let $\overline{q}$ be the projection of q onto the quotient algebra $\mathbb{C}[a_1, a_2, a_3, ...] \diagup I_v$. Let also $\overline{\mathfrak{i}}$ be the projection of the ideal $\mathfrak{i}$ onto the quotient algebra $\mathbb{C}[a_1, a_2, a_3, ...] \diagup I_v$. Clearly, $\overline{\mathfrak{i}} = (\overline{a_1}, \overline{a_2}, \overline{a_3}, ...)$.

For every $j > 0$, there exists some $i \in \mathbb{N}$ such that $a_j^i v = 0$ (since V has a locally nilpotent action of $\mathbb{C}[a_1, a_2, a_3, ...]$). Hence, for every $j > 0$, the element $\overline{a_j}$ of $\mathbb{C}[a_1, a_2, a_3, ...] \diagup I_v$ is nilpotent (because there exists some $i \in \mathbb{N}$ such that $a_j^i v = 0$, and thus this i satisfies $a_j^i \in I_v$, so that $\overline{a_j}^i = 0$). Hence, the ideal $\overline{\mathfrak{i}}$ is generated by nilpotent generators (since $\overline{\mathfrak{i}} = (\overline{a_1}, \overline{a_2}, \overline{a_3}, ...)$). Since we also know that $\overline{\mathfrak{i}}$ is finitely generated (since $\overline{\mathfrak{i}}$ is an ideal of the finite-dimensional algebra $\mathbb{C}[a_1, a_2, a_3, ...] \diagup I_v$), it follows that $\overline{\mathfrak{i}}$ is generated by *finitely many* nilpotent generators. But if an ideal of a commutative ring is generated by finitely many nilpotent generators, it must be nilpotent. Thus, $\overline{\mathfrak{i}}$ is nilpotent. In other words, there exists some $M \in \mathbb{N}$ such that $\overline{\mathfrak{i}}^M = 0$. Consider this M. Since $\overline{\mathfrak{i}}^M = 0$, we have $\mathfrak{i}^M \subseteq I_v$ and thus $\mathfrak{i}^M + I_v = I_v$.

Now, Lemma 2.2.17 (applied to $k = \mathbb{C}$, $B = \mathbb{C}[a_1, a_2, a_3, ...]$ and $I = I_v$) yields that there exists an $\mathfrak{i}$-coflag from $\mathbb{C}[a_1, a_2, a_3, ...]$ to $\mathfrak{i}^M + I_v$. Denote this $\mathfrak{i}$-coflag by $(J_0, J_1, ..., J_N)$. Since $\mathfrak{i}^M + I_v = I_v$, this $\mathfrak{i}$-coflag $(J_0, J_1, ..., J_N)$ thus is an $\mathfrak{i}$-coflag from $\mathbb{C}[a_1, a_2, a_3, ...]$ to I_v. Thus, $(J_0, J_1, ..., J_N)$ is a complete coflag from $\mathbb{C}[a_1, a_2, a_3, ...]$ to I_v. In other words:

- We have $J_0 \supseteq J_1 \supseteq ... \supseteq J_N$.
- Every $i \in \{0, 1, ..., N\}$ satisfies $\dim(\mathbb{C}[a_1, a_2, a_3, ...] \diagup J_i) = i$.

[28] We can even prove that there are exactly $\dim(\mathbb{C}[a_1, a_2, a_3, ...] \diagup I_v)$ composition factors.

- We have $J_0 = \mathbb{C}[a_1, a_2, a_3, ...]$ and $J_N = I_v$.

Besides, since $(J_0, J_1, ..., J_N)$ is an $\mathfrak{i}$-coflag, we have

$$\mathfrak{i} \cdot J_i \subseteq J_{i+1} \qquad \text{for every } i \in \{0, 1, ..., N-1\}. \tag{20}$$

For every $i \in \{0, 1, ..., N\}$, let $D_i = D(x_1, x_2, ...) \cdot J_i$. Then,

$$D_0 = D(x_1, x_2, ...) \cdot \underbrace{J_0}_{=\mathbb{C}[a_1, a_2, a_3, ...]} = D(x_1, x_2, ...)$$

and

$$D_N = D(x_1, x_2, ...) \cdot \underbrace{J_N}_{=I_v} = D(x_1, x_2, ...) \cdot I_v.$$

Hence, $D_0 \diagup D_N = D(x_1, x_2, ...) \diagup (D(x_1, x_2, ...) I_v) = \widetilde{W}$.

Now, we are going to prove that

$$D_i \diagup D_{i+1} \cong F \text{ or } D_i \diagup D_{i+1} = 0 \qquad \text{for every } i \in \{0, 1, ..., N-1\} \tag{21}$$

(where $\cong$ means isomorphism of $\mathcal{A}_0$-modules).

Proof of (21). Let $i \in \{0, 1, ..., N-1\}$. Since $\dim(\mathbb{C}[a_1, a_2, a_3, ...] \diagup J_i) = i$ and $\dim(\mathbb{C}[a_1, a_2, a_3, ...] \diagup J_{i+1}) = i+1$, there exists some $u \in J_i$ such that $J_i = u + J_{i+1}$. Consider this u. By abuse of notation, we also use the letter u to denote the element $1 \cdot u \in D(x_1, x_2, ...) \cdot J_i = D_i$. Then,

$$\begin{aligned} D_i &= D(x_1, x_2, ...) \cdot \underbrace{J_i}_{=u+J_{i+1}} = D(x_1, x_2, ...) \cdot (u + J_{i+1}) \\ &= D(x_1, x_2, ...) \cdot u + \underbrace{D(x_1, x_2, ...) \cdot J_{i+1}}_{=D_{i+1}} = D(x_1, x_2, ...) \cdot u + D_{i+1}. \end{aligned}$$

Thus,

$$D_i \diagup D_{i+1} = D(x_1, x_2, ...) \cdot u',$$

where u' denotes the residue class of $u \in D_i$ modulo D_{i+1}. For every $j > 0$, we have $\underbrace{a_j}_{\in \mathfrak{i}} \underbrace{u}_{\in J_i} \in \mathfrak{i} \cdot J_i \subseteq J_{i+1}$ (by (20)) and thus $a_j u \in D(x_1, x_2, ...) \cdot J_{i+1} = D_{i+1}$. In other words, for every $j > 0$, we have $a_j u' = 0$. Also, it is pretty clear that $Ku' = u'$. Thus, Lemma 2.2.12 (applied to $D_i \diagup D_{i+1}$ and u' instead of V and u) yields that there exists a homomorphism $\eta : F \to D_i \diagup D_{i+1}$ of $\mathcal{A}_0$-modules such that $\eta(1) = u'$. This homomorphism η must be surjective[29], and thus $D_i \diagup D_{i+1}$ is a factor module of F. Since F is irreducible, this yields that $D_i \diagup D_{i+1} \cong F$ or $D_i \diagup D_{i+1} = 0$. This proves (21).

Now, clearly, the $\mathcal{A}_0$-module $\widetilde{W} = D_0 \diagup D_N$ is filtered by the $\mathcal{A}_0$-modules $D_i \diagup D_N$ for $i \in \{0, 1, ..., N\}$. Due to (21), the subquotients of this filtration are all $\cong F$ or $= 0$,

[29]since its image is $\eta\left(\underbrace{F}_{=D(x_1,x_2,...)\cdot 1}\right) = D(x_1, x_2, ...) \cdot \underbrace{\eta(1)}_{=u'} = D(x_1, x_2, ...) \cdot u' = D_i \diagup D_{i+1}$

so that $\widetilde{W}$ is a finite-length $\mathcal{A}_0$-module with all composition factors isomorphic to F (since F is irreducible).

Since W is a quotient module of $\widetilde{W}$, this yields that W must also be a finite-length $\mathcal{A}_0$-module with all composition factors isomorphic to F.

Now forget that we fixed v. We have thus shown that for every $v \in V$, the $\mathcal{A}_0$-submodule $U(\mathcal{A}_0) \cdot v$ of V (this submodule is what we called W) is a finite-length module with composition factors isomorphic to F.

By the assumption (that for every $v \in V$, there exists some $N \in \mathbb{N}$ such that for every $n \geq N$, we have $a_n v = 0$), we can define an action of $E = \sum\limits_{i>0} a_{-i} a_i \in \widehat{\mathcal{A}}$ (the so-called *Euler field*) on V. Note that E acts on V in a locally finite way (this means that for any $v \in V$, the space $\mathbb{C}[E] \cdot v$ is finite-dimensional)[30]. Now, let us notice that the eigenvalues of the map $E\mid_V : V \to V$ (this is the action of E on V) are nonnegative integers.[31] Hence, we can write V as $V = \bigoplus\limits_{j \geq 0} V[j]$, where $V[j]$ is the generalized eigenspace of $E\mid_V$ with eigenvalue j for every $j \in \mathbb{N}$.

If some $v \in V$ satisfies $a_i v = 0$ for all $i > 0$, then $Ev = 0$ and thus $v \in V[0]$.

[30] *Proof.* Notice that E acts on F as $\sum\limits_{i>0} i x_i \dfrac{\partial}{\partial x_i}$, and thus E acts on F in a locally finite way (since the differential operator $\sum\limits_{i>0} i x_i \dfrac{\partial}{\partial x_i}$ preserves the degrees of polynomials), and thus also on V (because for every $v \in V$, the $\mathcal{A}_0$-submodule $U(\mathcal{A}_0) \cdot v$ of V is a finite-length module with composition factors isomorphic to F).

[31] *Proof.* Let ρ be an eigenvalue of $E\mid_V$. Then, there exists some nonzero eigenvector $v \in V$ to the eigenvalue ρ. Consider this v. Clearly, ρ must thus also be an eigenvalue of $E\mid_{U(\mathcal{A}_0)\cdot v}$ (because v is a nonzero eigenvector of $E\mid_V$ to the eigenvalue ρ and lies in $U(\mathcal{A}_0) \cdot v$). But the eigenvalues of $E\mid_{U(\mathcal{A}_0)\cdot v}$ are nonnegative integers (since we know that the $\mathcal{A}_0$-submodule $U(\mathcal{A}_0) \cdot v$ of V is a finite-length module with composition factors isomorphic to F, and we can easily check that the eigenvalues of $E\mid_F$ are nonnegative integers). Hence, ρ is a nonnegative integer. We have thus shown that every eigenvalue of $E\mid_V$ is a nonnegative integer, qed.

Conversely, if $v \in V[0]$, then $a_i v = 0$ for all $i > 0$. [32]

So we conclude that $V[0] = \operatorname{Ker} E = \bigcap\limits_{i \geq 1} \operatorname{Ker} a_i$.

Now, $F \otimes V[0]$ is an $\mathcal{A}_0$-module (where $\mathcal{A}_0$ acts only on the F tensorand, where $V[0]$ is considered just as a vector space). We will now construct an isomorphism $F \otimes V[0] \to V$ of $\mathcal{A}_0$-modules. This will prove Proposition 2.2.13.

For every $v \in V[0]$, there exists a homomorphism $\eta_v : F \to V$ of $\mathcal{A}_0$-modules such that $\eta_v(1) = v$ (according to Lemma 2.2.12, applied to v instead of u (since $a_i v = 0$ for all $i > 0$ and $Kv = v$)). Consider these homomorphisms η_v for various v. Clearly, every $v \in V[0]$ and $P \in F$ satisfy

$$\begin{aligned}
\eta_v(P) &= \eta_v(P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot 1) && (\text{since } P = P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot 1) \\
&= P(a_{-1}, a_{-2}, a_{-3}, ...) \underbrace{\eta_v(1)}_{=v} && (\text{since } \eta_v \text{ is an } \mathcal{A}_0\text{-module map}) \\
&= P(a_{-1}, a_{-2}, a_{-3}, ...) v.
\end{aligned}$$

Hence, we can define a $\mathbb{C}$-linear map $\rho : F \otimes V[0] \to V$ by

$$\rho(P \otimes v) = \eta_v(P) = P(a_{-1}, a_{-2}, a_{-3}, ...) v \qquad \text{for any } P \in F \text{ and } v \in V[0].$$

This map ρ is an $\mathcal{A}_0$-module map (because η_v is an $\mathcal{A}_0$-module map for every $v \in V[0]$).

The restriction of the map ρ to the subspace $\mathbb{C} \cdot 1 \otimes V[0]$ of $F \otimes V[0]$ is injective (since it maps every $1 \otimes v$ to v). Hence, the map ρ is injective[33]. Also, considering the quotient $\mathcal{A}_0$-module $V \diagup \rho(F \otimes V[0])$, we notice that $E \mid_{V \diagup \rho(F \otimes V[0])}$ has only strictly positive

[32] *Proof.* Let $v \in V[0]$. Let j be positive.

It is easy to check that $a_{-i} a_i a_j = a_j a_{-i} a_i - i \delta_{i,j} a_i$ for any positive i (here, we use that $j > 0$). Since $E = \sum\limits_{i > 0} a_{-i} a_i$, we have

$$\begin{aligned}
E a_j &= \sum_{i>0} \underbrace{a_{-i} a_i a_j}_{= a_j a_{-i} a_i - i \delta_{i,j} a_i} = \sum_{i>0} (a_j a_{-i} a_i - i \delta_{i,j} a_i) \\
&= a_j \underbrace{\sum_{i>0} a_{-i} a_i}_{=E} - \underbrace{\sum_{i>0} i \delta_{i,j} a_i}_{= j a_j} = a_j E - j a_j,
\end{aligned}$$

so that $(E + j) a_j = a_j E$. This yields (by induction over m) that $(E + j)^m a_j = a_j E^m$ for every $m \in \mathbb{N}$.

Now, since $v \in V[0] = (\text{generalized eigenspace of } E \mid_V \text{ with eigenvalue } 0)$, there exists an $m \in \mathbb{N}$ such that $E^m v = 0$. Consider this m. Then, from $(E + j)^m a_j = a_j E^m$, we obtain $(E + j)^m a_j v = a_j E^m v = 0$, so that

$$a_j v \in (\text{generalized eigenspace of } E \mid_V \text{ with eigenvalue } -j) = 0$$

(because the eigenvalues of the map $E \mid_V : V \to V$ are nonnegative integers, whereas $-j$ is not). In other words, $a_j v = 0$.

We have thus proven that $a_j v = 0$ for every positive j. In other words, $a_i v = 0$ for all $i > 0$, qed.

[33] This follows from the following general representation-theoretical fact (applied to $A = U(\mathcal{A}_0)$, $I = F$, $R = V[0]$, $S = V$, $i = 1$ and $\phi = \rho$):

Let A be a $\mathbb{C}$-algebra. Let I be an irreducible A-module, and let S be an A-module. Let R be a vector space. Let $i \in I$ be nonzero. Let $\phi : I \otimes R \to S$ be an A-module homomorphism such that the restriction of ϕ to $\mathbb{C} i \otimes R$ is injective. Then, ϕ is injective.

eigenvalues (since $\rho(F \otimes V[0]) \supseteq V[0]$, so that all eigenvectors of $E\mid_V$ to eigenvalue 0 have been killed when factoring modulo $\rho(F \otimes V[0])$), and thus $V \diagup \rho(F \otimes V[0]) = 0$ [34]. In other words, $V = \rho(F \otimes V[0])$, so that ρ is surjective. Since ρ is an injective and surjective $\mathcal{A}_0$-module map, we conclude that ρ is an $\mathcal{A}_0$-module isomorphism. Thus, $V \cong F \otimes V[0]$ as $\mathcal{A}_0$-modules. This proves Proposition 2.2.13.

2.2.4. Remark on $\mathcal{A}$-modules

We will not use this until much later, but here is an analogue of Lemma 2.2.12 for $\mathcal{A}$ instead of $\mathcal{A}_0$:

Lemma 2.2.18. Let V be an $\mathcal{A}$-module. Let $\mu \in \mathbb{C}$. Let $u \in V$ be such that $a_i u = 0$ for all $i > 0$, such that $a_0 u = \mu u$, and such that $Ku = u$. Then, there exists a homomorphism $\eta : F_\mu \to V$ of $\mathcal{A}$-modules such that $\eta(1) = u$. (This homomorphism η is unique, although we won't need this.)

Proof of Lemma 2.2.18. Let η be the map $F \to V$ which sends every polynomial $P \in F = \mathbb{C}[x_1, x_2, x_3, ...]$ to $P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot u \in V$. [35] Just as in the Second proof of Lemma 2.2.12, we can show that η is an $\mathcal{A}_0$-module homomorphism $F \to V$ such that $\eta(1) = u$. We are now going to prove that this η is also a homomorphism $F_\mu \to V$ of $\mathcal{A}$-modules. Clearly, in order to prove this, it is enough to show that $\eta(a_0 P) = a_0 \eta(P)$ for all $P \in F_\mu$.

Let $P \in F_\mu$. Since a_0 acts as multiplication by μ on F_μ, we have $a_0 P = \mu P$.

On the other hand, by the definition of η, we have $\eta(P) = P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot u$, so that

$$\begin{aligned} a_0 \eta(P) &= a_0 P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot u = P(a_{-1}, a_{-2}, a_{-3}, ...)\, a_0 \cdot u \\ &\qquad \left(\begin{array}{c} \text{since } a_0 \text{ lies in the center of } \mathcal{A}\text{, and thus in the center of } U(\mathcal{A}), \\ \text{and thus } a_0 P(a_{-1}, a_{-2}, a_{-3}, ...) = P(a_{-1}, a_{-2}, a_{-3}, ...)\, a_0 \end{array} \right) \\ &= P(a_{-1}, a_{-2}, a_{-3}, ...) \underbrace{a_0 u}_{=\mu u} = \mu \underbrace{P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot u}_{=\eta(P)} = \mu \eta(P) \\ &= \eta \left(\underbrace{\mu P}_{=a_0 P} \right) = \eta(a_0 P). \end{aligned}$$

Thus, we have shown that $\eta(a_0 P) = a_0 \eta(P)$ for all $P \in F_\mu$. This completes the proof of Lemma 2.2.18.

[34] *Proof.* Assume the contrary. Then, $V \diagup \rho(F \otimes V[0]) \neq 0$. Thus, there exists some nonzero $w \in V \diagup \rho(F \otimes V[0])$. Write w as $\overline{v}$, where v is an element of V and $\overline{v}$ denotes the residue class of v modulo $\rho(F \otimes V[0])$. As we know, the $\mathcal{A}_0$-submodule $U(\mathcal{A}_0) \cdot v$ of V is a finite-length module with composition factors isomorphic to F. Thus, the $\mathcal{A}_0$-module $U(\mathcal{A}_0) \cdot w$ (being a quotient module of $U(\mathcal{A}_0) \cdot v$) must also be a finite-length module with composition factors isomorphic to F. Hence, there exists a submodule of $U(\mathcal{A}_0) \cdot w$ isomorphic to F (since $w \neq 0$ and thus $U(\mathcal{A}_0) \cdot w \neq 0$). This submodule contains a nonzero eigenvector of E to eigenvalue 0 (because F contains a nonzero eigenvector of E to eigenvalue 0, namely 1). This is a contradiction to the fact that $E\mid_{V \diagup \rho(F \otimes V[0])}$ has only strictly positive eigenvalues. This contradiction shows that our assumption was wrong, so we do have $V \diagup \rho(F \otimes V[0]) = 0$, qed.

[35] Note that the term $P(a_{-1}, a_{-2}, a_{-3}, ...)$ denotes the evaluation of the polynomial P at $(x_1, x_2, x_3, ...) = (a_{-1}, a_{-2}, a_{-3}, ...)$. This evaluation is a well-defined element of $U(\mathcal{A}_0)$, since the elements a_{-1}, a_{-2}, a_{-3}, ... of $U(\mathcal{A}_0)$ commute.

2.2.5. A rescaled version of the Fock space

Here is a statement very similar to Corollary 2.2.4:

Corollary 2.2.19. The Lie algebra $\mathcal{A}_0$ has a representation $\widetilde{F} = \mathbb{C}[x_1, x_2, x_3, ...]$ which is given by

$$\begin{aligned} a_{-i} &\mapsto ix_i && \text{for every } i \geq 1; \\ a_i &\mapsto \frac{\partial}{\partial x_i} && \text{for every } i \geq 1, \\ K &\mapsto 1 \end{aligned}$$

(where "$a_{-i} \mapsto ix_i$" is just shorthand for "$a_{-i} \mapsto$ (multiplication by ix_i)"). For every $\mu \in \mathbb{C}$, we can upgrade $\widetilde{F}$ to a representation $\widetilde{F_\mu}$ of $\mathcal{A}$ by adding the condition that $a_0 \mid_{\widetilde{F_\mu}} = \mu \cdot \mathrm{id}$.

Note that the $\mathcal{A}_0$-module structure on $\widetilde{F}$ differs from that on F by a different choice of "where to put the i factor": in F it is in the action of a_i, while in $\widetilde{F}$ it is in the action of a_{-i} (where $i \geq 1$).

Definition 2.2.20. The representation $\widetilde{F}$ of $\mathcal{A}_0$ introduced in Corollary 2.2.19 will be called the *rescaled Fock module* or the *rescaled Fock representation*. For every $\mu \in \mathbb{C}$, the representation $\widetilde{F_\mu}$ of $\mathcal{A}$ introduced in Corollary 2.2.19 will be called the *rescaled μ-Fock representation* of $\mathcal{A}$. The vector space $\widetilde{F}$ itself, of course, is the same as the vector space F of Corollary 2.2.4, and thus we simply call it the Fock space.

Proposition 2.2.21. Let $\mathrm{resc} : \mathbb{C}[x_1, x_2, x_3, ...] \to \mathbb{C}[x_1, x_2, x_3, ...]$ be the $\mathbb{C}$-algebra homomorphism which sends x_i to ix_i for every $i \in \{1, 2, 3, ...\}$. (This homomorphism exists and is unique by the universal property of the polynomial algebra. It is clear that resc multiplies every monomial by some scalar.)

(a) Then, resc is an $\mathcal{A}_0$-module isomorphism $F \to \widetilde{F}$. Thus, $F \cong \widetilde{F}$ as $\mathcal{A}_0$-modules.

(b) Let $\mu \in \mathbb{C}$. Then, resc is an $\mathcal{A}$-module isomorphism $F_\mu \to \widetilde{F_\mu}$. Thus, $F_\mu \cong \widetilde{F_\mu}$ as $\mathcal{A}$-modules.

Corollary 2.2.19 and Proposition 2.2.21 are both very easy to prove: It is best to prove Proposition 2.2.21 first (without yet knowing that $\widetilde{F}$ and $\widetilde{F_\mu}$ are really an $\mathcal{A}_0$-module and an $\mathcal{A}$-module, respectively), and then use it to derive Corollary 2.2.19 from Corollary 2.2.4 by means of resc. We leave all details to the reader.

The modules $\widetilde{F}$ and F aren't that much different: They are isomorphic by an isomorphism which has diagonal form with respect to the monomial bases (due to Proposition 2.2.21). Nevertheless, it pays off to use different notations for them so as not to let confusion arise. We are going to work with F most of the time, except when $\widetilde{F}$ is easier to handle.

2.2.6. An involution on $\mathcal{A}$ and a bilinear form on the Fock space

The following fact is extremely easy to prove:

Proposition 2.2.22. Define a $\mathbb{C}$-linear map $\omega : \mathcal{A} \to \mathcal{A}$ by setting

$$\omega(K) = -K \qquad \text{and}$$
$$\omega(a_i) = -a_{-i} \qquad \text{for every } i \in \mathbb{Z}.$$

Then, ω is an automorphism of the Lie algebra $\mathcal{A}$. Also, ω is an involution (this means that $\omega^2 = \mathrm{id}$). Moreover, $\omega(\mathcal{A}[i]) = \mathcal{A}[-i]$ for all $i \in \mathbb{Z}$. Finally, $\omega \mid_{\mathcal{A}[0]} = -\mathrm{id}$.

Now, let us make a few conventions:

Convention 2.2.23. In the following, a map $\varphi : A \to \mathbb{N}$ (where A is some set) is said to be *finitely supported* if all but finitely many $a \in A$ satisfy $\varphi(a) = 0$. Sequences (finite, infinite, or two-sided infinite) are considered as maps (from finite sets, $\mathbb{N}$ or $\mathbb{Z}$, or occasionally other sets). Thus, a sequence is finitely supported if and only if all but finitely many of its elements are zero.

If A is a set, then $\mathbb{N}_{\text{fin}}^{A}$ will denote the set of all finitely supported maps $A \to \mathbb{N}$.

Proposition 2.2.24. Define a $\mathbb{C}$-bilinear form $(\cdot,\cdot) : F \times F \to \mathbb{C}$ by setting

$$\left(x_1^{n_1}x_2^{n_2}x_3^{n_3}..., x_1^{m_1}x_2^{m_2}x_3^{m_3}...\right) = \prod_{i=1}^{\infty} \delta_{n_i,m_i} \cdot \prod_{i=1}^{\infty} i^{n_i} \cdot \prod_{i=1}^{\infty} n_i!$$
$$\text{for all sequences } (n_1, n_2, n_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}} \text{ and } (m_1, m_2, m_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}.$$

(This is well-defined, because each of the infinite products $\prod\limits_{i=1}^{\infty} \delta_{n_i,m_i}$, $\prod\limits_{i=1}^{\infty} i^{n_i}$ and $\prod\limits_{i=1}^{\infty} n_i!$ has only finitely many terms distinct from 1, and thus is well-defined.)

(a) This form $(\cdot,\cdot)$ is symmetric and nondegenerate.

(b) Every polynomial $P \in F = \mathbb{C}[x_1, x_2, x_3, ...]$ satisfies $(1, P) = P(0, 0, 0, ...)$.

(c) Let $\mu \in \mathbb{C}$. Any $x \in \mathcal{A}$, $P \in F_\mu$ and $Q \in F_\mu$ satisfy $(xP, Q) = -(P, \omega(x)Q)$, where xP and $\omega(x)Q$ are evaluated in the $\mathcal{A}$-module F_μ.

(d) Let $\mu \in \mathbb{C}$. Any $x \in \mathcal{A}$, $P \in F_\mu$ and $Q \in F_\mu$ satisfy $(P, xQ) = -(\omega(x)P, Q)$, where xQ and $\omega(x)P$ are evaluated in the $\mathcal{A}$-module F_μ.

(e) Let $\mu \in \mathbb{C}$. Any $x \in \mathcal{A}$, $P \in \widetilde{F}_\mu$ and $Q \in \widetilde{F}_\mu$ satisfy $(xP, Q) = -(P, \omega(x)Q)$, where xP and $\omega(x)Q$ are evaluated in the $\mathcal{A}$-module $\widetilde{F}_\mu$.

(f) Let $\mu \in \mathbb{C}$. Any $x \in \mathcal{A}$, $P \in \widetilde{F}_\mu$ and $Q \in \widetilde{F}_\mu$ satisfy $(P, xQ) = -(\omega(x)P, Q)$, where xQ and $\omega(x)P$ are evaluated in the $\mathcal{A}$-module $\widetilde{F}_\mu$.

We are going to put the form $(\cdot,\cdot)$ from this proposition into a broader context in Proposition 2.9.12; indeed, we will see that it is an example of a contravariant form on a Verma module of a Lie algebra with involution. ("Contravariant" means that $(av, w) = -(v, \omega(a)w)$ and $(v, aw) = -(\omega(a)v, w)$ for all a in the Lie algebra and v and w in the module. In the case of our form $(\cdot,\cdot)$, the contravariantness of the form follows from Proposition 2.2.24 **(c)** and **(d)**.)

Proof of Proposition 2.2.24. **(a)** For any sequences $(n_1, n_2, n_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ and

$(m_1, m_2, m_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$, we have

$$\left(x_1^{n_1} x_2^{n_2} x_3^{n_3} ..., x_1^{m_1} x_2^{m_2} x_3^{m_3} ...\right) = \prod_{i=1}^{\infty} \delta_{n_i, m_i} \cdot \prod_{i=1}^{\infty} i^{n_i} \cdot \prod_{i=1}^{\infty} n_i!$$

and

$$\left(x_1^{m_1} x_2^{m_2} x_3^{m_3} ..., x_1^{n_1} x_2^{n_2} x_3^{n_3} ...\right) = \prod_{i=1}^{\infty} \delta_{m_i, n_i} \cdot \prod_{i=1}^{\infty} i^{m_i} \cdot \prod_{i=1}^{\infty} m_i!.$$

These two terms are equal in the case when $(n_1, n_2, n_3, ...) \neq (m_1, m_2, m_3, ...)$ (because in this case, they are both 0 due to the presence of the $\prod\limits_{i=1}^{\infty} \delta_{n_i, m_i}$ and $\prod\limits_{i=1}^{\infty} \delta_{m_i, n_i}$ factors), and are clearly equal in the case when $(n_1, n_2, n_3, ...) = (m_1, m_2, m_3, ...)$ as well. Hence, these two terms are always equal. In other words, any sequences $(n_1, n_2, n_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ and $(m_1, m_2, m_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ satisfy

$$\left(x_1^{n_1} x_2^{n_2} x_3^{n_3} ..., x_1^{m_1} x_2^{m_2} x_3^{m_3} ...\right) = \left(x_1^{m_1} x_2^{m_2} x_3^{m_3} ..., x_1^{n_1} x_2^{n_2} x_3^{n_3} ...\right).$$

This proves that the form $(\cdot, \cdot)$ is symmetric.

The space $F = \mathbb{C}[x_1, x_2, x_3, ...]$ has a basis consisting of monomials. With respect to this basis, the form $(\cdot, \cdot)$ is represented by a diagonal matrix (because whenever $(n_1, n_2, n_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ and $(m_1, m_2, m_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ are distinct, we have

$$\left(x_1^{n_1} x_2^{n_2} x_3^{n_3} ..., x_1^{m_1} x_2^{m_2} x_3^{m_3} ...\right) = \underbrace{\prod_{i=1}^{\infty} \delta_{n_i, m_i}}_{\substack{=0 \\ (\text{since } (n_1, n_2, n_3, ...) \neq (m_1, m_2, m_3, ...))}} \cdot \prod_{i=1}^{\infty} i^{n_i} \cdot \prod_{i=1}^{\infty} n_i! = 0$$

), whose diagonal entries are all nonzero (since every $(n_1, n_2, n_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ satisfies

$$\left(x_1^{n_1} x_2^{n_2} x_3^{n_3} ..., x_1^{n_1} x_2^{n_2} x_3^{n_3} ...\right) = \prod_{i=1}^{\infty} \underbrace{\delta_{n_i, n_i}}_{=1} \cdot \prod_{i=1}^{\infty} \underbrace{i^{n_i}}_{\neq 0} \cdot \prod_{i=1}^{\infty} \underbrace{n_i!}_{\neq 0} \neq 0$$

). Hence, this form is nondegenerate. Proposition 2.2.24 **(a)** is proven.

(b) We must prove that every polynomial $P \in F = \mathbb{C}[x_1, x_2, x_3, ...]$ satisfies $(1, P) = P(0, 0, 0, ...)$. In order to show this, it is enough to check that every **monomial** $P \in F = \mathbb{C}[x_1, x_2, x_3, ...]$ satisfies $(1, P) = P(0, 0, 0, ...)$ (because the equation $(1, P) = P(0, 0, 0, ...)$ is linear in P, and because the monomials span F). In other words, we must check that every $(m_1, m_2, m_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ satisfies $\left(1, x_1^{m_1} x_2^{m_2} x_3^{m_3} ...\right) = \left(x_1^{m_1} x_2^{m_2} x_3^{m_3} ...\right)(0, 0, 0, ...)$. But this is easy:

$$\left(\underbrace{1}_{=x_1^0 x_2^0 x_3^0 ...}, x_1^{m_1} x_2^{m_2} x_3^{m_3} ...\right) = \left(x_1^0 x_2^0 x_3^0 ..., x_1^{m_1} x_2^{m_2} x_3^{m_3} ...\right) = \prod_{i=1}^{\infty} \underbrace{\delta_{0, m_i}}_{=0^{m_i}} \cdot \prod_{i=1}^{\infty} \underbrace{i^0}_{=1} \cdot \prod_{i=1}^{\infty} \underbrace{0!}_{=1}$$

$$(\text{by the definition of } (\cdot, \cdot))$$

$$= \prod_{i=1}^{\infty} 0^{m_i} = 0^{m_1} 0^{m_2} 0^{m_3} ... = \left(x_1^{m_1} x_2^{m_2} x_3^{m_3} ...\right)(0, 0, 0, ...),$$

qed. Proposition 2.2.24 **(b)** is proven.

(c) We must prove that any $x \in \mathcal{A}$, $P \in F_\mu$ and $Q \in F_\mu$ satisfy $(xP, Q) = -(P, \omega(x) Q)$. Since this equation is linear in each of x, P and Q, we can WLOG assume that x is an element of the basis $\{a_n \mid n \in \mathbb{Z}\} \cup \{K\}$ of $\mathcal{A}$ and that P and Q are monomials (since monomials span F). So let us assume this.

Since x is an element of the basis $\{a_n \mid n \in \mathbb{Z}\} \cup \{K\}$ of $\mathcal{A}$, we have either $x = a_j$ for some $j \in \mathbb{Z}$, or $x = K$. Since the latter case is trivial (in fact, when $x = K$, then

$$(xP, Q) = (KP, Q) = (P, Q) \qquad \text{(since } K \text{ acts as 1 on } F_\mu\text{, so that } KP = P\text{)}$$

and

$$-\left(P, \omega\left(\underbrace{x}_{=K}\right) Q\right) = -\left(P, \underbrace{\omega(K)}_{=-K} Q\right) = -(P, -KQ) = (P, KQ) = (P, Q)$$
$$\text{(since } K \text{ acts as 1 on } F_\mu\text{, so that } KQ = Q\text{)},$$

so that $(xP, Q) = -(P, \omega(x) Q)$ is proven), we can WLOG assume that we are in the former case, i. e., that $x = a_j$ for some $j \in \mathbb{Z}$. Assume this, and consider this j.

Since P is a monomial, there exists a $(n_1, n_2, n_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ such that $P = x_1^{n_1} x_2^{n_2} x_3^{n_3} ...$. Consider this $(n_1, n_2, n_3, ...)$.

Since Q is a monomial, there exists a $(m_1, m_2, m_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ such that $Q = x_1^{m_1} x_2^{m_2} x_3^{m_3} ...$. Consider this $(m_1, m_2, m_3, ...)$.

We must prove that $(xP, Q) = -(P, \omega(x) Q)$. Since $(xP, Q) = (a_j P, Q)$ (because $x = a_j$) and $-(P, \omega(x) Q) = (P, a_{-j} Q)$ (because $-\left(P, \omega\left(\underbrace{x}_{=a_j}\right) Q\right) = -\left(P, \underbrace{\omega(a_j)}_{=-a_{-j}} Q\right) = -(P, -a_{-j} Q) = (P, a_{-j} Q)$), this rewrites as $(a_j P, Q) = (P, a_{-j} Q)$. Hence, we must only prove that $(a_j P, Q) = (P, a_{-j} Q)$.

We will distinguish between three cases:

Case 1: We have $j \geq 1$.

Case 2: We have $j = 0$.

Case 3: We have $j \leq -1$.

First, let us consider Case 1. In this case, by the definition of F_μ, we know that a_j acts on F_μ as $j \dfrac{\partial}{\partial x_j}$, whereas a_{-j} acts on F_μ as multiplication by x_j. Hence, $a_j P = j \dfrac{\partial}{\partial x_j} P$ and $a_{-j} Q = x_j Q$.

Since $Q = x_1^{m_1} x_2^{m_2} x_3^{m_3} ...$, we have $x_j Q = x_1^{m_1'} x_2^{m_2'} x_3^{m_3'} ...$, where the sequence $(m_1', m_2', m_3', ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ is defined by

$$m_i' = \begin{cases} m_i, & \text{if } i \neq j; \\ m_i + 1, & \text{if } i = j \end{cases} \qquad \text{for every } i \in \{1, 2, 3, ...\}.$$

Note that this definition immediately yields $m_j' = m_j + 1 \geq 1$, so that $\delta_{0, m_j'} = 0$.

As a consequence of the definition of $(m_1', m_2', m_3', ...)$, we have $m_i' - m_i = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j \end{cases}$ for every $i \in \{1, 2, 3, ...\}$.

Now, $(a_j P, Q) = (P, a_{-j} Q)$ is easily proven when $n_j = 0$ [36]. Hence, for the remaining part of Case 1, we can WLOG assume that $n_j \neq 0$. Let us assume this. Then, $n_j \geq 1$. Hence, since $P = x_1^{n_1} x_2^{n_2} x_3^{n_3} ...$, we have $\dfrac{\partial}{\partial x_j} P = n_j x_1^{n'_1} x_2^{n'_2} x_3^{n'_3} ...$, where the sequence $(n'_1, n'_2, n'_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ is defined by

$$n'_i = \begin{cases} n_i, & \text{if } i \neq j; \\ n_i - 1, & \text{if } i = j \end{cases} \qquad \text{for every } i \in \{1, 2, 3, ...\}.$$

From this definition, it is clear that the sequence $(n'_1, n'_2, n'_3, ...)$ differs from the sequence $(n_1, n_2, n_3, ...)$ only in the j-th term. Hence, the product $\prod\limits_{i=1}^{\infty} i^{n'_i}$ differs from the product $\prod\limits_{i=1}^{\infty} i^{n_i}$ only in the j-th factor. Thus,

$$\frac{\prod\limits_{i=1}^{\infty} i^{n_i}}{\prod\limits_{i=1}^{\infty} i^{n'_i}} = \frac{j^{n_j}}{j^{n'_j}} = \frac{j^{n_j}}{j^{n_j - 1}} \qquad \left(\text{since } n'_j = n_j - 1 \text{ by the definition of } (n'_1, n'_2, n'_3, ...)\right)$$
$$= j,$$

so that $\prod\limits_{i=1}^{\infty} i^{n_i} = j \prod\limits_{i=1}^{\infty} i^{n'_i}$. A similar argument (using the products $\prod\limits_{i=1}^{\infty} n'_i$ and $\prod\limits_{i=1}^{\infty} n_i$ instead of the products $\prod\limits_{i=1}^{\infty} i^{n'_i}$ and $\prod\limits_{i=1}^{\infty} i^{n_i}$) shows that $\prod\limits_{i=1}^{\infty} n_i! = n_j \prod\limits_{i=1}^{\infty} n'_i!$.

As a consequence of the definition of $(n'_1, n'_2, n'_3, ...)$, we have $n_i - n'_i = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j \end{cases}$ for every $i \in \{1, 2, 3, ...\}$. Thus, every $i \in \{1, 2, 3, ...\}$ satisfies

$$n_i - n'_i = \begin{cases} 0, & \text{if } i \neq j; \\ 1, & \text{if } i = j \end{cases} = m'_i - m_i,$$

[36] *Proof.* Assume that $n_j = 0$. Then, $P = x_1^{n_1} x_2^{n_2} x_3^{n_3} ...$ is a monomial that does not involve the indeterminate x_j; hence, $\dfrac{\partial}{\partial x_j} P = 0$, so that $a_j P = j \underbrace{\dfrac{\partial}{\partial x_j} P}_{=0} = 0$, and thus $(a_j P, Q) = (0, Q) = 0$.

On the other hand, since $n_j = 0$, we have $\delta_{n_j, m'_j} = \delta_{0, m'_j} = 0$ and thus $\prod\limits_{i=1}^{\infty} \delta_{n_i, m'_i} = 0$ (since the product $\prod\limits_{i=1}^{\infty} \delta_{n_i, m'_i}$ contains the factor δ_{n_j, m'_j}). Now, since $P = x_1^{n_1} x_2^{n_2} x_3^{n_3} ...$ and $a_{-j} Q = x_j Q = x_1^{m'_1} x_2^{m'_2} x_3^{m'_3} ...$, we have

$$(P, a_{-j} Q) = \left(x_1^{n_1} x_2^{n_2} x_3^{n_3} ..., x_1^{m'_1} x_2^{m'_2} x_3^{m'_3} ...\right)$$
$$= \underbrace{\prod_{i=1}^{\infty} \delta_{n_i, m'_i}}_{=0} \cdot \prod_{i=1}^{\infty} i^{n_i} \cdot \prod_{i=1}^{\infty} n_i! \qquad (\text{by the definition of } (\cdot, \cdot))$$
$$= 0 = (a_j P, Q).$$

Hence, $(a_j P, Q) = (P, a_{-j} Q)$ is proven when $n_j = 0$.

so that $n_i - m_i' = n_i' - m_i$, so that $\delta_{n_i - m_i',0} = \delta_{n_i' - m_i,0}$.

Now, since $P = x_1^{n_1} x_2^{n_2} x_3^{n_3} ...$ and $a_{-j} Q = x_j Q = x_1^{m_1'} x_2^{m_2'} x_3^{m_3'} ...$, we have

$$\begin{aligned}(P, a_{-j} Q) &= \left(x_1^{n_1} x_2^{n_2} x_3^{n_3} ..., x_1^{m_1'} x_2^{m_2'} x_3^{m_3'} ... \right) \\ &= \prod_{i=1}^{\infty} \underbrace{\delta_{n_i, m_i'}}_{= \delta_{n_i - m_i', 0} = \delta_{n_i' - m_i, 0} = \delta_{n_i', m_i}} \cdot \prod_{i=1}^{\infty} i^{n_i} \cdot \prod_{i=1}^{\infty} n_i! \qquad \text{(by the definition of } (\cdot, \cdot)) \\ &= \prod_{i=1}^{\infty} \delta_{n_i', m_i} \cdot \underbrace{\prod_{i=1}^{\infty} i^{n_i}}_{= j \prod_{i=1}^{\infty} i^{n_i'}} \cdot \underbrace{\prod_{i=1}^{\infty} n_i!}_{= n_j \prod_{i=1}^{\infty} n_i'!} = j n_j \cdot \prod_{i=1}^{\infty} \delta_{n_i', m_i} \cdot \prod_{i=1}^{\infty} i^{n_i'} \cdot \prod_{i=1}^{\infty} n_i'!.\end{aligned}$$

Compared with

$$\begin{aligned}(a_j P, Q) &= \left(j n_j x_1^{n_1'} x_2^{n_2'} x_3^{n_3'} ..., x_1^{m_1} x_2^{m_2} x_3^{m_3} ... \right) \\ &\qquad \left(\text{since } a_j P = j \underbrace{\frac{\partial}{\partial x_j} P}_{= n_j x_1^{n_1'} x_2^{n_2'} x_3^{n_3'} ...} = j n_j x_1^{n_1'} x_2^{n_2'} x_3^{n_3'} ... \text{ and } Q = x_1^{m_1} x_2^{m_2} x_3^{m_3} ... \right) \\ &= j n_j \underbrace{\left(x_1^{n_1'} x_2^{n_2'} x_3^{n_3'} ..., x_1^{m_1} x_2^{m_2} x_3^{m_3} ... \right)}_{\substack{= \prod_{i=1}^{\infty} \delta_{n_i', m_i} \cdot \prod_{i=1}^{\infty} i^{n_i'} \cdot \prod_{i=1}^{\infty} n_i'! \\ \text{(by the definition of } (\cdot, \cdot))}} = j n_j \cdot \prod_{i=1}^{\infty} \delta_{n_i', m_i} \cdot \prod_{i=1}^{\infty} i^{n_i'} \cdot \prod_{i=1}^{\infty} n_i'!,\end{aligned}$$

this yields $(a_j P, Q) = (P, a_{-j} Q)$. Thus, $(a_j P, Q) = (P, a_{-j} Q)$ is proven in Case 1. In other words, we have shown that

$$(a_j P, Q) = (P, a_{-j} Q) \qquad \text{for every integer } j \geq 1 \text{ and any monomials } P \text{ and } Q. \tag{22}$$

In Case 2, proving $(a_j P, Q) = (P, a_{-j} Q)$ is trivial (since a_0 acts on F_μ as $\mu \cdot \mathrm{id}$).

Now, let us consider Case 3. In this case, $j \leq -1$, so that $-j \geq 1$. Thus, (22) (applied to $-j$, Q and P instead of j, P and Q) yields $(a_{-j} Q, P) = \left(Q, a_{-(-j)} P \right)$. Now, since $(\cdot, \cdot)$ is symmetric, we have $(a_j P, Q) = \left(Q, \underbrace{a_j}_{= a_{-(-j)}} P \right) = \left(Q, a_{-(-j)} P \right) = (a_{-j} Q, P) = (P, a_{-j} Q)$ (again since $(\cdot, \cdot)$ is symmetric). Thus, $(a_j P, Q) = (P, a_{-j} Q)$ is proven in Case 3.

We have now proven $(a_j P, Q) = (P, a_{-j} Q)$ is each of the cases 1, 2 and 3. Since no other cases can occur, this completes the proof of $(a_j P, Q) = (P, a_{-j} Q)$. As we have explained above, this proves Proposition 2.2.24 **(c)**.

(d) Let $x \in \mathcal{A}$, $P \in F_\mu$ and $Q \in F_\mu$. Since the form $(\cdot, \cdot)$ is symmetric, we have $(P, xQ) = (xQ, P)$ and $(\omega(x) P, Q) = (Q, \omega(x) P)$. Proposition 2.2.24 **(c)** (applied to P and Q instead of Q and P) yields $(xQ, P) = -(Q, \omega(x) P)$. Thus, $(P, xQ) = (xQ, P) = -\underbrace{(Q, \omega(x) P)}_{=(\omega(x)P,Q)} = -(\omega(x) P, Q)$. This proves Proposition 2.2.24 **(d)**.

(e) and **(f)** The proofs of Proposition 2.2.24 **(e)** and **(f)** are analogous to those of Proposition 2.2.24 **(c)** and **(d)**, respectively, and thus will be omitted.

2.3. Representations of the Virasoro algebra Vir

We now come to the Virasoro algebra Vir. First, some notations:

Definition 2.3.1. **(a)** The notion "*Virasoro module*" will be a synonym for "Vir-module". Similarly, "Virasoro action" means "Vir-action".

(b) Let $c \in \mathbb{C}$. A Vir-module M is said to have *central charge* c if and only if the element C of Vir acts as $c \cdot \operatorname{id}$ on M.

Note that not every Vir-module has a central charge (and the zero module has infinitely many central charges), but Corollary 2.1.3 yields that every irreducible Vir-module of countable dimension has a (unique) central charge.

There are lots and lots of Virasoro modules in mathematics, and we will encounter them as this course progresses; the more complicated among them will require us to introduce a lot of machinery like Verma modules, semiinfinite wedges and affine Lie algebras. For now, we define one of the simplest families of representations of Vir: the "chargeless" Vir-modules $V_{\alpha,\beta}$ parametrized by pairs of complex numbers (α, β).

Proposition 2.3.2. Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. Let $V_{\alpha,\beta}$ be the vector space of formal expressions of the form $gt^{\alpha}(dt)^{\beta}$ with $g \in \mathbb{C}[t, t^{-1}]$ (where $\mathbb{C}[t, t^{-1}]$ is the ring of Laurent polynomials in the variable t). (Formally, this vector space $V_{\alpha,\beta}$ is defined to be a copy of the $\mathbb{C}$-vector space $\mathbb{C}[t, t^{-1}]$, but in which the element corresponding to any $g \in \mathbb{C}[t, t^{-1}]$ is denoted by $gt^{\alpha}(dt)^{\beta}$. For a geometric intuition, the elements of $V_{\alpha,\beta}$ can be seen as "tensor fields" of rank β and branching α on the punctured complex plane $\mathbb{C}^{\times}$.)

(a) The formula

$$f\partial \rightharpoonup \left(gt^{\alpha}(dt)^{\beta}\right) = \left(fg' + \alpha t^{-1} fg + \beta f' g\right) t^{\alpha}(dt)^{\beta} \tag{23}$$

defines an action of W on $V_{\alpha,\beta}$. Thus, $V_{\alpha,\beta}$ becomes a Vir-module with C acting as 0. (In other words, $V_{\alpha,\beta}$ becomes a Vir-module with central charge 0.)

(b) For every $k \in \mathbb{Z}$, let $v_k = t^{-k+\alpha}(dt)^{\beta} \in V_{\alpha,\beta}$. Here, for any $\ell \in \mathbb{Z}$, the term $t^{\ell+\alpha}(dt)^{\beta}$ denotes $t^{\ell} t^{\alpha}(dt)^{\beta}$. Then,

$$L_m v_k = (k - \alpha - \beta(m+1)) v_{k-m} \qquad \text{for every } m \in \mathbb{Z} \text{ and } k \in \mathbb{Z}. \tag{24}$$

Note that Proposition 2.3.2 was Homework Set 1 exercise 1, but the notation v_k had a slightly different meaning in Homework Set 1 exercise 1 than it has here.

The proof of this proposition consists of straightforward computations. We give it for the sake of completeness, slightly simplifying the calculation by introducing auxiliary functions.

Proof of Proposition 2.3.2. **(a)** In order to prove Proposition 2.3.2 **(a)**, we must show that the formula (23) defines an action of W on $V_{\alpha,\beta}$.

It is clear that $(fg' + \alpha t^{-1} fg + \beta f'g)\, t^\alpha (dt)^\beta$ depends linearly on each of f and g. Hence, we must only prove that, with the definition (23), we have

$$[f\partial, g\partial] \rightharpoonup \left(ht^\alpha (dt)^\beta\right) = f\partial \rightharpoonup \left(g\partial \rightharpoonup \left(ht^\alpha (dt)^\beta\right)\right) - g\partial \rightharpoonup \left(f\partial \rightharpoonup \left(ht^\alpha (dt)^\beta\right)\right) \tag{25}$$

for any Laurent polynomials f, g and h in $\mathbb{C}[t, t^{-1}]$.

So let f, g and h be any three Laurent polynomials in $\mathbb{C}[t, t^{-1}]$. Denote by p the Laurent polynomial $h' + \alpha t^{-1} h$. Denote by q the Laurent polynomial $fg' - gf'$. Then,[37]

$$f(g'h)' - g(f'h)' = q'h + qh' \tag{26}$$

[38] and

$$\begin{aligned}
& f \underbrace{(gp)'}_{\substack{=g'p+gp' \\ \text{(by the Leibniz rule)}}} - g \underbrace{(fp)'}_{\substack{=f'p+fp' \\ \text{(by the Leibniz rule)}}} \\
&= \underbrace{f(g'p + gp')}_{=fg'p+fgp'} - \underbrace{g(f'p + fp')}_{=gf'p+gfp'=gf'p+fgp'} \\
&= fg'p + fgp' - gf'p - fgp' = fg'p - gf'p = \underbrace{(fg' - gf')}_{=q} p = qp.
\end{aligned} \tag{27}$$

Also,

$$\begin{aligned}
\underbrace{[f\partial, g\partial]}_{=(fg'-gf')\partial} \rightharpoonup \left(ht^\alpha (dt)^\beta\right) &= \underbrace{(fg' - gf')}_{=q} \partial \rightharpoonup \left(ht^\alpha (dt)^\beta\right) = q\partial \rightharpoonup \left(ht^\alpha (dt)^\beta\right) \\
&= \left(\underbrace{qh' + \alpha t^{-1} qh}_{\substack{=q\left(h' + \alpha t^{-1} h\right) = qp \\ \text{(since } h' + \alpha t^{-1} h = p)}} + \beta q'h \right) t^\alpha (dt)^\beta = (qp + \beta q'h)\, t^\alpha (dt)^\beta .
\end{aligned}$$

---

[37] In the following computations, terms like $f(u)$ (where u is a subterm, usually a complicated one) have to be understood as $f \cdot u$ (the product of f with u) and not as $f(u)$ (the Laurent polynomial f applied to u).

[38] *Proof of (26):* Since $q = fg' - gf'$, we have $qh = (fg' - gf')\, h = fg'h - gf'h = f(g'h) - g(f'h)$, so that

$$\begin{aligned}
(qh)' = (f(g'h) - g(f'h))' &= \underbrace{(f(g'h))'}_{\substack{=f'(g'h)+f(g'h)' \\ \text{(by the Leibniz rule)}}} - \underbrace{(g(f'h))'}_{\substack{=g'(f'h)+g(f'h)' \\ \text{(by the Leibniz rule)}}} \\
&= \underbrace{f'(g'h)}_{=f'g'h} + f(g'h)' - \underbrace{g'(f'h)}_{=f'g'h} - g(f'h)' \\
&= f'g'h + f(g'h)' - f'g'h - g(f'h)' = f(g'h)' - g(f'h)' .
\end{aligned}$$

Since $(qh)' = q'h + qh'$ (by the Leibniz rule), this rewrites as $q'h + qh' = f(g'h)' - g(f'h)'$. This proves (26).

Moreover, $gh' + \alpha t^{-1} gh = g \underbrace{\left(h' + \alpha t^{-1} h\right)}_{=p} = gp$, and

$$g\partial \rightharpoonup \left(ht^{\alpha}\left(dt\right)^{\beta}\right) = \left(\underbrace{gh' + \alpha t^{-1} gh}_{=gp} + \beta g' h\right) t^{\alpha}\left(dt\right)^{\beta} = \left(gp + \beta g' h\right) t^{\alpha}\left(dt\right)^{\beta},$$

so that

$$\begin{aligned}
& f\partial \rightharpoonup \underbrace{\left(g\partial \rightharpoonup \left(ht^{\alpha}\left(dt\right)^{\beta}\right)\right)}_{=(gp+\beta g'h)t^{\alpha}(dt)^{\beta}} \\
&= f\partial \rightharpoonup \left(\left(gp + \beta g' h\right) t^{\alpha}\left(dt\right)^{\beta}\right) \\
&= \left(f \underbrace{\left(gp + \beta g' h\right)'}_{=(gp)'+\beta(g'h)'} + \underbrace{\alpha t^{-1} f\left(gp + \beta g' h\right)}_{=\alpha t^{-1} fgp + \alpha\beta t^{-1} fg'h} + \underbrace{\beta f'\left(gp + \beta g' h\right)}_{=\beta f' gp + \beta^{2} f' g' h}\right) t^{\alpha}\left(dt\right)^{\beta} \\
&= \left(\underbrace{f\left(\left(gp\right)' + \beta\left(g'h\right)'\right)}_{=f(gp)'+\beta f(g'h)'} + \alpha t^{-1} fgp + \alpha\beta t^{-1} fg'h + \beta f' gp + \beta^{2} f' g' h\right) t^{\alpha}\left(dt\right)^{\beta} \\
&= \left(f\left(gp\right)' + \beta f\left(g'h\right)' + \alpha t^{-1} fgp + \alpha\beta t^{-1} fg'h + \beta f' gp + \beta^{2} f' g' h\right) t^{\alpha}\left(dt\right)^{\beta}. \qquad (28)
\end{aligned}$$

Since the roles of f and g in our situation are symmetric, we can interchange f and g in (28), and obtain

$$\begin{aligned}
& g\partial \rightharpoonup \left(f\partial \rightharpoonup \left(ht^{\alpha}\left(dt\right)^{\beta}\right)\right) \\
&= \left(g\left(fp\right)' + \beta g\left(f'h\right)' + \alpha t^{-1} gfp + \alpha\beta t^{-1} gf'h + \beta g' fp + \beta^{2} g' f' h\right) t^{\alpha}\left(dt\right)^{\beta} \\
&= \left(g\left(fp\right)' + \beta g\left(f'h\right)' + \alpha t^{-1} fgp + \alpha\beta t^{-1} gf'h + \beta g' fp + \beta^{2} f' g' h\right) t^{\alpha}\left(dt\right)^{\beta}. \qquad (29)
\end{aligned}$$

Thus,

$$\underbrace{f\partial \rightharpoonup \left(g\partial \rightharpoonup \left(ht^{\alpha}\left(dt\right)^{\beta}\right)\right)}_{\substack{=\left(f(gp)'+\beta f(g'h)'+\alpha t^{-1}fgp+\alpha\beta t^{-1}fg'h+\beta f'gp+\beta^2 f'g'h\right)t^{\alpha}(dt)^{\beta} \\ \text{(by (28))}}} - \underbrace{g\partial \rightharpoonup \left(f\partial \rightharpoonup \left(ht^{\alpha}\left(dt\right)^{\beta}\right)\right)}_{\substack{=\left(g(fp)'+\beta g(f'h)'+\alpha t^{-1}fgp+\alpha\beta t^{-1}gf'h+\beta g'fp+\beta^2 f'g'h\right)t^{\alpha}(dt)^{\beta} \\ \text{(by (29))}}}$$

$$= \left(f\left(gp\right)' + \beta f\left(g'h\right)' + \alpha t^{-1}fgp + \alpha\beta t^{-1}fg'h + \beta f'gp + \beta^2 f'g'h\right) t^{\alpha}\left(dt\right)^{\beta} - \left(g\left(fp\right)' + \beta g\left(f'h\right)' + \alpha t^{-1}fgp + \alpha\beta t^{-1}gf'h + \beta g'fp + \beta^2 f'g'h\right) t^{\alpha}\left(dt\right)^{\beta}$$

$$= \left(\left(f\left(gp\right)' + \beta f\left(g'h\right)' + \alpha t^{-1}fgp + \alpha\beta t^{-1}fg'h + \beta f'gp + \beta^2 f'g'h\right) - \left(g\left(fp\right)' + \beta g\left(f'h\right)' + \alpha t^{-1}fgp + \alpha\beta t^{-1}gf'h + \beta g'fp + \beta^2 f'g'h\right)\right) t^{\alpha}\left(dt\right)^{\beta}$$

$$= \left(\underbrace{f\left(gp\right)' - g\left(fp\right)'}_{\substack{=qp \\ \text{(by (27))}}} + \underbrace{\beta f\left(g'h\right)' - \beta g\left(f'h\right)'}_{=\beta\left(f(g'h)'-g(f'h)'\right)} + \underbrace{\alpha\beta t^{-1}fg'h - \alpha\beta t^{-1}gf'h}_{=\alpha\beta t^{-1}(fg'-gf')h} + \underbrace{\beta f'gp - \beta g'fp}_{=\beta(f'g-g'f)p}\right) t^{\alpha}\left(dt\right)^{\beta}$$

$$= \left(qp + \beta\underbrace{\left(f\left(g'h\right)' - g\left(f'h\right)'\right)}_{\substack{=q'h+qh' \\ \text{(by (26))}}} + \alpha\beta t^{-1}\underbrace{\left(fg' - gf'\right)}_{=q}h + \beta\underbrace{\left(f'g - g'f\right)}_{\substack{=-q \\ \text{(since } q=fg'-gf'=g'f-f'g)}}p\right) t^{\alpha}\left(dt\right)^{\beta}$$

$$= \left(qp + \underbrace{\beta\left(q'h + qh'\right)}_{=\beta q'h+\beta qh'} + \alpha\beta t^{-1}qh + \underbrace{\beta\left(-q\right)p}_{=-\beta qp}\right) t^{\alpha}\left(dt\right)^{\beta}$$

$$= \left(qp + \beta q'h + \underbrace{\beta qh' + \alpha\beta t^{-1}qh}_{=\beta q(h'+\alpha t^{-1}h)} - \beta qp\right) t^{\alpha}\left(dt\right)^{\beta}$$

$$= \left(qp + \beta q'h + \beta q\underbrace{\left(h' + \alpha t^{-1}h\right)}_{=p} - \beta qp\right) t^{\alpha}\left(dt\right)^{\beta}$$

$$= \left(qp + \beta q'h + \underbrace{\beta qp - \beta qp}_{=0}\right) t^{\alpha}\left(dt\right)^{\beta} = \left(qp + \beta q'h\right) t^{\alpha}\left(dt\right)^{\beta} = \left[f\partial, g\partial\right] \rightharpoonup \left(ht^{\alpha}\left(dt\right)^{\beta}\right).$$

Thus, (25) is proven for any Laurent polynomials f, g and h. This proves that the formula (23) defines an action of W on $V_{\alpha,\beta}$. Hence, $V_{\alpha,\beta}$ becomes a W-module, i. e., a Vir-module with C acting as 0. (In other words, $V_{\alpha,\beta}$ becomes a Vir-module with central charge 0.) This proves Proposition 2.3.2 **(a)**.

(b) We only need to prove (24).

Let $m \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Then, $v_k = t^{-k+\alpha}\left(dt\right)^{\beta} = t^{-k}t^{\alpha}\left(dt\right)^{\beta}$ and $v_{k-m} = t^{-(k-m)+\alpha}\left(dt\right)^{\beta} =$

$t^{m-k}t^{\alpha}(dt)^{\beta}$. Thus,

$$\begin{aligned}
&\underbrace{L_m}_{=-t^{m+1}\partial} \rightharpoonup \underbrace{v_k}_{=t^{-k}t^{\alpha}(dt)^{\beta}} \\
&= \left(-t^{m+1}\partial\right) \rightharpoonup \left(t^{-k}t^{\alpha}(dt)^{\beta}\right) \\
&= \left(-t^{m+1}\underbrace{\left(t^{-k}\right)'}_{=-kt^{-k-1}} + \underbrace{\alpha t^{-1}\left(-t^{m+1}\right)t^{-k}}_{=-\alpha t^{-1}t^{m+1}t^{-k}} + \beta\underbrace{\left(-t^{m+1}\right)'}_{=-(m+1)t^{m}}t^{-k}\right) t^{\alpha}(dt)^{\beta} \\
&\qquad \left(\text{by (23), applied to } f = -t^{m+1} \text{ and } g = t^{-k}\right) \\
&= \left(-(-k)\underbrace{t^{m+1}t^{-k-1}}_{=t^{m-k}} - \alpha\underbrace{t^{-1}t^{m+1}t^{-k}}_{=t^{(-1)+(m+1)+(-k)}=t^{m-k}} + \beta(-(m+1))\underbrace{t^{m}t^{-k}}_{=t^{m-k}}\right) t^{\alpha}(dt)^{\beta} \\
&= \left(kt^{m-k} - \alpha t^{m-k} + \beta(-(m+1))t^{m-k}\right) t^{\alpha}(dt)^{\beta} \\
&= \underbrace{(k-\alpha+\beta(-(m+1)))}_{=k-\alpha-(m+1)\beta}\underbrace{t^{m-k}t^{\alpha}(dt)^{\beta}}_{=v_{k-m}} = (k-\alpha-(m+1)\beta)\,v_{k-m}.
\end{aligned}$$

This proves (24). Proposition 2.3.2 **(b)** is proven.

The representations $V_{\alpha,\beta}$ are not all pairwise non-isomorphic, but there are still uncountably many non-isomorphic ones among them. More precisely:

Proposition 2.3.3. **(a)** For every $\ell \in \mathbb{Z}$, $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$, the $\mathbb{C}$-linear map

$$\begin{aligned}
V_{\alpha,\beta} &\to V_{\alpha+\ell,\beta}, \\
gt^{\alpha}(dt)^{\beta} &\mapsto \left(gt^{-\ell}\right)t^{\alpha+\ell}(dt)^{\beta}
\end{aligned}$$

is an isomorphism of Vir-modules. (This map sends v_k to $v_{k+\ell}$ for every $k \in \mathbb{Z}$.)

(b) For every $\alpha \in \mathbb{C}$, the $\mathbb{C}$-linear map

$$\begin{aligned}
V_{\alpha,0} &\to V_{\alpha-1,1}, \\
gt^{\alpha}(dt)^{0} &\mapsto (-g't-\alpha g)\,t^{\alpha-1}(dt)^{1}
\end{aligned}$$

is a homomorphism of Vir-modules. (This map sends v_k to $(k-\alpha)\,v_k$ for every $k \in \mathbb{Z}$.) If $\alpha \notin \mathbb{Z}$, then this map is an isomorphism.

(c) Let $(\alpha,\beta,\alpha',\beta') \in \mathbb{C}^4$. Then, $V_{\alpha,\beta} \cong V_{\alpha',\beta'}$ as Vir-modules if and only if either ($\beta = \beta'$ and $\alpha - \alpha' \in \mathbb{Z}$) or ($\beta = 0$, $\beta' = 1$, $\alpha - \alpha' \in \mathbb{Z}$ and $\alpha \notin \mathbb{Z}$) or ($\beta = 1$, $\beta' = 0$, $\alpha - \alpha' \in \mathbb{Z}$ and $\alpha \notin \mathbb{Z}$).

Proof of Proposition 2.3.3 (sketched). **(a)** and **(b)** Very easy and left to the reader.

(c) The $\Longleftarrow$ direction is handled by parts **(a)** and **(b)**.

$\Longrightarrow$: Assume that $V_{\alpha,\beta} \cong V_{\alpha',\beta'}$ as Vir-modules. We must prove that either ($\beta = \beta'$ and $\alpha - \alpha' \in \mathbb{Z}$) or ($\beta = 0$, $\beta' = 1$, $\alpha - \alpha' \in \mathbb{Z}$ and $\alpha \notin \mathbb{Z}$) or ($\beta = 1$, $\beta' = 0$, $\alpha - \alpha' \in \mathbb{Z}$ and $\alpha \notin \mathbb{Z}$).

Let Φ be the Vir-module isomorphism $V_{\alpha,\beta} \to V_{\alpha',\beta'}$.

Applying (24) to $m = 0$, we obtain

$$L_0 v_k = (k-\alpha-\beta)\,v_k \text{ in } V_{\alpha,\beta} \qquad \text{for every } k \in \mathbb{Z}. \tag{30}$$

Hence, L_0 acts on $V_{\alpha,\beta}$ as a diagonal matrix with eigenvalues $k-\alpha-\beta$ for all $k \in \mathbb{Z}$, each eigenvalue appearing exactly once. Similarly, applying (24) to 0 and (α', β') instead of m and (α, β), we obtain

$$L_0 v_k = (k - \alpha' - \beta') v_k \text{ in } V_{\alpha',\beta'} \qquad \text{for every } k \in \mathbb{Z}. \tag{31}$$

Thus, L_0 acts on $V_{\alpha',\beta'}$ as a diagonal matrix with eigenvalues $k - \alpha' - \beta'$ for all $k \in \mathbb{Z}$, each eigenvalue appearing exactly once.

But since $V_{\alpha,\beta} \cong V_{\alpha',\beta'}$ as Vir-modules, the eigenvalues of L_0 acting on $V_{\alpha,\beta}$ must be the same as the eigenvalues of L_0 acting on $V_{\alpha',\beta'}$. In other words,

$$\{k - \alpha - \beta \mid k \in \mathbb{Z}\} = \{k - \alpha' - \beta' \mid k \in \mathbb{Z}\}$$

(because we know that the eigenvalues of L_0 acting on $V_{\alpha,\beta}$ are $k - \alpha - \beta$ for all $k \in \mathbb{Z}$, while the eigenvalues of L_0 acting on $V_{\alpha',\beta'}$ are $k - \alpha' - \beta'$ for all $k \in \mathbb{Z}$). Hence, $(\alpha + \beta) - (\alpha' + \beta') \in \mathbb{Z}$. Since we can shift α by an arbitrary integer without changing the isomorphism class of $V_{\alpha,\beta}$ (due to part **(a)**), we can thus WLOG assume that $\alpha + \beta = \alpha' + \beta'$.

Let us once again look at the equality (30). This equality tells us that, for each $k \in \mathbb{Z}$, the vector v_k is the unique (up to scaling) eigenvector of the operator L_0 with eigenvalue $k - \alpha - \beta$ in $V_{\alpha,\beta}$. The isomorphism Φ (being Vir-linear) must map this vector v_k to an eigenvector of the operator L_0 with eigenvalue $k-\alpha-\beta$ in $V_{\alpha',\beta'}$. Since $\alpha + \beta = \alpha' + \beta'$, this eigenvalue equals $k - \alpha' - \beta'$. But (due to (31)) the unique (up to scaling) eigenvector of the operator L_0 with eigenvalue $k - \alpha' - \beta'$ in $V_{\alpha',\beta'}$ is v_k. Hence, $\Phi(v_k)$ must equal v_k up to scaling, i. e., there exists a nonzero complex number λ_k such that $\Phi(v_k) = \lambda_k v_k$.

Now, let $m \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Then, in $V_{\alpha,\beta}$, we have

$$L_m v_k = (k - \alpha - \beta(m+1)) v_{k-m},$$

so that

$$\begin{aligned}\Phi(L_m v_k) &= \Phi((k - \alpha - \beta(m+1)) v_{k-m}) = (k - \alpha - \beta(m+1)) \underbrace{\Phi(v_{k-m})}_{=\lambda_{k-m} v_{k-m}} \\ &= \lambda_{k-m} (k - \alpha - \beta(m+1)) v_{k-m}\end{aligned}$$

in $V_{\alpha',\beta'}$. Compared with

$$\begin{aligned}\Phi(L_m v_k) &= L_m \underbrace{\Phi(v_k)}_{=\lambda_k v_k} \qquad \text{(since } \Phi \text{ is Vir-linear)} \\ &= \lambda_k \underbrace{L_m v_k}_{=(k-\alpha'-\beta'(m+1))v_{k-m}} = \lambda_k (k - \alpha' - \beta'(m+1)) v_{k-m}\end{aligned}$$

in $V_{\alpha',\beta'}$, this yields

$$\lambda_{k-m} (k - \alpha - \beta(m+1)) v_{k-m} = \lambda_k (k - \alpha' - \beta'(m+1)) v_{k-m}.$$

Since $v_{k-m} \neq 0$, this yields

$$\lambda_{k-m} (k - \alpha - \beta(m+1)) = \lambda_k (k - \alpha' - \beta'(m+1)). \tag{32}$$

Now, any $m \in \mathbb{Z}$, $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$ satisfy

$$\lambda_{k-(n+m)} (k - \alpha - \beta (m+1)) = \lambda_k (k - \alpha' - \beta' (n+m+1)) \tag{33}$$

(by (32), applied to $n+m$ instead of m) and

$$\lambda_{k-m-n} (k - m - \alpha - \beta (n+1)) = \lambda_{k-m} (k - m - \alpha' - \beta' (n+1)) \tag{34}$$

(by (32), applied to $k-m$ and n instead of k and m). Hence, any $m \in \mathbb{Z}$, $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$ satisfy

$$\begin{aligned}
&\lambda_k \lambda_{k-m} \lambda_{k-m-n} \cdot (k - \alpha' - \beta' (n+m+1)) \cdot (k - \alpha - \beta (m+1)) \cdot (k - m - \alpha - \beta (n+1)) \\
&= \underbrace{\lambda_k (k - \alpha' - \beta' (n+m+1))}_{\substack{=\lambda_{k-(n+m)}(k-\alpha-\beta(m+1)) \\ \text{(by (33))}}} \cdot \underbrace{\lambda_{k-m} (k - \alpha - \beta (m+1))}_{\substack{=\lambda_k(k-\alpha'-\beta'(m+1)) \\ \text{(by (32))}}} \cdot \underbrace{\lambda_{k-m-n} (k - m - \alpha - \beta (n+1))}_{\substack{=\lambda_{k-m}(k-m-\alpha'-\beta'(n+1)) \\ \text{(by (34))}}} \\
&= \lambda_{k-(n+m)} (k - \alpha - \beta (m+1)) \cdot \lambda_k (k - \alpha' - \beta' (m+1)) \cdot \lambda_{k-m} (k - m - \alpha' - \beta' (n+1)) \\
&= \lambda_k \lambda_{k-m} \underbrace{\lambda_{k-(n+m)}}_{=\lambda_{k-m-n}} \cdot (k - \alpha - \beta (n+m+1)) \cdot (k - \alpha' - \beta' (m+1)) \cdot (k - m - \alpha' - \beta' (n+1)) \\
&= \lambda_k \lambda_{k-m} \lambda_{k-m-n} \cdot (k - \alpha - \beta (n+m+1)) \cdot (k - \alpha' - \beta' (m+1)) \cdot (k - m - \alpha' - \beta' (n+1)) .
\end{aligned}$$

We can divide this equality by $\lambda_k \lambda_{k-m} \lambda_{k-m-n}$ (since $\lambda_i \neq 0$ for every $i \in \mathbb{Z}$, and therefore we have $\lambda_k \lambda_{k-m} \lambda_{k-m-n} \neq 0$), and thus obtain that any $m \in \mathbb{Z}$, $k \in \mathbb{Z}$ and $n \in \mathbb{Z}$ satisfy

$$\begin{aligned}
&(k - \alpha' - \beta' (n+m+1)) \cdot (k - \alpha - \beta (m+1)) \cdot (k - m - \alpha - \beta (n+1)) \\
&= (k - \alpha - \beta (n+m+1)) \cdot (k - \alpha' - \beta' (m+1)) \cdot (k - m - \alpha' - \beta' (n+1)) .
\end{aligned}$$

Since $\mathbb{Z}^3$ is Zariski-dense in $\mathbb{C}^3$, this yields that

$$\begin{aligned}
&(X - \alpha' - \beta' (Y+Z+1)) \cdot (X - \alpha - \beta (Z+1)) \cdot (X - Z - \alpha - \beta (Y+1)) \\
&= (X - \alpha - \beta (Y+Z+1)) \cdot (X - \alpha' - \beta' (Z+1)) \cdot (X - Z - \alpha' - \beta' (Y+1)) .
\end{aligned}$$

holds as a polynomial identity in the polynomial ring $\mathbb{C}[X, Y, Z]$.

If we compare coefficients before XYZ in this polynomial identity, we get an equation which easily simplifies to $(\beta - \beta')(\beta + \beta' - 1) = 0$. If we compare coefficients before YZ^2 in the same identity, we similarly obtain $\beta\beta'(\beta - \beta') = 0$.

If $\beta = \beta'$, then $\alpha = \alpha'$ (since $\alpha + \beta = \alpha' + \beta'$), and thus we are done. Hence, let us assume that $\beta \neq \beta'$ for the rest of this proof. Then, $(\beta - \beta')(\beta + \beta' - 1) = 0$ simplifies to $\beta + \beta' - 1 = 0$, and $\beta\beta'(\beta - \beta') = 0$ simplifies to $\beta\beta' = 0$. Combining these two equations, we see that either ($\beta = 0$ and $\beta' = 1$) or ($\beta = 1$ and $\beta' = 0$). Assume WLOG that ($\beta = 0$ and $\beta' = 1$) (otherwise, just switch (α, β) with (α', β')). From $\alpha + \beta = \alpha' + \beta'$, we obtain $\alpha - \alpha' = \underbrace{\beta'}_{=1} - \underbrace{\beta}_{=0} = 1 \in \mathbb{Z}$. If we are able to prove that $\alpha \notin \mathbb{Z}$, then we can conclude that ($\beta = 0$, $\beta' = 1$, $\alpha - \alpha' \in \mathbb{Z}$ and $\alpha \notin \mathbb{Z}$), and thus we are done. So let us show that $\alpha \notin \mathbb{Z}$.

In fact, assume the opposite. Then, $\alpha \in \mathbb{Z}$, so that v_α is well-defined in $V_{\alpha,\beta}$ and in $V_{\alpha',\beta'}$. Then, (24) yields that every $m \in \mathbb{Z}$ satisfies

$$L_m v_\alpha = \left(\underbrace{\alpha - \alpha}_{=0} - \underbrace{\beta}_{=0} (m+1) \right) v_{\alpha - m} = 0 \text{ in } V_{\alpha,\beta}.$$

Thus, every $m \in \mathbb{Z}$ satisfies $\Phi(L_m v_\alpha) = \Phi(0) = 0$, so that $0 = \Phi(L_m v_\alpha) = L_m \underbrace{\Phi(v_\alpha)}_{=\lambda_\alpha v_\alpha} = \lambda_\alpha L_m v_\alpha$ in $V_{\alpha',\beta'}$, and thus $0 = L_m v_\alpha$ in $V_{\alpha',\beta'}$ (since $\lambda_\alpha \neq 0$). But since (24) yields

$$L_m v_\alpha = \left(\alpha - \alpha' - \underbrace{\beta'}_{=1}(m+1)\right) \underbrace{v_{\alpha-\alpha}}_{=v_0} = (\alpha - \alpha' - (m+1)) v_0 \text{ in } V_{\alpha',\beta'},$$

this rewrites as $0 = (\alpha - \alpha' - (m+1)) v_0$, so that $0 = \alpha - \alpha' - (m+1)$. But this cannot hold for every $m \in \mathbb{Z}$. This contradiction shows that our assumption (that $\alpha \in \mathbb{Z}$) was wrong. Thus, $\alpha \notin \mathbb{Z}$, and our proof of the $\Longrightarrow$ direction is finally done. Proposition 2.3.3 **(c)** is finally proven.

Proving Proposition 2.3.3 was one part of Homework Set 1 exercise 2; the other was the following:

Proposition 2.3.4. Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. Then, the Vir-module $V_{\alpha,\beta}$ is not irreducible if and only if ($\alpha \in \mathbb{Z}$ and $\beta \in \{0,1\}$).

We will not prove this; the interested reader is referred to Proposition 1.1 in §1.2 of Kac-Raina.

Remark 2.3.5. Consider the Vir-module Vir (with the adjoint action). Since $\langle C \rangle$ is a Vir-submodule of Vir, we obtain a Vir-module $\operatorname{Vir} / \langle C \rangle$. This Vir-module is isomorphic to $V_{1,-1}$. More precisely, the $\mathbb{C}$-linear map

$$\operatorname{Vir} / \langle C \rangle \to V_{1,-1},$$
$$\overline{L_n} \mapsto v_{-n}$$

is a Vir-module isomorphism. Thus, $\operatorname{Vir} / \langle C \rangle \cong V_{1,-1} \cong V_{\alpha,-1}$ as Vir-modules for every $\alpha \in \mathbb{Z}$ (because of Proposition 2.3.3 **(a)**).

2.4. Some consequences of Poincaré-Birkhoff-Witt

We will now spend some time with generalities on Lie algebras and their universal enveloping algebras. These generalities will be applied later, and while these applications could be substituted by concrete computations, it appears to me that it is better for the sake of clarity to do them generally in here.

Proposition 2.4.1. Let k be a field. Let $\mathfrak{c}$ be a k-Lie algebra. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie subalgebras of $\mathfrak{c}$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{c}$. Notice that $\mathfrak{a} \cap \mathfrak{b}$ is also a Lie subalgebra of $\mathfrak{c}$.

Let $\rho : U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} U(\mathfrak{b}) \to U(\mathfrak{c})$ be the k-vector space homomorphism defined by

$$\rho\left(\alpha \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} \beta\right) = \alpha\beta \qquad \text{for all } \alpha \in U(\mathfrak{a}) \text{ and } \beta \in U(\mathfrak{b})$$

(this is clearly well-defined). Then, ρ is an isomorphism of filtered vector spaces, of left $U(\mathfrak{a})$-modules and of right $U(\mathfrak{b})$-modules.

Corollary 2.4.2. Let k be a field. Let $\mathfrak{c}$ be a k-Lie algebra. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie subalgebras of $\mathfrak{c}$ such that $\mathfrak{a} \oplus \mathfrak{b} = \mathfrak{c}$ (as vector spaces, not necessarily as Lie algebras). Let $\rho : U(\mathfrak{a}) \otimes_k U(\mathfrak{b}) \to U(\mathfrak{c})$ be the k-vector space homomorphism defined by

$$\rho(\alpha \otimes \beta) = \alpha\beta \qquad \text{for all } \alpha \in U(\mathfrak{a}) \text{ and } \beta \in U(\mathfrak{b})$$

(this is clearly well-defined). Then, ρ is an isomorphism of filtered vector spaces, of left $U(\mathfrak{a})$-modules and of right $U(\mathfrak{b})$-modules.

We give two proofs of Proposition 2.4.1. They are very similar (both use the Poincaré-Birkhoff-Witt theorem, albeit different versions thereof). The first is more conceptual (and more general), while the second is more down-to-earth.

First proof of Proposition 2.4.1. For any Lie algebra $\mathfrak{u}$, we have a k-algebra homomorphism $\mathrm{PBW}_{\mathfrak{u}} : S(\mathfrak{u}) \to \mathrm{gr}(U(\mathfrak{u}))$ which sends $u_1 u_2 ... u_\ell$ to $\overline{u_1 u_2 ... u_\ell} \in \mathrm{gr}_\ell(U(\mathfrak{u}))$ for every $\ell \in \mathbb{N}$ and every $u_1, u_2, ..., u_\ell \in \mathfrak{u}$. This homomorphism $\mathrm{PBW}_{\mathfrak{u}}$ is an isomorphism due to the Poincaré-Birkhoff-Witt theorem.

We can define a k-algebra homomorphism $f : \mathrm{gr}(U(\mathfrak{a})) \otimes_{\mathrm{gr}(U(\mathfrak{a}\cap\mathfrak{b}))} \mathrm{gr}(U(\mathfrak{b})) \to \mathrm{gr}\left(U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} U(\mathfrak{b})\right)$ by

$$f\left(\overline{u} \otimes_{\mathrm{gr}(U(\mathfrak{a}\cap\mathfrak{b}))} \overline{v}\right) = \overline{u \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} v} \in \mathrm{gr}_{k+\ell}\left(U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} U(\mathfrak{b})\right)$$

for any $k \in \mathbb{N}$, any $\ell \in \mathbb{N}$, any $u \in U_{\leq k}(\mathfrak{a})$ and $v \in U_{\leq \ell}(\mathfrak{b})$. This f is easily seen to be well-defined. Moreover, f is surjective[39].

It is easy to see that the isomorphisms $\mathrm{PBW}_{\mathfrak{a}} : S(\mathfrak{a}) \to \mathrm{gr}(U(\mathfrak{a}))$, $\mathrm{PBW}_{\mathfrak{b}} : S(\mathfrak{b}) \to \mathrm{gr}(U(\mathfrak{b}))$ and $\mathrm{PBW}_{\mathfrak{a}\cap\mathfrak{b}} : S(\mathfrak{a}\cap\mathfrak{b}) \to \mathrm{gr}(U(\mathfrak{a}\cap\mathfrak{b}))$ are "compatible" with each other in the sense that the diagrams

$$\begin{tikzcd}
S(\mathfrak{a}) \otimes S(\mathfrak{a}\cap\mathfrak{b}) \arrow[r, "\text{action of } S(\mathfrak{a}\cap\mathfrak{b}) \text{ on } S(\mathfrak{a})"] \arrow[d, "\mathrm{PBW}_{\mathfrak{a}} \otimes \mathrm{PBW}_{\mathfrak{a}\cap\mathfrak{b}}"', "\cong"] & S(\mathfrak{a}) \arrow[d, "\mathrm{PBW}_{\mathfrak{a}}"', "\cong"] \\
\mathrm{gr}(U(\mathfrak{a})) \otimes \mathrm{gr}(U(\mathfrak{a}\cap\mathfrak{b})) \arrow[r, "\text{action of } \mathrm{gr}(U(\mathfrak{a}\cap\mathfrak{b})) \text{ on } \mathrm{gr}(U(\mathfrak{a}))"'] & \mathrm{gr}(U(\mathfrak{a}))
\end{tikzcd}$$

[39] To show this, either notice that the image of f contains a generating set of $\mathrm{gr}\left(U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} U(\mathfrak{b})\right)$ (because the definition of f easily rewrites as

$$f\left(\overline{\alpha_1\alpha_2...\alpha_k} \otimes_{\mathrm{gr}(U(\mathfrak{a}\cap\mathfrak{b}))} \overline{\beta_1\beta_2...\beta_\ell}\right) = \overline{\alpha_1\alpha_2...\alpha_k \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} \beta_1\beta_2...\beta_\ell} \in \mathrm{gr}_{k+\ell}\left(U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} U(\mathfrak{b})\right)$$

for any $k \in \mathbb{N}$, any $\ell \in \mathbb{N}$, any $\alpha_1, \alpha_2, ..., \alpha_k \in \mathfrak{a}$ and $\beta_1, \beta_2, ..., \beta_\ell \in \mathfrak{b}$), or prove the more general fact that for any $\mathbb{Z}_+$-filtered algebra A, any filtered right A-module M and any filtered left A-module N, the canonical map

$$\mathrm{gr}(M) \otimes_{\mathrm{gr}(A)} \mathrm{gr}(N) \to \mathrm{gr}(M \otimes_A N),$$
$$\overline{\mu} \otimes_{\mathrm{gr}(A)} \overline{\nu} \mapsto \overline{\mu \otimes_A \nu} \in \mathrm{gr}_{m+n}(M \otimes_A N) \qquad (\text{for all } \mu \in M_m \text{ and } \nu \in N_n, \text{ for all } m, n \in \mathbb{N})$$

is well-defined and surjective (this is easy to prove).

and

$$\begin{tikzcd}
S\left(\mathfrak{a}\cap\mathfrak{b}\right)\otimes S\left(\mathfrak{b}\right) \arrow[r, "\text{action of } S(\mathfrak{a}\cap\mathfrak{b}) \text{ on } S(\mathfrak{b})"] \arrow[d, "\mathrm{PBW}_{\mathfrak{a}\cap\mathfrak{b}}\otimes\mathrm{PBW}_{\mathfrak{b}}"', "\cong"] & S\left(\mathfrak{b}\right) \arrow[d, "\mathrm{PBW}_{\mathfrak{b}}"', "\cong"] \\
\operatorname{gr}\left(U\left(\mathfrak{a}\cap\mathfrak{b}\right)\right)\otimes\operatorname{gr}\left(U\left(\mathfrak{b}\right)\right) \arrow[r, "\text{action of } \operatorname{gr}(U(\mathfrak{a}\cap\mathfrak{b})) \text{ on } \operatorname{gr}(U(\mathfrak{b}))"'] & \operatorname{gr}\left(U\left(\mathfrak{b}\right)\right)
\end{tikzcd}$$

commute[40]. Hence, they give rise to an isomorphism

$$S\left(\mathfrak{a}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)\to\operatorname{gr}\left(U\left(\mathfrak{a}\right)\right)\otimes_{\operatorname{gr}(U(\mathfrak{a}\cap\mathfrak{b}))}\operatorname{gr}\left(U\left(\mathfrak{b}\right)\right),$$
$$\alpha\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\beta\mapsto\left(\mathrm{PBW}_{\mathfrak{a}}\alpha\right)\otimes_{\operatorname{gr}(U(\mathfrak{a}\cap\mathfrak{b}))}\left(\mathrm{PBW}_{\mathfrak{b}}\beta\right).$$

Denote this isomorphism by $\left(\mathrm{PBW}_{\mathfrak{a}}\right)\otimes_{\mathrm{PBW}_{\mathfrak{a}\cap\mathfrak{b}}}\left(\mathrm{PBW}_{\mathfrak{b}}\right)$.

Finally, let $\sigma:S\left(\mathfrak{a}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)\to S\left(\mathfrak{c}\right)$ be the vector space homomorphism defined by

$$\sigma\left(\alpha\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\beta\right)=\alpha\beta\qquad\text{for all }\alpha\in S\left(\mathfrak{a}\right)\text{ and }\beta\in S\left(\mathfrak{b}\right).$$

This σ is rather obviously an algebra homomorphism. Now, it is easy to see that σ is an algebra isomorphism[41].

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[40] This is pretty easy to see from the definition of $\mathrm{PBW}_{\mathfrak{u}}$.

[41] *First proof that σ is an algebra isomorphism:* Since every subspace of a vector space has a complementary subspace, we can find a k-vector subspace $\mathfrak{d}$ of $\mathfrak{a}$ such that $\mathfrak{a}=\mathfrak{d}\oplus\left(\mathfrak{a}\cap\mathfrak{b}\right)$. Consider such a $\mathfrak{d}$.

Since $\mathfrak{a}=\mathfrak{d}\oplus\left(\mathfrak{a}\cap\mathfrak{b}\right)=\mathfrak{d}+\left(\mathfrak{a}\cap\mathfrak{b}\right)$, the fact that $\mathfrak{c}=\mathfrak{a}+\mathfrak{b}$ rewrites as $\mathfrak{c}=\mathfrak{d}+\underbrace{\left(\mathfrak{a}\cap\mathfrak{b}\right)+\mathfrak{b}}_{\substack{=\mathfrak{b}\\(\text{since }\mathfrak{a}\cap\mathfrak{b}\subseteq\mathfrak{b})}}=\mathfrak{d}+\mathfrak{b}$. Combined with $\underbrace{\mathfrak{d}}_{\substack{=\mathfrak{d}\cap\mathfrak{a}\\(\text{since }\mathfrak{d}\subseteq\mathfrak{a})}}\cap\mathfrak{b}\subseteq\mathfrak{d}\cap\mathfrak{a}\cap\mathfrak{b}=0$ (since $\mathfrak{d}\oplus\left(\mathfrak{a}\cap\mathfrak{b}\right)$ is a well-defined internal direct sum), this yields $\mathfrak{c}=\mathfrak{d}\oplus\mathfrak{b}$.

Recall a known fact from multilinear algebra: Any two k-vector spaces U and V satisfy $S\left(U\oplus V\right)\cong S\left(U\right)\otimes_k S\left(V\right)$ by the canonical algebra isomorphism. Hence, $S\left(\mathfrak{d}\oplus\mathfrak{b}\right)\cong S\left(\mathfrak{d}\right)\otimes_k S\left(\mathfrak{b}\right)$.

But $\mathfrak{a}=\mathfrak{d}\oplus\left(\mathfrak{a}\cap\mathfrak{b}\right)$ yields $S\left(\mathfrak{a}\right)=S\left(\mathfrak{d}\oplus\left(\mathfrak{a}\cap\mathfrak{b}\right)\right)\cong S\left(\mathfrak{d}\right)\otimes_k S\left(\mathfrak{a}\cap\mathfrak{b}\right)$ (by the above-quoted fact that any two k-vector spaces U and V satisfy $S\left(U\oplus V\right)\cong S\left(U\right)\otimes_k S\left(V\right)$ by the canonical algebra isomorphism). Hence,

$$\begin{aligned}S\left(\mathfrak{a}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)&\cong\left(S\left(\mathfrak{d}\right)\otimes_k S\left(\mathfrak{a}\cap\mathfrak{b}\right)\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)\\&\cong S\left(\mathfrak{d}\right)\otimes_k\underbrace{\left(S\left(\mathfrak{a}\cap\mathfrak{b}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)\right)}_{\cong S(\mathfrak{b})}\cong S\left(\mathfrak{d}\right)\otimes_k S\left(\mathfrak{b}\right)\cong S\left(\underbrace{\mathfrak{d}\oplus\mathfrak{b}}_{=\mathfrak{c}}\right)=S\left(\mathfrak{c}\right).\end{aligned}$$

Thus we have constructed an algebra isomorphism $S\left(\mathfrak{a}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)\to S\left(\mathfrak{c}\right)$. If we track down what happens to elements of $\mathfrak{d}$, $\mathfrak{a}\cap\mathfrak{b}$ and $\mathfrak{b}$ under this isomorphism, we notice that they just get sent to themselves, so this isomorphism must coincide with σ (because if two algebra homomorphisms from the same algebra coincide on a set of generators of said algebra, then these two algebra homomorphisms must be identical). Thus, σ is an algebra isomorphism, qed.

Second proof that σ is an algebra isomorphism: Define a map $\tau:\mathfrak{c}\to S\left(\mathfrak{a}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)$ as follows: For every $c\in\mathfrak{c}$, let $\tau\left(c\right)$ be $a\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}b$, where we have written c in the form $c=a+b$ with $a\in\mathfrak{a}$ and $b\in\mathfrak{b}$ (in fact, we can write c this way, because $\mathfrak{c}=\mathfrak{a}+\mathfrak{b}$). This map τ is well-defined, because the value of $a\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}b$ depends only on c and not on the exact values of a and b in the decomposition $c=a+b$. (In fact, if $c=a+b$ and $c=a'+b'$ are two different ways to decompose c into a sum of an element of $\mathfrak{a}$ with an element of $\mathfrak{b}$, then $a+b=c=a'+b'$, Now, it is easy to see (by elementwise checking) that the diagram

$$\begin{tikzcd}
\operatorname{gr}\left(U\left(\mathfrak{a}\right)\right)\otimes_{\operatorname{gr}\left(U\left(\mathfrak{a}\cap\mathfrak{b}\right)\right)}\operatorname{gr}\left(U\left(\mathfrak{b}\right)\right) \arrow[d, "f"] & & S\left(\mathfrak{a}\right)\otimes_{S\left(\mathfrak{a}\cap\mathfrak{b}\right)}S\left(\mathfrak{b}\right) \arrow[ll, "\left(\mathrm{PBW}_{\mathfrak{a}}\right)\otimes_{\mathrm{PBW}_{\mathfrak{a}\cap\mathfrak{b}}}\left(\mathrm{PBW}_{\mathfrak{b}}\right)"', "\cong"] \arrow[d, "\sigma", "\cong"'] \\
\operatorname{gr}\left(U\left(\mathfrak{a}\right)\otimes_{U\left(\mathfrak{a}\cap\mathfrak{b}\right)}U\left(\mathfrak{b}\right)\right) \arrow[drr, "\operatorname{gr}\rho"'] & & S\left(\mathfrak{c}\right) \arrow[d, "\mathrm{PBW}_{\mathfrak{c}}", "\cong"'] \\
& & \operatorname{gr}\left(U\left(\mathfrak{c}\right)\right)
\end{tikzcd}$$

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so that $a-a'=b'-b$, thus $a-a'\in\mathfrak{a}\cap\mathfrak{b}$ (because $a-a'\in\mathfrak{a}$ and $a-a'=b'-b\in\mathfrak{b}$), so that

$$\begin{aligned}
&\underbrace{a}_{=a'+(a-a')}\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}b\\
&=\left(a'+\left(a-a'\right)\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}b\\
&=a'\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+\underbrace{\left(a-a'\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1}_{\substack{=1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\left(a-a'\right)\\ (\text{since } a-a'\in\mathfrak{a}\cap\mathfrak{b}\subseteq S(\mathfrak{a}\cap\mathfrak{b}))}}+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}b\\
&=a'\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\underbrace{\left(a-a'\right)}_{=b'-b}+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}b\\
&=a'\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+\underbrace{1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\left(b'-b\right)+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}b}_{=1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}((b'-b)+b)}\\
&=a'\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\underbrace{\left(\left(b'-b\right)+b\right)}_{=b'}=a'\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}b'.
\end{aligned}$$

)

It is also easy to see that τ is a linear map. Thus, by the universal property of the symmetric algebra, the map $\tau:\mathfrak{c}\to S\left(\mathfrak{a}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)$ gives rise to a k-algebra homomorphism $\widehat{\tau}:S\left(\mathfrak{c}\right)\to S\left(\mathfrak{a}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)$ that lifts τ.

Any $\alpha\in\mathfrak{a}$ satisfies

$$\begin{aligned}
\left(\widehat{\tau}\circ\sigma\right)\left(\alpha\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1\right)&=\widehat{\tau}\left(\underbrace{\sigma\left(\alpha\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1\right)}_{\substack{=\alpha 1\\ (\text{by the definition of } \sigma)}}\right)=\widehat{\tau}\left(\alpha 1\right)=\widehat{\tau}\left(\alpha\right)=\tau\left(\alpha\right)\qquad\left(\text{since } \widehat{\tau} \text{ lifts } \tau\right)\\
&=\alpha\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1+1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}0\\
&\qquad\left(\begin{array}{c}\text{by the definition of } \tau\text{, since } \alpha=\alpha+0 \text{ is a decomposition of}\\ \alpha \text{ into a sum of an element of } \mathfrak{a} \text{ with an element of } \mathfrak{b}\end{array}\right)\\
&=\alpha\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1.
\end{aligned}$$

In other words, the map $\widehat{\tau}\circ\sigma$ fixes all tensors of the form $\alpha\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1$ with $\alpha\in\mathfrak{a}$. Similarly, the map $\widehat{\tau}\circ\sigma$ fixes all tensors of the form $1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\beta$ with $\beta\in\mathfrak{b}$. Combining the previous two sentences, we conclude that the map map $\widehat{\tau}\circ\sigma$ fixes all elements of the set $\left\{\alpha\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1\ \mid\ \alpha\in\mathfrak{a}\right\}\cup\left\{1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\beta\ \mid\ \beta\in\mathfrak{b}\right\}$. Thus, there is a generating set of the k-algebra $S\left(\mathfrak{a}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)$ such that the map $\widehat{\tau}\circ\sigma$ fixes all elements of this set (because $\left\{\alpha\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}1\ \mid\ \alpha\in\mathfrak{a}\right\}\cup\left\{1\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\beta\ \mid\ \beta\in\mathfrak{b}\right\}$ is a generating set of the k-algebra $S\left(\mathfrak{a}\right)\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}S\left(\mathfrak{b}\right)$). Since this map $\widehat{\tau}\circ\sigma$ is a k-algebra homomorphism (because $\widehat{\tau}$ and σ are k-algebra homomorphisms), this yields that the map $\widehat{\tau}\circ\sigma$ is the identity (since a k-algebra homomorphism which fixes a generating set of its domain must be the identity). In other words, we have shown that $\widehat{\tau}\circ\sigma=\mathrm{id}$. A slightly different but similarly simple argument shows that $\sigma\circ\widehat{\tau}=\mathrm{id}$. Combining $\sigma\circ\widehat{\tau}=\mathrm{id}$ with $\widehat{\tau}\circ\sigma=\mathrm{id}$, we conclude that $\widehat{\tau}$ is an inverse to σ, so that σ is an algebra isomorphism, qed.

is commutative.[42] Hence, $(\operatorname{gr}\rho)\circ f$ is an isomorphism, so that f is injective. Since f is also surjective, this yields that f is an isomorphism. Thus, $\operatorname{gr}\rho$ is an isomorphism (since $(\operatorname{gr}\rho)\circ f$ is an isomorphism). Since ρ is a filtered map and $\operatorname{gr}\rho$ is an isomorphism, it follows that ρ is an isomorphism of filtered vector spaces. Hence, ρ is an isomorphism of filtered vector spaces, of left $U(\mathfrak{a})$-modules and of right $U(\mathfrak{b})$-modules (since it is clear that ρ is a homomorphism of $U(\mathfrak{a})$-left modules and of $U(\mathfrak{b})$-right modules). This proves Proposition 2.4.1.

Second proof of Proposition 2.4.1. Let $(z_i)_{i\in I}$ be a basis of the k-vector space $\mathfrak{a}\cap\mathfrak{b}$. We extend this basis to a basis $(z_i)_{i\in I}\cup(x_j)_{j\in J}$ of the k-vector space $\mathfrak{a}$ and to a basis $(z_i)_{i\in I}\cup(y_\ell)_{\ell\in L}$ of the k-vector space $\mathfrak{b}$. Then, $(z_i)_{i\in I}\cup(x_j)_{j\in J}\cup(y_\ell)_{\ell\in L}$ is a basis of the k-vector space $\mathfrak{c}$. We endow this basis with a total ordering in such a way that every x_j is smaller than every z_i, and that every z_i is smaller than every y_ℓ. By the Poincaré-Birkhoff-Witt theorem, we have a basis of $U(\mathfrak{c})$ consisting of increasing products of elements of the basis $(z_i)_{i\in I}\cup(x_j)_{j\in J}\cup(y_\ell)_{\ell\in L}$. On the other hand, again by the Poincaré-Birkhoff-Witt theorem, we have a basis of $U(\mathfrak{a})$ consisting of increasing products of elements of the basis $(z_i)_{i\in I}\cup(x_j)_{j\in J}$. Note that the z_i accumulate at the right end of these products, while the x_j accumulate at the left end (because we defined the total ordering in such a way that every x_j is smaller than every z_i). Hence, $U(\mathfrak{a})$ is a free right $U(\mathfrak{a}\cap\mathfrak{b})$-module, with a basis (over $U(\mathfrak{a}\cap\mathfrak{b})$, not over k) consisting of increasing products of elements of the basis $(x_j)_{j\in J}$. Combined with the fact that $U(\mathfrak{b})$ is a free k-vector space with a basis consisting of increasing products of elements of the basis $(z_i)_{i\in I}\cup(y_\ell)_{\ell\in L}$ (again by Poincaré-Birkhoff-Witt), this yields that $U(\mathfrak{a})\otimes_{U(\mathfrak{a}\cap\mathfrak{b})}U(\mathfrak{b})$ is a free k-vector space with a basis consisting of tensors of the form

$$\left(\text{some increasing product of elements of the basis } (x_j)_{j\in J}\right)$$
$$\otimes_{U(\mathfrak{a}\cap\mathfrak{b})}\left(\text{some increasing product of elements of the basis } (z_i)_{i\in I}\cup(y_\ell)_{\ell\in L}\right).$$

The map ρ clearly maps such terms bijectively into increasing products of elements of the basis $(z_i)_{i\in I}\cup(x_j)_{j\in J}\cup(y_\ell)_{\ell\in L}$. Hence, ρ maps a basis of $U(\mathfrak{a})\otimes_{U(\mathfrak{a}\cap\mathfrak{b})}U(\mathfrak{b})$ bijectively to a basis of $U(\mathfrak{c})$. Thus, ρ is an isomorphism of vector spaces. Moreover, since both of our bases were filtered[43], and ρ respects this filtration on the bases, we can even conclude that ρ is an isomorphism of filtered vector spaces. Since it is clear that ρ is a homomorphism of $U(\mathfrak{a})$-left modules and of $U(\mathfrak{b})$-right modules, it follows that ρ is an isomorphism of filtered vector spaces, of left $U(\mathfrak{a})$-modules and of right $U(\mathfrak{b})$-modules. This proves Proposition 2.4.1.

Proof of Corollary 2.4.2. Corollary 2.4.2 immediately follows from Proposition 2.4.1 (since $\mathfrak{a}\oplus\mathfrak{b}=\mathfrak{c}$ yields $\mathfrak{a}\cap\mathfrak{b}=0$, thus $U(\mathfrak{a}\cap\mathfrak{b})=U(0)=k$).

Remark 2.4.3. While we have required k to be a field in Proposition 2.4.1 and Corollary 2.4.2, these two results hold in more general situations as well. For instance, Proposition 2.4.1 holds whenever k is a commutative ring, as long as $\mathfrak{a}$, $\mathfrak{b}$

[42] In fact, if we follow the pure tensor $\alpha_1\alpha_2...\alpha_k\otimes_{S(\mathfrak{a}\cap\mathfrak{b})}\beta_1\beta_2...\beta_\ell$ (with $k\in\mathbb{N}$, $\ell\in\mathbb{N}$, $\alpha_1,\alpha_2,...,\alpha_k\in\mathfrak{a}$ and $\beta_1,\beta_2,...,\beta_\ell\in\mathfrak{b}$) through this diagram, we get $\overline{\alpha_1\alpha_2...\alpha_k\beta_1\beta_2...\beta_\ell}\in\operatorname{gr}_{k+\ell}(U(\mathfrak{c}))$ both ways.

[43] A basis $\mathcal{B}$ of a filtered vector space V is said to be *filtered* if for every $n\in\mathbb{N}$, the subfamily of $\mathcal{B}$ consisting of those elements of $\mathcal{B}$ lying in the n-th filtration of V is a basis of the n-th filtration of V.

and $\mathfrak{a} \cap \mathfrak{b}$ are free k-modules, and $\mathfrak{a} \cap \mathfrak{b}$ is a direct summand of $\mathfrak{a}$ as a k-module. In fact, the first proof of Proposition 2.4.1 works in this situation (because the Poincaré-Birkhoff-Witt theorem holds for free modules). In a more restrictive situation (namely, when $\mathfrak{a} \cap \mathfrak{b}$ is a free k-module, and a direct summand of each of $\mathfrak{a}$ and $\mathfrak{b}$, with the other two summands also being free), the second proof of Proposition 2.4.1 works as well. As for Corollary 2.4.2, it holds whenever k is a commutative ring, as long as $\mathfrak{a}$ and $\mathfrak{b}$ are free k-modules.

This generality is more than enough for most applications of Proposition 2.4.1 and Corollary 2.4.2. Yet we can go even further using the appropriate generalizations of the Poincaré-Birkhoff-Witt theorem (for these, see, e. g., P. J. Higgins, *Baer Invariants and the Birkhoff-Witt theorem*, J. of Alg. 11, pp. 469-482, (1969), `http://www.sciencedirect.com/science/article/pii/0021869369900866`).

2.5. $\mathbb{Z}$-graded Lie algebras and Verma modules

2.5.1. $\mathbb{Z}$-graded Lie algebras

Let us show some general results about representations of $\mathbb{Z}$-graded Lie algebras – particularly of *nondegenerate* $\mathbb{Z}$-graded Lie algebras. This is a notion that encompasses many of the concrete Lie algebras that we want to study (among others, $\mathcal{A}$, $\mathcal{A}_0$, W and Vir), and thus by proving the properties of nondegenerate $\mathbb{Z}$-graded Lie algebras now we can avoid proving them separately in many different cases.

Definition 2.5.1. A *$\mathbb{Z}$-graded Lie algebra* is a Lie algebra $\mathfrak{g}$ with a decomposition $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ (as a vector space) such that $[\mathfrak{g}_n, \mathfrak{g}_m] \subseteq \mathfrak{g}_{n+m}$ for all $n, m \in \mathbb{Z}$. The family $(\mathfrak{g}_n)_{n\in\mathbb{Z}}$ is called the *grading* of this $\mathbb{Z}$-graded Lie algebra.[44]

Of course, every $\mathbb{Z}$-graded Lie algebra automatically is a $\mathbb{Z}$-graded vector space (by way of forgetting the Lie bracket and only keeping the grading). Note that if $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is a $\mathbb{Z}$-graded Lie algebra, then $\bigoplus_{n<0} \mathfrak{g}_n$, $\mathfrak{g}_0$ and $\bigoplus_{n>0} \mathfrak{g}_n$ are Lie subalgebras of $\mathfrak{g}$.

Example 2.5.2. We defined a grading on the Heisenberg algebra $\mathcal{A}$ in Definition 2.2.6. This makes $\mathcal{A}$ into a $\mathbb{Z}$-graded Lie algebra. Also, $\mathcal{A}_0$ is a $\mathbb{Z}$-graded Lie subalgebra of $\mathcal{A}$.

Example 2.5.3. We make the Witt algebra W into a $\mathbb{Z}$-graded Lie algebra by using the grading $(W[n])_{n\in\mathbb{Z}}$, where $W[n] = \langle L_n \rangle$ for every $n \in \mathbb{Z}$.

We make the Virasoro algebra Vir into a $\mathbb{Z}$-graded Lie algebra by using the grading $(\mathrm{Vir}[n])_{n\in\mathbb{Z}}$, where $\mathrm{Vir}[n] = \begin{cases} \langle L_n \rangle, & \text{if } n \neq 0; \\ \langle L_0, C \rangle, & \text{if } n = 0 \end{cases}$ for every $n \in \mathbb{Z}$.

[44] **Warning:** Some algebraists use the words "$\mathbb{Z}$-graded Lie algebra" to denote a $\mathbb{Z}$-graded Lie **super**algebra, where the even homogeneous components constitute the even part and the odd homogeneous components constitute the odd part. This is **not** how we understand the notion of a "$\mathbb{Z}$-graded Lie algebra" here. In particular, for us, a $\mathbb{Z}$-graded Lie algebra $\mathfrak{g}$ should satisfy $[x, x] = 0$ for all $x \in \mathfrak{g}$ (not just for x lying in even homogeneous components).

Definition 2.5.4. A $\mathbb{Z}$-graded Lie algebra $\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$ is said to be *nondegenerate* if

(1) the vector space $\mathfrak{g}_n$ is finite-dimensional for every $n \in \mathbb{Z}$;

(2) the Lie algebra $\mathfrak{g}_0$ is abelian;

(3) for every positive integer n, for generic $\lambda \in \mathfrak{g}_0^*$, the bilinear form $\mathfrak{g}_n \times \mathfrak{g}_{-n} \to \mathbb{C}$, $(a, b) \mapsto \lambda([a, b])$ is nondegenerate. ("Generic λ" means "λ lying in some dense open subset of $\mathfrak{g}_0^*$ with respect to the Zariski topology". This subset can depend on n.)

Note that condition **(3)** in Definition 2.5.4 implies that $\dim(\mathfrak{g}_n) = \dim(\mathfrak{g}_{-n})$ for all $n \in \mathbb{Z}$.

Here are some examples:

Proposition 2.5.5. The $\mathbb{Z}$-graded Lie algebras $\mathcal{A}$, $\mathcal{A}_0$, W and Vir are nondegenerate (with the gradings defined above).

Proposition 2.5.6. Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra. The following is a reasonable (although non-canonical) way to define a grading on $\mathfrak{g}$:

Using a Cartan subalgebra and the roots of $\mathfrak{g}$, we can present the Lie algebra $\mathfrak{g}$ as a Lie algebra with generators $e_1, e_2, ..., e_m, f_1, f_2, ..., f_m, h_1, h_2, ..., h_m$ (the so-called Chevalley generators) and some relations (among them the Serre relations). Then, we can define a grading on $\mathfrak{g}$ by setting

$$\deg(e_i) = 1, \qquad \deg(f_i) = -1 \qquad \text{and} \qquad \deg(h_i) = 0 \qquad \text{for all } i \in \{1, 2, ..., m\},$$

and extending this grading in such a way that $\mathfrak{g}$ becomes a graded Lie algebra. This grading is non-canonical, but it makes $\mathfrak{g}$ into a nondegenerate graded Lie algebra.

Proposition 2.5.7. If $\mathfrak{g}$ is a finite-dimensional simple Lie algebra, then the loop algebra $\mathfrak{g}[t, t^{-1}]$ and the affine Kac-Moody algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ can be graded as follows:

Fix Chevalley generators for $\mathfrak{g}$ and grade $\mathfrak{g}$ as in Proposition 2.5.6. Now let θ be the maximal root of $\mathfrak{g}$, i. e., the highest weight of the adjoint representation of $\mathfrak{g}$. Let e_θ and f_θ be the root elements corresponding to θ. The *Coxeter number* of $\mathfrak{g}$ is defined as $\deg(e_\theta) + 1$, and denoted by h. Now let us grade $\widehat{\mathfrak{g}}$ by setting $\deg K = 0$ and $\deg(at^m) = \deg a + mh$ for every homogeneous $a \in \mathfrak{g}$ and every $m \in \mathbb{Z}$. This grading satisfies $\deg(f_\theta t) = 1$ and $\deg(e_\theta t^{-1}) = -1$. Moreover, the map $\mathfrak{g}[t, t^{-1}] \to \mathfrak{g}[t, t^{-1}]$, $x \mapsto xt$ is homogeneous of degree h; this is often informally stated as "$\deg t = h$" (although t itself is not an element of $\widehat{\mathfrak{g}}$). It is easy to see that the elements of $\widehat{\mathfrak{g}}$ of positive degree span $\mathfrak{n}_+ \oplus t\mathfrak{g}[t]$.

The graded Lie algebra $\widehat{\mathfrak{g}}$ is nondegenerate. The loop algebra $\mathfrak{g}[t, t^{-1}]$, however, is not (with the grading defined in the same way).

If $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra, we can write

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n = \bigoplus_{n<0} \mathfrak{g}_n \oplus \mathfrak{g}_0 \oplus \bigoplus_{n>0} \mathfrak{g}_n.$$

We denote $\bigoplus_{n<0} \mathfrak{g}_n$ by $\mathfrak{n}_-$ and we denote $\bigoplus_{n>0} \mathfrak{g}_n$ by $\mathfrak{n}_+$. We also denote $\mathfrak{g}_0$ by $\mathfrak{h}$. Then, $\mathfrak{n}_-$, $\mathfrak{n}_+$ and $\mathfrak{h}$ are Lie subalgebras of $\mathfrak{g}$, and the above decomposition rewrites as $\mathfrak{g} =$

$\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ (but this is, of course, not a direct sum of Lie algebras). This is called the *triangular decomposition* of $\mathfrak{g}$.

It is easy to see that when $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra, the universal enveloping algebra $U(\mathfrak{g})$ canonically becomes a $\mathbb{Z}$-graded algebra.[45]

2.5.2. $\mathbb{Z}$-graded modules

Definition 2.5.8. Let $\mathfrak{g}$ be a Lie algebra over a field k. Let M be a $\mathfrak{g}$-module. Let U be a vector subspace of $\mathfrak{g}$. Let N be a vector subspace of M. Then, $U \rightharpoonup N$ will denote the k-linear span of all elements of the form $u \rightharpoonup n$ with $u \in U$ and $n \in N$. (Notice that this notation is analogous to the notation $[U, N]$ which is defined if U and N are both subspaces of $\mathfrak{g}$.)

Definition 2.5.9. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra with grading $(\mathfrak{g}_n)_{n\in\mathbb{Z}}$. A *$\mathbb{Z}$-graded $\mathfrak{g}$-module* means a $\mathbb{Z}$-graded vector space M equipped with a $\mathfrak{g}$-module structure such that any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ satisfy $\mathfrak{g}_i \rightharpoonup M_j \subseteq M_{i+j}$, where $(M_n)_{n\in\mathbb{Z}}$ denotes the grading of M.

The reader can easily check that when $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra, and M is a $\mathbb{Z}$-graded $\mathfrak{g}$-module, then M canonically becomes a $\mathbb{Z}$-graded $U(\mathfrak{g})$-module (by taking the canonical $U(\mathfrak{g})$-module structure on M and the given $\mathbb{Z}$-grading on M).

Examples of $\mathbb{Z}$-graded $\mathfrak{g}$-modules for various Lie algebras $\mathfrak{g}$ are easy to get by. For example, when $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra, then the adjoint representation $\mathfrak{g}$ itself is a $\mathbb{Z}$-graded $\mathfrak{g}$-module. For two more interesting examples:

Example 2.5.10. The action of the Heisenberg algebra $\mathcal{A}$ on the μ-Fock representation F_μ makes F_μ into a $\mathbb{Z}$-graded $\mathcal{A}$-module (i. e., it maps $\mathcal{A}[i] \otimes F_\mu[j]$ to $F_\mu[i+j]$ for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$). Here, we are using the $\mathbb{Z}$-grading on F_μ defined in Definition 2.2.7. (If we would use the alternative $\mathbb{Z}$-grading on F_μ defined in Remark 2.2.8, then the action of $\mathcal{A}$ on F_μ would still make F_μ into a $\mathbb{Z}$-graded $\mathcal{A}$-module.)

The action of $\mathcal{A}_0$ on the Fock module F makes F into a $\mathbb{Z}$-graded $\mathcal{A}_0$-module.

Example 2.5.11. Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. The Vir-module $V_{\alpha,\beta}$ defined in Proposition 2.3.2 becomes a $\mathbb{Z}$-graded Vir-module by means of the grading $(V_{\alpha,\beta}[n])_{n\in\mathbb{Z}}$, where $V_{\alpha,\beta}[n] = \langle v_{-n}\rangle$ for every $n \in \mathbb{Z}$.

Let us formulate a graded analogue of Lemma 2.2.12:

Lemma 2.5.12. Let V be a $\mathbb{Z}$-graded $\mathcal{A}_0$-module with grading $(V[n])_{n\in\mathbb{Z}}$. Let $u \in V[0]$ be such that $a_i u = 0$ for all $i > 0$, and such that $Ku = u$. Then, there exists a $\mathbb{Z}$-graded homomorphism $\eta : F \to V$ of $\mathcal{A}_0$-modules such that $\eta(1) = u$. (This homomorphism η is unique, although we won't need this.)

Proof of Lemma 2.5.12. Let η be the map $F \to V$ which sends every polynomial $P \in F = \mathbb{C}[x_1, x_2, x_3, ...]$ to $P(a_{-1}, a_{-2}, a_{-3}, ...) \cdot u \in V$. [46] Just as in the Second

[45] In fact, $U(\mathfrak{g})$ is defined as the quotient of the tensor algebra $T(\mathfrak{g})$ by a certain ideal. When $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra, this ideal is generated by homogeneous elements, and thus is a graded ideal.

[46] Note that the term $P(a_{-1}, a_{-2}, a_{-3}, ...)$ denotes the evaluation of the polynomial P at $(x_1, x_2, x_3, ...) = (a_{-1}, a_{-2}, a_{-3}, ...)$. This evaluation is a well-defined element of $U(\mathcal{A}_0)$, since the elements a_{-1}, a_{-2}, a_{-3}, ... of $U(\mathcal{A}_0)$ commute.

proof of Lemma 2.2.12, we can show that η is an $\mathcal{A}_0$-module homomorphism $F \to V$ such that $\eta(1) = u$. Hence, in order to finish the proof of Lemma 2.5.12, we only need to check that η is a $\mathbb{Z}$-graded map.

If A is a set, then $\mathbb{N}_{\text{fin}}^{A}$ will denote the set of all finitely supported maps $A \to \mathbb{N}$.

Let $n \in \mathbb{Z}$ and $P \in F[n]$. Then, we can write the polynomial P in the form

$$P = \sum_{\substack{(i_1,i_2,i_3,\ldots)\in\mathbb{N}_{\text{fin}}^{\{1,2,3,\ldots\}};\\ 1i_1+2i_2+3i_3+\ldots=-n}} \lambda_{(i_1,i_2,i_3,\ldots)} x_1^{i_1} x_2^{i_2} x_3^{i_3} \ldots \tag{35}$$

for some scalars $\lambda_{(i_1,i_2,i_3,\ldots)} \in \mathbb{C}$. Consider these $\lambda_{(i_1,i_2,i_3,\ldots)}$. From (35), it follows that

$$\begin{aligned} P(a_{-1}, a_{-2}, a_{-3}, \ldots) &= \sum_{\substack{(i_1,i_2,i_3,\ldots)\in\mathbb{N}_{\text{fin}}^{\{1,2,3,\ldots\}};\\ 1i_1+2i_2+3i_3+\ldots=-n}} \lambda_{(i_1,i_2,i_3,\ldots)} \underbrace{a_{-1}^{i_1} a_{-2}^{i_2} a_{-3}^{i_3} \ldots}_{\substack{\in U(\mathcal{A}_0)[i_1(-1)+i_2(-2)+i_3(-3)+\ldots] \\ \text{(since every positive integer } k \text{ satisfies} \\ a_{-k}\in\mathcal{A}_0[-k]\subseteq U(\mathcal{A}_0)[-k] \text{ and thus } a_{-k}^{i_k}\in U(\mathcal{A}_0)[i_k(-k)])}} \\ &\in \sum_{\substack{(i_1,i_2,i_3,\ldots)\in\mathbb{N}_{\text{fin}}^{\{1,2,3,\ldots\}};\\ 1i_1+2i_2+3i_3+\ldots=-n}} \lambda_{(i_1,i_2,i_3,\ldots)} U(\mathcal{A}_0) \left[\underbrace{i_1(-1) + i_2(-2) + i_3(-3) + \ldots}_{\substack{=-(1i_1+2i_2+3i_3+\ldots)=n \\ \text{(since } 1i_1+2i_2+3i_3+\ldots=-n)}} \right] \\ &= \sum_{\substack{(i_1,i_2,i_3,\ldots)\in\mathbb{N}_{\text{fin}}^{\{1,2,3,\ldots\}};\\ 1i_1+2i_2+3i_3+\ldots=-n}} \lambda_{(i_1,i_2,i_3,\ldots)} U(\mathcal{A}_0)[n] \subseteq U(\mathcal{A}_0)[n] \end{aligned}$$

(since $U(\mathcal{A}_0)[n]$ is a vector space). By the definition of η, we have

$$\eta(P) = \underbrace{P(a_{-1}, a_{-2}, a_{-3}, \ldots)}_{\in U(\mathcal{A}_0)[n]} \cdot \underbrace{u}_{\in V[0]} \in U(\mathcal{A}_0)[n] \cdot V[0] \subseteq V[n]$$

(since V is a $\mathbb{Z}$-graded $\mathcal{A}_0$-module and thus a $\mathbb{Z}$-graded $U(\mathcal{A}_0)$-module). Now forget that we fixed n and P. We have thus shown that every $n \in \mathbb{Z}$ and $P \in F[n]$ satisfy $\eta(P) \subseteq V[n]$. In other words, every $n \in \mathbb{Z}$ satisfies $\eta(F[n]) \subseteq V[n]$. In other words, η is $\mathbb{Z}$-graded. This proves Lemma 2.5.12.

And here is a graded analogue of Lemma 2.2.18:

Lemma 2.5.13. Let V be a graded $\mathcal{A}$-module with grading $(V[n])_{n\in\mathbb{Z}}$. Let $\mu \in \mathbb{C}$. Let $u \in V[0]$ be such that $a_i u = 0$ for all $i > 0$, such that $a_0 u = \mu u$, and such that $Ku = u$. Then, there exists a $\mathbb{Z}$-graded homomorphism $\eta : F_\mu \to V$ of $\mathcal{A}$-modules such that $\eta(1) = u$. (This homomorphism η is unique, although we won't need this.)

The proof of Lemma 2.5.13 is completely analogous to that of Lemma 2.5.12, but this time using Lemma 2.2.18 instead of Lemma 2.2.12.

2.5.3. Verma modules

Definition 2.5.14. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra (not necessarily nondegenerate). Let us work with the notations introduced above. Let $\lambda \in \mathfrak{h}^*$.

Let $\mathbb{C}_\lambda$ denote the $(\mathfrak{h} \oplus \mathfrak{n}_+)$-module which, as a $\mathbb{C}$-vector space, is the free vector space with basis $\left(v_\lambda^+\right)$ (thus, a 1-dimensional vector space), and whose $(\mathfrak{h} \oplus \mathfrak{n}_+)$-action is given by

$$\begin{aligned} hv_\lambda^+ &= \lambda(h)\, v_\lambda^+ \qquad &&\text{for every } h \in \mathfrak{h}; \\ \mathfrak{n}_+ v_\lambda^+ &= 0. \end{aligned}$$

The *Verma highest-weight module* M_λ^+ *of* $(\mathfrak{g}, \lambda)$ is defined by

$$M_\lambda^+ = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda.$$

The element $1 \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} v_\lambda^+$ of M_λ^+ will still be denoted by v_λ^+ by abuse of notation, and will be called the *defining vector* of M_λ^+. Since $U(\mathfrak{g})$ and $\mathbb{C}_\lambda$ are graded $U(\mathfrak{h} \oplus \mathfrak{n}_+)$-modules, their tensor product $U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda = M_\lambda^+$ becomes graded as well.

Let $\mathbb{C}_\lambda$ denote the $(\mathfrak{h} \oplus \mathfrak{n}_-)$-module which, as a $\mathbb{C}$-vector space, is the free vector space with basis $\left(v_\lambda^-\right)$ (thus, a 1-dimensional vector space), and whose $(\mathfrak{h} \oplus \mathfrak{n}_-)$-action is given by

$$\begin{aligned} hv_\lambda^- &= \lambda(h)\, v_\lambda^- \qquad &&\text{for every } h \in \mathfrak{h}; \\ \mathfrak{n}_- v_\lambda^- &= 0. \end{aligned}$$

(Note that we denote this $(\mathfrak{h} \oplus \mathfrak{n}_-)$-module by $\mathbb{C}_\lambda$, although we already have denoted an $(\mathfrak{h} \oplus \mathfrak{n}_+)$-module by $\mathbb{C}_\lambda$. This is ambiguous, but misunderstandings are unlikely to occur since these modules are modules over different Lie algebras, and their restrictions to $\mathfrak{h}$ are identical.)

The *Verma lowest-weight module* M_λ^- *of* $(\mathfrak{g}, \lambda)$ is defined by

$$M_\lambda^- = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_-)} \mathbb{C}_\lambda.$$

The element $1 \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_-)} v_\lambda^-$ of M_λ^- will still be denoted by v_λ^- by abuse of notation, and will be called the *defining vector* of M_λ^-. Since $U(\mathfrak{g})$ and $\mathbb{C}_\lambda$ are graded $U(\mathfrak{h} \oplus \mathfrak{n}_-)$-modules, their tensor product $U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_-)} \mathbb{C}_\lambda = M_\lambda^-$ becomes graded as well.

We notice some easy facts about these modules:

Proposition 2.5.15. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra (not necessarily nondegenerate). Let us work with the notations introduced above. Let $\lambda \in \mathfrak{h}^*$.

(a) As a graded $\mathfrak{n}_-$-module, $M_\lambda^+ = U(\mathfrak{n}_-)\, v_\lambda^+$; more precisely, there exists a graded $\mathfrak{n}_-$-module isomorphism $U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda \to M_\lambda^+$ which sends every $x \otimes t \in U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda$ to xtv_λ^+. The Verma module M_λ^+ is concentrated in nonpositive degrees:

$$M_\lambda^+ = \bigoplus_{n \geq 0} M_\lambda^+ [-n]; \qquad M_\lambda^+ [-n] = U(\mathfrak{n}_-)[-n]\, v_\lambda^+ \qquad \text{for every } n \geq 0.$$

Also, if $\dim \mathfrak{g}_j < \infty$ for all $j \leq -1$, we have

$$\sum_{n \geq 0} \dim \left(M_\lambda^+ [-n]\right) q^n = \frac{1}{\prod\limits_{j \leq -1} \left(1 - q^{-j}\right)^{\dim \mathfrak{g}_j}}.$$

(b) As a graded $\mathfrak{n}_+$-module, $M_\lambda^- = U(\mathfrak{n}_+) v_\lambda^-$; more precisely, there exists a graded $\mathfrak{n}_+$-module isomorphism $U(\mathfrak{n}_+) \otimes \mathbb{C}_\lambda \to M_\lambda^-$ which sends every $x \otimes t \in U(\mathfrak{n}_+) \otimes \mathbb{C}_\lambda$ to xtv_λ^-. The Verma module M_λ^- is concentrated in nonnegative degrees:

$$M_\lambda^- = \bigoplus_{n \geq 0} M_\lambda^- [n] ; \qquad M_\lambda^- [n] = U(\mathfrak{n}_+) [n] v_\lambda^- \qquad \text{for every } n \geq 0.$$

Also, if $\dim \mathfrak{g}_j < \infty$ for all $j \geq 1$, we have

$$\sum_{n \geq 0} \dim \left(M_\lambda^- [n] \right) q^n = \frac{1}{\prod\limits_{j \geq 1} (1 - q^j)^{\dim \mathfrak{g}_j}}.$$

Proof of Proposition 2.5.15. **(a)** Let $\rho : U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{h} \oplus \mathfrak{n}_+) \to U(\mathfrak{g})$ be the $\mathbb{C}$-vector space homomorphism defined by

$$\rho(\alpha \otimes \beta) = \alpha\beta \qquad \text{for all } \alpha \in U(\mathfrak{n}_-) \text{ and } \beta \in U(\mathfrak{h} \oplus \mathfrak{n}_+)$$

(this is clearly well-defined). By Corollary 2.4.2 (applied to $\mathfrak{a} = \mathfrak{n}_-$, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{c} = \mathfrak{g}$), this ρ is an isomorphism of filtered[47] vector spaces, of left $U(\mathfrak{n}_-)$-modules and of right $U(\mathfrak{h} \oplus \mathfrak{n}_+)$-modules. Also, it is a graded linear map[48] (this is clear from its definition), and thus an isomorphism of graded vector spaces (because if a vector space isomorphism of graded vector spaces is a graded linear map, then it must be an isomorphism of graded vector spaces[49]). Altogether, ρ is an isomorphism of graded filtered vector spaces, of left $U(\mathfrak{n}_-)$-modules and of right $U(\mathfrak{h} \oplus \mathfrak{n}_+)$-modules. Hence,

$$\begin{aligned} M_\lambda^+ &= \underbrace{U(\mathfrak{g})}_{\substack{\cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{h} \oplus \mathfrak{n}_+) \\ \text{(by the isomorphism } \rho)}} \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda \cong \left(U(\mathfrak{n}_-) \otimes_{\mathbb{C}} U(\mathfrak{h} \oplus \mathfrak{n}_+) \right) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda \\ &\cong U(\mathfrak{n}_-) \otimes_{\mathbb{C}} \underbrace{\left(U(\mathfrak{h} \oplus \mathfrak{n}_+) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_\lambda \right)}_{\cong \mathbb{C}_\lambda} \cong U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda \qquad \text{as graded } U(\mathfrak{n}_-)\text{-modules.} \end{aligned}$$

This gives us a graded $\mathfrak{n}_-$-module isomorphism $U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda \to M_\lambda^+$ which is easily seen to send every $x \otimes t \in U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda$ to xtv_λ^+. Hence, $M_\lambda^+ = U(\mathfrak{n}_-) v_\lambda^+$. Since $\mathfrak{n}_-$ is concentrated in negative degrees, it is clear that $U(\mathfrak{n}_-)$ is concentrated in nonpositive degrees. Hence, $U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda$ is concentrated in nonpositive degrees, and thus the same

[47] Filtered by the usual filtration on the universal enveloping algebra of a Lie algebra. This filtration does not take into account the grading on $\mathfrak{n}_-$, $\mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{g}$.

[48] Here we *do* take into account the grading on $\mathfrak{n}_-$, $\mathfrak{h} \oplus \mathfrak{n}_+$ and $\mathfrak{g}$.

[49] If you are wondering why this statement is more than a blatantly obvious tautology, let me add some clarifications:

A *graded linear map* is a morphism in the category of graded vector spaces. What I am stating here is that if a vector space isomorphism between graded vector spaces is at the same time a morphism in the category of graded vector spaces, then it must be an *isomorphism* in the category of graded vector spaces. This is very easy to show, but not a self-evident tautology. In fact, the analogous assertion about filtered vector spaces (i. e., the assertion that if a vector space isomorphism between filtered vector spaces is at the same time a morphism in the category of filtered vector spaces, then it must be an *isomorphism* in the category of filtered vector spaces) is wrong.

holds for M_λ^+ (since $M_\lambda^+ \cong U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda$ as graded $U(\mathfrak{n}_-)$-modules). In other words, $M_\lambda^+ = \bigoplus\limits_{n \geq 0} M_\lambda^+[-n]$.

Since the isomorphism $U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda \to M_\lambda^+$ which sends every $x \otimes t \in U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda$ to xtv_λ^+ is graded, it sends $U(\mathfrak{n}_-)[-n] \otimes \mathbb{C}_\lambda = (U(\mathfrak{n}_-) \otimes \mathbb{C}_\lambda)[-n]$ to $M_\lambda^+[-n]$ for every $n \geq 0$. Thus, $M_\lambda^+[-n] = U(\mathfrak{n}_-)[-n]\, v_\lambda^+$ for every $n \geq 0$. Hence,

$$\begin{aligned}\dim\left(M_\lambda^+[-n]\right) &= \dim\left(U(\mathfrak{n}_-)[-n]\, v_\lambda^+\right) = \dim\left(U(\mathfrak{n}_-)[-n]\right) = \dim\left(S(\mathfrak{n}_-)[-n]\right) \\ &\qquad \left(\begin{array}{c}\text{because } U(\mathfrak{n}_-) \cong S(\mathfrak{n}_-) \text{ as graded vector spaces} \\ \text{(by the Poincaré-Birkhoff-Witt theorem)}\end{array}\right)\end{aligned}$$

for every $n \geq 0$. Hence, if $\dim \mathfrak{g}_j < \infty$ for all $j \leq -1$, then

$$\sum_{n\geq 0} \dim\left(M_\lambda^+[-n]\right) q^n = \sum_{n\geq 0} \dim\left(S(\mathfrak{n}_-)[-n]\right) q^n = \frac{1}{\prod\limits_{j\leq -1} (1-q^{-j})^{\dim\left((\mathfrak{n}_-)_j\right)}} = \frac{1}{\prod\limits_{j\leq -1} (1-q^{-j})^{\dim \mathfrak{g}_j}}.$$

This proves Proposition 2.5.15 **(a)**.

(b) The proof of part **(b)** is analogous to that of **(a)**.

This proves Proposition 2.5.15.

We have already encountered an example of a Verma highest-weight module:

Proposition 2.5.16. Let $\mathfrak{g}$ be the Lie algebra $\mathcal{A}_0$. Consider the Fock module F over the Lie algebra $\mathcal{A}_0$. Then, there is a canonical isomorphism $M_1^+ \to F$ of $\mathcal{A}_0$-modules (where 1 is the element of $\mathfrak{h}^*$ which sends K to 1) which sends $v_1^+ \in M_1^+$ to $1 \in F$.

First proof of Proposition 2.5.16. As we showed in the First proof of Lemma 2.2.12, there exists a homomorphism $\overline{\eta}_{F,1} : \operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C} \to F$ of $\mathcal{A}_0$-modules such that $\overline{\eta}_{F,1}(1) = 1$. In the same proof, we also showed that this $\overline{\eta}_{F,1}$ is an isomorphism. We thus have an isomorphism $\overline{\eta}_{F,1} : \operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C} \to F$ of $\mathcal{A}_0$-modules such that $\overline{\eta}_{F,1}(1) = 1$. Since

$$\begin{aligned}\operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C} &= U(\mathcal{A}_0) \otimes_{U\left(\mathbb{C}K \oplus \mathcal{A}_0^+\right)} \mathbb{C} = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_1 \\ &\qquad \left(\text{since } \mathcal{A}_0 = \mathfrak{g},\ \mathbb{C}K = \mathfrak{h},\ \mathcal{A}_0^+ = \mathfrak{n}_+ \text{ and } \mathbb{C} = \mathbb{C}_1\right) \\ &= M_1^+,\end{aligned}$$

and since the element 1 of $\operatorname{Ind}_{\mathbb{C}K \oplus \mathcal{A}_0^+}^{\mathcal{A}_0} \mathbb{C}$ is exactly the element v_1^+ of M_1^+, this rewrites as follows: We have an isomorphism $\overline{\eta}_{F,1} : M_1^+ \to F$ of $\mathcal{A}_0$-modules such that $\overline{\eta}_{F,1}\left(v_1^+\right) = 1$. This proves Proposition 2.5.16.

Second proof of Proposition 2.5.16. It is clear from the definition of v_1^+ that $a_i v_1^+ = 0$ for all $i > 0$, and that $K v_1^+ = v_1^+$. Applying Lemma 2.2.12 to $u = v_1^+$ and $V = M_1^+$, we thus conclude that there exists a homomorphism $\eta : F \to M_1^+$ of $\mathcal{A}_0$-modules such that $\eta(1) = v_1^+$.

On the other hand, since $M_1^+ = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_1$ (by the definition of M_1^+), we can define an $U(\mathfrak{g})$-module homomorphism

$$M_1^+ \to F, \qquad \alpha \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} z \mapsto \alpha z.$$

Since $\mathfrak{g} = \mathcal{A}_0$, this is an $U(\mathcal{A}_0)$-module homomorphism, i. e., an $\mathcal{A}_0$-module homomorphism. Denote this homomorphism by ξ. We are going to prove that η and ξ are mutually inverse.

Since $v_1^+ = 1 \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} 1$, we have

$$\xi\left(v_1^+\right) = \xi\left(1 \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} 1\right) = 1 \cdot 1 \qquad \text{(by the definition of } \xi\text{)}$$
$$= 1.$$

Since $v_1^+ = \eta(1)$, this rewrites as $\xi(\eta(1)) = 1$. In other words, $(\xi \circ \eta)(1) = 1$. Since the vector 1 generates the $\mathcal{A}_0$-module F (because Lemma 2.2.10 yields $P = \underbrace{P(a_{-1}, a_{-2}, a_{-3}, ...)}_{\in U(\mathcal{A}_0)} \cdot 1 \in U(\mathcal{A}_0) \cdot 1$ for every $P \in F$), this yields that the $\mathcal{A}_0$-module homomorphisms $\xi \circ \eta : F \to F$ and $\mathrm{id} : F \to F$ are equal on a generating set of the $\mathcal{A}_0$-module F. Thus, $\xi \circ \eta = \mathrm{id}$.

Also, $(\eta \circ \xi)\left(v_1^+\right) = \eta\left(\underbrace{\xi\left(v_1^+\right)}_{=1}\right) = \eta(1) = v_1^+$. Since the vector v_1^+ generates M_1^+ as an $\mathcal{A}_0$-module (because $M_1^+ = U(\mathfrak{g}) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_1 = U(\mathcal{A}_0) \otimes_{U(\mathfrak{h} \oplus \mathfrak{n}_+)} \mathbb{C}_1$), this yields that the $\mathcal{A}_0$-module homomorphisms $\eta \circ \xi : M_1^+ \to M_1^+$ and $\mathrm{id} : M_1^+ \to M_1^+$ are equal on a generating set of the $\mathcal{A}_0$-module M_1^+. Thus, $\eta \circ \xi = \mathrm{id}$.

Since $\eta \circ \xi = \mathrm{id}$ and $\xi \circ \eta = \mathrm{id}$, the maps ξ and η are mutually inverse, so that ξ is an isomorphism $M_1^+ \to F$ of $\mathcal{A}_0$-modules. We know that ξ sends v_1^+ to $\xi\left(v_1^+\right) = 1$. Thus, there is a canonical isomorphism $M_1^+ \to F$ of $\mathcal{A}_0$-modules which sends $v_1^+ \in M_1^+$ to $1 \in F$. Proposition 2.5.16 is proven.

In analogy to the Second proof of Proposition 2.5.16, we can show:

Proposition 2.5.17. Let $\mathfrak{g}$ be the Lie algebra $\mathcal{A}$. Let $\mu \in \mathbb{C}$. Consider the μ-Fock module F_μ over the Lie algebra $\mathcal{A}$. Then, there is a canonical isomorphism $M_{1,\mu}^+ \to F_\mu$ of $\mathcal{A}$-modules (where $(1, \mu)$ is the element of $\mathfrak{h}^*$ which sends K to 1 and a_0 to μ) which sends $v_{1,\mu}^+ \in M_{1,\mu}^+$ to $1 \in F_\mu$.

2.5.4. Degree-0 forms

We introduce another simple notion:

Definition 2.5.18. Let V and W be two $\mathbb{Z}$-graded vector spaces over a field k. Let $\beta : V \times W \to k$ be a k-bilinear form. We say that the k-bilinear form β *has degree* 0 (or, equivalently, is a *degree*-0 bilinear form) if and only if it satisfies

$$\left(\beta(V_n \times W_m) = 0 \qquad \text{for all } (n, m) \in \mathbb{Z}^2 \text{ satisfying } n + m \neq 0\right).$$

(Here, V_n denotes the n-th homogeneous component of V, and W_m denotes the m-th homogeneous component of W.)

It is straightforward to see the following characterization of degree-0 bilinear forms:

Remark 2.5.19. Let V and W be two $\mathbb{Z}$-graded vector spaces over a field k. Let $\beta : V \times W \to k$ be a k-bilinear form. Let B be the linear map $V \otimes W \to k$ induced by the k-bilinear map $V \times W \to k$ using the universal property of the tensor product. Consider $V \otimes W$ as a $\mathbb{Z}$-graded vector space (in the usual way in which one defines a grading on the tensor product of two $\mathbb{Z}$-graded vector spaces), and consider k as a $\mathbb{Z}$-graded vector space (by letting the whole field k live in degree 0).

Then, β has degree 0 if and only if B is a graded map.

2.6. The invariant bilinear form on Verma modules

2.6.1. The invariant bilinear form

The study of the Verma modules rests on a $\mathfrak{g}$-bilinear form which connects a highest-weight Verma module with a lowest-weight Verma module for the opposite weight. First, let us prove its existence and basic properties:

Proposition 2.6.1. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra, and $\lambda \in \mathfrak{h}^*$.

(a) There exists a unique $\mathfrak{g}$-invariant bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_{-\lambda}^-\right) = 1$ (where we denote this bilinear form by $(\cdot, \cdot)$).

(b) This form has degree 0. (This means that if we consider this bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ as a linear map $M_\lambda^+ \otimes M_{-\lambda}^- \to \mathbb{C}$, then it is a graded map, where $M_\lambda^+ \otimes M_{-\lambda}^-$ is graded as a tensor product of graded vector spaces, and $\mathbb{C}$ is concentrated in degree 0.)

(c) Every $\mathfrak{g}$-invariant bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ is a scalar multiple of this form $(\cdot, \cdot)$.

Remark 2.6.2. Proposition 2.6.1 still holds when the ground field $\mathbb{C}$ is replaced by a commutative ring k, as long as some rather weak conditions hold (for instance, it is enough that $\mathfrak{n}_-$, $\mathfrak{n}_+$ and $\mathfrak{h}$ are free k-modules).

Definition 2.6.3. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra, and $\lambda \in \mathfrak{h}^*$. According to Proposition 2.6.1 **(a)**, there exists a unique $\mathfrak{g}$-invariant bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_{-\lambda}^-\right) = 1$ (where we denote this bilinear form by $(\cdot, \cdot)$). This form is going to be denoted by $(\cdot, \cdot)_\lambda$ (to stress its dependency on λ). (Later we will also denote this form by $(\cdot, \cdot)_\lambda^{\mathfrak{g}}$ to point out its dependency on both λ and $\mathfrak{g}$.)

To prove Proposition 2.6.1, we recall two facts about modules over Lie algebras:

Lemma 2.6.4. Let $\mathfrak{a}$ be a Lie algebra, and let $\mathfrak{b}$ be a Lie subalgebra of $\mathfrak{a}$. Let V be a $\mathfrak{b}$-module, and W be an $\mathfrak{a}$-module. Then, $(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W \cong \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} (V \otimes W)$ as $\mathfrak{a}$-modules (where the W on the right hand side is to be understood as $\operatorname{Res}_{\mathfrak{b}}^{\mathfrak{a}} W$). More precisely, there exists a canonical $\mathfrak{a}$-module isomorphism $(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W \to \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} (V \otimes W)$ which maps $\left(1 \otimes_{U(\mathfrak{b})} v\right) \otimes w$ to $1 \otimes_{U(\mathfrak{b})} (v \otimes w)$ for all $v \in V$ and $w \in W$.

Lemma 2.6.5. Let $\mathfrak{c}$ be a Lie algebra. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two Lie subalgebras of $\mathfrak{c}$ such that $\mathfrak{a} + \mathfrak{b} = \mathfrak{c}$. Notice that $\mathfrak{a} \cap \mathfrak{b}$ is also a Lie subalgebra of $\mathfrak{c}$. Let N be a $\mathfrak{b}$-module. Then, $\operatorname{Ind}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{a}} \left(\operatorname{Res}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{b}} N\right) \cong \operatorname{Res}_{\mathfrak{a}}^{\mathfrak{c}} (\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{c}} N)$ as $\mathfrak{a}$-modules.

We will give two proofs of Lemma 2.6.4: one which is direct and uses Hopf algebras; the other which is more elementary but less direct.

First proof of Lemma 2.6.4. Remember that $U(\mathfrak{a})$ is a Hopf algebra (a cocommutative one, actually; but we won't use this). Let us denote its antipode by S and use sumfree Sweedler notation.

Recalling that $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V = U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} V$ and $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W) = U(\mathfrak{a}) \otimes_{U(\mathfrak{b})} (V \otimes W)$, we define a $\mathbb{C}$-linear map $\phi : (\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W \to \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)$ by $\left(\alpha \otimes_{U(\mathfrak{b})} v\right) \otimes w \mapsto \alpha_{(1)} \otimes_{U(\mathfrak{b})} \left(v \otimes S\left(\alpha_{(2)}\right) w\right)$. This map is easily checked to be well-defined and $\mathfrak{a}$-linear. Also, we define a $\mathbb{C}$-linear map $\psi : \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W) \to (\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W$ by $\alpha \otimes_{U(\mathfrak{b})} (v \otimes w) \mapsto \left(\alpha_{(1)} \otimes_{U(\mathfrak{b})} v\right) \otimes \alpha_{(2)} w$. This map is easily checked to be well-defined. It is also easy to see that $\phi \circ \psi = \operatorname{id}$ and $\psi \circ \phi = \operatorname{id}$. Hence, ϕ and ψ are mutually inverse isomorphisms between the $\mathfrak{a}$-modules $(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W$ and $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)$. This proves that $(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W \cong \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)$ as $\mathfrak{a}$-modules. Moreover, the isomorphism $\phi : (\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W \to \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)$ is canonical and maps $\left(1 \otimes_{U(\mathfrak{b})} v\right) \otimes w$ to $1 \otimes_{U(\mathfrak{b})} (v \otimes w)$ for all $v \in V$ and $w \in W$. In other words, Lemma 2.6.4 is proven.

Second proof of Lemma 2.6.4. For every $\mathfrak{a}$-module Y, we have

$$\begin{aligned}
&\operatorname{Hom}_{\mathfrak{a}}\left((\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W, Y\right) \\
&= \left(\underbrace{\operatorname{Hom}_{\mathbb{C}}\left((\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W, Y\right)}_{\cong \operatorname{Hom}_{\mathbb{C}}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V, \operatorname{Hom}_{\mathbb{C}}(W,Y)\right)}\right)^{\mathfrak{a}} \\
&\cong \left(\operatorname{Hom}_{\mathbb{C}}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V, \operatorname{Hom}_{\mathbb{C}}(W, Y)\right)\right)^{\mathfrak{a}} = \operatorname{Hom}_{\mathfrak{a}}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V, \operatorname{Hom}_{\mathbb{C}}(W, Y)\right) \\
&\cong \operatorname{Hom}_{\mathfrak{b}}\left(V, \operatorname{Hom}_{\mathbb{C}}(W, Y)\right) \qquad \text{(by Frobenius reciprocity)} \\
&= \left(\underbrace{\operatorname{Hom}_{\mathbb{C}}\left(V, \operatorname{Hom}_{\mathbb{C}}(W, Y)\right)}_{\cong \operatorname{Hom}_{\mathbb{C}}(V \otimes W, Y)}\right)^{\mathfrak{b}} \cong \left(\operatorname{Hom}_{\mathbb{C}}(V \otimes W, Y)\right)^{\mathfrak{b}} \\
&= \operatorname{Hom}_{\mathfrak{b}}(V \otimes W, Y) \cong \operatorname{Hom}_{\mathfrak{a}}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W), Y\right) \qquad \text{(by Frobenius reciprocity)}.
\end{aligned}$$

Since this isomorphism is canonical, it gives us a natural isomorphism between the functors $\operatorname{Hom}_{\mathfrak{a}}\left((\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W, -\right)$ and $\operatorname{Hom}_{\mathfrak{a}}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W), -\right)$. By Yoneda's lemma, this yields that $(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W \cong \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)$ as $\mathfrak{a}$-modules. It is also rather clear that the $\mathfrak{a}$-module isomorphism $(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W \to \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)$ we have just obtained is canonical.

In order to check that this isomorphism maps $\left(1 \otimes_{U(\mathfrak{b})} v\right) \otimes w$ to $1 \otimes_{U(\mathfrak{b})} (v \otimes w)$ for all $v \in V$ and $w \in W$, we must retrace the proof of Yoneda's lemma. This proof proceeds by evaluating the natural isomorphism $\operatorname{Hom}_{\mathfrak{a}}\left((\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W, -\right) \to \operatorname{Hom}_{\mathfrak{a}}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W), -\right)$ at the object $\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)$, thus obtaining an isomorphism

$$\operatorname{Hom}_{\mathfrak{a}}\left((\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W, \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)\right) \to \operatorname{Hom}_{\mathfrak{a}}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W), \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)\right),$$

and taking the preimage of $\operatorname{id} \in \operatorname{Hom}_{\mathfrak{a}}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W), \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)\right)$ under this isomorphism. This preimage is our isomorphism $(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}} V) \otimes W \to \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{a}}(V \otimes W)$. Checking that this maps $\left(1 \otimes_{U(\mathfrak{b})} v\right) \otimes w$ to $1 \otimes_{U(\mathfrak{b})} (v \otimes w)$ for all $v \in V$ and $w \in W$ is a matter of routine now, and left to the reader. Lemma 2.6.4 is thus proven.

Proof of Lemma 2.6.5. Let $\rho : U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} U(\mathfrak{b}) \to U(\mathfrak{c})$ be the $\mathbb{C}$-vector space homomorphism defined by

$$\rho\left(\alpha \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} \beta\right) = \alpha\beta \qquad \text{for all } \alpha \in U(\mathfrak{a}) \text{ and } \beta \in U(\mathfrak{b})$$

(this is clearly well-defined). By Proposition 2.4.1, this map ρ is an isomorphism of left $U(\mathfrak{a})$-modules and of right $U(\mathfrak{b})$-modules. Hence, $U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} U(\mathfrak{b}) \cong U(\mathfrak{c})$ as left $U(\mathfrak{a})$-modules and simultaneously right $U(\mathfrak{b})$-modules. Now,

$$\begin{aligned}
\operatorname{Ind}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{a}}\left(\operatorname{Res}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{b}} \underbrace{N}_{\cong U(\mathfrak{b})\otimes_{U(\mathfrak{b})}N}\right) &\cong \operatorname{Ind}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{a}}\left(\underbrace{\operatorname{Res}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{b}}\left(U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} N\right)}_{\substack{=U(\mathfrak{b})\otimes_{U(\mathfrak{b})}N \\ \text{(as a } U(\mathfrak{a}\cap\mathfrak{b})\text{-module)}}}\right) \\
&= \operatorname{Ind}_{\mathfrak{a}\cap\mathfrak{b}}^{\mathfrak{a}}\left(U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} N\right) = U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} \left(U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} N\right) \\
&\cong \underbrace{\left(U(\mathfrak{a}) \otimes_{U(\mathfrak{a}\cap\mathfrak{b})} U(\mathfrak{b})\right)}_{\cong U(\mathfrak{c})} \otimes_{U(\mathfrak{b})} N \cong U(\mathfrak{c}) \otimes_{U(\mathfrak{b})} N \\
&= \operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{c}} N = \operatorname{Res}_{\mathfrak{a}}^{\mathfrak{c}}\left(\operatorname{Ind}_{\mathfrak{b}}^{\mathfrak{c}} N\right) \qquad \text{as } \mathfrak{a}\text{-modules.}
\end{aligned}$$

This proves Lemma 2.6.5.

Proof of Proposition 2.6.1. We have $M_\lambda^+ = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda = \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}} \mathbb{C}_\lambda$. Thus,

$$\begin{aligned}
&\operatorname{Hom}_{\mathfrak{g}}\left(M_\lambda^+ \otimes M_{-\lambda}^-, \mathbb{C}\right) \\
&= \operatorname{Hom}_{\mathfrak{g}}\left(\underbrace{\left(\operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}} \mathbb{C}_\lambda\right) \otimes M_{-\lambda}^-}_{\substack{\cong \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}}\left(\mathbb{C}_\lambda \otimes M_{-\lambda}^-\right) \\ \text{(by Lemma 2.6.4)}}}, \mathbb{C}\right) \cong \operatorname{Hom}_{\mathfrak{g}}\left(\operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}}\left(\mathbb{C}_\lambda \otimes M_{-\lambda}^-\right), \mathbb{C}\right) \\
&\cong \operatorname{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}\left(\mathbb{C}_\lambda \otimes \underbrace{M_{-\lambda}^-}_{\substack{= U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} \mathbb{C}_{-\lambda} \\ = \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}} \mathbb{C}_{-\lambda}}}, \mathbb{C}\right) \qquad \text{(by Frobenius reciprocity)} \\
&= \operatorname{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}\left(\underbrace{\mathbb{C}_\lambda \otimes \left(\operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}} \mathbb{C}_{-\lambda}\right)}_{\substack{\cong \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}\right) \\ \text{(by Lemma 2.6.4)}}}, \mathbb{C}\right) \cong \operatorname{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}\left(\operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}\right), \mathbb{C}\right) \\
&\cong \operatorname{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}\left(\operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{h}\oplus\mathfrak{n}_+}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}\right), \mathbb{C}\right) \\
&\qquad \left(\begin{array}{c} \text{since Lemma 2.6.5 (applied to } \mathfrak{c} = \mathfrak{g},\ \mathfrak{a} = \mathfrak{h} \oplus \mathfrak{n}_+,\ \mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_- \text{ and } N = \mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}) \\ \text{yields } \operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{h}\oplus\mathfrak{n}_+}\left(\operatorname{Res}_{\mathfrak{h}}^{\mathfrak{h}\oplus\mathfrak{n}_-}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}\right)\right) \cong \operatorname{Res}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}}\left(\operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}\right)\right), \\ \text{which rewrites as } \operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{h}\oplus\mathfrak{n}_+}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}\right) \cong \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}\right) \\ \text{(since we are suppressing the Res functors),} \\ \text{so that } \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_-}^{\mathfrak{g}}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}\right) \cong \operatorname{Ind}_{\mathfrak{h}}^{\mathfrak{h}\oplus\mathfrak{n}_+}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}\right) \text{ (as } (\mathfrak{h} \oplus \mathfrak{n}_+)\text{-modules)} \end{array}\right) \\
&\cong \operatorname{Hom}_{\mathfrak{h}}\left(\mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda}, \mathbb{C}\right) \qquad \text{(by Frobenius reciprocity)} \\
&\cong \mathbb{C} \qquad \left(\text{since } \mathbb{C}_\lambda \otimes \mathbb{C}_{-\lambda} \cong \mathbb{C} \text{ as } \mathfrak{h}\text{-modules (this is easy to see)}\right).
\end{aligned}$$

This isomorphism $\operatorname{Hom}_{\mathfrak{g}}\left(M_\lambda^+ \otimes M_{-\lambda}^-, \mathbb{C}\right) \to \mathbb{C}$ is easily seen to map every $\mathfrak{g}$-invariant bilinear form $(\cdot,\cdot) : M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ (seen as a linear map $M_\lambda^+ \otimes M_{-\lambda}^- \to \mathbb{C}$) to the value $\left(v_\lambda^+, v_{-\lambda}^-\right)$. Hence, there exists a unique $\mathfrak{g}$-invariant bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_{-\lambda}^-\right) = 1$ (where we denote this bilinear form by $(\cdot,\cdot)$), and every other $\mathfrak{g}$-invariant bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ must be a scalar multiple of this one. This proves Proposition 2.6.1 **(a)** and **(c)**.

Now, for the proof of **(b)**: Denote by $(\cdot,\cdot)$ the unique $\mathfrak{g}$-invariant bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_{-\lambda}^-\right) = 1$. Let us now prove that this bilinear form is of degree 0:

Consider the antipode $S : U(\mathfrak{g}) \to U(\mathfrak{g})$ of the Hopf algebra $U(\mathfrak{g})$. This S is a graded algebra antiautomorphism satisfying $S(x) = -x$ for every $x \in \mathfrak{g}$. It can be explicitly described by

$$S(x_1 x_2 ... x_m) = (-1)^m x_m x_{m-1} ... x_1 \qquad \text{for all } m \in \mathbb{N} \text{ and } x_1, x_2, ..., x_m \in \mathfrak{g}.$$

We can easily see by induction (using the $\mathfrak{g}$-invariance of the bilinear form $(\cdot,\cdot)$) that $(v, aw) = (S(a)v, w)$ for all $v \in M_\lambda^+$ and $w \in M_{-\lambda}^-$ and $a \in U(\mathfrak{g})$. In particular,

$$\left(av_\lambda^+, bv_{-\lambda}^-\right) = \left(S(b)\,av_\lambda^+, v_{-\lambda}^-\right) \qquad \text{for all } a \in U(\mathfrak{g}) \text{ and } b \in U(\mathfrak{g}).$$

Thus, $\left(av_\lambda^+, bv_{-\lambda}^-\right) = \left(S(b)\,av_\lambda^+, v_{-\lambda}^-\right) = 0$ whenever a and b are homogeneous elements of $U(\mathfrak{g})$ satisfying $\deg b > -\deg a$ (this is because any two homogeneous elements a and b of $U(\mathfrak{g})$ satisfying $\deg b > -\deg a$ satisfy $S(b)\,av_\lambda^+ = 0$ [50]). In other words, whenever $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ are integers satisfying $m > -n$, we have $\left(av_\lambda^+, bv_{-\lambda}^-\right) = 0$ for every $a \in U(\mathfrak{g})[n]$ and $b \in U(\mathfrak{g})[m]$. Since $M_\lambda^+[n] = \left\{av_\lambda^+ \mid a \in U(\mathfrak{g})[n]\right\}$ and $M_{-\lambda}^-[m] = \left\{bv_{-\lambda}^- \mid b \in U(\mathfrak{g})[m]\right\}$, this rewrites as follows: Whenever $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ are integers satisfying $m > -n$, we have $\left(M_\lambda^+[n], M_{-\lambda}^-[m]\right) = 0$.

Similarly, using the formula $(av, w) = (v, S(a)w)$ (which holds for all $v \in M_\lambda^+$ and $w \in M_{-\lambda}^-$ and $a \in U(\mathfrak{g})$), we can show that whenever $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ are integers satisfying $m < -n$, we have $\left(M_\lambda^+[n], M_{-\lambda}^-[m]\right) = 0$.

Thus we have $\left(M_\lambda^+[n], M_{-\lambda}^-[m]\right) = 0$ whenever $m > -n$ and whenever $m < -n$. Hence, $\left(M_\lambda^+[n], M_{-\lambda}^-[m]\right)$ can only be nonzero when $m = -n$. In other words, the form $(\cdot,\cdot)$ has degree 0. This proves Proposition 2.6.1. In this proof, we have not used any properties of $\mathbb{C}$ other than being a commutative ring over which $\mathfrak{n}_-$, $\mathfrak{n}_+$ and $\mathfrak{h}$ are free modules (the latter was only used for applying consequences of Poincaré-Birkhoff-Witt); we thus have also verified Remark 2.6.2.

2.6.2. Generic nondegeneracy: Statement of the fact

We will later (Theorem 2.7.3) see that the bilinear form $(\cdot,\cdot)_\lambda : M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ is nondegenerate if and only if the $\mathfrak{g}$-module M_λ^+ is irreducible. This makes the question of when the form $(\cdot,\cdot)_\lambda$ is nondegenerate an important question to study. It can, in many concrete cases, be answered by combinatorial computations. But let us first give a general result about how it is nondegenerate "if λ is in sufficiently general position":

Theorem 2.6.6. Assume that $\mathfrak{g}$ is a nondegenerate $\mathbb{Z}$-graded Lie algebra.

Let $(\cdot,\cdot)$ be the form $(\cdot,\cdot)_\lambda : M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$. (In other words, let $(\cdot,\cdot)$ be the unique $\mathfrak{g}$-invariant bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_{-\lambda}^-\right) = 1$. Such a form exists and is unique by Proposition 2.6.1 **(a)**.)

In every degree, the form $(\cdot,\cdot)$ is nondegenerate for generic λ. More precisely: For every $n \in \mathbb{N}$, the restriction of the form $(\cdot,\cdot) : M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ to $M_\lambda^+[-n] \times M_{-\lambda}^-[n]$ is nondegenerate for generic λ.

(What "generic λ" means here may depend on the degree. Thus, we cannot claim that "for generic λ, the form $(\cdot,\cdot)$ is nondegenerate in every degree"!)

The proof of this theorem will occupy the rest of Section 2.6. While the statement of Theorem 2.6.6 itself will never be used in this text, the proof involves several useful ideas and provides good examples of how to work with Verma modules computationally; moreover, the main auxiliary result (Proposition 2.6.17) will be used later in the text.

[50] *Proof.* Let a and b be homogeneous elements of $U(\mathfrak{g})$ satisfying $\deg b > -\deg a$. Then, $\deg b + \deg a > 0$, and thus the element $S(b)\,av_\lambda^+$ of M_λ^+ is a homogeneous element of positive degree (since $\deg v_\lambda^+ = 0$), but the only homogeneous element of M_λ^+ of positive degree is 0 (since M_λ^+ is concentrated in nonpositive degrees), so that $S(b)\,av_\lambda^+ = 0$.

[Note: The below proof has been written at nighttime and not been checked for mistakes. It also has not been checked for redundancies and readability.]

2.6.3. Proof of Theorem 2.6.6: Casting bilinear forms on coinvariant spaces

Before we start with the proof, a general fact from representation theory:

Lemma 2.6.7. Let k be a field, and let G be a finite group. Let $\Lambda \in k[G]$ be the element $\sum\limits_{g \in G} g$.

Let V and W be representations of G over k. Let $B : V \times W \to k$ be a G-invariant bilinear form.

(a) Then, there exists one and only one bilinear form $B' : V_G \times W_G \to k$ satisfying

$$B'(\overline{v}, \overline{w}) = B(\Lambda v, w) = B(v, \Lambda w) \qquad \text{for all } v \in V \text{ and } w \in W.$$

(Here, $\overline{v}$ denotes the projection of v onto V_G, and $\overline{w}$ denotes the projection of w onto W_G.)

(b) Assume that $|G|$ is invertible in k (in other words, assume that $\operatorname{char} k$ is either 0 or coprime to $|G|$). If the form B is nondegenerate, then the form B' constructed in Lemma 2.6.7 **(a)** is nondegenerate, too.

Proof of Lemma 2.6.7. Every $h \in G$ satisfies

$$\begin{aligned} h\Lambda &= h \sum_{g \in G} g \qquad \left(\text{since } \Lambda = \sum_{g \in G} g\right) \\ &= \sum_{g \in G} hg = \sum_{i \in G} i \qquad \left(\begin{array}{c}\text{here, we substituted } i \text{ for } hg \text{ in the sum, since the map} \\ G \to G,\ g \mapsto hg \text{ is a bijection}\end{array}\right) \\ &= \sum_{g \in G} g = \Lambda \end{aligned}$$

and similarly $\Lambda h = \Lambda$.

Also,

$$\begin{aligned} \sum_{g \in G} g^{-1} &= \sum_{g \in G} g \qquad \left(\begin{array}{c}\text{here, we substituted } g \text{ for } g^{-1} \text{ in the sum, since the map} \\ G \to G,\ g \mapsto g^{-1} \text{ is a bijection}\end{array}\right) \\ &= \Lambda. \end{aligned}$$

We further notice that the group G acts trivially on the G-modules k and W_G (this follows from the definitions of these modules), and thus G acts trivially on $\operatorname{Hom}(W_G, k)$ as well.

For every $v \in V$, the map

$$W \to k, \qquad w \mapsto B(\Lambda v, w)$$

is clearly G-equivariant (since it maps hw to

$$\begin{aligned}
B\left(\underbrace{\Lambda}_{=h\Lambda} v, hw\right) &= B\left(h\Lambda v, hw\right) = B\left(\Lambda v, w\right) \qquad \text{(since } B \text{ is } G\text{-invariant)} \\
&= hB\left(\Lambda v, w\right) \qquad \text{(since } G \text{ acts trivially on } k\text{)}
\end{aligned}$$

for every $h \in G$ and $w \in W$), and thus descends to a map

$$W_G \to k_G, \qquad \overline{w} \mapsto \overline{B\left(\Lambda v, w\right)}.$$

Hence, we have obtained a map

$$V \to \operatorname{Hom}\left(W_G, k_G\right), \qquad v \mapsto \left(\overline{w} \mapsto \overline{B\left(\Lambda v, w\right)}\right).$$

Since $k_G = k$ (because G acts trivially on k), this rewrites as a map

$$V \to \operatorname{Hom}\left(W_G, k\right), \qquad v \mapsto \left(\overline{w} \mapsto B\left(\Lambda v, w\right)\right).$$

This map, too, is G-equivariant (since it maps hv to the map

$$\begin{aligned}
&\left(W_G \to k, \qquad \overline{w} \mapsto B\left(\underbrace{\Lambda h}_{=\Lambda} v, w\right)\right) \\
&= \left(W_G \to k, \qquad \overline{w} \mapsto B\left(\Lambda v, w\right)\right) = h\left(W_G \to k, \qquad \overline{w} \mapsto B\left(\Lambda v, w\right)\right) \\
&\qquad \text{(since } G \text{ acts trivially on } \operatorname{Hom}\left(W_G, k\right)\text{)}
\end{aligned}$$

for every $h \in G$ and $v \in V$). Thus, it descends to a map

$$V_G \to \left(\operatorname{Hom}\left(W_G, k\right)\right)_G, \qquad \overline{v} \mapsto \overline{\left(\overline{w} \mapsto B\left(\Lambda v, w\right)\right)}.$$

Since $\left(\operatorname{Hom}\left(W_G, k\right)\right)_G = \operatorname{Hom}\left(W_G, k\right)$ (because G acts trivially on $\operatorname{Hom}\left(W_G, k\right)$), this rewrites as a map

$$V_G \to \operatorname{Hom}\left(W_G, k\right), \qquad \overline{v} \mapsto \left(\overline{w} \mapsto B\left(\Lambda v, w\right)\right).$$

This map can be rewritten as a bilinear form $V_G \times W_G \to k$ which maps $\left(\overline{v}, \overline{w}\right)$ to $B\left(\Lambda v, w\right)$ for all $v \in V$ and $w \in W$. Since

$$\begin{aligned}
B\left(\Lambda v, w\right) &= B\left(\sum_{g \in G} gv, w\right) \qquad \left(\text{since } \Lambda = \sum_{g \in G} g\right) \\
&= \sum_{g \in G} B\left(gv, \underbrace{w}_{=gg^{-1}w}\right) = \sum_{g \in G} \underbrace{B\left(gv, gg^{-1}w\right)}_{\substack{=B\left(v, g^{-1}w\right) \\ \text{(since } B \text{ is } G\text{-invariant)}}} = \sum_{g \in G} B\left(v, g^{-1}w\right) \\
&= B\left(v, \underbrace{\sum_{g \in G} g^{-1}}_{=\Lambda} w\right) = B\left(v, \Lambda w\right)
\end{aligned}$$

for all $v \in V$ and $w \in W$, we have thus proven that there exists a bilinear form $B' : V_G \times W_G \to k$ satisfying

$$B'(\overline{v}, \overline{w}) = B(\Lambda v, w) = B(v, \Lambda w) \qquad \text{for all } v \in V \text{ and } w \in W.$$

The uniqueness of such a form is self-evident. This proves Lemma 2.6.7 **(a)**.

(b) Assume that $|G|$ is invertible in k. Assume that the form B is nondegenerate. Consider the form B' constructed in Lemma 2.6.7 **(a)**.

Let $p \in V_G$ be such that $B'(p, W_G) = 0$. Since $p \in V_G$, there exists some $v \in V$ such that $p = \overline{v}$. Consider this v. Then, every $w \in W$ satisfies $B(\Lambda v, w) = 0$ (since $B(\Lambda v, w) = B'\left(\underbrace{\overline{v}}_{=p}, \underbrace{\overline{w}}_{\in W_G}\right) \in B'(p, W_G) = 0$). Hence, $\Lambda v = 0$ (since B is nondegenerate).

But since the projection of V to V_G is a G-module map, we have

$$\begin{aligned} \overline{\Lambda v} = \Lambda \overline{v} &= \sum_{g \in G} \underbrace{g\overline{v}}_{\substack{=\overline{v} \\ \text{(since } G \text{ acts} \\ \text{trivially on } V_G)}} \qquad \left(\text{since } \Lambda = \sum_{g \in G} g\right) \\ &= \sum_{g \in G} \overline{v} = |G|\,\overline{v}. \end{aligned}$$

Since $|G|$ is invertible in k, this yields $\overline{v} = \frac{1}{|G|}\overline{\Lambda v} = 0$ (since $\Lambda v = 0$), so that $p = \overline{v} = 0$.

We have thus shown that every $p \in V_G$ such that $B'(p, W_G) = 0$ must satisfy $p = 0$. In other words, the form B' is nondegenerate. Lemma 2.6.7 **(b)** is proven.

2.6.4. Proof of Theorem 2.6.6: The form $(\cdot,\cdot)^\circ_\lambda$

Let us formulate some standing assumptions:

> **Convention 2.6.8.** From now on until the end of Section 2.6, we let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra, and let $\lambda \in \mathfrak{h}^*$. We also require that $\mathfrak{g}_0$ is abelian (this is condition **(2)** of Definition 2.5.4), but we do *not* require $\mathfrak{g}$ to be nondegenerate (unless we explicitly state this).

As vector spaces, $M^+_\lambda = U(\mathfrak{n}_-)\, v^+_\lambda \cong U(\mathfrak{n}_-)$ (where the isomorphism maps v^+_λ to 1) and $M^-_{-\lambda} = U(\mathfrak{n}_+)\, v^-_{-\lambda} \cong U(\mathfrak{n}_+)$ (where the isomorphism maps $v^-_{-\lambda}$ to 1). Thus, the bilinear form $(\cdot,\cdot) = (\cdot,\cdot)_\lambda : M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ corresponds to a bilinear form $U(\mathfrak{n}_-) \times U(\mathfrak{n}_+) \to \mathbb{C}$.

For every $n \in \mathbb{N}$, let $(\cdot,\cdot)_{\lambda,n}$ denote the restriction of our form $(\cdot,\cdot) = (\cdot,\cdot)_\lambda : M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ to $M^+_\lambda[-n] \times M^-_{-\lambda}[n]$. In order to prove Theorem 2.6.6, it is enough to prove that for every $n \in \mathbb{N}$, when $\mathfrak{g}$ is nondegenerate, this form $(\cdot,\cdot)_{\lambda,n}$ is nondegenerate for generic λ.

We now introduce a $\mathbb{C}$-bilinear form, which will turn out to be, in some sense, the "highest term" of the form $(\cdot,\cdot)$ with respect to λ (what this exactly means will be explained in Proposition 2.6.17).

Proposition 2.6.9. For every $k \in \mathbb{N}$, there exists one and only one $\mathbb{C}$-bilinear form $\lambda_k : S^k(\mathfrak{n}_-) \times S^k(\mathfrak{n}_+) \to \mathbb{C}$ by

$$\lambda_k(\alpha_1\alpha_2...\alpha_k, \beta_1\beta_2...\beta_k) = \sum_{\sigma \in S_k} \lambda\left(\left[\alpha_1, \beta_{\sigma(1)}\right]\right) \lambda\left(\left[\alpha_2, \beta_{\sigma(2)}\right]\right) ...\lambda\left(\left[\alpha_k, \beta_{\sigma(k)}\right]\right)$$
$$\text{for all } \alpha_1, \alpha_2, ..., \alpha_k \in \mathfrak{n}_- \text{ and } \beta_1, \beta_2, ..., \beta_k \in \mathfrak{n}_+. \tag{36}$$

Here, we are using the following convention:

Convention 2.6.10. From now on until the end of Section 2.6, the map $\lambda : \mathfrak{g}_0 \to \mathbb{C}$ is extended to a linear map $\lambda : \mathfrak{g} \to \mathbb{C}$ by composing it with the canonical projection $\mathfrak{g} \to \mathfrak{g}_0$.

First proof of Proposition 2.6.9 (sketched). Let $k \in \mathbb{N}$. The value of

$$\sum_{\sigma \in S_k} \lambda\left(\left[\alpha_1, \beta_{\sigma(1)}\right]\right) \lambda\left(\left[\alpha_2, \beta_{\sigma(2)}\right]\right) ...\lambda\left(\left[\alpha_k, \beta_{\sigma(k)}\right]\right)$$

depends linearly on each of the $\alpha_1, \alpha_2, ..., \alpha_k$ and $\beta_1, \beta_2, ..., \beta_k$, and is invariant under any permutation of the $\alpha_1, \alpha_2, ..., \alpha_k$ and under any permutation of the $\beta_1, \beta_2, ..., \beta_k$ (as is easily checked). This readily shows that we can indeed define a $\mathbb{C}$-bilinear form $\lambda_k : S^k(\mathfrak{n}_-) \times S^k(\mathfrak{n}_+) \to \mathbb{C}$ by (36). This proves Proposition 2.6.9.

Second proof of Proposition 2.6.9. Let $G = S_k$. Let $\Lambda \in \mathbb{C}[G]$ be the element $\sum\limits_{g \in S_k} g = \sum\limits_{\sigma \in S_k} \sigma = \sum\limits_{\sigma \in S_k} \sigma^{-1}$. Let V and W be the canonical representations $\mathfrak{n}_-^{\otimes k}$ and $\mathfrak{n}_+^{\otimes k}$ of S_k (where S_k acts by permuting the tensorands). Let $B : V \times W \to \mathbb{C}$ be the $\mathbb{C}$-bilinear form defined as the k-th tensor power of the $\mathbb{C}$-bilinear form $\mathfrak{n}_- \times \mathfrak{n}_+ \to \mathbb{C}$, $(\alpha, \beta) \mapsto \lambda([\alpha, \beta])$. It is easy to see that this form is S_k-invariant (in fact, more generally, the k-th tensor power of any bilinear form is S_k-invariant). Thus, Lemma 2.6.7 **(a)** (applied to $\mathbb{C}$ instead of k) yields that there exists one and only one bilinear form $B' : V_G \times W_G \to \mathbb{C}$ satisfying

$$B'(\overline{v}, \overline{w}) = B(\Lambda v, w) = B(v, \Lambda w) \qquad \text{for all } v \in V \text{ and } w \in W \tag{37}$$

(where $\overline{v}$ denotes the projection of v onto $V_G = V_{S_k} = S^k(\mathfrak{n}_-)$, and $\overline{w}$ denotes the projection of w onto $W_G = W_{S_k} = S^k(\mathfrak{n}_+)$). Consider this form B'. All $\alpha_1, \alpha_2, ..., \alpha_k \in$

$\mathfrak{n}_-$ and $\beta_1, \beta_2, ..., \beta_k \in \mathfrak{n}_+$ satisfy

$$\begin{aligned}
&B'\left(\alpha_1\alpha_2...\alpha_k, \beta_1\beta_2...\beta_k\right) \\
&= B'\left(\overline{\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_k}, \overline{\beta_1 \otimes \beta_2 \otimes ... \otimes \beta_k}\right) \\
&\qquad \left(\text{since } \alpha_1\alpha_2...\alpha_k = \overline{\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_k} \text{ and } \beta_1\beta_2...\beta_k = \overline{\beta_1 \otimes \beta_2 \otimes ... \otimes \beta_k}\right) \\
&= B\left(\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_k, \Lambda\left(\beta_1 \otimes \beta_2 \otimes ... \otimes \beta_k\right)\right) \\
&\qquad \left(\text{by (37), applied to } v = \alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_k \text{ and } w = \beta_1 \otimes \beta_2 \otimes ... \otimes \beta_k\right) \\
&= B\left(\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_k, \sum_{\sigma \in S_k} \beta_{\sigma(1)} \otimes \beta_{\sigma(2)} \otimes ... \otimes \beta_{\sigma(k)}\right) \\
&\qquad \left(\begin{array}{c} \text{since } \Lambda = \sum_{\sigma \in S_k} \sigma^{-1} \text{ yields } \Lambda\left(\beta_1 \otimes \beta_2 \otimes ... \otimes \beta_k\right) = \sum_{\sigma \in S_k} \underbrace{\sigma^{-1}\left(\beta_1 \otimes \beta_2 \otimes ... \otimes \beta_k\right)}_{=\beta_{\sigma(1)} \otimes \beta_{\sigma(2)} \otimes ... \otimes \beta_{\sigma(k)}} \\ = \sum_{\sigma \in S_k} \beta_{\sigma(1)} \otimes \beta_{\sigma(2)} \otimes ... \otimes \beta_{\sigma(k)} \end{array}\right) \\
&= \sum_{\sigma \in S_k} \underbrace{B\left(\alpha_1 \otimes \alpha_2 \otimes ... \otimes \alpha_k, \beta_{\sigma(1)} \otimes \beta_{\sigma(2)} \otimes ... \otimes \beta_{\sigma(k)}\right)}_{\substack{=\lambda\left(\left[\alpha_1, \beta_{\sigma(1)}\right]\right)\lambda\left(\left[\alpha_2, \beta_{\sigma(2)}\right]\right)...\lambda\left(\left[\alpha_k, \beta_{\sigma(k)}\right]\right) \\ \text{(since } B \text{ is the } k\text{-th tensor power of the } \mathbb{C}\text{-bilinear form } \mathfrak{n}_- \times \mathfrak{n}_+ \to \mathbb{C},\ (\alpha,\beta) \mapsto \lambda([\alpha,\beta]))}} \\
&= \sum_{\sigma \in S_k} \lambda\left(\left[\alpha_1, \beta_{\sigma(1)}\right]\right) \lambda\left(\left[\alpha_2, \beta_{\sigma(2)}\right]\right) ... \lambda\left(\left[\alpha_k, \beta_{\sigma(k)}\right]\right).
\end{aligned}$$

Thus, there exists a $\mathbb{C}$-bilinear form $\lambda_k : S^k(\mathfrak{n}_-) \times S^k(\mathfrak{n}_+) \to \mathbb{C}$ satisfying (36) (namely, B'). On the other hand, there exists **at most one** $\mathbb{C}$-bilinear form $\lambda_k : S^k(\mathfrak{n}_-) \times S^k(\mathfrak{n}_+) \to \mathbb{C}$ satisfying (36) [51]. Hence, we can indeed define a $\mathbb{C}$-bilinear form $\lambda_k : S^k(\mathfrak{n}_-) \times S^k(\mathfrak{n}_+) \to \mathbb{C}$ by (36). And, moreover,

$$\text{this form } \lambda_k \text{ is the form } B' \text{ satisfying (37).} \tag{38}$$

Proposition 2.6.9 is thus proven.

Definition 2.6.11. For every $k \in \mathbb{N}$, let $\lambda_k : S^k(\mathfrak{n}_-) \times S^k(\mathfrak{n}_+) \to \mathbb{C}$ be the $\mathbb{C}$-bilinear form whose existence and uniqueness is guaranteed by Proposition 2.6.9. These forms can be added together, resulting in a bilinear form $\bigoplus_{k \geq 0} \lambda_k : S(\mathfrak{n}_-) \times S(\mathfrak{n}_+) \to \mathbb{C}$. It is very easy to see that this form is of degree 0 (where the grading on $S(\mathfrak{n}_-)$ and $S(\mathfrak{n}_+)$ is not the one that gives the k-th symmetric power the degree k for every $k \in \mathbb{N}$, but is the one induced by the grading on $\mathfrak{n}_-$ and $\mathfrak{n}_+$). Denote this form by $(\cdot, \cdot)^\circ_\lambda$.

2.6.5. Proof of Theorem 2.6.6: Generic nondegeneracy of $(\cdot, \cdot)^\circ_\lambda$

[51] *Proof.* The vector space $S^k(\mathfrak{n}_-)$ is spanned by products of the form $\alpha_1\alpha_2...\alpha_k$ with $\alpha_1, \alpha_2, ..., \alpha_k \in \mathfrak{n}_-$, whereas the vector space $S^k(\mathfrak{n}_+)$ is spanned by products of the form $\beta_1\beta_2...\beta_k$ with $\beta_1, \beta_2, ..., \beta_k \in \mathfrak{n}_+$. Hence, the equation (36) makes it possible to compute the value of $\lambda_k(A, B)$ for any $A \in S^k(\mathfrak{n}_-)$ and $B \in S^k(\mathfrak{n}_+)$. Thus, the equation (36) uniquely determines λ_k. In other words, there exists at most one $\mathbb{C}$-bilinear form $\lambda_k : S^k(\mathfrak{n}_-) \times S^k(\mathfrak{n}_+) \to \mathbb{C}$ satisfying (36).

Lemma 2.6.12. Let $\lambda \in \mathfrak{h}^*$ be such that the $\mathbb{C}$-bilinear form $\mathfrak{n}_- \times \mathfrak{n}_+ \to \mathbb{C}$, $(\alpha, \beta) \mapsto \lambda([\alpha, \beta])$ is nondegenerate. Then, the form $(\cdot, \cdot)^\circ_\lambda$ is nondegenerate.

Proof of Lemma 2.6.12. Let $k \in \mathbb{N}$. Introduce the same notations as in the Second proof of Proposition 2.6.9.

The $\mathbb{C}$-bilinear form $\mathfrak{n}_- \times \mathfrak{n}_+ \to \mathbb{C}$, $(\alpha, \beta) \mapsto \lambda([\alpha, \beta])$ is nondegenerate. Thus, the k-th tensor power of this form is also nondegenerate (since all tensor powers of a nondegenerate form are always nondegenerate). But the k-th tensor power of this form is B. Thus, B is nondegenerate. Hence, Lemma 2.6.7 **(b)** yields that the form B' is nondegenerate. Due to (38), this yields that the form λ_k is nondegenerate.

Forget that we fixed k. We thus have shown that for every $k \in \mathbb{N}$, the form λ_k is nondegenerate. Thus, the direct sum $\bigoplus\limits_{k \geq 0} \lambda_k$ of these forms is also nondegenerate. Since $\bigoplus\limits_{k \geq 0} \lambda_k = (\cdot, \cdot)^\circ_\lambda$, this yields that $(\cdot, \cdot)^\circ_\lambda$ is nondegenerate. This proves Lemma 2.6.12.

For every $n \in \mathbb{N}$, define $(\cdot, \cdot)^\circ_{\lambda,n} : S(\mathfrak{n}_-)[-n] \times S(\mathfrak{n}_+)[n] \to \mathbb{C}$ to be the restriction of this form $(\cdot, \cdot)^\circ_\lambda = \bigoplus\limits_{k \geq 0} \lambda_k : S(\mathfrak{n}_-) \times S(\mathfrak{n}_+) \to \mathbb{C}$ to $S(\mathfrak{n}_-)[-n] \times S(\mathfrak{n}_+)[n]$. We now need the following strengthening of Lemma 2.6.12:

Lemma 2.6.13. Let $n \in \mathbb{N}$ and $\lambda \in \mathfrak{h}^*$ be such that the bilinear form

$$\mathfrak{g}_{-k} \times \mathfrak{g}_k \to \mathbb{C}, \qquad (a, b) \mapsto \lambda([a, b])$$

is nondegenerate for every $k \in \{1, 2, ..., n\}$. Then, the form $(\cdot, \cdot)^\circ_{\lambda,n}$ must also be nondegenerate.

Proof of Lemma 2.6.13. For Lemma 2.6.12 to hold, we did not need $\mathfrak{g}$ to be a graded Lie algebra; we only needed that $\mathfrak{g}$ is a graded vector space with a well-defined bilinear map $[\cdot, \cdot] : \mathfrak{g}_{-k} \times \mathfrak{g}_k \to \mathfrak{g}_0$ for every positive integer k. This is a rather weak condition, and holds not only for $\mathfrak{g}$, but also for the graded subspace $\mathfrak{g}_{-n} \oplus \mathfrak{g}_{-n+1} \oplus ... \oplus \mathfrak{g}_n$ of $\mathfrak{g}$. Denote this graded subspace $\mathfrak{g}_{-n} \oplus \mathfrak{g}_{-n+1} \oplus ... \oplus \mathfrak{g}_n$ by $\mathfrak{g}'$, and let $\mathfrak{n}'_- \oplus \mathfrak{h}' \oplus \mathfrak{n}'_+$ be its triangular decomposition (thus, $\mathfrak{n}'_- = \mathfrak{g}_{-n} \oplus \mathfrak{g}_{-n+1} \oplus ... \oplus \mathfrak{g}_{-1}$, $\mathfrak{h}' = \mathfrak{g}_0 = \mathfrak{h}$ and $\mathfrak{n}'_+ = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus ... \oplus \mathfrak{g}_n$). The $\mathbb{C}$-bilinear form $\mathfrak{n}'_- \times \mathfrak{n}'_+ \to \mathbb{C}$, $(\alpha, \beta) \mapsto \lambda([\alpha, \beta])$ is nondegenerate (because the bilinear form $\mathfrak{g}_{-k} \times \mathfrak{g}_k \to \mathbb{C}$, $(a, b) \mapsto \lambda([a, b])$ is nondegenerate for every $k \in \{1, 2, ..., n\}$). Hence, by Lemma 2.6.12, the form $(\cdot, \cdot)^\circ_\lambda$ *defined for* $\mathfrak{g}'$ *instead of* $\mathfrak{g}$ is nondegenerate. Since this form is of degree 0, the restriction $(\cdot, \cdot)^\circ_{\lambda,n}$ of this form to $S(\mathfrak{n}'_-)[-n] \times S(\mathfrak{n}'_+)[n]$ must also be nondegenerate[52]. But since $S(\mathfrak{n}'_+)[n] =$

[52]This is because if V and W are two graded vector spaces, and $\phi : V \times W \to \mathbb{C}$ is a nondegenerate bilinear form of degree 0, then for every $n \in \mathbb{Z}$, the restriction of ϕ to $V[-n] \times W[n]$ must also be nondegenerate.

$S(\mathfrak{n}_+)[n]$ [53] and $S(\mathfrak{n}'_-)[-n] = S(\mathfrak{n}_-)[-n]$ [54], this restriction is exactly our form $(\cdot,\cdot)^\circ_{\lambda,n} : S(\mathfrak{n}_-)[-n] \times S(\mathfrak{n}_+)[n] \to \mathbb{C}$ (in fact, the form is clearly given by the same formula). Thus we have shown that our form $(\cdot,\cdot)^\circ_{\lambda,n} : S(\mathfrak{n}_-)[-n] \times S(\mathfrak{n}_+)[n] \to \mathbb{C}$ is nondegenerate. Lemma 2.6.13 is proven.

2.6.6. Proof of Theorem 2.6.6: $(\cdot,\cdot)^\circ_\lambda$ is the "highest term" of $(\cdot,\cdot)_\lambda$

Before we go on, let us sketch the direction in which we want to go. We want to study how, for a fixed $n \in \mathbb{N}$, the form $(\cdot,\cdot)_{\lambda,n}$ changes with λ. If V and W are two finite-dimensional vector spaces **of the same dimension**, and if we have chosen bases for these two vector spaces V and W, then we can represent every bilinear form $V \times W \to \mathbb{C}$ as a square matrix with respect to these two bases, and the bilinear form is nondegenerate if and only if this matrix has nonzero determinant. This suggests that we study how the determinant $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ of the form $(\cdot,\cdot)_{\lambda,n}$ with respect to some bases of $M^+_\lambda[-n]$ and $M^-_{-\lambda}[n]$ changes with λ (and, in particular, show that this determinant is nonzero for generic λ when $\mathfrak{g}$ is nondegenerate). Of course, speaking of this determinant $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ only makes sense when the bases of $M^+_\lambda[-n]$ and $M^-_{-\lambda}[n]$ have the same size (since only square matrices have determinants), but this is automatically satisfied if we have $\dim(\mathfrak{g}_n) = \dim(\mathfrak{g}_{-n})$ for every integer $n > 0$ (this condition is automatically satisfied when $\mathfrak{g}$ is a nondegenerate $\mathbb{Z}$-graded Lie algebra, but of course not only then).

Unfortunately, the spaces $M^+_\lambda[-n]$ and $M^-_{-\lambda}[n]$ themselves change with λ. Thus,

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[53] *Proof.* Since $\mathfrak{n}_+ = \sum_{i\geq 1} \mathfrak{g}_i$, we have $S(\mathfrak{n}_+) = \sum_{k\in\mathbb{N}} \sum_{\substack{(i_1,i_2,...,i_k)\in\mathbb{N}^k;\\ \text{each } i_j\geq 1}} \mathfrak{g}_{i_1}\mathfrak{g}_{i_2}...\mathfrak{g}_{i_k}$ and thus

$$S(\mathfrak{n}_+)[n] = \sum_{k\in\mathbb{N}} \sum_{\substack{(i_1,i_2,...,i_k)\in\mathbb{N}^k;\\ \text{each } i_j\geq 1;\\ i_1+i_2+...+i_k=n}} \mathfrak{g}_{i_1}\mathfrak{g}_{i_2}...\mathfrak{g}_{i_k}$$

(since $\mathfrak{g}_{i_1}\mathfrak{g}_{i_2}...\mathfrak{g}_{i_k} \subseteq S(\mathfrak{n}_+)[i_1+i_2+...+i_k]$ for all $(i_1,i_2,...,i_k) \in \mathbb{N}^k$). Similarly,

$$S(\mathfrak{n}'_+)[n] = \sum_{k\in\mathbb{N}} \sum_{\substack{(i_1,i_2,...,i_k)\in\mathbb{N}^k;\\ \text{each } i_j\geq 1;\\ \text{each } |i_j|\leq n;\\ i_1+i_2+...+i_k=n}} \mathfrak{g}_{i_1}\mathfrak{g}_{i_2}...\mathfrak{g}_{i_k}$$

(because $\mathfrak{g}'$ is obtained from $\mathfrak{g}$ by removing all $\mathfrak{g}_i$ with $|i| > n$). Thus,

$$\begin{aligned} S(\mathfrak{n}'_+)[n] &= \sum_{k\in\mathbb{N}} \sum_{\substack{(i_1,i_2,...,i_k)\in\mathbb{N}^k;\\ \text{each } i_j\geq 1;\\ \text{each } |i_j|\leq n;\\ i_1+i_2+...+i_k=n}} \mathfrak{g}_{i_1}\mathfrak{g}_{i_2}...\mathfrak{g}_{i_k} = \sum_{k\in\mathbb{N}} \sum_{\substack{(i_1,i_2,...,i_k)\in\mathbb{N}^k;\\ \text{each } i_j\geq 1;\\ i_1+i_2+...+i_k=n}} \mathfrak{g}_{i_1}\mathfrak{g}_{i_2}...\mathfrak{g}_{i_k} \\ &\qquad \left(\begin{array}{c}\text{here, we removed the condition (each } |i_j|\leq n\text{), because it was redundant}\\ \text{(since every } (i_1,i_2,...,i_k)\in\mathbb{N}^k \text{ satisfying } i_1+i_2+...+i_k=n \text{ automatically}\\ \text{satisfies (each } |i_j|\leq n\text{))}\end{array}\right) \\ &= S(\mathfrak{n}_+)[n], \end{aligned}$$

qed.

[54] for analogous reasons

if we want to pick some bases of $M_{\lambda}^{+}[-n]$ and $M_{-\lambda}^{-}[n]$ for all $\lambda \in \mathfrak{h}^*$, we have to pick new bases **for every** λ. If we just pick these bases randomly, then the determinant $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ can change very unpredictably (because the determinant depends on the choice of bases). Thus, if we want to say something interesting about how $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ changes with λ, then we should specify a reasonable choice of bases for all λ. Fortunately, this is not difficult: It is enough to choose Poincaré-Birkhoff-Witt bases for $U(\mathfrak{n}_-)[-n]$ and $U(\mathfrak{n}_+)[n]$, and thus obtain bases $M_{\lambda}^{+}[-n]$ and $M_{-\lambda}^{-}[n]$ due to the isomorphisms $M_{\lambda}^{+}[-n] \cong U(\mathfrak{n}_-)[-n]$ and $M_{-\lambda}^{-}[n] \cong U(\mathfrak{n}_+)[n]$. (See Convention 2.6.21 for details.) With bases chosen this way, the determinant $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ will depend on λ polynomially, and we will be able to conclude some useful properties of this polynomial.

So much for our roadmap. Let us first make a convention:

Convention 2.6.14. If V and W are two finite-dimensional vector spaces **of the same dimension**, and if we have chosen bases for these two vector spaces V and W, then we can represent every bilinear form $B : V \times W \to \mathbb{C}$ as a square matrix with respect to these two bases. The determinant of this matrix will be denoted by $\det B$ and called the *determinant of the form* B. Of course, this determinant $\det B$ depends on the bases chosen. A change of either basis induces a scaling of $\det B$ by a **nonzero** scalar. Thus, while the determinant $\det B$ itself depends on the choice of bases, the property of $\det B$ to be zero or nonzero does **not** depend on the choice of bases.

Let us now look at how the form $(\cdot,\cdot)_{\lambda,n}$ and its determinant $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ depend on λ. We want to show that this dependence is polynomial. In order to make sense of this, let us define what we mean by "polynomial" here:

Definition 2.6.15. Let V be a finite-dimensional vector space. A function $\phi : V \to \mathbb{C}$ is said to be a *polynomial function* (or just to be *polynomial* – but this is not the same as being *a polynomial*) if one of the following equivalent conditions holds:

(1) There exist a basis $(\beta_1, \beta_2, ..., \beta_m)$ of the dual space V^* and a polynomial $P \in \mathbb{C}[X_1, X_2, ..., X_m]$ such that

$$\text{every } v \in V \text{ satisfies } \phi(v) = P(\beta_1(v), \beta_2(v), ..., \beta_m(v)).$$

(2) For every basis $(\beta_1, \beta_2, ..., \beta_m)$ of the dual space V^*, there exists a polynomial $P \in \mathbb{C}[X_1, X_2, ..., X_m]$ such that

$$\text{every } v \in V \text{ satisfies } \phi(v) = P(\beta_1(v), \beta_2(v), ..., \beta_m(v)).$$

(3) There exist finitely many elements β_1, β_2, ..., β_m of the dual space V^* and a polynomial $P \in \mathbb{C}[X_1, X_2, ..., X_m]$ such that

$$\text{every } v \in V \text{ satisfies } \phi(v) = P(\beta_1(v), \beta_2(v), ..., \beta_m(v)).$$

Note that this is exactly the meaning of the word "polynomial function" that is used in Classical Invariant Theory. In our case (where the field is $\mathbb{C}$), polynomial functions

$V \to \mathbb{C}$ can be identified with elements of the symmetric algebra $\mathrm{S}(V^*)$, and in some sense are an "obsoleted version" of the latter.[55] For our goals, however, polynomial functions are enough. Let us define the notion of *homogeneous polynomial functions*:

Definition 2.6.16. Let V be a finite-dimensional vector space.

(a) Let $n \in \mathbb{N}$. A polynomial function $\phi : V \to \mathbb{C}$ is said to be *homogeneous of degree n* if and only if

$$\text{every } v \in V \text{ and every } \lambda \in \mathbb{C} \text{ satisfy } \phi(\lambda v) = \lambda^n \phi(v).$$

(b) A polynomial function $\phi : V \to \mathbb{C}$ is said to be *homogeneous* if and only if there exists some $n \in \mathbb{N}$ such that ϕ is homogeneous of degree n.

(c) It is easy to see that for every polynomial function $\phi : V \to \mathbb{C}$, there exists a unique sequence $(\phi_n)_{n\in\mathbb{N}}$ of polynomial functions $\phi_n : V \to \mathbb{C}$ such that all but finitely many $n \in \mathbb{N}$ satisfy $\phi_n = 0$, such that ϕ_n is homogeneous of degree n for every $n \in \mathbb{N}$, and such that $\phi = \sum\limits_{n\in\mathbb{N}} \phi_n$. This sequence is said to be the *graded decomposition* of ϕ. For every $n \in \mathbb{N}$, its member ϕ_n is called the *n-th homogeneous component* of ϕ. If N is the highest $n \in \mathbb{N}$ such that $\phi_n \neq 0$, then ϕ_N is said to be the *leading term* of ϕ.

Note that Definition 2.6.16 **(c)** defines the "leading term" of a polynomial as its highest-degree nonzero homogeneous component. This "leading term" may (and usually will) contain more than one monomial, so this notion of a "leading term" is not the same as the notion of a "leading term" commonly used, e. g., in Gröbner basis theory.

We now state the following crucial fact:

Proposition 2.6.17. Let $n \in \mathbb{N}$. Assume that $\mathfrak{g}$ is a nondegenerate $\mathbb{Z}$-graded Lie algebra. As a consequence, $\dim \mathfrak{h} = \dim(\mathfrak{g}_0) \neq \infty$, so that $\dim(\mathfrak{h}^*) \neq \infty$, and thus the notion of a polynomial function $\mathfrak{h}^* \to \mathbb{C}$ is well-defined.

There is an appropriate way of choosing bases of the vector spaces $S(\mathfrak{n}_-)[-n]$ and $S(\mathfrak{n}_+)[n]$ and bases of the vector spaces $M_\lambda^+[-n]$ and $M_{-\lambda}^-[n]$ for all $\lambda \in \mathfrak{h}^*$ such that the following holds:

(a) The determinants $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ and $\det\left((\cdot,\cdot)_{\lambda,n}^\circ\right)$ (these determinants are defined with respect to the chosen bases of $S(\mathfrak{n}_-)[-n]$, $S(\mathfrak{n}_+)[n]$, $M_\lambda^+[-n]$ and $M_{-\lambda}^-[n]$) depend polynomially on λ. By this, we mean that the functions

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)_{\lambda,n}\right)$$

and

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)_{\lambda,n}^\circ\right)$$

are polynomial functions.

[55] The identification of polynomial functions $V \to \mathbb{C}$ with elements of the symmetric algebra $\mathrm{S}(V^*)$ works similarly over any *infinite* field instead of $\mathbb{C}$. It breaks down over finite fields, however (because different elements of $\mathrm{S}(V^*)$ may correspond to the same polynomial function over a finite field).

(b) The leading term of the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)_{\lambda,n}\right)$$

is

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)^\circ_{\lambda,n}\right).$$

Remark 2.6.18. We can extend Proposition 2.6.17 to the case when $\mathfrak{g}$ is no longer nondegenerate. However, this requires the following changes to Proposition 2.6.17:

Replace the requirement that $\mathfrak{g}$ be nondegenerate by the requirement that $\mathfrak{g}$ satisfy the conditions **(1)** and **(2)** in Definition 2.5.4 as well as the condition that $\dim(\mathfrak{g}_n) = \dim(\mathfrak{g}_{-n})$ for every integer $n > 0$ (this condition is a weakening of condition **(3)** in Definition 2.5.4). Replace the claim that "The leading term of the polynomial function $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ is $\det\left((\cdot,\cdot)^\circ_{\lambda,n}\right)$, up to multiplication by a nonzero scalar" by the claim that "There exists some $k \in \mathbb{N}$ such that the polynomial function $\det\left((\cdot,\cdot)^\circ_{\lambda,n}\right)$ is the k-th homogeneous component of the polynomial function $\det\left((\cdot,\cdot)_{\lambda,n}\right)$, and such that the ℓ-th homogeneous component of the polynomial function $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ is 0 for all $\ell > k$". Note that this does not imply that $\det\left((\cdot,\cdot)^\circ_{\lambda,n}\right)$ is not identically zero, and indeed $\det\left((\cdot,\cdot)^\circ_{\lambda,n}\right)$ can be identically zero.

Before we prove Proposition 2.6.17, let us show how it completes the proof of Theorem 2.6.6:

Proof of Theorem 2.6.6. Fix a positive $n \in \mathbb{N}$. For generic λ, the bilinear form

$$\mathfrak{g}_{-k} \times \mathfrak{g}_k \to \mathbb{C}, \qquad (a,b) \mapsto \lambda([a,b])$$

is nondegenerate for every $k \in \{1,2,...,n\}$ (because $\mathfrak{g}$ is nondegenerate). Thus, for generic λ, the form $(\cdot,\cdot)^\circ_{\lambda,n}$ must also be nondegenerate (by Lemma 2.6.13), so that $\det\left((\cdot,\cdot)^\circ_{\lambda,n}\right) \neq 0$. Since the leading term of the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)_{\lambda,n}\right)$$

is

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)^\circ_{\lambda,n}\right)$$

(by Proposition 2.6.17), this yields that $\det\left((\cdot,\cdot)_{\lambda,n}\right) \neq 0$ for generic λ. In other words, the form $(\cdot,\cdot)_{\lambda,n}$ is nondegenerate for generic λ. But this form $(\cdot,\cdot)_{\lambda,n}$ is exactly the restriction of the form $(\cdot,\cdot) : M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ to $M^+_\lambda[-n] \times M^-_{-\lambda}[n]$. Hence, the restriction of the form $(\cdot,\cdot) : M^+_\lambda \times M^-_{-\lambda} \to \mathbb{C}$ to $M^+_\lambda[-n] \times M^-_{-\lambda}[n]$ is nondegenerate for generic λ. This proves Theorem 2.6.6.

So all that remains to finish the proof of Theorem 2.6.6 is verifying Proposition 2.6.17.

2.6.7. Proof of Theorem 2.6.6: Polynomial maps

We already defined the notion of a polynomial function in Definition 2.6.15. Let us give a definition of a notion of a "polynomial map" which is tailored for our proof of Theorem 2.6.6. I cannot guarantee that it is the same as what other people call "polynomial map", but it should be very close.

Definition 2.6.19. Let V be a finite-dimensional vector space. Let W be a vector space. A map $\phi : V \to W$ is said to be a *polynomial map* if and only if there exist:
- some $n \in \mathbb{N}$;
- n vectors w_1, w_2, ..., w_n in W;
- n polynomial functions P_1, P_2, ..., P_n from V to $\mathbb{C}$

such that

$$\text{every } v \in V \text{ satisfies } \phi(v) = \sum_{i=1}^{n} P_i(v) w_i.$$

Note that it is clear that:

- If V is a finite-dimensional vector space and W is a vector space, then any $\mathbb{C}$-linear combination of polynomial maps $V \to W$ is a polynomial map.
- If V is a finite-dimensional vector space and W is a $\mathbb{C}$-algebra, then any product of polynomial maps $V \to W$ is a polynomial map.
- If V is a finite-dimensional vector space, then polynomial maps $V \to \mathbb{C}$ are exactly the same as polynomial functions $V \to \mathbb{C}$ (since $\mathbb{C}$-linear combinations of polynomial functions are polynomial functions).

2.6.8. Proof of Theorem 2.6.6: The deformed Lie algebra $\mathfrak{g}^\varepsilon$

Before we go on, here is a rough plan of how we will attack Proposition 2.6.17:

In order to gain a foothold on $\det\left((\cdot,\cdot)_{\lambda,n}\right)$, we are going to consider not just one Lie algebra $\mathfrak{g}$ but a whole family $(\mathfrak{g}^\varepsilon)_{\varepsilon\in\mathbb{C}}$ of its "deformations" at the same time. Despite all of these deformations being isomorphic as Lie algebras with one exception, they will give us useful information: we will show that the bilinear forms $(\cdot,\cdot)_{\lambda,n}^{\mathfrak{g}^\varepsilon}$ they induce, in some sense, depend "polynomially" on λ and ε. We will have to restrain from speaking directly of the bilinear form $(\cdot,\cdot)_{\lambda,n}^{\mathfrak{g}^\varepsilon}$ as depending polynomially on λ, since this makes no sense (the domain of the bilinear form $(\cdot,\cdot)_{\lambda,n}^{\mathfrak{g}^\varepsilon}$ changes with λ), but instead we will sample this form on particular elements of the Verma modules coming from appropriately chosen Poincaré-Birkhoff-Witt bases of $U\left(\mathfrak{n}_-^\varepsilon\right)$ and $U\left(\mathfrak{n}_+^\varepsilon\right)$. These sampled values of the form will turn out to depend polynomially on λ and ε, and thus the determinant $\det\left((\cdot,\cdot)_{\lambda,n}^\varepsilon\right)$ will be a polynomial function in λ and ε. This polynomial function will turn out to have some kind of "homogeneity with respect to λ and ε^2" (this is not a standard notion, but see Corollary 2.6.27 for what exactly this means in our context), so that the leading term of λ will be the term with smallest power of ε (and, as it will turn out, this will be the power ε^0, so this term will be obtainable by setting ε to 0). Once this all is formalized and proven, we will explicitly show that (more or less) $(\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n} = (\cdot,\cdot)^{\circ}_{\lambda,n}$ (again this does not literally hold but must be correctly interpreted), and we know the form $(\cdot,\cdot)^{\circ}_{\lambda,n}$ to be nondegenerate (by Lemma 2.6.13), so that the form $(\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n}$ will be nondegenerate, and this will quickly yield the nondegeneracy of $\det\left((\cdot,\cdot)^{\varepsilon}_{\lambda,n}\right)$ for generic λ and ε, and thus the nondegeneracy of $\det\left((\cdot,\cdot)_{\lambda,n}\right)$ for generic λ.

Now, to the details. Consider the situation of Proposition 2.6.17. In particular, this means that (from now on until the end of Section 2.6) the Lie algebra $\mathfrak{g}$ will be assumed nondegenerate.

First, let us define $(\mathfrak{g}^{\varepsilon})_{\varepsilon\in\mathbb{C}}$.

For every $\varepsilon \in \mathbb{C}$, let us define a new Lie bracket $[\cdot,\cdot]^{\varepsilon}$ on the vector space $\mathfrak{g}$ by the formula

$$[x,y]^{\varepsilon} = \varepsilon[x,y] + (1-\varepsilon)\,\pi([x,y]) - \varepsilon(1-\varepsilon)[x,\pi(y)] - \varepsilon(1-\varepsilon)[\pi(x),y] \qquad (39)$$
$$\text{for all } x\in\mathfrak{g} \text{ and } y\in\mathfrak{g},$$

where π is the canonical projection $\mathfrak{g}\to\mathfrak{g}_0$. In other words, let us define a new Lie bracket $[\cdot,\cdot]^{\varepsilon}$ on the vector space $\mathfrak{g}$ by

$$[x,y]^{\varepsilon} = \varepsilon^{\delta_{n,0}+\delta_{m,0}+1-\delta_{n+m,0}}[x,y] \qquad (40)$$
$$\text{for all } n\in\mathbb{Z},\ m\in\mathbb{Z},\ x\in\mathfrak{g}_n \text{ and } y\in\mathfrak{g}_m$$

(note that the right hand side of this equation makes sense since $1-\delta_{n+m,0}\geq 0$ for all $n\in\mathbb{Z}$ and $m\in\mathbb{Z}$) [56]. It is easy to prove that this Lie bracket $[\cdot,\cdot]^{\varepsilon}$ is antisymmetric and satisfies the Jacobi identity[57] and is graded. Thus, this Lie bracket $[\cdot,\cdot]^{\varepsilon}$ defines a graded Lie algebra structure on $\mathfrak{g}$. Let us denote this Lie algebra by $\mathfrak{g}^{\varepsilon}$. Thus, $\mathfrak{g}^{\varepsilon}$ is identical with $\mathfrak{g}$ as a vector space, but the Lie bracket on $\mathfrak{g}^{\varepsilon}$ is $[\cdot,\cdot]^{\varepsilon}$ rather than $[\cdot,\cdot]$.

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[56] Proving that these two definitions of $[\cdot,\cdot]^{\varepsilon}$ are equivalent is completely straightforward: just assume WLOG that x and y are homogeneous, so that $x\in\mathfrak{g}_n$ and $y\in\mathfrak{g}_m$ for $n\in\mathbb{Z}$ and $m\in\mathbb{Z}$, and distinguish between the following four cases:

Case 1: We have $n=0$ and $m=0$.

Case 2: We have $n\neq 0$ and $m\neq 0$ but $n+m=0$.

Case 3: We have $n\neq 0$, $m\neq 0$ and $n+m\neq 0$.

Case 4: Exactly one of n and m is 0.

In Case 1, the assumption that $\mathfrak{g}_0$ is abelian must be used.

[57] *Proof.* Antisymmetry is obvious. As for the Jacobi identity, it can be proven in a straightforward way:

We must show the equality $[x,[y,z]^{\varepsilon}]^{\varepsilon} + [y,[z,x]^{\varepsilon}]^{\varepsilon} + [z,[x,y]^{\varepsilon}]^{\varepsilon} = 0$ for all $x,y,z\in\mathfrak{g}$. Since this equality is linear in each of x, y and z, it is enough to prove it for homogeneous $x,y,z\in\mathfrak{g}$. So let $x,y,z\in\mathfrak{g}$ be homogeneous. Then, there exist $n,m,p\in\mathbb{Z}$ such that $x\in\mathfrak{g}_n$, $y\in\mathfrak{g}_m$ and $z\in\mathfrak{g}_p$. Consider these n, m and p. Then, by (40) (applied to y, z, m and p instead of x, y, n and m), we have $[y,z]^{\varepsilon} = \varepsilon^{\delta_{m,0}+\delta_{p,0}+1-\delta_{m+p,0}}[y,z]$. Thus,

$$\begin{aligned}
&[x,[y,z]^{\varepsilon}]^{\varepsilon}\\
&= \left[x,\varepsilon^{\delta_{m,0}+\delta_{p,0}+1-\delta_{m+p,0}}[y,z]\right]^{\varepsilon} = \varepsilon^{\delta_{m,0}+\delta_{p,0}+1-\delta_{m+p,0}}[x,[y,z]]^{\varepsilon}\\
&= \varepsilon^{\delta_{m,0}+\delta_{p,0}+1-\delta_{m+p,0}}\varepsilon^{\delta_{n,0}+\delta_{m+p,0}+1-\delta_{n+m+p,0}}[x,[y,z]]\\
&\qquad\left(\begin{array}{c}\text{because (40) (applied to } [y,z] \text{ and } m+p \text{ instead of } y \text{ and } m) \text{ yields}\\ [x,[y,z]]^{\varepsilon} = \varepsilon^{\delta_{n,0}+\delta_{m+p,0}+1-\delta_{n+m+p,0}}[x,[y,z]] \text{ (since } [y,z]\in\mathfrak{g}_{m+p} \text{ (since } y\in\mathfrak{g}_m \text{ and } z\in\mathfrak{g}_p))\end{array}\right)\\
&= \varepsilon^{\delta_{m,0}+\delta_{p,0}+1-\delta_{m+p,0}+\delta_{n,0}+\delta_{m+p,0}+1-\delta_{n+m+p,0}}[x,[y,z]] = \varepsilon^{\delta_{n,0}+\delta_{m,0}+\delta_{p,0}+2-\delta_{n+m+p,0}}[x,[y,z]].
\end{aligned}$$

Trivially, $\mathfrak{g}^1 = \mathfrak{g}$ (this is an actual equality, not only an isomorphism) and $[\cdot,\cdot]^1 = [\cdot,\cdot]$.

For every $\varepsilon \in \mathbb{C}$, define a $\mathbb{C}$-linear map $J_\varepsilon : \mathfrak{g}^\varepsilon \to \mathfrak{g}$ by

$$J_\varepsilon (x) = \varepsilon^{1+\delta_{n,0}} x \qquad \text{for every } n \in \mathbb{Z} \text{ and } x \in \mathfrak{g}_n.$$

Then, J_ε is a Lie algebra homomorphism[58]. Also, J_ε is a vector space isomorphism when $\varepsilon \neq 0$. Hence, J_ε is a Lie algebra isomorphism when $\varepsilon \neq 0$. Moreover, $J_1 = \mathrm{id}$.

For every $\varepsilon \in \mathbb{C}$, we are going to denote by $\mathfrak{n}_-^\varepsilon$, $\mathfrak{n}_+^\varepsilon$ and $\mathfrak{h}^\varepsilon$ the vector spaces $\mathfrak{n}_-$, $\mathfrak{n}_+$ and $\mathfrak{h}$ **as Lie subalgebras of** $\mathfrak{g}^\varepsilon$. Note that $\mathfrak{h}^\varepsilon = \mathfrak{h}$ as Lie algebras (because $\mathfrak{h}$ and $\mathfrak{h}^\varepsilon$ are abelian Lie algebras), but the equalities $\mathfrak{n}_-^\varepsilon = \mathfrak{n}_-$ and $\mathfrak{n}_+^\varepsilon = \mathfrak{n}_+$ hold only as equalities of vector spaces (unless we are in some rather special situation). Since the grading of $\mathfrak{g}^\varepsilon$ is the same as the grading of $\mathfrak{g}$, the triangular decomposition of $\mathfrak{g}^\varepsilon$ is $\mathfrak{n}_-^\varepsilon \oplus \mathfrak{h}^\varepsilon \oplus \mathfrak{n}_+^\varepsilon$ for every $\varepsilon \in \mathbb{C}$.

Now, we are dealing with several Lie algebras on the same vector space, and we are going to be dealing with their Verma modules. In order not to confuse them, let us introduce a notation:

Convention 2.6.20. In the following, whenever $\mathfrak{e}$ is a $\mathbb{Z}$-graded Lie algebra, and $\lambda \in \mathfrak{e}_0^*$, we are going to denote by $M_\lambda^{+\mathfrak{e}}$ the Verma highest-weight module of $(\mathfrak{e}, \lambda)$, and we are going to denote by $M_\lambda^{-\mathfrak{e}}$ the Verma lowest-weight module of $(\mathfrak{e}, \lambda)$. We

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Similarly,

$$[y,[z,x]^\varepsilon]^\varepsilon = \varepsilon^{\delta_{n,0}+\delta_{m,0}+\delta_{p,0}+2-\delta_{n+m+p,0}} [y,[z,x]] \qquad \text{and}$$
$$[z,[x,y]^\varepsilon]^\varepsilon = \varepsilon^{\delta_{n,0}+\delta_{m,0}+\delta_{p,0}+2-\delta_{n+m+p,0}} [z,[x,y]].$$

Adding up these three equations yields

$$\begin{aligned}
&[x,[y,z]^\varepsilon]^\varepsilon + [y,[z,x]^\varepsilon]^\varepsilon + [z,[x,y]^\varepsilon]^\varepsilon \\
&= \varepsilon^{\delta_{n,0}+\delta_{m,0}+\delta_{p,0}+2-\delta_{n+m+p,0}} [x,[y,z]] + \varepsilon^{\delta_{n,0}+\delta_{m,0}+\delta_{p,0}+2-\delta_{n+m+p,0}} [y,[z,x]] + \varepsilon^{\delta_{n,0}+\delta_{m,0}+\delta_{p,0}+2-\delta_{n+m+p,0}} [z,[x,y]] \\
&= \varepsilon^{\delta_{n,0}+\delta_{m,0}+\delta_{p,0}+2-\delta_{n+m+p,0}} \underbrace{([x,[y,z]] + [y,[z,x]] + [z,[x,y]])}_{=0 \text{ (since } \mathfrak{g} \text{ is a Lie algebra)}} = 0.
\end{aligned}$$

This proves the Jacobi identity for the Lie bracket $[\cdot,\cdot]^\varepsilon$, qed.

[58] *Proof.* We must show that $J_\varepsilon([x,y]^\varepsilon) = [J_\varepsilon(x), J_\varepsilon(y)]$ for all $x,y \in \mathfrak{g}$. In order to show this, it is enough to prove that $J_\varepsilon([x,y]^\varepsilon) = [J_\varepsilon(x), J_\varepsilon(y)]$ for all homogeneous $x,y \in \mathfrak{g}$ (because of linearity). So let $x,y \in \mathfrak{g}$ be homogeneous. Thus, there exist $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ such that $x \in \mathfrak{g}_n$ and $y \in \mathfrak{g}_m$. Consider these n and m. Then, $[x,y] \in \mathfrak{g}_{n+m}$. Now, $J_\varepsilon(x) = \varepsilon^{1+\delta_{n,0}} x$ and $J_\varepsilon(y) = \varepsilon^{1+\delta_{m,0}} y$ by the definition of J_ε. Thus,

$$[J_\varepsilon(x), J_\varepsilon(y)] = \left[\varepsilon^{1+\delta_{n,0}} x, \varepsilon^{1+\delta_{m,0}} y\right] = \varepsilon^{1+\delta_{n,0}} \varepsilon^{1+\delta_{m,0}} [x,y] = \varepsilon^{2+\delta_{n,0}+\delta_{m,0}} [x,y].$$

Compared with

$$\begin{aligned}
J_\varepsilon([x,y]^\varepsilon) &= J_\varepsilon\left(\varepsilon^{\delta_{n,0}+\delta_{m,0}+1-\delta_{n+m,0}} [x,y]\right) \qquad \text{(by (40))} \\
&= \varepsilon^{\delta_{n,0}+\delta_{m,0}+1-\delta_{n+m,0}} \underbrace{J_\varepsilon([x,y])}_{\substack{=\varepsilon^{1+\delta_{n+m,0}}[x,y] \\ \text{(by the definition of } J_\varepsilon, \\ \text{since } [x,y] \in \mathfrak{g}_{n+m})}} = \varepsilon^{\delta_{n,0}+\delta_{m,0}+1-\delta_{n+m,0}} \varepsilon^{1+\delta_{n+m,0}} [x,y] \\
&= \varepsilon^{2+\delta_{n,0}+\delta_{m,0}} [x,y],
\end{aligned}$$

this yields $J_\varepsilon([x,y]^\varepsilon) = [J_\varepsilon(x), J_\varepsilon(y)]$, qed.

will furthermore denote by $v_\lambda^{+\mathfrak{e}}$ the defining vector of $M_\lambda^{+\mathfrak{e}}$, and we will denote by $v_\lambda^{-\mathfrak{e}}$ the defining vector of $M_\lambda^{-\mathfrak{e}}$.

Further, we denote by $(\cdot,\cdot)_\lambda^{\mathfrak{e}}$ and $(\cdot,\cdot)_{\lambda,n}^{\mathfrak{e}}$ the forms $(\cdot,\cdot)_\lambda$ and $(\cdot,\cdot)_{\lambda,n}$ defined for the Lie algebra $\mathfrak{e}$ instead of $\mathfrak{g}$.

Thus, for instance, the Verma highest-weight module of $(\mathfrak{g},\lambda)$ (which we have always denoted by M_λ^+) can now be called $M_\lambda^{+\mathfrak{g}}$, and thus can be discerned from the Verma highest-weight module $M_\lambda^{+\mathfrak{g}^\varepsilon}$ of $(\mathfrak{g}^\varepsilon,\lambda)$.

Convention 2.6.21. For every $n \in \mathbb{Z}$, let $(e_{n,i})_{i\in\{1,2,...,m_n\}}$ be a basis of the vector space $\mathfrak{g}_n$ (such a basis exists since $\dim(\mathfrak{g}_n) < \infty$). Then, $(e_{n,i})_{(n,i)\in E}$ is a basis of the vector space $\mathfrak{g}$, where $E = \{(n,i) \mid n \in \mathbb{Z};\ i \in \{1,2,...,m_n\}\}$.

For every integer $n > 0$, we have $\dim(\mathfrak{g}_n) = m_n$ (since $(e_{n,i})_{i\in\{1,2,...,m_n\}}$ is a basis of the vector space $\mathfrak{g}_n$) and $\dim(\mathfrak{g}_{-n}) = m_{-n}$ (similarly), so that $m_n = \dim(\mathfrak{g}_n) = \dim(\mathfrak{g}_{-n}) = m_{-n}$. Of course, this yields that $m_n = m_{-n}$ for every integer n (whether positive or not).

We totally order the set E lexicographically. Let $\operatorname{Seq} E$ be the set of all finite sequences of elements of E. For every $\mathbf{i} \in \operatorname{Seq} E$ and every $\varepsilon \in \mathbb{C}$, we define an element $e_\mathbf{i}^\varepsilon$ of $U(\mathfrak{g}^\varepsilon)$ by

$$e_\mathbf{i}^\varepsilon = e_{n_1,i_1}e_{n_2,i_2}...e_{n_\ell,i_\ell}, \qquad \text{where we write } \mathbf{i} \text{ in the form } ((n_1,i_1),(n_2,i_2),...,(n_\ell,i_\ell)).$$

For every $\mathbf{i} \in \operatorname{Seq} E$, we define the *length* $\operatorname{len}\mathbf{i}$ of $\mathbf{i}$ to be the number of members of $\mathbf{i}$ (in other words, we set $\operatorname{len}\mathbf{i} = \ell$, where we write $\mathbf{i}$ in the form $((n_1,i_1),(n_2,i_2),...,(n_\ell,i_\ell))$), and we define the *degree* $\deg\mathbf{i}$ of $\mathbf{i}$ to be the sum $n_1 + n_2 + ... + n_\ell$, where we write $\mathbf{i}$ in the form $((n_1,i_1),(n_2,i_2),...,(n_\ell,i_\ell))$. It is clear that $e_\mathbf{i}^\varepsilon \in U(\mathfrak{g}^\varepsilon)[\deg\mathbf{i}]$.

Let $\operatorname{Seq}_+ E$ be the set of all **nondecreasing** sequences $((n_1,i_1),(n_2,i_2),...,(n_\ell,i_\ell)) \in \operatorname{Seq} E$ such that all of n_1, n_2, ..., n_ℓ are **positive**. By the Poincaré-Birkhoff-Witt theorem (applied to the Lie algebra $\mathfrak{n}_+^\varepsilon$), the family $\left(e_\mathbf{j}^\varepsilon\right)_{\mathbf{j}\in\operatorname{Seq}_+ E}$ is a basis of the vector space $U\left(\mathfrak{n}_+^\varepsilon\right)$. Moreover, it is a graded basis, i. e., the family $\left(e_\mathbf{j}^\varepsilon\right)_{\mathbf{j}\in\operatorname{Seq}_+ E;\ \deg\mathbf{j}=n}$ is a basis of the vector space $U\left(\mathfrak{n}_+^\varepsilon\right)[n]$ for every $n \in \mathbb{Z}$. Hence, $\left(e_\mathbf{j}^\varepsilon v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_{\mathbf{j}\in\operatorname{Seq}_+ E;\ \deg\mathbf{j}=n}$ is a basis of the vector space $M_{-\lambda}^{-\mathfrak{g}^\varepsilon}[n]$ for every $n \in \mathbb{Z}$ and $\lambda \in \mathfrak{h}^*$.

Let $\operatorname{Seq}_- E$ be the set of all **nonincreasing** sequences $((n_1,i_1),(n_2,i_2),...,(n_\ell,i_\ell)) \in \operatorname{Seq} E$ such that all of n_1, n_2, ..., n_ℓ are **negative**. By the Poincaré-Birkhoff-Witt theorem (applied to the Lie algebra $\mathfrak{n}_-^\varepsilon$), the family $(e_\mathbf{i}^\varepsilon)_{\mathbf{i}\in\operatorname{Seq}_- E}$ is a basis of the vector space $U\left(\mathfrak{n}_-^\varepsilon\right)$. Moreover, it is a graded basis, i. e., the family $(e_\mathbf{i}^\varepsilon)_{\mathbf{i}\in\operatorname{Seq}_- E;\ \deg\mathbf{i}=-n}$ is a basis of the vector space $U\left(\mathfrak{n}_-^\varepsilon\right)[-n]$ for every $n \in \mathbb{Z}$. Hence, $\left(e_\mathbf{i}^\varepsilon v_\lambda^{+\mathfrak{g}^\varepsilon}\right)_{\mathbf{i}\in\operatorname{Seq}_- E;\ \deg\mathbf{i}=-n}$ is a basis of the vector space $M_\lambda^{+\mathfrak{g}^\varepsilon}[-n]$ for every $n \in \mathbb{Z}$ and $\lambda \in \mathfrak{h}^*$.

We can define a bijection

$$E \to E,$$
$$(n,i) \mapsto (-n, m_n + 1 - i)$$

(because $m_n = m_{-n}$ for every $n \in \mathbb{Z}$). This bijection reverses the order on E. Hence, this bijection canonically induces a bijection $\operatorname{Seq} E \to \operatorname{Seq} E$, which maps $\operatorname{Seq}_+ E$ to $\operatorname{Seq}_- E$ and vice versa, and reverses the degree of every sequence while keeping the length of every sequence invariant. One consequence of this bijection is that for every $n \in \mathbb{Z}$, the number of all $\mathbf{j} \in \operatorname{Seq}_+ E$ satisfying $\deg \mathbf{j} = n$ equals the number of all $\mathbf{i} \in \operatorname{Seq}_- E$ satisfying $\deg \mathbf{i} = -n$. Another consequence is that $\sum\limits_{\substack{\mathbf{i} \in \operatorname{Seq}_- E; \\ \deg \mathbf{i} = -n}} \operatorname{len} \mathbf{i} = \sum\limits_{\substack{\mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{j} = n}} \operatorname{len} \mathbf{j}$.

For every positive integer n, we represent the bilinear form $(\cdot, \cdot)_{\lambda,n}^{\mathfrak{g}^\varepsilon} : M_\lambda^{+\mathfrak{g}^\varepsilon}[-n] \times M_{-\lambda}^{-\mathfrak{g}^\varepsilon}[n] \to \mathbb{C}$ by its matrix with respect to the bases $\left(e_{\mathbf{i}}^\varepsilon v_\lambda^{+\mathfrak{g}^\varepsilon}\right)_{\mathbf{i} \in \operatorname{Seq}_- E;\ \deg \mathbf{i} = -n}$ and $\left(e_{\mathbf{j}}^\varepsilon v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_{\mathbf{j} \in \operatorname{Seq}_+ E;\ \deg \mathbf{j} = n}$ of $M_\lambda^{+\mathfrak{g}^\varepsilon}[-n]$ and $M_{-\lambda}^{-\mathfrak{g}^\varepsilon}[n]$, respectively. This is the matrix

$$\left(\left(e_{\mathbf{i}}^\varepsilon v_\lambda^{+\mathfrak{g}^\varepsilon}, e_{\mathbf{j}}^\varepsilon v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_{\lambda,n}^{\mathfrak{g}^\varepsilon}\right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}}.$$

This matrix is a square matrix (since the number of all $\mathbf{j} \in \operatorname{Seq}_+ E$ satisfying $\deg \mathbf{j} = n$ equals the number of all $\mathbf{i} \in \operatorname{Seq}_- E$ satisfying $\deg \mathbf{i} = -n$), and its determinant is what we are going to denote by $\det\left((\cdot, \cdot)_{\lambda,n}^{\mathfrak{g}^\varepsilon}\right)$.

A few words about tensor algebras:

Convention 2.6.22. In the following, we let T denote the tensor algebra functor. Hence, for every vector space V, we denote by $T(V)$ the tensor algebra of V.

We notice that $T(V)$ is canonically graded even if V is not. In fact, $T(V) = \bigoplus\limits_{i \in \mathbb{N}} V^{\otimes i}$, so that we get a grading on $T(V)$ if we set $V^{\otimes i}$ to be the i-th homogeneous component of $T(V)$. This grading is called the *tensor length grading* on $T(V)$. It makes $T(V)$ concentrated in nonnegative degrees.

If V itself is a graded vector space, then we can also grade $T(V)$ by canonically extending the grading on V to $T(V)$ (this means that whenever v_1, v_2, ..., v_n are homogeneous elements of V of degrees d_1, d_2, ..., d_n, then the pure tensor $v_1 \otimes v_2 \otimes ... \otimes v_n$ has degree $d_1 + d_2 + ... + d_n$). This grading is called the *internal grading* on $T(V)$. It is different from the tensor length grading (unless V is concentrated in degree 1).

Hence, if V is a graded vector space, then $T(V)$ becomes a bigraded vector space (i. e., a vector space with two gradings). Let us agree to denote by $T(V)[n, m]$ the intersection of the n-th homogeneous component in the internal grading with the m-th homogeneous component in the tensor length grading (i. e., with $V^{\otimes m}$).

Let us notice that **as vector spaces**, we have $\mathfrak{g} = \mathfrak{g}^\varepsilon$, $\mathfrak{n}_- = \mathfrak{n}_-^\varepsilon$, $\mathfrak{n}_+ = \mathfrak{n}_+^\varepsilon$ and $\mathfrak{h} = \mathfrak{h}^\varepsilon$ for every $\varepsilon \in \mathbb{C}$. Hence, $T(\mathfrak{g}) = T(\mathfrak{g}^\varepsilon)$, $T(\mathfrak{n}_-) = T\left(\mathfrak{n}_-^\varepsilon\right)$, $T(\mathfrak{n}_+) = T\left(\mathfrak{n}_+^\varepsilon\right)$ and $T(\mathfrak{h}) = T(\mathfrak{h}^\varepsilon)$.

Definition 2.6.23. In the following, for every Lie algebra $\mathfrak{a}$ and every element $x \in T(\mathfrak{a})$, we denote by $\operatorname{env}_{\mathfrak{a}} x$ the projection of x onto the factor algebra $U(\mathfrak{a})$ of $T(\mathfrak{a})$.

Let us again stress that $T(\mathfrak{g}) = T(\mathfrak{g}^\varepsilon)$, so that $T(\mathfrak{g}^\varepsilon)$ does not depend on ε, whereas $U(\mathfrak{g}^\varepsilon)$ does. Hence, if we want to study the form $(\cdot, \cdot)_{\lambda,n}^{\mathfrak{g}^\varepsilon}$ as it changes with ε, the easiest thing to do is to study the values of $\left((\operatorname{env}_{\mathfrak{g}^\varepsilon} a)\, v_\lambda^{+\mathfrak{g}^\varepsilon}, (\operatorname{env}_{\mathfrak{g}^\varepsilon} b)\, v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_{\lambda,n}^{\mathfrak{g}^\varepsilon}$ for fixed $a \in T(\mathfrak{g}) = T(\mathfrak{g}^\varepsilon)$ and $b \in T(\mathfrak{g}) = T(\mathfrak{g}^\varepsilon)$. Here is the polynomiality lemma that we want to have:

Lemma 2.6.24. Let $\mathbf{i} \in \operatorname{Seq} E$ and $\mathbf{j} \in \operatorname{Seq} E$. Then, there exists a polynomial function $Q_{\mathbf{i},\mathbf{j}} : \mathfrak{h}^* \times \mathbb{C} \to \mathbb{C}$ such that every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$\left(e_{\mathbf{i}}^\varepsilon v_\lambda^{+\mathfrak{g}^\varepsilon}, e_{\mathbf{j}}^\varepsilon v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_\lambda^{\mathfrak{g}^\varepsilon} = Q_{\mathbf{i},\mathbf{j}}(\lambda, \varepsilon).$$

To prove this lemma, we show something more general:

Lemma 2.6.25. For every $n \in \mathbb{Z}$ and $c \in T(\mathfrak{g})[n]$, there exists a polynomial map $d : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[n]$ such that every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$(\operatorname{env}_{\mathfrak{g}^\varepsilon} c)\, v_\lambda^{+\mathfrak{g}^\varepsilon} = (\operatorname{env}_{\mathfrak{g}^\varepsilon} (d(\lambda, \varepsilon)))\, v_\lambda^{+\mathfrak{g}^\varepsilon}.$$

To get some intuition about Lemma 2.6.25, recall that the Verma highest-weight module $M_\lambda^{+\mathfrak{g}^\varepsilon}$ was defined as $U(\mathfrak{g}^\varepsilon) \otimes_{U(\mathfrak{h}^\varepsilon \oplus \mathfrak{n}_+^\varepsilon)} \mathbb{C}_\lambda$, but turned out to be $U(\mathfrak{n}_-^\varepsilon)\, v_\lambda^{+\mathfrak{g}^\varepsilon}$ (as a vector space), so that every term of the form $x v_\lambda^{+\mathfrak{g}^\varepsilon}$ with $x \in U(\mathfrak{g}^\varepsilon)$ can be reduced to the form $y v_\lambda^{+\mathfrak{g}^\varepsilon}$ with $y \in U(\mathfrak{n}_-^\varepsilon)$. Lemma 2.6.25 says that, if x is given as the projection $\operatorname{env}_{\mathfrak{g}^\varepsilon} c$ of some tensor $c \in T(\mathfrak{g})[n]$ onto $U(\mathfrak{g}^\varepsilon)$, then y can be found as the projection of some tensor $d(\lambda, \varepsilon) \in T(\mathfrak{n}_-)[n]$ onto $U(\mathfrak{n}_-^\varepsilon)$ which depends polynomially on λ and ε. This is not particularly surprising, since y is found from x by picking a tensorial representation[59] of x and "gradually" stratifying it[60], and the λ's and ε's which appear during this stratification process don't appear "randomly", but rather appear at foreseeable places. The following proof of Lemma 2.6.25 will formalize this idea.

Proof of Lemma 2.6.25. First some notations:

If $n \in \mathbb{Z}$, then a tensor $c \in T(\mathfrak{g})[n]$ is said to be *n-stratifiable* if there exists a polynomial map $d : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[n]$ such that every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$(\operatorname{env}_{\mathfrak{g}^\varepsilon} c)\, v_\lambda^{+\mathfrak{g}^\varepsilon} = (\operatorname{env}_{\mathfrak{g}^\varepsilon} (d(\lambda, \varepsilon)))\, v_\lambda^{+\mathfrak{g}^\varepsilon}.$$

[59] By a "tensorial representation" of x, I mean a tensor $c \in T(\mathfrak{g})$ such that $\operatorname{env}_{\mathfrak{g}^\varepsilon} c = x$.

[60] By "stratifying" a tensorial representation of x, I mean writing it as a linear combination of pure tensors, and whenever such a pure tensor has a negative tensorand (i. e., a tensorand in $\mathfrak{n}_-$) standing directly before a positive tensorand (i. e., a tensorand in $\mathfrak{n}_+$), applying the $xy - yx = [x, y]^\varepsilon$ relations in $U(\mathfrak{g}^\varepsilon)$ to move the negative tensorand past the positive one. As soon as a positive tensorand hits the right end of the tensor, the tensor can be thrown away since $\mathfrak{n}_+ v_\lambda^{+\mathfrak{g}^\varepsilon} = 0$. For instance, in Example 2.9.8 further below, we compute $L_1 L_{-1} v_\lambda^+$ by stratifying the tensorial representation $L_1 \otimes L_{-1}$ of $L_1 L_{-1}$, and we compute $L_1^2 L_{-1}^2 v_\lambda^+$ by stratifying the tensorial representation $L_1 \otimes L_1 \otimes L_{-1} \otimes L_{-1}$ of $L_1^2 L_{-1}^2$.

Lemma 2.6.25 states that for every $n \in \mathbb{Z}$, every tensor $c \in T(\mathfrak{g})[n]$ is n-stratifiable. We will now prove that

$$\text{for every } n \in \mathbb{Z} \text{ and every } m \in \mathbb{N}, \text{ every tensor } c \in T(\mathfrak{g})[n,m] \text{ is } n\text{-stratifiable.} \qquad (41)$$

Before we start proving this, let us formulate two easy observations about stratifiable tensors:

Observation 1: For any fixed n, any $\mathbb{C}$-linear combination of n-stratifiable tensors is n-stratifiable. (In fact, we can just take the corresponding $\mathbb{C}$-linear combination of the corresponding polynomial maps d.)

Observation 2: If an integer n, a negative integer ν, a vector $x \in \mathfrak{g}_\nu$ and a tensor $y \in T(\mathfrak{g})[n-\nu]$ are such that y is $(n-\nu)$-stratifiable, then $x \otimes y \in T(\mathfrak{g})[n]$ is n-stratifiable.[61]

We are now going to prove (41) by induction on m:

Induction base: We have $T(\mathfrak{g})[n,0] = \mathbb{C}[n]$. Hence, every tensor $c \in T(\mathfrak{g})[n,0]$ is n-stratifiable (because we can define the polynomial map $d : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[n]$ by

$$d(\lambda,\varepsilon) = c \qquad \text{for all } (\lambda,\varepsilon) \in \mathfrak{h}^* \times \mathbb{C}$$

). In other words, (41) is proven for $m = 0$. In other words, the induction base is complete.

Induction step: Let $m \in \mathbb{N}$ be positive. We must show that (41) holds for this m, using the assumption that (41) holds for $m-1$ instead of m.

Let $n \in \mathbb{Z}$. Let $\pi_n : T(\mathfrak{g}) \to T(\mathfrak{g})[n]$ denote the canonical projection of $T(\mathfrak{g})$ to the n-th homogeneous component with respect to the internal grading.

Let $c \in T(\mathfrak{g})[n,m]$. We must prove that c is n-stratifiable.

We have $c \in T(\mathfrak{g})[n,m] \subseteq \mathfrak{g}^{\otimes m}$, and since the m-th tensor power is generated by pure tensors, this yields that c is a $\mathbb{C}$-linear combination of pure tensors. In other words, c is a $\mathbb{C}$-linear combination of finitely many pure tensors of the form $x_1 \otimes x_2 \otimes ... \otimes x_m$

[61] *Proof of Observation 2.* Let an integer n, a negative integer ν, a vector $x \in \mathfrak{g}_\nu$ and a tensor $y \in T(\mathfrak{g})[n-\nu]$ be such that y is $(n-\nu)$-stratifiable. Then, there exists a polynomial map $\widetilde{d} : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[n-\nu]$ such that every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$(\operatorname{env}_{\mathfrak{g}^\varepsilon} y)\, v_\lambda^{+\mathfrak{g}^\varepsilon} = \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d}(\lambda,\varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}$$

(by the definition of "$(n-\nu)$-stratifiable"). Now, define a map $d : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[n]$ by

$$d(\lambda,\varepsilon) = x \otimes \widetilde{d}(\lambda,\varepsilon) \qquad \text{for every } (\lambda,\varepsilon) \in \mathfrak{h}^* \times \mathbb{C}.$$

(This is well-defined, since $x \in \mathfrak{g}_\nu \subseteq \mathfrak{n}_-$ (since ν is negative).) This map d is clearly polynomial (since $\widetilde{d}$ is a polynomial map), and every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$\underbrace{(\operatorname{env}_{\mathfrak{g}^\varepsilon}(x \otimes y))}_{= x \cdot \operatorname{env}_{\mathfrak{g}^\varepsilon} y} v_\lambda^{+\mathfrak{g}^\varepsilon} = x \cdot \underbrace{(\operatorname{env}_{\mathfrak{g}^\varepsilon} y)\, v_\lambda^{+\mathfrak{g}^\varepsilon}}_{= \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d}(\lambda,\varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}} = \underbrace{x \cdot \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d}(\lambda,\varepsilon)\right)\right)}_{= \operatorname{env}_{\mathfrak{g}^\varepsilon}\left(x \otimes \widetilde{d}(\lambda,\varepsilon)\right)} v_\lambda^{+\mathfrak{g}^\varepsilon}$$

$$= \left(\operatorname{env}_{\mathfrak{g}^\varepsilon} \underbrace{\left(x \otimes \widetilde{d}(\lambda,\varepsilon)\right)}_{= d(\lambda,\varepsilon)}\right) v_\lambda^{+\mathfrak{g}^\varepsilon} = (\operatorname{env}_{\mathfrak{g}^\varepsilon}(d(\lambda,\varepsilon)))\, v_\lambda^{+\mathfrak{g}^\varepsilon}.$$

Hence, $x \otimes y$ is n-stratifiable (by the definition of "n-stratifiable"). This proves Observation 2.

with $x_1, x_2, ..., x_m \in \mathfrak{g}$. We can WLOG assume that, in each of these pure tensors, the elements $x_1, x_2, ..., x_m$ are homogeneous (since otherwise we can break each of $x_1, x_2, ..., x_m$ into homogeneous components, and thus the pure tensors $x_1 \otimes x_2 \otimes ... \otimes x_m$ break into smaller pieces which are still pure tensors). So we can write c as a $\mathbb{C}$-linear combination of finitely many pure tensors of the form $x_1 \otimes x_2 \otimes ... \otimes x_m$ with **homogeneous** $x_1, x_2, ..., x_m \in \mathfrak{g}$. If we apply the projection π_n to this, then c remains invariant (since $c \in T(\mathfrak{g})[n, m] \subseteq T(\mathfrak{g})[n]$), and the terms of the form $x_1 \otimes x_2 \otimes ... \otimes x_m$ with **homogeneous** $x_1, x_2, ..., x_m \in \mathfrak{g}$ satisfying $\deg(x_1) + \deg(x_2) + ... + \deg(x_m) = n$ remain invariant as well (since they also lie in $T(\mathfrak{g})[n]$), whereas the terms of the form $x_1 \otimes x_2 \otimes ... \otimes x_m$ with **homogeneous** $x_1, x_2, ..., x_m \in \mathfrak{g}$ satisfying $\deg(x_1) + \deg(x_2) + ... + \deg(x_m) \neq n$ are mapped to 0 (since they lie in homogeneous components of $T(\mathfrak{g})$ other than $T(\mathfrak{g})[n]$). Hence, we write c as a $\mathbb{C}$-linear combination of finitely many pure tensors of the form $x_1 \otimes x_2 \otimes ... \otimes x_m$ with **homogeneous** $x_1, x_2, ..., x_m \in \mathfrak{g}$ **satisfying** $\deg(x_1) + \deg(x_2) + ... + \deg(x_m) = n$.

Therefore, in proving (41), we can WLOG assume that c **is** a pure tensor of the form $x_1 \otimes x_2 \otimes ... \otimes x_m$ with homogeneous $x_1, x_2, ..., x_m \in \mathfrak{g}$ satisfying $\deg(x_1) + \deg(x_2) + ... + \deg(x_m) = n$ (because, clearly, once Lemma 2.6.25 is proven for certain values of $c \in T(\mathfrak{g})[n, m]$, it must clearly also hold for all their $\mathbb{C}$-linear combinations[62]). Let us now assume this.

So we have $c = x_1 \otimes x_2 \otimes ... \otimes x_m$ with homogeneous $x_1, x_2, ..., x_m \in \mathfrak{g}$ satisfying $\deg(x_1) + \deg(x_2) + ... + \deg(x_m) = n$. We must now prove that c is n-stratifiable.

For every $i \in \{1, 2, ..., m\}$, let n_i be the degree of x_i (this is well-defined since x_i is homogeneous). Thus, $x_i \in \mathfrak{g}_{n_i}$.

We have

$$\deg(x_2) + \deg(x_3) + ... + \deg(x_m) = \underbrace{(\deg(x_1) + \deg(x_2) + ... + \deg(x_m))}_{=n} - \underbrace{\deg(x_1)}_{=n_1} = n - n_1,$$

so that $x_2 \otimes x_3 \otimes ... \otimes x_m \in T(\mathfrak{g})[n - n_1]$ and thus $x_2 \otimes x_3 \otimes ... \otimes x_m \in T(\mathfrak{g})[n - n_1, m - 1]$. Since we have assumed that (41) holds for $m - 1$ instead of m, we can thus apply (41) to $n - n_1$, $m - 1$ and $x_2 \otimes x_3 \otimes ... \otimes x_m$ instead of n, m and c. We conclude that $x_2 \otimes x_3 \otimes ... \otimes x_m$ is $(n - n_1)$-stratifiable. In other words, there exists a polynomial map $\widetilde{d} : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[n - n_1]$ such that every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$\left(\operatorname{env}_{\mathfrak{g}^\varepsilon}(x_2 \otimes x_3 \otimes ... \otimes x_m)\right) v_\lambda^{+\mathfrak{g}^\varepsilon} = \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d}(\lambda, \varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}.$$

[62] due to Observation 1

We notice that $c = x_1 \otimes x_2 \otimes ... \otimes x_m$, so that

$$\begin{aligned}
&\operatorname{env}_{\mathfrak{g}^\varepsilon} c \\
&= x_1 x_2 ... x_m \\
&= \sum_{i=1}^{m-1} \underbrace{\left(x_2 x_3 ... x_{i-1} x_i \cdot x_1 \cdot x_{i+1} x_{i+2} ... x_m - x_2 x_3 ... x_i x_{i+1} \cdot x_1 \cdot x_{i+2} x_{i+3} ... x_m\right)}_{= x_2 x_3 ... x_{i-1} x_i (x_1 x_{i+1} - x_{i+1} x_1) x_{i+2} x_{i+3} ... x_m} + x_2 x_3 ... x_m \cdot x_1 \\
&\qquad \left(\begin{array}{c} \text{since the sum } \sum\limits_{i=1}^{m-1} \left(x_2 x_3 ... x_{i-1} x_i \cdot x_1 \cdot x_{i+1} x_{i+2} ... x_m - x_2 x_3 ... x_i x_{i+1} \cdot x_1 \cdot x_{i+2} x_{i+3} ... x_m\right) \\ \text{telescopes to } x_1 x_2 ... x_m - x_2 x_3 ... x_m \cdot x_1 \end{array} \right) \\
&= \sum_{i=1}^{m-1} x_2 x_3 ... x_{i-1} x_i \underbrace{\left(x_1 x_{i+1} - x_{i+1} x_1\right)}_{\substack{= [x_1, x_{i+1}]^\varepsilon \\ \text{(since we are in } U(\mathfrak{g}^\varepsilon))}} x_{i+2} x_{i+3} ... x_m + x_2 x_3 ... x_m \cdot x_1 \\
&= \sum_{i=1}^{m-1} x_2 x_3 ... x_{i-1} x_i \underbrace{\left[x_1, x_{i+1}\right]^\varepsilon}_{\substack{= \varepsilon^{\delta_{n_1,0} + \delta_{n_{i+1},0} + 1 - \delta_{n_1 + n_{i+1},0}} [x_1, x_{i+1}] \\ \text{(by (40) (applied to } x_1 \text{ and } x_{i+1} \text{ instead of } x \text{ and } y), \\ \text{since } x_1 \in \mathfrak{g}_{n_1} \text{ and } x_{i+1} \in \mathfrak{g}_{n_{i+1}})}} x_{i+2} x_{i+3} ... x_m + x_2 x_3 ... x_m \cdot x_1 \\
&= \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0} + \delta_{n_{i+1},0} + 1 - \delta_{n_1 + n_{i+1},0}} \underbrace{x_2 x_3 ... x_{i-1} x_i \left[x_1, x_{i+1}\right] x_{i+2} x_{i+3} ... x_m}_{= \operatorname{env}_{\mathfrak{g}^\varepsilon} (x_2 \otimes x_3 \otimes ... \otimes x_{i-1} \otimes x_i \otimes [x_1, x_{i+1}] \otimes x_{i+2} \otimes x_{i+3} \otimes ... \otimes x_m)} \\
&\qquad + \underbrace{x_2 x_3 ... x_m}_{= \operatorname{env}_{\mathfrak{g}^\varepsilon} (x_2 \otimes x_3 \otimes ... \otimes x_m)} \cdot x_1 \\
&= \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0} + \delta_{n_{i+1},0} + 1 - \delta_{n_1 + n_{i+1},0}} \operatorname{env}_{\mathfrak{g}^\varepsilon} \left(x_2 \otimes x_3 \otimes ... \otimes x_{i-1} \otimes x_i \otimes \left[x_1, x_{i+1}\right] \otimes x_{i+2} \otimes x_{i+3} \otimes ... \otimes x_m\right) \\
&\qquad + \operatorname{env}_{\mathfrak{g}^\varepsilon} \left(x_2 \otimes x_3 \otimes ... \otimes x_m\right) \cdot x_1. \qquad (42)
\end{aligned}$$

Now, for every $i \in \{1, 2, ..., m-1\}$, denote the element $x_2 \otimes x_3 \otimes ... \otimes x_{i-1} \otimes x_i \otimes [x_1, x_{i+1}] \otimes x_{i+2} \otimes x_{i+3} \otimes ... \otimes x_m$ by c_i. It is easily seen that $c_i \in T(\mathfrak{g})[n, m-1]$. Since $c_i = x_2 \otimes x_3 \otimes ... \otimes x_{i-1} \otimes x_i \otimes [x_1, x_{i+1}] \otimes x_{i+2} \otimes x_{i+3} \otimes ... \otimes x_m$, the equality (42) rewrites as

$$\begin{aligned}
&\operatorname{env}_{\mathfrak{g}^\varepsilon} c \\
&= \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0} + \delta_{n_{i+1},0} + 1 - \delta_{n_1 + n_{i+1},0}} \operatorname{env}_{\mathfrak{g}^\varepsilon} (c_i) + \operatorname{env}_{\mathfrak{g}^\varepsilon} \left(x_2 \otimes x_3 \otimes ... \otimes x_m\right) \cdot x_1. \qquad (43)
\end{aligned}$$

For every $i \in \{1, 2, ..., m-1\}$, we can apply (41) to $m-1$ and c_i instead of m and c (since $c_i \in T(\mathfrak{g})[n, m-1]$, and since we have assumed that (41) holds for $m-1$ instead of m). We conclude that c_i is n-stratifiable for every $i \in \{1, 2, ..., m-1\}$. In other words, for every $i \in \{1, 2, ..., m-1\}$, there exists a polynomial map $\widetilde{d_i} : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[n]$ such that every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$\left(\operatorname{env}_{\mathfrak{g}^\varepsilon} (c_i)\right) v_\lambda^{+\mathfrak{g}^\varepsilon} = \left(\operatorname{env}_{\mathfrak{g}^\varepsilon} \left(\widetilde{d_i}(\lambda, \varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}.$$

We now distinguish between three cases:

Case 1: We have $n_1 > 0$.

Case 2: We have $n_1 = 0$.

Case 3: We have $n_1 < 0$.

First, let us consider Case 1. In this case, $n_1 > 0$. Thus, $x_1 \in \mathfrak{n}_+$ (since $x_1 \in \mathfrak{g}_{n_1}$), so that $x_1 v_\lambda^{+\mathfrak{g}^\varepsilon} \in \mathfrak{n}_+^\varepsilon v_\lambda^{+\mathfrak{g}^\varepsilon} = 0$ and thus $x_1 v_\lambda^{+\mathfrak{g}^\varepsilon} = 0$. Now, (43) yields

$$\begin{aligned}
&\left(\operatorname{env}_{\mathfrak{g}^\varepsilon} c\right) v_\lambda^{+\mathfrak{g}^\varepsilon} \\
&= \left(\sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \operatorname{env}_{\mathfrak{g}^\varepsilon}\left(c_i\right) + \operatorname{env}_{\mathfrak{g}^\varepsilon}\left(x_2 \otimes x_3 \otimes \dots \otimes x_m\right) \cdot x_1 \right) v_\lambda^{+\mathfrak{g}^\varepsilon} \\
&= \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \underbrace{\left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(c_i\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}}_{=\left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d_i}(\lambda,\varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}} + \operatorname{env}_{\mathfrak{g}^\varepsilon}\left(x_2 \otimes x_3 \otimes \dots \otimes x_m\right) \cdot \underbrace{x_1 v_\lambda^{+\mathfrak{g}^\varepsilon}}_{=0} \\
&= \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d_i}\left(\lambda,\varepsilon\right)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon} \\
&= \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \widetilde{d_i}\left(\lambda,\varepsilon\right)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}.
\end{aligned} \tag{44}$$

If we define a map $d : \mathfrak{h}^* \times \mathbb{C} \to T\left(\mathfrak{n}_-\right)[n]$ by

$$d\left(\lambda,\varepsilon\right) = \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \widetilde{d_i}\left(\lambda,\varepsilon\right) \qquad \text{for every } \left(\lambda,\varepsilon\right) \in \mathfrak{h}^* \times \mathbb{C},$$

then this map d is polynomial (since $\widetilde{d_i}$ are polynomial maps for all i), and (44) becomes

$$\begin{aligned}
&\left(\operatorname{env}_{\mathfrak{g}^\varepsilon} c\right) v_\lambda^{+\mathfrak{g}^\varepsilon} \\
&= \left(\operatorname{env}_{\mathfrak{g}^\varepsilon} \underbrace{\left(\sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \widetilde{d_i}\left(\lambda,\varepsilon\right)\right)}_{=d(\lambda,\varepsilon)}\right) v_\lambda^{+\mathfrak{g}^\varepsilon} = \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(d\left(\lambda,\varepsilon\right)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}.
\end{aligned}$$

Hence, c is n-stratifiable (by the definition of "n-stratifiable").

Next, let us consider Case 2. In this case, $n_1 = 0$. Thus, $x_1 \in \mathfrak{h}$ (since $x_1 \in \mathfrak{g}_{n_1}$), so that $x_1 v_\lambda^{+\mathfrak{g}^\varepsilon} = \lambda(x_1) v_\lambda^{+\mathfrak{g}^\varepsilon}$. Now, (43) yields

$$\begin{aligned}
&\left(\operatorname{env}_{\mathfrak{g}^\varepsilon} c\right) v_\lambda^{+\mathfrak{g}^\varepsilon} \\
&= \left(\sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \operatorname{env}_{\mathfrak{g}^\varepsilon}(c_i) + \operatorname{env}_{\mathfrak{g}^\varepsilon}(x_2 \otimes x_3 \otimes \ldots \otimes x_m) \cdot x_1\right) v_\lambda^{+\mathfrak{g}^\varepsilon} \\
&= \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \underbrace{\left(\operatorname{env}_{\mathfrak{g}^\varepsilon}(c_i)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}}_{=\left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d_i}(\lambda,\varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}} + \operatorname{env}_{\mathfrak{g}^\varepsilon}(x_2 \otimes x_3 \otimes \ldots \otimes x_m) \cdot \underbrace{x_1 v_\lambda^{+\mathfrak{g}^\varepsilon}}_{=\lambda(x_1) v_\lambda^{+\mathfrak{g}^\varepsilon}} \\
&= \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d_i}(\lambda,\varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon} \\
&\qquad + \lambda(x_1) \underbrace{\left(\operatorname{env}_{\mathfrak{g}^\varepsilon}(x_2 \otimes x_3 \otimes \ldots \otimes x_m)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}}_{=\left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d}(\lambda,\varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}} \\
&= \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d_i}(\lambda,\varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon} + \lambda(x_1) \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\widetilde{d}(\lambda,\varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon} \\
&= \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}\left(\sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \widetilde{d_i}(\lambda,\varepsilon) + \lambda(x_1) \widetilde{d}(\lambda,\varepsilon)\right)\right) v_\lambda^{+\mathfrak{g}^\varepsilon}. \qquad (45)
\end{aligned}$$

If we define a map $d : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[n]$ by

$$d(\lambda,\varepsilon) = \sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \widetilde{d_i}(\lambda,\varepsilon) + \lambda(x_1) \widetilde{d}(\lambda,\varepsilon) \qquad \text{for every } (\lambda,\varepsilon) \in \mathfrak{h}^* \times \mathbb{C}$$

(this map is well-defined, since $\widetilde{d}(\lambda,\varepsilon) \in T(\mathfrak{n}_-)[n-n_1] = T(\mathfrak{n}_-)[n]$ (due to $n_1 = 0$)), then this map d is polynomial (since $\widetilde{d_i}$ are polynomial maps for all i, and since $\widetilde{d}$ is polynomial), and (45) becomes

$$\begin{aligned}
&\left(\operatorname{env}_{\mathfrak{g}^\varepsilon} c\right) v_\lambda^{+\mathfrak{g}^\varepsilon} \\
&= \left(\operatorname{env}_{\mathfrak{g}^\varepsilon} \underbrace{\left(\sum_{i=1}^{m-1} \varepsilon^{\delta_{n_1,0}+\delta_{n_{i+1},0}+1-\delta_{n_1+n_{i+1},0}} \widetilde{d_i}(\lambda,\varepsilon) + \lambda(x_1) \widetilde{d}(\lambda,\varepsilon)\right)}_{=d(\lambda,\varepsilon)}\right) v_\lambda^{+\mathfrak{g}^\varepsilon} = \left(\operatorname{env}_{\mathfrak{g}^\varepsilon}(d(\lambda,\varepsilon))\right) v_\lambda^{+\mathfrak{g}^\varepsilon}.
\end{aligned}$$

Hence, c is n-stratifiable (by the definition of "n-stratifiable").

Now, let us consider Case 3. In this case, $n_1 < 0$. Thus, we can apply Observation 2 to x_1, $x_2 \otimes x_3 \otimes \ldots \otimes x_m$ and n_1 instead of x, y and ν, and conclude that $x_1 \otimes (x_2 \otimes x_3 \otimes \ldots \otimes x_m)$ is n-stratifiable (since $x_2 \otimes x_3 \otimes \ldots \otimes x_m$ is $(n-n_1)$-stratifiable). Since $x_1 \otimes (x_2 \otimes x_3 \otimes \ldots \otimes x_m) = x_1 \otimes x_2 \otimes \ldots \otimes x_m = c$, this shows that c is n-stratifiable.

Hence, in each of the cases 1, 2 and 3, we have shown that c is n-stratifiable. Thus, c is always n-stratifiable.

Forget that we fixed c. We thus have shown that c is n-stratifiable for every tensor $c \in T(\mathfrak{g})[n,m]$. In other words, we have proven (41) for our m. This completes the induction step.

Thus, (41) is proven by induction.

Now, let $n \in \mathbb{Z}$. Then, every $c \in T(\mathfrak{g})[n]$ is a $\mathbb{C}$-linear combination of elements of $T(\mathfrak{g})[n,m]$ for varying $m \in \mathbb{N}$ (since $T(\mathfrak{g})[n] = \bigoplus_{m\in\mathbb{N}} T(\mathfrak{g})[n,m]$), and thus every $c \in T(\mathfrak{g})[n]$ is n-stratifiable (since (41) shows that every element of $T(\mathfrak{g})[n,m]$ is n-stratifiable, and due to Observation 1).

Now forget that we fixed n. We have thus proven that for every $n \in \mathbb{Z}$, every $c \in T(\mathfrak{g})[n]$ is n-stratifiable. In other words, we have proved Lemma 2.6.25.

Proof of Lemma 2.6.24. We have $e_{\mathbf{i}}^{\varepsilon} \in U(\mathfrak{g}^{\varepsilon})[\deg \mathbf{i}]$ and thus $e_{\mathbf{i}}^{\varepsilon} v_{\lambda}^{+\mathfrak{g}^{\varepsilon}} \in M_{\lambda}^{+\mathfrak{g}^{\varepsilon}}[\deg \mathbf{i}]$. Similarly, $e_{\mathbf{j}}^{\varepsilon} v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}} \in M_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}[\deg \mathbf{j}]$. Hence, if $\deg \mathbf{i} + \deg \mathbf{j} \neq 0$, then $\left(e_{\mathbf{i}}^{\varepsilon} v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}, e_{\mathbf{j}}^{\varepsilon} v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}} \in \left(M_{\lambda}^{+\mathfrak{g}^{\varepsilon}}[\deg \mathbf{i}], M_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}[\deg \mathbf{j}]\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}} = 0$ (because the form $(\cdot,\cdot)_{\lambda}^{\mathfrak{g}^{\varepsilon}}$ is of degree 0, while $\deg \mathbf{i} + \deg \mathbf{j} \neq 0$) and thus $\left(e_{\mathbf{i}}^{\varepsilon} v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}, e_{\mathbf{j}}^{\varepsilon} v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}} = 0$. Thus, if $\deg \mathbf{i} + \deg \mathbf{j} \neq 0$, then Lemma 2.6.24 trivially holds (because we can then just take $Q_{\mathbf{i},\mathbf{j}} = 0$). Thus, for the rest of the proof of Lemma 2.6.24, we can WLOG assume that we *don't* have $\deg \mathbf{i} + \deg \mathbf{j} \neq 0$. Hence, we have $\deg \mathbf{i} + \deg \mathbf{j} = 0$.

Write the sequence $\mathbf{j}$ in the form $((m_1, j_1), (m_2, j_2), ..., (m_k, j_k))$. Then, $e_{\mathbf{j}}^{\varepsilon} = e_{m_1,j_1} e_{m_2,j_2} ... e_{m_k,j_k}$ and $\deg \mathbf{j} = m_1 + m_2 + ... + m_k = m_k + m_{k-1} + ... + m_1$.

Since $e_{\mathbf{j}}^{\varepsilon} = e_{m_1,j_1} e_{m_2,j_2} ... e_{m_k,j_k}$, we have

$$\begin{aligned}
&\left(e_{\mathbf{i}}^{\varepsilon} v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}, e_{\mathbf{j}}^{\varepsilon} v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}} \\
&= \left(e_{\mathbf{i}}^{\varepsilon} v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}, e_{m_1,j_1} e_{m_2,j_2} ... e_{m_k,j_k} v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}} = (-1)^k \left(e_{m_k,j_k} e_{m_{k-1},j_{k-1}} ... e_{m_1,j_1} \cdot e_{\mathbf{i}}^{\varepsilon} v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}, v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}} \\
&\qquad \left(\text{here, we applied the } \mathfrak{g}^{\varepsilon}\text{-invariance of the form } (\cdot,\cdot)_{\lambda}^{\mathfrak{g}^{\varepsilon}} \text{ for a total of } k \text{ times}\right) \\
&= \left((-1)^k e_{m_k,j_k} e_{m_{k-1},j_{k-1}} ... e_{m_1,j_1} \cdot e_{\mathbf{i}}^{\varepsilon} v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}, v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}}.
\end{aligned} \tag{46}$$

Write the sequence $\mathbf{i}$ in the form $((n_1, i_1), (n_2, i_2), ..., (n_\ell, i_\ell))$. Then, $e_{\mathbf{i}}^{\varepsilon} = e_{n_1,i_1} e_{n_2,i_2} ... e_{n_\ell,i_\ell}$ and $\deg \mathbf{i} = n_1 + n_2 + ... + n_\ell$. Now,

$$\begin{aligned}
&(-1)^k e_{m_k,j_k} e_{m_{k-1},j_{k-1}} ... e_{m_1,j_1} \cdot \underbrace{e_{\mathbf{i}}^{\varepsilon}}_{= e_{n_1,i_1} e_{n_2,i_2} ... e_{n_\ell,i_\ell}} \\
&= (-1)^k e_{m_k,j_k} e_{m_{k-1},j_{k-1}} ... e_{m_1,j_1} \cdot e_{n_1,i_1} e_{n_2,i_2} ... e_{n_\ell,i_\ell} \\
&= \operatorname{env}_{\mathfrak{g}^{\varepsilon}} \left((-1)^k e_{m_k,j_k} \otimes e_{m_{k-1},j_{k-1}} \otimes ... \otimes e_{m_1,j_1} \otimes e_{n_1,i_1} \otimes e_{n_2,i_2} \otimes ... \otimes e_{n_\ell,i_\ell}\right).
\end{aligned} \tag{47}$$

Denote the tensor $(-1)^k e_{m_k,j_k} \otimes e_{m_{k-1},j_{k-1}} \otimes ... \otimes e_{m_1,j_1} \otimes e_{n_1,i_1} \otimes e_{n_2,i_2} \otimes ... \otimes e_{n_\ell,i_\ell}$ by c. Then, (47) rewrites as

$$(-1)^k e_{m_k,j_k} e_{m_{k-1},j_{k-1}} ... e_{m_1,j_1} \cdot e_{\mathbf{i}}^{\varepsilon} = \operatorname{env}_{\mathfrak{g}^{\varepsilon}} c. \tag{48}$$

Since

$$\begin{aligned} c &= (-1)^k e_{m_k,j_k} \otimes e_{m_{k-1},j_{k-1}} \otimes ... \otimes e_{m_1,j_1} \otimes e_{n_1,i_1} \otimes e_{n_2,i_2} \otimes ... \otimes e_{n_\ell,i_\ell} \\ &\in T(\mathfrak{g})[m_k + m_{k-1} + ... + m_1 + n_1 + n_2 + ... + n_\ell] \\ &\qquad \left(\begin{array}{c} \text{since } e_{m_k,j_k} \in \mathfrak{g}_{m_k},\ e_{m_{k-1},j_{k-1}} \in \mathfrak{g}_{m_{k-1}},\ ...,\ e_{m_1,j_1} \in \mathfrak{g}_{m_1} \\ \text{and } e_{n_1,i_1} \in \mathfrak{g}_{n_1},\ e_{n_2,i_2} \in \mathfrak{g}_{n_2},\ ...,\ e_{n_\ell,i_\ell} \in \mathfrak{g}_{n_\ell} \end{array}\right) \\ &= T(\mathfrak{g})[0] \\ &\qquad \left(\text{since } \underbrace{m_k + m_{k-1} + ... + m_1}_{=\deg \mathbf{j}} + \underbrace{n_1 + n_2 + ... + n_\ell}_{=\deg \mathbf{i}} = \deg \mathbf{j} + \deg \mathbf{i} = \deg \mathbf{i} + \deg \mathbf{j} = 0\right), \end{aligned}$$

we can apply Lemma 2.6.25 to $n = 0$. We conclude that there exists a polynomial map $d : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[0]$ such that every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$(\operatorname{env}_{\mathfrak{g}^\varepsilon} c)\, v_\lambda^{+\mathfrak{g}^\varepsilon} = (\operatorname{env}_{\mathfrak{g}^\varepsilon}(d(\lambda,\varepsilon)))\, v_\lambda^{+\mathfrak{g}^\varepsilon}. \tag{49}$$

Since $T(\mathfrak{n}_-)[0] = \mathbb{C}$ (because $\mathfrak{n}_-$ is concentrated in negative degrees), this polynomial map $d : \mathfrak{h}^* \times \mathbb{C} \to T(\mathfrak{n}_-)[0]$ is a polynomial function $d : \mathfrak{h}^* \times \mathbb{C} \to \mathbb{C}$. Denote this function d by $Q_{\mathbf{i},\mathbf{j}}$. Then, every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy $d(\lambda,\varepsilon) = Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon)$ and thus $\operatorname{env}_{\mathfrak{g}^\varepsilon}(d(\lambda,\varepsilon)) = \operatorname{env}_{\mathfrak{g}^\varepsilon}(Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon)) = Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon)$ (since $Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon) \in \mathbb{C}$). Thus, every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$\begin{aligned} (\operatorname{env}_{\mathfrak{g}^\varepsilon} c)\, v_\lambda^{+\mathfrak{g}^\varepsilon} &= \underbrace{(\operatorname{env}_{\mathfrak{g}^\varepsilon}(d(\lambda,\varepsilon)))}_{=Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon)} v_\lambda^{+\mathfrak{g}^\varepsilon} \qquad (\text{by } (49)) \\ &= Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon) \cdot v_\lambda^{+\mathfrak{g}^\varepsilon}. \end{aligned} \tag{50}$$

Now, every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$\begin{aligned} \left(e_{\mathbf{i}}^\varepsilon v_\lambda^{+\mathfrak{g}^\varepsilon}, e_{\mathbf{j}}^\varepsilon v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_\lambda^{\mathfrak{g}^\varepsilon} &= \left(\underbrace{(-1)^k e_{m_k,j_k} e_{m_{k-1},j_{k-1}} ... e_{m_1,j_1} \cdot e_{\mathbf{i}}^\varepsilon}_{\substack{=\operatorname{env}_{\mathfrak{g}^\varepsilon} c \\ (\text{by } (48))}} v_\lambda^{+\mathfrak{g}^\varepsilon}, v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_\lambda^{\mathfrak{g}^\varepsilon} \qquad (\text{by } (46)) \\ &= \left(\underbrace{(\operatorname{env}_{\mathfrak{g}^\varepsilon} c)\, v_\lambda^{+\mathfrak{g}^\varepsilon}}_{\substack{=Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon) \cdot v_\lambda^{+\mathfrak{g}^\varepsilon} \\ (\text{by } (50))}}, v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_\lambda^{\mathfrak{g}^\varepsilon} = \left(Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon) \cdot v_\lambda^{+\mathfrak{g}^\varepsilon}, v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_\lambda^{\mathfrak{g}^\varepsilon} \\ &= Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon) \cdot \underbrace{\left(v_\lambda^{+\mathfrak{g}^\varepsilon}, v_{-\lambda}^{-\mathfrak{g}^\varepsilon}\right)_\lambda^{\mathfrak{g}^\varepsilon}}_{=1} = Q_{\mathbf{i},\mathbf{j}}(\lambda,\varepsilon). \end{aligned}$$

This proves Lemma 2.6.24.

We shall now take a closer look at the polynomial function $Q_{\mathbf{i},\mathbf{j}}$ of Lemma 2.6.24:

Lemma 2.6.26. Let $\mathbf{i} \in \operatorname{Seq}_- E$ and $\mathbf{j} \in \operatorname{Seq}_+ E$. Consider the polynomial function $Q_{\mathbf{i},\mathbf{j}} : \mathfrak{h}^* \times \mathbb{C} \to \mathbb{C}$ of Lemma 2.6.24. Then, every $\lambda \in \mathfrak{h}^*$ and every nonzero $\varepsilon \in \mathbb{C}$ satisfy

$$Q_{\mathbf{i},\mathbf{j}}(\lambda, \varepsilon) = \varepsilon^{\operatorname{len}\mathbf{i} + \operatorname{len}\mathbf{j}} Q_{\mathbf{i},\mathbf{j}}\left(\lambda/\varepsilon^2, 1\right).$$

Note that Lemma 2.6.26 does not really need the conditions $\mathbf{i} \in \operatorname{Seq}_- E$ and $\mathbf{j} \in \operatorname{Seq}_+ E$. It is sufficient that $\mathbf{i} \in \operatorname{Seq} E$ is such that no element (n, i) of the sequence $\mathbf{i}$ satisfies $n = 0$, and that a similar condition holds for $\mathbf{j}$. But since we will only use Lemma 2.6.26 in the case when $\mathbf{i} \in \operatorname{Seq}_- E$ and $\mathbf{j} \in \operatorname{Seq}_+ E$, we would not gain much from thus generalizing it.

Proof of Lemma 2.6.26. We recall that the definition of $Q_{\mathbf{i},\mathbf{j}}$ said that

$$\left(e_{\mathbf{i}}^{\varepsilon} v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}, e_{\mathbf{j}}^{\varepsilon} v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}} = Q_{\mathbf{i},\mathbf{j}}(\lambda, \varepsilon) \qquad \text{for all } \lambda \in \mathfrak{h}^* \text{ and } \varepsilon \in \mathbb{C}. \tag{51}$$

Let $\lambda \in \mathfrak{h}^*$ be arbitrary, and let $\varepsilon \in \mathbb{C}$ be nonzero. Since $\varepsilon \neq 0$, the Lie algebra isomorphism $J_\varepsilon : \mathfrak{g}^\varepsilon \to \mathfrak{g}$ exists and satisfies $(\lambda/\varepsilon^2) \circ J_\varepsilon = \lambda$. Hence, we have an isomorphism $J_\varepsilon : (\mathfrak{g}^\varepsilon, \lambda) \to (\mathfrak{g}, \lambda/\varepsilon^2)$ in the category of pairs of a $\mathbb{Z}$-graded Lie algebra and a linear form on its 0-th homogeneous component (where the morphisms in this category are defined in the obvious way). This isomorphism induces a corresponding isomorphism $M_{\lambda}^{+\mathfrak{g}^{\varepsilon}} \to M_{\lambda/\varepsilon^2}^{+\mathfrak{g}}$ of Verma modules which sends $x v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}$ to $(U(J_\varepsilon))(x) v_{\lambda/\varepsilon^2}^{+\mathfrak{g}}$ for every $x \in U(\mathfrak{g}^\varepsilon)$ (where $U(J_\varepsilon)$ is the isomorphism $U(\mathfrak{g}^\varepsilon) \to U(\mathfrak{g})$ canonically induced by the Lie algebra isomorphism $J_\varepsilon : \mathfrak{g}^\varepsilon \to \mathfrak{g}$). Similarly, we get an isomorphism $M_{-\lambda}^{-\mathfrak{g}^{\varepsilon}} \to M_{-\lambda/\varepsilon^2}^{-\mathfrak{g}}$ of Verma modules which sends $y v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}$ to $(U(J_\varepsilon))(y) v_{-\lambda/\varepsilon^2}^{-\mathfrak{g}}$ for every $y \in U(\mathfrak{g}^\varepsilon)$. Since the bilinear form $(\cdot, \cdot)_{\mu}^{\mathfrak{c}}$ depends functorially on a $\mathbb{Z}$-graded Lie algebra $\mathfrak{c}$ and a linear form $\mu : \mathfrak{c}_0 \to \mathbb{C}$, these isomorphisms leave the bilinear form unchanged, i. e., we have

$$\left((U(J_\varepsilon))(x) v_{\lambda/\varepsilon^2}^{+\mathfrak{g}}, (U(J_\varepsilon))(y) v_{-\lambda/\varepsilon^2}^{-\mathfrak{g}}\right)_{\lambda/\varepsilon^2}^{\mathfrak{g}} = \left(x v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}, y v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}}$$

for every $x \in U(\mathfrak{g}^\varepsilon)$ and $y \in U(\mathfrak{g}^\varepsilon)$. Applied to $x = e_{\mathbf{i}}^{\varepsilon}$ and $y = e_{\mathbf{j}}^{\varepsilon}$, this yields

$$\left((U(J_\varepsilon))\left(e_{\mathbf{i}}^{\varepsilon}\right) v_{\lambda/\varepsilon^2}^{+\mathfrak{g}}, (U(J_\varepsilon))\left(e_{\mathbf{j}}^{\varepsilon}\right) v_{-\lambda/\varepsilon^2}^{-\mathfrak{g}}\right)_{\lambda/\varepsilon^2}^{\mathfrak{g}} = \left(e_{\mathbf{i}}^{\varepsilon} v_{\lambda}^{+\mathfrak{g}^{\varepsilon}}, e_{\mathbf{j}}^{\varepsilon} v_{-\lambda}^{-\mathfrak{g}^{\varepsilon}}\right)_{\lambda}^{\mathfrak{g}^{\varepsilon}} = Q_{\mathbf{i},\mathbf{j}}(\lambda, \varepsilon) \tag{52}$$

(by the definition of $Q_{\mathbf{i},\mathbf{j}}$).

But we have $(U(J_\varepsilon))(e_{\mathbf{i}}^{\varepsilon}) = \varepsilon^{\operatorname{len}\mathbf{i}} e_{\mathbf{i}}^{1}$ [63] and similarly $(U(J_\varepsilon))\left(e_{\mathbf{j}}^{\varepsilon}\right) = \varepsilon^{\operatorname{len}\mathbf{j}} e_{\mathbf{j}}^{1}$. Hence,

$$\begin{aligned}
&\left(\left(U(J_\varepsilon)\right)(e_{\mathbf{i}}^{\varepsilon}) v_{\lambda/\varepsilon^2}^{+\mathfrak{g}}, \left(U(J_\varepsilon)\right)\left(e_{\mathbf{j}}^{\varepsilon}\right) v_{-\lambda/\varepsilon^2}^{-\mathfrak{g}}\right)_{\lambda/\varepsilon^2}^{\mathfrak{g}} \\
&= \left(\varepsilon^{\operatorname{len}\mathbf{i}} e_{\mathbf{i}}^{1} v_{\lambda/\varepsilon^2}^{+\mathfrak{g}}, \varepsilon^{\operatorname{len}\mathbf{j}} e_{\mathbf{j}}^{1} v_{-\lambda/\varepsilon^2}^{-\mathfrak{g}}\right)_{\lambda/\varepsilon^2}^{\mathfrak{g}} = \varepsilon^{\operatorname{len}\mathbf{i}+\operatorname{len}\mathbf{j}} \left(e_{\mathbf{i}}^{1} v_{\lambda/\varepsilon^2}^{+\mathfrak{g}}, e_{\mathbf{j}}^{1} v_{-\lambda/\varepsilon^2}^{-\mathfrak{g}}\right)_{\lambda/\varepsilon^2}^{\mathfrak{g}} \\
&= \varepsilon^{\operatorname{len}\mathbf{i}+\operatorname{len}\mathbf{j}} \underbrace{\left(e_{\mathbf{i}}^{1} v_{\lambda/\varepsilon^2}^{+\mathfrak{g}^1}, e_{\mathbf{j}}^{1} v_{-\lambda/\varepsilon^2}^{-\mathfrak{g}^1}\right)_{\lambda/\varepsilon^2}^{\mathfrak{g}^1}}_{\substack{=Q_{\mathbf{i},\mathbf{j}}\left(\lambda/\varepsilon^2, 1\right) \\ \text{(by (51), applied to } \lambda/\varepsilon^1 \text{ and } 1 \text{ instead of } \lambda \text{ and } \varepsilon)}} \qquad \left(\text{since } \mathfrak{g} = \mathfrak{g}^1\right) \\
&= \varepsilon^{\operatorname{len}\mathbf{i}+\operatorname{len}\mathbf{j}} Q_{\mathbf{i},\mathbf{j}}\left(\lambda/\varepsilon^2, 1\right).
\end{aligned}$$

Compared to (52), this yields $Q_{\mathbf{i},\mathbf{j}}(\lambda, \varepsilon) = \varepsilon^{\operatorname{len}\mathbf{i}+\operatorname{len}\mathbf{j}} Q_{\mathbf{i},\mathbf{j}}(\lambda/\varepsilon^2, 1)$. This proves Lemma 2.6.26.

Here is the consequence of Lemmas 2.6.24 and 2.6.26 that we will actually use:

Corollary 2.6.27. Let $n \in \mathbb{N}$. Let $\operatorname{LEN} n = \sum\limits_{\substack{\mathbf{i} \in \operatorname{Seq}_- E; \\ \deg \mathbf{i} = -n}} \operatorname{len}\mathbf{i} = \sum\limits_{\substack{\mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{j} = n}} \operatorname{len}\mathbf{j}$ (we are using the fact that $\sum\limits_{\substack{\mathbf{i} \in \operatorname{Seq}_- E; \\ \deg \mathbf{i} = -n}} \operatorname{len}\mathbf{i} = \sum\limits_{\substack{\mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{j} = n}} \operatorname{len}\mathbf{j}$, which we proved above).

Then, there exists a polynomial function $Q_n : \mathfrak{h}^* \times \mathbb{C} \to \mathbb{C}$ such that every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$\det\left((\cdot, \cdot)_{\lambda,n}^{\mathfrak{g}^\varepsilon}\right) = Q_n(\lambda, \varepsilon). \tag{53}$$

This function Q_n satisfies

$$Q_n(\lambda, \varepsilon) = \varepsilon^{2\operatorname{LEN} n} Q_n\left(\lambda/\varepsilon^2, 1\right) \qquad \text{for every } \lambda \in \mathfrak{h}^* \text{ and every nonzero } \varepsilon \in \mathbb{C}.$$

[63] *Proof.* Write the sequence $\mathbf{i}$ in the form $((n_1, i_1), (n_2, i_2), ..., (n_\ell, i_\ell))$. Since $\mathbf{i} \in \operatorname{Seq}_- E$, all of the numbers n_1, n_2, ..., n_ℓ are negative, so that none of them is 0. As a consequence, $\delta_{n_u,0} = 0$ for every $u \in \{1, 2, ..., \ell\}$. By the definition of J_ε, we have

$$\begin{aligned}
J_\varepsilon(e_{n_u,i_u}) &= \underbrace{\varepsilon^{1+\delta_{n_u,0}}}_{\substack{=\varepsilon \\ (\text{since } \delta_{n_u,0}=0)}} e_{n_u,i_u} \qquad (\text{since } e_{n_u,i_u} \in \mathfrak{g}_{n_u}) \\
&= \varepsilon e_{n_u,i_u}
\end{aligned}$$

for every $u \in \{1, 2, ..., \ell\}$.

Now, $e_{\mathbf{i}}^{\varepsilon}$ is defined as the product $e_{n_1,i_1} e_{n_2,i_2} ... e_{n_\ell,i_\ell}$ in $U(\mathfrak{g}^\varepsilon)$, and $e_{\mathbf{i}}^{1}$ is defined as the product $e_{n_1,i_1} e_{n_2,i_2} ... e_{n_\ell,i_\ell}$ in $U\left(\mathfrak{g}^1\right)$. Hence,

$$\begin{aligned}
(U(J_\varepsilon))(e_{\mathbf{i}}^{\varepsilon}) &= (U(J_\varepsilon))(e_{n_1,i_1} e_{n_2,i_2} ... e_{n_\ell,i_\ell}) \qquad (\text{since } e_{\mathbf{i}}^{\varepsilon} = e_{n_1,i_1} e_{n_2,i_2} ... e_{n_\ell,i_\ell}) \\
&= J_\varepsilon(e_{n_1,i_1}) J_\varepsilon(e_{n_2,i_2}) ... J_\varepsilon(e_{n_\ell,i_\ell}) \\
&= \varepsilon e_{n_1,i_1} \cdot \varepsilon e_{n_2,i_2} \cdot ... \cdot \varepsilon e_{n_\ell,i_\ell} \qquad (\text{since } J_\varepsilon(e_{n_u,i_u}) = \varepsilon e_{n_u,i_u} \text{ for every } u \in \{1, 2, ..., \ell\}) \\
&= \varepsilon^\ell \underbrace{e_{n_1,i_1} e_{n_2,i_2} ... e_{n_\ell,i_\ell}}_{=e_{\mathbf{i}}^{1}} = \varepsilon^\ell e_{\mathbf{i}}^{1} = \varepsilon^{\operatorname{len}\mathbf{i}} e_{\mathbf{i}}^{1} \\
&\qquad (\text{since } \ell = \operatorname{len}\mathbf{i} \text{ by the definition of } \operatorname{len}\mathbf{i}),
\end{aligned}$$

qed.

Proof of Corollary 2.6.27. For any $\mathbf{i} \in \operatorname{Seq}_- E$ satisfying $\deg \mathbf{i} = -n$, and any $\mathbf{j} \in \operatorname{Seq}_+ E$ satisfying $\deg \mathbf{j} = n$, consider the polynomial function $Q_{\mathbf{i},\mathbf{j}} : \mathfrak{h}^* \times \mathbb{C} \to \mathbb{C}$ of Lemma 2.6.24. Define a polynomial function $Q_n : \mathfrak{h}^* \times \mathbb{C} \to \mathbb{C}$ by

$$Q_n = \det \left(\left(Q_{\mathbf{i},\mathbf{j}} \right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}} \right).$$

Then, every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy

$$Q_n (\lambda, \varepsilon) = \det \left(\left(Q_{\mathbf{i},\mathbf{j}} (\lambda, \varepsilon) \right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}} \right) = \det \left(\left(\left(e^\varepsilon_{\mathbf{i}} v^{+\mathfrak{g}^\varepsilon}_\lambda , e^\varepsilon_{\mathbf{j}} v^{-\mathfrak{g}^\varepsilon}_{-\lambda} \right)^{\mathfrak{g}^\varepsilon}_{\lambda,n} \right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}} \right)$$

$$\left(\begin{array}{c} \text{since Lemma 2.6.24 yields} \\ Q_{\mathbf{i},\mathbf{j}} (\lambda, \varepsilon) = \left(e^\varepsilon_{\mathbf{i}} v^{+\mathfrak{g}^\varepsilon}_\lambda , e^\varepsilon_{\mathbf{j}} v^{-\mathfrak{g}^\varepsilon}_{-\lambda} \right)^{\mathfrak{g}^\varepsilon}_{\lambda} = \left(e^\varepsilon_{\mathbf{i}} v^{+\mathfrak{g}^\varepsilon}_\lambda , e^\varepsilon_{\mathbf{j}} v^{-\mathfrak{g}^\varepsilon}_{-\lambda} \right)^{\mathfrak{g}^\varepsilon}_{\lambda,n} \\ \text{(since } \deg \mathbf{i} = -n \text{ yields } e^\varepsilon_{\mathbf{i}} \in U(\mathfrak{g}^\varepsilon)[-n] \text{ and thus } e^\varepsilon_{\mathbf{i}} v^{+\mathfrak{g}^\varepsilon}_\lambda \in M^{+\mathfrak{g}^\varepsilon}_\lambda [-n] \\ \text{and similarly } e^\varepsilon_{\mathbf{j}} v^{-\mathfrak{g}^\varepsilon}_{-\lambda} \in M^{-\mathfrak{g}^\varepsilon}_{-\lambda} [n] \text{)} \end{array} \right)$$

$$= \det \left((\cdot, \cdot)^{\mathfrak{g}^\varepsilon}_{\lambda,n} \right).$$

We have thus proven that every $\lambda \in \mathfrak{h}^*$ and every $\varepsilon \in \mathbb{C}$ satisfy $\det \left((\cdot, \cdot)^{\mathfrak{g}^\varepsilon}_{\lambda,n} \right) = Q_n (\lambda, \varepsilon)$.

Now, it remains to show that this function Q_n satisfies $Q_n (\lambda, \varepsilon) = \varepsilon^{2 \operatorname{LEN} n} Q_n (\lambda / \varepsilon^2, 1)$ for every $\lambda \in \mathfrak{h}^*$ and every nonzero $\varepsilon \in \mathbb{C}$. In order to do this, we let $\lambda \in \mathfrak{h}^*$ be arbitrary and $\varepsilon \in \mathbb{C}$ be nonzero. Then,

$$Q_n (\lambda, \varepsilon) = \det \left(\left(Q_{\mathbf{i},\mathbf{j}} (\lambda, \varepsilon) \right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}} \right) = \det \left(\left(\varepsilon^{\operatorname{len} \mathbf{i}} \varepsilon^{\operatorname{len} \mathbf{j}} Q_{\mathbf{i},\mathbf{j}} (\lambda / \varepsilon^2, 1) \right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}} \right) \tag{54}$$

(since Lemma 2.6.26 yields $Q_{\mathbf{i},\mathbf{j}} (\lambda, \varepsilon) = \varepsilon^{\operatorname{len} \mathbf{i} + \operatorname{len} \mathbf{j}} Q_{\mathbf{i},\mathbf{j}} (\lambda / \varepsilon^2, 1) = \varepsilon^{\operatorname{len} \mathbf{i}} \varepsilon^{\operatorname{len} \mathbf{j}} Q_{\mathbf{i},\mathbf{j}} (\lambda / \varepsilon^2, 1)$ for all $\mathbf{i} \in \operatorname{Seq}_- E$ and $\mathbf{j} \in \operatorname{Seq}_+ E$).

Now, recall that if we multiply a row of a square matrix by some scalar, then the determinant of the matrix is also multiplied by the same scalar. A similar fact holds for the columns. Thus,

$$\det \left(\left(\varepsilon^{\operatorname{len} \mathbf{i}} \varepsilon^{\operatorname{len} \mathbf{j}} Q_{\mathbf{i},\mathbf{j}} (\lambda / \varepsilon^2, 1) \right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}} \right)$$
$$= \left(\prod_{\substack{\mathbf{i} \in \operatorname{Seq}_- E; \\ \deg \mathbf{i} = -n}} \varepsilon^{\operatorname{len} \mathbf{i}} \right) \cdot \left(\prod_{\substack{\mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{j} = n}} \varepsilon^{\operatorname{len} \mathbf{j}} \right) \cdot \det \left(\left(Q_{\mathbf{i},\mathbf{j}} (\lambda / \varepsilon^2, 1) \right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}} \right)$$

(because the matrix $\left(\varepsilon^{\operatorname{len} \mathbf{i}} \varepsilon^{\operatorname{len} \mathbf{j}} Q_{\mathbf{i},\mathbf{j}} (\lambda / \varepsilon^2, 1) \right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}}$ is obtained from the matrix $\left(Q_{\mathbf{i},\mathbf{j}} (\lambda / \varepsilon^2, 1) \right)_{\substack{\mathbf{i} \in \operatorname{Seq}_- E;\ \mathbf{j} \in \operatorname{Seq}_+ E; \\ \deg \mathbf{i} = -n;\ \deg \mathbf{j} = n}}$ by multiplying every row $\mathbf{i}$ by the scalar $\varepsilon^{\operatorname{len} \mathbf{i}}$ and multiplying every column $\mathbf{j}$ by the scalar $\varepsilon^{\operatorname{len}\mathbf{j}}$). Hence, (54) becomes

$$Q_n(\lambda,\varepsilon) = \left(\prod_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\\ \deg\mathbf{i}=-n}} \varepsilon^{\operatorname{len}\mathbf{i}}\right) \cdot \left(\prod_{\substack{\mathbf{j}\in\operatorname{Seq}_+ E;\\ \deg\mathbf{j}=n}} \varepsilon^{\operatorname{len}\mathbf{j}}\right) \cdot \det\left(\left(Q_{\mathbf{i},\mathbf{j}}\left(\lambda/\varepsilon^2,1\right)\right)_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\ \mathbf{j}\in\operatorname{Seq}_+ E;\\ \deg\mathbf{i}=-n;\ \deg\mathbf{j}=n}}\right). \tag{55}$$

Now, since $\operatorname{LEN} n = \sum\limits_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\\ \deg\mathbf{i}=-n}} \operatorname{len}\mathbf{i}$, we have $\varepsilon^{\operatorname{LEN} n} = \prod\limits_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\\ \deg\mathbf{i}=-n}} \varepsilon^{\operatorname{len}\mathbf{i}}$. Also, since $\operatorname{LEN} n = \sum\limits_{\substack{\mathbf{j}\in\operatorname{Seq}_+ E;\\ \deg\mathbf{j}=n}} \operatorname{len}\mathbf{j}$, we have $\varepsilon^{\operatorname{LEN} n} = \prod\limits_{\substack{\mathbf{j}\in\operatorname{Seq}_+ E;\\ \deg\mathbf{j}=n}} \varepsilon^{\operatorname{len}\mathbf{j}}$. Thus,

$$\underbrace{\left(\prod_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\\ \deg\mathbf{i}=-n}} \varepsilon^{\operatorname{len}\mathbf{i}}\right)}_{=\varepsilon^{\operatorname{LEN} n}} \cdot \underbrace{\left(\prod_{\substack{\mathbf{j}\in\operatorname{Seq}_+ E;\\ \deg\mathbf{j}=n}} \varepsilon^{\operatorname{len}\mathbf{j}}\right)}_{=\varepsilon^{\operatorname{LEN} n}} = \varepsilon^{\operatorname{LEN} n}\varepsilon^{\operatorname{LEN} n} = \varepsilon^{2\operatorname{LEN} n}. \tag{56}$$

On the other hand, since $Q_n = \det\left(\left(Q_{\mathbf{i},\mathbf{j}}\right)_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\ \mathbf{j}\in\operatorname{Seq}_+ E;\\ \deg\mathbf{i}=-n;\ \deg\mathbf{j}=n}}\right)$, we have

$$Q_n\left(\lambda/\varepsilon^2,1\right) = \det\left(\left(Q_{\mathbf{i},\mathbf{j}}\left(\lambda/\varepsilon^2,1\right)\right)_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\ \mathbf{j}\in\operatorname{Seq}_+ E;\\ \deg\mathbf{i}=-n;\ \deg\mathbf{j}=n}}\right). \tag{57}$$

Hence, (55) becomes

$$\begin{aligned} &Q_n(\lambda,\varepsilon)\\ &= \underbrace{\left(\prod_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\\ \deg\mathbf{i}=-n}} \varepsilon^{\operatorname{len}\mathbf{i}}\right) \cdot \left(\prod_{\substack{\mathbf{j}\in\operatorname{Seq}_+ E;\\ \deg\mathbf{j}=n}} \varepsilon^{\operatorname{len}\mathbf{j}}\right)}_{\substack{=\varepsilon^{2\operatorname{LEN} n}\\ (\text{by (56)})}} \cdot \underbrace{\det\left(\left(Q_{\mathbf{i},\mathbf{j}}\left(\lambda/\varepsilon^2,1\right)\right)_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\ \mathbf{j}\in\operatorname{Seq}_+ E;\\ \deg\mathbf{i}=-n;\ \deg\mathbf{j}=n}}\right)}_{\substack{=Q_n\left(\lambda/\varepsilon^2,1\right)\\ (\text{by (57)})}}\\ &= \varepsilon^{2\operatorname{LEN} n} \cdot Q_n\left(\lambda/\varepsilon^2,1\right). \end{aligned}$$

We have thus proven that $Q_n(\lambda,\varepsilon) = \varepsilon^{2\operatorname{LEN} n} Q_n\left(\lambda/\varepsilon^2,1\right)$ for every $\lambda\in\mathfrak{h}^*$ and every nonzero $\varepsilon\in\mathbb{C}$. This concludes the proof of Corollary 2.6.27.

2.6.9. Proof of Theorem 2.6.6: On leading terms of pseudo-homogeneous polynomial maps

The following lemma about polynomial maps could be an easy exercise in any algebra text. Unfortunately I do not see a quick way to prove it, so the proof is going to take a few pages. Reading it will probably waste more of the reader's time than proving it on her own.

Lemma 2.6.28. Let V be a finite-dimensional $\mathbb{C}$-vector space. Let $k \in \mathbb{N}$. Let $\phi : V \times \mathbb{C} \to \mathbb{C}$ be a polynomial function such that every $\lambda \in V$ and every nonzero $\varepsilon \in \mathbb{C}$ satisfy

$$\phi(\lambda, \varepsilon) = \varepsilon^{2k} \phi\left(\lambda / \varepsilon^2, 1\right).$$

Then:

(a) The polynomial function

$$V \to \mathbb{C}, \qquad \lambda \mapsto \phi(\lambda, 0)$$

is homogeneous of degree k.

(b) For every integer $N > k$, the N-th homogeneous component of the polynomial function

$$V \to \mathbb{C}, \qquad \lambda \mapsto \phi(\lambda, 1)$$

is zero.

(c) The k-th homogeneous component of the polynomial function

$$V \to \mathbb{C}, \qquad \lambda \mapsto \phi(\lambda, 1)$$

is the polynomial function

$$V \to \mathbb{C}, \qquad \lambda \mapsto \phi(\lambda, 0).$$

Proof of Lemma 2.6.28. **(a)** Let $(v_1, v_2, ..., v_n)$ be a basis of the vector space V^*. Let $\pi_V : V \times \mathbb{C} \to V$ and $\pi_{\mathbb{C}} : V \times \mathbb{C} \to \mathbb{C}$ be the canonical projections. Then, $(v_1 \circ \pi_V, v_2 \circ \pi_V, ..., v_n \circ \pi_V, \pi_{\mathbb{C}})$ is a basis of the vector space $(V \times \mathbb{C})^*$.

Therefore, since ϕ is a polynomial function, there exists a polynomial $P \in \mathbb{C}[X_1, X_2, ..., X_n, X_{n+1}]$ such that every $w \in V \times \mathbb{C}$ satisfies

$$\phi(w) = P((v_1 \circ \pi_V)(w), (v_2 \circ \pi_V)(w), ..., (v_n \circ \pi_V)(w), \pi_{\mathbb{C}}(w)).$$

In other words, every $(\lambda, \varepsilon) \in V \times \mathbb{C}$ satisfies

$$\phi(\lambda, \varepsilon) = P(v_1(\lambda), v_2(\lambda), ..., v_n(\lambda), \varepsilon). \tag{58}$$

Now, it is easy to see that for every $(x_1, x_2, ..., x_n) \in \mathbb{C}^n$ and nonzero $\varepsilon \in \mathbb{C}$, we have

$$P(x_1, x_2, ..., x_n, \varepsilon) = \varepsilon^{2k} P\left(x_1/\varepsilon^2, x_2/\varepsilon^2, ..., x_n/\varepsilon^2, 1\right). \tag{59}$$

[64]

Now, since $P \in \mathbb{C}[X_1, X_2, ..., X_n, X_{n+1}] \cong (\mathbb{C}[X_1, X_2, ..., X_n])[X_{n+1}]$, we can write the polynomial P as a polynomial in the variable X_{n+1} over the ring $\mathbb{C}[X_1, X_2, ..., X_n]$. In other words, we can write the polynomial P in the form $P = \sum\limits_{i \in \mathbb{N}} P_i \cdot X_{n+1}^i$ for some polynomials P_0, P_1, P_2, ... in $\mathbb{C}[X_1, X_2, ..., X_n]$ such that all but finitely many $i \in \mathbb{N}$ satisfy $P_i = 0$. Consider these P_0, P_1, P_2, ...

[64] *Proof of (59).* Let $(x_1, x_2, ..., x_n) \in \mathbb{C}^n$ be arbitrary, and let $\varepsilon \in \mathbb{C}$ be nonzero.

Let $\lambda \in V$ be a vector satisfying

$$v_i(\lambda) = x_i \qquad \text{for every } i \in \{1, 2, ..., n\}$$

Since all but finitely many $i \in \mathbb{N}$ satisfy $P_i = 0$, there exists a $d \in \mathbb{N}$ such that every integer $i > d$ satisfies $P_i = 0$. Consider this d. Then, $P = \sum\limits_{i\in\mathbb{N}} P_i \cdot X_{n+1}^i = \sum\limits_{i=0}^{d} P_i \cdot X_{n+1}^i$ (here, we have removed all the terms with $i > d$ from the sum, because every integer $i > d$ satisfies $P_i = 0$ and thus $P_i \cdot X_{n+1}^i = 0$).

For every $i \in \mathbb{N}$ and every $j \in \mathbb{N}$, let $Q_{i,j}$ be the j-th homogeneous component of the polynomial P_i. Then, $P_i = \sum\limits_{j\in\mathbb{N}} Q_{i,j}$ for every $i \in \mathbb{N}$, and each $Q_{i,j}$ is homogeneous of degree j.

Hence,

$$P = \sum_{i\in\mathbb{N}} \underbrace{P_i}_{=\sum\limits_{j\in\mathbb{N}} Q_{i,j}} \cdot X_{n+1}^i = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} Q_{i,j} X_{n+1}^i. \tag{60}$$

Now, we are going to show the following fact: We have

$$Q_{u,v} = 0 \qquad \text{for all } (u,v) \in \mathbb{N}\times\mathbb{N} \text{ which don't satisfy } u + 2v = 2k. \tag{61}$$

Proof of (61). Let $(u,v) \in \mathbb{N}\times\mathbb{N}$ be such that $u + 2v \neq 2k$. We must prove that $Q_{u,v} = 0$.

If $u > d$, then $Q_{u,v} = 0$ is clear (because $Q_{u,v}$ is the v-th homogeneous component of P_u, but we have $P_u = 0$ since $u > d$). Hence, for the rest of the proof of $Q_{u,v} = 0$, we can WLOG assume that $u \leq d$.

We have

$$P = \sum_{i=0}^{d} \underbrace{P_i}_{=\sum\limits_{j\in\mathbb{N}} Q_{i,j}} \cdot X_{n+1}^i = \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} Q_{i,j} X_{n+1}^i.$$

Let $(x_1, x_2, ..., x_n) \in \mathbb{C}^n$ and $\varepsilon \in \mathbb{C} \diagdown \{0\}$. Then, ε is nonzero, and we have

$$\begin{aligned} P(x_1, x_2, ..., x_n, 1/\varepsilon) &= \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} Q_{i,j}(x_1, x_2, ..., x_n) \underbrace{(1/\varepsilon)^i}_{=\varepsilon^{d-i}/\varepsilon^d} \qquad \left(\text{since } P = \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} Q_{i,j} X_{n+1}^i\right) \\ &= \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} Q_{i,j}(x_1, x_2, ..., x_n)\, \varepsilon^{d-i}/\varepsilon^d = \frac{1}{\varepsilon^d} \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} Q_{i,j}(x_1, x_2, ..., x_n)\, \varepsilon^{d-i} \end{aligned}$$

(such a vector λ exists since $(v_1, v_2, ..., v_n)$ is a basis of V^*). Then,

$$\begin{aligned} P(x_1, x_2, ..., x_n, \varepsilon) &= P(v_1(\lambda), v_2(\lambda), ..., v_n(\lambda), \varepsilon) \qquad (\text{since } x_i = v_i(\lambda) \text{ for every } i \in \{1,2,...,n\}) \\ &= \phi(\lambda, \varepsilon) \qquad (\text{by } (58)) \\ &= \varepsilon^{2k} \underbrace{\phi(\lambda/\varepsilon^2, 1)}_{\substack{=P\left(v_1(\lambda/\varepsilon^2), v_2(\lambda/\varepsilon^2), ..., v_n(\lambda/\varepsilon^2), 1\right) \\ (\text{by (58), applied to } (\lambda/\varepsilon^2, 1) \text{ instead of } (\lambda,\varepsilon))}} \\ &= \varepsilon^{2k} P\left(v_1(\lambda/\varepsilon^2), v_2(\lambda/\varepsilon^2), ..., v_n(\lambda/\varepsilon^2), 1\right) = \varepsilon^{2k} P\left(x_1/\varepsilon^2, x_2/\varepsilon^2, ..., x_n/\varepsilon^2, 1\right) \\ &\qquad \left(\text{since } v_i(\lambda/\varepsilon^2) = \underbrace{v_i(\lambda)}_{=x_i}/\varepsilon^2 = x_i/\varepsilon^2 \text{ for every } i \in \{1,2,...,n\}\right). \end{aligned}$$

This proves (59).

and

$$P\left(\varepsilon^2 x_1, \varepsilon^2 x_2, ..., \varepsilon^2 x_n, 1\right) = \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} \underbrace{Q_{i,j}\left(\varepsilon^2 x_1, \varepsilon^2 x_2, ..., \varepsilon^2 x_n\right)}_{\substack{=\left(\varepsilon^2\right)^j Q_{i,j}(x_1,x_2,...,x_n) \\ \text{(since } Q_{i,j} \text{ is homogeneous of degree } j\text{)}}} \underbrace{1^i}_{=1}$$

$$\left(\text{since } P = \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} Q_{i,j} X_{n+1}^i\right)$$

$$= \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} \underbrace{\left(\varepsilon^2\right)^j}_{=\varepsilon^{2j}} Q_{i,j}\left(x_1, x_2, ..., x_n\right) = \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} \varepsilon^{2j} Q_{i,j}\left(x_1, x_2, ..., x_n\right).$$

Now,

$$\frac{1}{\varepsilon^d} \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} Q_{i,j}\left(x_1, x_2, ..., x_n\right) \varepsilon^{d-i}$$

$$= P\left(x_1, x_2, ..., x_n, 1/\varepsilon\right) = \left(1/\varepsilon\right)^{2k} \underbrace{P\left(x_1/\left(\frac{1}{\varepsilon}\right)^2, x_2/\left(\frac{1}{\varepsilon}\right)^2, ..., x_n/\left(\frac{1}{\varepsilon}\right)^2, 1\right)}_{=P(\varepsilon^2 x_1,\varepsilon^2 x_2,...,\varepsilon^2 x_n,1)=\sum_{i=0}^{d}\sum_{j\in\mathbb{N}} \varepsilon^{2j} Q_{i,j}(x_1,x_2,...,x_n)}$$

$$\text{(by (59), applied to } 1/\varepsilon \text{ instead of } \varepsilon\text{)}$$

$$= \left(1/\varepsilon\right)^{2k} \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} \varepsilon^{2j} Q_{i,j}\left(x_1, x_2, ..., x_n\right),$$

so that

$$\varepsilon^{2k} \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} Q_{i,j}\left(x_1, x_2, ..., x_n\right) \varepsilon^{d-i} = \varepsilon^d \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} \varepsilon^{2j} Q_{i,j}\left(x_1, x_2, ..., x_n\right).$$

For fixed ε, this is a polynomial identity in $\left(x_1, x_2, ..., x_n\right) \in \mathbb{C}^n$. Since it holds for all $\left(x_1, x_2, ..., x_n\right) \in \mathbb{C}^n$ (as we just have shown), it thus must hold as a formal identity, i. e., we must have

$$\varepsilon^{2k} \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} Q_{i,j} \varepsilon^{d-i} = \varepsilon^d \sum_{i=0}^{d} \sum_{j\in\mathbb{N}} \varepsilon^{2j} Q_{i,j} \qquad \text{in } \mathbb{C}\left[X_1, X_2, ..., X_n\right].$$

Let us take the v-th homogeneous components of both sides of this equation. Since each $Q_{i,j}$ is homogeneous of degree j, this amounts to removing all $Q_{i,j}$ with $j \neq v$, and leaving the $Q_{i,j}$ with $j = v$ unchanged. Thus, we obtain

$$\varepsilon^{2k} \sum_{i=0}^{d} Q_{i,v} \varepsilon^{d-i} = \varepsilon^d \sum_{i=0}^{d} \varepsilon^{2v} Q_{i,v} \qquad \text{in } \mathbb{C}\left[X_1, X_2, ..., X_n\right]. \tag{62}$$

Now, let $\left(x_1, x_2, ..., x_n\right) \in \mathbb{C}^n$ be arbitrary again. Then, evaluating the identity (62) at $\left(X_1, X_2, ..., X_n\right) = \left(x_1, x_2, ..., x_n\right)$, we obtain

$$\varepsilon^{2k} \sum_{i=0}^{d} Q_{i,v}\left(x_1, x_2, ..., x_n\right) \varepsilon^{d-i} = \varepsilon^d \sum_{i=0}^{d} \varepsilon^{2v} Q_{i,v}\left(x_1, x_2, ..., x_n\right).$$

For fixed $(x_1, x_2, ..., x_n)$, this is a polynomial identity in ε (since $d - i \geq 0$ for all $i \in \{0, 1, ..., d\}$). Since it holds for all nonzero $\varepsilon \in \mathbb{C}$ (as we just have shown), it thus must hold as a formal identity (since any polynomial in one variable which evaluates to zero at all nonzero complex numbers must be the zero polynomial). In other words, we must have

$$E^{2k} \sum_{i=0}^{d} Q_{i,v}(x_1, x_2, ..., x_n) E^{d-i} = E^d \sum_{i=0}^{d} E^{2v} Q_{i,v}(x_1, x_2, ..., x_n) \qquad \text{in } \mathbb{C}[E]$$

(where $\mathbb{C}[E]$ denotes the polynomial ring over $\mathbb{C}$ in one variable E). Let us compare the coefficients of E^{2k+d-u} on both sides of this equation: The coefficient of E^{2k+d-u} on the left hand side of this equation is clearly $Q_{u,v}(x_1, x_2, ..., x_n)$, while the coefficient of E^{2k+d-u} on the right hand side is 0 (in fact, the only coefficient on the right hand side of the equation which is not trivially zero is the coefficient of E^{d+2v}, but $d + 2v \neq 2k + d - u$ (since $u + 2v \neq 2k$ and thus $2v \neq 2k - u$)). Hence, comparison yields $Q_{u,v}(x_1, x_2, ..., x_n) = 0$. Since this holds for all $(x_1, x_2, ..., x_n) \in \mathbb{C}^n$, we thus obtain $Q_{u,v} = 0$ (because any polynomial which vanishes on the whole $\mathbb{C}^n$ must be the zero polynomial). This proves (61).

Now, (60) rewrites as

$$\begin{aligned} P &= \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} Q_{i,j} X_{n+1}^i = \sum_{u\in\mathbb{N}} \sum_{v\in\mathbb{N}} Q_{u,v} X_{n+1}^u \qquad (\text{here, we renamed the indices } i \text{ and } j \text{ as } u \text{ and } v) \\ &= \sum_{(u,v)\in\mathbb{N}\times\mathbb{N}} Q_{u,v} X_{n+1}^u = \sum_{\substack{(u,v)\in\mathbb{N}\times\mathbb{N};\\ u+2v=2k}} Q_{u,v} X_{n+1}^u \\ &\qquad \left(\begin{array}{c} \text{here, we removed from our sum all terms for } (u,v) \in \mathbb{N}\times\mathbb{N} \text{ which} \\ \text{don't satisfy } u+2v=2k \text{ (because (61) shows that these terms} \\ \text{don't contribute anything to the sum)} \end{array}\right) \\ &= \sum_{v=0}^{k} Q_{2k-2v,v} X_{n+1}^{2k-2v} \qquad (\text{here, we substituted } (2k-2v, v) \text{ for } (u,v) \text{ in the sum}). \end{aligned}$$

Now, for every $v \in \{0, 1, ..., k\}$, let $\psi_v : V \to \mathbb{C}$ be the polynomial map defined by

$$\psi_v(\lambda) = Q_{2k-2v,v}(v_1(\lambda), v_2(\lambda), ..., v_n(\lambda)) \qquad \text{for every } \lambda \in V.$$

Then, ψ_v is homogeneous of degree v (since $Q_{2k-2v,v}$ is homogeneous of degree v). In particular, this yields that ψ_k is homogeneous of degree k.

Every $(\lambda, \varepsilon) \in V \times \mathbb{C}$ satisfies

$$\begin{aligned} \phi(\lambda, \varepsilon) &= P(v_1(\lambda), v_2(\lambda), ..., v_n(\lambda), \varepsilon) \\ &= \sum_{v=0}^{k} \underbrace{Q_{2k-2v,v}(v_1(\lambda), v_2(\lambda), ..., v_n(\lambda))}_{=\psi_v(\lambda)} \varepsilon^{2k-2v} \qquad \left(\text{since } P = \sum_{v=0}^{k} Q_{2k-2v,v} X_{n+1}^{2k-2v}\right) \\ &= \sum_{v=0}^{k} \psi_v(\lambda) \varepsilon^{2k-2v}. \end{aligned} \tag{63}$$

Applied to $\varepsilon = 0$, this yields

$$\phi(\lambda, 0) = \sum_{v=0}^{k} \psi_v(\lambda) 0^{2k-2v} = \psi_k(\lambda) \qquad \left(\text{since } 0^{2k-2v} = 0 \text{ for all } v < k\right)$$

for every $\lambda \in V$. Hence, the polynomial function $V \to \mathbb{C}$, $\lambda \mapsto \phi(\lambda, 0)$ equals the polynomial function ψ_k, and thus is homogeneous of degree k (since ψ_k is homogeneous of degree k). This proves Lemma 2.6.28 **(a)**.

Applying (63) to $\varepsilon = 1$, we obtain

$$\phi(\lambda, 1) = \sum_{v=0}^{k} \psi_v(\lambda) \underbrace{1^{2k-2v}}_{=1} = \sum_{v=0}^{k} \psi_v(\lambda).$$

Hence, the polynomial function $V \to \mathbb{C}$, $\lambda \mapsto \phi(\lambda, 1)$ equals the sum $\sum\limits_{v=0}^{k} \psi_v$. Since we know that the polynomial function ψ_v is homogeneous of degree v for every $v \in \{0, 1, ..., k\}$, this yields that, for every integer $N > k$, the N-th homogeneous component of the polynomial function $V \to \mathbb{C}$, $\lambda \mapsto \phi(\lambda, 1)$ is zero. This proves Lemma 2.6.28 **(b)**.

Finally, recall that the polynomial function $V \to \mathbb{C}$, $\lambda \mapsto \phi(\lambda, 1)$ equals the sum $\sum\limits_{v=0}^{k} \psi_v$, and the polynomial function ψ_v is homogeneous of degree v for every $v \in \{0, 1, ..., k\}$. Hence, for every $v \in \{0, 1, ..., k\}$, the v-th homogeneous component of the polynomial function $V \to \mathbb{C}$, $\lambda \mapsto \phi(\lambda, 1)$ is ψ_v. In particular, the k-th homogeneous component of the polynomial function $V \to \mathbb{C}$, $\lambda \mapsto \phi(\lambda, 1)$ is ψ_k. Since ψ_k equals the function $V \to \mathbb{C}$, $\lambda \mapsto \phi(\lambda, 0)$, this rewrites as follows: The k-th homogeneous component of the polynomial function $V \to \mathbb{C}$, $\lambda \mapsto \phi(\lambda, 1)$ is the function $V \to \mathbb{C}$, $\lambda \mapsto \phi(\lambda, 0)$. This proves Lemma 2.6.28 **(c)**.

2.6.10. Proof of Theorem 2.6.6: The Lie algebra $\mathfrak{g}^0$

Consider the polynomial function Q_n of Corollary 2.6.27. Due to Corollary 2.6.27, it satisfies the condition of Lemma 2.6.28 for $k = \operatorname{LEN} n$. Hence, Lemma 2.6.28 suggests that we study the Lie algebra $\mathfrak{g}^0$, since this will show us what the function $\mathfrak{h}^* \to \mathbb{C}$, $\lambda \mapsto Q_n(\lambda, 0)$ looks like.

First, let us reformulate the definition of $\mathfrak{g}^0$ as follows: As a vector space, $\mathfrak{g}^0 = \mathfrak{g}$, but the bracket on $\mathfrak{g}^0$ is given by

$$[\cdot, \cdot]^0 : \mathfrak{g}_i \otimes \mathfrak{g}_j \to \mathfrak{g}_{i+j} \text{ is } \left\{ \begin{array}{c} \text{zero if } i + j \neq 0; \\ \text{the Lie bracket } [\cdot, \cdot] \text{ of } \mathfrak{g} \text{ if } i + j = 0 \end{array} \right. . \tag{64}$$

It is very easy to see (from this) that $[\mathfrak{n}_-, \mathfrak{n}_-]^0 = 0$, $[\mathfrak{n}_+, \mathfrak{n}_+]^0 = 0$, $[\mathfrak{n}_-, \mathfrak{n}_+]^0 = [\mathfrak{n}_+, \mathfrak{n}_-]^0 \subseteq \mathfrak{h}$ and that $\mathfrak{h} \subseteq Z(\mathfrak{g}^0)$.

We notice that $\mathfrak{n}_-^0 = \mathfrak{n}_-$, $\mathfrak{n}_+^0 = \mathfrak{n}_+$ and $\mathfrak{h}^0 = \mathfrak{h}$ as vector spaces.

Since $\left[\mathfrak{n}_-^0, \mathfrak{n}_-^0\right]^0 = [\mathfrak{n}_-, \mathfrak{n}_-]^0 = 0$, the Lie algebra $\mathfrak{n}_-^0$ is abelian, so that $U\left(\mathfrak{n}_-^0\right) = S\left(\mathfrak{n}_-^0\right) = S(\mathfrak{n}_-)$. Similarly, $U\left(\mathfrak{n}_+^0\right) = S\left(\mathfrak{n}_+^0\right) = S(\mathfrak{n}_+)$.

We notice that

$$\lambda\left([x, y]^0\right) = \lambda([x, y]) \qquad \text{for any } x \in \mathfrak{g} \text{ and } y \in \mathfrak{g}. \tag{65}$$

[65]

In the following, we will use the form $(\cdot,\cdot)^{\circ}_{\lambda}$ defined in Definition 2.6.11. We will only consider this form for the Lie algebra $\mathfrak{g}$, not for the Lie algebras $\mathfrak{g}^{\varepsilon}$ and $\mathfrak{g}^{0}$; thus we don't have any reason to rename it as $(\cdot,\cdot)^{\circ\mathfrak{g}}_{\lambda}$.

Lemma 2.6.29. We have

$$\left(av_{\lambda}^{+\mathfrak{g}^0}, bv_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda}^{\mathfrak{g}^0} = (a,b)^{\circ}_{\lambda} \qquad \text{for all } a \in S(\mathfrak{n}_-) \text{ and } b \in S(\mathfrak{n}_+). \tag{66}$$

Here, $av_{\lambda}^{+\mathfrak{g}^0}$ and $bv_{-\lambda}^{-\mathfrak{g}^0}$ are elements of $M_{\lambda}^{+\mathfrak{g}^0}$ and $M_{-\lambda}^{-\mathfrak{g}^0}$, respectively (because $a \in S(\mathfrak{n}_-) = U\left(\mathfrak{n}_-^0\right)$ and $b \in S(\mathfrak{n}_+) = U\left(\mathfrak{n}_+^0\right)$).

Proof of Lemma 2.6.29. Let $a \in S(\mathfrak{n}_-)$ and $b \in S(\mathfrak{n}_+)$ be arbitrary. Since the claim that $\left(av_{\lambda}^{+\mathfrak{g}^0}, bv_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda}^{\mathfrak{g}^0} = (a,b)^{\circ}_{\lambda}$ is linear in each of a and b, we can WLOG assume that $a = a_1a_2...a_u$ for some homogeneous $a_1,a_2,...,a_u \in \mathfrak{n}_-$ and that $b = b_1b_2...b_v$ for some homogeneous $b_1,b_2,...,b_v \in \mathfrak{n}_+$ (because every element of $S(\mathfrak{n}_-)$ is a $\mathbb{C}$-linear combination of products of the form $a_1a_2...a_u$ with homogeneous $a_1,a_2,...,a_u \in \mathfrak{n}_-$, and because every element of $S(\mathfrak{n}_+)$ is a $\mathbb{C}$-linear combination of products of the form $b_1b_2...b_v$ with homogeneous $b_1,b_2,...,b_v \in \mathfrak{n}_+$).

WLOG assume that $v \geq u$. (Else, the proof is analogous.)

Recall the equality $\left(av_{\lambda}^{+}, bv_{-\lambda}^{-}\right) = \left(S(b)\,av_{\lambda}^{+}, v_{-\lambda}^{-}\right)$ shown during the proof of Proposition 2.6.1. Applied to $\mathfrak{g}^0$ instead of $\mathfrak{g}$, this yields $\left(av_{\lambda}^{+\mathfrak{g}^0}, bv_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda}^{\mathfrak{g}^0} = \left(S(b)\,av_{\lambda}^{+\mathfrak{g}^0}, v_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda}^{\mathfrak{g}^0}$.

Since $\mathfrak{h} \subseteq Z(\mathfrak{g}^0)$, we have $\mathfrak{h} \subseteq Z(U(\mathfrak{g}^0))$ (because the center of a Lie algebra always lies in the center of its universal enveloping algebra).

Since $b = b_1b_2...b_v$, we have $S(b) = (-1)^v b_vb_{v-1}...b_1$. Combined with $a = a_1a_2...a_u$, this yields

$$S(b)\,a = (-1)^v b_vb_{v-1}...b_1a_1a_2...a_u,$$

so that

$$\left(av_{\lambda}^{+\mathfrak{g}^0}, bv_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda}^{\mathfrak{g}^0} = \left(\underbrace{S(b)\,a}_{=(-1)^v b_vb_{v-1}...b_1a_1a_2...a_u} v_{\lambda}^{+\mathfrak{g}^0}, v_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda}^{\mathfrak{g}^0} = (-1)^v \left(b_vb_{v-1}...b_1a_1a_2...a_uv_{\lambda}^{+\mathfrak{g}^0}, v_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda}^{\mathfrak{g}^0}. \tag{67}$$

We will now prove some identities in order to simplify the $b_vb_{v-1}...b_1a_1a_2...a_uv_{\lambda}^{+\mathfrak{g}^0}$ term here.

[65] *Proof of (65).* Let $x \in \mathfrak{g}$ and $y \in \mathfrak{g}$. Since the equation (65) is linear in each of x and y, we can WLOG assume that x and y are homogeneous (since every element of $\mathfrak{g}$ is a sum of homogeneous elements). So we can assume that $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$ for some $i \in \mathbb{N}$ and $j \in \mathbb{N}$. Consider these i and j. If $i+j \neq 0$, then $[x,y]^0 = 0$ (by (64)) and $\lambda([x,y]) = 0$ (since $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$ yield $[x,y] \in \mathfrak{g}_{i+j}$, and due to $i+j \neq 0$ the form λ annihilates $\mathfrak{g}_{i+j}$), so that (65) trivially holds in this case. If $i+j=0$, then $[x,y]^0 = [x,y]$ (again by (64)), and thus (65) holds in this case as well. We have thus proven (65) both in the case $i+j \neq 0$ and in the case $i+j=0$. These cases cover all possibilities, and thus (65) is proven.

First: In the Verma highest-weight module $M_{\lambda}^{+\mathfrak{g}^0}$ of $(\mathfrak{g}^0, \lambda)$, we have

$$\beta\alpha_1\alpha_2...\alpha_\ell v_\lambda^{+\mathfrak{g}^0} = \sum_{p=1}^{\ell} \lambda\left([\beta,\alpha_p]\right) \alpha_1\alpha_2...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_\ell v_\lambda^{+\mathfrak{g}^0} \qquad (68)$$

$$\text{for every } \ell \in \mathbb{N},\ \alpha_1,\alpha_2,...,\alpha_\ell \in \mathfrak{n}_- \text{ and } \beta \in \mathfrak{n}_+.$$

[66]

[66] *Proof of (68).* We will prove (68) by induction over ℓ:

Induction base: For $\ell = 0$, the left hand side of (68) is $\beta v_\lambda^{+\mathfrak{g}^0} = 0$ (since $\beta \in \mathfrak{n}_+ = \mathfrak{n}_+^0$), and the right hand side of (68) is (empty sum) $= 0$. Thus, for $\ell = 0$, the equality (68) holds. This completes the induction base.

Induction step: Let $m \in \mathbb{N}$ be positive. Assume that (68) holds for $\ell = m-1$. We now must show that (68) holds for $\ell = m$.

Let $\alpha_1, \alpha_2, ..., \alpha_m \in \mathfrak{n}_-$ and $\beta \in \mathfrak{n}_+$.

Since (68) holds for $\ell = m-1$, we can apply (68) to $m-1$ and $(\alpha_2, \alpha_3, ..., \alpha_m)$ instead of ℓ and $(\alpha_1, \alpha_2, ..., \alpha_\ell)$, and thus obtain

$$\begin{aligned}\beta\alpha_2\alpha_3...\alpha_m v_\lambda^{+\mathfrak{g}^0} &= \sum_{p=1}^{m-1} \lambda\left([\beta,\alpha_{p+1}]\right) \alpha_2\alpha_3...\alpha_{p-1+1}\alpha_{p+1+1}\alpha_{p+2+1}...\alpha_m v_\lambda^{+\mathfrak{g}^0} \\ &= \sum_{p=2}^{m} \lambda\left([\beta,\alpha_p]\right) \alpha_2\alpha_3...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m v_\lambda^{+\mathfrak{g}^0} \\ &\qquad \left(\text{here, we substituted } p \text{ for } p+1 \text{ in the sum}\right).\end{aligned}$$

Now, we notice that $\beta \in \mathfrak{n}_+$ and $\alpha_1 \in \mathfrak{n}_-$, so that $[\beta,\alpha_1]^0 \in [\mathfrak{n}_+,\mathfrak{n}_-]^0 \subseteq \mathfrak{h} \subseteq Z\left(U\left(\mathfrak{g}^0\right)\right)$. Thus, $[\beta,\alpha_1]^0 \alpha_2\alpha_3...\alpha_m = \alpha_2\alpha_3...\alpha_m [\beta,\alpha_1]^0$. But since $[\beta,\alpha_1]^0 \in \mathfrak{h} = \mathfrak{h}^0$, we also have $[\beta,\alpha_1]^0 v_\lambda^{+\mathfrak{g}^0} = \lambda\left([\beta,\alpha_1]^0\right) v_\lambda^{+\mathfrak{g}^0} = \lambda\left([\beta,\alpha_1]\right) v_\lambda^{+\mathfrak{g}^0}$ (since $\lambda\left([\beta,\alpha_1]^0\right) = \lambda\left([\beta,\alpha_1]\right)$ by (65)).

We now compute:

$$\begin{aligned}\beta\alpha_1\alpha_2...\alpha_m v_\lambda^{+\mathfrak{g}^0} &= \underbrace{\beta\alpha_1}_{\substack{=\alpha_1\beta+[\beta,\alpha_1]^0 \\ (\text{since we are in } U(\mathfrak{g}^0))}} \alpha_2\alpha_3...\alpha_m v_\lambda^{+\mathfrak{g}^0} = \left(\alpha_1\beta + [\beta,\alpha_1]^0\right)\alpha_2\alpha_3...\alpha_m v_\lambda^{+\mathfrak{g}^0} \\ &= \alpha_1 \underbrace{\beta\alpha_2\alpha_3...\alpha_m v_\lambda^{+\mathfrak{g}^0}}_{=\sum\limits_{p=2}^{m}\lambda([\beta,\alpha_p])\alpha_2\alpha_3...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m v_\lambda^{+\mathfrak{g}^0}} + \underbrace{[\beta,\alpha_1]^0\alpha_2\alpha_3...\alpha_m}_{=\alpha_2\alpha_3...\alpha_m[\beta,\alpha_1]^0} v_\lambda^{+\mathfrak{g}^0} \\ &= \underbrace{\alpha_1 \sum_{p=2}^{m} \lambda\left([\beta,\alpha_p]\right) \alpha_2\alpha_3...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m v_\lambda^{+\mathfrak{g}^0}}_{=\sum\limits_{p=2}^{m}\lambda([\beta,\alpha_p])\alpha_1\alpha_2\alpha_3...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m v_\lambda^{+\mathfrak{g}^0}} + \alpha_2\alpha_3...\alpha_m \underbrace{[\beta,\alpha_1]^0 v_\lambda^{+\mathfrak{g}^0}}_{=\lambda([\beta,\alpha_1])v_\lambda^{+\mathfrak{g}^0}} \\ &= \sum_{p=2}^{m} \lambda\left([\beta,\alpha_p]\right) \alpha_1\alpha_2\alpha_3...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m v_\lambda^{+\mathfrak{g}^0} + \lambda\left([\beta,\alpha_1]\right)\alpha_2\alpha_3...\alpha_m v_\lambda^{+\mathfrak{g}^0} \\ &= \sum_{p=1}^{m} \lambda\left([\beta,\alpha_p]\right) \alpha_1\alpha_2\alpha_3...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m v_\lambda^{+\mathfrak{g}^0} \\ &= \sum_{p=1}^{m} \lambda\left([\beta,\alpha_p]\right) \alpha_1\alpha_2...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m v_\lambda^{+\mathfrak{g}^0}.\end{aligned}$$

Thus, (68) holds for $\ell = m$. This completes the induction step. Thus, (68) is proven.

Next we will show that in the Verma highest-weight module $M_\lambda^{+\mathfrak{g}^0}$ of $(\mathfrak{g}^0, \lambda)$, we have

$$\beta_\ell \beta_{\ell-1} ... \beta_1 \alpha_1 \alpha_2 ... \alpha_\ell v_\lambda^{+\mathfrak{g}^0} = (-1)^\ell \sum_{\sigma \in S_\ell} \lambda\left(\left[\alpha_1, \beta_{\sigma(1)}\right]\right) \lambda\left(\left[\alpha_2, \beta_{\sigma(2)}\right]\right) ... \lambda\left(\left[\alpha_\ell, \beta_{\sigma(\ell)}\right]\right) v_\lambda^{+\mathfrak{g}^0} \tag{69}$$

for every $\ell \in \mathbb{N}$, $\alpha_1, \alpha_2, ..., \alpha_\ell \in \mathfrak{n}_-$ and $\beta_1, \beta_2, ..., \beta_\ell \in \mathfrak{n}_+$.

Proof of (69). We will prove (69) by induction over ℓ:

Induction base: For $\ell = 0$, we have $\underbrace{\beta_\ell \beta_{\ell-1} ... \beta_1}_{\text{empty product}} \underbrace{\alpha_1 \alpha_2 ... \alpha_\ell}_{\text{empty product}} v_\lambda^{+\mathfrak{g}^0} = v_\lambda^{+\mathfrak{g}^0}$ and $\underbrace{(-1)^\ell}_{=1} \underbrace{\sum_{\sigma \in S_\ell}}_{\text{sum over 1 element}} \underbrace{\lambda\left(\left[\alpha_1, \beta_{\sigma(1)}\right]\right) \lambda\left(\left[\alpha_2, \beta_{\sigma(2)}\right]\right) ... \lambda\left(\left[\alpha_\ell, \beta_{\sigma(\ell)}\right]\right)}_{\text{empty product}} v_\lambda^{+\mathfrak{g}^0} = v_\lambda^{+\mathfrak{g}^0}$. Thus, for $\ell = 0$, the equality (69) holds. This completes the induction base.

Induction step: Let $m \in \mathbb{N}$ be positive. Assume that (69) holds for $\ell = m-1$. We now must show that (69) holds for $\ell = m$.

Let $\alpha_1, \alpha_2, ..., \alpha_m \in \mathfrak{n}_-$ and $\beta_1, \beta_2, ..., \beta_m \in \mathfrak{n}_+$.

For every $p \in \{1, 2, ..., m\}$, let c_p denote the permutation in S_m which is written in row form as $(1, 2, ..., p-1, p+1, p+2, ..., m, p)$. (This is the permutation with cycle decomposition $(1)(2)...(p-1)(p, p+1, ..., m)$.) Since (69) holds for $\ell = m-1$, we can apply (69) to $m-1$ and $\left(\alpha_{c_p(1)}, \alpha_{c_p(2)}, ..., \alpha_{c_p(m-1)}\right)$ instead of ℓ and $(\alpha_1, \alpha_2, ..., \alpha_\ell)$. This results in

$$\begin{aligned}
&\beta_{m-1}\beta_{m-2}...\beta_1 \alpha_{c_p(1)} \alpha_{c_p(2)} ... \alpha_{c_p(m-1)} v_\lambda^{+\mathfrak{g}^0} \\
&= (-1)^{m-1} \sum_{\sigma \in S_{m-1}} \underbrace{\lambda\left(\left[\alpha_{c_p(1)}, \beta_{\sigma(1)}\right]\right) \lambda\left(\left[\alpha_{c_p(2)}, \beta_{\sigma(2)}\right]\right) ... \lambda\left(\left[\alpha_{c_p(m-1)}, \beta_{\sigma(m-1)}\right]\right)}_{\substack{= \prod_{i \in \{1,2,...,m-1\}} \lambda\left(\left[\alpha_{c_p(i)}, \beta_{\sigma(i)}\right]\right) = \prod_{i \in \{1,2,...,m\} \setminus \{p\}} \lambda\left(\left[\alpha_i, \beta_{\sigma\left(c_p^{-1}(i)\right)}\right]\right) \\ \text{(here, we substituted } i \text{ for } c_p(i) \text{ in the product)}}} v_\lambda^{+\mathfrak{g}^0} \\
&= (-1)^{m-1} \sum_{\sigma \in S_{m-1}} \prod_{i \in \{1,2,...,m\} \setminus \{p\}} \lambda\left(\left[\alpha_i, \underbrace{\beta_{\sigma\left(c_p^{-1}(i)\right)}}_{=\beta_{\left(\sigma \circ c_p^{-1}\right)(i)}}\right]\right) v_\lambda^{+\mathfrak{g}^0} \\
&= (-1)^{m-1} \sum_{\sigma \in S_{m-1}} \prod_{i \in \{1,2,...,m\} \setminus \{p\}} \lambda\left(\left[\alpha_i, \beta_{\left(\sigma \circ c_p^{-1}\right)(i)}\right]\right) v_\lambda^{+\mathfrak{g}^0} \\
&= (-1)^{m-1} \sum_{\sigma \in S_m;\ \sigma(m)=m} \prod_{i \in \{1,2,...,m\} \setminus \{p\}} \lambda\left(\left[\alpha_i, \beta_{\left(\sigma \circ c_p^{-1}\right)(i)}\right]\right) v_\lambda^{+\mathfrak{g}^0} \\
&\qquad \left(\begin{array}{c}\text{here, we identified the permutations in } S_{m-1} \text{ with the permutations} \\ \sigma \in S_m \text{ satisfying } \sigma(m) = m\end{array}\right) \\
&= (-1)^{m-1} \sum_{\sigma \in S_m;\ \sigma(p)=m} \prod_{i \in \{1,2,...,m\} \setminus \{p\}} \lambda\left(\left[\alpha_i, \beta_{\sigma(i)}\right]\right) v_\lambda^{+\mathfrak{g}^0} \\
&\qquad \left(\text{here, we substituted } \sigma \text{ for } \sigma \circ c_p^{-1} \text{ in the sum}\right).
\end{aligned}$$

The elements β_m, β_{m-1}, ..., β_1 all lie in $\mathfrak{n}_+$ and thus commute in $U(\mathfrak{g}^0)$ (since

$[\mathfrak{n}_+, \mathfrak{n}_+]^0 = 0)$. Thus, $\beta_m\beta_{m-1}...\beta_1 = \beta_{m-1}\beta_{m-2}...\beta_1\beta_m$ in $U(\mathfrak{g}^0)$, so that

$$\begin{aligned}
&\beta_m\beta_{m-1}...\beta_1\alpha_1\alpha_2...\alpha_m v_\lambda^{+\mathfrak{g}^0} \\
&= \beta_{m-1}\beta_{m-2}...\beta_1 \underbrace{\beta_m\alpha_1\alpha_2...\alpha_m v_\lambda^{+\mathfrak{g}^0}}_{\substack{=\sum\limits_{p=1}^{m}\lambda([\beta_m,\alpha_p])\alpha_1\alpha_2...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m v_\lambda^{+\mathfrak{g}^0} \\ \text{(by (68), applied to } \beta=\beta_m \text{ and } \ell=m)}} \\
&= \beta_{m-1}\beta_{m-2}...\beta_1 \sum_{p=1}^{m} \lambda([\beta_m, \alpha_p])\, \alpha_1\alpha_2...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m v_\lambda^{+\mathfrak{g}^0} \\
&= \sum_{p=1}^{m} \underbrace{\lambda([\beta_m, \alpha_p])}_{=\lambda(-[\alpha_p,\beta_m])=-\lambda([\alpha_p,\beta_m])} \beta_{m-1}\beta_{m-2}...\beta_1 \underbrace{\alpha_1\alpha_2...\alpha_{p-1}\alpha_{p+1}\alpha_{p+2}...\alpha_m}_{\substack{=\alpha_{c_p(1)}\alpha_{c_p(2)}...\alpha_{c_p(m-1)} \\ \text{(by the definition of } c_p)}} v_\lambda^{+\mathfrak{g}^0} \\
&= -\sum_{p=1}^{m} \lambda([\alpha_p, \beta_m]) \underbrace{\beta_{m-1}\beta_{m-2}...\beta_1\alpha_{c_p(1)}\alpha_{c_p(2)}...\alpha_{c_p(m-1)} v_\lambda^{+\mathfrak{g}^0}}_{=(-1)^{m-1}\sum\limits_{\sigma\in S_m;\ \sigma(p)=m}\prod\limits_{i\in\{1,2,...,m\}\setminus\{p\}}\lambda\left(\left[\alpha_i,\beta_{\sigma(i)}\right]\right)v_\lambda^{+\mathfrak{g}^0}} \\
&= \underbrace{-(-1)^{m-1}}_{=(-1)^m} \sum_{p=1}^{m} \sum_{\sigma\in S_m;\ \sigma(p)=m} \lambda\left(\left[\alpha_p, \underbrace{\beta_m}_{\substack{=\beta_{\sigma(p)} \\ \text{(since } \sigma(p)=m)}}\right]\right) \prod_{i\in\{1,2,...,m\}\setminus\{p\}} \lambda\left(\left[\alpha_i, \beta_{\sigma(i)}\right]\right) v_\lambda^{+\mathfrak{g}^0} \\
&= (-1)^m \sum_{p=1}^{m} \sum_{\sigma\in S_m;\ \sigma(p)=m} \underbrace{\lambda\left(\left[\alpha_p, \beta_{\sigma(p)}\right]\right) \prod_{i\in\{1,2,...,m\}\setminus\{p\}} \lambda\left(\left[\alpha_i, \beta_{\sigma(i)}\right]\right)}_{\substack{=\prod\limits_{i\in\{1,2,...,m\}}\lambda\left(\left[\alpha_i,\beta_{\sigma(i)}\right]\right) \\ =\lambda\left(\left[\alpha_1,\beta_{\sigma(1)}\right]\right)\lambda\left(\left[\alpha_2,\beta_{\sigma(2)}\right]\right)...\lambda\left(\left[\alpha_m,\beta_{\sigma(m)}\right]\right)}} v_\lambda^{+\mathfrak{g}^0} \\
&= (-1)^m \underbrace{\sum_{p=1}^{m} \sum_{\sigma\in S_m;\ \sigma(p)=m}}_{=\sum\limits_{\sigma\in S_m}} \lambda\left(\left[\alpha_1, \beta_{\sigma(1)}\right]\right) \lambda\left(\left[\alpha_2, \beta_{\sigma(2)}\right]\right) ...\lambda\left(\left[\alpha_m, \beta_{\sigma(m)}\right]\right) v_\lambda^{+\mathfrak{g}^0} \\
&= (-1)^m \sum_{\sigma\in S_m} \lambda\left(\left[\alpha_1, \beta_{\sigma(1)}\right]\right) \lambda\left(\left[\alpha_2, \beta_{\sigma(2)}\right]\right) ...\lambda\left(\left[\alpha_m, \beta_{\sigma(m)}\right]\right) v_\lambda^{+\mathfrak{g}^0}.
\end{aligned}$$

In other words, (69) is proven for $\ell = m$. This completes the induction step. Thus, the induction proof of (69) is done.

Now, back to proving $\left(av_\lambda^{+\mathfrak{g}^0}, bv_{-\lambda}^{-\mathfrak{g}^0}\right)_\lambda^{\mathfrak{g}^0} = (a,b)_\lambda^\circ$. Applying (69) to $\ell = u$, $\alpha_i = a_i$ and $\beta_i = b_i$, we obtain

$$b_u b_{u-1}...b_1 a_1 a_2...a_u v_\lambda^{+\mathfrak{g}^0} = (-1)^u \sum_{\sigma\in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right) \lambda\left(\left[a_2, b_{\sigma(2)}\right]\right) ...\lambda\left(\left[a_u, b_{\sigma(u)}\right]\right) v_\lambda^{+\mathfrak{g}^0}.$$

Hence, if $v > u$, then

$$\begin{aligned}
& b_v b_{v-1} ... b_1 a_1 a_2 ... a_u v_\lambda^{+\mathfrak{g}^0} \\
& = b_v b_{v-1} ... b_{u+2} b_{u+1} \underbrace{b_u b_{u-1} ... b_1 a_1 a_2 ... a_u v_\lambda^{+\mathfrak{g}^0}}_{= (-1)^u \sum\limits_{\sigma \in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right) \lambda\left(\left[a_2, b_{\sigma(2)}\right]\right) ... \lambda\left(\left[a_u, b_{\sigma(u)}\right]\right) v_\lambda^{+\mathfrak{g}^0}} \\
& = b_v b_{v-1} ... b_{u+2} b_{u+1} (-1)^u \sum_{\sigma \in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right) \lambda\left(\left[a_2, b_{\sigma(2)}\right]\right) ... \lambda\left(\left[a_u, b_{\sigma(u)}\right]\right) v_\lambda^{+\mathfrak{g}^0} \\
& = (-1)^u \sum_{\sigma \in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right) \lambda\left(\left[a_2, b_{\sigma(2)}\right]\right) ... \lambda\left(\left[a_u, b_{\sigma(u)}\right]\right) b_v b_{v-1} ... b_{u+2} \underbrace{b_{u+1} v_\lambda^{+\mathfrak{g}^0}}_{\substack{=0 \\ (\text{since } b_{u+1} \in \mathfrak{n}_+ = \mathfrak{n}_+^0)}} \\
& = 0,
\end{aligned}$$

and thus

$$\begin{aligned}
& \left(a v_\lambda^{+\mathfrak{g}^0}, b v_{-\lambda}^{-\mathfrak{g}^0}\right)_\lambda^{\mathfrak{g}^0} \\
& = (-1)^v \left(\underbrace{b_v b_{v-1} ... b_1 a_1 a_2 ... a_u v_\lambda^{+\mathfrak{g}^0}}_{=0}, v_{-\lambda}^{-\mathfrak{g}^0}\right)_\lambda^{\mathfrak{g}^0} \qquad (\text{by } (67)) \\
& = 0 = (a, b)_\lambda^\circ \\
& \qquad \left(\begin{array}{c} \text{because the form } (\cdot, \cdot)_\lambda^\circ \text{ was defined as a restriction of a sum} \\ \bigoplus\limits_{k \geq 0} \lambda_k : S(\mathfrak{n}_-) \times S(\mathfrak{n}_+) \to \mathbb{C} \text{ of bilinear forms } \lambda_k : S^k(\mathfrak{n}_-) \times S^k(\mathfrak{n}_+) \to \mathbb{C}, \\ \text{and thus } (S^u(\mathfrak{n}_-), S^v(\mathfrak{n}_+))_\lambda^\circ = 0 \text{ for } u \neq v \text{, so that } (a, b)_\lambda^\circ = 0 \\ (\text{since } a \in S^u(\mathfrak{n}_-) \text{ and } b \in S^v(\mathfrak{n}_+) \text{ and } u \neq v) \end{array}\right).
\end{aligned}$$

We thus have proven $\left(a v_\lambda^{+\mathfrak{g}^0}, b v_{-\lambda}^{-\mathfrak{g}^0}\right)_\lambda^{\mathfrak{g}^0} = (a, b)_\lambda^\circ$ in the case when $v > u$. It remains to prove that $\left(a v_\lambda^{+\mathfrak{g}^0}, b v_{-\lambda}^{-\mathfrak{g}^0}\right)_\lambda^{\mathfrak{g}^0} = (a, b)_\lambda^\circ$ in the case when $v = u$. So let us assume that $v = u$. In this case,

$$\begin{aligned}
b_v b_{v-1} ... b_1 a_1 a_2 ... a_u v_\lambda^{+\mathfrak{g}^0} & = b_u b_{u-1} ... b_1 a_1 a_2 ... a_u v_\lambda^{+\mathfrak{g}^0} \\
& = (-1)^u \sum_{\sigma \in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right) \lambda\left(\left[a_2, b_{\sigma(2)}\right]\right) ... \lambda\left(\left[a_u, b_{\sigma(u)}\right]\right) v_\lambda^{+\mathfrak{g}^0},
\end{aligned}$$

so that

$$\begin{aligned}
&\left(av_\lambda^{+\mathfrak{g}^0}, bv_{-\lambda}^{-\mathfrak{g}^0}\right)_\lambda^{\mathfrak{g}^0} \\
&= \underbrace{(-1)^v}_{\substack{=(-1)^u \\ \text{(since } v=u\text{)}}} \left(\underbrace{b_v b_{v-1} ... b_1 a_1 a_2 ... a_u v_\lambda^{+\mathfrak{g}^0}}_{=(-1)^u \sum\limits_{\sigma\in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right)\lambda\left(\left[a_2, b_{\sigma(2)}\right]\right)...\lambda\left(\left[a_u, b_{\sigma(u)}\right]\right) v_\lambda^{+\mathfrak{g}^0}}, v_{-\lambda}^{-\mathfrak{g}^0} \right)_\lambda^{\mathfrak{g}^0} \\
&= (-1)^u \left((-1)^u \sum_{\sigma\in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right)\lambda\left(\left[a_2, b_{\sigma(2)}\right]\right)...\lambda\left(\left[a_u, b_{\sigma(u)}\right]\right) v_\lambda^{+\mathfrak{g}^0}, v_{-\lambda}^{-\mathfrak{g}^0} \right)_\lambda^{\mathfrak{g}^0} \\
&= \underbrace{(-1)^u (-1)^u}_{\substack{=(-1)^{u+u}=(-1)^{2u}=1 \\ \text{(since } 2u \text{ is even)}}} \sum_{\sigma\in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right)\lambda\left(\left[a_2, b_{\sigma(2)}\right]\right)...\lambda\left(\left[a_u, b_{\sigma(u)}\right]\right) \underbrace{\left(v_\lambda^{+\mathfrak{g}^0}, v_{-\lambda}^{-\mathfrak{g}^0}\right)_\lambda^{\mathfrak{g}^0}}_{=1} \\
&= \sum_{\sigma\in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right)\lambda\left(\left[a_2, b_{\sigma(2)}\right]\right)...\lambda\left(\left[a_u, b_{\sigma(u)}\right]\right).
\end{aligned}$$

Compared to

$$\begin{aligned}
\left(\underbrace{a}_{=a_1a_2...a_u}, \underbrace{b}_{\substack{=b_1b_2...b_v=b_1b_2...b_u \\ \text{(since } v=u\text{)}}} \right)_\lambda^\circ &= \left(a_1a_2...a_u, b_1b_2...b_u\right)_\lambda^\circ = \lambda_u\left(a_1a_2...a_u, b_1b_2...b_u\right) \\
&= \sum_{\sigma\in S_u} \lambda\left(\left[a_1, b_{\sigma(1)}\right]\right)\lambda\left(\left[a_2, b_{\sigma(2)}\right]\right)...\lambda\left(\left[a_u, b_{\sigma(u)}\right]\right),
\end{aligned}$$

this yields $\left(av_\lambda^{+\mathfrak{g}^0}, bv_{-\lambda}^{-\mathfrak{g}^0}\right)_\lambda^{\mathfrak{g}^0} = (a,b)_\lambda^\circ$. Now that $\left(av_\lambda^{+\mathfrak{g}^0}, bv_{-\lambda}^{-\mathfrak{g}^0}\right)_\lambda^{\mathfrak{g}^0} = (a,b)_\lambda^\circ$ is proven in each of the cases $v > u$ and $v = u$ (and the case $v < u$ is analogous), we are done with proving (66).

This proves Proposition 2.6.29.

Corollary 2.6.30. Let $n \in \mathbb{N}$. Recall that the family $\left(e_\mathbf{i}^0\right)_{\mathbf{i}\in \operatorname{Seq}_- E;\ \deg \mathbf{i}=-n}$ is a basis of the vector space $U\left(\mathfrak{n}_-^0\right)[-n] = S\left(\mathfrak{n}_-\right)[-n]$, and that the family $\left(e_\mathbf{j}^0\right)_{\mathbf{j}\in \operatorname{Seq}_+ E;\ \deg \mathbf{j}=n}$ is a basis of the vector space $U\left(\mathfrak{n}_+^0\right)[n] = S\left(\mathfrak{n}_+\right)[n]$. Thus, let us represent the bilinear form $(\cdot,\cdot)_{\lambda,n}^\circ : S\left(\mathfrak{n}_-\right)[-n] \times S\left(\mathfrak{n}_+\right)[n]$ by its matrix with respect to the bases $\left(e_\mathbf{i}^0\right)_{\mathbf{i}\in \operatorname{Seq}_- E;\ \deg \mathbf{i}=-n}$ and $\left(e_\mathbf{j}^0\right)_{\mathbf{j}\in \operatorname{Seq}_+ E;\ \deg \mathbf{j}=n}$ of $S\left(\mathfrak{n}_-\right)[-n]$ and $S\left(\mathfrak{n}_+\right)[n]$, respectively. This is the matrix

$$\left(\left(e_\mathbf{i}^0, e_\mathbf{j}^0\right)_{\lambda,n}^\circ\right)_{\substack{\mathbf{i}\in \operatorname{Seq}_- E;\ \mathbf{j}\in \operatorname{Seq}_+ E; \\ \deg \mathbf{i}=-n;\ \deg \mathbf{j}=n}}.$$

This matrix is a square matrix (since the number of all $\mathbf{j} \in \operatorname{Seq}_+ E$ satisfying $\deg \mathbf{j} = n$ equals the number of all $\mathbf{i} \in \operatorname{Seq}_- E$ satisfying $\deg \mathbf{i} = -n$), and its determinant is what we are going to denote by $\det\left((\cdot,\cdot)_{\lambda,n}^\circ\right)$.

Then,

$$\det\left((\cdot,\cdot)_{\lambda,n}^{\mathfrak{g}^0}\right) = \det\left((\cdot,\cdot)_{\lambda,n}^\circ\right).$$

Proof of Corollary 2.6.30. For every $\mathbf{i} \in \operatorname{Seq}_- E$ satisfying $\deg \mathbf{i} = -n$, and every $\mathbf{j} \in \operatorname{Seq}_+ E$ satisfying $\deg \mathbf{j} = n$, we have

$$\begin{aligned}\left(e_{\mathbf{i}}^0 v_\lambda^{+\mathfrak{g}^0}, e_{\mathbf{j}}^0 v_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda,n}^{\mathfrak{g}^0} = \left(e_{\mathbf{i}}^0 v_\lambda^{+\mathfrak{g}^0}, e_{\mathbf{j}}^0 v_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda}^{\mathfrak{g}^0} &= \left(e_{\mathbf{i}}^0, e_{\mathbf{j}}^0\right)_\lambda^\circ \\ &\qquad \left(\text{by Lemma 2.6.29, applied to } a = e_{\mathbf{i}}^0 \text{ and } b = e_{\mathbf{j}}^0\right) \\ &= \left(e_{\mathbf{i}}^0, e_{\mathbf{j}}^0\right)_{\lambda,n}^\circ .\end{aligned}$$

Thus,

$$\det\left(\left(\left(e_{\mathbf{i}}^0 v_\lambda^{+\mathfrak{g}^0}, e_{\mathbf{j}}^0 v_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda,n}^{\mathfrak{g}^0}\right)_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\ \mathbf{j}\in\operatorname{Seq}_+ E; \\ \deg\mathbf{i}=-n;\ \deg\mathbf{j}=n}}\right) = \det\left(\left(\left(e_{\mathbf{i}}^0, e_{\mathbf{j}}^0\right)_{\lambda,n}^\circ\right)_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\ \mathbf{j}\in\operatorname{Seq}_+ E; \\ \deg\mathbf{i}=-n;\ \deg\mathbf{j}=n}}\right).$$

Now,

$$\begin{aligned}\det\left((\cdot,\cdot)_{\lambda,n}^\circ\right) &= \det\left(\left(\left(e_{\mathbf{i}}^0, e_{\mathbf{j}}^0\right)_{\lambda,n}^\circ\right)_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\ \mathbf{j}\in\operatorname{Seq}_+ E; \\ \deg\mathbf{i}=-n;\ \deg\mathbf{j}=n}}\right) \\ &= \det\left(\left(\left(e_{\mathbf{i}}^0 v_\lambda^{+\mathfrak{g}^0}, e_{\mathbf{j}}^0 v_{-\lambda}^{-\mathfrak{g}^0}\right)_{\lambda,n}^{\mathfrak{g}^0}\right)_{\substack{\mathbf{i}\in\operatorname{Seq}_- E;\ \mathbf{j}\in\operatorname{Seq}_+ E; \\ \deg\mathbf{i}=-n;\ \deg\mathbf{j}=n}}\right) = \det\left((\cdot,\cdot)_{\lambda,n}^{\mathfrak{g}^0}\right).\end{aligned}$$

This proves Corollary 2.6.30.

2.6.11. Proof of Theorem 2.6.6: Joining the threads

Proof of Proposition 2.6.17. Consider the polynomial function $Q_n : \mathfrak{h}^* \times \mathbb{C} \to \mathbb{C}$ introduced in Corollary 2.6.27. Due to Corollary 2.6.27, every $\lambda \in V$ and every nonzero $\varepsilon \in \mathbb{C}$ satisfy

$$Q_n(\lambda, \varepsilon) = \varepsilon^{2\operatorname{LEN} n} Q_n\left(\lambda/\varepsilon^2, 1\right).$$

Hence, we can apply Lemma 2.6.28 to $V = \mathfrak{h}^*$, $\phi = Q_n$ and $k = \operatorname{LEN} n$. Thus, we obtain the following three observations:

Observation 1: The polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto Q_n(\lambda, 0)$$

is homogeneous of degree k. (This follows from Lemma 2.6.28 **(a)**.)

Observation 2: For every integer $N > k$, the N-th homogeneous component of the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto Q_n(\lambda, 1)$$

is zero. (This follows from Lemma 2.6.28 **(b)**.)

Observation 3: The k-th homogeneous component of the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto Q_n(\lambda, 1)$$

is the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto Q_n(\lambda, 0).$$

(This follows from Lemma 2.6.28 **(c)**.)

Since every $\lambda \in \mathfrak{h}^*$ satisfies

$$\begin{aligned} Q_n(\lambda, 1) &= \det\left((\cdot,\cdot)^{\mathfrak{g}^1}_{\lambda,n}\right) && \left(\begin{array}{c}\text{since (53) (applied to } \varepsilon = 1)\\ \text{yields } \det\left((\cdot,\cdot)^{\mathfrak{g}^1}_{\lambda,n}\right) = Q_n(\lambda, 1)\end{array}\right) \\ &= \det\left((\cdot,\cdot)_{\lambda,n}\right) && \left(\text{since } \mathfrak{g}^1 = \mathfrak{g} \text{ and thus } (\cdot,\cdot)^{\mathfrak{g}^1}_{\lambda,n} = (\cdot,\cdot)^{\mathfrak{g}}_{\lambda,n} = (\cdot,\cdot)_{\lambda,n}\right), \end{aligned}$$

the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto Q_n(\lambda, 1)$$

is the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)_{\lambda,n}\right).$$

This yields that

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)_{\lambda,n}\right)$$

is a polynomial function.

Since every $\lambda \in \mathfrak{h}^*$ satisfies

$$\begin{aligned} Q_n(\lambda, 0) &= \det\left((\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n}\right) && \left(\begin{array}{c}\text{since (53) (applied to } \varepsilon = 0)\\ \text{yields } \det\left((\cdot,\cdot)^{\mathfrak{g}^0}_{\lambda,n}\right) = Q_n(\lambda, 0)\end{array}\right) \\ &= \det\left((\cdot,\cdot)^{\circ}_{\lambda,n}\right) && (\text{by Corollary 2.6.30}), \end{aligned}$$

the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto Q_n(\lambda, 0)$$

is the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)^{\circ}_{\lambda,n}\right).$$

This yields that

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)^{\circ}_{\lambda,n}\right)$$

is a polynomial function. This polynomial function is not identically zero[67].

[67] *Proof.* Since $\mathfrak{g}$ is nondegenerate, there exists $\lambda \in \mathfrak{h}^*$ such that the bilinear form

$$\mathfrak{g}_{-k} \times \mathfrak{g}_k \to \mathbb{C}, \qquad (a, b) \mapsto \lambda([a, b])$$

is nondegenerate for every $k \in \{1, 2, ..., n\}$. For such λ, the form $(\cdot,\cdot)^{\circ}_{\lambda,n}$ must be nondegenerate (by Lemma 2.6.13), so that $\det\left((\cdot,\cdot)^{\circ}_{\lambda,n}\right) \neq 0$. Hence, there exists $\lambda \in \mathfrak{h}^*$ such that $\det\left((\cdot,\cdot)^{\circ}_{\lambda,n}\right) \neq 0$. In other words, the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)^{\circ}_{\lambda,n}\right)$$

is not identically zero, qed.

Since $Q_n(\lambda, 1) = \det\left((\cdot,\cdot)_{\lambda,n}\right)$ for every $\lambda \in \mathfrak{h}^*$, Observation 2 rewrites as follows:

Observation 2': For every integer $n > k$, the n-th homogeneous component of the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)_{\lambda,n}\right)$$

is zero.

Since $Q_n(\lambda, 1) = \det\left((\cdot,\cdot)_{\lambda,n}\right)$ and $Q_n(\lambda, 0) = \det\left((\cdot,\cdot)^\circ_{\lambda,n}\right)$ for every $\lambda \in \mathfrak{h}^*$, Observation 3 rewrites as follows:

Observation 3': The k-th homogeneous component of the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)_{\lambda,n}\right)$$

is the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)^\circ_{\lambda,n}\right).$$

Combining Observations 2' and 3' and the fact that the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)^\circ_{\lambda,n}\right)$$

is not identically zero, we conclude that the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)^\circ_{\lambda,n}\right)$$

is the leading term of the polynomial function

$$\mathfrak{h}^* \to \mathbb{C}, \qquad \lambda \mapsto \det\left((\cdot,\cdot)_{\lambda,n}\right).$$

This proves Proposition 2.6.17.

Now that Proposition 2.6.17 is proven, the proof of Theorem 2.6.6 is also complete (because we have already proven Theorem 2.6.6 using Proposition 2.6.17).

2.7. The irreducible quotients of the Verma modules

We will now use the form $(\cdot,\cdot)_\lambda$ to develop the representation theory of $\mathfrak{g}$. In the following, we assume that $\mathfrak{g}$ is nondegenerate.

Definition 2.7.1. Let $(\cdot,\cdot)$ denote the form $(\cdot,\cdot)_\lambda$. Let $J^\pm_\lambda$ be the kernel of $(\cdot,\cdot)$ on $M^\pm_\lambda$. This is a graded $\mathfrak{g}$-submodule of $M^\pm_\lambda$ (since the form $(\cdot,\cdot)$ is $\mathfrak{g}$-invariant). Let $L^\pm_\lambda$ be the quotient module $M^\pm_\lambda \diagup J^\pm_\lambda$. Then, $(\cdot,\cdot)$ descends to a nondegenerate pairing $L^+_\lambda \times L^-_{-\lambda} \to \mathbb{C}$.

Remark 2.7.2. For Weil-generic λ (away from a countable union of hypersurfaces), we have $J^\pm_\lambda = 0$ (by Theorem 2.6.6) and thus $L^\pm_\lambda = M^\pm_\lambda$.

Theorem 2.7.3. **(i)** The $\mathfrak{g}$-module $L_\lambda^\pm$ is irreducible.

(ii) The $\mathfrak{g}$-module $J_\lambda^\pm$ is the maximal proper graded submodule of $M_\lambda^\pm$. (This means that $J_\lambda^\pm$ contains all proper graded submodules in $M_\lambda^\pm$.)

(iii) Assume that there exists some $L \in \mathfrak{g}_0$ such that every $n \in \mathbb{Z}$ satisfies

$$(\operatorname{ad} L) \mid_{\mathfrak{g}_n} = n \cdot \operatorname{id} \mid_{\mathfrak{g}_n} .$$

(In this case it is said that *the grading on* $\mathfrak{g}$ *is internal*, i. e., comes from bracketing with some $L \in \mathfrak{g}_0$.) Then $J_\lambda^\pm$ is the maximal proper submodule of $M_\lambda^\pm$.

Remark 2.7.4. Here are two examples of cases when the grading on $\mathfrak{g}$ is internal:

(a) If $\mathfrak{g}$ is a simple finite-dimensional Lie algebra, then we know (from Proposition 2.5.6) that choosing a Cartan subalgebra $\mathfrak{h}$ and corresponding Chevalley generators e_1, e_2, ..., e_m, f_1, f_2, ..., f_m, h_1, h_2, ..., h_m of $\mathfrak{g}$ endows $\mathfrak{g}$ with a grading. This grading is internal. In fact, in this case, we can take $L = \rho^\vee$, where $\rho^\vee$ is defined as the element of $\mathfrak{h}$ satisfying $\alpha_i(\rho^\vee) = 1$ for all i (where α_i are the simple roots of $\mathfrak{g}$). Since the actions of the α_i on $\mathfrak{h}$ are a basis of $\mathfrak{h}^*$, this $\rho^\vee$ is well-defined and unique. (But it depends on the choice of $\mathfrak{h}$ and the Chevalley generators, of course.)

(b) If $\mathfrak{g} = \operatorname{Vir}$, then the grading on $\mathfrak{g}$ is internal. In fact, in this case, we can take $L = -L_0$.

On the other hand, if $\mathfrak{g}$ is the affine Kac-Moody algebra $\widehat{\mathfrak{g}}_\omega$ of Definition 1.7.6, then the grading on $\mathfrak{g}$ is not internal.

Proof of Theorem 2.7.3. **(i)** Let us show that L_λ^- is irreducible (the proof for L_λ^+ will be similar).

In fact, assume the contrary. Then, there exists a nonzero $w \in L_\lambda^-$ such that $U(\mathfrak{g}) \cdot w \neq L_\lambda^-$. Since L_λ^- is graded by *nonnegative* integers, we can choose w to have the smallest possible degree m (without necessarily being homogeneous). Clearly, $m > 0$. Thus we can write $w = w_0 + w_1 + ... + w_m$, where each w_i is homogeneous of degree $\deg w_i = i$ and $w_m \neq 0$.

Let $a \in \mathfrak{g}_j$ for some $j < 0$. Then $aw = 0$ (since $\deg(aw) < \deg w$, but still $U(\mathfrak{g}) \cdot aw \neq L_\lambda^-$ (since $U(\mathfrak{g}) \cdot aw \subseteq U(\mathfrak{g}) \cdot w$ and $U(\mathfrak{g}) \cdot w \neq L_\lambda^-$), and we have chosen w to have the smallest possible degree). By homogeneity, this yields $aw_m = 0$ (since aw_m is the $(m+j)$-th homogeneous component of aw).

For every $u \in L_{-\lambda}^+[-m-j]$, the term (au, w_m) is well-defined (since $au \in L_{-\lambda}^+$ and $w_m \in L_\lambda^-$). Since the form $(\cdot,\cdot)$ is $\mathfrak{g}$-invariant, it satisfies $(au, w_m) = -\left(u, \underbrace{aw_m}_{=0}\right) = 0$.

But since $m > 0$, we have $L_{-\lambda}^+[-m] = \sum\limits_{j<0} \mathfrak{g}_j \cdot L_{-\lambda}^+[-m-j]$ (because Proposition 2.5.15 **(a)** yields $M_{-\lambda}^+ = U(\mathfrak{n}_-) v_\lambda^+$, so that $L_{-\lambda}^+ = U(\mathfrak{n}_-) \overline{v_\lambda^+}$, thus

$$L_{-\lambda}^+[-m] = \underbrace{U(\mathfrak{n}_-)[-m]}_{=\sum\limits_{j<0}(\mathfrak{n}_-)[j]\cdot U(\mathfrak{n}_-)[-m-j]} \overline{v_\lambda^+} = \sum_{j<0} \underbrace{(\mathfrak{n}_-)[j]}_{=\mathfrak{g}[j]=\mathfrak{g}_j} \cdot \underbrace{U(\mathfrak{n}_-)[-m-j]\,\overline{v_\lambda^+}}_{\substack{=L_{-\lambda}^+[-m-j] \\ (\text{since } U(\mathfrak{n}_-)\overline{v_\lambda^+}=L_{-\lambda}^+)}} = \sum_{j<0} \mathfrak{g}_j \cdot L_{-\lambda}^+[-m-j]$$

). Hence, any element of $L_{-\lambda}^+[-m]$ is a linear combination of elements of the form au with $a \in \mathfrak{g}_j$ (for $j < 0$) and $u \in L_{-\lambda}^+[-m-j]$. Thus, since we know that $(au, w_m) = 0$

for every $a \in \mathfrak{g}_j$ and $u \in L^+_{-\lambda}[-m-j]$, we conclude that $\left(L^+_{-\lambda}[-m], w_m\right) = 0$. As a consequence, $\left(L^+_{-\lambda}, w_m\right) = 0$ (because the form $(\cdot,\cdot) : L^+_{-\lambda} \times L^-_\lambda \to \mathbb{C}$ is of degree 0, and thus $\left(L^+_{-\lambda}[j], w_m\right) = 0$ for all $j \neq -m$). Since the form $(\cdot,\cdot) : L^+_{-\lambda} \times L^-_\lambda \to \mathbb{C}$ is nondegenerate, this yields $w_m = 0$. This is a contradiction to $w_m \neq 0$. This contradiction shows that our assumption was wrong. Thus, L^-_λ is irreducible. Similarly, L^+_λ is irreducible.

(ii) First let us prove that the $\mathfrak{g}$-module J^+_λ is the maximal proper graded submodule of M^+_λ.

Let $K \subseteq M^+_\lambda$ be a proper graded submodule, and let $\overline{K}$ be its image in L^+_λ. Then, K lives in strictly negative degrees (because it is graded, so if it would have a component in degrees ≥ 0, it would contain v^+_λ and thus contain everything, and thus not be proper). Hence, $\overline{K}$ also lives in strictly negative degrees, and thus is proper. Hence, by **(i)**, we have $\overline{K} = 0$, thus $K \subseteq J^+_\lambda$. This shows that J^+_λ is the maximal proper graded submodule of M^+_λ. The proof of the corresponding statement for J^-_λ and M^-_λ is similar.

(iii) Assume that there exists some $L \in \mathfrak{g}_0$ such that every $n \in \mathbb{Z}$ satisfies

$$(\operatorname{ad} L)\mid_{\mathfrak{g}_n} = n \cdot \operatorname{id}\mid_{\mathfrak{g}_n}.$$

Consider this L. It is easy to prove (by induction) that $[L, a] = na$ for every $a \in U(\mathfrak{g})[n]$.

We are now going to show that all $\mathfrak{g}$-submodules of M^+_λ are automatically graded.

In fact, it is easy to see that $M^+_\lambda[n] \subseteq \operatorname{Ker}\left(L\mid_{M^+_\lambda} - (\lambda(L)+n)\operatorname{id}\right)$ for every $n \in \mathbb{Z}$. [68] In other words, for every $n \in \mathbb{Z}$, the n-th homogeneous component $M^+_\lambda[n]$ of M^+_λ is contained in the eigenspace of the operator $L\mid_{M^+_\lambda}$ for the eigenvalue $\lambda(L)+n$. Now,

$$M^+_\lambda = \bigoplus_{n\in\mathbb{Z}} M^+_\lambda[n] = \sum_{n\in\mathbb{Z}} \underbrace{M^+_\lambda[n]}_{\substack{\subseteq \operatorname{Ker}\left(L\mid_{M^+_\lambda} - (\lambda(L)+n)\operatorname{id}\right) \\ = \left(\text{eigenspace of the operator } L\mid_{M^+_\lambda} \text{ for the eigenvalue } \lambda(L)+n\right)}}$$
$$\subseteq \sum_{n\in\mathbb{Z}} \left(\text{eigenspace of the operator } L\mid_{M^+_\lambda} \text{ for the eigenvalue } \lambda(L)+n\right).$$

---

[68] *Proof.* Let $n \in \mathbb{Z}$. Let $a \in U(\mathfrak{n}_-)[n]$. Then, $a \in U(\mathfrak{g})[n]$, so that $[L, a] = na$ and thus $La = aL + \underbrace{[L,a]}_{=na} = aL + na$. Thus,

$$\left(L\mid_{M^+_\lambda}\right)\left(av^+_\lambda\right) = \underbrace{La}_{=aL+na} v^+_\lambda = (aL+na)\,v^+_\lambda = a \underbrace{Lv^+_\lambda}_{=\lambda(L)v^+_\lambda} + nav^+_\lambda = \lambda(L)\,av^+_\lambda + nav^+_\lambda$$
$$= (\lambda(L)+n)\,av^+_\lambda,$$

so that $av^+_\lambda \in \operatorname{Ker}\left(L\mid_{M^+_\lambda} - (\lambda(L)+n)\operatorname{id}\right)$. Forget that we fixed $a \in U(\mathfrak{n}_-)[n]$. Thus we have showed that every $a \in U(\mathfrak{n}_-)[n]$ satisfies $av^+_\lambda \in \operatorname{Ker}\left(L\mid_{M^+_\lambda} - (\lambda(L)+n)\operatorname{id}\right)$. In other words, $\left\{av^+_\lambda \mid a \in U(\mathfrak{n}_-)[n]\right\} \subseteq \operatorname{Ker}\left(L\mid_{M^+_\lambda} - (\lambda(L)+n)\operatorname{id}\right)$. Since $\left\{av^+_\lambda \mid a \in U(\mathfrak{n}_-)[n]\right\} = U(\mathfrak{n}_-)[n] \cdot v^+_\lambda = M^+_\lambda[n]$, this becomes $M^+_\lambda[n] \subseteq \operatorname{Ker}\left(L\mid_{M^+_\lambda} - (\lambda(L)+n)\operatorname{id}\right)$, qed.

Since all eigenspaces of $L\mid_{M_\lambda^+}$ are clearly contained in M_λ^+, this rewrites as

$$M_\lambda^+ = \sum_{n\in\mathbb{Z}} \left(\text{eigenspace of the operator } L\mid_{M_\lambda^+} \text{ for the eigenvalue } \lambda(L)+n\right).$$

Since eigenspaces of an operator corresponding to distinct eigenvalues are linearly disjoint, the sum $\sum\limits_{n\in\mathbb{Z}} \left(\text{eigenspace of the operator } L\mid_{M_\lambda^+} \text{ for the eigenvalue } \lambda(L)+n\right)$ must be a direct sum, so this becomes

$$M_\lambda^+ = \bigoplus_{n\in\mathbb{Z}} \left(\text{eigenspace of the operator } L\mid_{M_\lambda^+} \text{ for the eigenvalue } \lambda(L)+n\right). \qquad (70)$$

As a consequence of this, the map $L\mid_{M_\lambda^+}$ is diagonalizable, and all of its eigenvalues belong to the set $\{\lambda(L)+n \mid n\in\mathbb{Z}\}$.

So for every $n\in\mathbb{Z}$, we have the inclusion

$$\begin{aligned} M_\lambda^+[n] &\subseteq \operatorname{Ker}\left(L\mid_{M_\lambda^+} - (\lambda(L)+n)\operatorname{id}\right) \\ &= \left(\text{eigenspace of the operator } L\mid_{M_\lambda^+} \text{ for the eigenvalue } \lambda(L)+n\right), \end{aligned}$$

but the direct sum of these inclusions over all $n\in\mathbb{Z}$ is an equality (since

$$\bigoplus_{n\in\mathbb{Z}} M_\lambda^+[n] = M_\lambda^+ = \bigoplus_{n\in\mathbb{Z}} \left(\text{eigenspace of the operator } L\mid_{M_\lambda^+} \text{ for the eigenvalue } \lambda(L)+n\right)$$

by (70)). Hence, each of these inclusions must be an equality. In other words,

$$M_\lambda^+[n] = \left(\text{eigenspace of the operator } L\mid_{M_\lambda^+} \text{ for the eigenvalue } \lambda(L)+n\right) \qquad \text{for every } n\in\mathbb{Z}. \qquad (71)$$

Now, let K be a $\mathfrak{g}$-submodule of M_λ^+. Then, $L\mid_K$ is a restriction of $L\mid_{M_\lambda^+}$ to K. Hence, map $L\mid_K$ is diagonalizable, and all of its eigenvalues belong to the set $\{\lambda(L)+n \mid n\in\mathbb{Z}\}$ (because we know that the map $L\mid_{M_\lambda^+}$ is diagonalizable, and all of its eigenvalues belong to the set $\{\lambda(L)+n \mid n\in\mathbb{Z}\}$). In other words,

$$\begin{aligned} K &= \bigoplus_{n\in\mathbb{Z}} \underbrace{\left(\text{eigenspace of the operator } L\mid_K \text{ for the eigenvalue } \lambda(L)+n\right)}_{=K\cap\left(\text{eigenspace of the operator } L\mid_{M_\lambda^+} \text{ for the eigenvalue } \lambda(L)+n\right)} \\ &= \bigoplus_{n\in\mathbb{Z}} \left(K \cap \underbrace{\left(\text{eigenspace of the operator } L\mid_{M_\lambda^+} \text{ for the eigenvalue } \lambda(L)+n\right)}_{=M_\lambda^+[n]}\right) \\ &= \bigoplus_{n\in\mathbb{Z}} \left(K\cap M_\lambda^+[n]\right). \end{aligned}$$

Hence, K is graded. We thus have shown that every $\mathfrak{g}$-submodule of M_λ^+ is graded. Similarly, every $\mathfrak{g}$-submodule of M_λ^- is graded. Thus, Theorem 2.7.3 **(iii)** follows from Theorem 2.7.3 **(ii)**.

Remark 2.7.5. Theorem 2.7.3 **(ii)** does not hold if the word "graded" is removed. In fact, here is a counterexample: Let $\mathfrak{g}$ be the 3-dimensional Heisenberg algebra. (This is the Lie algebra with vector-space basis (x, K, y) and with Lie bracket given by $[y, x] = K$, $[x, K] = 0$ and $[y, K] = 0$. It can be considered as a Lie subalgebra of the oscillator algebra $\mathcal{A}$ defined in Definition 1.1.4.) It is easy to see that $\mathfrak{g}$ becomes a nondegenerate $\mathbb{Z}$-graded Lie algebra by setting $\mathfrak{g}_{-1} = \langle x \rangle$, $\mathfrak{g}_0 = \langle K \rangle$, $\mathfrak{g}_1 = \langle y \rangle$ and $\mathfrak{g}_i = 0$ for every $i \in \mathbb{Z} \diagdown \{-1, 0, 1\}$. Then, on the Verma highest-weight module $M_0^+ = \mathbb{C}[x] v_0^+$, both K and y act as 0 (and x acts as multiplication with x), so that Iv_0^+ is a $\mathfrak{g}$-submodule of M_0^+ for every ideal $I \subseteq \mathbb{C}[x]$, but not all of these ideals are graded, and not all of them are contained in J_0^+ (as can be easily checked).

Corollary 2.7.6. For Weil-generic λ (this means a λ outside of countably many hypersurfaces in $\mathfrak{h}^*$), the $\mathfrak{g}$-modules M_λ^+ and M_λ^- are irreducible.

Definition 2.7.7. Let Y be a $\mathfrak{g}$-module. A vector $w \in Y$ is called a *singular vector of weight* $\mu \in \mathfrak{h}^*$ (here, recall that $\mathfrak{h} = \mathfrak{g}_0$) if it satisfies

$$hw = \mu(h) w \qquad \text{for every } h \in \mathfrak{h}$$

and

$$aw = 0 \qquad \text{for every } a \in \mathfrak{g}_i \text{ for every } i > 0.$$

We denote by $\operatorname{Sing}_\mu(Y)$ the space of singular vectors of Y of weight μ.

When people talk about "singular vectors", they usually mean nonzero singular vectors in negative degrees. We are not going to adhere to this convention, though.

Lemma 2.7.8. Let Y be a $\mathfrak{g}$-module. Then there is a canonical isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}\left(M_\lambda^+, Y\right) &\to \operatorname{Sing}_\lambda Y, \\ \phi &\mapsto \phi\left(v_\lambda^+\right). \end{aligned}$$

Proof of Lemma 2.7.8. We have $M_\lambda^+ = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_\lambda = \operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}} \mathbb{C}_\lambda$, so that

$$\operatorname{Hom}_{\mathfrak{g}}\left(M_\lambda^+, Y\right) = \operatorname{Hom}_{\mathfrak{g}}\left(\operatorname{Ind}_{\mathfrak{h}\oplus\mathfrak{n}_+}^{\mathfrak{g}} \mathbb{C}_\lambda, Y\right) \cong \operatorname{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda, Y) \qquad \text{(by Frobenius reciprocity)}.$$

But $\operatorname{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda, Y) \cong \operatorname{Sing}_\lambda Y$ (because every $\mathbb{C}$-linear map $\mathbb{C}_\lambda \to Y$ is uniquely determined by the image of v_λ^+, and this map is a $(\mathfrak{h} \oplus \mathfrak{n}_+)$-module map if and only if this image is a singular vector of Y of weight λ). Thus, $\operatorname{Hom}_{\mathfrak{g}}\left(M_\lambda^+, Y\right) \cong \operatorname{Hom}_{\mathfrak{h}\oplus\mathfrak{n}_+}(\mathbb{C}_\lambda, Y) \cong \operatorname{Sing}_\lambda Y$. If we make this isomorphism explicit, we notice that it sends every ϕ to $\phi\left(v_\lambda^+\right)$, so that Lemma 2.7.8 is proven.

Corollary 2.7.9. The representation M_λ^+ is irreducible if and only if it does not have nonzero singular vectors in negative degrees. Here, a vector in M_λ^+ is said to be "in negative degrees" if its projection on the 0-th homogeneous component $M_\lambda^+[0]$ is zero.

Proof of Corollary 2.7.9. $\Longleftarrow$: Assume that M_λ^+ does not have nonzero singular vectors in negative degrees.

We must then show that M_λ^+ is irreducible.

In fact, assume the contrary. Then, M_λ^+ is not irreducible. Hence, there exists a nonzero *homogeneous* $v \in M_\lambda^+$ such that $U(\mathfrak{g}) \cdot v \neq M_\lambda^+$. [69] Consider this v. Then, $U(\mathfrak{g}) \cdot v$ is a proper graded submodule of M_λ^+, and thus is contained in J_λ^+. Hence, $J_\lambda^+ \neq 0$.

There exist some $d \in \mathbb{Z}$ such that $J_\lambda^+[d] \neq 0$ (since $J_\lambda^+ \neq 0$ and since J_λ^+ is graded). All such d are nonpositive (since J_λ^+ is nonpositively graded). Thus, there exists a highest integer d such that $J_\lambda^+[d] \neq 0$. Consider this d. Clearly, $d < 0$ (since the bilinear form $(\cdot,\cdot) : M_\lambda^+ \times M_{-\lambda}^-$ is obviously nondegenerate on $M_\lambda^+[0] \times M_{-\lambda}^-[0]$, so that $J_\lambda^+[0] = 0$).

Every $i > 0$ satisfies

$$\begin{aligned}\mathfrak{g}_i \cdot \left(J_\lambda^+[d]\right) &\subseteq J_\lambda^+[i+d] && \left(\text{since } J_\lambda^+ \text{ is a graded } \mathfrak{g}\text{-module}\right)\\ &= 0 && \left(\text{since } i+d > d, \text{ but } d \text{ was the highest integer such that } J_\lambda^+[d] \neq 0\right).\end{aligned}$$

By Conditions **(1)** and **(2)** of Definition 2.5.4, the Lie algebra $\mathfrak{g}_0$ is abelian and finite-dimensional. Hence, every nonzero $\mathfrak{g}_0$-module has a one-dimensional submodule[70].

[69] *Proof.* Notice that M_λ^+ is a graded $U(\mathfrak{g})$-module (since M_λ^+ is a graded $\mathfrak{g}$-module).

Since M_λ^+ is not irreducible, there exists a nonzero $w \in M_\lambda^+$ such that $U(\mathfrak{g}) \cdot w \neq M_\lambda^+$. Since M_λ^+ is graded by *nonpositive* integers, we can write w in the form $w = \sum\limits_{j=0}^{m} w_j$, where each w_i is homogeneous of degree $\deg w_i = -i$ and $m \in \mathbb{Z}$. Now,

$$\underbrace{U(\mathfrak{g})}_{=\sum\limits_{i\in\mathbb{Z}} U(\mathfrak{g})[i]} \cdot \underbrace{w}_{=\sum\limits_{j=0}^{m} w_j} = \left(\sum_{i\in\mathbb{Z}} U(\mathfrak{g})[i]\right) \cdot \left(\sum_{j=0}^{m} w_j\right) = \sum_{i\in\mathbb{Z}} \sum_{j=0}^{m} U(\mathfrak{g})[i] \cdot w_j.$$

Hence, for every $n \in \mathbb{Z}$, we have

$$(U(\mathfrak{g}) \cdot w)[n] = \left(\sum_{i\in\mathbb{Z}} \sum_{j=0}^{m} U(\mathfrak{g})[i] \cdot w_j\right)[n] = \sum_{j=0}^{m} \underbrace{\left(\sum_{i\in\mathbb{Z}} U(\mathfrak{g})[i] \cdot w_j\right)}_{\substack{\subseteq U(\mathfrak{g})[i-j] \\ (\text{since } \deg w_j = -j \text{ and since} \\ M_\lambda^+ \text{ is a graded } U(\mathfrak{g})\text{-module})}} [n]$$

$$= \sum_{j=0}^{m} U(\mathfrak{g})[n+j] \cdot w_j.$$

Now, since $U(\mathfrak{g}) \cdot w \neq M_\lambda^+$, there exists at least one $n \in \mathbb{Z}$ such that $(U(\mathfrak{g}) \cdot w)[n] \neq M_\lambda^+[n]$. Consider such an n. Then, $M_\lambda^+[n] \neq (U(\mathfrak{g}) \cdot w)[n] = \sum\limits_{j=0}^{m} U(\mathfrak{g})[n+j] \cdot w_j$. Thus, $U(\mathfrak{g})[n+j] \cdot w_j \neq M_\lambda^+[n]$ for all $j \in \{0, 1, ..., m\}$. But some $j \in \{0, 1, ..., m\}$ satisfies $w_j \neq 0$ (since $\sum\limits_{j=0}^{m} w_j = w \neq 0$). Consider this j. Then, w_j is a nonzero homogeneous element of M_λ^+ satisfying $U(\mathfrak{g}) \cdot w_j \neq M_\lambda^+$ (because $(U(\mathfrak{g}) \cdot w_j)[n] = U(\mathfrak{g})[n+j] \cdot w_j \neq M_\lambda^+[n]$). This proves that there exists a nonzero *homogeneous* $v \in M_\lambda^+$ such that $U(\mathfrak{g}) \cdot v \neq M_\lambda^+$. Qed.

[70] *Proof.* This is because of the following fact:

Thus, the nonzero $\mathfrak{g}_0$-module $J_\lambda^+[d]$ has a one-dimensional submodule. Let w be the generator of this submodule. Then, this submodule is $\langle w \rangle$.

For every $h \in \mathfrak{h}$, the vector hw is a scalar multiple of w (since $h \in \mathfrak{h} = \mathfrak{g}_0$, so that hw lies in the $\mathfrak{g}_0$-submodule of $J_\lambda^+[d]$ generated by w, but this submodule is $\langle w \rangle$). Thus, we can write $hw = \lambda_h w$ for some $\lambda_h \in \mathbb{C}$. This λ_h is uniquely determined (since $w \neq 0$), so we can define a map $\mu : \mathfrak{h} \to \mathbb{C}$ such that $\mu(h) = \lambda_h$ for every $h \in \mathfrak{h}$. This map μ is easily seen to be $\mathbb{C}$-linear, so that we have found a $\mu \in \mathfrak{h}^*$ such that

$$hw = \mu(h) w \qquad \text{for every } h \in \mathfrak{h}.$$

Also,

$$aw = 0 \qquad \text{for every } a \in \mathfrak{g}_i \text{ for every } i > 0$$

(since $\underbrace{a}_{\in \mathfrak{g}_i} \underbrace{w}_{\in J_\lambda^+[d]} \in \mathfrak{g}_i \cdot \left(J_\lambda^+[d]\right) \subseteq 0$). Thus, w is a nonzero singular vector. Since $w \in J_\lambda^+[d]$ and $d < 0$, this vector w is in negative degrees. This contradicts to the assumption that M_λ^+ does not have nonzero singular vectors in negative degrees. This contradiction shows that our assumption was wrong, so that M_λ^+ is irreducible. This proves the $\Longleftarrow$ direction of Corollary 2.7.9.

$\Longrightarrow$: Assume that M_λ^+ is irreducible.

We must then show that M_λ^+ does not have nonzero singular vectors in negative degrees.

Let v be a singular vector of M_λ^+ in negative degrees. Let it be a singular vector of weight μ for some $\mu \in \mathfrak{h}^*$.

By Lemma 2.7.8 (applied to μ and M_λ^+ instead of λ and Y), we have an isomorphism

$$\operatorname{Hom}_{\mathfrak{g}}\left(M_\mu^+, M_\lambda^+\right) \to \operatorname{Sing}_\mu\left(M_\lambda^+\right),$$
$$\phi \mapsto \phi\left(v_\mu^+\right).$$

Let ϕ be the preimage of v under this isomorphism. Then, $v = \phi\left(v_\mu^+\right)$.

Since v is in negative degrees, we have $v \in \sum\limits_{n<0} M_\lambda^+[n]$. Now, $M_\mu^+ = U(\mathfrak{n}_-) v_\mu^+ = \sum\limits_{m \leq 0} U(\mathfrak{n}_-)[m] v_\mu^+$ (since M_μ^+ is nonpositively graded), so that

$$\begin{aligned}
\phi\left(M_\mu^+\right) &= \phi\left(\sum_{m\leq 0} U(\mathfrak{n}_-)[m] v_\mu^+\right) = \sum_{m \leq 0} U(\mathfrak{n}_-)[m] \underbrace{\phi\left(v_\mu^+\right)}_{= v \in \sum\limits_{n<0} M_\lambda^+[n]} \qquad \left(\text{since } \phi \in \operatorname{Hom}_{\mathfrak{g}}\left(M_\mu^+, M_\lambda^+\right)\right) \\
&\in \sum_{m\leq 0} U(\mathfrak{n}_-)[m] \sum_{n<0} M_\lambda^+[n] = \sum_{m\leq 0}\sum_{n<0} \underbrace{U(\mathfrak{n}_-)[m] \cdot M_\lambda^+[n]}_{\substack{\subseteq M_\lambda^+[m+n] \\ (\text{since } M_\lambda^+ \text{ is a graded } \mathfrak{g}\text{-module})}} \\
&\subseteq \sum_{m\leq 0}\sum_{n<0} M_\lambda^+[m+n] \subseteq \sum_{r<0} M_\lambda^+[r].
\end{aligned}$$

Every nonzero finite-dimensional module over an abelian finite-dimensional Lie algebra has a one-dimensional submodule. (This is just a restatement of the fact that a finite set of pairwise commuting matrices on a finite-dimensional nonzero $\mathbb{C}$-vector space has a common nonzero eigenvector.)

Thus, the projection of $\phi\left(M_{\mu}^{+}\right)$ onto the 0-th degree of M_{λ}^{+} is 0. Hence, $\phi\left(M_{\mu}^{+}\right)$ is a proper $\mathfrak{g}$-submodule of M_{λ}^{+}. Therefore, $\phi\left(M_{\mu}^{+}\right) = 0$ (since M_{λ}^{+} is irreducible). Thus, $v = \phi\left(v_{\mu}^{+}\right) \in \phi\left(M_{\mu}^{+}\right) = 0$, so that $v = 0$.

We have thus proven: Whenever v is a singular vector of M_{λ}^{+} in negative degrees, we have $v = 0$. In other words, M_{λ}^{+} does not have nonzero singular vectors in negative degrees. This proves the $\Longrightarrow$ direction of Corollary 2.7.9.

Here is a variation on Corollary 2.7.9:

Corollary 2.7.10. The representation M_{λ}^{+} is irreducible if and only if it does not have nonzero homogeneous singular vectors in negative degrees.

Proof of Corollary 2.7.10. $\Longrightarrow$: This follows from the $\Longrightarrow$ direction of Corollary 2.7.9.

$\Longleftarrow$: Repeat the proof of the $\Longleftarrow$ direction of Corollary 2.7.9, noticing that w is homogeneous (since $w \in J_{\lambda}^{+}[d]$).

Corollary 2.7.10 is thus proven.

2.8. Highest/lowest-weight modules

Definition 2.8.1. A *highest-weight module* with highest weight $\lambda \in \mathfrak{h}^*$ means a quotient V of the graded $\mathfrak{g}$-module M_{λ}^{+} by a proper graded submodule. The projection of $v_{\lambda}^{+} \in M_{\lambda}^{+}$ onto this quotient will be called a *highest-weight vector* of V. (Note that a highest-weight module may have several highest-weight vectors: in fact, every nonzero vector in its 0-th homogeneous component is a highest-weight vector.) The notion "highest-weight representation" is also used as a synonym for "highest-weight module".

A *lowest-weight module* with lowest weight $\lambda \in \mathfrak{h}^*$ means a quotient V of the graded $\mathfrak{g}$-module M_{λ}^{-} by a proper graded submodule. The projection of $v_{\lambda}^{-} \in M_{\lambda}^{-}$ onto this quotient will be called a *lowest-weight vector* of V. (Note that a lowest-weight module may have several lowest-weight vectors: in fact, every nonzero vector in its 0-th homogeneous component is a lowest-weight vector.) The notion "lowest-weight representation" is also used as a synonym for "lowest-weight module".

If Y is a highest-weight module with highest weight λ, then we have an exact sequence $M_{\lambda}^{+} \twoheadrightarrow Y \twoheadrightarrow L_{\lambda}^{+}$ (by Theorem 2.7.3 **(ii)**).

If Y is a lowest-weight module with lowest weight λ, then we have an exact sequence $M_{\lambda}^{-} \twoheadrightarrow Y \twoheadrightarrow L_{\lambda}^{-}$ (by Theorem 2.7.3 **(ii)**).

2.9. Categories $\mathcal{O}^{+}$ and $\mathcal{O}^{-}$

The category of all $\mathfrak{g}$-modules for a graded Lie algebra is normally not particularly well-behaved: modules can be too big. One could restrict one's attention to finite-dimensional modules, but this is often too much of a sacrifice (e. g., the Heisenberg algebra $\mathcal{A}$ has no finite-dimensional modules which are not direct sums of 1-dimensional ones). A balance between nontriviality and tamability is achieved by considering the so-called *Category* $\mathcal{O}$. Actually, there are two of these categories, $\mathcal{O}^{+}$ and $\mathcal{O}^{-}$, which are antiequivalent to each other (in general) and equivalent to each other (in some more restrictive cases). There are several definitions for each of these categories, and some of them are not even equivalent to each other, although they mostly differ in minor technicalities. Here are the definitions that we are going to use:

Definition 2.9.1. The objects of *category* $\mathcal{O}^+$ will be $\mathbb{C}$-graded $\mathfrak{g}$-modules M such that:

(1) all degrees lie in a halfplane $\operatorname{Re} z < a$ and fall into finitely many arithmetic progressions with step 1;

(2) for every $d \in \mathbb{C}$, the space $M[d]$ is finite-dimensional.

The *morphisms of category* $\mathcal{O}^+$ will be graded $\mathfrak{g}$-module homomorphisms.

Definition 2.9.2. The objects of *category* $\mathcal{O}^-$ will be $\mathbb{C}$-graded $\mathfrak{g}$-modules M such that:

(1) all degrees lie in a halfplane $\operatorname{Re} z > a$ and fall into finitely many arithmetic progressions with step 1;

(2) for every $d \in \mathbb{C}$, the space $M[d]$ is finite-dimensional.

The *morphisms of category* $\mathcal{O}^-$ will be graded $\mathfrak{g}$-module homomorphisms.

It is rather clear that for a nondegenerate $\mathbb{Z}$-graded Lie algebra (or, more generally, for a $\mathbb{Z}$-graded Lie algebra satisfying conditions **(1)** and **(2)** of Definition 2.5.4), the Verma highest-weight module M_λ^+ lies in category $\mathcal{O}^+$ for every $\lambda \in \mathfrak{h}^*$, and the Verma lowest-weight module M_λ^- lies in category $\mathcal{O}^-$ for every $\lambda \in \mathfrak{h}^*$.

Definition 2.9.3. Let V and W be two $\mathbb{C}$-graded vector spaces, and $x \in \mathbb{C}$. A map $f : V \to W$ is said to be *homogeneous of degree* x if and only if every $z \in \mathbb{C}$ satisfies $f(V[z]) \subseteq W[z+x]$. (For example, this yields that a map is homogeneous of degree 0 if and only if it is graded.)

Proposition 2.9.4. The irreducible modules in category $\mathcal{O}^\pm$ (up to homogeneous isomorphism) are $L_\lambda^\pm$ for varying $\lambda \in \mathbb{C}$.

Proof of Proposition 2.9.4. First of all, for every $\lambda \in \mathfrak{h}^*$, the $\mathfrak{g}$-module L_λ^+ has a unique singular vector (up to scaling), and this vector is a singular vector of weight λ. [71] Thus, the $\mathfrak{g}$-modules L_λ^+ are pairwise nonisomorphic for varying λ. Similarly, the $\mathfrak{g}$-modules L_λ^- are pairwise nonisomorphic for varying λ.

Let Y be any irreducible module in category $\mathcal{O}^+$. We are now going to prove that $Y \cong L_\lambda^+$ for some $\lambda \in \mathfrak{h}^*$.

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[71] *Proof.* It is clear that $\overline{v_\lambda^+} \in L_\lambda^+$ is a singular vector of weight λ. Now we must prove that it is the only singular vector (up to scaling).

In fact, assume the opposite. Then, there exists a singular vector in L_λ^+ which is not a scalar multiple of $\overline{v_\lambda^+}$. This singular vector must have a nonzero d-th homogeneous component for some $d < 0$ (because it is not a scalar multiple of $\overline{v_\lambda^+}$), and this component itself must be a singular vector (since any homogeneous component of a singular vector must itself be a singular vector). So the module L_λ^+ has a nonzero homogeneous singular vector w of degree d.

Now, repeat the proof of the $\Longrightarrow$ part of Corollary 2.7.9, with M_λ^+ replaced by L_λ^+ (using the fact that L_λ^+ is irreducible). As a consequence, it follows that L_λ^+ does not have nonzero singular vectors in negative degrees. This contradicts the fact that the module L_λ^+ has a nonzero homogeneous singular vector w of degree $d < 0$. This contradiction shows that our assumption was wrong, so that indeed, $\overline{v_\lambda^+}$ is the only singular vector of L_λ^+ (up to scaling), qed.

Let d be a complex number such that $Y[d] \neq 0$ and $Y[d+j] = 0$ for all $j \geq 1$. (Such a complex number exists due to condition **(1)** in Definition 2.9.1.) For every $v \in Y[d]$, we have $av = 0$ for every $a \in \mathfrak{g}_i$ for every $i > 0$ [72].

By Conditions **(1)** and **(2)** of Definition 2.5.4, the Lie algebra $\mathfrak{g}_0$ is abelian and finite-dimensional. Hence, every nonzero $\mathfrak{g}_0$-module has a one-dimensional submodule[73]. Thus, the nonzero $\mathfrak{g}_0$-module $Y[d]$ has a one-dimensional submodule. Let w be the generator of this submodule. Then, this submodule is $\langle w \rangle$.

For every $h \in \mathfrak{h}$, the vector hw is a scalar multiple of w (since $h \in \mathfrak{h} = \mathfrak{g}_0$, so that hw lies in the $\mathfrak{g}_0$-submodule of $Y[d]$ generated by w, but this submodule is $\langle w \rangle$). Thus, we can write $hw = \lambda_h w$ for some $\lambda_h \in \mathbb{C}$. This λ_h is uniquely determined by h (since $w \neq 0$), so we can define a map $\lambda : \mathfrak{h} \to \mathbb{C}$ such that $\lambda(h) = \lambda_h$ for every $h \in \mathfrak{h}$. This map λ is easily seen to be $\mathbb{C}$-linear, so that we have found a $\lambda \in \mathfrak{h}^*$ such that

$$hw = \lambda(h) w \qquad \text{for every } h \in \mathfrak{h}.$$

Also,

$$aw = 0 \qquad \text{for every } a \in \mathfrak{g}_i \text{ for every } i > 0$$

(since $av = 0$ for every $v \in Y[d]$ and every $a \in \mathfrak{g}_i$ for every $i > 0$). Thus, w is a nonzero singular vector of weight λ.

By Lemma 2.7.8, we have an isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathfrak{g}}\left(M_\lambda^+, Y\right) &\to \operatorname{Sing}_\lambda Y, \\ \phi &\mapsto \phi\left(v_\lambda^+\right). \end{aligned}$$

Let ϕ be the preimage of w under this isomorphism. Then, $w = \phi\left(v_\lambda^+\right)$. Since $w \in Y[d]$, it is easy to see that ϕ is a homogeneous homomorphism of degree d (in fact, every $n \in \mathbb{Z}$ satisfies $M_\lambda^+[n] = U(\mathfrak{n}_-)[n] \cdot v_\lambda^+$, so that

$$\begin{aligned} \phi\left(M_\lambda^+[n]\right) &= \phi\left(U(\mathfrak{n}_-)[n] \cdot v_\lambda^+\right) = U(\mathfrak{n}_-)[n] \cdot \underbrace{\phi\left(v_\lambda^+\right)}_{= w \in Y[d]} \qquad (\text{since } \phi \text{ is } \mathfrak{g}\text{-linear}) \\ &\subseteq U(\mathfrak{n}_-)[n] \cdot Y[d] \subseteq Y[n+d] \end{aligned}$$

). This homomorphism ϕ must be surjective, since Y is irreducible. Thus, we have a homogeneous isomorphism $M_\lambda^+ \diagup (\operatorname{Ker}\phi) \cong Y$. Also, $\operatorname{Ker}\phi$ is a proper graded submodule of M_λ^+, thus a submodule of J_λ^+ (by Theorem 2.7.3 **(ii)**). Hence, we have a projection $M_\lambda^+ \diagup (\operatorname{Ker}\phi) \to M_\lambda^+ \diagup J_\lambda^+$. Since $M_\lambda^+ \diagup (\operatorname{Ker}\phi) \cong Y$ is irreducible, this projection must either be an isomorphism or the zero map. It cannot be the zero map (since it is a projection onto the nonzero module $M_\lambda^+ \diagup J_\lambda^+$), so it therefore is an isomorphism. Thus, $M_\lambda^+ \diagup J_\lambda^+ \cong M_\lambda^+ \diagup (\operatorname{Ker}\phi) \cong Y$, so we have a homogeneous isomorphism $Y \cong M_\lambda^+ \diagup J_\lambda^+ = L_\lambda^+$.

We thus have showed that any irreducible module in category $\mathcal{O}^+$ is isomorphic to L_λ^+ for some $\lambda \in \mathfrak{h}^*$. Similarly, the analogous assertion holds for $\mathcal{O}^-$. Proposition 2.9.4 is thus proven.

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[72] *Proof.* Let $i > 0$ and $a \in \mathfrak{g}_i$. Then, $i \geq 1$. Now, $a \in \mathfrak{g}_i$ and $v \in Y[d]$ yield $av \in \mathfrak{g}_i \cdot Y[d] \subseteq Y[d+i] = 0$ (since $Y[d+j] = 0$ for all $j \geq 1$), so that $av = 0$, qed.

[73] *Proof.* This is because of the following fact:

Every nonzero finite-dimensional module over an abelian finite-dimensional Lie algebra has a one-dimensional submodule. (This is just a restatement of the fact that a finite set of pairwise commuting matrices on a finite-dimensional nonzero $\mathbb{C}$-vector space has a common nonzero eigenvector.)

Definition 2.9.5. Let M be a module in category $\mathcal{O}^+$. We define the *character* $\operatorname{ch} M$ of M as follows:

Write $M = \bigoplus_d M[d]$. Then, define $\operatorname{ch} M$ by

$$\operatorname{ch} M = \sum_d q^{-d} \operatorname{tr}_{M[d]}(e^x) \qquad \text{as a power series in } q$$

for every $x \in \mathfrak{h}$. We also write $(\operatorname{ch} M)(q, x)$ for this, so it becomes a formal power series in both q and x. (Note that this power series can contain noninteger powers of q, but due to $M \in \mathcal{O}^+$, the exponents in these powers are bounded from above in their real part, and fall into infinitely many arithmetic progressions with step 1.)

Proposition 2.9.6. Here is an example:

$$\left(\operatorname{ch} M_\lambda^+\right)(x) = \frac{1}{\prod\limits_{j>0} \det_{\mathfrak{g}[-j]}\left(1 - q^j e^{\operatorname{ad}(x)}\right)}.$$

(To prove this, use Molien's identity which states that, for every linear map $A : V \to V$, we have

$$\sum_{n \in \mathbb{N}} q^n \operatorname{Tr}_{S^n V}(S^n A) = \frac{1}{\det(1 - qA)},$$

where $S^n A$ denotes the n-th symmetric power of the operator A.)

Let us consider some examples:

Example 2.9.7. Let $\mathfrak{g} = \mathfrak{sl}_2$. We can write this Lie algebra in terms of Chevalley generators and their relations (this is a particular case of what we did in Proposition 2.5.6). The most traditional way to do this is by setting $e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; then, $\mathfrak{g}$ is generated by e, f and h as a Lie algebra, and these generators satisfy $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. Also, (e, f, h) is a basis of the vector space $\mathfrak{g}$. In accordance with Proposition 2.5.6, we grade $\mathfrak{g}$ by setting $\deg e = 1$, $\deg f = -1$ and $\deg h = 0$. Then, $\mathfrak{n}_+ = \langle e \rangle$, $\mathfrak{n}_- = \langle f \rangle$ and $\mathfrak{h} = \langle h \rangle$. Hence, linear maps $\lambda : \mathfrak{h} \to \mathbb{C}$ are in 1-to-1 correspondence with complex numbers (namely, the images $\lambda(h)$ of h under these maps). Thus, we can identify any linear map $\lambda : \mathfrak{h} \to \mathbb{C}$ with the image $\lambda(h) \in \mathbb{C}$.

Consider any $\lambda \in \mathfrak{h}^*$. Since $\mathfrak{n}_- = \langle f \rangle$, the universal enveloping algebra $U(\mathfrak{n}_-)$ is the polynomial algebra $\mathbb{C}[f]$, and Proposition 2.5.15 **(a)** yields $M_\lambda^+ = \underbrace{U(\mathfrak{n}_-)}_{=\mathbb{C}[f]} v_\lambda^+ = \mathbb{C}[f] v_\lambda^+$. Similarly, $M_{-\lambda}^- = \mathbb{C}[e] v_{-\lambda}^-$. In order to compute the bilinear form $(\cdot, \cdot)$ on $M_\lambda^+ \times M_{-\lambda}^-$, it is thus enough to compute $\left(f^n v_\lambda^+, e^n v_{-\lambda}^-\right)$ for all $n \in \mathbb{N}$. (The values $\left(f^n v_\lambda^+, e^m v_{-\lambda}^-\right)$ for $n \neq m$ are zero since the form has degree 0.) In order to do this, we notice that

$e^n f^n v_\lambda^+ = n!\lambda(\lambda-1)\dots(\lambda-n+1)v_\lambda^+$ [74] and thus

$$\begin{aligned}\left(f^n v_\lambda^+, e^n v_{-\lambda}^-\right) &= \left(\underbrace{S(e^n)}_{=(-1)^n e^n} f^n v_\lambda^+, v_{-\lambda}^-\right) = \left((-1)^n \underbrace{e^n f^n v_\lambda^+}_{=n!\lambda(\lambda-1)\dots(\lambda-n+1)v_\lambda^+}, v_{-\lambda}^-\right) \\ &= \left((-1)^n n!\lambda(\lambda-1)\dots(\lambda-n+1)v_\lambda^+, v_{-\lambda}^-\right) \\ &= (-1)^n n!\lambda(\lambda-1)\dots(\lambda-n+1)\underbrace{\left(v_\lambda^+, v_{-\lambda}^-\right)}_{=1} \\ &= (-1)^n n!\lambda(\lambda-1)\dots(\lambda-n+1).\end{aligned} \tag{72}$$

So M_λ^+ is irreducible if $\lambda \notin \mathbb{Z}_+$. If $\lambda \in \mathbb{Z}_+$, then $J_\lambda^+ = \left\langle f^n v_\lambda^+ \mid n \geq \lambda+1\right\rangle = \mathbb{C}[f]\cdot\left(f^{\lambda+1}v_\lambda^+\right)$, and the irreducible $\mathfrak{g}$-module $L_\lambda^+ = \left\langle \overline{v_\lambda^+}, \overline{fv_\lambda^+}, ..., \overline{f^\lambda v_\lambda^+}\right\rangle$ has dimension $\dim \lambda + 1$. [75]

Example 2.9.8. Let $\mathfrak{g} = \mathrm{Vir}$. With the grading that we have defined on Vir, we have $\mathfrak{h} = \mathfrak{g}_0 = \langle L_0, C\rangle$. Thus, linear maps $\lambda : \mathfrak{h} \to \mathbb{C}$ can be uniquely described by the images of L_0 and C under these maps. We thus identify every linear map $\lambda : \mathfrak{h} \to \mathbb{C}$ with the pair $(\lambda(L_0), \lambda(C))$.

For every $\lambda = (\lambda(L_0), \lambda(C))$, the number $\lambda(L_0)$ is denoted by h and called the *conformal weight* of λ, and the number $\lambda(C)$ is denoted by c and called the *central charge* of λ. Thus, λ is identified with the pair (h, c). As a consequence, the Verma modules M_λ^+ and M_λ^- are often denoted by $M_{h,c}^+$ and $M_{h,c}^-$, respectively, and the modules L_λ^+ and L_λ^- are often denoted by $L_{h,c}^+$ and $L_{h,c}^-$, respectively.

(Note, of course, that the central charge of λ is the central charge of each of the Vir-modules M_λ^+, M_λ^-, L_λ^+ and L_λ^-.)

Consider any $\lambda \in \mathfrak{h}^*$. Let us compute the bilinear form $(\cdot,\cdot)$ on $M_\lambda^+ \times M_{-\lambda}^-$. Note first that $L_0 v_\lambda^+ = \underbrace{\lambda(L_0)}_{=h} v_\lambda^+ = hv_\lambda^+$ and $Cv_\lambda^+ = \underbrace{\lambda(C)}_{=c} v_\lambda^+ = cv_\lambda^+$.

In order to compute $\left(L_{-1}v_\lambda^+, L_1 v_{-\lambda}^-\right)$, we notice that

$$\underbrace{L_1 L_{-1}}_{=L_{-1}L_1+[L_1,L_{-1}]} v_\lambda^+ = L_{-1}\underbrace{L_1 v_\lambda^+}_{=0} + \underbrace{[L_1, L_{-1}]}_{=2L_0} v_\lambda^+ = 2\underbrace{L_0 v_\lambda^+}_{=hv_\lambda^+} = 2hv_\lambda^+,$$

[74] *Proof.* Here is a sketch of the proof. (If you want to see it in details, read the proof of Lemma 4.6.1 **(a)** below; this lemma yields the equality $e^n f^n v_\lambda^+ = n!\lambda(\lambda-1)\dots(\lambda-n+1)v_\lambda^+$ by substituting $x = v_\lambda^+$.)

First show that $hf^m v_\lambda^+ = (\lambda - 2m) f^m v_\lambda^+$ for every $m \in \mathbb{N}$. (This follows easily by induction over m, using $hf - fh = [h, f] = -2f$.)

Next show that $ef^n v_\lambda^+ = n(\lambda - n + 1) f^{n-1} v_\lambda^+$ for every positive $n \in \mathbb{N}$. (This is again an easy induction proof using the equalities $ef - fe = [e, f] = h$, $hv_\lambda^+ = \underbrace{\lambda(h)}_{=\lambda} v_\lambda^+ = \lambda v_\lambda^+$ and $ev_\lambda^+ = 0$, and using the equality $hf^m v_\lambda^+ = (\lambda - 2m) f^m v_\lambda^+$ applied to $m = n - 1$.)

Now show that $e^n f^n v_\lambda^+ = n!\lambda(\lambda-1)\dots(\lambda-n+1)v_\lambda^+$ for every $n \in \mathbb{N}$. (For this, again use induction.)

[75] If you know the representation theory of $\mathfrak{sl}_2$, you probably recognize this module L_λ^+ as the $(\dim\lambda)$-th symmetric power of the vector module $\mathbb{C}^2$ (as there is only one irreducible $\mathfrak{sl}_2$-module of every dimension).

so that

$$\left(L_{-1}v_\lambda^+, L_1 v_{-\lambda}^-\right) = \left(-\underbrace{L_1 L_{-1} v_\lambda^+}_{=2hv_\lambda^+}, v_{-\lambda}^-\right) = \left(-2hv_\lambda^+, v_{-\lambda}^-\right) = -2h\underbrace{\left(v_\lambda^+, v_{-\lambda}^-\right)}_{=1} = -2h.$$

Since $\left(L_{-1}v_\lambda^+\right)$ is a basis of $M_\lambda^+[-1]$ and $\left(L_1 v_{-\lambda}^-\right)$ is a basis of $M_{-\lambda}^-[1]$, this yields $\det\left((\cdot,\cdot)_1\right) = 2h$ (where $(\cdot,\cdot)_1$ denotes the restriction of the form $(\cdot,\cdot)$ to $M_\lambda^+[-1] \times M_{-\lambda}^-[1]$). This vanishes for $h = 0$.

In degree 2, the form is somewhat more complicated: With respect to the basis $\left(L_{-1}^2 v_\lambda^+, L_{-2}v_\lambda^+\right)$ of $M_\lambda^+[-2]$, and the basis $\left(L_1^2 v_{-\lambda}^-, L_2 v_{-\lambda}^-\right)$ of $M_{-\lambda}^-[2]$, the restriction $(\cdot,\cdot)_2$ of the form $(\cdot,\cdot)$ to $M_\lambda^+[-2] \times M_{-\lambda}^-[2]$ is given by the matrix

$$\begin{pmatrix} \left(L_{-1}^2 v_\lambda^+, L_1^2 v_{-\lambda}^-\right) & \left(L_{-1}^2 v_\lambda^+, L_2 v_{-\lambda}^-\right) \\ \left(L_{-2}v_\lambda^+, L_1^2 v_{-\lambda}^-\right) & \left(L_{-2}v_\lambda^+, L_2 v_{-\lambda}^-\right) \end{pmatrix}.$$

Let us compute, as an example, the lower right entry of this matrix, that is, the entry $\left(L_{-2}v_\lambda^+, L_2 v_{-\lambda}^-\right)$. We have

$$\begin{aligned}\underbrace{L_2 L_{-2}}_{=L_{-2}L_2+[L_2,L_{-2}]} v_\lambda^+ &= L_{-2}\underbrace{L_2 v_\lambda^+}_{=0} + \underbrace{[L_2, L_{-2}]}_{=4L_0+\frac{1}{2}C} v_\lambda^+ = \left(4L_0 + \frac{1}{2}C\right) v_\lambda^+ = 4\underbrace{L_0 v_\lambda^+}_{=hv_\lambda^+} + \frac{1}{2}\underbrace{Cv_\lambda^+}_{=cv_\lambda^+} \\ &= 4hv_\lambda^+ + \frac{1}{2}cv_\lambda^+ = \left(4h + \frac{1}{2}c\right) v_\lambda^+,\end{aligned}$$

so that

$$\begin{aligned}\left(L_{-2}v_\lambda^+, L_2 v_{-\lambda}^-\right) &= \left(-\underbrace{L_2 L_{-2} v_\lambda^+}_{=\left(4h+\frac{1}{2}c\right)v_\lambda^+}, v_{-\lambda}^-\right) = \left(-\left(4h + \frac{1}{2}c\right) v_\lambda^+, v_{-\lambda}^-\right) \\ &= -\left(4h + \frac{1}{2}c\right)\underbrace{\left(v_\lambda^+, v_{-\lambda}^-\right)}_{=1} = -\left(4h + \frac{1}{2}c\right).\end{aligned}$$

As a further (more complicated) example, let us compute the upper left entry of the matrix, namely $\left(L_{-1}^2 v_\lambda^+, L_1^2 v_{-\lambda}^-\right)$. We have

$$\begin{aligned}L_1^2 L_{-1}^2 v_\lambda^+ &= L_1 \underbrace{L_1 L_{-1}}_{=L_{-1}L_1+[L_1,L_{-1}]} L_{-1}v_\lambda^+ = L_1 L_{-1}\underbrace{L_1 L_{-1} v_\lambda^+}_{=2hv_\lambda^+} + L_1\underbrace{[L_1, L_{-1}]}_{=2L_0} L_{-1}v_\lambda^+ \\ &= 2h\underbrace{L_1 L_{-1} v_\lambda^+}_{=2hv_\lambda^+} + 2L_1 \underbrace{L_0 L_{-1}}_{\substack{=L_{-1}L_0+[L_0,L_{-1}] \\ =L_{-1}L_0+L_{-1} \\ (\text{since } [L_0,L_{-1}]=L_{-1})}} v_\lambda^+ = 4h^2 v_\lambda^+ + 2L_1 L_{-1}\underbrace{L_0 v_\lambda^+}_{=hv_\lambda^+} + 2\underbrace{L_1 L_{-1} v_\lambda^+}_{=2hv_\lambda^+} \\ &= 4h^2 v_\lambda^+ + 2h\underbrace{L_1 L_{-1} v_\lambda^+}_{=2hv_\lambda^+} + 4hv_\lambda^+ = 4h^2 v_\lambda^+ + 4h^2 v_\lambda^+ + 4hv_\lambda^+ = \left(8h^2 + 4h\right) v_\lambda^+\end{aligned}$$

and thus

$$\left(L_{-1}^2 v_\lambda^+, L_1^2 v_{-\lambda}^-\right) = \left(-L_1 L_{-1}^2 v_\lambda^+, L_1 v_{-\lambda}^-\right) = \left(\underbrace{L_1^2 L_{-1}^2 v_\lambda^+}_{=(8h^2+4h)v_\lambda^+}, v_{-\lambda}^-\right) = \left(\left(8h^2+4h\right) v_\lambda^+, v_{-\lambda}^-\right)$$

$$= \left(8h^2+4h\right) \underbrace{\left(v_\lambda^+, v_{-\lambda}^-\right)}_{=1} = 8h^2+4h.$$

Similarly, we compute the other two entries of the matrix. The matrix thus becomes

$$\begin{pmatrix} 8h^2+4h & 6h \\ -6h & -\left(4h+\dfrac{1}{2}c\right) \end{pmatrix}.$$

The determinant of this matrix is

$$\det\left((\cdot,\cdot)_2\right) = \left(8h^2+4h\right)\left(-\left(4h+\frac{1}{2}c\right)\right) - 6h\left(-6h\right) = -4h\left((2h+1)\left(4h+\frac{1}{2}c\right) - 9h\right).$$

Notice the term $(2h+1)\left(4h+\frac{1}{2}c\right) - 9h$: The set of zeroes of this term is a hyperbola[76]. The determinant of $(\cdot,\cdot)_2$ thus vanishes on the union of a line and a hyperbola. For every point (h,c) lying on this hyperbola, the highest-weight module $M_{h,c}^+$ has a nonzero singular vector in degree -2 (this means a nonzero singular vector of the form $\alpha L_{-2} v_\lambda^+ + \beta L_{-1}^2 v_\lambda^+$ for some $\alpha, \beta \in \mathbb{C}$).

We will later discuss $\det\left((\cdot,\cdot)_n\right)$ for generic n. In fact, there is an explicit formula for this determinant, namely the so-called Kac determinant formula.

2.9.1. Restricted dual modules

Definition 2.9.9. Let $V = \bigoplus_{i\in I} V[i]$ be an I-graded vector space, where I is some set (for example, I can be $\mathbb{Z}$, $\mathbb{N}$ or $\mathbb{C}$). The *restricted dual* $V^\vee$ of V is defined to be the direct sum $\bigoplus_{i\in I} V[i]^*$. This is a vector subspace of the dual V^* of V, but (in general) not the same as V^* unless the direct sum is finite.

One can make the restricted dual $V^\vee$ into an I-graded vector space by defining $V^\vee[i] = V[i]^*$ for every $i \in I$. But when I is an abelian group, one can also make the restricted dual $V^\vee$ into an I-graded vector space by defining $V^\vee[i] = V[-i]^*$ for every $i \in I$. These two constructions result in two (generally) **different** gradings on $V^\vee$; both of these gradings are used in algebra.

Using either of these two gradings on $V^\vee$, we can make sense of the restricted dual $V^{\vee\vee}$ of $V^\vee$. This restricted dual $V^{\vee\vee}$ does not depend on which of the two gradings on $V^\vee$ has been chosen. There is a canonical injection $V \to V^{\vee\vee}$. If $V[i]$ is finite-dimensional for every $i \in I$, then this injection $V \to V^{\vee\vee}$ is an isomorphism (so that $V^{\vee\vee} \cong V$ canonically).

[76] Here, a *hyperbola* means an affine conic over $\mathbb{C}$ which is defined over $\mathbb{R}$ and whose restriction to $\mathbb{R}$ is a hyperbola.

If $\mathfrak{g}$ is a $\mathbb{Z}$-graded Lie algebra, and V is a $\mathbb{C}$-graded $\mathfrak{g}$-module, then $V^\vee$ canonically becomes a $\mathbb{C}$-graded $\mathfrak{g}$-module if the grading on $V^\vee$ is defined by $V^\vee [i] = V [-i]^*$ for every $i \in \mathbb{C}$. (Note that the grading defined by $V^\vee [i] = V [i]^*$ for every $i \in \mathbb{C}$ would **not** (in general) make $V^\vee$ into a $\mathbb{C}$-graded $\mathfrak{g}$-module.)

It is clear that:

Proposition 2.9.10. We have two mutually inverse antiequivalences of categories $\mathcal{O}^+ \overset{\vee}{\to} \mathcal{O}^-$ and $\mathcal{O}^- \overset{\vee}{\to} \mathcal{O}^+$, each defined by mapping every $\mathfrak{g}$-module in one category to its restricted dual.

We can view the form $(\cdot,\cdot) : M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ as a linear map $M_\lambda^+ \to \left(M_{-\lambda}^-\right)^\vee$. The kernel of this map is J_λ^+, and therefore, when $\mathfrak{g}$ is nondegenerate, this map is an isomorphism for Weil-generic λ (by Theorem 2.6.6). In general, this map factors as $M_\lambda^+ \twoheadrightarrow L_\lambda^+ \xrightarrow{\cong} \left(L_{-\lambda}^-\right)^\vee \hookrightarrow \left(M_{-\lambda}^-\right)^\vee$.

2.9.2. Involutions

In many applications, we are not just working with a graded Lie algebra $\mathfrak{g}$. Very often we additionally have a degree-reversing involution:

Definition 2.9.11. Let $\mathfrak{g}$ be a graded Lie algebra. Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism of the Lie algebra $\mathfrak{g}$ ("involutive" means $\omega^2 = \mathrm{id}$) such that $\omega(\mathfrak{g}_i) = \mathfrak{g}_{-i}$ for all $i \in \mathbb{Z}$ and such that $\omega \mid_{\mathfrak{g}_0} = -\mathrm{id}$. Then, for every graded $\mathfrak{g}$-module M, we can define a graded $\mathfrak{g}$-module M^c as being the $\mathfrak{g}$-module M^ω with opposite grading (i. e., the grading on M^c is defined by $M^c[i] = M^\omega[-i]$ for every i). Then, we have an equivalence of categories $\mathcal{O}^+ \overset{\omega}{\to} \mathcal{O}^-$ which sends every $\mathfrak{g}$-module $M \in \mathcal{O}^+$ to the $\mathfrak{g}$-module $M^c \in \mathcal{O}^-$, and the quasiinverse equivalence of categories $\mathcal{O}^- \overset{\omega}{\to} \mathcal{O}^+$ which does the same thing.

So the functor $\mathcal{O}^+ \overset{\vee}{\to} \mathcal{O}^- \overset{\omega}{\to} \mathcal{O}^+$ is an antiequivalence, called the *functor of contragredient module*. This functor allows us to identify $\left(M_{-\lambda}^-\right)^\omega$ with M_λ^+ (via the isomorphism $M_\lambda^+ \to \left(M_{-\lambda}^-\right)^\omega$ which sends $x \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} v_\lambda^+$ to $(U(\omega))(x) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} v_{-\lambda}^-$ for every $x \in U(\mathfrak{g})$), and thus to view the form $(\cdot,\cdot)$ as a form $(\cdot,\cdot) : M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}$. But this form is not $\mathfrak{g}$-invariant; it is contravariant; this means that any $a \in \mathfrak{g}$, $v \in M_\lambda^+$ and $w \in M_\lambda^+$ satisfy $(av, w) = -(v, \omega(a) w)$ and $(v, aw) = -(\omega(a) v, w)$.

This form can be viewed as a linear map $M_\lambda^+ \to \left(M_\lambda^+\right)^c$, which factors into $M_\lambda^+ \twoheadrightarrow L_\lambda^+ \xrightarrow{\cong} \left(L_\lambda^+\right)^c \hookrightarrow \left(M_\lambda^+\right)^c$.

Notice that this form $(\cdot,\cdot)$ is a contravariant form $M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_\lambda^+\right) = 1$. Of course, this yields that the transpose of $(\cdot,\cdot)$ is also such a form. Since there exists a **unique** contravariant form $M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_\lambda^+\right) = 1$ (because contravariant forms $M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}$ are in 1-to-1 correspondence with $\mathfrak{g}$-invariant bilinear forms $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$, and for the latter we have Proposition 2.6.1 **(a)**), this yields that the form $(\cdot,\cdot)$ and its transpose must be identical. In other words, the form $(\cdot,\cdot)$ is symmetric.

Involutive automorphisms of $\mathfrak{g}$ satisfying the conditions of Definition 2.9.11 are not uncommon; here are four examples:

Proposition 2.9.12. The $\mathbb{C}$-linear map $\omega : \mathcal{A} \to \mathcal{A}$ defined by $\omega(K) = -K$ and $\omega(a_i) = -a_{-i}$ for every $i \in \mathbb{Z}$ is an involutive automorphism of the Lie algebra $\mathcal{A}$. This automorphism ω satisfies the conditions of Definition 2.9.11 (for $\mathfrak{g} = \mathcal{A}$). We already know this from Proposition 2.2.22. Moreover, if we let $\lambda = (1, \mu)$ for a complex number μ, then $M_\lambda^+ \cong F_\mu$ (by Proposition 2.5.17), and thus we can regard the contravariant form $M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}$ from Definition 2.9.11 as a contravariant form $F_\mu \times F_\mu \to \mathbb{C}$. This contravariant form $F_\mu \times F_\mu \to \mathbb{C}$ is exactly the form $(\cdot, \cdot)$ of Proposition 2.2.24. (This is because the form $(\cdot, \cdot)$ of Proposition 2.2.24 is contravariant (due to Proposition 2.2.24 **(c)** and **(d)**) and satisfies $(1, 1) = 1$.)

Proposition 2.9.13. The $\mathbb{C}$-linear map $\omega : \mathrm{Vir} \to \mathrm{Vir}$ defined by $\omega(C) = -C$ and $\omega(L_i) = -L_{-i}$ for every $i \in \mathbb{Z}$ is an involutive automorphism of the Lie algebra Vir. This automorphism ω satisfies the conditions of Definition 2.9.11 (for $\mathfrak{g} = \mathrm{Vir}$).

Proposition 2.9.14. Let $\mathfrak{g}$ be a simple Lie algebra, graded and presented as in Proposition 2.5.6. Then, there exists a unique Lie algebra homomorphism $\omega : \mathfrak{g} \to \mathfrak{g}$ satisfying $\omega(e_i) = -f_i$, $\omega(h_i) = -h_i$ and $\omega(f_i) = -e_i$ for every $i \in \{1, 2, ..., m\}$. This automorphism ω satisfies the conditions of Definition 2.9.11.

Proposition 2.9.15. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra, graded and presented as in Proposition 2.5.6. Let $\widehat{\mathfrak{g}}$ be the Kac-Moody Lie algebra defined in Definition 1.7.6. Let K denote the element $(0, 1)$ of $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} = \widehat{\mathfrak{g}}$. Consider the $\mathbb{Z}$-grading on $\widehat{\mathfrak{g}}$ defined in Proposition 2.5.7.

Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be defined as in Proposition 2.9.14. Then, the $\mathbb{C}$-linear map $\widehat{\omega} : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ defined by $\widehat{\omega}(a \cdot t^j) = \omega(a) t^{-j}$ for every $a \in \mathfrak{g}$ and $j \in \mathbb{Z}$, and $\widehat{\omega}(K) = -K$, is an involutive automorphism of the Lie algebra $\widehat{\mathfrak{g}}$. This automorphism $\widehat{\omega}$ satisfies the conditions of Definition 2.9.11 (for $\widehat{\mathfrak{g}}$ and $\widehat{\omega}$ instead of $\mathfrak{g}$ and ω).

More generally:

Proposition 2.9.16. Let $\mathfrak{g}$ be a Lie algebra equipped with a $\mathfrak{g}$-invariant symmetric bilinear form $(\cdot, \cdot)$ of degree 0. Let $\widehat{\mathfrak{g}}$ be the Lie algebra defined in Definition 1.7.1. Let K denote the element $(0, 1)$ of $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C} = \widehat{\mathfrak{g}}$.

Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be an involutive automorphism of the Lie algebra $\mathfrak{g}$ (not to be confused with the 2-cocycle ω of Definition 1.7.1). Then, the $\mathbb{C}$-linear map $\widehat{\omega} : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ defined by $\widehat{\omega}(a \cdot t^j) = \omega(a) t^{-j}$ for every $a \in \mathfrak{g}$ and $j \in \mathbb{Z}$, and $\widehat{\omega}(K) = -K$, is an involutive automorphism of the Lie algebra $\widehat{\mathfrak{g}}$.

Assume now that the Lie algebra $\mathfrak{g}$ is graded and that the automorphism ω satisfies the conditions of Definition 2.9.11. Assume further that we extend the grading of $\mathfrak{g}$ to a grading on $\widehat{\mathfrak{g}}$ in such a way that K is homogeneous of degree 0, and that the multiplications by t and t^{-1} are homogeneous linear maps (that is, linear maps which shift the degree by a fixed integer). Then, the automorphism $\widehat{\omega}$ of $\widehat{\mathfrak{g}}$ satisfies $\widehat{\omega}(\widehat{\mathfrak{g}}_i) = \widehat{\mathfrak{g}}_{-i}$ for all $i \in \mathbb{Z}$. (But in general, $\widehat{\omega}$ does not necessarily satisfy $\widehat{\omega} \mid_{\widehat{\mathfrak{g}}_0} = -\mathrm{id}$.)

2.9.3. [unfinished] Unitary structures

Important Notice 2.9.17. **The parts of these notes concerned with unitary/Hermitian/real structures are in an unfinished state and contain mistakes which I don't know how to fix.**

For instance, if we define $\mathfrak{g}_{\mathbb{R}}$ by $\mathfrak{g}_{\mathbb{R}} = \left\{a \in \mathfrak{g} \mid a^\dagger = -a\right\}$, and define $\mathfrak{g}_{0\mathbb{R}}^*$ by $\mathfrak{g}_{0\mathbb{R}}^* = \{f \in \mathfrak{g}_0^* \mid f(\mathfrak{g}_{0\mathbb{R}}) \subseteq \mathbb{R}\}$ (as I do below), and define the antilinear $\mathbb{R}$-antiinvolution $\dagger : \mathrm{Vir} \to \mathrm{Vir}$ on Vir by $L_i^\dagger = L_{-i}$ for all $i \in \mathbb{Z}$, and $C^\dagger = C$, then $\mathrm{Vir}_{0\mathbb{R}}^*$ is **not** the set of all weights (h, c) satisfying $h, c \in \mathbb{R}$, but it is the set of all weights (h, c) satisfying $ih, ic \in \mathbb{R}$ (because the definition of $\dagger$ that we gave leads to $\mathrm{Vir}_{0\mathbb{R}} = \langle iC, iL_0\rangle_{\mathbb{R}}$). This is not what we want later. Probably it is possible to fix these issues by correcting some signs, but I do not know how. If you know a consistent way to correct these definitions and results, please drop me a mail (AB`@gmail.com` where A=`darij` and B=`grinberg`).

Over $\mathbb{C}$, it makes sense to study not only linear but also antilinear maps. Sometimes, the latter actually enjoy even better properties of the former (e. g., Hermitian forms are better behaved than complex-symmetric forms).

Definition 2.9.18. If $\mathfrak{g}$ and $\mathfrak{h}$ are two Lie algebras over a field k, then a *k-antihomomorphism* from $\mathfrak{g}$ to $\mathfrak{h}$ means a k-linear map $f : \mathfrak{g} \to \mathfrak{h}$ such that $f([x, y]) = -[f(x), f(y)]$ for all $x, y \in \mathfrak{g}$.

Definition 2.9.19. In the following, an *k-antiinvolution* of a Lie algebra $\mathfrak{g}$ over a field k means a k-antihomomorphism from $\mathfrak{g}$ to $\mathfrak{g}$ which is simultaneously an involution.

Definition 2.9.20. Let $\mathfrak{g}$ be a complex Lie algebra. Let $\dagger : \mathfrak{g} \to \mathfrak{g}$ be an antilinear $\mathbb{R}$-antiinvolution. This means that $\dagger$ is an $\mathbb{R}$-linear map and satisfies the relations

$$\begin{aligned} \dagger^2 &= \mathrm{id}; \\ (za)^\dagger &= \overline{z}a^\dagger && \text{for all } z \in \mathbb{C} \text{ and } a \in \mathfrak{g}; \\ [a, b]^\dagger &= -\left[a^\dagger, b^\dagger\right] && \text{for all } a, b \in \mathfrak{g}. \end{aligned}$$

(Here and in the following, we write $c^\dagger$ for the image of an element $c \in \mathfrak{g}$ under $\dagger$.) Such a map $\dagger$ is called a *real structure*, for the following reason: If $\dagger$ is such a map, then we can define an $\mathbb{R}$-vector subspace $\mathfrak{g}_{\mathbb{R}} = \left\{a \in \mathfrak{g} \mid a^\dagger = -a\right\}$ of $\mathfrak{g}$, and this $\mathfrak{g}_{\mathbb{R}}$ is a real Lie algebra such that $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ as complex Lie algebras. (It is said that $\mathfrak{g}_{\mathbb{R}}$ is a *real form* of $\mathfrak{g}$.)

Definition 2.9.21. Let $\mathfrak{g}$ be a complex Lie algebra with a real structure $\dagger$. If V is a $\mathfrak{g}$-module, we say that V is *Hermitian* if V is equipped with a nondegenerate Hermitian form $(\cdot, \cdot)$ satisfying

$$(av, w) = \left(v, a^\dagger w\right) \qquad \text{for all } a \in \mathfrak{g},\ v \in V \text{ and } w \in V.$$

The $\mathfrak{g}$-module V is said to be *unitary* if this form is positive definite.

The real Lie algebra $\mathfrak{g}_{\mathbb{R}}$ acts on a Hermitian module by skew-Hermitian operators.

Remark 2.9.22. While we will not be studying Lie groups in this course, here are some facts about them that explain why unitary $\mathfrak{g}$-modules are called "unitary":

If $\mathfrak{g}$ is a finite-dimensional Lie algebra, and V is a unitary $\mathfrak{g}$-module, then the Hilbert space completion of V is a unitary representation of the Lie group $G_{\mathbb{R}} = \exp(\mathfrak{g}_{\mathbb{R}})$ corresponding to $\mathfrak{g}_{\mathbb{R}}$ by Lie's Third Theorem. (Note that this Hilbert space completion of V is V itself if $\dim V < \infty$.) This even holds for some infinite-dimensional $\mathfrak{g}$ under sufficiently restrictive conditions.

So let us consider this situation. Two definitions:

Definition 2.9.23. Let $\mathfrak{g}$ be a complex Lie algebra with a real structure $\dagger$. Let V be a $\mathfrak{g}$-module. A Hermitian form $(\cdot,\cdot)$ on V is said to be $\dagger$-*invariant* if and only if

$$(av, w) = \left(v, a^\dagger w\right) \qquad \text{for all } a \in \mathfrak{g},\ v \in V \text{ and } w \in V.$$

Definition 2.9.24. Let $\mathfrak{g}$ be a complex Lie algebra with a real structure $\dagger$. For every $f \in \mathfrak{g}^*$, we denote by $f^\dagger$ the map $\mathfrak{g}_0 \to \mathbb{C}$, $x \mapsto \overline{f\left(x^\dagger\right)}$ (this map $f^\dagger$ is easily seen to be $\mathbb{C}$-linear). Let $\mathfrak{g}_{\mathbb{R}}^\star$ be the subset $\left\{f \in \mathfrak{g}^* \mid f^\dagger = -f\right\}$ of $\mathfrak{g}^\star$. Then, it is easily seen that

$$\mathfrak{g}_{\mathbb{R}}^\star = \{f \in \mathfrak{g}^* \mid f(\mathfrak{g}_{\mathbb{R}}) \subseteq \mathbb{R}\}.$$

Hence, we get an $\mathbb{R}$-bilinear form $\mathfrak{g}_{\mathbb{R}}^\star \times \mathfrak{g}_{\mathbb{R}} \to \mathbb{R}$, $(f, a) \mapsto f(a)$. This form is nondegenerate and thus enables us to identify $\mathfrak{g}_{\mathbb{R}}^\star$ with the dual space of the $\mathbb{R}$-vector space $\mathfrak{g}_{\mathbb{R}}$. (More precisely, we have an isomorphism from $\mathfrak{g}_{\mathbb{R}}^\star$ to the dual space of the $\mathbb{R}$-vector space $\mathfrak{g}_{\mathbb{R}}$. This isomorphism sends every $f \in \mathfrak{g}_{\mathbb{R}}^\star$ to the map $f\mid_{\mathfrak{g}_{\mathbb{R}}}$ (with target restricted to $\mathbb{R}$), and conversely, the preimage of any $\mathbb{R}$-linear map $F : \mathfrak{g}_{\mathbb{R}} \to \mathbb{R}$ is the $\mathbb{C}$-linear map $f \in \mathfrak{g}_{\mathbb{R}}^\star$ given by

$$f(a) = F\left(\frac{a - a^\dagger}{2}\right) + iF\left(\frac{a + a^\dagger}{2i}\right) \qquad \text{for all } a \in \mathfrak{g}.$$

) We can thus write $\mathfrak{g}_{\mathbb{R}}^*$ for $\mathfrak{g}_{\mathbb{R}}^\star$.

The elements of $\mathfrak{g}_{\mathbb{R}}^*$ are said to be the *real* elements of $\mathfrak{g}^*$.

Proposition 2.9.25. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra with real structure $\dagger$. Assume that the map $\dagger$ reverses the degree (i. e., every $j \in \mathbb{Z}$ satisfies $\dagger(\mathfrak{g}_j) \subseteq \mathfrak{g}_{-j}$). In particular, $\dagger(\mathfrak{g}_0) \subseteq \mathfrak{g}_0$. Also, assume that $\mathfrak{g}_0$ is an abelian Lie algebra (but let us not require $\mathfrak{g}$ to be nondegenerate). Note that $\mathfrak{g}_0$ itself is a Lie algebra, and thus Definition 2.9.24 can be applied to $\mathfrak{g}_0$ in lieu of $\mathfrak{g}$.

If $\lambda \in \mathfrak{g}_{0\mathbb{R}}^*$, then the $\mathfrak{g}$-module M_λ^+ carries a $\dagger$-invariant Hermitian form $(\cdot,\cdot)$ satisfying $\left(v_\lambda^+, v_\lambda^+\right) = 1$.

Proof of Proposition 2.9.25. In the following, whenever U is a $\mathbb{C}$-vector space, we will denote by $\overline{U}$ the $\mathbb{C}$-vector space which is identical to U as a set, but with the $\mathbb{C}$-vector space structure twisted by complex conjugation.

The antilinear $\mathbb{R}$-Lie algebra homomorphism $-\dagger : \mathfrak{g} \to \mathfrak{g}$ can be viewed as a $\mathbb{C}$-Lie algebra homomorphism $-\dagger : \mathfrak{g} \to \overline{\mathfrak{g}}$, and thus induces a $\mathbb{C}$-algebra homomorphism $U(-\dagger) : U(\mathfrak{g}) \to U(\overline{\mathfrak{g}})$. Since $U(\overline{\mathfrak{g}}) \cong \overline{U(\mathfrak{g})}$ canonically as $\mathbb{C}$-algebras (because taking the universal enveloping algebra commutes with base change)[77], we can thus consider this $U(-\dagger)$ as a $\mathbb{C}$-algebra homomorphism $U(\mathfrak{g}) \to \overline{U(\mathfrak{g})}$. This, in turn, can be viewed as an antilinear $\mathbb{R}$-algebra homomorphism $U(-\dagger) : U(\mathfrak{g}) \to U(\mathfrak{g})$.

Let $\lambda \in \mathfrak{g}_{0\mathbb{R}}^*$. Let $\left(M_{-\lambda}^-\right)^{-\dagger}$ be the $\mathfrak{g}$-module $M_{-\lambda}^-$ twisted by the isomorphism $-\dagger : \mathfrak{g} \to \mathfrak{g}$ of $\mathbb{R}$-Lie algebras. Then, $\left(M_{-\lambda}^-\right)^{-\dagger}$ is a module over the $\mathbb{R}$-Lie algebra $\mathfrak{g}$, but not a module over the $\mathbb{C}$-Lie algebra $\mathfrak{g}$, since it satisfies $(za) \rightharpoonup v = \overline{z}(a \rightharpoonup v)$ (rather than $(za) \rightharpoonup v = z(a \rightharpoonup v)$) for all $z \in \mathbb{C}$, $a \in \mathfrak{g}$ and $v \in M_{-\lambda}^-$ (where $\rightharpoonup$ denotes the action of $\mathfrak{g}$). However, this can be easily transformed into a $\mathbb{C}$-Lie algebra action: Namely, $\overline{\left(M_{-\lambda}^-\right)^{-\dagger}}$ is a module over the $\mathbb{C}$-Lie algebra $\mathfrak{g}$.

We have an isomorphism

$$\overline{\left(M_{-\lambda}^-\right)^{-\dagger}} \to M_\lambda^+,$$
$$x \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} z v_{-\lambda}^- \mapsto U(-\dagger)(x) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} \overline{z} v_\lambda^+$$

of modules over the $\mathbb{C}$-Lie algebra $\mathfrak{g}$. [78] Hence, $M_{-\lambda}^- \cong \overline{\left(M_\lambda^+\right)^{-\dagger}}$.

Hence, our bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ can be viewed as a bilinear form $M_\lambda^+ \times \overline{M_\lambda^+} \to \mathbb{C}$, id est, as a sesquilinear form $M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}$. This sesquilinear form is the unique sesquilinear Hermitian form $M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_\lambda^+\right) = 1$ [79]. As a consequence, this sesquilinear form can be easily seen to be Hermitian symmetric, i. e., to satisfy

$$(v, w) = \overline{(w, v)} \qquad \text{for all } v \in M_\lambda^+ \text{ and } w \in M_\lambda^+.$$

[80]

However, this form can be degenerate. Its kernel is J_λ^+, so it descends to a nondegenerate Hermitian form on L_λ^+. Thus, we get:

Proposition 2.9.26. If λ is real (this means that $\lambda \in \mathfrak{g}_{0\mathbb{R}}^*$), then L_λ^+ carries a $\dagger$-invariant nondegenerate Hermitian form. Different degrees in L_λ^+ are orthogonal with respect to this form.

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[77] Warning: This isomorphism $U(\overline{\mathfrak{g}}) \to \overline{U(\mathfrak{g})}$ sends $i \cdot 1_{U(\overline{\mathfrak{g}})}$ to $-i \cdot 1_{U(\mathfrak{g})}$.

[78] Here are some details on the definition of this isomorphism:

As $\mathbb{R}$-vector spaces, $\overline{\left(M_{-\lambda}^-\right)^{-\dagger}} = M_{-\lambda}^- = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} \mathbb{C}_{-\lambda}$ and $M_\lambda^+ = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} \mathbb{C}_\lambda$.

Hence, we can define an $\mathbb{R}$-linear map $\overline{\left(M_{-\lambda}^-\right)^{-\dagger}} \to M_\lambda^+$ that sends $x \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} z v_{-\lambda}^-$ to $U(-\dagger)(x) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} \overline{z} v_\lambda^+$ for every $x \in U(\mathfrak{g})$ and $z \in \mathbb{C}$ if we are able to show that

$$U(-\dagger)(xw) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} \overline{z} v_\lambda^+ = U(-\dagger)(x) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} \overline{wz} v_\lambda^+ \qquad \text{for all } x \in U(\mathfrak{g}),\ w \in U(\mathfrak{h} \oplus \mathfrak{n}_+) \text{ and } z \in \mathbb{C}.$$

But showing this is rather easy (left to the reader), and thus we get an $\mathbb{R}$-linear map $\overline{\left(M_{-\lambda}^-\right)^{-\dagger}} \to M_\lambda^+$ that sends $x \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_+)} z v_{-\lambda}^-$ to $U(-\dagger)(x) \otimes_{U(\mathfrak{h}\oplus\mathfrak{n}_-)} \overline{z} v_\lambda^+$ for every $x \in U(\mathfrak{g})$ and $z \in \mathbb{C}$. This map is easily seen to be $\mathfrak{g}$-linear and $\mathbb{C}$-linear, so it is a homomorphism of modules over $\mathbb{C}$-Lie algebra $\mathfrak{g}$. Showing that it is an isomorphism is easy as well (one just has to construct its inverse).

[79] This can be easily derived from Proposition 2.6.1 **(a)**, which claims that our form $(\cdot, \cdot) : M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ is the unique $\mathfrak{g}$-invariant bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_{-\lambda}^-\right) = 1$.

[80] In fact, the form which sends $v \times w$ to $\overline{(w, v)}$ is also a sesquilinear Hermitian form $M_\lambda^+ \times M_\lambda^+ \to \mathbb{C}$ satisfying $\left(v_\lambda^+, v_\lambda^+\right) = 1$, so that by uniqueness, it must be identical with the form which sends $v \times w$ to (v, w).

A reasonable (and, in most cases, difficult and interesting) question to ask is the following: For which λ is L_λ^+ unitary?

We are going to address this question in some cases and give hints in some others, leaving many more unanswered.

First, let us give several examples of complex Lie algebras $\mathfrak{g}$ with antilinear $\mathbb{R}$-antiinvolutions $\dagger : \mathfrak{g} \to \mathfrak{g}$:

Proposition 2.9.27. We can define an antilinear map $\dagger : \mathcal{A} \to \mathcal{A}$ by $K^\dagger = K$ and $a_i^\dagger = a_{-i}$ for all $i \in \mathbb{Z}$. This map is an antilinear $\mathbb{R}$-antiinvolution of the Heisenberg algebra $\mathcal{A}$.

Proposition 2.9.28. One can define an antilinear map $\dagger : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ by $e^\dagger = f,\ f^\dagger = e,\ h^\dagger = h$. This map is an antilinear $\mathbb{R}$-antiinvolution of the Lie algebra $\mathfrak{sl}_2$.

More generally:

Proposition 2.9.29. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Using the Chevalley generators e_1, e_2, ..., e_m, f_1, f_2, ..., f_m, h_1, h_2, ..., h_m of Proposition 2.5.6, we can define an antilinear map $\dagger : \mathfrak{g} \to \mathfrak{g}$ by $e_i^\dagger = f_i,\ f_i^\dagger = e_i,\ h_i^\dagger = h_i$ for all $i \in \{1, 2, ..., m\}$. This map is an antilinear $\mathbb{R}$-antiinvolution of the Lie algebra $\mathfrak{g}$.

Proposition 2.9.30. We can define an antilinear map $\dagger : \mathrm{Vir} \to \mathrm{Vir}$ by $L_i^\dagger = L_{-i}$ for all $i \in \mathbb{Z}$, and $C^\dagger = C$. This map is an antilinear $\mathbb{R}$-antiinvolution of the Virasoro algebra Vir.

Proposition 2.9.31. If $\mathfrak{g}$ is a Lie algebra with an antilinear $\mathbb{R}$-antiinvolution $\dagger : \mathfrak{g} \to \mathfrak{g}$ and with a symmetric $\mathfrak{g}$-invariant bilinear form $(\cdot, \cdot)$ of degree 0, then we can define an antilinear map $\dagger : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ (where $\widehat{\mathfrak{g}}$ is the Lie algebra defined in Definition 1.7.1) by $(at^n)^\dagger = a^\dagger \cdot t^{-n}$ for every $a \in \mathfrak{g}$ and $n \in \mathbb{Z}$, and by $K^\dagger = K$ (where K denotes the element $(0, 1)$ of $\mathfrak{g}\left[t, t^{-1}\right] \oplus \mathbb{C} = \widehat{\mathfrak{g}}$). This map $\dagger$ is an antilinear involution of the Lie algebra $\widehat{\mathfrak{g}}$.

As for examples of Hermitian modules: The Vir-module $L_{h,c}^+$ (see Example 2.9.8 for the definition of this module) for $h, c \in \mathbb{R}$ has a $\dagger$-invariant nondegenerate Hermitian form. (This is because the requirement $h, c \in \mathbb{R}$ forces the form $\lambda \in \mathfrak{g}_0^*$ which corresponds to the pair (h, c) to lie in $\mathfrak{g}_{0\mathbb{R}}^*$, and thus we can apply Proposition 2.9.26.)

But now, back to the general case:

Proposition 2.9.32. Let V be a unitary representation in Category $\mathcal{O}^+$. Then, V is completely reducible (i. e., the representation V is a direct sum of irreducible representations).

To prove this, we will use a lemma:

Lemma 2.9.33. If V is a highest-weight representation, and V has a nondegenerate $\dagger$-invariant Hermitian form, then V is irreducible. (We recall that a "highest-weight representation" means a quotient of M_λ^+ by a proper graded submodule for some λ.)

Proof of Lemma 2.9.33. Let V be a highest-weight representation having a nondegenerate $\dagger$-invariant Hermitian form. Since V is a highest-weight representation, V is a quotient of M_λ^+ by a proper graded submodule P for some λ. The nondegenerate $\dagger$-invariant Hermitian form on V thus induces a $\dagger$-invariant Hermitian form on M_λ^+ whose kernel is P. It is easy to see that λ is real. Thus, this $\dagger$-invariant Hermitian form on M_λ^+ can be rewritten as a $\mathfrak{g}$-invariant bilinear form $M_\lambda^+ \times M_{-\lambda}^- \to \mathbb{C}$, which still has kernel P. Such a form is unique up to scaling (by Proposition 2.6.1 **(c)**), and thus must be the form defined in Proposition 2.6.1 **(a)**. But the kernel of this form is J_λ^+. Thus, the kernel of this form is, at the same time, P and J_λ^+. Hence, $P = J_\lambda^+$, so that $V = L_\lambda^+$ (since V is the quotient of M_λ^+ by P), and thus V is irreducible. Lemma 2.9.33 is proven.

Proof of Proposition 2.9.32. Take a nonzero homogeneous vector $v \in V$ of maximal degree. ("Maximal" means "maximal in real part". Such a maximal degree exists by the definition of Category $\mathcal{O}^+$.) Let v be an eigenvector of $\mathfrak{g}_0$ with eigenvalue λ. Consider the submodule of V generated by v. This submodule is highest-weight (since $\mathfrak{g}_j v = 0$ for $j > 0$). Hence, by Lemma 2.9.33, this submodule is irreducible and therefore $\cong L_{\lambda_1}^+$ for some $\lambda_1 \in \mathfrak{h}^*$. Let V_1 be the orthogonal complement of $L_{\lambda_1}^+$. Then, $V = L_{\lambda_1}^+ \oplus V_1$. Now take a vector in V_1, and so on. Since the degrees of V lie in finitely many arithmetic progressions, and homogeneous subspaces have finite dimension, this process is exhaustive, so we obtain $V = L_{\lambda_1}^+ \oplus L_{\lambda_2}^+ \oplus ....$

Remark 2.9.34. In this decomposition, every irreducible object of Category $\mathcal{O}^+$ occurs finitely many times.

3. Representation theory: concrete examples

3.1. Some lemmata about exponentials and commutators

This section is devoted to some elementary lemmata about power series and iterated commutators over noncommutative rings. These lemmata are well-known in geometrical contexts (in these contexts they tend to appear in Lie groups textbooks), but here we will formulate and prove them purely algebraically. We will not use these lemmata until Theorem 3.11.2, but I prefer to put them here in order not to interrupt the flow of representation-theoretical arguments later.

We start with easy things:

Lemma 3.1.1. Let K be a commutative ring. If α and β are two elements of a topological K-algebra R such that $[\alpha, \beta]$ commutes with β, then $[\alpha, P(\beta)] = [\alpha, \beta] \cdot P'(\beta)$ for every power series $P \in K[[X]]$ for which the series $P(\beta)$ and $P'(\beta)$ converge.

Proof of Lemma 3.1.1. Let $\gamma = [\alpha, \beta]$. Then, γ commutes with β (since we know that $[\alpha, \beta]$ commutes with β), so that $\gamma\beta = \beta\gamma$.

Write P in the form $P = \sum\limits_{i=0}^{\infty} u_i X^i$ for some $(u_0, u_1, u_2, ...) \in K^{\mathbb{N}}$. Then, $P' = \sum\limits_{i=1}^{\infty} i u_i X^{i-1}$, so that $P'(\beta) = \sum\limits_{i=1}^{\infty} i u_i \beta^{i-1}$. On the other hand, $P = \sum\limits_{i=0}^{\infty} u_i X^i$ shows that

$P(\beta) = \sum_{i=0}^{\infty} u_i \beta^i$ and thus

$$[\alpha, P(\beta)] = \left[\alpha, \sum_{i=0}^{\infty} u_i \beta^i\right] = \sum_{i=0}^{\infty} u_i \left[\alpha, \beta^i\right] = u_0 \underbrace{\left[\alpha, \beta^0\right]}_{\substack{=0 \\ (\text{since } \beta^0 = 1 \in Z(R))}} + \sum_{i=1}^{\infty} u_i \left[\alpha, \beta^i\right] = \sum_{i=1}^{\infty} u_i \left[\alpha, \beta^i\right].$$

Now, it is easy to prove that every positive $i \in \mathbb{N}$ satisfies $[\alpha, \beta^i] = i\gamma\beta^{i-1}$ [81]. Hence,

$$[\alpha, P(\beta)] = \sum_{i=1}^{\infty} u_i \underbrace{\left[\alpha, \beta^i\right]}_{= i\gamma\beta^{i-1}} = \sum_{i=1}^{\infty} u_i i \gamma \beta^{i-1} = \underbrace{\gamma}_{=[\alpha,\beta]} \underbrace{\sum_{i=1}^{\infty} i u_i \beta^{i-1}}_{=P'(\beta)} = [\alpha, \beta] \cdot P'(\beta).$$

Lemma 3.1.1 is proven.

Corollary 3.1.2. If α and β are two elements of a topological $\mathbb{Q}$-algebra R such that $[\alpha, \beta]$ commutes with β, then $[\alpha, \exp \beta] = [\alpha, \beta] \cdot \exp \beta$ whenever the power series $\exp \beta$ converges.

Proof of Corollary 3.1.2. Applying Lemma 3.1.1 to $P = \exp X$ and $K = \mathbb{Q}$, and recalling that $\exp' = \exp$, we obtain $[\alpha, \exp \beta] = [\alpha, \beta] \cdot \exp \beta$. This proves Corollary 3.1.2.

In Lemma 3.1.1 and Corollary 3.1.2, we had to require convergence of certain power series in order for the results to make sense. In the following, we will prove some results for which such requirements are not sufficient anymore[82]; instead we need more global conditions. A standard condition to require in such cases is that all the elements

[81] *Proof.* We will prove this by induction over i:

Induction base: For $i = 1$, we have $\left[\alpha, \beta^i\right] = \left[\alpha, \beta^1\right] = [\alpha, \beta] = \gamma$ and $\underbrace{i}_{=1} \gamma \underbrace{\beta^{i-1}}_{=\beta^{1-1}=1} = \gamma$, so that $\left[\alpha, \beta^i\right] = \gamma = i\gamma\beta^{i-1}$. This proves $\left[\alpha, \beta^i\right] = i\gamma\beta^{i-1}$ for $i = 1$, and thus the induction base is complete.

Induction step: Let $j \in \mathbb{N}$ be positive. Assume that $\left[\alpha, \beta^i\right] = i\gamma\beta^{i-1}$ is proven for $i = j$. We must then prove $\left[\alpha, \beta^i\right] = i\gamma\beta^{i-1}$ for $i = j + 1$.

Since $\left[\alpha, \beta^i\right] = i\gamma\beta^{i-1}$ is proven for $i = j$, we have $\left[\alpha, \beta^j\right] = j\gamma\beta^{j-1}$.

Now,

$$\begin{aligned} \left[\alpha, \underbrace{\beta^{j+1}}_{=\beta\beta^j}\right] &= \left[\alpha, \beta\beta^j\right] = \alpha\beta\beta^j - \beta\beta^j\alpha = \underbrace{\left(\alpha\beta\beta^j - \beta\alpha\beta^j\right)}_{=(\alpha\beta-\beta\alpha)\beta^j} + \underbrace{\left(\beta\alpha\beta^j - \beta\beta^j\alpha\right)}_{=\beta(\alpha\beta^j - \beta^j\alpha)} \\ &= \underbrace{(\alpha\beta - \beta\alpha)}_{=[\alpha,\beta]=\gamma} \beta^j + \beta \underbrace{\left(\alpha\beta^j - \beta^j\alpha\right)}_{=[\alpha,\beta^j]=j\gamma\beta^{j-1}} = \gamma\beta^j + \beta j\gamma\beta^{j-1} = \gamma\beta^j + j \underbrace{\beta\gamma}_{=\gamma\beta} \beta^{j-1} \\ &= \gamma\beta^j + j\gamma \underbrace{\beta\beta^{j-1}}_{=\beta^j} = \gamma\beta^j + j\gamma\beta^j = (j+1)\gamma\beta^j = (j+1)\gamma\beta^{(j+1)-1}. \end{aligned}$$

In other words, $\left[\alpha, \beta^i\right] = i\gamma\beta^{i-1}$ holds for $i = j + 1$. This completes the induction step, and thus by induction we have proven that $\left[\alpha, \beta^i\right] = i\gamma\beta^{i-1}$ for every positive $i \in \mathbb{N}$.

[82] At least they are not sufficient for my proofs...

to which we apply power series lie in some ideal I of R such that R is complete and Hausdorff with respect to the I-adic topology. Under this condition, things work nicely, due to the following fact (which is one part of the universal property of the power series ring $K[[X]]$):

Proposition 3.1.3. Let K be a commutative ring. Let R be a K-algebra, and I be an ideal of R such that R is complete and Hausdorff with respect to the I-adic topology. Then, for every power series $P \in K[[X]]$ and every $\alpha \in I$, there is a well-defined element $P(\alpha) \in R$ (which is defined as the limit $\lim\limits_{n\to\infty} \sum\limits_{i=0}^{n} u_i \alpha^i$ (with respect to the I-adic topology), where the power series P is written in the form $P = \sum\limits_{i=0}^{\infty} u_i X^i$ for some $(u_0, u_1, u_2, ...) \in K^{\mathbb{N}}$). For every $\alpha \in I$, the map $K[[X]] \to R$ which sends every $P \in K[[X]]$ to $P(\alpha)$ is a continuous K-algebra homomorphism (where the topology on $K[[X]]$ is the standard one, and the topology on R is the I-adic one).

Theorem 3.1.4. Let R be a $\mathbb{Q}$-algebra, and let I be an ideal of R such that R is complete and Hausdorff with respect to the I-adic topology. Let $\alpha \in I$ and $\beta \in I$ be such that $\alpha\beta = \beta\alpha$. Then, $\exp\alpha$, $\exp\beta$ and $\exp(\alpha+\beta)$ are well-defined (by Proposition 3.1.3) and satisfy $\exp(\alpha+\beta) = (\exp\alpha)\cdot(\exp\beta)$.

Proof of Theorem 3.1.4. We know that $\alpha\beta = \beta\alpha$. That is, α and β commute, so that we can apply the binomial formula to α and β.

Comparing

$$\exp(\alpha+\beta) = \sum_{n=0}^{\infty} \frac{(\alpha+\beta)^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{(\alpha+\beta)^n}_{\substack{=\sum_{i=0}^{n}\binom{n}{i}\alpha^i\beta^{n-i} \\ \text{(by the binomial formula,} \\ \text{since } \alpha \text{ and } \beta \text{ commute)}}} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta^{n-i}$$

with

$$\underbrace{(\exp\alpha)}_{=\sum_{i=0}^{\infty}\frac{\alpha^i}{i!}} \cdot \underbrace{(\exp\beta)}_{=\sum_{j=0}^{\infty}\frac{\beta^j}{j!}} = \left(\sum_{i=0}^{\infty} \frac{\alpha^i}{i!}\right) \cdot \left(\sum_{j=0}^{\infty} \frac{\beta^j}{j!}\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{\alpha^i\beta^j}{i!j!} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{1}{i!j!}\alpha^i\beta^j$$

$$= \underbrace{\sum_{i=0}^{\infty}\sum_{n=i}^{\infty}}_{=\sum_{n=0}^{\infty}\sum_{i=0}^{n}} \underbrace{\frac{1}{i!\,(n-i)!}}_{\substack{=\frac{1}{n!}\binom{n}{i} \\ \left(\text{since } \binom{n}{i} = \frac{n!}{i!\,(n-i)!}\right)}} \alpha^i\beta^{n-i}$$

(here, we substituted n for $i+j$ in the second sum)

$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n} \frac{1}{n!}\binom{n}{i}\alpha^i\beta^{n-i} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta^{n-i},$$

we obtain $\exp(\alpha+\beta) = (\exp\alpha)\cdot(\exp\beta)$. This proves Theorem 3.1.4.

Corollary 3.1.5. Let R be a $\mathbb{Q}$-algebra, and let I be an ideal of R such that R is complete and Hausdorff with respect to the I-adic topology. Let $\gamma \in I$. Then, $\exp \gamma$ and $\exp(-\gamma)$ are well-defined (by Proposition 3.1.3) and satisfy $(\exp \gamma) \cdot (\exp(-\gamma)) = 1$.

Proof of Corollary 3.1.5. By Theorem 3.1.4 (applied to $\alpha = \gamma$ and $\beta = -\gamma$), we have $\exp(\gamma + (-\gamma)) = (\exp \gamma) \cdot (\exp(-\gamma))$, thus

$$(\exp \gamma) \cdot (\exp(-\gamma)) = \exp \underbrace{(\gamma + (-\gamma))}_{=0} = \exp 0 = 1.$$

This proves Corollary 3.1.5.

Theorem 3.1.6. Let R be a $\mathbb{Q}$-algebra, and let I be an ideal of R such that R is complete and Hausdorff with respect to the I-adic topology. Let $\alpha \in I$. Denote by $\operatorname{ad} \alpha$ the map $R \to R,\ x \mapsto [\alpha, x]$ (where $[\alpha, x]$ denotes the commutator $\alpha x - x\alpha$).

(a) Then, the infinite series $\sum\limits_{n=0}^{\infty} \dfrac{(\operatorname{ad} \alpha)^n}{n!}$ converges pointwise (i. e., for every $x \in R$, the infinite series $\sum\limits_{n=0}^{\infty} \dfrac{(\operatorname{ad} \alpha)^n}{n!}(x)$ converges). Denote the value of this series by $\exp(\operatorname{ad} \alpha)$.

(b) We have $(\exp \alpha) \cdot \beta \cdot (\exp(-\alpha)) = (\exp(\operatorname{ad} \alpha))(\beta)$ for every $\beta \in R$.

To prove this, we will use a lemma:

Lemma 3.1.7. Let R be a ring. Let α and β be elements of R. Denote by $\operatorname{ad} \alpha$ the map $R \to R,\ x \mapsto [\alpha, x]$ (where $[\alpha, x]$ denotes the commutator $\alpha x - x\alpha$). Let $n \in \mathbb{N}$. Then,

$$(\operatorname{ad} \alpha)^n (\beta) = \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta (-\alpha)^{n-i}.$$

Proof of Lemma 3.1.7. Let L_α denote the map $R \to R,\ x \mapsto \alpha x$. Let R_α denote the map $R \to R,\ x \mapsto x\alpha$. Then, every $x \in R$ satisfies

$$(L_\alpha - R_\alpha)(x) = \underbrace{L_\alpha(x)}_{\substack{=\alpha x \\ \text{(by the definition of } L_\alpha)}} - \underbrace{R_\alpha(x)}_{\substack{=x\alpha \\ \text{(by the definition of } R_\alpha)}} = \alpha x - x\alpha = [\alpha, x] = (\operatorname{ad} \alpha)(x).$$

Hence, $L_\alpha - R_\alpha = \operatorname{ad} \alpha$.

Also, every $x \in R$ satisfies

$$(L_\alpha \circ R_\alpha)(x) = L_\alpha \underbrace{(R_\alpha(x))}_{\substack{=x\alpha \\ \text{(by the definition of } R_\alpha)}} = L_\alpha(x\alpha) = \alpha x \alpha$$

(by the definition of L_α) and

$$(R_\alpha \circ L_\alpha)(x) = R_\alpha \underbrace{(L_\alpha(x))}_{\substack{=\alpha x \\ \text{(by the definition of } L_\alpha)}} = R_\alpha(\alpha x) = \alpha x \alpha$$

(by the definition of R_α), so that $(L_\alpha \circ R_\alpha)(x) = (R_\alpha \circ L_\alpha)(x)$. Hence, $L_\alpha \circ R_\alpha = R_\alpha \circ L_\alpha$. In other words, the maps L_α and R_α commute. Thus, we can apply the binomial formula to L_α and R_α, and conclude that $(L_\alpha - R_\alpha)^n = \sum\limits_{i=0}^{n} (-1)^{n-i} \dbinom{n}{i} L_\alpha^i \circ R_\alpha^{n-i}$. Since $L_\alpha - R_\alpha = \operatorname{ad}\alpha$, this rewrites as $(\operatorname{ad}\alpha)^n = \sum\limits_{i=0}^{n} (-1)^{n-i} \dbinom{n}{i} L_\alpha^i \circ R_\alpha^{n-i}$.

Now, it is easy to see (by induction over j) that

$$L_\alpha^j y = \alpha^j y \qquad \text{for every } j \in \mathbb{N} \text{ and } y \in R. \tag{73}$$

Also, it is easy to see (by induction over j) that

$$R_\alpha^j y = y\alpha^j \qquad \text{for every } j \in \mathbb{N} \text{ and } y \in R. \tag{74}$$

Now, since $(\operatorname{ad}\alpha)^n = \sum\limits_{i=0}^{n} (-1)^{n-i} \dbinom{n}{i} L_\alpha^i \circ R_\alpha^{n-i}$, we have

$$\begin{aligned}
(\operatorname{ad}\alpha)^n(\beta) &= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \underbrace{\left(L_\alpha^i \circ R_\alpha^{n-i}\right)(\beta)}_{\substack{=L_\alpha^i\left(R_\alpha^{n-i}\beta\right)=\alpha^i R_\alpha^{n-i}\beta \\ \text{(by (73), applied to } j=i \text{ and } y=R_\alpha^{n-i}\beta\text{)}}} \\
&= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \alpha^i \underbrace{R_\alpha^{n-i}\beta}_{\substack{=\beta\alpha^{n-i} \\ \text{(by (74), applied to } j=n-i \text{ and } y=\beta\text{)}}} \\
&= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \alpha^i \beta \alpha^{n-i} = \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta (-\alpha)^{n-i}.
\end{aligned}$$

This proves Lemma 3.1.7.

Proof of Theorem 3.1.6. **(a)** For every $x \in R$ and every $n \in \mathbb{N}$, we have $(\operatorname{ad}\alpha)^n(x) \in I^n$ (this can be easily proven by induction over n, using the fact that I is an ideal) and thus $\dfrac{(\operatorname{ad}\alpha)^n}{n!}(x) = \dfrac{1}{n!} \underbrace{(\operatorname{ad}\alpha)^n(x)}_{\in I^n} \in I^n$. Hence, for every $x \in R$, the infinite series $\sum\limits_{n=0}^{\infty} \dfrac{(\operatorname{ad}\alpha)^n}{n!}(x)$ converges (because R is complete and Hausdorff with respect to the I-adic topology). In other words, the infinite series $\sum\limits_{n=0}^{\infty} \dfrac{(\operatorname{ad}\alpha)^n}{n!}$ converges pointwise. Theorem 3.1.6 **(a)** is proven.

(b) Let $\beta \in R$. By the definition of of $\exp(\operatorname{ad}\alpha)$, we have

$$\begin{aligned}
(\exp(\operatorname{ad}\alpha))(\beta) &= \sum_{n=0}^{\infty} \frac{(\operatorname{ad}\alpha)^n}{n!}(\beta) = \sum_{n=0}^{\infty} \frac{1}{n!} \underbrace{(\operatorname{ad}\alpha)^n(\beta)}_{\substack{=\sum\limits_{i=0}^{n} \binom{n}{i} \alpha^i\beta(-\alpha)^{n-i} \\ \text{(by Lemma 3.1.7)}}} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^{n} \binom{n}{i} \alpha^i \beta (-\alpha)^{n-i}.
\end{aligned}$$

Compared with

$$\underbrace{(\exp\alpha)}_{=\sum_{i=0}^{\infty}\frac{\alpha^i}{i!}}\cdot\beta\cdot\underbrace{(\exp(-\alpha))}_{=\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}} = \left(\sum_{i=0}^{\infty}\frac{\alpha^i}{i!}\right)\cdot\beta\cdot\left(\sum_{j=0}^{\infty}\frac{(-\alpha)^j}{j!}\right)$$
$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\alpha^i\beta(-\alpha)^j}{i!j!} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{i!j!}\alpha^i\beta(-\alpha)^j$$
$$= \underbrace{\sum_{i=0}^{\infty}\sum_{n=i}^{\infty}}_{=\sum_{n=0}^{\infty}\sum_{i=0}^{n}} \underbrace{\frac{1}{i!(n-i)!}}_{\substack{=\frac{1}{n!}\binom{n}{i} \\ \left(\text{since }\binom{n}{i}=\frac{n!}{i!(n-i)!}\right)}} \alpha^i\beta(-\alpha)^{n-i}$$
$$\text{(here, we substituted } n \text{ for } i+j \text{ in the second sum)}$$
$$= \sum_{n=0}^{\infty}\sum_{i=0}^{n}\frac{1}{n!}\binom{n}{i}\alpha^i\beta(-\alpha)^{n-i} = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{i=0}^{n}\binom{n}{i}\alpha^i\beta(-\alpha)^{n-i},$$

this yields $(\exp\alpha)\cdot\beta\cdot(\exp(-\alpha)) = (\exp(\operatorname{ad}\alpha))(\beta)$. This proves Theorem 3.1.6 **(b)**.

Corollary 3.1.8. Let R be a $\mathbb{Q}$-algebra, and let I be an ideal of R such that R is complete and Hausdorff with respect to the I-adic topology. Let $\alpha \in I$. Denote by $\operatorname{ad}\alpha$ the map $R \to R$, $x \mapsto [\alpha, x]$ (where $[\alpha, x]$ denotes the commutator $\alpha x - x\alpha$).

As we know from Theorem 3.1.6 **(a)**, the infinite series $\sum_{n=0}^{\infty}\frac{(\operatorname{ad}\alpha)^n}{n!}$ converges pointwise. Denote the value of this series by $\exp(\operatorname{ad}\alpha)$.

We have $(\exp\alpha)\cdot(\exp\beta)\cdot(\exp(-\alpha)) = \exp((\exp(\operatorname{ad}\alpha))(\beta))$ for every $\beta \in I$.

Proof of Corollary 3.1.8. Corollary 3.1.5 (applied to $\gamma = -\alpha$) yields $(\exp(-\alpha))\cdot(\exp(-(-\alpha))) = 1$. Since $-(-\alpha) = \alpha$, this rewrites as $(\exp(-\alpha))\cdot(\exp\alpha) = 1$.

Let $\beta \in I$. Let T denote the map $R \to R$, $x \mapsto (\exp\alpha)\cdot x\cdot(\exp(-\alpha))$. Clearly, this map T is $\mathbb{Q}$-linear. It also satisfies

$$\begin{aligned} T(1) &= (\exp\alpha)\cdot 1\cdot(\exp(-\alpha)) && \text{(by the definition of } T\text{)} \\ &= (\exp\alpha)\cdot(\exp(-\alpha)) = 1, \end{aligned}$$

and any $x \in R$ and $y \in R$ satisfy

$$\underbrace{T(x)}_{\substack{=(\exp\alpha)\cdot x\cdot(\exp(-\alpha)) \\ \text{(by the definition of } T\text{)}}}\cdot\underbrace{T(y)}_{\substack{=(\exp\alpha)\cdot y\cdot(\exp(-\alpha)) \\ \text{(by the definition of } T\text{)}}} = (\exp\alpha)\cdot x\cdot\underbrace{(\exp(-\alpha))\cdot(\exp\alpha)}_{=1}\cdot y\cdot(\exp(-\alpha))$$
$$= (\exp\alpha)\cdot xy\cdot(\exp(-\alpha)) = T(xy)$$

(since $T(xy) = (\exp\alpha)\cdot xy\cdot(\exp(-\alpha))$ by the definition of T). Hence, T is a $\mathbb{Q}$-algebra homomorphism. Also, T is continuous (with respect to the I-adic topology).

Thus, T is a continuous $\mathbb{Q}$-algebra homomorphism, and hence commutes with the application of power series. Thus, $T(\exp\beta) = \exp(T(\beta))$. But since $T(\exp\beta) = (\exp\alpha)\cdot(\exp\beta)\cdot(\exp(-\alpha))$ (by the definition of T) and

$$\begin{aligned} T(\beta) &= (\exp\alpha)\cdot\beta\cdot(\exp(-\alpha)) && \text{(by the definition of } T\text{)} \\ &= (\exp(\operatorname{ad}\alpha))(\beta) && \text{(by Theorem 3.1.6 } \textbf{(b)}\text{)}, \end{aligned}$$

this rewrites as $(\exp\alpha)\cdot(\exp\beta)\cdot(\exp(-\alpha)) = \exp((\exp(\operatorname{ad}\alpha))(\beta))$. This proves Corollary 3.1.8.

Lemma 3.1.9. Let R be a $\mathbb{Q}$-algebra, and let I be an ideal of R such that R is complete and Hausdorff with respect to the I-adic topology. Let $\alpha\in I$ and $\beta\in I$. Assume that $[\alpha,\beta]$ commutes with each of α and β. Then, $(\exp\alpha)\cdot(\exp\beta) = (\exp\beta)\cdot(\exp\alpha)\cdot(\exp[\alpha,\beta])$.

First we give two short proofs of this lemma.

First proof of Lemma 3.1.9. Define the map $\operatorname{ad}\alpha$ as in Corollary 3.1.8. Then, $(\operatorname{ad}\alpha)^2(\beta) = [\alpha,[\alpha,\beta]] = 0$ (since $[\alpha,\beta]$ commutes with α). Hence, $(\operatorname{ad}\alpha)^n(\beta) = 0$ for every integer $n\geq 2$. Now, by the definition of $\exp(\operatorname{ad}\alpha)$, we have

$$\begin{aligned} (\exp(\operatorname{ad}\alpha))(\beta) &= \sum_{n=0}^{\infty}\frac{(\operatorname{ad}\alpha)^n}{n!}(\beta) = \sum_{n=0}^{\infty}\frac{1}{n!}(\operatorname{ad}\alpha)^n(\beta) \\ &= \underbrace{\frac{1}{0!}}_{=1}\underbrace{(\operatorname{ad}\alpha)^0}_{=\operatorname{id}}(\beta) + \underbrace{\frac{1}{1!}}_{=1}\underbrace{(\operatorname{ad}\alpha)^1}_{=\operatorname{ad}\alpha}(\beta) + \sum_{n=2}^{\infty}\frac{1}{n!}\underbrace{(\operatorname{ad}\alpha)^n(\beta)}_{\substack{=0\\ (\text{since } n\geq 2)}} \\ &= \underbrace{\operatorname{id}(\beta)}_{=\beta} + \underbrace{(\operatorname{ad}\alpha)(\beta)}_{=[\alpha,\beta]} + \underbrace{\sum_{n=2}^{\infty}\frac{1}{n!}0}_{=0} = \beta + [\alpha,\beta]. \end{aligned}$$

By Corollary 3.1.8, we now have

$$(\exp\alpha)\cdot(\exp\beta)\cdot(\exp(-\alpha)) = \exp\underbrace{((\exp(\operatorname{ad}\alpha))(\beta))}_{=\beta+[\alpha,\beta]} = \exp(\beta+[\alpha,\beta]).$$

But β and $[\alpha,\beta]$ commute, so that $\beta[\alpha,\beta] = [\alpha,\beta]\beta$. Hence, Theorem 3.1.4 (applied to β and $[\alpha,\beta]$ instead of α and β) yields $\exp(\beta+[\alpha,\beta]) = (\exp\beta)\cdot(\exp[\alpha,\beta])$.

On the other hand,

$$\begin{aligned} (\exp\alpha)\cdot(\exp\beta)\cdot(\exp(-\alpha))\cdot\left(\exp\underbrace{\alpha}_{=-(-\alpha)}\right) &= (\exp\alpha)\cdot(\exp\beta)\cdot\underbrace{(\exp(-\alpha))\cdot(\exp(-(-\alpha)))}_{\substack{=1\\ (\text{by Corollary 3.1.5, applied to } \gamma=-\alpha)}} \\ &= (\exp\alpha)\cdot(\exp\beta). \end{aligned}$$

Compared with

$$\underbrace{(\exp\alpha)\cdot(\exp\beta)\cdot(\exp(-\alpha))}_{=\exp(\beta+[\alpha,\beta])=(\exp\beta)\cdot(\exp[\alpha,\beta])}\cdot(\exp\alpha) = (\exp\beta)\cdot(\exp[\alpha,\beta])\cdot(\exp\alpha),$$

this yields
$$(\exp\alpha)\cdot(\exp\beta)=(\exp\beta)\cdot(\exp[\alpha,\beta])\cdot(\exp\alpha). \tag{75}$$

Besides, α and $[\alpha,\beta]$ commute, so that $\alpha[\alpha,\beta]=[\alpha,\beta]\alpha$. Hence, Theorem 3.1.4 (applied to $[\alpha,\beta]$ instead of β) yields $\exp(\alpha+[\alpha,\beta])=(\exp\alpha)\cdot(\exp[\alpha,\beta])$.

On the other hand, α and $[\alpha,\beta]$ commute, so that $[\alpha,\beta]\alpha=\alpha[\alpha,\beta]$. Hence, Theorem 3.1.4 (applied to $[\alpha,\beta]$ and α instead of α and β) yields $\exp([\alpha,\beta]+\alpha)=(\exp[\alpha,\beta])\cdot(\exp\alpha)$.

Thus, $(\exp[\alpha,\beta])\cdot(\exp\alpha)=\exp\underbrace{([\alpha,\beta]+\alpha)}_{=\alpha+[\alpha,\beta]}=\exp(\alpha+[\alpha,\beta])=(\exp\alpha)\cdot(\exp[\alpha,\beta])$.

Now, (75) becomes
$$(\exp\alpha)\cdot(\exp\beta)=(\exp\beta)\cdot\underbrace{(\exp[\alpha,\beta])\cdot(\exp\alpha)}_{=(\exp\alpha)\cdot(\exp[\alpha,\beta])}=(\exp\beta)\cdot(\exp\alpha)\cdot(\exp[\alpha,\beta]).$$

This proves Lemma 3.1.9.

Second proof of Lemma 3.1.9. Clearly, $[\beta,\alpha]=-[\alpha,\beta]$ commutes with each of α and β (since $[\alpha,\beta]$ commutes with each of α and β).

The Baker-Campbell-Hausdorff formula has the form
$$(\exp\alpha)\cdot(\exp\beta)=\exp\left(\alpha+\beta+\frac{1}{2}[\alpha,\beta]+(\text{higher terms})\right),$$
where the "higher terms" on the right hand side mean $\mathbb{Q}$-linear combinations of nested Lie brackets of three or more α's and β's. Since $[\alpha,\beta]$ commutes with each of α and β, all of these higher terms are zero, and thus the Baker-Campbell-Hausdorff formula simplifies to
$$(\exp\alpha)\cdot(\exp\beta)=\exp\left(\alpha+\beta+\frac{1}{2}[\alpha,\beta]\right). \tag{76}$$
Applying this to β and α instead of α and β, we obtain
$$(\exp\beta)\cdot(\exp\alpha)=\exp\left(\beta+\alpha+\frac{1}{2}[\beta,\alpha]\right).$$
Since $[\beta,\alpha]=-[\alpha,\beta]$, this becomes
$$(\exp\beta)\cdot(\exp\alpha)=\exp\left(\beta+\alpha+\frac{1}{2}\underbrace{[\beta,\alpha]}_{=-[\alpha,\beta]}\right)=\exp\left(\beta+\alpha-\frac{1}{2}[\alpha,\beta]\right). \tag{77}$$

Now, $[\alpha,\beta]$ commutes with each of α and β (by the assumptions of the lemma) and also with $[\alpha,\beta]$ itself (clearly). Hence, $[\alpha,\beta]$ commutes with $\beta+\alpha-\frac{1}{2}[\alpha,\beta]$. In other words, $\left(\beta+\alpha-\frac{1}{2}[\alpha,\beta]\right)[\alpha,\beta]=[\alpha,\beta]\left(\beta+\alpha-\frac{1}{2}[\alpha,\beta]\right)$. Hence, Theorem 3.1.4 (applied to $\beta+\alpha-\frac{1}{2}[\alpha,\beta]$ and $[\alpha,\beta]$ instead of α and β) yields $\exp\left(\beta+\alpha-\frac{1}{2}[\alpha,\beta]+[\alpha,\beta]\right)=$

$\left(\exp\left(\beta+\alpha-\frac{1}{2}[\alpha,\beta]\right)\right)\cdot(\exp[\alpha,\beta])$. Now,

$$\underbrace{(\exp\beta)\cdot(\exp\alpha)}_{\substack{=\exp\left(\beta+\alpha-\frac{1}{2}[\alpha,\beta]\right)\\ \text{(by (77))}}}\cdot(\exp[\alpha,\beta]) = \left(\exp\left(\beta+\alpha-\frac{1}{2}[\alpha,\beta]\right)\right)\cdot(\exp[\alpha,\beta])$$

$$= \exp\underbrace{\left(\beta+\alpha-\frac{1}{2}[\alpha,\beta]+[\alpha,\beta]\right)}_{=\alpha+\beta+\frac{1}{2}[\alpha,\beta]}$$

$$= \exp\left(\alpha+\beta+\frac{1}{2}[\alpha,\beta]\right) = (\exp\alpha)\cdot(\exp\beta)$$

(by (76)). Lemma 3.1.9 is proven.

We are going to also present a third, very elementary (term-by-term) proof of Lemma 3.1.9. It relies on the following proposition, which can also be applied in some other contexts (e. g., computing in universal enveloping algebras):

Proposition 3.1.10. Let R be a ring. Let $\alpha\in R$ and $\beta\in R$. Assume that $[\alpha,\beta]$ commutes with each of α and β. Then, for every $i\in\mathbb{N}$ and $j\in\mathbb{N}$, we have

$$\alpha^j\beta^i = \sum_{\substack{k\in\mathbb{N};\\ k\leq i;\ k\leq j}} k!\binom{i}{k}\binom{j}{k}\beta^{i-k}\alpha^{j-k}[\alpha,\beta]^k .$$

Proof of Proposition 3.1.10. Let γ denote $[\alpha,\beta]$. Then, γ commutes with each of α and β (since $[\alpha,\beta]$ commutes with each of α and β). In other words, $\gamma\alpha=\alpha\gamma$ and $\gamma\beta=\beta\gamma$.

As we showed in the proof of Lemma 3.1.1, every positive $i\in\mathbb{N}$ satisfies $[\alpha,\beta^i]=i\gamma\beta^{i-1}$. Since $\gamma=[\alpha,\beta]$, this rewrites as follows:

$$\text{every positive } i\in\mathbb{N} \text{ satisfies } \left[\alpha,\beta^i\right]=i[\alpha,\beta]\beta^{i-1}. \tag{78}$$

Since $[\beta,\alpha]=-\underbrace{[\alpha,\beta]}_{=\gamma}=-\gamma$, we see that $\underbrace{[\beta,\alpha]}_{=-\gamma}\alpha=-\underbrace{\gamma\alpha}_{=\alpha\gamma}=-\alpha\gamma=\alpha\underbrace{(-\gamma)}_{=[\beta,\alpha]}=\alpha[\beta,\alpha]$ and $\underbrace{[\beta,\alpha]}_{=-\gamma}\beta=-\underbrace{\gamma\beta}_{=\beta\gamma}=-\beta\gamma=\beta\underbrace{(-\gamma)}_{=[\beta,\alpha]}=\beta[\beta,\alpha]$. In other words, $[\beta,\alpha]$ commutes with each of α and β. Therefore, the roles of α and β are symmetric, and thus we can apply (78) to β and α instead of α and β, and conclude that

$$\text{every positive } i\in\mathbb{N} \text{ satisfies } \left[\beta,\alpha^i\right]=i[\beta,\alpha]\alpha^{i-1}. \tag{79}$$

Thus, every positive $i\in\mathbb{N}$ satisfies $\beta\alpha^i-\alpha^i\beta=[\beta,\alpha^i]=i\underbrace{[\beta,\alpha]}_{=-\gamma}\alpha^{i-1}=-i\gamma\alpha^{i-1}$, so that $\beta\alpha^i=\alpha^i\beta-i\gamma\alpha^{i-1}$ and thus $\alpha^i\beta=\beta\alpha^i+i\gamma\alpha^{i-1}$. We have thus proven that

$$\text{every positive } i\in\mathbb{N} \text{ satisfies } \alpha^i\beta=\beta\alpha^i+i\gamma\alpha^{i-1}. \tag{80}$$

Now, we are going to prove that every $i \in \mathbb{N}$ and $j \in \mathbb{N}$ satisfy

$$\alpha^j \beta^i = \sum_{\substack{k\in\mathbb{N};\\ k\leq i;\ k\leq j}} k! \binom{i}{k} \binom{j}{k} \beta^{i-k} \alpha^{j-k} \gamma^k. \tag{81}$$

Proof of (81): We will prove (81) by induction over i:

Induction base: Let $j \in \mathbb{N}$ be arbitrary. For $i = 0$, we have $\alpha^j \beta^i = \alpha^j \underbrace{\beta^0}_{=1} = \alpha^j$ and

$$\sum_{\substack{k\in\mathbb{N};\\ k\leq i;\ k\leq j}} k! \binom{i}{k} \binom{j}{k} \beta^{i-k} \alpha^{j-k} \gamma^k = \underbrace{\sum_{\substack{k\in\mathbb{N};\\ k\leq 0;\ k\leq j}}}_{=\sum\limits_{k\in\{0\}}} k! \binom{0}{k} \binom{j}{k} \beta^{0-k} \alpha^{j-k} \gamma^k = \sum_{k\in\{0\}} k! \binom{0}{k} \binom{j}{k} \beta^{0-k} \alpha^{j-k} \gamma^k$$

$$= \underbrace{0!}_{=1} \underbrace{\binom{0}{0}}_{=1} \underbrace{\binom{j}{0}}_{=1} \underbrace{\beta^{0-0}}_{=1} \underbrace{\alpha^{j-0}}_{=\alpha^j} \underbrace{\gamma^0}_{=1} = \alpha^j.$$

Hence, for $i = 0$, we have $\alpha^j \beta^i = \alpha^j = \sum\limits_{\substack{k\in\mathbb{N};\\ k\leq i;\ k\leq j}} k! \binom{i}{k} \binom{j}{k} \beta^{i-k} \alpha^{j-k} \gamma^k$. Thus, (81) holds for $i = 0$, so that the induction base is complete.

Induction step: Let $u \in \mathbb{N}$. Assume that (81) holds for $i = u$. We must now prove that (81) holds for $i = u + 1$.

Since (81) holds for $i = u$, we have

$$\alpha^j \beta^u = \sum_{\substack{k\in\mathbb{N};\\ k\leq u;\ k\leq j}} k! \binom{u}{k} \binom{j}{k} \beta^{u-k} \alpha^{j-k} \gamma^k \qquad \text{for every } j \in \mathbb{N}. \tag{82}$$

Now, let $j \in \mathbb{N}$ be positive. Then, $j-1 \in \mathbb{N}$. Now,

$$\alpha^j \underbrace{\beta^{u+1}}_{=\beta\beta^u}$$
$$= \underbrace{\alpha^j \beta}_{\substack{=\beta\alpha^j + j\gamma\alpha^{j-1} \\ \text{(by (80),} \\ \text{applied to } j \text{ instead of } i)}} \beta^u = \left(\beta\alpha^j + j\gamma\alpha^{j-1}\right)\beta^u = \beta\alpha^j\beta^u + j \underbrace{\gamma\alpha^{j-1}\beta^u}_{\substack{=\alpha^{j-1}\beta^u\gamma \\ \text{(since } \gamma \text{ commutes with} \\ \text{each of } \beta \text{ and } \alpha)}}$$
$$= \beta \underbrace{\alpha^j\beta^u}_{\substack{=\sum\limits_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j}} k!\binom{u}{k}\binom{j}{k}\beta^{u-k}\alpha^{j-k}\gamma^k \\ \text{(by (82))}}} + j \underbrace{\alpha^{j-1}\beta^u}_{\substack{=\sum\limits_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j-1}} k!\binom{u}{k}\binom{j-1}{k}\beta^{u-k}\alpha^{j-1-k}\gamma^k \\ \text{(by (82), applied to } j-1 \\ \text{instead of } j \text{ (since } j-1\in\mathbb{N}))}} \gamma$$
$$= \beta \sum_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j}} k!\binom{u}{k}\binom{j}{k}\beta^{u-k}\alpha^{j-k}\gamma^k + j\left(\sum_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j-1}} k!\binom{u}{k}\binom{j-1}{k}\beta^{u-k}\alpha^{j-1-k}\gamma^k\right)\gamma$$
$$= \sum_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j}} k!\binom{u}{k}\binom{j}{k}\underbrace{\beta\beta^{u-k}}_{=\beta^{u+1-k}}\alpha^{j-k}\gamma^k + j\sum_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j-1}} k!\binom{u}{k}\binom{j-1}{k}\beta^{u-k}\alpha^{j-1-k}\underbrace{\gamma^k\gamma}_{=\gamma^{k+1}}$$
$$= \sum_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j}} k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j + j\sum_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j-1}} k!\binom{u}{k}\binom{j-1}{k}\beta^{u-k}\alpha^{j-1-k}\gamma^{k+1}. \tag{83}$$

Let us separately simplify the two addends on the right hand side of this equation.

First of all, every $k \in \mathbb{N}$ which satisfies $k \le u+1$ and $k \le j$ but does **not** satisfy $k \le u$ must satisfy $k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j = 0$ (because this k does not satisfy $k \le u$, so that we have $k > u$, and thus $\binom{u}{k} = 0$). Thus, $\sum\limits_{\substack{k\in\mathbb{N};\\ k\le u+1;\ (\text{not } k\le u);\ k\le j}} k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j = \sum\limits_{\substack{k\in\mathbb{N};\\ k\le u+1;\ (\text{not } k\le u);\ k\le j}} 0 = 0$. Hence,

$$\sum_{\substack{k\in\mathbb{N};\\ k\le u+1;\ k\le j}} k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j$$
$$= \underbrace{\sum_{\substack{k\in\mathbb{N};\\ k\le u+1;\ k\le u;\ k\le j}}}_{=\sum\limits_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j}}} k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j + \underbrace{\sum_{\substack{k\in\mathbb{N};\\ k\le u+1;\ (\text{not } k\le u);\ k\le j}} k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j}_{=0}$$
$$= \sum_{\substack{k\in\mathbb{N};\\ k\le u;\ k\le j}} k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j. \tag{84}$$

On the other hand,

$$\sum_{\substack{k\in\mathbb{N};\\ k\leq u;\ k\leq j-1}} k!\binom{u}{k}\binom{j-1}{k}\beta^{u-k}\alpha^{j-1-k}\gamma^{k+1}$$
$$= \sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} (k-1)!\binom{u}{k-1}\binom{j-1}{k-1}\underbrace{\beta^{u-(k-1)}}_{=\beta^{u+1-k}}\underbrace{\alpha^{j-1-(k-1)}}_{=\alpha^{j-k}}\underbrace{\gamma^{(k-1)+1}}_{=\gamma^{k}}$$
(here, we substituted $k-1$ for k in the sum)
$$= \sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} (k-1)!\binom{u}{k-1}\binom{j-1}{k-1}\beta^{u+1-k}\alpha^{j-k}\gamma^{j}. \tag{85}$$

But every $k\in\mathbb{N}$ satisfying $k\geq 1$ and $k\leq j$ satisfies

$$\binom{j-1}{k-1} = \frac{(j-1)!}{(k-1)!\,((j-1)-(k-1))!} = \frac{(j-1)!}{(k-1)!\,(j-k)!}.$$

Hence, every $k\in\mathbb{N}$ satisfying $k\geq 1$ and $k\leq j$ satisfies

$$(k-1)!\binom{u}{k-1}\underbrace{\binom{j-1}{k-1}}_{=\frac{(j-1)!}{(k-1)!\,(j-k)!}} = (k-1)!\binom{u}{k-1}\frac{(j-1)!}{(k-1)!\,(j-k)!}$$
$$= \binom{u}{k-1}\frac{(j-1)!}{(j-k)!}. \tag{86}$$

But multiplying both sides of (85) with j, we obtain

$$j\sum_{\substack{k\in\mathbb{N};\\ k\leq u;\ k\leq j-1}} k!\binom{u}{k}\binom{j-1}{k}\beta^{u-k}\alpha^{j-1-k}\gamma^{k+1}$$
$$= j\sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} \underbrace{(k-1)!\binom{u}{k-1}\binom{j-1}{k-1}}_{\substack{=\binom{u}{k-1}\frac{(j-1)!}{(j-k)!}\\ \text{(by (86))}}}\beta^{u+1-k}\alpha^{j-k}\gamma^{j}$$
$$= j\sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} \binom{u}{k-1}\frac{(j-1)!}{(j-k)!}\beta^{u+1-k}\alpha^{j-k}\gamma^{j} = \sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} \binom{u}{k-1}j\frac{(j-1)!}{(j-k)!}\beta^{u+1-k}\alpha^{j-k}\gamma^{j}. \tag{87}$$

But every $k\in\mathbb{N}$ satisfying $k\geq 1$ and $k\leq j$ satisfies

$$\binom{j}{k} = \frac{j!}{k!\,(j-k)!} = \frac{j\,(j-1)!}{k!\,(j-k)!} \qquad (\text{since } j! = j\,(j-1)!)$$
$$= \frac{1}{k!}\cdot j\frac{(j-1)!}{(j-k)!}.$$

Hence, every $k \in \mathbb{N}$ satisfying $k \geq 1$ and $k \leq j$ satisfies

$$k!\binom{j}{k} = j\frac{(j-1)!}{(j-k)!}. \tag{88}$$

Thus, (87) becomes

$$\begin{aligned}
& j \sum_{\substack{k\in\mathbb{N};\\ k\leq u;\ k\leq j-1}} k!\binom{u}{k}\binom{j-1}{k}\beta^{u-k}\alpha^{j-1-k}\gamma^{k+1} \\
&= \sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} \binom{u}{k-1}\underbrace{j\frac{(j-1)!}{(j-k)!}}_{\substack{=k!\binom{j}{k} \\ \text{(by (88))}}}\beta^{u+1-k}\alpha^{j-k}\gamma^{j} \\
&= \sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} \binom{u}{k-1}k!\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j} = \sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j}.
\end{aligned} \tag{89}$$

But every $k \in \mathbb{N}$ which satisfies $k \leq u+1$ and $k \leq j$ but does **not** satisfy $k \geq 1$ must satisfy $k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j} = 0$ (because this k does not satisfy $k \geq 1$, so that we have $k < 1$, and thus $\binom{u}{k-1} = 0$). Thus, $\sum_{\substack{k\in\mathbb{N};\ (\text{not } k\geq 1);\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j} = \sum_{\substack{k\in\mathbb{N};\ (\text{not } k\geq 1);\\ k\leq u+1;\ k\leq j}} 0 = 0$. Hence,

$$\begin{aligned}
& \sum_{\substack{k\in\mathbb{N}\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j} \\
&= \sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j} + \underbrace{\sum_{\substack{k\in\mathbb{N};\ (\text{not } k\geq 1);\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j}}_{=0} \\
&= \sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j}.
\end{aligned} \tag{90}$$

Thus, (89) becomes

$$\begin{aligned}
& j \sum_{\substack{k\in\mathbb{N};\\ k\leq u;\ k\leq j-1}} k!\binom{u}{k}\binom{j-1}{k}\beta^{u-k}\alpha^{j-1-k}\gamma^{k+1} \\
&= \sum_{\substack{k\in\mathbb{N};\ k\geq 1;\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j} = \sum_{\substack{k\in\mathbb{N};\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^{j}
\end{aligned} \tag{91}$$

(by (90)).

Also, notice that every $k \in \mathbb{N}$ satisfies

$$k!\binom{u}{k} + k!\binom{u}{k-1} = k! \underbrace{\left(\binom{u}{k} + \binom{u}{k-1}\right)}_{\substack{=\binom{u+1}{k} \\ \text{(by the recurrence equation} \\ \text{of the binomial coefficients)}}} = k!\binom{u+1}{k}. \tag{92}$$

Now, (83) becomes

$$\begin{aligned}
\alpha^j \beta^{u+1} &= \underbrace{\sum_{\substack{k\in\mathbb{N};\\ k\leq u;\ k\leq j}} k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j}_{\substack{=\sum\limits_{\substack{k\in\mathbb{N};\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j \\ \text{(by (84))}}} + \underbrace{j\sum_{\substack{k\in\mathbb{N};\\ k\leq u;\ k\leq j-1}} k!\binom{u}{k}\binom{j-1}{k}\beta^{u-k}\alpha^{j-1-k}\gamma^{k+1}}_{\substack{=\sum\limits_{\substack{k\in\mathbb{N};\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j \\ \text{(by (91))}}} \\
&= \sum_{\substack{k\in\mathbb{N};\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j + \sum_{\substack{k\in\mathbb{N};\\ k\leq u+1;\ k\leq j}} k!\binom{u}{k-1}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j \\
&= \sum_{\substack{k\in\mathbb{N};\\ k\leq u+1;\ k\leq j}} \underbrace{\left(k!\binom{u}{k} + k!\binom{u}{k-1}\right)}_{\substack{=k!\binom{u+1}{k} \\ \text{(by (92))}}} \binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j \\
&= \sum_{\substack{k\in\mathbb{N};\\ k\leq u+1;\ k\leq j}} k!\binom{u+1}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j.
\end{aligned}$$

Now, forget that we fixed j. We thus have shown that

$$\alpha^j \beta^{u+1} = \sum_{\substack{k\in\mathbb{N};\\ k\leq u+1;\ k\leq j}} k!\binom{u+1}{k}\binom{j}{k}\beta^{u+1-k}\alpha^{j-k}\gamma^j \tag{93}$$

holds for every positive $j \in \mathbb{N}$. Since it is easy to see that (93) also holds for $j = 0$ (the proof is similar to our induction base above), this yields that (93) holds for every $j \in \mathbb{N}$. In other words, (81) holds for $i = u + 1$. Thus, the induction step is complete. Hence, we have proven (81) by induction over i.

Since $\gamma = [\alpha, \beta]$, the (now proven) identity (81) rewrites as

$$\alpha^j \beta^i = \sum_{\substack{k\in\mathbb{N};\\ k\leq i;\ k\leq j}} k!\binom{i}{k}\binom{j}{k}\beta^{i-k}\alpha^{j-k}\underbrace{\gamma}_{=[\alpha,\beta]}{}^k = \sum_{\substack{k\in\mathbb{N};\\ k\leq i;\ k\leq j}} k!\binom{i}{k}\binom{j}{k}\beta^{i-k}\alpha^{j-k}[\alpha,\beta]^k.$$

Proposition 3.1.10 is thus proven.

Third proof of Lemma 3.1.9. By the definition of the exponential, we have $\exp[\alpha,\beta] = \sum_{k\in\mathbb{N}} \frac{[\alpha,\beta]^k}{k!}$, $\exp\alpha = \sum_{j\in\mathbb{N}} \frac{\alpha^j}{j!}$ and $\exp\beta = \sum_{i\in\mathbb{N}} \frac{\beta^i}{i!}$. Multiplying the last two of these three equalities, we obtain

$$
\begin{aligned}
&(\exp\alpha)\cdot(\exp\beta)\\
&= \left(\sum_{j\in\mathbb{N}} \frac{\alpha^j}{j!}\right)\cdot\left(\sum_{i\in\mathbb{N}} \frac{\beta^i}{i!}\right) = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \frac{\alpha^j}{j!}\cdot\frac{\beta^i}{i!} = \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \frac{1}{i!j!} \underbrace{\alpha^j\beta^i}_{\substack{=\sum\limits_{\substack{k\in\mathbb{N};\\ k\leq i;\ k\leq j}} k!\binom{i}{k}\binom{j}{k}\beta^{i-k}\alpha^{j-k}[\alpha,\beta]^k \\ \text{(by Proposition 3.1.10)}}}\\
&= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \frac{1}{i!j!} \sum_{\substack{k\in\mathbb{N};\\ k\leq i;\ k\leq j}} k!\binom{i}{k}\binom{j}{k}\beta^{i-k}\alpha^{j-k}[\alpha,\beta]^k\\
&= \underbrace{\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \sum_{\substack{k\in\mathbb{N};\\ k\leq i;\ k\leq j}}}_{=\sum\limits_{k\in\mathbb{N}}\sum\limits_{\substack{i\in\mathbb{N};\\ k\leq i}}\sum\limits_{\substack{j\in\mathbb{N};\\ k\leq j}}} \underbrace{\frac{1}{i!j!}k!\binom{i}{k}\binom{j}{k}}_{\substack{=\frac{1}{(i-k)!(j-k)!k!}\\ \text{(by easy computations)}}} \beta^{i-k}\alpha^{j-k}[\alpha,\beta]^k\\
&= \sum_{k\in\mathbb{N}}\sum_{\substack{i\in\mathbb{N};\\ k\leq i}}\sum_{\substack{j\in\mathbb{N};\\ k\leq j}} \frac{1}{(i-k)!(j-k)!k!}\beta^{i-k}\alpha^{j-k}[\alpha,\beta]^k = \sum_{k\in\mathbb{N}}\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}} \frac{1}{i!j!k!}\beta^i\alpha^j[\alpha,\beta]^k\\
&\qquad \left(\begin{array}{c}\text{here, we substituted } i \text{ for } i-k \text{ in the second sum,}\\ \text{and we substituted } j \text{ for } j-k \text{ in the third sum}\end{array}\right)\\
&= \sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}\sum_{k\in\mathbb{N}} \frac{\beta^i}{i!}\cdot\frac{\alpha^j}{j!}\cdot\frac{[\alpha,\beta]^k}{k!} = \underbrace{\left(\sum_{i\in\mathbb{N}}\frac{\beta^i}{i!}\right)}_{=\exp\beta}\cdot\underbrace{\left(\sum_{j\in\mathbb{N}}\frac{\alpha^j}{j!}\right)}_{=\exp\alpha}\cdot\underbrace{\left(\sum_{k\in\mathbb{N}}\frac{[\alpha,\beta]^k}{k!}\right)}_{=\exp[\alpha,\beta]}\\
&= (\exp\beta)\cdot(\exp\alpha)\cdot(\exp[\alpha,\beta]).
\end{aligned}
$$

This proves Lemma 3.1.9 once again.

3.2. Representations of Vir on F_μ

3.2.1. The Lie-algebraic semidirect product: the general case

Let us define the "full-fledged" version of the Lie-algebraic semidirect product, although it will not be central to what we will later do:

Definition 3.2.1. Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{h}$ be a vector space equipped with both a Lie algebra structure and a $\mathfrak{g}$-module structure.

(a) Let $\rho : \mathfrak{g} \to \operatorname{End}\mathfrak{h}$ be the map representing the action of $\mathfrak{g}$ on $\mathfrak{h}$. We say that $\mathfrak{g}$ *acts on* $\mathfrak{h}$ *by derivations* if $\rho(\mathfrak{g}) \subseteq \operatorname{Der}\mathfrak{h}$, or, equivalently, if the map

$$\mathfrak{h}\to\mathfrak{h}, \qquad x\mapsto a \rightharpoonup x$$

is a derivation for every $a \in \mathfrak{g}$. (Here and in the following, the symbol $\rightharpoonup$ means action; i. e., a term like $c \rightharpoonup h$ (with $c \in \mathfrak{g}$ and $h \in \mathfrak{h}$) means the action of c on h.)

(b) Assume that $\mathfrak{g}$ acts on $\mathfrak{h}$ by derivations. Then, we define the *semidirect product* $\mathfrak{g} \ltimes \mathfrak{h}$ to be the Lie algebra which, as a vector space, is $\mathfrak{g} \oplus \mathfrak{h}$, but whose Lie bracket is defined by

$$[(a, \alpha), (b, \beta)] = ([a, b], [\alpha, \beta] + a \rightharpoonup \beta - b \rightharpoonup \alpha) \qquad \text{for all } a \in \mathfrak{g},\ \alpha \in \mathfrak{h},\ b \in \mathfrak{g} \text{ and } \beta \in \mathfrak{h} .$$

Thus, the canonical injection $\mathfrak{g} \to \mathfrak{g} \ltimes \mathfrak{h}$, $a \mapsto (a, 0)$ is a Lie algebra homomorphism, and so is the canonical projection $\mathfrak{g} \ltimes \mathfrak{h} \to \mathfrak{g}$, $(a, \alpha) \mapsto a$. Also, the canonical injection $\mathfrak{h} \to \mathfrak{g} \ltimes \mathfrak{h}$, $\alpha \mapsto (0, \alpha)$ is a Lie algebra homomorphism.

All statements made in Definition 3.2.1 (including the tacit statement that the Lie bracket on $\mathfrak{g} \ltimes \mathfrak{h}$ defined in Definition 3.2.1 satisfies antisymmetry and the Jacobi identity) are easy to verify by computation.

Remark 3.2.2. If $\mathfrak{g}$ is a Lie algebra, and $\mathfrak{h}$ is an **abelian** Lie algebra with any $\mathfrak{g}$-module structure, then $\mathfrak{g}$ automatically acts on $\mathfrak{h}$ by derivations (because any endomorphism of the vector space $\mathfrak{h}$ is a derivation), and thus Definition 3.2.1 **(b)** defines a semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$. In this case, this semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$ coincides with the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$ defined in Definition 1.7.7 (applied to $M = \mathfrak{h}$). However, when $\mathfrak{h}$ is not abelian, the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$ defined in Definition 3.2.1 (in general) differs from that defined in Definition 1.7.7 (since the former depends on the Lie algebra structure on $\mathfrak{h}$, while the latter does not). Care must therefore be taken when speaking of semidirect products.

An example for the semidirect product construction given in Definition 3.2.1 **(b)** is given by the following proposition:

Proposition 3.2.3. Consider the Witt algebra W, the Virasoro algebra Vir and the Heisenberg algebra $\mathcal{A}$.

(a) In Lemma 1.4.3, we constructed a homomorphism $\eta : W \to \operatorname{Der} \mathcal{A}$ of Lie algebras. This homomorphism η makes $\mathcal{A}$ into a W-module, and W acts on $\mathcal{A}$ by derivations. Therefore, a Lie algebra $W \ltimes \mathcal{A}$ is defined (according to Definition 3.2.1 **(b)**).

(b) There is a natural homomorphism $\widetilde{\eta} : \mathrm{Vir} \to \operatorname{Der} \mathcal{A}$ of Lie algebras given by

$$(\widetilde{\eta}(f\partial + \lambda K))(g, \alpha) = (fg', 0) \qquad \text{for all } f \in \mathbb{C}\left[t, t^{-1}\right],\ g \in \mathbb{C}\left[t, t^{-1}\right],\ \lambda \in \mathbb{C} \text{ and } \alpha \in \mathbb{C}.$$

This homomorphism $\widetilde{\eta}$ is simply the extension of the homomorphism $\eta : W \to \operatorname{Der} \mathcal{A}$ (defined in Lemma 1.4.3) to Vir by means of requiring that $\widetilde{\eta}(K) = 0$.

This homomorphism $\widetilde{\eta}$ makes $\mathcal{A}$ a Vir-module, and Vir acts on $\mathcal{A}$ by derivations. Therefore, a Lie algebra $\mathrm{Vir} \ltimes \mathcal{A}$ is defined (according to Definition 3.2.1 **(b)**).

The proof of Proposition 3.2.3 is straightforward and left to the reader.

3.2.2. The action of Vir on F_μ

Let us now return to considering the Witt and Heisenberg algebras.

According to Proposition 3.2.3 **(a)**, we have a Lie algebra $W \ltimes \mathcal{A}$, of which $\mathcal{A}$ is a Lie subalgebra. Now, recall (from Definition 2.2.5) that, for every $\mu \in \mathbb{C}$, we have a representation F_μ of the Lie algebra $\mathcal{A}$ on the Fock space F.

Can we extend this representation F_μ of $\mathcal{A}$ to a representation of the semidirect product $W \ltimes \mathcal{A}$?

This question splits into two questions:

Question 1: Can we find linear operators $L_n : F_\mu \to F_\mu$ for all $n \in \mathbb{Z}$ such that $[L_n, a_m] = -ma_{n+m}$? (Note that there are several abuses of notation in this question. First, we denote the sought operators $L_n : F_\mu \to F_\mu$ by the same letters as the elements L_n of W because our intuition for the L_n is as if they would form a representation of W, although we do not actually require them to form a representation of W in Question 1. Second, in the equation $[L_n, a_m] = -ma_{n+m}$, we use a_m and a_{n+m} as abbreviations for $a_m \mid_{F_\mu}$ and $a_{n+m} \mid_{F_\mu}$, respectively (so that this equation actually means $\left[L_n, a_m \mid_{F_\mu}\right] = -ma_{n+m} \mid_{F_\mu}$).)

Question 2: Do the operators $L_n : F_\mu \to F_\mu$ that answer Question 1 also satisfy $[L_n, L_m] = (n-m) L_{n+m}$? (In other words, do they really form a representation of W ?)

The answers to these questions are the following:

Answer to Question 1: Yes, and moreover, these operators are unique up to adding a constant (a new constant for each operator). (The uniqueness is rather easy to prove: If we have two families $(L'_n)_{n\in\mathbb{Z}}$ and $(L''_n)_{n\in\mathbb{Z}}$ of linear maps $F_\mu \to F_\mu$ satisfying $[L'_n, a_m] = -ma_{n+m}$ and $[L''_n, a_m] = -ma_{n+m}$, then every $L'_n - L''_n$ commutes with all a_m, and thus is constant by Dixmier's lemma.)

Answer to Question 2: No, but almost. Our operators L_n satisfy $[L_n, L_m] = (n-m) L_{n+m}$ whenever $n + m \neq 0$, but the $n + m = 0$ case requires a correction term. This correction term (as a function of (L_n, L_m)) happens to be the 2-cocycle ω of Theorem 1.5.2. So the $\mathcal{A}$-module F_μ does not extend to a $W \ltimes \mathcal{A}$-module, but extends to a $\mathrm{Vir} \ltimes \mathcal{A}$-module, where $\mathrm{Vir} \ltimes \mathcal{A}$ is defined as in Proposition 3.2.3 **(b)**.

Now we are going to prove the answers to Questions 1 and 2 formulated above. First, we must define our operators L_n. "Formally" (in the sense of "not caring about divergence of sums"), one could try to define L_n by

$$L_n = \frac{1}{2} \sum_{m\in\mathbb{Z}} a_{-m} a_{n+m} \qquad \text{for all } n \in \mathbb{Z} \tag{94}$$

(where a_ℓ is shorthand notation for $a_\ell \mid_{F_\mu}$ for every $\ell \in \mathbb{Z}$), and this would "formally" make F_μ into a $W \ltimes \mathcal{A}$-module (in the sense that if the sums were not divergent, one could manipulate them to "prove" that $[L_n, a_m] = -ma_{n+m}$ and $[L_n, L_m] = (n-m) L_{n+m}$ for all n and m). But the problem with this "formal" approach is that the sum $\sum\limits_{m\in\mathbb{Z}} a_{-m} a_{n+m}$ does not make sense for $n = 0$: it is an infinite sum, and infinitely many of its terms yield nonzero values when applied to a given vector.[83] So we

[83]In fact, assume that this sum would make sense for $n = 0$. Thus we would have $L_0 = \frac{1}{2} \sum\limits_{m\in\mathbb{Z}} a_{-m} a_m$.

are not allowed to make the definition (94), and we cannot rescue it just by defining a more liberal notion of convergence. Instead, we must modify this "definition".

In order to modify it, we define the so-called *normal ordering*:

Definition 3.2.4. For any two integers m and n, define the *normal ordered product* $: a_m a_n :$ in the universal enveloping algebra $U(\mathcal{A})$ by

$$: a_m a_n : \ = \begin{cases} a_m a_n, & \text{if } m \leq n; \\ a_n a_m, & \text{if } m > n \end{cases}.$$

More generally, for any integers n_1, n_2, ..., n_k, define the *normal ordered product* $: a_{n_1} a_{n_2} ... a_{n_k} :$ in the universal enveloping algebra $U(\mathcal{A})$ by

$$: a_{n_1} a_{n_2} ... a_{n_k} : \ = \left(\begin{array}{c} \text{the product of the elements } a_{n_1}, a_{n_2}, ..., a_{n_k} \text{ of } U(\mathcal{A}), \\ \text{rearranged in such a way that the subscripts are in increasing order} \end{array} \right).$$

(More formally, this normal ordered product $: a_{n_1} a_{n_2} ... a_{n_k} :$ is defined as the product $a_{m_1} a_{m_2} ... a_{m_k}$, where $(m_1, m_2, ..., m_k)$ is the permutation of the list $(n_1, n_2, ..., n_k)$ satisfying $m_1 \leq m_2 \leq ... \leq m_k$.)

Note that we have thus defined only normal ordered products of elements of the form a_n for $n \in \mathbb{Z}$. Normal ordered products of basis elements of other Lie algebras are not always defined by the same formulas (although sometimes they are).

Remark 3.2.5. If m and n are integers such that $m \neq -n$, then $: a_m a_n : = a_m a_n$. (This is because $[a_m, a_n] = 0$ in $\mathcal{A}$ when $m \neq -n$.)

Normal ordered products have the property of being commutative:

Remark 3.2.6. (a) Any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ satisfy $: a_m a_n : = : a_n a_m :$.

(b) Any integers n_1, n_2, ..., n_k and any permutation $\pi \in S_k$ satisfy $: a_{n_1} a_{n_2} ... a_{n_k} : = : a_{n_{\pi(1)}} a_{n_{\pi(2)}} ... a_{n_{\pi(k)}} :$.

The proof of this is trivial.

By Remark 3.2.5 (and by the rather straightforward generalization of this fact to many integers), normal ordered products are rarely different from the usual products. But even when they are different, they don't differ much:

Applied to the vector $1 \in F_0$, this would give $L_0 1 = \frac{1}{2} \sum\limits_{m \in \mathbb{Z}} a_{-m} a_m 1$. The terms for $m > 0$ will get killed (since $a_m 1 = 0$ for $m > 0$), but the terms for $m \leq 0$ will survive. The sum would become

$$\begin{aligned} L_0 1 &= \frac{1}{2} \left(a_0 a_{-0} 1 + a_1 a_{-1} 1 + a_2 a_{-2} 1 + a_3 a_{-3} 1 + ... \right) \\ &= \frac{1}{2} \left(\mu^2 1 + 1 \frac{\partial}{\partial x_1} x_1 + 2 \frac{\partial}{\partial x_2} x_2 + 3 \frac{\partial}{\partial x_3} x_3 + ... \right) = \frac{1}{2} \left(\mu^2 + 1 + 2 + 3 + ... \right). \end{aligned}$$

Unless we interpret $1 + 2 + 3 + ...$ as $-\frac{1}{12}$ (which we are going to do in some sense: the modified formulae further below include $-\frac{1}{12}$ factors), this makes no sense.

Remark 3.2.7. Let m and n be integers.

(a) Then, $: a_m a_n : = a_m a_n + n [m > 0] \delta_{m,-n} K$. Here, when $\mathfrak{A}$ is an assertion, we denote by $[\mathfrak{A}]$ the truth value of $\mathfrak{A}$ (that is, the number $\left(\begin{cases} 1, & \text{if } \mathfrak{A} \text{ is true;} \\ 0, & \text{if } \mathfrak{A} \text{ is false} \end{cases} \right)$.

(b) For any $x \in U(\mathcal{A})$, we have $[x, : a_m a_n :] = [x, a_m a_n]$ (where $[\cdot, \cdot]$ denotes the commutator in $U(\mathcal{A})$).

Note that when we denote by $[\cdot, \cdot]$ the commutator in $U(\mathcal{A})$, we are seemingly risking a confusion with the notation $[\cdot, \cdot]$ for the Lie bracket of $\mathcal{A}$ (because we embed $\mathcal{A}$ in $U(\mathcal{A})$). However, this confusion is harmless, because the very definition of $U(\mathcal{A})$ ensures that the commutator of two elements of $\mathcal{A}$, taken in $U(\mathcal{A})$, equals to their Lie bracket in $\mathcal{A}$.

Proof of Remark 3.2.7. **(a)** We distinguish between three cases:

Case 1: We have $m \neq -n$.

Case 2: We have $m = -n$ and $m > 0$.

Case 3: We have $m = -n$ and $m \leq 0$.

In Case 1, we have $m \neq -n$, so that $\delta_{m,-n} = 0$ and thus

$$a_m a_n + n [m > 0] \underbrace{\delta_{m,-n}}_{=0} K = a_m a_n = : a_m a_n : \qquad \text{(by Remark 3.2.5)}.$$

Hence, Remark 3.2.7 **(a)** is proven in Case 1.

In Case 2, we have $m = -n$ and $m > 0$, so that $m > n$, and thus

$$\begin{aligned} : a_m a_n : &= \begin{cases} a_m a_n, & \text{if } m \leq n; \\ a_n a_m, & \text{if } m > n \end{cases} = a_n a_m \qquad (\text{since } m > n) \\ &= a_m a_n + \underbrace{[a_n, a_m]}_{= n \delta_{n,-m} K = n 1 \delta_{m,-n} K} = a_m a_n + n \underbrace{1}_{\substack{=[m>0] \\ (\text{since } m>0)}} \delta_{m,-n} K = a_m a_n + n [m > 0] \delta_{m,-n} K. \end{aligned}$$

Hence, Remark 3.2.7 **(a)** is proven in Case 2.

In Case 3, we have $m = -n$ and $m \leq 0$, so that $m \leq n$, and thus

$$\begin{aligned} : a_m a_n : &= \begin{cases} a_m a_n, & \text{if } m \leq n; \\ a_n a_m, & \text{if } m > n \end{cases} = a_m a_n \qquad (\text{since } m \leq n) \\ &= a_m a_n + \underbrace{0}_{\substack{= n [m>0] \delta_{m,-n} K \\ (\text{since } m \leq 0, \text{ so that (not } m>0), \text{ thus} \\ [m>0]=0 \text{ and hence } n[m>0]\delta_{m,-n}K=0)}} = a_m a_n + n [m > 0] \delta_{m,-n} K. \end{aligned}$$

Hence, Remark 3.2.7 **(a)** is proven in Case 3.

Thus, we have proven Remark 3.2.7 **(a)** in all three possible cases. This completes the proof of Remark 3.2.7 **(a)**.

(b) We have $K \in Z(\mathcal{A}) \subseteq Z(U(\mathcal{A}))$ (since the center of a Lie algebra is contained in the center of its universal enveloping algebra). Hence, $[x, K] = 0$ for any $x \in U(\mathcal{A})$.

Since $: a_m a_n : = a_m a_n + n [m > 0] \delta_{m,-n} K$, we have

$$\begin{aligned} [x, : a_m a_n :] &= [x, a_m a_n + n [m > 0] \delta_{m,-n} K] \\ &= [x, a_m a_n] + n [m > 0] \delta_{m,-n} \underbrace{[x, K]}_{=0} = [x, a_m a_n] \end{aligned}$$

for every $x \in U(\mathcal{A})$. This proves Remark 3.2.7 **(b)**.

Now, the true definition of our maps $L_n : F_\mu \to F_\mu$ will be the following:

Definition 3.2.8. For every $n \in \mathbb{Z}$ and $\mu \in \mathbb{C}$, define a linear map $L_n : F_\mu \to F_\mu$ by

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_{-m} a_{n+m} : \tag{95}$$

(where a_ℓ is shorthand notation for $a_\ell \mid_{F_\mu}$ for every $\ell \in \mathbb{Z}$). This sum $\sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{n+m} :$ is an infinite sum, but it is well-defined in the following sense: For any vector $v \in F_\mu$, applying $\sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{n+m} :$ to the vector v gives the sum $\sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{n+m} : v$, which has only finitely many nonzero addends (because of Lemma 3.2.10 **(c)** below) and thus has a well-defined value.

Note that we have not defined the meaning of the sum $\sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{n+m} :$ in the universal enveloping algebra $U(\mathcal{A})$ itself, but only its meaning as an endomorphism of F_μ. However, if we wanted, we could also define the sum $\sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{n+m} :$ as an element of a suitable completion of the universal enveloping algebra $U(\mathcal{A})$ (although not in $U(\mathcal{A})$ itself). We don't really have a reason to do so here, however.

Convention 3.2.9. During the rest of Section 3.2, we are going to use the labels L_n for the maps $L_n : F_\mu \to F_\mu$ introduced in Definition 3.2.8, and **not** for the eponymous elements of the Virasoro algebra Vir or of the Witt algebra W, unless we explicitly refer to "the element L_n of Vir" or "the element L_n of W" or something similarly unambiguous.

(While it is correct that the maps $L_n : F_\mu \to F_\mu$ satisfy the same relations as the eponymous elements L_n of Vir (but not the eponymous elements L_n of W), this is a nontrivial fact that needs to be proven, and until it is proven we must avoid any confusion between these different meanings of L_n.)

Let us first show that Definition 3.2.8 makes sense:

Lemma 3.2.10. Let $n \in \mathbb{Z}$ and $\mu \in \mathbb{C}$. Let $v \in F_\mu$. Then:

(a) If $m \in \mathbb{Z}$ is sufficiently high, then $: a_{-m} a_{n+m} : v = 0$.

(b) If $m \in \mathbb{Z}$ is sufficiently low, then $: a_{-m} a_{n+m} : v = 0$.

(c) All but finitely many $m \in \mathbb{Z}$ satisfy $: a_{-m} a_{n+m} : v = 0$.

Proof of Lemma 3.2.10. **(a)** Since $v \in F_\mu \in \mathbb{C}[x_1, x_2, x_3, ...]$, the vector v is a polynomial in infinitely many variables. Since every polynomial contains only finitely many variables, there exists an integer $N \in \mathbb{N}$ such that no variable x_r with $r > N$ occurs in v. Consider this N. Then,

$$\frac{\partial}{\partial x_r} v = 0 \qquad \text{for every integer } r > N. \tag{96}$$

Now, let $m \geq \max\left\{-n + N + 1, -\frac{1}{2}n\right\}$. Then, $m \geq -n + N + 1$ and $m \geq -\frac{1}{2}n$.

Since $m \geq -\frac{1}{2}n$, we have $2m \geq -n$, so that $-m \leq n+m$.

From $m \geq -n+N+1$, we get $n+m \geq N+1$, so that $n+m > 0$. Hence, $a_{n+m} \mid_{F_\mu} = (n+m)\frac{\partial}{\partial x_{n+m}}$, so that $a_{n+m}v = (n+m)\frac{\partial}{\partial x_{n+m}}v$. Since $\frac{\partial}{\partial x_{n+m}}v = 0$ (by (96), applied to $r = n+m$ (since $n+m \geq N+1 > N$)), we thus have $a_{n+m}v = 0$.

By Definition 3.2.4, we have

$$: a_{-m}a_{n+m} : \ = \begin{cases} a_{-m}a_{n+m}, & \text{if } -m \leq n+m; \\ a_{n+m}a_{-m}, & \text{if } -m > n+m \end{cases}.$$

Since $-m \leq n+m$, this rewrites as $: a_{-m}a_{n+m} : = a_{-m}a_{n+m}$. Thus, $: a_{-m}a_{n+m} : v = a_{-m}\underbrace{a_{n+m}v}_{=0} = 0$, and Lemma 3.2.10 **(a)** is proven.

(b) Applying Lemma 3.2.10 **(a)** to $-n-m$ instead of m, we see that, if $m \in \mathbb{Z}$ is sufficiently low, then $: a_{-(-n-m)}a_{n+(-n-m)} : v = 0$. Since

$$: a_{-(-n-m)}a_{n+(-n-m)} : = : a_{n+m}a_{-m} : = : a_{-m}a_{n+m} : \qquad \text{(by Remark 3.2.6 **(a)**)},$$

this rewrites as follows: If $m \in \mathbb{Z}$ is sufficiently low, then $: a_{-m}a_{n+m} : v = 0$. This proves Lemma 3.2.10 **(b)**.

(c) Lemma 3.2.10 **(c)** follows immediately by combining Lemma 3.2.10 **(a)** and Lemma 3.2.10 **(b)**.

Remark 3.2.11. **(a)** If $n \neq 0$, then the operator L_n defined in Definition 3.2.8 can be rewritten as

$$L_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} a_{-m}a_{n+m}.$$

In other words, for $n \neq 0$, our old definition (94) of L_n makes sense and is equivalent to the new definition (Definition 3.2.8).

(b) But when $n = 0$, the formula (94) is devoid of sense, whereas Definition 3.2.8 is legit. However, we can rewrite the definition of L_0 without using normal ordered products: Namely, we have

$$L_0 = \sum_{m>0} a_{-m}a_m + \frac{a_0^2}{2} = \sum_{m>0} a_{-m}a_m + \frac{\mu^2}{2}.$$

(c) Let us grade the space F_μ as in Definition 2.2.7. (Recall that this is the grading which gives every variable x_i the degree $-i$ and makes $F_\mu = \mathbb{C}[x_1, x_2, x_3, ...]$ into a graded $\mathbb{C}$-algebra. This is **not** the modified grading that we gave to the space F_μ in Remark 2.2.8.) Let $d \in \mathbb{N}$. Then, every homogeneous polynomial $f \in F_\mu$ of degree d (with respect to this grading) satisfies $L_0 f = \left(\frac{\mu^2}{2} - d\right) f$.

(d) Consider the grading on F_μ defined in part **(c)**. For every $n \in \mathbb{Z}$, the map $L_n : F_\mu \to F_\mu$ is homogeneous of degree n. (The notion "homogeneous of degree n" we are using here is that defined in Definition 3.3.8 **(a)**, not the one defined in Definition 2.6.16 **(a)**.)

Proof of Remark 3.2.11. **(a)** Let $n \neq 0$. Then, every $m \in \mathbb{Z}$ satisfies $-m \neq -(n+m)$ and thus $: a_{-m}a_{n+m} : = a_{-m}a_{n+m}$ (by Remark 3.2.5, applied to $-m$ and $n+m$ instead of m and n). Hence, the formula $L_n = \frac{1}{2}\sum\limits_{m\in\mathbb{Z}} : a_{-m}a_{n+m} :$ (which is how we defined L_n) rewrites as $L_n = \frac{1}{2}\sum\limits_{m\in\mathbb{Z}} a_{-m}a_{n+m}$. This proves Remark 3.2.11 **(a)**.

(b) By the definition of L_0 (in Definition 3.2.8), we have

$$\begin{aligned}
L_0 &= \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_{-m}a_{0+m} : = \frac{1}{2}\sum_{m\in\mathbb{Z}} : a_{-m}a_m : \\
&= \frac{1}{2}\left(\sum_{m<0} \underbrace{: a_{-m}a_m :}_{\substack{=a_m a_{-m} \\ \text{(by the definition of } :a_{-m}a_m: \\ \text{(since } m<0 \text{ and thus } -m>m))}} + \underbrace{: a_{-0}a_0 :}_{\substack{=:a_0a_0:=a_0a_0 \\ \text{(by the definition of } :a_0a_0: \\ \text{(since } 0\leq 0))}} + \sum_{m>0} \underbrace{: a_{-m}a_m :}_{\substack{=a_{-m}a_m \\ \text{(by the definition of } :a_{-m}a_m: \\ \text{(since } m>0 \text{ and thus } -m\leq m))}}\right) \\
&= \frac{1}{2}\left(\underbrace{\sum_{m<0} a_m a_{-m}}_{\substack{=\sum\limits_{m>0} a_{-m}a_m \\ \text{(here, we substituted } m \text{ for } -m \text{ in the sum)}}} + \underbrace{a_0a_0}_{=a_0^2} + \sum_{m>0} a_{-m}a_m\right) \\
&= \frac{1}{2}\left(\sum_{m>0} a_{-m}a_m + a_0^2 + \sum_{m>0} a_{-m}a_m\right) = \frac{1}{2}\left(2\sum_{m>0} a_{-m}a_m + a_0^2\right) = \sum_{m>0} a_{-m}a_m + \frac{a_0^2}{2} \\
&= \sum_{m>0} a_{-m}a_m + \frac{\mu^2}{2} \qquad \text{(since } a_0 \text{ acts as multiplication with } \mu \text{ on } F_\mu)
\end{aligned}$$

on F_μ. This proves Remark 3.2.11 **(b)**.

(c) We must prove the equation $L_0 f = \left(\frac{\mu^2}{2} - d\right) f$ for every homogeneous polynomial $f \in F_\mu$ of degree d. Since this equation is linear in f, it is clearly enough to prove this for the case of f being a monomial[84] of degree d. So let f be a monomial of degree d. Then, f can be written in the form $f = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}...$ for a sequence $(\alpha_1, \alpha_2, \alpha_3, ...)$ of nonnegative integers such that $\sum\limits_{m>0}(-m)\alpha_m = d$ (the $-m$ coefficient comes from $\deg(x_m) = -m$) and such that all but finitely many $i \in \{1, 2, 3, ...\}$ satisfy $\alpha_i = 0$. Consider this sequence. Clearly, $\sum\limits_{m>0}(-m)\alpha_m = d$ yields $\sum\limits_{m>0} m\alpha_m = -d$.

By Remark 3.2.11 **(b)**, we have $L_0 = \sum\limits_{m>0} a_{-m}a_m + \frac{\mu^2}{2}$. Since $a_m = m\frac{\partial}{\partial x_m}$ and $a_{-m} = x_m$ for every integer $m > 0$ (by the definition of the action of a_m on F_μ), this

[84]Here, "monomial" means "monomial without coefficient".

rewrites as $L_0 = \sum\limits_{m>0} x_m m \dfrac{\partial}{\partial x_m} + \dfrac{\mu^2}{2}$. Now, since $f = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} ...$, every $m > 0$ satisfies

$$x_m m \frac{\partial}{\partial x_m} f = x_m m \underbrace{\frac{\partial}{\partial x_m}\left(x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} ...\right)}_{\substack{= \alpha_m x_1^{\alpha_1} x_2^{\alpha_2} ... x_{m-1}^{\alpha_{m-1}} x_m^{\alpha_m - 1} x_{m+1}^{\alpha_{m+1}} x_{m+2}^{\alpha_{m+2}} ... \\ \text{(this term should be understood as 0 if } \alpha_m = 0)}}$$
$$= x_m m \alpha_m x_1^{\alpha_1} x_2^{\alpha_2} ... x_{m-1}^{\alpha_{m-1}} x_m^{\alpha_m - 1} x_{m+1}^{\alpha_{m+1}} x_{m+2}^{\alpha_{m+2}} ...$$
$$= m\alpha_m \cdot \underbrace{x_m \cdot x_1^{\alpha_1} x_2^{\alpha_2} ... x_{m-1}^{\alpha_{m-1}} x_m^{\alpha_m - 1} x_{m+1}^{\alpha_{m+1}} x_{m+2}^{\alpha_{m+2}} ...}_{= x_1^{\alpha_1} x_2^{\alpha_2} ... x_{m-1}^{\alpha_{m-1}} x_m^{\alpha_m} x_{m+1}^{\alpha_{m+1}} x_{m+2}^{\alpha_{m+2}} ... = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} ... = f} = m\alpha_m f.$$

Hence,

$$L_0 f = \sum_{m>0} \underbrace{x_m m \frac{\partial}{\partial x_m} f}_{= m\alpha_m f} + \frac{\mu^2}{2} f \qquad \left(\text{since } L_0 = \sum_{m>0} x_m m \frac{\partial}{\partial x_m} + \frac{\mu^2}{2}\right)$$
$$= \underbrace{\sum_{m>0} m\alpha_m}_{= -d} f + \frac{\mu^2}{2} f = -df + \frac{\mu^2}{2} f = \left(\frac{\mu^2}{2} - d\right) f.$$

We thus have proven the equation $L_0 f = \left(\dfrac{\mu^2}{2} - d\right) f$ for every monomial f of degree d. As we said above, this completes the proof of Remark 3.2.11 **(c)**.

(d) For every $m \in \mathbb{Z}$,

$$\text{the map } a_m : F_\mu \to F_\mu \text{ is homogeneous of degree } m. \tag{97}$$

(In fact, this is easily seen from the definition of how a_m acts on F_μ.)

Thus, for every $u \in \mathbb{Z}$ and $v \in \mathbb{Z}$, the map $: a_u a_v :$ is homogeneous of degree $u + v$ [85]. Applied to $u = -m$ and $v = n + m$, this yields: For every $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, the map $: a_{-m} a_{n+m} :$ is homogeneous of degree $(-m) + (n + m) = n$. Now, the map

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} \underbrace{: a_{-m} a_{n+m} :}_{\text{this map is homogeneous of degree } n}$$

must be homogeneous of degree n. This proves Remark 3.2.11 **(d)**.

Now it turns out that the operators L_n that we have defined give a positive answer to question **1)**:

---

[85] *Proof.* Let $u \in \mathbb{Z}$ and $v \in \mathbb{Z}$. By (97) (applied to $m = u$), the map a_u is homogeneous of degree u. Similarly, the map a_v is homogeneous of degree v. Thus, the map $a_u a_v$ is homogeneous of degree $u + v$. Similarly, the map $a_v a_u$ is homogeneous of degree $v + u = u + v$.

Since $: a_u a_v : = \begin{cases} a_u a_v, & \text{if } u \leq v; \\ a_v a_u, & \text{if } u > v \end{cases}$ (by the definition of normal ordered products), the map $: a_u a_v :$ equals one of the maps $a_u a_v$ and $a_v a_u$. Since both of these maps $a_u a_v$ and $a_v a_u$ are homogeneous of degree $u + v$, this yields that $: a_u a_v :$ is homogeneous of degree $u + v$, qed.

Proposition 3.2.12. Let $n \in \mathbb{Z}$, $m \in \mathbb{Z}$ and $\mu \in \mathbb{C}$. Then, $[L_n, a_m] = -m a_{n+m}$ (where L_n is defined as in Definition 3.2.8, and a_ℓ is shorthand notation for $a_\ell \mid_{F_\mu}$).

Proof of Proposition 3.2.12. Since

$$L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_{-m} a_{n+m} : = \frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j} a_{n+j} : ,$$

we have

$$\begin{aligned}
[L_n, a_m] &= \left[\frac{1}{2} \sum_{j \in \mathbb{Z}} : a_{-j} a_{n+j} : , a_m \right] = \frac{1}{2} \sum_{j \in \mathbb{Z}} \underbrace{[: a_{-j} a_{n+j} : , a_m]}_{= -[a_m, : a_{-j} a_{n+j} :]} \\
&= -\frac{1}{2} \sum_{j \in \mathbb{Z}} \underbrace{[a_m, : a_{-j} a_{n+j} :]}_{\substack{= [a_m, a_{-j} a_{n+j}] \\ \text{(by Remark 3.2.7 (b), applied} \\ \text{to } a_m, \, -j \text{ and } n+j \text{ instead of } x, \, m \text{ and } n)}} = -\frac{1}{2} \sum_{j \in \mathbb{Z}} \underbrace{[a_m, a_{-j} a_{n+j}]}_{= [a_m, a_{-j}] a_{n+j} + a_{-j} [a_m, a_{n+j}]} \\
&= -\frac{1}{2} \sum_{j \in \mathbb{Z}} \left(\underbrace{[a_m, a_{-j}]}_{= m \delta_{m, -(-j)} K} a_{n+j} + a_{-j} \underbrace{[a_m, a_{n+j}]}_{= m \delta_{m, -(n+j)} K} \right) \\
&= -\frac{1}{2} \sum_{j \in \mathbb{Z}} \left(m \underbrace{\delta_{m, -(-j)}}_{= \delta_{m,j}} K a_{n+j} + a_{-j} m \underbrace{\delta_{m, -(n+j)}}_{= \delta_{-m, n+j} = \delta_{-m-n, j}} K \right) \\
&= -\frac{1}{2} \sum_{j \in \mathbb{Z}} \left(m \delta_{m,j} K a_{n+j} + a_{-j} m \delta_{-m-n,j} K \right) . \qquad (98)
\end{aligned}$$

But each of the two sums $\sum_{j \in \mathbb{Z}} m \delta_{m,j} K a_{n+j}$ and $\sum_{j \in \mathbb{Z}} a_{-j} m \delta_{-m-n,j} K$ is convergent[86]. Hence, we can split the sum $\sum_{j \in \mathbb{Z}} \left(m \delta_{m,j} K a_{n+j} + a_{-j} m \delta_{-m-n,j} K \right)$ into $\sum_{j \in \mathbb{Z}} m \delta_{m,j} K a_{n+j} +$

[86] In fact, due to the factors $\delta_{m,j}$ and $\delta_{-m-n,j}$ in the addends, it is clear that in each of these two sums, only at most one addend can be nonzero. Concretely:

$$\sum_{j \in \mathbb{Z}} m \delta_{m,j} K a_{n+j} = m K a_{n+m} \qquad \text{and} \qquad \sum_{j \in \mathbb{Z}} a_{-j} m \delta_{-m-n,j} K = a_{-(-m-n)} m K.$$

$\sum_{j\in\mathbb{Z}} a_{-j} m\delta_{-m-n,j} K$. Thus, (98) becomes

$$[L_n, a_m] = -\frac{1}{2}\left(\underbrace{\sum_{j\in\mathbb{Z}} m\delta_{m,j} K a_{n+j}}_{=mKa_{n+m}} + \underbrace{\sum_{j\in\mathbb{Z}} a_{-j} m\delta_{-m-n,j} K}_{=a_{-(-m-n)}mK}\right) = -\frac{1}{2}\left(mKa_{n+m} + a_{-(-m-n)}mK\right)$$
$$= -\frac{1}{2}\left(ma_{n+m} + a_{-(-m-n)}m\right) \qquad \text{(since } K \text{ acts as id on } F_\mu\text{)}$$
$$= -\frac{1}{2}m\left(a_{n+m} + \underbrace{a_{-(-m-n)}}_{=a_{m+n}=a_{n+m}}\right) = -\frac{1}{2}m\left(a_{n+m} + a_{n+m}\right) = -ma_{n+m}.$$

This proves Proposition 3.2.12.

Now let us check whether our operators L_n answer Question **2)**, or at least try to do so. We are going to make some "dirty" arguments; cleaner ones can be found in the proof of Proposition 3.2.13 that we give below.

First, it is easy to see that any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfy

$$[[L_n, L_m] - (n-m) L_{n+m}, a_k] = 0 \qquad \text{for any } k \in \mathbb{Z}$$

[87]. Hence, for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, the endomorphism $[L_n, L_m] - (n-m) L_{n+m}$ of F_μ is an $\mathcal{A}$-module homomorphism (since $[[L_n, L_m] - (n-m) L_{n+m}, K] = 0$ also holds,

[87] *Proof.* Let $n \in \mathbb{Z}$, $m \in \mathbb{Z}$ and $k \in \mathbb{Z}$. Then,

$$[[L_n, L_m] - (n-m) L_{n+m}, a_k]$$
$$= \underbrace{[[L_n, L_m], a_k]}_{\substack{=[[L_n,a_k],L_m]+[L_n,[L_m,a_k]] \\ \text{(by the Leibniz identity for commutators)}}} - (n-m)[L_{n+m}, a_k]$$
$$= \left[\underbrace{[L_n, a_k]}_{\substack{=-ka_{n+k} \\ \text{(by Proposition 3.2.12,} \\ \text{applied to } k \text{ instead of } m)}}, L_m\right] + \left[L_n, \underbrace{[L_m, a_k]}_{\substack{=-ka_{m+k} \\ \text{(by Proposition 3.2.12,} \\ \text{applied to } m \text{ and } k \\ \text{instead of } n \text{ and } m)}}\right] - (n-m)\underbrace{[L_{n+m}, a_k]}_{\substack{=-ka_{n+m+k} \\ \text{(by Proposition 3.2.12,} \\ \text{applied to } n+m \text{ and } k \\ \text{instead of } n \text{ and } m)}}$$
$$= -k\underbrace{[a_{n+k}, L_m]}_{=-[L_m, a_{n+k}]} - k[L_n, a_{m+k}] + (n-m)ka_{n+m+k}$$
$$= k\underbrace{[L_m, a_{n+k}]}_{\substack{=-(n+k)a_{m+n+k} \\ \text{(by Proposition 3.2.12,} \\ \text{applied to } m \text{ and } n+k \text{ instead of } n \text{ and } m)}} - k\underbrace{[L_n, a_{m+k}]}_{\substack{=-(m+k)a_{n+m+k} \\ \text{(by Proposition 3.2.12,} \\ \text{applied to } m+k \text{ instead of } m)}} + (n-m)ka_{n+m+k}$$
$$= -k(n+k)\underbrace{a_{m+n+k}}_{=a_{n+m+k}} + k(m+k)a_{n+m+k} + (n-m)ka_{n+m+k}$$
$$= -k(n+k)a_{n+m+k} + k(m+k)a_{n+m+k} + (n-m)ka_{n+m+k}$$
$$= \underbrace{(-k(n+k) + k(m+k) + (n-m)k)}_{=0} a_{n+m+k} = 0.$$

Qed.

for obvious reasons). Since F_μ is an irreducible $\mathcal{A}$-module of countable dimension, this yields (by Lemma 2.1.1) that, for any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, the map $[L_n, L_m] - (n-m) L_{n+m} : F_\mu \to F_\mu$ is a scalar multiple of the identity. But since this map $[L_n, L_m] - (n-m) L_{n+m}$ must also be homogeneous of degree $n+m$ (by an application of Remark 3.2.11 **(d)**), this yields that $[L_n, L_m] - (n-m) L_{n+m} = 0$ whenever $n+m \neq 0$ (because any homogeneous map of degree $\neq 0$ which is, at the same time, a scalar multiple of the identity, must be the 0 map). Thus, for every $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we can write

$$[L_n, L_m] - (n-m) L_{n+m} = \gamma_n \delta_{n,-m} \operatorname{id} \qquad \text{for some } \gamma_n \in \mathbb{C} \text{ depending on } n. \quad (99)$$

We can get some more information about these γ_n if we consider the Lie algebra with basis $(L_n)_{n\in\mathbb{Z}} \cup (\mathrm{id})$ [88]. (Note that, according to Convention 3.2.9, these L_n still denote maps from F_μ to F_μ, rather than elements of Vir or W. Of course, this Lie algebra with basis $(L_n)_{n\in\mathbb{Z}} \cup (\mathrm{id})$ **will** turn out to be isomorphic to Vir, but we have not yet proven this.) This Lie algebra, due to the formula (99) and to the fact that id commutes with everything, must be a 1-dimensional central extension of the Witt algebra. Hence, the map

$$W \times W \to \mathbb{C}, \qquad (L_n, L_m) \mapsto \gamma_n \delta_{n,-m}$$

(where L_n and L_m really mean the elements L_n and L_m of W this time) must be a 2-cocycle on W. But since we know (from Theorem 1.5.2) that every 2-cocycle on W is a scalar multiple of the 2-cocycle ω defined in Theorem 1.5.2 modulo the 2-coboundaries, this yields that this 2-cocycle is a scalar multiple of ω modulo the 2-coboundaries. In other words, there exist $c \in \mathbb{C}$ and $\xi \in W^*$ such that

$$\gamma_n \delta_{n,-m} = c\omega(L_n, L_m) + \xi([L_n, L_m]) \qquad \text{for all } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$

Since $\omega(L_n, L_m) = \dfrac{n^3 - n}{6} \delta_{n,-m}$, this rewrites as

$$\gamma_n \delta_{n,-m} = c\frac{n^3 - n}{6} \delta_{n,-m} + \xi([L_n, L_m]) \qquad \text{for all } n \in \mathbb{Z} \text{ and } m \in \mathbb{Z}.$$

Applied to $m = -n$, this yields

$$\gamma_n = c\frac{n^3 - n}{6} + \xi\left(\underbrace{[L_n, L_{-n}]}_{=2nL_0}\right) = c\frac{n^3 - n}{6} + 2n\xi(L_0). \quad (100)$$

All that remains now, in order to get the values of $[L_n, L_m] - (n-m) L_{n+m}$, is to compute the scalars c and $\xi(L_0)$. For this, we only need to compute γ_1 and γ_2 (because this will give 2 linear equations for c and L_0). In order to do this, we will evaluate the endomorphisms $[L_1, L_{-1}] - 2L_0$ and $[L_2, L_{-2}] - 4L_0$ at the element 1 of F_μ.

By Remark 3.2.11 **(c)** (applied to $d = 0$ and $f = 1$), we get $L_0 1 = \left(\dfrac{\mu^2}{2} - 0\right) 1 = \dfrac{\mu^2}{2}$.

[88] This is well-defined because (as the reader can easily check) the family $(L_n)_{n\in\mathbb{Z}} \cup (\mathrm{id})$ of operators on F_μ is linearly independent.

Since $L_1 = \frac{1}{2} \sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{1+m} :$, we have $L_1 1 = \frac{1}{2} \sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{1+m} : 1 = 0$ (because, as it is easily seen, $: a_{-m} a_{1+m} : 1 = 0$ for every $m \in \mathbb{Z}$). Similarly, $L_2 1 = 0$.

Since $L_{-1} = \frac{1}{2} \sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{-1+m} :$, we have $L_{-1} 1 = \frac{1}{2} \sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{-1+m} : 1$. It is easy to see that the only $m \in \mathbb{Z}$ for which $: a_{-m} a_{-1+m} : 1$ is nonzero are $m = 0$ and $m = 1$. Hence,

$$\sum_{m \in \mathbb{Z}} : a_{-m} a_{-1+m} : 1 = \underbrace{: a_{-0} a_{-1+0} : 1}_{= : a_0 a_{-1} : 1 = a_{-1} a_0 1 = x_1 \cdot \mu 1 = \mu x_1} + \underbrace{: a_{-1} a_{-1+1} : 1}_{= : a_{-1} a_0 : 1 = a_{-1} a_0 1 = x_1 \cdot \mu 1 = \mu x_1} = \mu x_1 + \mu x_1 = 2 \mu x_1,$$

so that $L_{-1} 1 = \frac{1}{2} \underbrace{\sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{-1+m} : 1}_{= 2 \mu x_1} = \mu x_1$. Thus,

$$\begin{aligned} L_1 L_{-1} 1 &= L_1 \mu x_1 = \mu \underbrace{L_1}_{= \frac{1}{2} \sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{1+m} :} x_1 = \mu \cdot \frac{1}{2} \underbrace{\sum_{m \in \mathbb{Z}} : a_{-m} a_{1+m} : x_1}_{\substack{= : a_{-(-1)} a_{1+(-1)} : x_1 + : a_{-0} a_{1+0} : x_1 \\ \text{(in fact, it is easy to see that the only} \\ m \in \mathbb{Z} \text{ for which } : a_{-m} a_{1+m} : x_1 \neq 0 \text{ are } m = -1 \text{ and } m = 0)}} \\ &= \mu \cdot \frac{1}{2} \left(\underbrace{: a_{-(-1)} a_{1+(-1)} : x_1}_{= : a_1 a_0 : x_1 = a_0 a_1 x_1 = \mu \cdot 1 \frac{\partial}{\partial x_1} x_1 = \mu} + \underbrace{: a_{-0} a_{1+0} : x_1}_{= : a_0 a_1 : x_1 = \mu \cdot 1 \frac{\partial}{\partial x_1} x_1 = \mu} \right) \\ &= \mu \cdot \frac{1}{2} (\mu + \mu) = \mu^2. \end{aligned}$$

A similar (but messier) computation works for $L_2 L_{-2} 1$: Since $L_{-2} = \frac{1}{2} \sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{-2+m} :$, we have $L_{-2} 1 = \frac{1}{2} \sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{-2+m} : 1$. It is easy to see that the only $m \in \mathbb{Z}$ for which $: a_{-m} a_{-2+m} : 1$ is nonzero are $m = 0$, $m = 1$ and $m = 2$. This allows us to simplify $L_{-2} 1 = \frac{1}{2} \sum\limits_{m \in \mathbb{Z}} : a_{-m} a_{-2+m} : 1$ to $L_{-2} 1 = \mu x_2 + \frac{1}{2} x_1^2$ (the details are left to the reader). Thus,

$$L_2 L_{-2} 1 = L_2 \left(\mu x_2 + \frac{1}{2} x_1^2 \right) = \mu L_2 x_2 + \frac{1}{2} L_2 x_1^2.$$

Straightforward computations, which I omit, show that $L_2 x_2 = 2\mu$ and $L_2 x_1^2 = 1$. Hence,

$$L_2 L_{-2} 1 = \mu \underbrace{L_2 x_2}_{= 2\mu} + \frac{1}{2} \underbrace{L_2 x_1^2}_{= 1} = 2 \mu^2 + \frac{1}{2}.$$

Now,

$$([L_1, L_{-1}] - 2 L_0) 1 = \underbrace{L_1 L_{-1} 1}_{= \mu^2} - L_{-1} \underbrace{L_1 1}_{= 0} - 2 \underbrace{L_0 1}_{= \frac{\mu^2}{2}} = \mu^2 - 0 - 2 \cdot \frac{\mu^2}{2} = 0.$$

Since

$$\begin{aligned}
[L_1, L_{-1}] - 2L_0 &= \gamma_1 \underbrace{\delta_{1,-(-1)}}_{=1} \qquad \text{(by (99), applied to } n = 1 \text{ and } m = -1\text{)} \\
&= \gamma_1 = c \underbrace{\frac{1^3 - 1}{6}}_{=0} + 2 \cdot 1 \cdot \xi(L_0) \qquad \text{(by (100), applied to } n = 1\text{)} \\
&= 0 + 2 \cdot 1 \cdot \xi(L_0) = 2\xi(L_0),
\end{aligned}$$

this rewrites as $2\xi(L_0) \cdot 1 = 0$, so that $\xi(L_0) = 0$.

On the other hand,

$$([L_2, L_{-2}] - 4L_0) 1 = \underbrace{L_2 L_{-2} 1}_{=2\mu^2 + \frac{1}{2}} - L_{-2} \underbrace{L_2 1}_{=0} - 4 \underbrace{L_0 1}_{=\frac{\mu^2}{2}} = \left(2\mu^2 + \frac{1}{2}\right) - 0 - 4 \cdot \frac{\mu^2}{2} = \frac{1}{2}.$$

Since

$$\begin{aligned}
([L_2, L_{-2}] - 4L_0) 1 &= \gamma_2 \underbrace{\delta_{2,-(-2)}}_{=1} \qquad \text{(by (99), applied to } n = 2 \text{ and } m = -2\text{)} \\
&= \gamma_2 = c \underbrace{\frac{2^3 - 2}{6}}_{=1} + 2 \cdot 2 \cdot \underbrace{\xi(L_0)}_{=0} \qquad \text{(by (100), applied to } n = 2\text{)} \\
&= c + 0 = c,
\end{aligned}$$

this rewrites as $c = \frac{1}{2}$.

Due to $\xi(L_0) = 0$ and $c = \frac{1}{2}$, we can rewrite (100) as

$$\gamma_n = \frac{1}{2} \cdot \frac{n^3 - n}{6} + 2n0 = \frac{n^3 - n}{12}.$$

Hence, (99) becomes

$$[L_n, L_m] - (n - m) L_{n+m} = \frac{n^3 - n}{12} \delta_{n,-m} \operatorname{id}.$$

We have thus proven:

Proposition 3.2.13. For any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$, we have

$$[L_n, L_m] = (n - m) L_{n+m} + \frac{n^3 - n}{12} \delta_{n,-m} \operatorname{id} \tag{101}$$

(where L_n and L_m are maps $F_\mu \to F_\mu$ as explained in Convention 3.2.9). Thus, we can make F_μ a representation of Vir by letting the element L_n of Vir act as the map $L_n : F_\mu \to F_\mu$ for every $n \in \mathbb{Z}$, and letting the element C of Vir act as id.

Due to Proposition 3.2.12, this Vir-action harmonizes with the $\mathcal{A}$-action on F_μ:

Proposition 3.2.14. The $\mathcal{A}$-action on F_μ extends (essentially uniquely) to an action of $\operatorname{Vir} \ltimes \mathcal{A}$ on F_μ with C acting as 1.

This is the reason why the construction of the Virasoro algebra involved the 2-cocycle $\frac{1}{2}\omega$ rather than ω (or, actually, rather than simpler-looking 2-cocycles like $(L_n, L_m) \mapsto n^3 \delta_{n,-m}$).

Our proof of Proposition 3.2.13 above was rather insidious and nonconstructive: We used the Dixmier theorem to prove (what boils down to) an algebraic identity, and later we used Theorem 1.5.2 (which is constructive but was applied in a rather unexpected way) to reduce our computations to two concrete cases. We will now show a different, more direct proof of Proposition 3.2.13:[89]

Second proof of Proposition 3.2.13. Let $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$. By (95) (with the index m renamed as ℓ), we have $L_n = \frac{1}{2} \sum\limits_{\ell \in \mathbb{Z}} : a_{-\ell} a_{n+\ell} :$. Hence,

$$\begin{aligned}
[L_n, L_m] &= \left[\frac{1}{2} \sum_{\ell \in \mathbb{Z}} : a_{-\ell} a_{n+\ell} : , L_m \right] = \frac{1}{2} \sum_{\ell \in \mathbb{Z}} \underbrace{[: a_{-\ell} a_{n+\ell} : , L_m]}_{= -[L_m, : a_{-\ell} a_{n+\ell} :]} \\
&= -\frac{1}{2} \sum_{\ell \in \mathbb{Z}} [L_m, : a_{-\ell} a_{n+\ell} :] . \qquad (102)
\end{aligned}$$

Now, let $\ell \in \mathbb{Z}$. Then, we obtain $[L_m, : a_{-\ell} a_{n+\ell} :] = [L_m, a_{-\ell} a_{n+\ell}]$ (more or less by applying Remark 3.2.7 **(b)** to L_m, $-\ell$ and $n + \ell$ instead of x, m and n [90]), so that

$$\begin{aligned}
[L_m, : a_{-\ell} a_{n+\ell} :] &= [L_m, a_{-\ell} a_{n+\ell}] \\
&= \underbrace{[L_m, a_{-\ell}]}_{\substack{= -(-\ell) a_{m+(-\ell)} \\ \text{(by Proposition 3.2.12} \\ \text{(applied to } m \text{ and } -\ell \text{ instead of } n \text{ and } m))}} a_{n+\ell} + a_{-\ell} \underbrace{[L_m, a_{n+\ell}]}_{\substack{= -(n+\ell) a_{m+(n+\ell)} \\ \text{(by Proposition 3.2.12} \\ \text{(applied to } m \text{ and } n+\ell \text{ instead of } n \text{ and } m))}} \\
&= \underbrace{-(-\ell)}_{= \ell} \underbrace{a_{m+(-\ell)}}_{= a_{m-\ell}} a_{n+\ell} + \underbrace{a_{-\ell} \left(-(n+\ell) a_{m+(n+\ell)} \right)}_{= -(n+\ell) a_{-\ell} a_{m+n+\ell}} \\
&= \ell a_{m-\ell} a_{n+\ell} - (n+\ell) a_{-\ell} a_{m+n+\ell}.
\end{aligned}$$

[89] The following proof is a slight variation of the proof given in the Kac-Raina book (where our Proposition 3.2.13 is Proposition 2.3).

[90] I am saying "more or less" because this is not completely correct: We cannot apply Remark 3.2.7 **(b)** to L_m, $-\ell$ and $n + \ell$ instead of x, m and n (since L_m does not lie in $U(\mathcal{A})$). However, there are two ways to get around this obstruction:

One way is to generalize Remark 3.2.7 **(b)** to a suitable completion of $U(\mathcal{A})$. We will not do this here.

Another way is to notice that we can replace $U(\mathcal{A})$ by $\operatorname{End}(F_\mu)$ throughout Remark 3.2.7. (This, of course, means that a_n and a_m have to be reinterpreted as endomorphisms of F_μ rather than elements of $\mathcal{A}$; but since the action of $\mathcal{A}$ on F_μ is a Lie algebra representation, all equalities that hold in $U(\mathcal{A})$ remain valid in $\operatorname{End}(F_\mu)$.) The proof of Remark 3.2.7 still works after this replacement (except that $[x, K] = 0$ should no longer be proven using the argument $K \in Z(\mathcal{A}) \subseteq Z(U(\mathcal{A}))$, but simply follows from the fact that K acts as the identity on F_μ). Now, after this replacement, we **can** apply Remark 3.2.7 **(b)** to L_m, $-\ell$ and $n + \ell$ instead of x, m and n, and we obtain $[L_m, : a_{-\ell} a_{n+\ell} :] = [L_m, a_{-\ell} a_{n+\ell}]$.

Since $a_{m-\ell}a_{n+\ell} = : a_{m-\ell}a_{n+\ell} : -(n+\ell)\,[\ell<m]\,\delta_{m,-n}\operatorname{id}$ [91] and $a_{-\ell}a_{m+n+\ell} = : a_{-\ell}a_{m+n+\ell} : - \ell\,[\ell<0]\,\delta_{m,-n}\operatorname{id}$ [92], this equation rewrites as

$$\begin{aligned}
&[L_m, : a_{-\ell}a_{n+\ell} :] \\
&= \ell \underbrace{a_{m-\ell}a_{n+\ell}}_{=:a_{m-\ell}a_{n+\ell}: -(n+\ell)[\ell<m]\delta_{m,-n}\operatorname{id}} - (n+\ell) \underbrace{a_{-\ell}a_{m+n+\ell}}_{=:a_{-\ell}a_{m+n+\ell}: -\ell[\ell<0]\delta_{m,-n}\operatorname{id}} \\
&= \ell\,(: a_{m-\ell}a_{n+\ell} : - (n+\ell)\,[\ell<m]\,\delta_{m,-n}\operatorname{id}) - (n+\ell)\,(: a_{-\ell}a_{m+n+\ell} : - \ell\,[\ell<0]\,\delta_{m,-n}\operatorname{id}) \\
&= \ell : a_{m-\ell}a_{n+\ell} : - \ell\,(n+\ell)\,[\ell<m]\,\delta_{m,-n}\operatorname{id} - (n+\ell) : a_{-\ell}a_{m+n+\ell} : + (n+\ell)\,\ell\,[\ell<0]\,\delta_{m,-n}\operatorname{id} \\
&= \ell : a_{m-\ell}a_{n+\ell} : - \underbrace{(n+\ell)}_{=(n-m)+(m+\ell)} : a_{-\ell}a_{m+n+\ell} : \\
&\qquad + \underbrace{(n+\ell)\,\ell\,[\ell<0]\,\delta_{m,-n}\operatorname{id} - \ell\,(n+\ell)\,[\ell<m]\,\delta_{m,-n}\operatorname{id}}_{=\ell(n+\ell)([\ell<0]-[\ell<m])\delta_{m,-n}\operatorname{id}} \\
&= \ell : a_{m-\ell}a_{n+\ell} : - \underbrace{((n-m)+(m+\ell)) : a_{-\ell}a_{m+n+\ell} :}_{=(n-m):a_{-\ell}a_{m+n+\ell}:+(m+\ell):a_{-\ell}a_{m+n+\ell}:} \\
&\qquad + \ell\,(n+\ell)\,([\ell<0]-[\ell<m])\,\delta_{m,-n}\operatorname{id} \\
&= \ell : a_{m-\ell}a_{n+\ell} : - (n-m) : a_{-\ell}a_{m+n+\ell} : - (m+\ell) : a_{-\ell}a_{m+n+\ell} : \\
&\qquad + \ell\,(n+\ell)\,([\ell<0]-[\ell<m])\,\delta_{m,-n}\operatorname{id}
\end{aligned} \tag{103}$$

Now forget that we fixed ℓ. We want to use the equality (103) in order to split the infinite sum $\sum\limits_{\ell\in\mathbb{Z}} [L_m, : a_{-\ell}a_{n+\ell} :]$ on the right hand side of (102) into

$$\sum_{\ell\in\mathbb{Z}} \ell : a_{m-\ell}a_{n+\ell} : - (n-m)\sum_{\ell\in\mathbb{Z}} : a_{-\ell}a_{m+n+\ell} : - \sum_{\ell\in\mathbb{Z}} (m+\ell) : a_{-\ell}a_{m+n+\ell} :$$
$$+ \sum_{\ell\in\mathbb{Z}} \ell\,(n+\ell)\,([\ell<0]-[\ell<m])\,\delta_{m,-n}\operatorname{id}.$$

[91] because Remark 3.2.7 **(a)** (applied to $m-\ell$ and $n+\ell$ instead of m and n) yields

$$\begin{aligned}
: a_{m-\ell}a_{n+\ell} : &= a_{m-\ell}a_{n+\ell} + (n+\ell) \underbrace{[m-\ell>0]}_{=[\ell<m]} \underbrace{\delta_{m-\ell,-(n+\ell)}}_{=\delta_{m-\ell,-n-\ell}=\delta_{m,-n}} \underbrace{K}_{\substack{=\operatorname{id} \\ \text{(since } K \text{ acts as id on } F_\mu)}} \\
&= a_{m-\ell}a_{n+\ell} + (n+\ell)\,[\ell<m]\,\delta_{m,-n}\operatorname{id}
\end{aligned}$$

[92] because Remark 3.2.7 **(a)** (applied to ℓ and $n+m+\ell$ instead of m and n) yields

$$\begin{aligned}
: a_{-\ell}a_{m+n+\ell} : &= a_{-\ell}a_{m+n+\ell} + (m+n+\ell) \underbrace{[-\ell>0]}_{=[\ell<0]} \underbrace{\delta_{-\ell,-(m+n+\ell)}}_{=\delta_{-\ell,-m-n-\ell}=\delta_{m,-n}} \underbrace{K}_{\substack{=\operatorname{id} \\ \text{(since } K \text{ acts as id on } F_\mu)}} \\
&= a_{-\ell}a_{m+n+\ell} + \underbrace{(m+n+\ell)\,[\ell<0]}_{=[\ell<0](m+n+\ell)}\,\delta_{m,-n}\operatorname{id} \\
&= a_{-\ell}a_{m+n+\ell} + [\ell<0] \underbrace{(m+n+\ell)\,\delta_{m,-n}}_{\substack{=\ell\delta_{m,-n} \\ \text{(this can be easily proven by treating} \\ \text{the cases of } m=-n \text{ and of } m\neq -n \text{ separately)}}} \operatorname{id} \\
&= a_{-\ell}a_{m+n+\ell} + \underbrace{[\ell<0]\,\ell}_{=\ell[\ell<0]}\,\delta_{m,-n}\operatorname{id} = a_{-\ell}a_{m+n+\ell} + \ell\,[\ell<0]\,\delta_{m,-n}\operatorname{id}
\end{aligned}$$

But before we can do this, we must check that this splitting is allowed (since infinite sums cannot always be split: e. g., the sum $\sum\limits_{\ell\in\mathbb{Z}}(1-1)$ is well-defined (and has value 0), but splitting it into $\sum\limits_{\ell\in\mathbb{Z}}1-\sum\limits_{\ell\in\mathbb{Z}}1$ is not allowed). Clearly, in order to check this, it is enough to check that the four infinite sums $\sum\limits_{\ell\in\mathbb{Z}}\ell : a_{m-\ell}a_{n+\ell}:$, $\sum\limits_{\ell\in\mathbb{Z}} : a_{-\ell}a_{m+n+\ell}:$, $\sum\limits_{\ell\in\mathbb{Z}}(m+\ell) : a_{-\ell}a_{m+n+\ell}:$ and $\sum\limits_{\ell\in\mathbb{Z}}\ell\,(n+\ell)\,([\ell<0]-[\ell<m])\,\delta_{m,-n}\operatorname{id}$ converge.

Before we do this, let us formalize what we mean by "converge": We consider the product topology on the set $(F_\mu)^{F_\mu}$ (the set of all maps $F_\mu\to F_\mu$) by viewing this set as $\prod\limits_{v\in F_\mu}F_\mu$, where each F_μ is endowed with the discrete topology. With respect to this topology, a net $(f_i)_{i\in I}$ of maps $f_i : F_\mu\to F_\mu$ converges to a map $f : F_\mu\to F_\mu$ if and only if

$$\left(\begin{array}{c}\text{for every } v\in F_\mu\text{, the net of values } (f_i(v))_{i\in I}\text{ converges to } f(v)\in F_\mu\\ \text{with respect to the discrete topology on } F_\mu\end{array}\right).$$

Hence, with respect to this topology, an infinite sum $\sum\limits_{\ell\in\mathbb{Z}}f_\ell$ of maps $f_\ell : F_\mu\to F_\mu$ converges if and only if

$$\left(\text{for every } v\in F_\mu\text{, all but finitely many } \ell\in\mathbb{Z}\text{ satisfy } f_\ell(v)=0\right).$$

Hence, this is exactly the notion of convergence which we used in Definition 3.2.8 to make sense of the infinite sum $\sum\limits_{m\in\mathbb{Z}} : a_{-m}a_{n+m}:$.

Now, we are going to show that the infinite sums $\sum\limits_{\ell\in\mathbb{Z}}\ell : a_{m-\ell}a_{n+\ell}:$, $\sum\limits_{\ell\in\mathbb{Z}} : a_{-\ell}a_{m+n+\ell}:$, $\sum\limits_{\ell\in\mathbb{Z}}(m+\ell) : a_{-\ell}a_{m+n+\ell}:$ and $\sum\limits_{\ell\in\mathbb{Z}}\ell\,(n+\ell)\,([\ell<0]-[\ell<m])\,\delta_{m,-n}\operatorname{id}$ converge with respect to this topology.

Proof of the convergence of $\sum\limits_{\ell\in\mathbb{Z}} : a_{-\ell}a_{m+n+\ell}:$*:* For every $v\in F_\mu$, all but finitely many $\ell\in\mathbb{Z}$ satisfy $: a_{-\ell}a_{m+n+\ell}: v=0$ (by Lemma 3.2.10 **(c)**, applied to $m+n$ and ℓ instead of n and m). Hence, the sum $\sum\limits_{\ell\in\mathbb{Z}} : a_{-\ell}a_{m+n+\ell}:$ converges.

Proof of the convergence of $\sum\limits_{\ell\in\mathbb{Z}}(m+\ell) : a_{-\ell}a_{m+n+\ell}:$*:* For every $v\in F_\mu$, all but finitely many $\ell\in\mathbb{Z}$ satisfy $: a_{-\ell}a_{m+n+\ell}: v=0$ (by Lemma 3.2.10 **(c)**, applied to $m+n$ and ℓ instead of n and m). Hence, for every $v\in F_\mu$, all but finitely many $\ell\in\mathbb{Z}$ satisfy $(m+\ell) : a_{-\ell}a_{m+n+\ell}: =0$. Thus, the sum $\sum\limits_{\ell\in\mathbb{Z}}(m+\ell) : a_{-\ell}a_{m+n+\ell}:$ converges.

Proof of the convergence of $\sum\limits_{\ell\in\mathbb{Z}}\ell : a_{m-\ell}a_{n+\ell}:$*:* We know that the sum $\sum\limits_{\ell\in\mathbb{Z}}(m+\ell) : a_{-\ell}a_{m+n+\ell}:$ converges. Thus, we have

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}}(m+\ell) : a_{-\ell}a_{m+n+\ell}: &= \sum_{\ell\in\mathbb{Z}}\underbrace{(m+(\ell-m))}_{=\ell} : \underbrace{a_{-(\ell-m)}}_{=a_{m-\ell}}\underbrace{a_{m+n+(\ell-m)}}_{=a_{n+\ell}}:\\ &\qquad\text{(here, we substituted } \ell-m \text{ for } \ell \text{ in the sum)}\\ &= \sum_{\ell\in\mathbb{Z}}\ell : a_{m-\ell}a_{n+\ell}: . \qquad (104)\end{aligned}$$

Hence, the sum $\sum\limits_{\ell\in\mathbb{Z}}\ell : a_{m-\ell}a_{n+\ell}:$ converges.

Proof of the convergence of $\sum_{\ell\in\mathbb{Z}} \ell(n+\ell)([\ell<0]-[\ell<m])\delta_{m,-n}\operatorname{id}$*:* It is easy to see that:

- Every sufficiently small $\ell \in \mathbb{Z}$ satisfies $\ell(n+\ell)([\ell<0]-[\ell<m])\delta_{m,-n}\operatorname{id} = 0$. [93]
- Every sufficiently high $\ell \in \mathbb{Z}$ satisfies $\ell(n+\ell)([\ell<0]-[\ell<m])\delta_{m,-n}\operatorname{id} = 0$. [94]

Combining these two results, we conclude that all but finitely many $\ell \in \mathbb{Z}$ satisfy $\ell(n+\ell)([\ell<0]-[\ell<m])\delta_{m,-n}\operatorname{id} = 0$. The sum $\sum_{\ell\in\mathbb{Z}} \ell(n+\ell)([\ell<0]-[\ell<m])\delta_{m,-n}\operatorname{id}$ therefore converges.

We now know that all four sums that we care about converge, and that two of them have the same value (by (104)). Let us compute the other two of the sums:

First of all, by (95) (with the index m renamed as ℓ), we have $L_n = \frac{1}{2}\sum_{\ell\in\mathbb{Z}} : a_{-\ell}a_{n+\ell} :$. Applying this to $m+n$ instead of n, we get

$$L_{m+n} = \frac{1}{2}\sum_{\ell\in\mathbb{Z}} : a_{-\ell}a_{m+n+\ell} : . \tag{105}$$

This gives us the value of one of the sums we need.

Finally, let us notice that

$$\sum_{\ell\in\mathbb{Z}} \ell(n+\ell)([\ell<0]-[\ell<m])\delta_{m,-n}\operatorname{id} = -\frac{n^3-n}{6}\delta_{m,-n}\operatorname{id}. \tag{106}$$

In fact, proving this is a completely elementary computation exercise[95].

---

[93] *Proof.* Every sufficiently small $\ell \in \mathbb{Z}$ satisfies $\ell < 0$ and $\ell < m$ and thus

$$\ell(n+\ell)\left(\underbrace{[\ell<0]}_{=1 \text{ (since } \ell<0)} - \underbrace{[\ell<m]}_{=1 \text{ (since } \ell<m)}\right)\delta_{m,-n}\operatorname{id} = \ell(n+\ell)\underbrace{(1-1)}_{=0}\delta_{m,-n}\operatorname{id} = 0.$$

[94] *Proof.* Every sufficiently high $\ell \in \mathbb{Z}$ satisfies $\ell \geq 0$ and $\ell \geq m$ and thus

$$\ell(n+\ell)\left(\underbrace{[\ell<0]}_{=0 \text{ (since } \ell\geq0)} - \underbrace{[\ell<m]}_{=0 \text{ (since } \ell\geq m)}\right)\delta_{m,-n}\operatorname{id} = \ell(n+\ell)\underbrace{(0-0)}_{=0}\delta_{m,-n}\operatorname{id} = 0.$$

[95] Indeed, both sides of this equation are 0 when $m \neq -n$, so the only nontrivial case is the case when $m = -n$. This case splits further into two subcases: $m \geq 0$ and $m < 0$. In the first of these two subcases, the left hand side of (106) simplifies as $-\sum_{\ell=0}^{m-1} \ell(n+\ell)\operatorname{id}$; in the second, it simplifies as $\sum_{\ell=m}^{-1} \ell(n+\ell)\operatorname{id}$. The rest is straightforward computation.

Now, since (103) holds for every $\ell \in \mathbb{Z}$, we have

$$\sum_{\ell\in\mathbb{Z}} [L_m, :a_{-\ell}a_{n+\ell}:]$$
$$= \sum_{\ell\in\mathbb{Z}} \Big(\ell :a_{m-\ell}a_{n+\ell}: - (n-m) :a_{-\ell}a_{m+n+\ell}: - (m+\ell) :a_{-\ell}a_{m+n+\ell}:$$
$$+\ell(n+\ell)([\ell<0]-[\ell<m])\,\delta_{m,-n}\,\mathrm{id}\Big)$$
$$= \sum_{\ell\in\mathbb{Z}} \ell :a_{m-\ell}a_{n+\ell}: - (n-m) \underbrace{\sum_{\ell\in\mathbb{Z}} :a_{-\ell}a_{m+n+\ell}:}_{\substack{=2L_{m+n}\\ \text{(by (105))}}} - \underbrace{\sum_{\ell\in\mathbb{Z}} (m+\ell) :a_{-\ell}a_{m+n+\ell}:}_{\substack{=\sum_{\ell\in\mathbb{Z}} \ell :a_{m-\ell}a_{n+\ell}: \\ \text{(by (104))}}}$$
$$+ \underbrace{\sum_{\ell\in\mathbb{Z}} \ell(n+\ell)([\ell<0]-[\ell<m])\,\delta_{m,-n}\,\mathrm{id}}_{\substack{=-\frac{n^3-n}{6}\delta_{m,-n}\,\mathrm{id} \\ \text{(by (106))}}}$$
$$\left(\begin{array}{c}\text{here, we have split the sum; this was allowed, since the infinite sums} \\ \sum\limits_{\ell\in\mathbb{Z}} \ell :a_{m-\ell}a_{n+\ell}:,\ \sum\limits_{\ell\in\mathbb{Z}} :a_{-\ell}a_{m+n+\ell}:,\ \sum\limits_{\ell\in\mathbb{Z}} (m+\ell) :a_{-\ell}a_{m+n+\ell}: \\ \text{and } \sum\limits_{\ell\in\mathbb{Z}} \ell(n+\ell)([\ell<0]-[\ell<m])\,\delta_{m,-n}\,\mathrm{id} \text{ converge}\end{array}\right)$$
$$= \sum_{\ell\in\mathbb{Z}} \ell :a_{m-\ell}a_{n+\ell}: - (n-m)\cdot 2L_{m+n} - \sum_{\ell\in\mathbb{Z}} \ell :a_{m-\ell}a_{n+\ell}: - \frac{n^3-n}{6}\delta_{m,-n}\,\mathrm{id}$$
$$= -(n-m)\cdot 2L_{m+n} - \frac{n^3-n}{6}\delta_{m,-n}\,\mathrm{id}.$$

Hence, (102) becomes

$$[L_n, L_m] = -\frac{1}{2} \underbrace{\sum_{\ell\in\mathbb{Z}} [L_m, :a_{-\ell}a_{n+\ell}:]}_{=-(n-m)\cdot 2L_{m+n}-\frac{n^3-n}{6}\delta_{m,-n}\,\mathrm{id}}$$
$$= -\frac{1}{2}\left(-(n-m)\cdot 2L_{m+n} - \frac{n^3-n}{6}\delta_{m,-n}\,\mathrm{id}\right)$$
$$= (n-m)\underbrace{L_{m+n}}_{=L_{n+m}} - \frac{n^3-n}{12}\underbrace{\delta_{m,-n}}_{=\delta_{n,-m}}\mathrm{id} = (n-m)L_{n+m} - \frac{n^3-n}{12}\delta_{n,-m}\,\mathrm{id}.$$

This proves Proposition 3.2.13.

We can generalize our family $(L_n)_{n\in\mathbb{Z}}$ of operators on F_μ as follows (the so-called *Fairlie construction*):

Theorem 3.2.15. Let $\mu \in \mathbb{C}$ and $\lambda \in \mathbb{C}$. We can define a linear map $\widetilde{L}_n : F_\mu \to F_\mu$ for every $n \in \mathbb{Z}$ as follows: For $n \neq 0$, define the map $\widetilde{L}_n$ by

$$\widetilde{L}_n = \frac{1}{2}\sum_{m\in\mathbb{Z}} :a_{-m}a_{m+n}: + i\lambda n a_n$$

(where i stands for the complex number $\sqrt{-1}$). Define the map $\widetilde{L}_0$ by

$$\widetilde{L}_0 = \frac{\mu^2}{2} + \frac{\lambda^2}{2} + \sum_{j>0} a_{-j} a_j.$$

Then, this defines an action of Vir on F_μ with $c = 1 + 12\lambda^2$ (by letting $L_n \in \mathrm{Vir}$ act as the operator $\widetilde{L}_n$, and by letting $C \in \mathrm{Vir}$ acting as $(1 + 12\lambda^2)\,\mathrm{id}$). Moreover, it satisfies $\left[\widetilde{L}_n, a_m\right] = -m a_{n+m} + i\lambda n^2 \delta_{n,-m}\,\mathrm{id}$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

Proving this proposition was exercise 1 in homework problem set 2. It is rather easy now that we have proven Propositions 3.2.12 and 3.2.13 and thus left to the reader.

3.2.3. [unfinished] Unitarity properties of the Fock module

Proposition 3.2.16. Let $\mu \in \mathbb{R}$. Consider the representation F_μ of $\mathcal{A}$. Let $\langle \cdot, \cdot \rangle : F_\mu \times F_\mu \to \mathbb{C}$ be the unique Hermitian form satisfying $\langle 1, 1 \rangle = 1$ and

$$\langle av, w \rangle = \left\langle v, a^\dagger w \right\rangle \qquad \text{for all } a \in \mathcal{A},\ v \in F_\mu \text{ and } w \in F_\mu \tag{107}$$

(this is the usual Hermitian form on F_μ). Then, equipped with this form, F_μ is a unitary representation of $\mathcal{A}$.

Proof. We must prove that the form $\langle \cdot, \cdot \rangle$ is positive definite.

Let $\overrightarrow{n} = (n_1, n_2, n_3, ...)$ and $\overrightarrow{m} = (m_1, m_2, m_3, ...)$ be two sequences of nonnegative integers, each of them containing only finitely many nonzero entries. We are going to compute the value $\langle x_1^{n_1} x_2^{n_2} x_3^{n_3} ..., x_1^{m_1} x_2^{m_2} x_3^{m_3} ... \rangle$. This will give us the matrix that represents the Hermitian form $\langle \cdot, \cdot \rangle$ with respect to the monomial basis of F_μ.

If $n_1 + n_2 + n_3 + ... \neq m_1 + m_2 + m_3 + ...$, then this value is clearly zero, because the Hermitian form $\langle \cdot, \cdot \rangle$ is of degree 0 (as can be easily seen). Thus, we can WLOG assume that $n_1 + n_2 + n_3 + ... = m_1 + m_2 + m_3 + ...$

Let k be a positive integer such that every $i > k$ satisfies $n_i = 0$ and $m_i = 0$. (Such a k clearly exists.) Then, $n_1 + n_2 + ... + n_k = n_1 + n_2 + n_3 + ...$ and $m_1 + m_2 + ... + m_k = m_1 + m_2 + m_3 + ...$ Hence, the equality $n_1 + n_2 + n_3 + ... = m_1 + m_2 + m_3 + ...$ (which we know to hold) rewrites as $n_1 + n_2 + ... + n_k = m_1 + m_2 + ... + m_k$. Now, since every

$i > k$ satisfies $n_i = 0$ and $m_i = 0$, we have

$$\begin{aligned}
&\left\langle x_1^{n_1} x_2^{n_2} x_3^{n_3} ..., x_1^{m_1} x_2^{m_2} x_3^{m_3} ...\right\rangle \\
&= \left\langle x_1^{n_1} x_2^{n_2} ... x_k^{n_k}, \underbrace{x_1^{m_1} x_2^{m_2} ... x_k^{m_k}}_{\substack{= a_{-1}^{m_1} a_{-2}^{m_2} ... a_{-k}^{m_k} 1 \\ = \left(a_1^\dagger\right)^{m_1} \left(a_2^\dagger\right)^{m_2} ... \left(a_k^\dagger\right)^{m_k} 1}} \right\rangle = \left\langle x_1^{n_1} x_2^{n_2} ... x_k^{n_k}, \left(a_1^\dagger\right)^{m_1} \left(a_2^\dagger\right)^{m_2} ... \left(a_k^\dagger\right)^{m_k} 1 \right\rangle \\
&= \left\langle a_k^{m_k} a_{k-1}^{m_{k-1}} ... a_1^{m_1} x_1^{n_1} x_2^{n_2} ... x_k^{n_k}, 1 \right\rangle \qquad \text{(due to (107), applied several times)} \\
&= \left\langle \underbrace{\left(k \frac{\partial}{\partial x_k}\right)^{m_k} \left((k-1) \frac{\partial}{\partial x_{k-1}}\right)^{m_{k-1}} ... \left(1 \frac{\partial}{\partial x_1}\right)^{m_1} x_1^{n_1} x_2^{n_2} ... x_k^{n_k}}_{\substack{\text{this is a constant polynomial,} \\ \text{since } n_1+n_2+...+n_k = m_1+m_2+...+m_k}}, 1 \right\rangle \\
&= \left(k \frac{\partial}{\partial x_k}\right)^{m_k} \left((k-1) \frac{\partial}{\partial x_{k-1}}\right)^{m_{k-1}} ... \left(1 \frac{\partial}{\partial x_1}\right)^{m_1} x_1^{n_1} x_2^{n_2} ... x_k^{n_k} \\
&= \prod_{j=1}^{k} j^{m_j} \cdot \underbrace{\left(\frac{\partial}{\partial x_k}\right)^{m_k} \left(\frac{\partial}{\partial x_{k-1}}\right)^{m_{k-1}} ... \left(\frac{\partial}{\partial x_1}\right)^{m_1} x_1^{n_1} x_2^{n_2} ... x_k^{n_k}}_{\substack{= \delta_{\overrightarrow{n}, \overrightarrow{m}} \cdot \prod_{j=1}^{k} m_j! \\ (\text{since } n_1+n_2+...+n_k = m_1+m_2+...+m_k)}} = \delta_{\overrightarrow{n}, \overrightarrow{m}} \cdot \prod_{j=1}^{k} j^{m_j} \prod_{j=1}^{k} m_j!.
\end{aligned}$$

This term is 0 when $\overrightarrow{n} \neq \overrightarrow{m}$, and a positive integer when $\overrightarrow{n} = \overrightarrow{m}$. Thus, the matrix which represents the form $\langle \cdot, \cdot \rangle$ with respect to the monomial basis of F_μ is diagonal with positive diagonal entries. This form is therefore positive definite. Proposition 3.2.16 is proven.

Corollary 3.2.17. If $\mu, \lambda \in \mathbb{R}$, then the Vir-representation on F_μ given by $\widetilde{L}_n$ is unitary.

Proof. For $n \neq 0$, we have

$$\begin{aligned}
\widetilde{L}_n^\dagger &= \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_{-m} a_{n+m} :^\dagger + (i\lambda n a_n)^\dagger \\
&= \frac{1}{2} \sum_{m \in \mathbb{Z}} : a_m a_{-n-m} : - i\lambda n a_{-n} = \widetilde{L}_{-n}.
\end{aligned}$$

Corollary 3.2.18. The Vir-representation F_μ is completely reducible for $\mu \in \mathbb{R}$.

Now, $L_0 1 = \dfrac{\mu^2 + \lambda^2}{2} 1$ and $C1 = (1 + 12\lambda^2)\, 1$. Thus, the Verma module $M_{h,c} := M_{h,c}^+$ of the Virasoro algebra Vir for $h = \dfrac{\mu^2 + \lambda^2}{2}$ and $c = 1 + 12\lambda^2$ maps to F_μ with $v_{h,c} \mapsto 1$.

Proposition 3.2.19. For Weil generic μ and λ, this is an isomorphism.

Proof. The dimension of the degree-n part of both modules is $p(n)$. The map has degree 0. Hence, if it is injective, it is surjective. But for Weil generic μ and λ, the Vir-module $M_{h,c}$ is irreducible, so the map is injective.

Corollary 3.2.20. For Weil generic μ and λ in $\mathbb{R}$, the representation $M_{\frac{\mu^2+\lambda^2}{2}, 1+12\lambda^2}$ is unitary.

For any μ and λ in $\mathbb{R}$, the representation $L_{\frac{\mu^2+\lambda^2}{2}, 1+12\lambda^2}$ is unitary.

In other words, $L_{h,c}$ is unitary if $c \geq 1$ and $h \geq \frac{c-1}{24}$.

3.3. Power series and quantum fields

In this section, we are going to study different kinds of power series: polynomials, formal power series, Laurent polynomials, Laurent series and, finally, a notion of "formal power series" which can be infinite "in both directions". Each of these kinds of power series will later be used in our work; it is important to know the properties and the shortcomings of each of them.

3.3.1. Definitions

Parts of the following definition should sound familiar to the reader (indeed, we have already been working with polynomials, formal power series and Laurent polynomials), although maybe not in this generality.

Definition 3.3.1. For every vector space B and symbol z, we make the following definitions:

(a) We denote by $B[z]$ the vector space of all sequences $(b_n)_{n\in\mathbb{N}} \in B^{\mathbb{N}}$ such that only finitely many $n \in \mathbb{N}$ satisfy $b_n \neq 0$. Such a sequence $(b_n)_{n\in\mathbb{N}}$ is denoted by $\sum\limits_{n\in\mathbb{N}} b_n z^n$. The elements of $B[z]$ are called *polynomials in the indeterminate z over B* (even when B is not a ring).

(b) We denote by $B[[z]]$ the vector space of all sequences $(b_n)_{n\in\mathbb{N}} \in B^{\mathbb{N}}$. Such a sequence $(b_n)_{n\in\mathbb{N}}$ is denoted by $\sum\limits_{n\in\mathbb{N}} b_n z^n$. The elements of $B[[z]]$ are called *formal power series in the indeterminate z over B* (even when B is not a ring).

(c) We denote by $B[z, z^{-1}]$ the vector space of all two-sided sequences $(b_n)_{n\in\mathbb{Z}} \in B^{\mathbb{Z}}$ such that only finitely many $n \in \mathbb{Z}$ satisfy $b_n \neq 0$. (A *two-sided sequence* means a sequence indexed by integers, not just nonnegative integers.) Such a sequence $(b_n)_{n\in\mathbb{Z}}$ is denoted by $\sum\limits_{n\in\mathbb{Z}} b_n z^n$. The elements of $B[z, z^{-1}]$ are called *Laurent polynomials in the indeterminate z over B* (even when B is not a ring).

(d) We denote by $B((z))$ the vector space of all two-sided sequences $(b_n)_{n\in\mathbb{Z}} \in B^{\mathbb{Z}}$ such that only finitely many among the negative $n \in \mathbb{Z}$ satisfy $b_n \neq 0$. (A *two-sided sequence* means a sequence indexed by integers, not just nonnegative integers.) Such a sequence $(b_n)_{n\in\mathbb{Z}}$ is denoted by $\sum\limits_{n\in\mathbb{Z}} b_n z^n$. Sometimes, $B((z))$ is also denoted by $B[[z, z^{-1}]$. The elements of $B((z))$ are called *formal Laurent series in the indeterminate z over B* (even when B is not a ring).

(e) We denote by $B[[z, z^{-1}]]$ the vector space of all two-sided sequences $(b_n)_{n\in\mathbb{Z}} \in B^{\mathbb{Z}}$. Such a sequence $(b_n)_{n\in\mathbb{Z}}$ is denoted by $\sum\limits_{n\in\mathbb{Z}} b_n z^n$.

All five of these spaces $B[z]$, $B[[z]]$, $B[z,z^{-1}]$, $B((z))$ and $B[[z,z^{-1}]]$ are $\mathbb{C}[z]$-modules. (Here, the $\mathbb{C}[z]$-module structure on $B[[z,z^{-1}]]$ is given by

$$\left(\sum_{n\in\mathbb{N}} c_n z^n\right)\cdot\left(\sum_{n\in\mathbb{Z}} b_n z^n\right)=\sum_{n\in\mathbb{Z}}\left(\sum_{m\in\mathbb{N}} c_m\cdot b_{n-m}\right)z^n \tag{108}$$

for all $\sum\limits_{n\in\mathbb{Z}} b_n z^n\in B[[z,z^{-1}]]$ and $\sum\limits_{n\in\mathbb{N}} c_n z^n\in\mathbb{C}[z]$, and the $\mathbb{C}[z]$-module structures on the other four spaces are defined similarly.) Besides, $B[[z]]$ and $B((z))$ are $\mathbb{C}[[z]]$-modules (defined in a similar way to (108)). Also, $B((z))$ is a $\mathbb{C}((z))$-module (in a similar way). Besides, $B[z,z^{-1}]$, $B((z))$ and $B[[z,z^{-1}]]$ are $\mathbb{C}[z,z^{-1}]$-modules (defined analogously to (108)).

Of course, if B is a $\mathbb{C}$-algebra, then the above-defined spaces $B[z]$, $B[z,z^{-1}]$, $B[[z]]$ and $B((z))$ are $\mathbb{C}$-algebras themselves (with the multiplication defined similarly to (108)), and in fact $B[z]$ is the algebra of polynomials in the variable z over B, and $B[z,z^{-1}]$ is the algebra of Laurent polynomials in the variable z over B, and $B[[z]]$ is the algebra of formal power series in the variable z over B.

It should be noticed that $B[z]\cong B\otimes\mathbb{C}[z]$ and $B[z,z^{-1}]\cong B\otimes\mathbb{C}[z,z^{-1}]$ canonically, but such isomorphisms do **not** hold for $B[[z]]$, $B((z))$ and $B[[z,z^{-1}]]$ unless B is finite-dimensional.

We regard the obvious injections $B[z]\to B[z,z^{-1}]$, $B[z^{-1}]\to B[z,z^{-1}]$ (this is the map sending $z^{-1}\in B[z^{-1}]$ to $z^{-1}\in B[z,z^{-1}]$), $B[z]\to B[[z]]$, $B[z^{-1}]\to B[[z^{-1}]]$, $B[[z]]\to B((z))$, $B[[z^{-1}]]\to B((z^{-1}))$, $B[z,z^{-1}]\to B((z))$, $B[z,z^{-1}]\to B((z^{-1}))$, $B((z))\to B[[z,z^{-1}]]$ and $B((z^{-1}))\to B[[z,z^{-1}]]$ as inclusions.

Clearly, all five spaces $B[z]$, $B[[z]]$, $B[z,z^{-1}]$, $B((z))$ and $B[[z,z^{-1}]]$ depend functorially on B.

Before we do anything further with these notions, let us give three warnings:

1) Given Definition 3.3.1, one might expect $B[[z,z^{-1}]]$ to canonically become a $\mathbb{C}[[z,z^{-1}]]$-algebra. But this is not true even for $B=\mathbb{C}$ (because there is no reasonable way to define a product of two elements of $\mathbb{C}[[z,z^{-1}]]$ [96]). This also answers why $B[[z,z^{-1}]]$ does not become a ring when B is a $\mathbb{C}$-algebra. Nor is $B[[z,z^{-1}]]$, in general, a $B[[z]]$-module.

2) The $\mathbb{C}[z,z^{-1}]$-module $B[[z,z^{-1}]]$ usually has torsion. For example, $(1-z)\cdot\sum\limits_{n\in\mathbb{Z}} z^n=0$ in $\mathbb{C}[[z,z^{-1}]]$ despite $\sum\limits_{n\in\mathbb{Z}} z^n\neq 0$. As a consequence, working in $B[[z,z^{-1}]]$ requires extra care.

3) Despite the suggestive notation $B((z))$, it is of course not true that $B((z))$ is a field whenever B is a commutative ring. However, $B((z))$ is a field whenever B is a field.

Convention 3.3.2. Let B be a vector space, and z a symbol. By analogy with the notations $B[z]$, $B[[z]]$ and $B((z))$ introduced in Definition 3.3.1, we will occasionally

[96] If we would try the natural way, we would get nonsense results. For instance, if we tried to compute the coefficient of $\left(\sum\limits_{n\in\mathbb{Z}} 1z^n\right)\cdot\left(\sum\limits_{n\in\mathbb{Z}} 1z^n\right)$ before z^0, we would get $\sum\limits_{\substack{(n,m)\in\mathbb{Z}^2;\\ n+m=0}} 1\cdot 1$, which is not a convergent series.

also use the notations $B\left[z^{-1}\right]$, $B\left[\left[z^{-1}\right]\right]$ and $B\left(\left(z^{-1}\right)\right)$. For example, $B\left[z^{-1}\right]$ will mean the vector space of all "reverse sequences" $(b_n)_{n\in -\mathbb{N}}$ such that only finitely many $n \in -\mathbb{N}$ satisfy $b_n \neq 0$ [97]. Of course, $B[z] \cong B\left[z^{-1}\right]$ as vector spaces, but $B[z]$ and $B\left[z^{-1}\right]$ are two different subspaces of $B\left[z, z^{-1}\right]$, so it is useful to distinguish between $B[z]$ and $B\left[z^{-1}\right]$.

Now, let us extend Definition 3.3.1 to several variables. The reader is advised to only skim through the following definition, as there is nothing unexpected in it:

Definition 3.3.3. Let $m \in \mathbb{N}$. Let $z_1, z_2, ..., z_m$ be m symbols. For every vector space B, we make the following definitions:

(a) We denote by $B[z_1, z_2, ..., z_m]$ the vector space of all families $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{N}^m} \in B^{\mathbb{N}^m}$ such that only finitely many $(n_1, n_2, ..., n_m) \in \mathbb{N}^m$ satisfy $b_{(n_1,n_2,...,n_m)} \neq 0$. Such a family $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{N}^m}$ is denoted by $\sum\limits_{(n_1,n_2,...,n_m)\in\mathbb{N}^m} b_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m}$. The elements of $B[z_1, z_2, ..., z_m]$ are called *polynomials in the indeterminates $z_1, z_2, ..., z_m$ over B* (even when B is not a ring).

(b) We denote by $B[[z_1, z_2, ..., z_m]]$ the vector space of all families $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{N}^m} \in B^{\mathbb{N}^m}$. Such a family $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{N}^m}$ is denoted by $\sum\limits_{(n_1,n_2,...,n_m)\in\mathbb{N}^m} b_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m}$. The elements of $B[[z_1, z_2, ..., z_m]]$ are called *formal power series in the indeterminates $z_1, z_2, ..., z_m$ over B* (even when B is not a ring).

(c) We denote by $B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$ the vector space of all families $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m} \in B^{\mathbb{Z}^m}$ such that only finitely many $(n_1, n_2, ..., n_m) \in \mathbb{Z}^m$ satisfy $b_{(n_1,n_2,...,n_m)} \neq 0$. Such a family $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m}$ is denoted by $\sum\limits_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m} b_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m}$. The elements of $B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$ are called *Laurent polynomials in the indeterminates $z_1, z_2, ..., z_m$ over B* (even when B is not a ring).

(d) We denote by $B((z_1, z_2, ..., z_m))$ the vector space of all families $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m} \in B^{\mathbb{Z}^m}$ for which there exists an $N \in \mathbb{Z}$ such that every $(n_1, n_2, ..., n_m) \in \mathbb{Z}^m \setminus \{N, N+1, N+2, \ldots\}^m$ satisfies $b_{(n_1,n_2,...,n_m)} = 0$. Such a family $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m}$ is denoted by $\sum\limits_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m} b_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m}$. The elements of $B((z_1, z_2, ..., z_m))$ are called *formal Laurent series in the indeterminates $z_1, z_2, ..., z_m$ over B* (even when B is not a ring).

(e) We denote by $B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ the vector space of all families $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m} \in B^{\mathbb{Z}^m}$. Such a family $\left(b_{(n_1,n_2,...,n_m)}\right)_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m}$ is denoted by $\sum\limits_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m} b_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m}$.

All five of these spaces $B[z_1, z_2, ..., z_m]$, $B[[z_1, z_2, ..., z_m]]$, $B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$, $B((z_1, z_2, ..., z_m))$ and $B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ are $\mathbb{C}[z_1, z_2, ..., z_m]$-modules. (Here, the $\mathbb{C}[z_1, z_2, ..., z_m]$-module structure on

[97] Here, $-\mathbb{N}$ denotes the set $\{0, -1, -2, -3, ...\}$, and a "reverse sequence" is a family indexed by elements of $-\mathbb{N}$.

$B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ is given by

$$\left(\sum_{(n_1,n_2,...,n_m)\in\mathbb{N}^m} c_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m}\right) \cdot \left(\sum_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m} b_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m}\right)$$
$$= \sum_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m} \left(\sum_{(m_1,m_2,...,m_m)\in\mathbb{N}^m} c_{(m_1,m_2,...,m_m)} \cdot b_{(n_1-m_1,n_2-m_2,...,n_m-m_m)}\right) z_1^{n_1} z_2^{n_2} ... z_m^{n_m} \tag{109}$$

for all $\sum\limits_{(n_1,n_2,...,n_m)\in\mathbb{Z}^m} b_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m} \in B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ and $\sum\limits_{(n_1,n_2,...,n_m)\in\mathbb{N}^m} c_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m} \in \mathbb{C}[z_1, z_2, ..., z_m]$, and the $\mathbb{C}[z_1, z_2, ..., z_m]$-module structures on the other four spaces are defined similarly.) Besides, $B[[z_1, z_2, ..., z_m]]$ and $B((z_1, z_2, ..., z_m))$ are $\mathbb{C}[[z_1, z_2, ..., z_m]]$-modules (defined in a similar fashion to (109)). Also, $B((z_1, z_2, ..., z_m))$ is a $\mathbb{C}((z_1, z_2, ..., z_m))$-module (defined in analogy to (109)). Besides, $B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$, $B((z_1, z_2, ..., z_m))$ and $B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ are $\mathbb{C}\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$-modules (in a similar way).

Of course, if B is a $\mathbb{C}$-algebra, then the above-defined spaces $B[z_1, z_2, ..., z_m]$, $B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$, $B[[z_1, z_2, ..., z_m]]$ and $B((z_1, z_2, ..., z_m))$ are $\mathbb{C}$-algebras themselves (with multiplication defined by a formula analogous to (109) again), and in fact $B[z_1, z_2, ..., z_m]$ is the algebra of polynomials in the variables $z_1, z_2, ..., z_m$ over B, and $B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$ is the algebra of Laurent polynomials in the variables $z_1, z_2, ..., z_m$ over B, and $B[[z_1, z_2, ..., z_m]]$ is the algebra of formal power series in the variables $z_1, z_2, ..., z_m$ over B.

It should be noticed that $B[z_1, z_2, ..., z_m] \cong B \otimes \mathbb{C}[z_1, z_2, ..., z_m]$ and $B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right] \cong B \otimes \mathbb{C}\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$ canonically, but such isomorphisms do **not** hold for $B[[z_1, z_2, ..., z_m]]$, $B((z_1, z_2, ..., z_m))$ and $B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ unless B is finite-dimensional or $m = 0$.

There are several obvious injections (analogous to the ones listed in Definition 3.3.1) which we regard as inclusions. For example, one of these is the injection $B[z_1, z_2, ..., z_m] \to B[[z_1, z_2, ..., z_m]]$; we won't list the others here.

Clearly, all five spaces $B[z_1, z_2, ..., z_m]$, $B[[z_1, z_2, ..., z_m]]$, $B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$, $B((z_1, z_2, ..., z_m))$ and $B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ depend functorially on B.

Clearly, when $m = 1$, Definition 3.3.3 is equivalent to Definition 3.3.1.

Definition 3.3.3 can be extended to infinitely many indeterminates; this is left to the reader.

Our definition of $B((z_1, z_2, ..., z_m))$ is rather intricate. The reader might gain a better understanding from the following equivalent definition: The set $B((z_1, z_2, ..., z_m))$ is the subset of $B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ consisting of those $p \in B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ for which there exists an $(a_1, a_2, \dots, a_m) \in \mathbb{Z}^m$ such that $z_1^{a_1} z_2^{a_2} ... z_m^{a_m} \cdot p \in B[[z_1, z_2, ..., z_m]]$. It is easy to show that $B((z_1, z_2, ..., z_m))$ is isomorphic to the localization of the ring $B[[z_1, z_2, ..., z_m]]$ at the multiplicatively closed subset consisting of all monomials.

The reader should be warned that if B is a field, m is an integer > 1, and z_1, z_2,

..., z_m are m symbols, then the ring $B\left(\left(z_1, z_2, ..., z_m\right)\right)$ is **not** a field (unlike in the case $m = 1$); for example, it does not contain an inverse to $z_1 - z_2$. This is potentially confusing and I would not be surprised if some texts define $B\left(\left(z_1, z_2, ..., z_m\right)\right)$ to mean a different ring which actually is a field.

When B is a vector space and z is a symbol, there is an operator we can define on each of the five spaces $B[z]$, $B[[z]]$, $B[z, z^{-1}]$, $B((z))$ and $B[[z, z^{-1}]]$: derivation with respect to z:

Definition 3.3.4. For every vector space B and symbol z, we make the following definitions:

Define a linear map $\dfrac{d}{dz} : B[z] \to B[z]$ by the formula

$$\frac{d}{dz}\left(\sum_{n\in\mathbb{N}} b_n z^n\right) = \sum_{n\in\mathbb{N}} (n+1)\, b_{n+1} z^n \qquad \text{for every } \sum_{n\in\mathbb{N}} b_n z^n \in B[z]. \tag{110}$$

Define a linear map $\dfrac{d}{dz} : B[[z]] \to B[[z]]$ by the very same formula, and define linear maps $\dfrac{d}{dz} : B[z, z^{-1}] \to B[z, z^{-1}]$, $\dfrac{d}{dz} : B((z)) \to B((z))$ and $\dfrac{d}{dz} : B[[z, z^{-1}]] \to B[[z, z^{-1}]]$ by analogous formulas (more precisely, by formulas which differ from (110) only in that the sums range over $\mathbb{Z}$ instead of over $\mathbb{N}$).

For every $f \in B[[z, z^{-1}]]$, the image $\dfrac{d}{dz} f$ of f under the linear map $\dfrac{d}{dz}$ will be denoted by $\dfrac{df}{dz}$ or by f' and called the *z-derivative* of f (or, briefly, the *derivative* of f). The operator $\dfrac{d}{dz}$ itself (on any of the five vector spaces $B[z]$, $B[[z]]$, $B[z, z^{-1}]$, $B((z))$ and $B[[z, z^{-1}]]$) will be called the *differentiation with respect to z*.

An analogous definition can be made for several variables:

Definition 3.3.5. Let $m \in \mathbb{N}$. Let $z_1, z_2, ..., z_m$ be m symbols. Let $i \in \{1, 2, ..., m\}$. For every vector space B, we make the following definitions:

Define a linear map $\dfrac{\partial}{\partial z_i} : B[z_1, z_2, ..., z_m] \to B[z_1, z_2, ..., z_m]$ by the formula

$$\begin{aligned} &\frac{\partial}{\partial z_i}\left(\sum_{(n_1,n_2,...,n_m)\in\mathbb{N}^m} b_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m}\right) \\ &= \sum_{(n_1,n_2,...,n_m)\in\mathbb{N}^m} (n_i+1)\, b_{(n_1,n_2,...,n_{i-1},n_i+1,n_{i+1},n_{i+2},...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m} \\ &\qquad \text{for every } \sum_{(n_1,n_2,...,n_m)\in\mathbb{N}^m} b_{(n_1,n_2,...,n_m)} z_1^{n_1} z_2^{n_2} ... z_m^{n_m} \in B[z_1, z_2, ..., z_m]. \end{aligned} \tag{111}$$

Define a linear map $\dfrac{\partial}{\partial z_i} : B[[z_1, z_2, ..., z_m]] \to B[[z_1, z_2, ..., z_m]]$ by the very same formula, and define linear maps $\dfrac{\partial}{\partial z_i} : B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right] \to$

$B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$, $\dfrac{\partial}{\partial z_i} : B\left(\left(z_1, z_2, ..., z_m\right)\right) \to B\left(\left(z_1, z_2, ..., z_m\right)\right)$ and $\dfrac{\partial}{\partial z_i} : B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right] \to B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$ by analogous formulas (more precisely, by formulas which differ from (111) only in that the sums range over $\mathbb{Z}^m$ instead of over $\mathbb{N}^m$).

For every $f \in B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$, the image $\dfrac{\partial}{\partial z_i} f$ of f under the linear map $\dfrac{\partial}{\partial z_i}$ will be denoted by $\dfrac{\partial f}{\partial z_i}$ and called the *z_i-derivative* of f (or the *partial derivative of f with respect to z_i*). The operator $\dfrac{\partial}{\partial z_i}$ itself (on any of the five vector spaces $B\left[z_1, z_2, ..., z_m\right]$, $B\left[\left[z_1, z_2, ..., z_m\right]\right]$, $B\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]$, $B\left(\left(z_1, z_2, ..., z_m\right)\right)$ and $B\left[\left[z_1, z_1^{-1}, z_2, z_2^{-1}, ..., z_m, z_m^{-1}\right]\right]$) will be called the *differentiation with respect to z_i*.

Again, it is straightforward (and left to the reader) to extend this definition to infinitely many indeterminates.

3.3.2. Quantum fields

Formal power series which are infinite "in both directions" might seem like a perverse and artificial notion; their failure to form a ring certainly does not suggest them to be useful. Nevertheless, they prove very suitable when studying infinite-dimensional Lie algebras. Let us explain how.

For us, when we study Lie algebras, we are mainly concerned with their elements, usually basis elements (e. g., the a_n in $\mathcal{A}$). For physicists, instead, certain generating functions built of these objects are objects of primary concern, since they are closer to what they observe. They are called *quantum fields*.

Now, what are quantum fields?

For example, in $\mathcal{A}$, let us set $a(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^{-n-1}$, where z is a formal variable. This sum $\sum\limits_{n\in\mathbb{Z}} a_n z^{-n-1}$ is a formal sum which is infinite in both directions, so it is not an element of any of the rings $U(\mathcal{A})[[z]]$ or $U(\mathcal{A})((z))$, but only an element of $U(\mathcal{A})[[z, z^{-1}]]$.

As we said, the vector space $U(\mathcal{A})[[z, z^{-1}]]$ is **not** a ring (even though $U(\mathcal{A})$ is a $\mathbb{C}$-algebra), so we cannot multiply two "sums" like $a(z)$ in general. **However**, in the following, we are going to learn about several things that we **can** do with such "sums". One first thing that we notice about our concrete "sum" $a(z) = \sum\limits_{n\in\mathbb{Z}} a_n z^{-n-1}$ is that if we apply $a(z)$ to some vector v in F_μ (by evaluating the term $(a(z))v$ componentwise[98]), then we get a sum $\sum\limits_{n\in\mathbb{Z}} z^{-n-1} a_n v$ which evaluates to an element of $F_\mu((z))$ (because every sufficiently large $n \in \mathbb{Z}$ satisfies $z^{-n-1} \underbrace{a_n v}_{=0} = 0$). As a consequence, $a(z)$ "acts"

[98]By "evaluating" a term like $(a(z))v$ at a vector v "componentwise", we mean evaluating $\sum\limits_{n\in\mathbb{Z}} \left(a_n z^{-n-1}\right)(v)$. Here, the variable z is decreed to commute with everything else, so that $\left(a_n z^{-n-1}\right)(v)$ means $z^{-n-1} a_n v$.

on F_μ. I am saying "acts" in quotation marks, since this "action" is not a map $F_\mu \to F_\mu$ but a map $F_\mu \to F_\mu((z))$, and since $a(z)$ does not lie in a ring (as I said, $U(\mathcal{A})[[z, z^{-1}]]$ is **not** a ring).

Physicists call $a(z)$ a *quantum field* (more precisely, a free bosonic field).

While we cannot take the square $(a(z))^2$ of our "sum" $a(z)$ (since $U(\mathcal{A})[[z, z^{-1}]]$ is not a ring), we can multiply two sums "with different variables"; e. g., we can multiply $a(z)$ and $a(w)$, where z and w are two distinct formal variables. The product $a(z)a(w)$ is defined as the formal sum $\sum\limits_{(n,m)\in\mathbb{Z}^2} a_n a_m z^{-n-1} w^{-m-1} \in U(\mathcal{A})[[z, z^{-1}]][[w, w^{-1}]]$. Note that elements of $U(\mathcal{A})[[z, z^{-1}]][[w, w^{-1}]]$ are two-sided sequences of two-sided sequences of elements of $U(\mathcal{A})$; of course, we can interpret them as maps $\mathbb{Z}^2 \to U(\mathcal{A})$.

It is easy to see that $[a(z), a(w)] = \sum\limits_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1}$. This identity, in the first place, holds on the level of formal sums (where $\sum\limits_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1}$ is a shorthand notation for a particular sequence of sequences: namely, the one whose j-th element is the sequence whose i-th element is $\delta_{i+j+2,0}(j+1)$), but if we evaluate it on an element v of F_μ, then we get an identity $[a(z), a(w)]v = \sum\limits_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1} v$ which holds in the space $F_\mu((z))((w))$.

We can obtain the "series" $[a(z), a(w)] = \sum\limits_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1}$ by differentiating a more basic "series":

$$\delta(w-z) := \sum_{n\in\mathbb{Z}} z^{-n-1} w^n.$$

This, again, is a formal series infinite in both directions. Why do we call it $\delta(w-z)$? Because in analysis, the delta-"function" (actually a distribution) satisfies the formula $\int \delta(x-y) f(y)\, dy = f(x)$ for every function f, whereas our series $\delta(w-z)$ satisfies a remarkably similar property[99]. And now, $[a(z), a(w)] = \sum\limits_{n\in\mathbb{Z}} n z^{-n-1} w^{n-1}$ becomes $[a(z), a(w)] = \partial_w \delta(w-z) =: \delta'(w-z)$.

Something more interesting comes out for the Witt algebra: Set $T(z) = \sum\limits_{n\in\mathbb{Z}} L_n z^{-n-2}$

[99]Namely, if we define the "formal residue" $\dfrac{1}{2\pi i} \oint\limits_{|z|=1} q(z)\, dz$ of an element $q(z) \in B((z))$ (for B being some vector space) to be the coefficient of $q(z)$ before z^{-1}, then every $f = \sum\limits_{n\in\mathbb{Z}} f_n z^n$ (with $f_n \in B$) satisfies $\dfrac{1}{2\pi i} \oint\limits_{|z|=1} z^{-n-1} f(z)\, dz = f_n$, and thus $\dfrac{1}{2\pi i} \oint\limits_{|z|=1} \delta(w-z) f(z)\, dz = f(w)$.

in the Witt algebra. Then, we have

$$\begin{aligned}
& [T(z), T(w)] \\
&= \sum_{(n,m)\in\mathbb{Z}^2} (n-m) L_{n+m} z^{-n-2} w^{-m-2} = \sum_{(k,m)\in\mathbb{Z}^2} L_k \underbrace{(k-2m)}_{=(k+2)+2(-m-1)} z^{m-k-2} w^{-m-2} \\
&= \underbrace{\left(\sum_{k\in\mathbb{Z}} L_k (k+2) z^{-k-3}\right)}_{=-T'(z)} \underbrace{\left(\sum_{m\in\mathbb{Z}} z^{m+1} w^{-m-2}\right)}_{=\delta(w-z)} \\
&\qquad + 2 \underbrace{\left(\sum_{k\in\mathbb{Z}} L_k z^{-k-2}\right)}_{=T(z)} \underbrace{\left(\sum_{m\in\mathbb{Z}} (-m-1) z^m w^{-m-2}\right)}_{=\delta'(w-z)} \\
&= -T'(z)\,\delta(w-z) + 2T(z)\,\delta'(w-z).
\end{aligned}$$

Note that this formula uniquely determines the Lie bracket of the Witt algebra. This is how physicists would define the Witt algebra.

Now, let us set $T(z) = \sum\limits_{n\in\mathbb{Z}} L_n z^{-n-2}$ **in the Virasoro algebra**. (This power series T looks exactly like the one before, but note that the L_n now mean elements of the Virasoro algebra rather than the Witt algebra.) Then, our previous computation of $[T(z), T(w)]$ must be modified by adding a term of $\sum\limits_{n\in\mathbb{Z}} \frac{n^3-n}{12} C z^{-n-2} w^{n-2} = \frac{C}{12}\delta'''(w-z)$. So we get

$$[T(z), T(w)] = -T'(z)\,\delta(w-z) + 2T(z)\,\delta'(w-z) + \frac{C}{12}\delta'''(w-z).$$

Exercise: Check that, if we interpret L_n and a_m as the actions of $L_n \in \operatorname{Vir}$ and $a_m \in \mathcal{A}$ on the $\operatorname{Vir} \ltimes \mathcal{A}$-module F_μ, then the following identity between maps $F_\mu \to F_\mu((z))((w))$ holds:

$$[T(z), a(w)] = a(z)\,\delta'(w-z).$$

Recall

$$: a_m a_n : = \begin{cases} a_m a_n, & \text{if } m \leq n; \\ a_n a_m, & \text{if } m > n \end{cases}.$$

So we can reasonably define the "normal ordered" product $: a(z)\,a(w) :$ to be

$$\sum_{(n,m)\in\mathbb{Z}^2} : a_n a_m : z^{-n-1} w^{-m-1} \in U(\mathcal{A}) \left[\left[z, z^{-1}\right]\right] \left[\left[w, w^{-1}\right]\right].$$

This definition of $: a(z)\,a(w) :$ is equivalent to the definition given in Problem 2 of Problem Set 3.

That $: a(z)\,a(w) :$ is well-defined is not a surprise: the variables z and w are distinct, so there are no terms to collect in the sum $\sum\limits_{(n,m)\in\mathbb{Z}^2} : a_n a_m : z^{-n-1} w^{-m-1}$, and thus there is no danger of obtaining an infinite sum which makes no sense (like what we would get if we would try to define $a(z)^2$). [100] But it is more interesting that (although we cannot define $a(z)^2$) we can define a "normal ordered" square $:a(z)^2:$ (or, what is the same, $:a(z)\,a(z):$), although it will not be an element of $U(\mathcal{A})[[z,z^{-1}]]$ but rather of a suitable completion. We are not going to do elaborate on how to choose this completion here; but for us it will be enough to notice that, if we reinterpret the a_n as endomorphisms of F_μ (using the action of $\mathcal{A}$ on F_μ) rather than elements of $U(\mathcal{A})$, then the "normal ordered" square $:a(z)^2:$ is a well-defined element of $(\operatorname{End} F_\mu)[[z,z^{-1}]]$. Namely:

$$\begin{aligned}
&:a(z)^2: \\
&= \sum_{(n,m)\in\mathbb{Z}^2} :a_n a_m: z^{-n-1} z^{-m-1} = \sum_{k\in\mathbb{Z}} \left(\sum_{\substack{(n,m)\in\mathbb{Z}^2;\\ n+m=k}} :a_n a_m: \right) z^{-k-2} \\
&\qquad \left(\begin{array}{c} \text{this is how power series are always multiplied; but we don't yet} \\ \text{know that the sum } \sum\limits_{\substack{(n,m)\in\mathbb{Z}^2;\\ n+m=k}} :a_n a_m: \text{ makes sense for all } k \\ \text{(although we will see in a few lines that it does)} \end{array} \right) \\
&= \sum_{k\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} :a_m a_{k-m}: \right) z^{-k-2} \qquad \text{(here, we substituted } (m,k-m) \text{ for } (n,m)\text{)} \\
&= \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} :a_{-m} a_{n+m}: \right) z^{-n-2} \qquad \left(\begin{array}{c} \text{here, we substituted } k \text{ by } n \text{ in the first sum,} \\ \text{and we substituted } m \text{ by } -m \text{ in the second sum} \end{array} \right),
\end{aligned}$$

and the sums $\sum\limits_{m\in\mathbb{Z}} :a_{-m}a_{n+m}:$ are well-defined for all $n\in\mathbb{Z}$ (by Lemma 3.2.10 **(c)**). We can simplify this result if we also reinterpret the $L_n \in \operatorname{Vir}$ as endomorphisms of F_μ (using the action of Vir on F_μ that was introduced in Proposition 3.2.13) rather than elements of $U(\operatorname{Vir})$. In fact, the "series" $T(z) = \sum\limits_{n\in\mathbb{Z}} L_n z^{-n-2}$ then becomes

$$\begin{aligned}
T(z) &= \sum_{n\in\mathbb{Z}} L_n z^{-n-2} = \sum_{n\in\mathbb{Z}} \frac{1}{2} \left(\sum_{m\in\mathbb{Z}} :a_{-m}a_{n+m}: \right) z^{-n-2} \qquad \text{(by (95))} \\
&= \frac{1}{2} \underbrace{\sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} :a_{-m}a_{n+m}: \right) z^{-n-2}}_{=:a(z)^2:} = \frac{1}{2} :a(z)^2: .
\end{aligned}$$

Remark 3.3.6. In Definition 3.2.4, we have defined the normal ordered product $:a_m a_n:$ in the universal enveloping algebra of the Heisenberg algebra. This is not the only situation in which we can define a normal ordered product, but in other situations the definition can happen to be different. For example, in Proposition 3.4.4, we will define a normal ordered product (on a different algebra) which will not be commutative, and not even "super-commutative". There is no general rule to define normal ordered products; it is done on a case-by-case basis.

[100]For the same reason, the product $a(z)\,a(w)$ (without normal ordering) is well-defined.

However, the definition of the normal ordered product of two **quantum fields** given in Problem 2 of Problem Set 3 is general, i. e., it is defined not only for quantum fields over $U(\mathcal{A})$.

Exercise 1. For any $\beta \in \mathbb{C}$, the formula $T(z) = \frac{1}{2} : a(z)^2 : + \beta a'(z)$ defines a representation of Vir on F_μ with $c = 1 - 12\beta^2$.

Exercise 2. For any $\beta \in \mathbb{C}$, there is a homomorphism $\varphi_\beta : \text{Vir} \to \text{Vir} \ltimes \mathcal{A}$ (a splitting of the projection $\text{Vir} \ltimes \mathcal{A} \to \text{Vir}$) given by

$$\varphi_\beta(L_n) = L_n + \beta a_n, \qquad n \neq 0;$$
$$\varphi_\beta(L_0) = L_0 + \beta a_0 + \frac{\beta^2}{2} K,$$
$$\varphi_\beta(C) = C.$$

Exercise 3. If we twist the action of Exercise 1 by this map, we recover the action of problem 1 of Homework 2 for $\beta = i\lambda$.

3.3.3. Recognizing exponential series

Here is a simple property of power series (actually, an algebraic analogue of the well-known fact from analysis that the solutions of the differential equation $f' = \alpha f$ are scalar multiples of the function $x \mapsto \exp(\alpha x)$):

Proposition 3.3.7. Let R be a commutative $\mathbb{Q}$-algebra. Let U be an R-module. Let $(\alpha_1, \alpha_2, \alpha_3, ...)$ be a sequence of elements of R. Let $P \in U[[x_1, x_2, x_3, ...]]$ is a formal power series with coefficients in U (where $x_1, x_2, x_3, ...$ are symbols) such that every $i > 0$ satisfies $\frac{\partial P}{\partial x_i} = \alpha_i P$. Then, there exists some $f \in U$ such that $P = f \cdot \exp\left(\sum\limits_{j>0} x_j \alpha_j\right)$.

The proof of Proposition 3.3.7 is easy (just let f be the constant term of the power series P, and prove by induction that every monomial of P equals the corresponding monomial of $f \cdot \exp\left(\sum\limits_{j>0} x_j \alpha_j\right)$).

3.3.4. Homogeneous maps and equigraded series

The discussion we will be doing now is only vaguely related to power series (let alone quantum fields); it is meant as a preparation for a later proof (namely, that of Theorem 3.11.2), where it will provide "convergence" assertions (in a certain sense).

A well-known nuisance in the theory of $\mathbb{Z}$-graded vector spaces is the fact that the endomorphism ring of a $\mathbb{Z}$-graded vector space is not (in general) $\mathbb{Z}$-graded. It does, however, contain a $\mathbb{Z}$-graded subring, which we will introduce now:

Definition 3.3.8. **(a)** Let V and W be two $\mathbb{Z}$-graded vector spaces, with gradings $(V[n])_{n\in\mathbb{Z}}$ and $(W[n])_{n\in\mathbb{Z}}$, respectively. Let $f : V \to W$ be a linear map. Let $m \in \mathbb{Z}$. Then, f is said to be a *homogeneous linear map of degree m* if every $n \in \mathbb{Z}$ satisfies $f(V[n]) \subseteq W[n+m]$.

(It is important not to confuse this notion of "homogeneous linear maps of degree m" with the notion of "homogeneous polynomial maps of degree n" defined in Definition 2.6.16 **(a)**; the former of these notions is not a particular case of the latter.)

Note that the homogeneous linear maps of degree 0 are exactly the graded linear maps.

(b) Let V and W be two $\mathbb{Z}$-graded vector spaces. For every $m \in \mathbb{Z}$, let $\mathrm{Hom}_{\mathrm{hg}=m}(V,W)$ denote the vector space of all homogeneous linear maps $V \to W$ of degree m. This $\mathrm{Hom}_{\mathrm{hg}=m}(V,W)$ is a vector subspace of $\mathrm{Hom}(V,W)$ for every $m \in \mathbb{Z}$. Moreover, $\bigoplus\limits_{m\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{hg}=m}(V,W)$ is a well-defined internal direct sum, and will be denoted by $\mathrm{Hom}_{\mathrm{hg}}(V,W)$. This $\mathrm{Hom}_{\mathrm{hg}}(V,W)$ is a vector subspace of $\mathrm{Hom}(V,W)$, and is canonically a $\mathbb{Z}$-graded vector space, with its m-th graded component being $\mathrm{Hom}_{\mathrm{hg}=m}(V,W)$.

(c) Let V be a $\mathbb{Z}$-graded vector space. Then, let $\mathrm{End}_{\mathrm{hg}} V$ denote the $\mathbb{Z}$-graded vector subspace $\mathrm{Hom}_{\mathrm{hg}}(V,V)$ of $\mathrm{Hom}(V,V) = \mathrm{End}\, V$. Then, $\mathrm{End}_{\mathrm{hg}} V$ is a subalgebra of $\mathrm{End}\, V$, and a $\mathbb{Z}$-graded algebra. Moreover, the canonical action of $\mathrm{End}_{\mathrm{hg}} V$ on V (obtained by restricting the action of $\mathrm{End}\, V$ on V to $\mathrm{End}_{\mathrm{hg}} V$) makes V into a $\mathbb{Z}$-graded $\mathrm{End}_{\mathrm{hg}} V$-module.

We next need a relatively simple notion for a special kind of power series. I (Darij) call them "equigraded power series", though noone else seems to use this nomenclature.

Definition 3.3.9. Let B be a $\mathbb{Z}$-graded vector space, and z a symbol. An element $\sum\limits_{n\in\mathbb{Z}} b_n z^n$ of $B[[z,z^{-1}]]$ (with $b_n \in B$ for every $n \in \mathbb{Z}$) is said to be *equigraded* if every $n \in \mathbb{Z}$ satisfies $b_n \in B[n]$ (where $(B[m])_{m\in\mathbb{Z}}$ denotes the grading on B). Since $B[[z]]$ and $B((z))$ are vector subspaces of $B[[z,z^{-1}]]$, it clearly makes sense to speak of equigraded elements of $B[[z]]$ or of $B((z))$. We will denote by $B[[z,z^{-1}]]_{\mathrm{equi}}$ the set of all equigraded elements of $B[[z,z^{-1}]]$. It is easy to see that $B[[z,z^{-1}]]_{\mathrm{equi}}$ is a vector subspace of $B[[z,z^{-1}]]$.

Elementary properties of equigraded elements are:

Proposition 3.3.10. **(a)** Let B be a $\mathbb{Z}$-graded vector space, and z a symbol. Then,

$$\begin{aligned}
&\{f \in B[z] \mid f \text{ is equigraded}\}, \qquad && \left\{f \in B\left[z,z^{-1}\right] \mid f \text{ is equigraded}\right\}, \\
&\{f \in B[[z]] \mid f \text{ is equigraded}\}, && \{f \in B((z)) \mid f \text{ is equigraded}\}, \\
&\left\{f \in B\left[\left[z,z^{-1}\right]\right] \mid f \text{ is equigraded}\right\} = B\left[\left[z,z^{-1}\right]\right]_{\mathrm{equi}} &&
\end{aligned}$$

are vector spaces.

(b) Let B be a $\mathbb{Z}$-graded algebra. Then, $\{f \in B[[z]] \mid f \text{ is equigraded}\}$ is a subalgebra of $B[[z]]$ and closed with respect to the usual topology on $B[[z]]$.

(c) Let B be a $\mathbb{Z}$-graded algebra. If $f \in B[[z]]$ is an equigraded power series and invertible in the ring $B[[z]]$, then f^{-1} also is an equigraded power series.

We will only use parts **(a)** and **(b)** of this proposition, and these are completely straightforward to prove. (Part **(c)** is less straightforward but still an easy exercise.)

Equigradedness of power series sometimes makes their actions on modules more manageable. Here is an example:

Proposition 3.3.11. Let A be a $\mathbb{Z}$-graded algebra, and let M be a $\mathbb{Z}$-graded A-module. Assume that M is concentrated in nonnegative degrees. Let u be a symbol.

(a) It is clear that for any $f \in A[[u, u^{-1}]]$ and any $x \in M[u, u^{-1}]$, the product fx is a well-defined element of $M[[u, u^{-1}]]$.

(b) For any **equigraded** $f \in A[[u, u^{-1}]]$ and any $x \in M[u, u^{-1}]$, the product fx is a well-defined element of $M((u))$ (and not only of $M[[u, u^{-1}]]$).

(c) For any **equigraded** $f \in A[[u^{-1}]]$ and any $x \in M[u^{-1}]$, the product fx is a well-defined element of $M[u^{-1}]$ (and not only of $M[[u^{-1}]]$).

The proof of this proposition is quick and straightforward. (The only idea is that for any fixed $x \in M[u, u^{-1}]$, any sufficiently low-degree element of A annihilates x due to the "concentrated in nonnegative degrees" assumption, but sufficiently low-degree monomials in f come with sufficiently low-degree coefficients due to f being equigraded.)

3.4. [unfinished] More on unitary representations

Let us consider the Verma modules of the Virasoro algebra.

Last time: $L_{\frac{\mu^2+\lambda^2}{2}, 1+12\lambda^2}$ is unitary (for $\lambda, \mu \in \mathbb{R}$), so the Vir-module $L_{h,c}$ is unitary if $c \geq 1$ and $h \geq \frac{c-1}{24}$.

We can extend this as follows: $L_{0,1}^{\otimes m-1} \otimes L_{h,c}$ is unitary and has a highest-weight vector $v_{0,1}^{\otimes m-1} \otimes v_{h,c}$ which has weight $(h, c+m-1)$. Hence, the representation $L_{h,c+m-1}$ is unitary [why? use irreducibility of unitary modules and stuff].

Hence, $L_{h,c}$ is unitary if $c \geq m$ and $h \geq \frac{c-m}{24}$.

Theorem 3.4.1. In fact, $L_{h,c}$ is unitary if $c \geq 1$ and $h \geq 0$.

But this is harder to show.

This is still not an only-if. For example, $L_{0,0}$ is unitary (and 1-dimensional).

Proposition 3.4.2. If $L_{h,c}$ is unitary, then $h \geq 0$ and $c \geq 0$.

Proof of Proposition 3.4.2. Assume that $L_{h,c}$ is unitary. Then, $(L_{-n}v_{h,c}, L_{-n}v_{h,c}) \geq 0$

for every $n \in \mathbb{Z}$. But every positive $n \in \mathbb{Z}$ satisfies

$$(L_{-n}v_{h,c}, L_{-n}v_{h,c}) = \left(\underbrace{L_n L_{-n}}_{=[L_n,L_{-n}]+L_{-n}L_n} v_{h,c}, v_{h,c} \right) = \left(\underbrace{([L_n, L_{-n}] + L_{-n}L_n) v_{h,c}}_{\substack{=[L_n,L_{-n}]v_{h,c} \\ (\text{since } L_{-n}L_n v_{h,c}=0)}}, v_{h,c} \right)$$

$$= \left(\underbrace{[L_n, L_{-n}]}_{=2nL_0+\frac{n^3-n}{12}C} v_{h,c}, v_{h,c} \right) = 2nh + \frac{n^3-n}{12}c.$$

Thus, $2nh + \frac{n^3-n}{12}c \geq 0$ for every positive $n \in \mathbb{Z}$. From this, by taking $n \to \infty$, we obtain $c \geq 0$. By taking $n = 1$, we get $h \geq 0$. This proves Proposition 3.4.2.

Definition 3.4.3. Let $\delta \in \left\{0, \frac{1}{2}\right\}$. Let C_δ be the $\mathbb{C}$-algebra with generators $\{\psi_j \mid j \in \delta + \mathbb{Z}\}$ and relations

$$\psi_j\psi_k + \psi_k\psi_j = \delta_{k,-j} \qquad \text{for all } j, k \in \delta + \mathbb{Z}.$$

This $\mathbb{C}$-algebra C_δ is an infinite-dimensional Clifford algebra (namely, the Clifford algebra of the free vector space with basis $\{\psi_j \mid j \in \delta + \mathbb{Z}\}$ and bilinear form $(\psi_j, \psi_k) \mapsto \frac{1}{2}\delta_{k,-j}$). The algebra C_δ is called an *algebra of free fermions*. For $\delta = 0$, it is called the *Ramond sector*; for $\delta = \frac{1}{2}$ it is called *Neveu-Schwarz sector*.

Let us now construct a representation V_δ of C_δ: Let V_δ be the $\mathbb{C}$-algebra $\wedge\left(\xi_n \mid n \in (\delta + \mathbb{Z})_{\geq 0}\right)$. For any $i \in \delta + \mathbb{Z}$, define an operator $\frac{\partial}{\partial \xi_i} : V_\delta \to V_\delta$ by

$$\frac{\partial}{\partial \xi_i}\left(\xi_{j_1} \wedge \xi_{j_2} \wedge ... \wedge \xi_{j_k}\right)$$
$$= \begin{cases} 0, & \text{if } i \notin \{j_1, j_2, ..., j_k\}; \\ (-1)^{\ell-1} \xi_{j_1} \wedge \xi_{j_2} \wedge ... \wedge \xi_{j_{\ell-1}} \wedge \xi_{j_{\ell+1}} \wedge \xi_{j_{\ell+2}} \wedge ... \wedge \xi_{j_k}, & \text{if } i \in \{j_1, j_2, ..., j_k\} \end{cases}$$
$$\text{for all } j_1 < j_2 < ... < j_k \text{ in } \delta + \mathbb{Z},$$

where, in the case when $i \in \{j_1, j_2, ..., j_k\}$, we denote by ℓ the element u of $\{1, 2, ..., k\}$ satisfying $j_\ell = u$. (Note the $(-1)^{\ell-1}$ sign, which distinguishes this "differentiation" from differentiation in the commutative case. This is a particular case of the Koszul sign rule.)

Define an action of C_δ on V_δ by

$$\psi_{-n} \mapsto \xi_n \qquad \text{for } n<0;$$
$$\psi_n \mapsto \frac{\partial}{\partial \xi_n} \qquad \text{for } n>0;$$
$$\psi_0 \mapsto \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \xi_0}+\xi_0\right) \qquad \text{(this is only relevant if } \delta=0\text{)}.$$

This indeed defines a representation of C_δ (exercise!). This is an infinite-dimensional analogue of the well-known spinor representation of Clifford algebras.

From Homework Set 2 problem 2, we know:

Proposition 3.4.4. Let $\delta \in \left\{0, \frac{1}{2}\right\}$. For every $k \in \mathbb{Z}$, define an endomorphism L_k of V_δ by

$$L_k = \delta_{k,0}\frac{1-2\delta}{16} + \frac{1}{2}\sum_{j\in\delta+\mathbb{Z}} j : \psi_{-j}\psi_{j+k} : ,$$

where the normal ordered product is defined as follows:

$$: \psi_n\psi_m : \ = \begin{cases} -\psi_m\psi_n, & \text{if } m\leq n; \\ \psi_n\psi_m, & \text{if } m>n \end{cases}.$$

Then:

(a) Every $m \in \delta+\mathbb{Z}$ and $k \in \mathbb{Z}$ satisfy $[\psi_m, L_k] = \left(m+\frac{k}{2}\right)\psi_{m+k}$.

(b) Every $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfy $[L_n, L_m] = (n-m)\,L_{n+m} + \delta_{n,-m}\frac{m^3-m}{24}$.

(Hence, V_δ is a representation of Vir with central charge $c=\frac{1}{2}$).

Now this representation V_δ of Vir is unitary. In fact, consider the Hermitian form under which all monomials in ψ_i are orthonormal (positive definite). Then it is easy to see that $\psi_j^\dagger = \psi_{-j}$. Thus, $L_n^\dagger = L_{-n}$.

But these representations V_δ are reducible. In fact, we can define a $(\mathbb{Z}\diagup 2\mathbb{Z})$-grading on V_δ by giving each ξ_n the degree $\overline{1}$, and then the operators L_n preserve parity (i. e., degree under this grading), so that the representation V_δ can be decomposed as a direct sum $V_\delta = V_\delta^+ \oplus V_\delta^-$, where V_δ^+ is the set of the even elements of V_δ, and V_δ^- is the set of the odd elements of V_δ.

Theorem 3.4.5. These subrepresentations V_δ^+ and V_δ^- are irreducible Virasoro modules.

We will not prove this.

What are the highest weights of V_δ^+ and V_δ^- ?

First consider the case $\delta=0$. The highest-weight vector of V_δ^+ is 1, with weight $\left(\frac{1}{16}, \frac{1}{2}\right)$. That of V_δ^- is ξ_0, with weight $\left(\frac{1}{16}, \frac{1}{2}\right)$. Thus, $V_\delta^+ \cong V_\delta^-$ by action of ψ_0 (since $\psi_0^2 = \frac{1}{2}$).

Now consider the case $\delta = \frac{1}{2}$. The highest-weight vector of V_δ^+ is 1, with weight $\left(0, \frac{1}{2}\right)$. That of V_δ^- is $\xi_{1/2}$, with weight $\left(\frac{1}{2}, \frac{1}{2}\right)$.

Corollary 3.4.6. The representation $L_{h, \frac{1}{2}}$ is unitary if $h = 0$, $h = \frac{1}{16}$ or $h = \frac{1}{2}$.

(In physics: Ising model.)

We will not prove:

Proposition 3.4.7. This is an only-if as well.

General answer for $c < 1$: for $c = 1 - \frac{6}{(m+2)(m+3)}$ for $m \in \mathbb{N}$, there are finitely many h where $L_{h,c}$ is unitary. For other values of c, there are no such values.

Definition 3.4.8. The *character* $\operatorname{ch}_V(q)$ of a Vir-module V from category $\mathcal{O}^+$ is $\operatorname{Tr}_V\left(q^{L_0}\right) = \sum (\dim V_\lambda)\, q^\lambda$ for V_λ =generalized eigenspace of L_0 with eigenvalue λ.

This is related to the old definition of character [how?]

What are the characters of the above modules? Since $V_\delta^+ = \wedge(\xi_1, \xi_2, \xi_3, ...)^+$, we have

$$\operatorname{ch}_{L_{\frac{1}{16}, \frac{1}{2}}}(q) = q^{1/16}(1+q)\left(1+q^2\right)\left(1+q^3\right)... = q^{1/16} \prod_{n \geq 1}(1+q^n)$$

(because

$$\begin{aligned} 2\operatorname{ch}_{L_{\frac{1}{16}, \frac{1}{2}}}(q) &= \operatorname{ch}_{V_0}(q) = q^{1/16}(1+1)(1+q)\left(1+q^2\right)\left(1+q^3\right)... \\ &= 2q^{1/16}(1+q)\left(1+q^2\right)\left(1+q^3\right)... \end{aligned}$$

).

Now

$$\begin{aligned} \operatorname{ch}_{L_{0, \frac{1}{2}}}(q) + \operatorname{ch}_{L_{\frac{1}{2}, \frac{1}{2}}}(q) = \operatorname{ch}_{V_{\frac{1}{2}}}(q) &= \left(1+q^{1/2}\right)\left(1+q^{3/2}\right)\left(1+q^{5/2}\right)... \\ &= \prod_{n \in \frac{1}{2}+\mathbb{N}} (1+q^n). \end{aligned}$$

Thus, $\operatorname{ch}_{L_{0, \frac{1}{2}}}(q)$ is the integer part of the product $\prod_{n \in \frac{1}{2}+\mathbb{N}} (1+q^n)$, and $\operatorname{ch}_{L_{\frac{1}{2}, \frac{1}{2}}}(q)$ is the half-integer part of the product $\prod_{n \in \frac{1}{2}+\mathbb{N}} (1+q^n)$.

With this, we conclude our study of V_δ.

Convention 3.4.9. The notation ψ_j for the generators of C_δ introduced in Definition 3.4.3 will not be used in the following. (Instead, we will use the notation ψ_j for some completely different objects.)

3.5. The Lie algebra $\mathfrak{gl}_\infty$ and its representations

For every $n \in \mathbb{N}$, we can define a Lie algebra $\mathfrak{gl}_n$ of $n \times n$-matrices over $\mathbb{C}$. One can wonder how this can be generalized to the "$n = \infty$ case", i. e., to infinite matrices. Obviously, not every pair of infinite matrices has a reasonable commutator (because not any such pair can be multiplied), but there are certain restrictions on infinite matrices which allow us to multiply them and form their commutators. These restrictions can be used to define various Lie algebras consisting of infinite matrices. We will be concerned with some such Lie algebras; the first of them is $\mathfrak{gl}_\infty$:

Definition 3.5.1. We define $\mathfrak{gl}_\infty$ to be the vector space of infinite matrices whose rows and columns are labeled by integers (not only positive integers!) such that only finitely many entries of the matrix are nonzero. This vector space $\mathfrak{gl}_\infty$ is an associative algebra *without unit* (by matrix multiplication); we can thus make $\mathfrak{gl}_\infty$ into a Lie algebra by the commutator in this associative algebra.

We will study the representations of this $\mathfrak{gl}_\infty$. The theory of these representations will extend the well-known (Schur-Weyl) theory of representations of $\mathfrak{gl}_n$.

Definition 3.5.2. The *vector representation* V of $\mathfrak{gl}_\infty$ is defined as the vector space $\mathbb{C}^{(\mathbb{Z})} = \left\{ (x_i)_{i\in\mathbb{Z}} \mid x_i \in \mathbb{C}; \text{ only finitely many } x_i \text{ are nonzero} \right\}$. The Lie algebra $\mathfrak{gl}_\infty$ acts on the vector representation V in the obvious way: namely, for any $a \in \mathfrak{gl}_\infty$ and $v \in V$, we let $a \rightharpoonup v$ be the product of the matrix a with the column vector v.

Here, every element $(x_i)_{i\in\mathbb{Z}}$ of V is identified with the column vector $\begin{pmatrix} ... \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ ... \end{pmatrix}$.

For every $j \in \mathbb{Z}$, let v_j be the vector $(\delta_{i,j})_{i\in\mathbb{Z}} \in V$. Then, $(v_j)_{j\in\mathbb{Z}}$ is a basis of the vector space V.

Convention 3.5.3. When we draw infinite matrices whose rows and columns are labeled by integers, the index of the rows is supposed to increase as we go from left to right, and the index of the columns is supposed to increase as we go from top to bottom.

Remark 3.5.4. In Definition 3.5.2, we used the following (very simple) fact: For every $a \in \mathfrak{gl}_\infty$ and every $v \in V$, the product av of the matrix a with the column vector v is a well-defined element of V. This fact can be generalized: If a is an infinite matrix (whose rows and columns are labeled by integers) such that every column of a has only finitely many nonzero entries, and v is an element of V, then the product av is a well-defined element of V. However, this does **no longer** hold if we drop the condition that every column of a have only finitely many nonzero entries. (For example, if a would be the matrix whose all entries equal 1, then the product av_0 would **not** be an element of V, but rather the element $\begin{pmatrix} ... \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ ... \end{pmatrix}$ of the **larger** vector space $\mathbb{C}^{\mathbb{Z}} = \left\{ (x_i)_{i\in\mathbb{Z}} \mid x_i \in \mathbb{C} \right\}$. Besides, the product $a \begin{pmatrix} ... \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ ... \end{pmatrix}$ would not make any sense at all, not even in $\mathbb{C}^{\mathbb{Z}}$.)

We can consider the representation $\wedge^i V$ of $\mathfrak{gl}_\infty$ for every $i \in \mathbb{N}$. More generally, we have the so-called *Schur modules*:

Definition 3.5.5. If $\pi \in \operatorname{Irr} S_n$, then we can define a representation $S_\pi(V)$ of $\mathfrak{gl}_\infty$ by $S_\pi(V) = \operatorname{Hom}_{S_n}(\pi, V^{\otimes n})$ (where S_n acts on $V^{\otimes n}$ by permuting the tensorands). This $S_\pi(V)$ is called the *π-th Schur module* of V.

This definition mimics the well-known definition (or, more precisely, one of the definitions) of the Schur modules of a finite-dimensional vector space.

Proposition 3.5.6. For every $\pi \in \operatorname{Irr} S_n$, the representation $S_\pi(V)$ of $\mathfrak{gl}_\infty$ is irreducible.

Proof of Proposition 3.5.6. The following is not a self-contained proof; it is just a way to reduce Proposition 3.5.6 to the similar fact about finite-dimensional vector spaces (which is a well-known fact in the representation theory of $\mathfrak{gl}_m$).

For every vector subspace $W \subseteq V$, we can canonically identify $S_\pi(W)$ with a vector subspace of $S_\pi(V)$.

For every subset I of $\mathbb{Z}$, let W_I be the subset of V generated by all v_i with $i \in I$. Clearly, whenever two subsets I and J of $\mathbb{Z}$ satisfy $I \subseteq J$, we have $W_I \subseteq W_J$. Also, whenever I is a finite subset of $\mathbb{Z}$, the vector space W_I is finite-dimensional.

For every tensor $u \in V^{\otimes n}$, there exists a finite subset I of $\mathbb{Z}$ such that $u \in (W_I)^{\otimes n}$. [101] Denote this subset I by $I(u)$. Thus, $u \in \left(W_{I(u)}\right)^{\otimes n}$ for every $u \in V^{\otimes n}$.

[101] *Proof.* The family $(v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_n})_{(i_1,i_2,...,i_n)\in\mathbb{Z}^n}$ is a basis of $V^{\otimes n}$ (since $(v_i)_{i\in\mathbb{Z}}$ is a basis of V). Thus, we can write the tensor $u \in V^{\otimes n}$ as a $\mathbb{C}$-linear combination of finitely many tensors of the form $v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_n}$ with $(i_1, i_2, ..., i_n) \in \mathbb{Z}^n$. Let I be the union of the sets $\{i_1, i_2, ..., i_n\}$ over all the tensors which appear in this linear combination. Since only finitely many tensors appear in this linear combination, the set I is finite. Every tensor $v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_n}$ which appears in this linear combination satisfies $\{i_1, i_2, ..., i_n\} \subseteq I$ (by the construction of I) and thus $v_{i_1} \otimes v_{i_2} \otimes ... \otimes v_{i_n} \in (W_I)^{\otimes n}$. Thus, u must lie in $(W_I)^{\otimes n}$, too (because u is the value of this linear combination). Hence, we have found a finite subset I of $\mathbb{Z}$ such that $u \in (W_I)^{\otimes n}$. Qed.

For every $w \in S_\pi(V)$, there exists some finite subset I of $\mathbb{Z}$ such that $w \in S_\pi(W_I)$. [102] Denote this subset I by $I(w)$. Thus, $w \in S_\pi\left(W_{I(w)}\right)$ for every $w \in S_\pi(V)$.

Let w and w' be two vectors in $S_\pi(V)$ such that $w \neq 0$. We are going to prove that $w' \in U(\mathfrak{gl}_\infty) w$. Once this is proven, it will be obvious that $S_\pi(V)$ is irreducible, and we will be done.

There exists a finite subset I of $\mathbb{Z}$ such that $w \in S_\pi(W_I)$ and $w' \in S_\pi(W_I)$. [103] Consider this I.

Since I is finite, the vector space W_I is finite-dimensional. Thus, by the analogue of Proposition 3.5.6 for representations of $\mathfrak{gl}_m$, the representation $S_\pi(W_I)$ of the Lie algebra $\mathfrak{gl}(W_I)$ is irreducible. Hence, $w' \in U(\mathfrak{gl}(W_I)) w$.

Now, we have a canonical injective Lie algebra homomorphism $\mathfrak{gl}(W_I) \to \mathfrak{gl}_\infty$ [104]. Thus, we can view $\mathfrak{gl}(W_I)$ as a Lie subalgebra of $\mathfrak{gl}_\infty$ in a canonical way. Moreover, the classical action $\mathfrak{gl}(W_I) \times S_\pi(W_I) \to S_\pi(W_I)$ of the Lie algebra $\mathfrak{gl}(W_I)$ on the Schur module $S_\pi(W_I)$ can be viewed as the restriction of the action $\mathfrak{gl}_\infty \times S_\pi(V) \to S_\pi(V)$ to $\mathfrak{gl}(W_I) \times S_\pi(W_I)$. Hence, $U(\mathfrak{gl}(W_I)) w \subseteq U(\mathfrak{gl}_\infty) w$. Since we know that $w' \in U(\mathfrak{gl}(W_I)) w$, we thus conclude $w' \in U(\mathfrak{gl}_\infty) w$. This completes the proof of Proposition 3.5.6.

On the other hand, we can define so-called *highest-weight representations*. Before we do so, let us make $\mathfrak{gl}_\infty$ into a graded Lie algebra:

Definition 3.5.7. For every $i \in \mathbb{Z}$, let $\mathfrak{gl}_\infty^i$ be the subspace of $\mathfrak{gl}_\infty$ which consists of matrices which have nonzero entries only on the i-th diagonal. (The *i-th diagonal* consists of the entries in the (α, β)-th places with $\beta - \alpha = i$.)

Then, $\mathfrak{gl}_\infty = \bigoplus\limits_{i \in \mathbb{Z}} \mathfrak{gl}_\infty^i$, and this makes $\mathfrak{gl}_\infty$ into a $\mathbb{Z}$-graded Lie algebra. Note that $\mathfrak{gl}_\infty^0$ is abelian. Let $\mathfrak{gl}_\infty = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\mathfrak{gl}_\infty$, so

[102] *Proof.* Let $w \in S_\pi(V)$. Then, $w \in S_\pi(V) = \operatorname{Hom}_{S_n}(\pi, V^{\otimes n})$. But since π is a finite-dimensional vector space, the image $w(\pi)$ must be finite-dimensional. Hence, $w(\pi)$ is a finite-dimensional vector subspace of $V^{\otimes n}$. Thus, $w(\pi)$ is generated by some elements $u_1, u_2, ..., u_k \in V^{\otimes n}$. Let I be the union $\bigcup\limits_{j=1}^{k} I(u_j)$. Then, I is finite (because for every $j \in \{1, 2, ..., k\}$, the set $I(u_j)$ is finite) and satisfies $I(u_j) \subseteq I$ for every $j \in \{1, 2, ..., k\}$.

Recall that every $u \in V^{\otimes n}$ satisfies $u \in \left(W_{I(u)}\right)^{\otimes n}$. Thus, every $j \in \{1, 2, ..., k\}$ satisfies $u_j \in \left(W_{I(u_j)}\right)^{\otimes n} \subseteq (W_I)^{\otimes n}$ (since $I(u_j) \subseteq I$ and thus $W_{I(u_j)} \subseteq W_I$). In other words, all k elements $u_1, u_2, ..., u_k$ lie in the vector space $(W_I)^{\otimes n}$. Since the elements $u_1, u_2, ..., u_k$ generate the subspace $w(\pi)$, this yields that $w(\pi) \subseteq (W_I)^{\otimes n}$. Hence, the map $w : \pi \to V^{\otimes n}$ factors through a map $\pi \to (W_I)^{\otimes n}$. In other words, $w \in \operatorname{Hom}_{S_n}(\pi, V^{\otimes n})$ is contained in $\operatorname{Hom}_{S_n}\left(\pi, (W_I)^{\otimes n}\right) = S_\pi(W_I)$, qed.

[103] *Proof.* Let $I = I(w) \cup I(w')$. Then, I is a finite subset of $\mathbb{Z}$ (since $I(w)$ and $I(w')$ are finite subsets of $\mathbb{Z}$), and $I(w) \subseteq I$ and $I(w') \subseteq I$. We have $w \in S_\pi\left(W_{I(w)}\right) \subseteq S_\pi(W_I)$ (since $I(w) \subseteq I$ and thus $W_{I(w)} \subseteq W_I$) and similarly $w' \in S_\pi(W_I)$. Thus, there exists a finite subset I of $\mathbb{Z}$ such that $w \in S_\pi(W_I)$ and $w' \in S_\pi(W_I)$, qed.

[104] Here is how it is defined: For every linear map $A \in \mathfrak{gl}(W_I)$, we define a linear map $A' \in \mathfrak{gl}(V)$ by setting

$$A' v_i = \begin{cases} A v_i, & \text{if } i \in I; \\ 0, & \text{if } i \notin I \end{cases} \qquad \text{for all } i \in \mathbb{Z}.$$

This linear map A' is represented (with respect to the basis $(v_i)_{i \in \mathbb{Z}}$ of V) by an infinite matrix whose rows and columns are labeled by integers. This matrix lies in $\mathfrak{gl}_\infty$.

Thus, we have assigned to every $A \in \mathfrak{gl}(W_I)$ a matrix in $\mathfrak{gl}_\infty$. This defines an injective Lie algebra homomorphism $\mathfrak{gl}(W_I) \to \mathfrak{gl}_\infty$.

that the subspace $\mathfrak{n}_- = \bigoplus_{i<0} \mathfrak{gl}_\infty^i$ is the space of all strictly lower-triangular matrices in $\mathfrak{gl}_\infty$, the subspace $\mathfrak{h} = \mathfrak{gl}_\infty^0$ is the space of all diagonal matrices in $\mathfrak{gl}_\infty$, and the subspace $\mathfrak{n}_+ = \bigoplus_{i>0} \mathfrak{gl}_\infty^i$ is the space of all strictly upper-triangular matrices in $\mathfrak{gl}_\infty$.

Definition 3.5.8. For every $i, j \in \mathbb{Z}$, let $E_{i,j}$ be the matrix (with rows and columns labeled by integers) whose (i, j)-th entry is 1 and whose all other entries are 0. Then, $(E_{i,j})_{(i,j)\in\mathbb{Z}^2}$ is a basis of the vector space $\mathfrak{gl}_\infty$.

Definition 3.5.9. For every $\lambda \in \mathfrak{h}^*$, let M_λ be the highest-weight Verma module M_λ^+ (as defined in Definition 2.5.14). Let $J_\lambda = \operatorname{Ker}(\cdot, \cdot) \subseteq M_\lambda$ be the maximal proper graded submodule. Let L_λ be the quotient module $M_\lambda \diagup J_\lambda = M_\lambda^+ \diagup J_\lambda^+ = L_\lambda^+$; then, L_λ is irreducible (as we know).

Definition 3.5.10. We can define an antilinear $\mathbb{R}$-antiinvolution $\dagger : \mathfrak{gl}_\infty \to \mathfrak{gl}_\infty$ on $\mathfrak{gl}_\infty$ by setting

$$E_{i,j}^\dagger = E_{j,i} \qquad \text{for all } (i, j) \in \mathbb{Z}^2.$$

(Thus, $\dagger : \mathfrak{gl}_\infty \to \mathfrak{gl}_\infty$ is the operator which transposes a matrix and then applies complex conjugation to each of its entries.) Thus we can speak of Hermitian and unitary $\mathfrak{gl}_\infty$-modules.

A very important remark:

For the Lie algebra $\mathfrak{gl}_n$, the highest-weight modules are the Schur modules up to tensoring with a power of the determinant module. (More precisely: For $\mathfrak{gl}_n$, every finite-dimensional irreducible representation and any unitary irreducible representation is of the form $S_\pi(V_n) \otimes (\wedge^n(V_n^*))^{\otimes j}$ for some partition π and some $j \in \mathbb{N}$, where V_n is the $\mathfrak{gl}_n$-module $\mathbb{C}^n$.)

Nothing like this is true for $\mathfrak{gl}_\infty$. Instead, exterior powers of V and highest-weight representations live "in different worlds". This is because V is composed of infinite-dimensional vectors which have "no top or bottom"; V has no highest or lowest weight and does not lie in category $\mathcal{O}^+$ or $\mathcal{O}^-$.

This is important, because many beautiful properties of representations of $\mathfrak{gl}_n$ come from the equality of the highest-weight and Schur module representations.

A way to marry these two worlds is by considering so-called *semiinfinite wedges*.

3.5.1. Semiinfinite wedges

Let us first give an informal definition of semiinfinite wedges and the semiinfinite wedge space $\wedge^{\frac{\infty}{2}} V$ (we will later define these things formally):

An *elementary semiinfinite wedge* will mean a formal infinite "wedge product" $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots$ with $(i_0, i_1, i_2, \ldots)$ being a sequence of integers satisfying $i_0 > i_1 > i_2 > \ldots$ and $i_{k+1} = i_k - 1$ for all sufficiently large k. (At the moment, we consider this wedge product $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots$ just as a fancy symbol for the sequence $(i_0, i_1, i_2, \ldots)$.)

The *semiinfinite wedge space* $\wedge^{\frac{\infty}{2}} V$ is defined as the free vector space with basis given by elementary semiinfinite wedges.

Note that, despite the notation $\wedge^{\frac{\infty}{2}} V$, the semiinfinite wedge space is not a functor in the vector space V. We could replace our definition of $\wedge^{\frac{\infty}{2}} V$ by a somewhat more functorial one, which doesn't use the basis $(v_i)_{i\in\mathbb{Z}}$ of V anymore. But it would still need a topology on V (which makes V locally linearly compact), and some working with formal Laurent series. It proceeds through the semiinfinite Grassmannian, and will not be done in these lectures.[105] For us, the definition using the basis will be enough.

The space $\wedge^{\frac{\infty}{2}} V$ is countably dimensional. More precisely, we can write $\wedge^{\frac{\infty}{2}} V$ as

$$\wedge^{\frac{\infty}{2}} V = \bigoplus_{m\in\mathbb{Z}} \wedge^{\frac{\infty}{2},m} V, \qquad \text{where}$$
$$\wedge^{\frac{\infty}{2},m} V = \operatorname{span}\left\{ v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \mid i_k + k = m \text{ for sufficiently large } k \right\}.$$

The space $\wedge^{\frac{\infty}{2},m} V$ has basis $\left\{ v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \mid i_k + k = m \text{ for sufficiently large } k \right\}$, which is easily seen to be countable. We will see later that this basis can be naturally labeled by partitions (of all integers, not just of m).

3.5.2. The action of $\mathfrak{gl}_\infty$ on $\wedge^{\frac{\infty}{2}} V$

For every $m \in \mathbb{Z}$, we want to define an action of the Lie algebra $\mathfrak{gl}_\infty$ on the space $\wedge^{\frac{\infty}{2},m} V$ which is given "by the usual Leibniz rule", i. e., satisfies the equation

$$a \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{k\geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...$$

for all $a \in \mathfrak{gl}_\infty$ and all elementary semiinfinite wedges $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ (where, of course, $a \rightharpoonup v_{i_k}$ is the same as av_{i_k} due to our definition of the action of $\mathfrak{gl}_\infty$ on V). Of course, it is not immediately clear how to interpret the infinite wedge products $v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...$ on the right hand side of this equation, since they are (in general) not elementary semiinfinite wedges anymore. We must find a reasonable definition for such wedge products. What properties should a wedge product

[105] Some pointers to the more functorial definition:

Consider the field $\mathbb{C}((t))$ of formal Laurent series over $\mathbb{C}$ as a $\mathbb{C}$-vector space.

Let $\operatorname{Gr} = \left\{ U \text{ vector subspace of } \mathbb{C}((t)) \mid \begin{pmatrix} U \supseteq t^n\mathbb{C}[[t]] \text{ and} \\ \dim\left(U \diagup \left(t^n\mathbb{C}[[t]]\right)\right) < \infty \end{pmatrix} \text{ for some sufficiently high } n \right\}$.

For every $U \in \operatorname{Gr}$, define an integer $\operatorname{sdim} U$ by $\operatorname{sdim} U = \dim\left(U \diagup \left(t^n\mathbb{C}[[t]]\right)\right) - n$ for any $n \in \mathbb{Z}$ satisfying $U \supseteq t^n\mathbb{C}[[t]]$. Note that this integer does not depend on n as long as n is sufficiently high to satisfy $U \supseteq t^n\mathbb{C}[[t]]$.

This Grassmannian Gr is the disjoint union $\coprod \operatorname{Gr}_n$.

There is something called a determinant line bundle on Gr. The space of semiinfinite wedges is then defined as the space of regular sections of this line bundle (in the sense of algebraic geometry).

See the book by Pressley and Segal about loop groups for explanations of these matters.

(infinite as it is) satisfy? It should be multilinear[106] and antisymmetric[107]. These properties make it possible to compute any wedge product of the form $b_0 \wedge b_1 \wedge b_2 \wedge ...$ with $b_0, b_1, b_2, ...$ being vectors in V which satisfy

$$b_i = v_{m-i} \qquad \text{for sufficiently large } i.$$

In fact, whenever we are given such vectors $b_0, b_1, b_2, ...$, we can compute the wedge product $b_0 \wedge b_1 \wedge b_2 \wedge ...$ by the following procedure:

- Find an integer $M \in \mathbb{N}$ such that every $i \geq M$ satisfies $b_i = v_{m-i}$. (This M exists by the condition that $b_i = v_{m-i}$ for sufficiently large i.)
- Expand each of the vectors $b_0, b_1, ..., b_{M-1}$ as a $\mathbb{C}$-linear combination of the basis vectors v_ℓ.
- Using these expansions and the multilinearity of the wedge product, reduce the computation of $b_0 \wedge b_1 \wedge b_2 \wedge ...$ to the computation of finitely many wedge products of basis vectors.
- Each wedge product of basis vectors can now be computed as follows: If two of the basis vectors are equal, then it must be 0 (by antisymmetry of the wedge product). If not, reorder the basis vectors in such a way that their indices decrease (this is possible, because "most" of these basis vectors are already in order, and only the first few must be reordered). Due to the antisymmetry of the wedge product, the wedge product of the basis vectors before reordering must be $(-1)^\pi$ times the wedge product of the basis vectors after reordering, where π is the permutation which corresponds to our reordering. But the wedge product of the basis vectors after reordering is an elementary semiinfinite wedge, and thus we know how to compute it.

This procedure is not exactly a formal definition, and it is not immediately clear that the value of $b_0 \wedge b_1 \wedge b_2 \wedge ...$ that it computes is independent of, e. g., the choice of M. In the following subsection (Subsection 3.5.3), we will give a formal version of this definition.

3.5.3. The $\mathfrak{gl}_\infty$-module $\wedge^{\frac{\infty}{2}} V$: a formal definition

Before we formally define the value of $b_0 \wedge b_1 \wedge b_2 \wedge ...$, let us start from scratch and repeat the definitions of $\wedge^{\frac{\infty}{2}} V$ and $\wedge^{\frac{\infty}{2},m} V$ in a cleaner fashion than how we defined them above.

[106] i. e., it should satisfy

$$\begin{aligned} & b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (\lambda b + \lambda' b') \wedge b_{k+1} \wedge b_{k+2} \wedge ... \\ & = \lambda b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge b \wedge b_{k+1} \wedge b_{k+2} \wedge ... + \lambda' b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge b' \wedge b_{k+1} \wedge b_{k+2} \wedge ... \end{aligned}$$

for all $k \in \mathbb{N}$, $b_0, b_1, b_2, ... \in V$, $b, b' \in V$ and $\lambda, \lambda' \in \mathbb{C}$ for which the right hand side is well-defined

[107] i. e., a well-defined wedge product $b_0 \wedge b_1 \wedge b_2 \wedge ...$ should be 0 whenever two of the b_k are equal

Warning 3.5.11. Some of the nomenclature defined in the following (particularly, the notions of "m-degression" and "straying m-degression") is mine (=Darij's). I don't know whether there are established names for these things.

First, we introduce the notion of *m-degressions* and formalize the definitions of $\wedge^{\frac{\infty}{2}} V$ and $\wedge^{\frac{\infty}{2},m} V$.

Definition 3.5.12. Let $m \in \mathbb{Z}$. An *m-degression* will mean a strictly decreasing sequence $(i_0, i_1, i_2, ...)$ of integers such that every sufficiently high $k \in \mathbb{N}$ satisfies $i_k + k = m$. It is clear that any m-degression $(i_0, i_1, i_2, ...)$ automatically satisfies $i_k - i_{k+1} = 1$ for all sufficiently high k.

For any m-degression $(i_0, i_1, i_2, ...)$, we introduce a new symbol $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ....$ This symbol is, for the time being, devoid of any meaning. The symbol $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ will be called an *elementary semiinfinite wedge*.

Definition 3.5.13. **(a)** Let $\wedge^{\frac{\infty}{2}} V$ denote the free $\mathbb{C}$-vector space with basis $(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)_{m\in\mathbb{Z};\ (i_0,i_1,i_2,...) \text{ is an } m\text{-degression}}$. We will refer to $\wedge^{\frac{\infty}{2}} V$ as the *semi-infinite wedge space*.

(b) For every $m \in \mathbb{Z}$, define a $\mathbb{C}$-vector subspace $\wedge^{\frac{\infty}{2},m} V$ of $\wedge^{\frac{\infty}{2}} V$ by

$$\wedge^{\frac{\infty}{2},m} V = \operatorname{span}\left\{v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \ \mid \ (i_0, i_1, i_2, ...) \text{ is an } m\text{-degression}\right\}.$$

Clearly, $\wedge^{\frac{\infty}{2},m} V$ has basis $(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)_{(i_0,i_1,i_2,...) \text{ is an } m\text{-degression}}$.

Obviously, $\wedge^{\frac{\infty}{2}} V = \bigoplus_{m\in\mathbb{Z}} \wedge^{\frac{\infty}{2},m} V$.

Now, let us introduce the (more flexible) notion of *straying m-degressions*. This notion is obtained from the notion of m-degressions by dropping the "strictly decreasing" condition:

Definition 3.5.14. Let $m \in \mathbb{Z}$. A *straying m-degression* will mean a sequence $(i_0, i_1, i_2, ...)$ of integers such that every sufficiently high $k \in \mathbb{N}$ satisfies $i_k + k = m$.

As a consequence, a straying m-degression is strictly decreasing from some point onwards, but needs not be strictly decreasing from the beginning (it can "stray", whence the name). A strictly decreasing straying m-degression is exactly the same as an m-degression. Thus, every m-degression is a straying m-degression.

Definition 3.5.15. Let S be a (possibly infinite) set. Recall that a *permutation* of S means a bijection from S to S.

A *finitary permutation* of S means a bijection from S to S which fixes all but finitely many elements of S. (Thus, all permutations of S are finitary permutations if S is finite.)

Notice that the finitary permutations of a given set S form a group (under composition).

Definition 3.5.16. Let $m \in \mathbb{Z}$. Let $(i_0, i_1, i_2, ...)$ be a straying m-degression. If no two elements of this sequence $(i_0, i_1, i_2, ...)$ are equal, then there exists a unique finitary permutation π of $\mathbb{N}$ such that $\left(i_{\pi^{-1}(0)}, i_{\pi^{-1}(1)}, i_{\pi^{-1}(2)}, ...\right)$ is an m-degression. This finitary permutation π is called the *straightening permutation* of $(i_0, i_1, i_2, ...)$.

Definition 3.5.17. Let $m \in \mathbb{Z}$. Let $(i_0, i_1, i_2, ...)$ be a straying m-degression. We define the meaning of the term $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ as follows:

- If some two elements of the sequence $(i_0, i_1, i_2, ...)$ are equal, then $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ is defined to mean the element 0 of $\wedge^{\frac{\infty}{2},m} V$.

- If no two elements of the sequence $(i_0, i_1, i_2, ...)$ are equal, then $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ is defined to mean the element $(-1)^{\pi} v_{i_{\pi^{-1}(0)}} \wedge v_{i_{\pi^{-1}(1)}} \wedge v_{i_{\pi^{-1}(2)}} \wedge ...$ of $\wedge^{\frac{\infty}{2},m} V$, where π is the straightening permutation of $(i_0, i_1, i_2, ...)$.

Note that whenever $(i_0, i_1, i_2, ...)$ is an m-degression (not just a straying one), then the value of $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ defined according to Definition 3.5.17 is exactly the symbol $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ of Definition 3.5.12 (because no two elements of the sequence $(i_0, i_1, i_2, ...)$ are equal, and the straightening permutation of $(i_0, i_1, i_2, ...)$ is id). Hence, Definition 3.5.17 does not conflict with Definition 3.5.12.

Definition 3.5.18. Let $m \in \mathbb{Z}$. Let $b_0, b_1, b_2, ...$ be vectors in V which satisfy

$$b_i = v_{m-i} \qquad \text{for sufficiently large } i.$$

Then, let us define the wedge product $b_0 \wedge b_1 \wedge b_2 \wedge ... \in \wedge^{\frac{\infty}{2},m} V$ as follows:

Find an integer $M \in \mathbb{N}$ such that every $i \geq M$ satisfies $b_i = v_{m-i}$. (This M exists by the condition that $b_i = v_{m-i}$ for sufficiently large i.)

For every $i \in \{0, 1, ..., M-1\}$, write the vector b_i as a $\mathbb{C}$-linear combination $\sum\limits_{j \in \mathbb{Z}} \lambda_{i,j} v_j$ (with $\lambda_{i,j} \in \mathbb{C}$ for all j).

Now, define $b_0 \wedge b_1 \wedge b_2 \wedge ...$ to be the element

$$\sum_{(j_0, j_1, ..., j_{M-1}) \in \mathbb{Z}^M} \lambda_{0,j_0} \lambda_{1,j_1} ... \lambda_{M-1,j_{M-1}} v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{M-1}} \wedge v_{m-M} \wedge v_{m-M-1} \wedge v_{m-M-2} \wedge ...$$

of $\wedge^{\frac{\infty}{2},m} V$. Here, $v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{M-1}} \wedge v_{m-M} \wedge v_{m-M-1} \wedge v_{m-M-2} \wedge ...$ is well-defined, since $(j_0, j_1, ..., j_{M-1}, m-M, m-M-1, m-M-2, ...)$ is a straying m-degression.

Note that this element $b_0 \wedge b_1 \wedge b_2 \wedge ...$ is well-defined (according to Proposition 3.5.19 **(a)** below).

We refer to $b_0 \wedge b_1 \wedge b_2 \wedge ...$ as the *(infinite) wedge product* of the vectors b_0, b_1, b_2, ...

Note that, for any straying m-degression $(i_0, i_1, i_2, ...)$, the value of $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ defined according to Definition 3.5.18 equals the value of $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ defined according to Definition 3.5.17. Hence, Definition 3.5.18 does not conflict with Definition 3.5.17.

We have the following easily verified properties of the infinite wedge product:

Proposition 3.5.19. Let $m \in \mathbb{Z}$. Let $b_0, b_1, b_2, ...$ be vectors in V which satisfy

$$b_i = v_{m-i} \qquad \text{for sufficiently large } i.$$

(a) The wedge product $b_0 \wedge b_1 \wedge b_2 \wedge ...$ as defined in Definition 3.5.18 is well-defined (i. e., does not depend on the choice of M).

(b) For any straying m-degression $(i_0, i_1, i_2, ...)$, the value of $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ defined according to Definition 3.5.18 equals the value of $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ defined according to Definition 3.5.17.

(c) The infinite wedge product is multilinear. That is, we have

$$\begin{aligned} & b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (\lambda b + \lambda' b') \wedge b_{k+1} \wedge b_{k+2} \wedge ... \\ &= \lambda b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge b \wedge b_{k+1} \wedge b_{k+2} \wedge ... \\ &\qquad + \lambda' b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge b' \wedge b_{k+1} \wedge b_{k+2} \wedge ... \end{aligned} \tag{112}$$

for all $k \in \mathbb{N}$, $b_0, b_1, b_2, ... \in V$, $b, b' \in V$ and $\lambda, \lambda' \in \mathbb{C}$ which satisfy ($b_i = v_{m-i}$ for sufficiently large i).

(d) The infinite wedge product is antisymmetric. This means that if $b_0, b_1, b_2, ... \in V$ are such that ($b_i = v_{m-i}$ for sufficiently large i) and (two of the vectors $b_0, b_1, b_2, ...$ are equal), then

$$b_0 \wedge b_1 \wedge b_2 \wedge ... = 0. \tag{113}$$

In other words, when (at least) two of the vectors forming a well-defined infinite wedge product are equal, then this wedge product is 0.

(e) As a consequence, the wedge product $b_0 \wedge b_1 \wedge b_2 \wedge ...$ gets multiplied by -1 when we switch b_i with b_j for any two distinct $i \in \mathbb{N}$ and $j \in \mathbb{N}$.

(f) If π is a finitary permutation of $\mathbb{N}$ and $b_0, b_1, b_2, ... \in V$ are vectors such that ($b_i = v_{m-i}$ for sufficiently large i), then the infinite wedge product $b_{\pi(0)} \wedge b_{\pi(1)} \wedge b_{\pi(2)} \wedge ...$ is well-defined and satisfies

$$b_{\pi(0)} \wedge b_{\pi(1)} \wedge b_{\pi(2)} \wedge ... = (-1)^{\pi} \cdot b_0 \wedge b_1 \wedge b_2 \wedge .... \tag{114}$$

Now, we can define the action of $\mathfrak{gl}_\infty$ on $\wedge^{\frac{\infty}{2},m} V$ just as we wanted to:

Definition 3.5.20. Let $m \in \mathbb{Z}$. Define an action of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$ by the equation

$$a \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{k \geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...$$

for all $a \in \mathfrak{gl}_\infty$ and all m-degressions $(i_0, i_1, i_2, ...)$ (and by linear extension). (Recall that $a \rightharpoonup v = av$ for every $a \in \mathfrak{gl}_\infty$ and $v \in V$, due to how we defined the $\mathfrak{gl}_\infty$-module V.)

Of course, this definition is only justified after showing that this indeed is an action. But this is rather easy. Let us state this as a proposition:

Proposition 3.5.21. Let $m \in \mathbb{Z}$. Then, Definition 3.5.20 really defines a representation of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$. In other words, there exists one and only one action of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$ such that all $a \in \mathfrak{gl}_\infty$ and all m-degressions $(i_0, i_1, i_2, ...)$ satisfy

$$a \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{k \geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ....$$

The proof of this proposition (using the multilinearity and the antisymmetry of our wedge product) is rather straightforward and devoid of surprises. I will show it nevertheless, if only because I assume every other text leaves it to the reader. Due to its length, it is postponed until Subsection 3.5.4.

Proposition 3.5.21 shows that the action of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$ in Definition 3.5.20 is well-defined. This makes $\wedge^{\frac{\infty}{2},m} V$ into a $\mathfrak{gl}_\infty$-module. Computations in this module can be somewhat simplified by the following "comparably basis-free" formula[108]:

Proposition 3.5.22. Let $m \in \mathbb{Z}$. Let $b_0, b_1, b_2, ...$ be vectors in V which satisfy

$$b_i = v_{m-i} \qquad \text{for all sufficiently large } i.$$

Then, every $a \in \mathfrak{gl}_\infty$ satisfies

$$a \rightharpoonup (b_0 \wedge b_1 \wedge b_2 \wedge ...) = \sum_{k \geq 0} b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge ....$$

We can also explicitly describe this action on elementary matrices and semiinfinite wedges:

Proposition 3.5.23. Let $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$. Let $m \in \mathbb{Z}$. Let $(i_0, i_1, i_2, ...)$ be a straying m-degression (so that $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \in \wedge^{\frac{\infty}{2},m} V$).

(a) If $j \notin \{i_0, i_1, i_2, ...\}$, then $E_{i,j} \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = 0$.

(b) If there exists a **unique** $\ell \in \mathbb{N}$ such that $j = i_\ell$, then for this ℓ we have

$$E_{i,j} \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\ell-1}} \wedge v_i \wedge v_{i_{\ell+1}} \wedge v_{i_{\ell+2}} \wedge ...$$

(In words: If v_j appears exactly once as a factor in the wedge product $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$, then $E_{i,j} \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$ is the wedge product which is obtained from $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ by replacing this factor by v_i.)

[108] I'm saying "comparably" because the condition that $b_i = v_{m-i}$ for all sufficiently large i is not basis-free. But this should not come as a surprise, as the definition of $\wedge^{\frac{\infty}{2},m} V$ itself is not basis-free to begin with.

Since we have given $\wedge^{\frac{\infty}{2},m} V$ a $\mathfrak{gl}_\infty$-module structure for every $m \in \mathbb{Z}$, it is clear that $\wedge^{\frac{\infty}{2}} V = \bigoplus_{m\in\mathbb{Z}} \wedge^{\frac{\infty}{2},m} V$ also becomes a $\mathfrak{gl}_\infty$-module.

3.5.4. Proofs

Here are proofs of some of the unproven statements made in Subsection 3.5.3:

Proof of Proposition 3.5.21. The first thing we need to check is the following:

> *Assertion 3.5.21.0:* Let $a \in \mathfrak{gl}_\infty$. Let $b_0, b_1, b_2, ...$ be vectors in V which satisfy
>
> $$b_i = v_{m-i} \qquad \text{for all sufficiently large } i.$$
>
> **(a)** For every $k \in \mathbb{N}$, the infinite wedge product $b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge ...$ is well-defined.
>
> **(b)** All but finitely many $k \in \mathbb{N}$ satisfy $b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge ... = 0$. (In other words, the sum
>
> $$\sum_{k\geq 0} b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge ...$$
>
> converges in the discrete topology.)

The proof of Assertion 3.5.21.0 can easily be supplied by the reader. (Part **(a)** is clear, since the property of the sequence $(b_0, b_1, b_2, ...)$ to satisfy ($b_i = v_{m-i}$ for all sufficiently large i) does not change if we modify one entry of the sequence. Part **(b)** requires showing that $b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge ... = 0$ for all sufficiently large k; but this follows from $a \in \mathfrak{gl}_\infty$ being a matrix with only finitely many nonzero entries, and from the condition that $b_i = v_{m-i}$ for all sufficiently large i.)

Now that Assertion 3.5.21.0 is proven, we can make the following definition:

For every $a \in \mathfrak{gl}_\infty$, let us define a $\mathbb{C}$-linear map $F_a : \wedge^{\frac{\infty}{2},m} V \to \wedge^{\frac{\infty}{2},m} V$ as follows: For every m-degression $(i_0, i_1, i_2, ...)$, set

$$F_a (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{k\geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \quad (115)$$

[109]. Thus, we have specified the values of the map F_a on the basis $(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)_{(i_0,i_1,i_2,...) \text{ is an } m\text{-degression}}$ of $\wedge^{\frac{\infty}{2},m} V$. Therefore, the map F_a is uniquely determined (and exists) by linearity.

We are going to prove various properties of this map now. First, we will prove that the formula (115) which we used to define $F_a (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$ for m-degressions $(i_0, i_1, i_2, ...)$ can also be applied when $(i_0, i_1, i_2, ...)$ is just a straying m-degression:

[109]The right hand side of (115) is indeed well-defined. This follows from applying Assertion 3.5.21.0 **(b)** to v_{i_i} instead of b_i.

Assertion 3.5.21.1: Let $a \in \mathfrak{gl}_\infty$. Then, every **straying** m-degression $(j_0, j_1, j_2, ...)$ satisfies

$$F_a (v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...) = \sum_{k \geq 0} v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{k-1}} \wedge (a \rightharpoonup v_{j_k}) \wedge v_{j_{k+1}} \wedge v_{j_{k+2}} \wedge .... \tag{116}$$

Proof of Assertion 3.5.21.1 (sketched): Let $(j_0, j_1, j_2, ...)$ be a straying m-degression. Thus, every sufficiently large $i \in \mathbb{N}$ satisfies $j_i + i = m$. We must prove that (116) holds.

Now, we distinguish between two cases:

Case 1: Some two elements of the sequence $(j_0, j_1, j_2, ...)$ are equal.

Case 2: No two elements of the sequence $(j_0, j_1, j_2, ...)$ are equal.

Let us first consider Case 1. In this case, some two elements of the sequence $(j_0, j_1, j_2, ...)$ are equal. Hence, $v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... = 0$ (by the definition of $v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...$), and thus the left hand side of (116) vanishes. We now need to show that so does the right hand side.

We know that some two elements of the sequence $(j_0, j_1, j_2, ...)$ are equal. Let j_p and j_q be two such elements, with $p \neq q$. So we have $p \neq q$ and $j_p = j_q$.

The right hand side of (116) is a sum over all $k \geq 0$. Each of its addends with $k \notin \{p, q\}$ is 0 (because it is an infinite wedge product with two equal factors v_{j_p} and v_{j_q}). So we need to check that the addend with $k = p$ and the addend with $k = q$ cancel each other. In other words, we need to prove that

$$\begin{aligned} & v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{p-1}} \wedge \left(a \rightharpoonup v_{j_p}\right) \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge ... \\ & = -v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{q-1}} \wedge \left(a \rightharpoonup v_{j_q}\right) \wedge v_{j_{q+1}} \wedge v_{j_{q+1}} \wedge .... \end{aligned} \tag{117}$$

We recall that an infinite wedge product of the form $b_0 \wedge b_1 \wedge b_2 \wedge ...$ (where $b_0, b_1, b_2, ...$ are vectors in V such that ($b_i = v_{m-i}$ for all sufficiently large i)) gets multiplied by -1 when we switch b_i with b_j for any two distinct $i \in \mathbb{N}$ and $j \in \mathbb{N}$ [110]. Thus, the infinite wedge product

$$v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{p-1}} \wedge \left(a \rightharpoonup v_{j_p}\right) \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge ... \wedge v_{j_{q-1}} \wedge v_{j_q} \wedge v_{j_{q+1}} \wedge v_{j_{q+2}} \wedge ...$$

gets multiplied by -1 when we switch $a \rightharpoonup v_{j_p}$ with v_{j_q} (since $p \in \mathbb{N}$ and $q \in \mathbb{N}$ are distinct). In other words,

$$\begin{aligned} & v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{p-1}} \wedge v_{j_q} \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge ... \wedge v_{j_{q-1}} \wedge \left(a \rightharpoonup v_{j_p}\right) \wedge v_{j_{q+1}} \wedge v_{j_{q+1}} \wedge ... \\ & = -v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{p-1}} \wedge \left(a \rightharpoonup v_{j_p}\right) \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge ... \wedge v_{j_{q-1}} \wedge v_{j_q} \wedge v_{j_{q+1}} \wedge v_{j_{q+2}} \wedge .... \end{aligned}$$

Thus,

$$\begin{aligned} & v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{p-1}} \wedge \left(a \rightharpoonup v_{j_p}\right) \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge ... \wedge v_{j_{q-1}} \wedge v_{j_q} \wedge v_{j_{q+1}} \wedge v_{j_{q+2}} \wedge ... \\ & = -v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{p-1}} \wedge v_{j_q} \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge ... \wedge v_{j_{q-1}} \wedge \left(a \rightharpoonup v_{j_p}\right) \wedge v_{j_{q+1}} \wedge v_{j_{q+2}} \wedge .... \end{aligned} \tag{118}$$

[110] This is a particular case of Proposition 3.5.19 **(f)** (namely, the case when π is the transposition (i, j)).

Now,

$$\begin{aligned}
& v_{j_0} \wedge v_{j_1} \wedge \ldots \wedge v_{j_{p-1}} \wedge \left(a \rightharpoonup v_{j_p}\right) \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge \ldots \\
& = v_{j_0} \wedge v_{j_1} \wedge \ldots \wedge v_{j_{p-1}} \wedge \left(a \rightharpoonup v_{j_p}\right) \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge \ldots \wedge v_{j_{q-1}} \wedge v_{j_q} \wedge v_{j_{q+1}} \wedge v_{j_{q+2}} \wedge \ldots \\
& = -v_{j_0} \wedge v_{j_1} \wedge \ldots \wedge v_{j_{p-1}} \wedge v_{j_q} \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge \ldots \wedge v_{j_{q-1}} \wedge \left(a \rightharpoonup v_{j_p}\right) \wedge v_{j_{q+1}} \wedge v_{j_{q+2}} \wedge \ldots \\
& \qquad \text{(by (118))} \\
& = -v_{j_0} \wedge v_{j_1} \wedge \ldots \wedge v_{j_{p-1}} \wedge v_{j_p} \wedge v_{j_{p+1}} \wedge v_{j_{p+2}} \wedge \ldots \wedge v_{j_{q-1}} \wedge \left(a \rightharpoonup v_{j_q}\right) \wedge v_{j_{q+1}} \wedge v_{j_{q+2}} \wedge \ldots \\
& \qquad \text{(since } j_q = j_p \text{ and } j_p = j_q\text{)} \\
& = -v_{j_0} \wedge v_{j_1} \wedge \ldots \wedge v_{j_{q-1}} \wedge \left(a \rightharpoonup v_{j_q}\right) \wedge v_{j_{q+1}} \wedge v_{j_{q+2}} \wedge \ldots.
\end{aligned}$$

This proves (117). The proof of (116) in Case 1 is thus complete.

Now, let us consider Case 2. In this case, no two elements of the sequence $(j_0, j_1, j_2, \ldots)$ are equal. Thus, the straightening permutation of the straying m-degression $(j_0, j_1, j_2, \ldots)$ is well-defined. Let π be this straightening permutation. Then, $\left(j_{\pi^{-1}(0)}, j_{\pi^{-1}(1)}, j_{\pi^{-1}(2)}, \ldots\right)$ is an m-degression.

Let $\sigma = \pi^{-1}$. Then, σ is a finitary permutation of $\mathbb{N}$, thus a bijective map $\mathbb{N} \to \mathbb{N}$. From $\sigma = \pi^{-1}$, we obtain $\sigma\pi = \mathrm{id}$, thus $(-1)^{\sigma\pi} = 1$.

We know that $\left(j_{\pi^{-1}(0)}, j_{\pi^{-1}(1)}, j_{\pi^{-1}(2)}, \ldots\right)$ is an m-degression. Since $\pi^{-1} = \sigma$, this rewrites as follows: The sequence $\left(j_{\sigma(0)}, j_{\sigma(1)}, j_{\sigma(2)}, \ldots\right)$ is an m-degression.

By the definition of $v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge \ldots$ (in Definition 3.5.17), we have

$$v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge \ldots = (-1)^{\pi} v_{j_{\pi^{-1}(0)}} \wedge v_{j_{\pi^{-1}(1)}} \wedge v_{j_{\pi^{-1}(2)}} \wedge \ldots = (-1)^{\pi} v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge v_{j_{\sigma(2)}} \wedge \ldots$$

(since $\pi^{-1} = \sigma$). Thus,

$$\begin{aligned}
& F_a \left(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge \ldots\right) \\
& = F_a \left((-1)^{\pi} v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge v_{j_{\sigma(2)}} \wedge \ldots\right) = (-1)^{\pi} \cdot F_a \left(v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge v_{j_{\sigma(2)}} \wedge \ldots\right)
\end{aligned}$$

(since F_a is linear). Multiplying this equality with $(-1)^{\sigma}$, we obtain

$$\begin{aligned}
& (-1)^{\sigma} \cdot F_a \left(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge \ldots\right) \\
& = \underbrace{(-1)^{\sigma} \cdot (-1)^{\pi}}_{=(-1)^{\sigma\pi}=1} \cdot F_a \left(v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge v_{j_{\sigma(2)}} \wedge \ldots\right) = F_a \left(v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge v_{j_{\sigma(2)}} \wedge \ldots\right) \\
& = \sum_{k \geq 0} v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge \ldots \wedge v_{j_{\sigma(k-1)}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k+1)}} \wedge v_{j_{\sigma(k+2)}} \wedge \ldots \qquad (119) \\
& \qquad \left(\begin{array}{c} \text{by the definition of } F_a \left(v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge v_{j_{\sigma(2)}} \wedge \ldots\right), \\ \text{since } \left(j_{\sigma(0)}, j_{\sigma(1)}, j_{\sigma(2)}, \ldots\right) \text{ is an } m\text{-degression} \end{array}\right).
\end{aligned}$$

On the other hand, for every $k \in \mathbb{N}$, we have

$$\begin{aligned}
& v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge \ldots \wedge v_{j_{\sigma(k-1)}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k+1)}} \wedge v_{j_{\sigma(k+2)}} \wedge \ldots \\
& = (-1)^{\sigma} \cdot v_{j_0} \wedge v_{j_1} \wedge \ldots \wedge v_{j_{\sigma(k)-1}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k)+1}} \wedge v_{j_{\sigma(k)+2}} \wedge \ldots. \qquad (120)
\end{aligned}$$

[111] Hence, (119) becomes

$$\begin{aligned}
&(-1)^{\sigma} \cdot F_a \left(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...\right) \\
&= \sum_{k \geq 0} \underbrace{v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge ... \wedge v_{j_{\sigma(k-1)}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k+1)}} \wedge v_{j_{\sigma(k+2)}} \wedge ...}_{\substack{=(-1)^{\sigma} \cdot v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{\sigma(k)-1}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k)+1}} \wedge v_{j_{\sigma(k)+2}} \wedge ... \\ \text{(by (120))}}} \\
&= \sum_{k \geq 0} (-1)^{\sigma} \cdot v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{\sigma(k)-1}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k)+1}} \wedge v_{j_{\sigma(k)+2}} \wedge ....
\end{aligned}$$

Dividing this equality by $(-1)^{\sigma}$, we obtain

$$\begin{aligned}
&F_a \left(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...\right) \\
&= \sum_{k \geq 0} v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{\sigma(k)-1}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k)+1}} \wedge v_{j_{\sigma(k)+2}} \wedge ... \\
&= \sum_{k \geq 0} v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{k-1}} \wedge \left(a \rightharpoonup v_{j_k}\right) \wedge v_{j_{k+1}} \wedge v_{j_{k+2}} \wedge ... \\
&\qquad \left(\text{here, we substituted } k \text{ for } \sigma(k) \text{ in the sum (since } \sigma \text{ is bijective)}\right).
\end{aligned}$$

Thus, (116) is proven in Case 2.

We have now proven (116) in each of the two Cases 1 and 2, hence in all situations. In other words, Assertion 3.5.21.1 is proven.

Our next goal is the following assertion:

---

[111] *Proof of (120):* Let $k \in \mathbb{N}$. Define a sequence $(c_0, c_1, c_2, ...)$ of elements of V by

$$(c_0, c_1, c_2, ...) = \left(v_{j_0}, v_{j_1}, ..., v_{j_{\sigma(k)-1}}, a \rightharpoonup v_{j_{\sigma(k)}}, v_{j_{\sigma(k)+1}}, v_{j_{\sigma(k)+2}}, ...\right).$$

Then,

$$c_0 \wedge c_1 \wedge c_2 \wedge ... = v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{\sigma(k)-1}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k)+1}} \wedge v_{j_{\sigma(k)+2}} \wedge ....$$

But according to Proposition 3.5.19 **(f)** (applied to $(c_0, c_1, c_2, ...)$ instead of $(b_0, b_1, b_2, ...)$), the infinite wedge product $c_{\sigma(0)} \wedge c_{\sigma(1)} \wedge c_{\sigma(2)} \wedge ...$ is well-defined and satisfies

$$c_{\sigma(0)} \wedge c_{\sigma(1)} \wedge c_{\sigma(2)} \wedge ... = (-1)^{\sigma} \cdot c_0 \wedge c_1 \wedge c_2 \wedge ....$$

But it is easy to see that

$$\left(c_{\sigma(0)}, c_{\sigma(1)}, c_{\sigma(2)}, ...\right) = \left(v_{j_{\sigma(0)}}, v_{j_{\sigma(1)}}, ..., v_{j_{\sigma(k-1)}}, a \rightharpoonup v_{j_{\sigma(k)}}, v_{j_{\sigma(k+1)}}, v_{j_{\sigma(k+2)}}, ...\right),$$

so that

$$c_{\sigma(0)} \wedge c_{\sigma(1)} \wedge c_{\sigma(2)} \wedge ... = v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge ... \wedge v_{j_{\sigma(k-1)}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k+1)}} \wedge v_{j_{\sigma(k+2)}} \wedge ....$$

Hence,

$$\begin{aligned}
&v_{j_{\sigma(0)}} \wedge v_{j_{\sigma(1)}} \wedge ... \wedge v_{j_{\sigma(k-1)}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k+1)}} \wedge v_{j_{\sigma(k+2)}} \wedge ... \\
&= c_{\sigma(0)} \wedge c_{\sigma(1)} \wedge c_{\sigma(2)} \wedge ... = (-1)^{\sigma} \cdot \underbrace{c_0 \wedge c_1 \wedge c_2 \wedge ...}_{=v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{\sigma(k)-1}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k)+1}} \wedge v_{j_{\sigma(k)+2}} \wedge ...} \\
&= (-1)^{\sigma} \cdot v_{j_0} \wedge v_{j_1} \wedge ... \wedge v_{j_{\sigma(k)-1}} \wedge \left(a \rightharpoonup v_{j_{\sigma(k)}}\right) \wedge v_{j_{\sigma(k)+1}} \wedge v_{j_{\sigma(k)+2}} \wedge ....
\end{aligned}$$

This proves (120).

Assertion 3.5.21.2: Let $a \in \mathfrak{gl}_\infty$. Let $b_0, b_1, b_2, ...$ be vectors in V which satisfy

$$b_i = v_{m-i} \qquad \text{for all sufficiently large } i.$$

Then,

$$F_a (b_0 \wedge b_1 \wedge b_2 \wedge ...) = \sum_{k \geq 0} b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge ....$$

Proof of Assertion 3.5.21.2 (sketched): We have $b_i = v_{m-i}$ for all sufficiently large i. In other words, there exists a $K \in \mathbb{N}$ such that every $i \geq K$ satisfies $b_i = v_{m-i}$. Fix such a K.

We have to prove the equality

$$F_a (b_0 \wedge b_1 \wedge b_2 \wedge ...) = \sum_{k \geq 0} b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge .... \tag{121}$$

This equality is clearly linear in **each** of the variables b_0, b_1, ..., b_{K-1} (and also in each of the variables b_K, b_{K+1}, b_{K+2}, ..., but we don't care about them). Hence, in proving it, we can WLOG assume that each of the vectors b_0, b_1, ..., b_{K-1} belongs to the basis $(v_j)_{j \in \mathbb{Z}}$ of V. [112] Assume this. Of course, the remaining vectors b_K, b_{K+1}, b_{K+2}, ... also belong to the basis $(v_j)_{j \in \mathbb{Z}}$ of V (because every $i \geq K$ satisfies $b_i = v_{m-i}$). Hence, all the vectors b_0, b_1, b_2, ... belong to the basis $(v_j)_{j \in \mathbb{Z}}$ of V. Hence, there exists a sequence $(j_0, j_1, j_2, ...) \in \mathbb{Z}^{\mathbb{N}}$ such that every $i \in \mathbb{N}$ satisfies $b_i = v_{j_i}$. Therefore, the equality that we need to prove, (121), will immediately follow from Assertion 3.5.21.1 once we can show that $(j_0, j_1, j_2, ...)$ is a straying m-degression. But the latter is obvious (since every $i \geq K$ satisfies $v_{j_i} = b_i = v_{m-i}$ and thus $j_i = m - i$, so that $j_i + i = m$). Hence, (121) is proven. That is, Assertion 3.5.21.2 is proven.

Next, here's something obvious that we are going to use a few times in the proof:

Assertion 3.5.21.4: Let f and g be two endomorphisms of the $\mathbb{C}$-vector space $\wedge^{\frac{\infty}{2},m} V$. If every m-degression $(i_0, i_1, i_2, ...)$ satisfies $f (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = g (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$, then $f = g$.

This follows from the fact that $(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)_{(i_0, i_1, i_2, ...) \text{ is an } m\text{-degression}}$ is a basis of the $\mathbb{C}$-vector space $\wedge^{\frac{\infty}{2},m} V$.

Next, we notice the following easy fact:

Assertion 3.5.21.5: Let $a \in \mathfrak{gl}_\infty$ and $b \in \mathfrak{gl}_\infty$. Let $\lambda \in \mathbb{C}$ and $\mu \in \mathbb{C}$. Then, $\lambda F_a + \mu F_b = F_{\lambda a + \mu b}$ in the Lie algebra $\mathfrak{gl} \left(\wedge^{\frac{\infty}{2},m} V \right)$.

This follows very quickly from the linearity of the definition of F_a with respect to a (the details are left to the reader).

Here is something rather simple:

[112] Note that this assumption is allowed because b_0, b_1, ..., b_{K-1} are **finitely many** vectors. In contrast, if we wanted to WLOG assume that each of the (infinitely many) vectors b_0, b_1, b_2, ... belongs to the basis $(v_j)_{j \in \mathbb{Z}}$ of V, then we would have to need more justification for such an assumption.

Assertion 3.5.21.6: Let $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$. Let $m \in \mathbb{Z}$. Let $(i_0, i_1, i_2, ...)$ be a straying m-degression.

(a) For every $\ell \in \mathbb{N}$, the sequence $(i_0, i_1, ..., i_{\ell-1}, i, i_{\ell+1}, i_{\ell+2}, ...)$ is a straying m-degression.

(b) If $j \notin \{i_0, i_1, i_2, ...\}$, then $F_{E_{i,j}} (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = 0$.

(c) If there exists a **unique** $\ell \in \mathbb{N}$ such that $j = i_\ell$, then we have

$$F_{E_{i,j}} (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{L-1}} \wedge v_i \wedge v_{i_{L+1}} \wedge v_{i_{L+2}} \wedge ...,$$

where L is the unique $\ell \in \mathbb{N}$ such that $j = i_\ell$.

The proof of Assertion 3.5.21.6 is as straightforward as one would expect: it is a matter of substituting $a = E_{i,j}$ and $b_k = v_{i_k}$ into Assertion 3.5.21.2 and taking care of the few addends which are not 0.

Now here is something less obvious:

Assertion 3.5.21.7: Every $a \in \mathfrak{gl}_\infty$ and $b \in \mathfrak{gl}_\infty$ satisfy $[F_a, F_b] = F_{[a,b]}$ in the Lie algebra $\mathfrak{gl}\left(\wedge^{\frac{\infty}{2},m} V\right)$.

There are two possible approaches to proving Assertion 3.5.21.7.

First proof of Assertion 3.5.21.7 (sketched): In order to prove Assertion 3.5.21.7, it is enough to show that

$$\left[F_a, F_b\right]\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right) = F_{[a,b]}\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right) \tag{122}$$

for every m-degression $(i_0, i_1, i_2, ...)$. (Indeed, once this is done, $[F_a, F_b] = F_{[a,b]}$ will follow from Assertion 3.5.21.4.) So let $(i_0, i_1, i_2, ...)$ be any m-degression. Then,

$$\begin{aligned}
& F_a\left(F_b\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right)\right) \\
&= F_a\left(\sum_{k\geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(b \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...\right) \\
&\qquad \left(\text{since } F_b\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right) \text{ is defined as } \sum_{k\geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(b \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...\right) \\
&= \sum_{k\geq 0} F_a\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(b \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...\right) \\
&= \sum_{q\geq 0} \underbrace{F_a\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{q-1}} \wedge \left(b \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge ...\right)}_{\substack{= \sum\limits_{k\geq 0} \begin{cases} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(b \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \wedge v_{i_{q-1}} \wedge \left(a \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}}, & \text{if } k < q; \\ v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{q-1}} \wedge \left(\left(ab\right) \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge ..., & \text{if } k = q; \\ v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{q-1}} \wedge \left(a \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge ... \wedge v_{i_{k-1}} \wedge \left(b \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}}, & \text{if } k > q \end{cases} \\ \text{(by an application of Assertion 3.5.21.2)}}} \\
&\qquad \left(\text{here, we renamed the summation index } k \text{ as } q\right) \\
&= \sum_{q\geq 0} \sum_{k\geq 0} \begin{cases} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(b \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \wedge v_{i_{q-1}} \wedge \left(a \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge ..., & \text{if } k < q; \\ v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{q-1}} \wedge \left(\left(ab\right) \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge ..., & \text{if } k = q; \\ v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{q-1}} \wedge \left(a \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge ... \wedge v_{i_{k-1}} \wedge \left(b \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ..., & \text{if } k > q \end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{q\geq 0} \sum_{\substack{k\geq 0;\\ k<q}} v_{i_0} \wedge v_{i_1} \wedge \ldots \wedge v_{i_{k-1}} \wedge (b \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge \ldots \wedge v_{i_{q-1}} \wedge \left(a \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge \ldots \\
&\qquad + \sum_{q\geq 0} v_{i_0} \wedge v_{i_1} \wedge \ldots \wedge v_{i_{q-1}} \wedge \left((ab) \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge \ldots \\
&\qquad + \sum_{q\geq 0} \sum_{\substack{k\geq 0;\\ k>q}} v_{i_0} \wedge v_{i_1} \wedge \ldots \wedge v_{i_{q-1}} \wedge \left(a \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge \ldots \wedge v_{i_{k-1}} \wedge (b \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge \ldots \\
&= \sum_{q\geq 0} \sum_{\substack{p\geq 0;\\ p<q}} v_{i_0} \wedge v_{i_1} \wedge \ldots \wedge v_{i_{p-1}} \wedge \left(b \rightharpoonup v_{i_p}\right) \wedge v_{i_{p+1}} \wedge v_{i_{p+2}} \wedge \ldots \wedge v_{i_{q-1}} \wedge \left(a \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge \ldots \\
&\qquad + \sum_{k\geq 0} v_{i_0} \wedge v_{i_1} \wedge \ldots \wedge v_{i_{k-1}} \wedge ((ab) \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge \ldots \\
&\qquad + \sum_{p\geq 0} \sum_{\substack{q\geq 0;\\ q>p}} v_{i_0} \wedge v_{i_1} \wedge \ldots \wedge v_{i_{p-1}} \wedge \left(a \rightharpoonup v_{i_p}\right) \wedge v_{i_{p+1}} \wedge v_{i_{p+2}} \wedge \ldots \wedge v_{i_{q-1}} \wedge \left(b \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge \ldots
\end{aligned}
\tag{123}
$$

[113]. Similarly,

$$
\begin{aligned}
& F_b \left(F_a \left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots\right)\right) \\
&= \sum_{q\geq 0} \sum_{\substack{p\geq 0;\\ p<q}} v_{i_0} \wedge v_{i_1} \wedge \ldots \wedge v_{i_{p-1}} \wedge \left(a \rightharpoonup v_{i_p}\right) \wedge v_{i_{p+1}} \wedge v_{i_{p+2}} \wedge \ldots \wedge v_{i_{q-1}} \wedge \left(b \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge \ldots \\
&\qquad + \sum_{k\geq 0} v_{i_0} \wedge v_{i_1} \wedge \ldots \wedge v_{i_{k-1}} \wedge ((ba) \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge \ldots \\
&\qquad + \sum_{p\geq 0} \sum_{\substack{q\geq 0;\\ q>p}} v_{i_0} \wedge v_{i_1} \wedge \ldots \wedge v_{i_{p-1}} \wedge \left(b \rightharpoonup v_{i_p}\right) \wedge v_{i_{p+1}} \wedge v_{i_{p+2}} \wedge \ldots \wedge v_{i_{q-1}} \wedge \left(a \rightharpoonup v_{i_q}\right) \wedge v_{i_{q+1}} \wedge v_{i_{q+2}} \wedge \ldots .
\end{aligned}
\tag{124}
$$

[113]In the last step of this computation, we did the following substitutions:

- We renamed the index k as p in the second sum.
- We renamed the index q as k in the third sum.
- We switched the meanings of the indices p and q in the fourth and fifth sums.

Now, let us subtract (124) from (123). I am claiming that the first term on the right hand side of (124) cancels against the third term on the right hand side of (123). Indeed, in order to see this, one needs to check that one can interchange the order of summation in the sum

$$\sum_{q\geq 0}\sum_{\substack{p\geq 0;\\ p<q}} v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{p-1}}\wedge \left(b\rightharpoonup v_{i_p}\right)\wedge v_{i_{p+1}}\wedge v_{i_{p+2}}\wedge ...\wedge v_{i_{q-1}}\wedge \left(a\rightharpoonup v_{i_q}\right)\wedge v_{i_{q+1}}\wedge v_{i_{q+2}}\wedge ...,$$

i. e., replace $\sum\limits_{q\geq 0}\sum\limits_{\substack{p\geq 0;\\ p<q}}$ by $\sum\limits_{p\geq 0}\sum\limits_{\substack{q\geq 0;\\ q>p}}$. This is easy to see (indeed, one must show that $v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{p-1}}\wedge \left(b\rightharpoonup v_{i_p}\right)\wedge v_{i_{p+1}}\wedge v_{i_{p+2}}\wedge ...\wedge v_{i_{q-1}}\wedge \left(a\rightharpoonup v_{i_q}\right)\wedge v_{i_{q+1}}\wedge v_{i_{q+2}}\wedge ... = 0$ for all but finitely many **pairs** $(i,j)\in\mathbb{N}^2$), but not trivial a priori[114]. So we know that the first term on the right hand side of (124) cancels against the third term on the right hand side of (123). Similarly, the third term on the right hand side of (124) cancels against the first term on the right hand side of (123). Thus, when we subtract (124) from (123), on the right hand side only the second terms of both equations remain, and we obtain

$$\begin{aligned}
&F_a\left(F_b\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right) - F_b\left(F_a\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\\
&=\sum_{k\geq 0} v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge \left((ab)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\\
&\qquad\qquad -\sum_{k\geq 0} v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge \left((ba)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\\
&=\sum_{k\geq 0} v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge \left((ab-ba)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\\
&\qquad\qquad \text{(by the multilinearity of the infinite wedge product)}\\
&=F_{ab-ba}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right) = F_{[a,b]}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right).
\end{aligned}$$

This proves (122), and thus Assertion 3.5.21.7. Filling the details of this proof is left to the reader.

Second proof of Assertion 3.5.21.7 (sketched): Due to Assertion 3.5.21.5, the value of F_c for $c\in\mathfrak{gl}_\infty$ depends $\mathbb{C}$-linearly on c.

But we must prove the equality $[F_a,F_b]=F_{[a,b]}$ for all $a\in\mathfrak{gl}_\infty$ and $b\in\mathfrak{gl}_\infty$. This equality is $\mathbb{C}$-linear in a and b (since the value of F_c for $c\in\mathfrak{gl}_\infty$ depends $\mathbb{C}$-linearly

[114] Here is a cautionary tale on why one cannot always interchange summation in infinite sums. Define a family $(\alpha_{p,q})_{(p,q)\in\mathbb{N}^2}$ of integers by $\alpha_{p,q}=\begin{cases}1, & \text{if } p=q;\\ -1, & \text{if } p=q+1\end{cases}$. Then, every $q\in\mathbb{N}$ satisfies $\sum\limits_{p\geq 0}\alpha_{p,q}=0$. Hence, $\sum\limits_{q\geq 0}\sum\limits_{p\geq 0}\alpha_{p,q}=0$. On the other hand, every $p\in\mathbb{N}$ satisfies $\sum\limits_{q\geq 0}\alpha_{p,q}=\delta_{p,0}$. Hence, $\sum\limits_{p\geq 0}\sum\limits_{q\geq 0}\alpha_{p,q}=1\neq 0=\sum\limits_{q\geq 0}\sum\limits_{p\geq 0}\alpha_{p,q}$. So the two summation signs in this situation cannot be interchanged, even though all sums (both inner and outer) converge in the discrete topology. Generally, for a family $(\lambda_{p,q})_{(p,q)\in\mathbb{N}^2}$ of elements of an additive group, we are guaranteed to have $\sum\limits_{p\geq 0}\sum\limits_{q\geq 0}\lambda_{p,q}=\sum\limits_{q\geq 0}\sum\limits_{p\geq 0}\lambda_{p,q}$ if the **double sum** $\sum\limits_{(p,q)\in\mathbb{N}^2}\lambda_{p,q}$ still converges in the discrete topology (this is analogous to Fubini's theorem). But the double sum $\sum\limits_{(p,q)\in\mathbb{N}^2}\alpha_{p,q}$ does not converge in the discrete topology, so $\sum\limits_{p\geq 0}\sum\limits_{q\geq 0}\alpha_{p,q}\neq\sum\limits_{q\geq 0}\sum\limits_{p\geq 0}\alpha_{p,q}$ should not come as a surprise.

on c), so it is enough to show it only when a and b belong to the basis $(E_{i,j})_{(i,j)\in\mathbb{Z}^2}$ of $\mathfrak{gl}_\infty$. But in this case, one can check this equality by verifying that every m-degression $(i_0, i_1, i_2, ...)$ satisfies

$$[F_a, F_b] (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = F_{[a,b]} (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) .$$

This can be done (using Assertion 3.5.21.6) by a straightforward distinction of cases (the cases depend on whether some indices belong to $\{i_0, i_1, i_2, ...\}$ or not, and whether some indices are equal or not). The reader should not have much of a trouble supplying these arguments, but they are as unenlightening as one would expect. There **is** a somewhat better way to do this verification (better in the sense that less cases have to be considered) by means of exploiting some symmetry; this relies on checking the following assertion:

Assertion 3.5.21.8: Let r, s, u and v be integers. Let $m \in \mathbb{Z}$. Let $(i_0, i_1, i_2, ...)$ be an m-degression. Let I denote the set $\{i_0, i_1, i_2, ...\}$.

(a) If $v \notin I$, then

$$\left(F_{E_{r,s}} F_{E_{u,v}} - \delta_{s,u} F_{E_{r,v}}\right) (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = 0.$$

(b) If $s = v$, then

$$\left(F_{E_{r,s}} F_{E_{u,v}} - \delta_{s,u} F_{E_{r,v}}\right) (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = 0.$$

(c) Assume that $s \neq v$. Let $\mathbf{w} : \mathbb{Z} \to \mathbb{Z}$ be the function defined by

$$\left(\mathbf{w}(k) = \begin{cases} r, & \text{if } k = s; \\ u, & \text{if } k = v; \\ k, & \text{otherwise} \end{cases} \qquad \text{for all } k \in \mathbb{Z} \right).$$

[115] Then, $(\mathbf{w}(i_0), \mathbf{w}(i_1), \mathbf{w}(i_2), ...)$ is a straying m-degression, and satisfies

$$\begin{aligned} &\left(F_{E_{r,s}} F_{E_{u,v}} - \delta_{s,u} F_{E_{r,v}}\right) (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) \\ &= [s \in I] \cdot [v \in I] \cdot v_{\mathbf{w}(i_0)} \wedge v_{\mathbf{w}(i_1)} \wedge v_{\mathbf{w}(i_2)} \wedge .... \end{aligned}$$

Here, whenever $\mathcal{A}$ is an assertion, we denote by $[\mathcal{A}]$ the integer $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is wrong} \end{cases}$.

The proof of this assertion, as well as the derivation of Assertion 3.5.21.7 from it (Assertion 3.5.21.8 must be applied twice), is left to the reader.

We are now ready for the endgame:

[115] Here, the term $\begin{cases} r, & \text{if } k = s; \\ u, & \text{if } k = v; \\ k, & \text{otherwise} \end{cases}$ makes sense, since $s \neq v$.

Assertion 3.5.21.9: There exists **at least** one action of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$ such that all $a \in \mathfrak{gl}_\infty$ and all m-degressions $(i_0, i_1, i_2, ...)$ satisfy

$$a \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{k \geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge .... \tag{125}$$

Assertion 3.5.21.10: There exists **at most** one action of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$ such that all $a \in \mathfrak{gl}_\infty$ and all m-degressions $(i_0, i_1, i_2, ...)$ satisfy

$$a \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{k \geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge .... \tag{126}$$

Proof of Assertion 3.5.21.9: Let ρ be the map

$$\mathfrak{gl}_\infty \to \mathfrak{gl}\left(\wedge^{\frac{\infty}{2},m} V\right),$$
$$c \mapsto F_c.$$

This map ρ is $\mathbb{C}$-linear (by Assertion 3.5.21.5) and hence a Lie algebra homomorphism (by Assertion 3.5.21.7). Hence, ρ is an action of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$. Let us write this action in infix notation (i. e., let us write $c \rightharpoonup w$ for $(\rho(c))\, w$ whenever $c \in \mathfrak{gl}_\infty$ and $w \in \wedge^{\frac{\infty}{2},m} V$). Then, all $c \in \mathfrak{gl}_\infty$ and $w \in \wedge^{\frac{\infty}{2},m} V$ satisfy

$$c \rightharpoonup w = \underbrace{(\rho(c))}_{\substack{=F_c \\ \text{(by the definition of } \rho(c))}} w = F_c(w).$$

Hence, all $a \in \mathfrak{gl}_\infty$ and all m-degressions $(i_0, i_1, i_2, ...)$ satisfy

$$a \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = F_a(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$$
$$= \sum_{k \geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...$$

(by the definition of $F_a(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$). In other words, all $a \in \mathfrak{gl}_\infty$ and all m-degressions $(i_0, i_1, i_2, ...)$ satisfy (125).

We have thus constructed an action of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$ such that all $a \in \mathfrak{gl}_\infty$ and all m-degressions $(i_0, i_1, i_2, ...)$ satisfy (125). Therefore, there exists **at least** one such action. This proves Assertion 3.5.21.9.

Proof of Assertion 3.5.21.10: Given an action of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$ such that all $a \in \mathfrak{gl}_\infty$ and all m-degressions $(i_0, i_1, i_2, ...)$ satisfy (126), it is clear that the value of $a \rightharpoonup w$ is uniquely determined for every $a \in \mathfrak{gl}_\infty$ and $w \in \wedge^{\frac{\infty}{2},m} V$ (by the bilinearity of the action, because w can be written as a $\mathbb{C}$-linear combination of elementary semiinfinite wedges $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$). Hence, there exists **at most** one such action. This proves Assertion 3.5.21.10.

Combining Assertion 3.5.21.9 with Assertion 3.5.21.10, we see that there exists **one and only one** action of the Lie algebra $\mathfrak{gl}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$ such that all $a \in \mathfrak{gl}_\infty$ and all m-degressions $(i_0, i_1, i_2, ...)$ satisfy

$$a \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{k \geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...$$

In other words, Proposition 3.5.21 is proven.

Proof of Proposition 3.5.22 and Proposition 3.5.23. Both Proposition 3.5.22 and Proposition 3.5.23 boil down to facts that have been proven during our proof of Proposition 3.5.21 (indeed, Proposition 3.5.22 boils down to Assertion 3.5.21.2, and Proposition 3.5.23 to parts **(b)** and **(c)** of Assertion 3.5.21.6).

3.5.5. Properties of $\wedge^{\frac{\infty}{2},m} V$

There is an easy way to define a grading on $\wedge^{\frac{\infty}{2},m} V$. To do it, we notice that:

Proposition 3.5.24. For every m-degression $(i_0, i_1, i_2, ...)$, the sequence $(i_k + k - m)_{k \geq 0}$ is a partition (i. e., a nonincreasing sequence of nonnegative integers such that all but finitely many of its elements are 0). In particular, every integer $k \geq 0$ satisfies $i_k + k - m \geq 0$, and only finitely many integers $k \geq 0$ satisfy $i_k + k - m \neq 0$. Hence, the sum $\sum\limits_{k \geq 0} (i_k + k - m)$ is well-defined and equals a nonnegative integer.

The proof of this is very easy and left to the reader. As a consequence of this proposition, we have:

Definition 3.5.25. Let $m \in \mathbb{Z}$. We define a grading on the $\mathbb{C}$-vector space $\wedge^{\frac{\infty}{2},m} V$ by setting

$$\left(\wedge^{\frac{\infty}{2},m} V \right) [d] = \left\langle v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \mid (i_0, i_1, i_2, ...) \text{ is an } m\text{-degression satisfying } \sum_{k \geq 0} (i_k + k - m) = -d \right\rangle \quad \text{for every } d \in \mathbb{Z}.$$

In other words, we define a grading on the $\mathbb{C}$-vector space $\wedge^{\frac{\infty}{2},m} V$ by setting

$$\deg (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = - \sum_k (i_k + k - m)$$

for every m-degression $(i_0, i_1, i_2, ...)$.

This grading satisfies $\wedge^{\frac{\infty}{2},m} V = \bigoplus_{d \leq 0} \left(\wedge^{\frac{\infty}{2},m} V \right) [d]$ (since Proposition 3.5.24 yields that $\sum_{k \geq 0} (i_k + k - m)$ is nonnegative for every m-degression $(i_0, i_1, i_2, ...)$). In other words, $\wedge^{\frac{\infty}{2},m} V$ is nonpositively graded.

Note that, for every given $m \in \mathbb{Z}$, the m-degressions are in a 1-to-1 correspondence with the partitions. This correspondence maps any m-degression $(i_0, i_1, i_2, ...)$ to the sequence $(i_k + k - m)_{k \geq 0}$ (this sequence is a partition due to Proposition 3.5.24). The degree $\deg (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$ of the semiinfinite wedge $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ equals minus the sum of the parts of this partition.

It is easy to check that:

Proposition 3.5.26. Let $m \in \mathbb{Z}$. With the grading defined in Definition 3.5.25, the $\mathfrak{gl}_\infty$-module $\wedge^{\frac{\infty}{2},m} V$ is graded (where the grading on $\mathfrak{gl}_\infty$ is the one from Definition 3.5.7).

Let us say more about this module:

Proposition 3.5.27. Let $m \in \mathbb{Z}$. The graded $\mathfrak{gl}_\infty$-module $\wedge^{\frac{\infty}{2},m} V$ is the irreducible highest-weight representation L_{ω_m} of $\mathfrak{gl}_\infty$ with highest weight $\omega_m = (..., 1, 1, 0, 0, ...)$, where the last 1 is on place m and the first 0 is on place $m+1$. Moreover, L_{ω_m} is unitary.

Before we prove this, let us define the vectors that will turn out to be the highest-weight vectors:

Definition 3.5.28. For every $m \in \mathbb{Z}$, we denote by ψ_m the vector $v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ... \in \wedge^{\frac{\infty}{2},m} V$. (This is well-defined since the infinite sequence $(m, m-1, m-2, ...)$ is an m-degression.)

(Let us repeat that we are no longer using the notations of Definition 3.4.3, so that this ψ_m has nothing to do with the ψ_j from Definition 3.4.3.)

Note that $\psi_m \in \left(\wedge^{\frac{\infty}{2},m} V \right) [0]$ by the definition of the grading on $\wedge^{\frac{\infty}{2},m} V$.

Proof of Proposition 3.5.27. It is easy to see that $\mathfrak{n}_+ \cdot \psi_m = 0$. (In fact, if $E_{i,j} \in \mathfrak{n}_+$ then $i < j$ and thus indices are replaced by smaller indices when computing $E_{i,j} \rightharpoonup \psi_m$... For an alternative proof, just use the fact that $\psi_m \in \left(\wedge^{\frac{\infty}{2},m} V \right) [0]$ and that $\wedge^{\frac{\infty}{2},m} V$ is concentrated in nonpositive degrees.) Moreover, every $h \in \mathfrak{h}$ satisfies $h\psi_m = \omega_m (h) \psi_m$ (in fact, test at $h = E_{i,i}$). Also, ψ_m generates the $\mathfrak{gl}_\infty$-module $\wedge^{\frac{\infty}{2},m} V$. Thus, $\wedge^{\frac{\infty}{2},m} V$

is a highest-weight representation with highest weight ω_m (and highest-weight vector ψ_m).

Next let us prove that it is unitary. This will yield that it is irreducible.[116]

The unitarity is because the form in which the wedges are orthonormal is †-invariant. Thus, irreducible. (We used Lemma 2.9.33.) Proposition 3.5.27 is proven.

Corollary 3.5.29. For every finite sum $\sum\limits_{i\in\mathbb{Z}} k_i\omega_i$ with $k_i \in \mathbb{N}$, the representation $L_{\sum\limits_{i\in\mathbb{Z}} k_i\omega_i}$ is unitary.

Proof. Take the module $\bigotimes\limits_i L_{\omega_i}^{\otimes k_i}$, and let v be the tensor product of their respective highest-weight vectors. Let L be the submodule generated by v. Then, L is a highest-weight module, and is unitary since it is a submodule of a unitary module. Hence it is irreducible, and thus $L \cong L_{\sum\limits_i k_i\omega_i}$, qed.

3.6. $\overline{\mathfrak{a}_\infty}$

The Lie algebra $\mathfrak{gl}_\infty$ is fairly small (it doesn't even contain the identity matrix) - too small for several applications. Here is a larger Lie algebra with roughly similar properties:

Definition 3.6.1. We define $\overline{\mathfrak{a}_\infty}$ to be the vector space of infinite matrices with rows and columns labeled by integers (not only positive integers) such that only finitely many **diagonals** are nonzero. This is an associative algebra with 1 (due to Remark 3.6.4 **(a)** below), and thus, by the commutator, a Lie algebra.

We can think of the elements of $\overline{\mathfrak{a}_\infty}$ as difference operators:

Consider V as the space of sequences[117] with finitely many nonzero entries. One very important endomorphism of V is defined as follows:

Definition 3.6.2. Let $T : V \to V$ be the linear map given by

$$(Tx)_n = x_{n+1} \qquad \text{for all } x \in V \text{ and } n \in \mathbb{Z}.$$

This map T is called the *shift operator*. It satisfies $Tv_{i+1} = v_i$ for every $i \in \mathbb{Z}$.

We can also write T in the form $T = \sum\limits_{i\in\mathbb{Z}} E_{i,i+1}$, where the sum is infinite but makes sense entrywise (i. e., for every $(a, b) \in \mathbb{Z}^2$, there are only finitely many $i \in \mathbb{Z}$ for which the matrix $E_{i,i+1}$ has nonzero (a, b)-th entry).

Note that:

Proposition 3.6.3. The shift operator T is invertible. Every $j \in \mathbb{Z}$ satisfies $T^j = \sum\limits_{i\in\mathbb{Z}} E_{i,i+j}$.

[116] We could also show the irreducibility more directly, by showing that every sum of wedges can be used to get back ψ_m.

[117] In the following, "sequences" means "sequences labeled by integers".

A *difference operator* is an operator of the form $A = \sum\limits_{i=p}^{q} \gamma_i(n) T^i$, where p and q are some integers, and $\gamma_i : \mathbb{Z} \to \mathbb{C}$ are some functions.[118] Then, $\overline{\mathfrak{a}_\infty}$ is the algebra of all such operators. (These operators also act on the space of *all* sequences, not only on the space of sequences with finitely many nonzero entries.) In particular, $T \in \overline{\mathfrak{a}_\infty}$, and $T^i \in \overline{\mathfrak{a}_\infty}$ for every $i \in \mathbb{Z}$.

Note that $\overline{\mathfrak{a}_\infty}$ is no longer countably dimensional. The family $(E_{i,j})_{(i,j)\in\mathbb{Z}^2}$ is no longer a vector space basis, but it is a topological basis in an appropriately defined topology.

Let us make a remark on multiplication of infinite matrices:

Remark 3.6.4. **(a)** For every $A \in \overline{\mathfrak{a}_\infty}$ and $B \in \overline{\mathfrak{a}_\infty}$, the matrix AB is well-defined and lies in $\overline{\mathfrak{a}_\infty}$.

(b) For every $A \in \overline{\mathfrak{a}_\infty}$ and $B \in \mathfrak{gl}_\infty$, the matrix AB is well-defined and lies in $\mathfrak{gl}_\infty$.

Proof of Remark 3.6.4. **(a)** Let $A \in \overline{\mathfrak{a}_\infty}$ and $B \in \overline{\mathfrak{a}_\infty}$. Write the matrix A in the form $(a_{i,j})_{(i,j)\in\mathbb{Z}^2}$, and write the matrix B in the form $(b_{i,j})_{(i,j)\in\mathbb{Z}^2}$.

Since $A \in \overline{\mathfrak{a}_\infty}$, only finitely many diagonals of A are nonzero. Hence, there exists a finite subset $\mathfrak{A}$ of $\mathbb{Z}$ such that

$$\text{for every } u \in \mathbb{Z} \diagdown \mathfrak{A}, \text{ the } u\text{-th diagonal of } A \text{ is zero.} \tag{127}$$

Consider this $\mathfrak{A}$.

Since $B \in \overline{\mathfrak{a}_\infty}$, only finitely many diagonals of B are nonzero. Hence, there exists a finite subset $\mathfrak{B}$ of $\mathbb{Z}$ such that

$$\text{for every } v \in \mathbb{Z} \diagdown \mathfrak{B}, \text{ the } v\text{-th diagonal of } B \text{ is zero.} \tag{128}$$

Consider this $\mathfrak{B}$.

For every $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$, the infinite sum $\sum\limits_{k\in\mathbb{Z}} a_{i,k} b_{k,j}$ has a well-defined value, because all but finitely many addends of this sum are zero[119]. Hence, the matrix AB is well-defined (because the matrix AB is defined as the matrix whose (i,j)-th entry is $\sum\limits_{k\in\mathbb{Z}} a_{i,k} b_{k,j}$ for all $(i,j) \in \mathbb{Z}^2$), and satisfies

$$((i,j)\text{-th entry of the matrix } AB) = \sum_{k\in\mathbb{Z}} a_{i,k} b_{k,j}$$

[118] The sum $\sum\limits_{i=p}^{q} \gamma_i(n) T^i$ has to be understood as the linear map $X : V \to V$ given by

$$(Xx)_n = \sum_{i=p}^{q} \gamma_i(n) x_{n+i} \qquad \text{for all } x \in V \text{ and } n \in \mathbb{Z}.$$

[119] *Proof.* Every $k \in \mathbb{Z}$ such that $k - i \notin \mathfrak{A}$ satisfies $a_{i,k} = 0$ (because $k - i \notin \mathfrak{A}$, so that $k - i \in \mathbb{Z} \diagdown \mathfrak{A}$, and thus (127) (applied to $u = k - i$) yields that the $(k-i)$-th diagonal of A is zero, and thus $a_{i,k}$ (being an entry in this diagonal) must be $= 0$). Hence, every $k \in \mathbb{Z}$ such that $k - i \notin \mathfrak{A}$ satisfies $a_{i,k} b_{k,j} = 0 b_{k,j} = 0$. Since $\mathfrak{A}$ is a finite set, all but finitely many $k \in \mathbb{Z}$ satisfy $k - i \notin \mathfrak{A}$, and thus all but finitely many $k \in \mathbb{Z}$ satisfy $a_{i,k} b_{k,j} = 0$ (because every $k \in \mathbb{Z}$ such that $k - i \notin \mathfrak{A}$ satisfies $a_{i,k} b_{k,j} = 0$). In other words, all but finitely many addends of the sum $\sum\limits_{k\in\mathbb{Z}} a_{i,k} b_{k,j}$ are zero, qed.

for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.

Now we must show that $AB \in \overline{\mathfrak{a}_\infty}$.

Let $\mathfrak{A} + \mathfrak{B}$ denote the set $\{a + b \mid (a, b) \in \mathfrak{A} \times \mathfrak{B}\}$. Clearly, $\mathfrak{A} + \mathfrak{B}$ is a finite set (since $\mathfrak{A}$ and $\mathfrak{B}$ are finite). Now, for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ satisfying $j - i \notin \mathfrak{A} + \mathfrak{B}$, every $k \in \mathbb{Z}$ satisfies $a_{i,k}b_{k,j} = 0$ [120]. Thus, for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ satisfying $j - i \notin \mathfrak{A} + \mathfrak{B}$, we have

$$((i, j)\text{-th entry of the matrix } AB) = \sum_{k \in \mathbb{Z}} \underbrace{a_{i,k}b_{k,j}}_{\substack{=0 \\ (\text{since } j-i \notin \mathfrak{A}+\mathfrak{B})}} = \sum_{k \in \mathbb{Z}} 0 = 0.$$

Thus, for every integer $w \notin \mathfrak{A} + \mathfrak{B}$, and any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ satisfying $j - i = w$, we have $((i, j)\text{-th entry of the matrix } AB) = 0$ (since $j - i = w \notin \mathfrak{A} + \mathfrak{B}$). In other words, for every integer $w \notin \mathfrak{A} + \mathfrak{B}$, the w-th diagonal of AB is zero. Since $\mathfrak{A} + \mathfrak{B}$ is a finite set, this yields that all but finitely many diagonals of AB are zero. In other words, only finitely many diagonals of AB are nonzero. In other words, $AB \in \overline{\mathfrak{a}_\infty}$. This proves Remark 3.6.4 **(a)**.

(b) We know from Remark 3.6.4 **(a)** that the matrix AB is well-defined (since $B \in \mathfrak{gl}_\infty \subseteq \overline{\mathfrak{a}_\infty}$).

The matrix B lies in $\mathfrak{gl}_\infty$ and thus has only finitely many nonzero entries. Hence, B has only finitely many nonzero rows. In other words, there exists a finite subset $\mathfrak{R}$ of $\mathbb{Z}$ such that

$$\text{for every } x \in \mathbb{Z} \diagdown \mathfrak{R}, \text{ the } x\text{-th row of } B \text{ is zero.} \tag{129}$$

Also, B has only finitely many nonzero entries, and thus only finitely many nonzero columns. In other words, there exists a finite subset $\mathfrak{C}$ of $\mathbb{Z}$ such that

$$\text{for every } y \in \mathbb{Z} \diagdown \mathfrak{C}, \text{ the } y\text{-th column of } B \text{ is zero.} \tag{130}$$

Define $\mathfrak{A}$ as in the proof of Remark 3.6.4 **(a)**. Let $\mathfrak{R} - \mathfrak{A}$ denote the set $\{r - a \mid (r, a) \in \mathfrak{R} \times \mathfrak{A}\}$. Clearly, $\mathfrak{R} - \mathfrak{A}$ is a finite set (since $\mathfrak{A}$ and $\mathfrak{R}$ are finite), and thus $(\mathfrak{R} - \mathfrak{A}) \times \mathfrak{C}$ is a finite set (since $\mathfrak{C}$, too, is finite). Now, for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ satisfying $(i, j) \notin (\mathfrak{R} - \mathfrak{A}) \times \mathfrak{C}$, we have $\sum_{k \in \mathbb{Z}} a_{i,k}b_{k,j} = 0$ [121]. Hence, for any $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ satisfying $(i, j) \notin (\mathfrak{R} - \mathfrak{A}) \times \mathfrak{C}$,

[120] *Proof.* Let $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ satisfy $j - i \notin \mathfrak{A} + \mathfrak{B}$, and let $k \in \mathbb{Z}$. Assume that $a_{i,k}b_{k,j} \neq 0$. Then, $a_{i,k} \neq 0$ and $b_{k,j} \neq 0$.

Since $a_{i,k}$ is an entry of the $(k - i)$-th diagonal of A, we see that some entry of the $(k - i)$-th diagonal of A is nonzero (since $a_{i,k} \neq 0$). Hence, the $(k - i)$-th diagonal of A is nonzero. Thus, $k - i \notin \mathbb{Z} \diagdown \mathfrak{A}$ (because otherwise, we would have $k - i \in \mathbb{Z} \diagdown \mathfrak{A}$, so that (127) (applied to $u = k - i$) would yield that the $(k - i)$-th diagonal of A is zero, contradicting the fact that it is nonzero), so that $k - i \in \mathfrak{A}$.

Since $b_{k,j}$ is an entry of the $(j - k)$-th diagonal of B, we see that some entry of the $(j - k)$-th diagonal of B is nonzero (since $b_{k,j} \neq 0$). Hence, the $(j - k)$-th diagonal of B is nonzero. Thus, $j - k \notin \mathbb{Z} \diagdown \mathfrak{B}$ (because otherwise, we would have $j - k \in \mathbb{Z} \diagdown \mathfrak{B}$, so that (128) (applied to $v = j - k$) would yield that the $(j - k)$-th diagonal of B is zero, contradicting the fact that it is nonzero), so that $j - k \in \mathfrak{B}$.

Now, $j - i = \underbrace{(k - i)}_{\in \mathfrak{A}} + \underbrace{(j - k)}_{\in \mathfrak{B}} \in \mathfrak{A} + \mathfrak{B}$. This contradicts $j - i \notin \mathfrak{A} + \mathfrak{B}$. Thus, our assumption that $a_{i,k}b_{k,j} \neq 0$ must have been wrong. Hence, $a_{i,k}b_{k,j} = 0$, qed.

[121] *Proof.* Let $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ be such that $(i, j) \notin (\mathfrak{R} - \mathfrak{A}) \times \mathfrak{C}$. Assume that $\sum_{k \in \mathbb{Z}} a_{i,k}b_{k,j} \neq 0$. Then,

we have

$$((i,j)\text{-th entry of the matrix } AB) = \sum_{k\in\mathbb{Z}} a_{i,k}b_{k,j} = 0.$$

Since $(\mathfrak{R} - \mathfrak{A}) \times \mathfrak{C}$ is a finite set, this yields that all but finitely many entries of the matrix AB are zero. In other words, AB has only finitely many nonzero entries. Thus, $AB \in \mathfrak{gl}_\infty$. Remark 3.6.4 **(b)** is proven.

Let us make $\overline{\mathfrak{a}_\infty}$ into a graded Lie algebra:

Definition 3.6.5. For every $i \in \mathbb{Z}$, let $\overline{\mathfrak{a}_\infty^i}$ be the subspace of $\overline{\mathfrak{a}_\infty}$ which consists of matrices which have nonzero entries only on the i-th diagonal. (The *i-th diagonal* consists of the entries in the (α, β)-th places with $\beta - \alpha = i$.)

Then, $\overline{\mathfrak{a}_\infty} = \bigoplus_{i\in\mathbb{Z}} \overline{\mathfrak{a}_\infty^i}$, and this makes $\overline{\mathfrak{a}_\infty}$ into a $\mathbb{Z}$-graded Lie algebra. Note that $\overline{\mathfrak{a}_\infty^0}$ is abelian. Let $\overline{\mathfrak{a}_\infty} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the triangular decomposition of $\overline{\mathfrak{a}_\infty}$, so that the subspace $\mathfrak{n}_- = \bigoplus_{i<0} \overline{\mathfrak{a}_\infty^i}$ is the space of all strictly lower-triangular matrices in $\overline{\mathfrak{a}_\infty}$, the subspace $\mathfrak{h} = \overline{\mathfrak{a}_\infty^0}$ is the space of all diagonal matrices in $\overline{\mathfrak{a}_\infty}$, and the subspace $\mathfrak{n}_+ = \bigoplus_{i>0} \overline{\mathfrak{a}_\infty^i}$ is the space of all strictly upper-triangular matrices in $\overline{\mathfrak{a}_\infty}$.

Note that this was completely analogous to Definition 3.5.7.

3.7. $\mathfrak{a}_\infty$ and its action on $\wedge^{\frac{\infty}{2},m} V$

Definition 3.7.1. Let $m \in \mathbb{Z}$. Let $\rho : \mathfrak{gl}_\infty \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ be the representation of $\mathfrak{gl}_\infty$ on $\wedge^{\frac{\infty}{2},m} V$ defined in Definition 3.5.20.

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there exists some $k \in \mathbb{Z}$ such that $a_{i,k}b_{k,j} \neq 0$. Consider this k.

Since $a_{i,k}b_{k,j} \neq 0$, we have $a_{i,k} \neq 0$ and $b_{k,j} \neq 0$.

Since $a_{i,k}$ is an entry of the $(k-i)$-th diagonal of A, we see that some entry of the $(k-i)$-th diagonal of A is nonzero (since $a_{i,k} \neq 0$). Hence, the $(k-i)$-th diagonal of A is nonzero. Thus, $k - i \notin \mathbb{Z} \diagdown \mathfrak{A}$ (because otherwise, we would have $k - i \in \mathbb{Z} \diagdown \mathfrak{A}$, so that (127) (applied to $u = k - i$) would yield that the $(k-i)$-th diagonal of A is zero, contradicting the fact that it is nonzero), so that $k - i \in \mathfrak{A}$.

Since $b_{k,j}$ is an entry of the k-th row of B, we see that some entry of the k-th row of B is nonzero (since $b_{k,j} \neq 0$). Hence, the k-th row of B is nonzero. Thus, $k \notin \mathbb{Z} \diagdown \mathfrak{R}$ (because otherwise, we would have $k \in \mathbb{Z} \diagdown \mathfrak{R}$, so that (129) (applied to $x = k$) would yield that the k-th row of B is zero, contradicting the fact that it is nonzero), so that $k \in \mathfrak{R}$.

Thus, $i = \underbrace{k}_{\in\mathfrak{R}} - \underbrace{(k-i)}_{\in\mathfrak{A}} \in \mathfrak{R} - \mathfrak{A}$.

Since $b_{k,j}$ is an entry of the j-th column of B, we see that some entry of the j-th column of B is nonzero (since $b_{k,j} \neq 0$). Hence, the j-th column of B is nonzero. Thus, $j \notin \mathbb{Z} \diagdown \mathfrak{C}$ (because otherwise, we would have $j \in \mathbb{Z} \diagdown \mathfrak{C}$, so that (130) (applied to $y = j$) would yield that the j-th column of B is zero, contradicting the fact that it is nonzero), so that $j \in \mathfrak{C}$. Combined with $i \in \mathfrak{R} - \mathfrak{A}$, this yields $(i,j) \in (\mathfrak{R} - \mathfrak{A}) \times \mathfrak{C}$, contradicting $(i,j) \notin (\mathfrak{R} - \mathfrak{A}) \times \mathfrak{C}$. Hence, the assumption that $\sum_{k\in\mathbb{Z}} a_{i,k}b_{k,j} \neq 0$ must have been wrong. In other words, $\sum_{k\in\mathbb{Z}} a_{i,k}b_{k,j} = 0$, qed.

The following question poses itself naturally now: Can we extend this representation ρ to a representation of $\overline{\mathfrak{a}_\infty}$ in a reasonable way?

This question depends on what we mean by "reasonable". One way to concretize this is by noticing that $\overline{\mathfrak{a}_\infty} = \bigoplus\limits_{i\in\mathbb{Z}} \overline{\mathfrak{a}_\infty^i}$, where $\overline{\mathfrak{a}_\infty^i}$ is the space of all matrices with nonzero entries only on the i-th diagonal. For each $i \in \mathbb{Z}$, the vector space $\overline{\mathfrak{a}_\infty^i}$ can be given the product topology (i. e., the topology in which a net $(s_z)_{z\in Z}$ of matrices converges to a matrix s if and only if for any $(m,n) \in \mathbb{Z}^2$ satisfying $n-m=i$, the net of the (m,n)-th entries of the matrices s_z converge to the (m,n)-th entry of s in the discrete topology). Then, $\mathfrak{gl}_\infty^i$ in dense in $\overline{\mathfrak{a}_\infty^i}$ for every $i \in \mathbb{Z}$. We can also make $\wedge^{\frac{\infty}{2},m} V$ into a topological space by using the discrete topology. Our question can now be stated as follows: Can we extend ρ by continuity to a representation of $\overline{\mathfrak{a}_\infty}$ (where "continuous" means "continuous on each $\overline{\mathfrak{a}_\infty^i}$", since we have not defined a topology on the whole space $\overline{\mathfrak{a}_\infty}$) ?

Answer: Almost, but not precisely. We cannot make $\overline{\mathfrak{a}_\infty}$ act on $\wedge^{\frac{\infty}{2},m} V$ in such a way that its action extends ρ continuously, but we can make a central extension of $\overline{\mathfrak{a}_\infty}$ act on $\wedge^{\frac{\infty}{2},m} V$ in a way that only slightly differs from ρ.

Let us first see what goes wrong if we try to find an extension of ρ to $\overline{\mathfrak{a}_\infty}$ by continuity:

For $i \neq 0$, a typical element $X \in \overline{\mathfrak{a}_\infty^i}$ is of the form $X = \sum\limits_{j\in\mathbb{Z}} z_j E_{j,j+i}$ with $z_j \in \mathbb{C}$. Now we can define $\rho(X) v = \sum\limits_{j\in\mathbb{Z}} z_j \rho(E_{j,j+i}) v$ for every $v \in \wedge^{\frac{\infty}{2},m} V$; this sum has only finitely many nonzero addends[122] and thus makes sense.

But when $i=0$, we run into a problem with this approach: $\rho\left(\sum\limits_{j\in\mathbb{Z}} z_j E_{j,j}\right) v = \sum\limits_{j\in\mathbb{Z}} z_j \rho(E_{j,j}) v$ is an infinite sum which may very well have infinitely many nonzero

[122]*Proof.* We must prove that, for every $v \in \wedge^{\frac{\infty}{2},m} V$, the sum $\sum\limits_{j\in\mathbb{Z}} z_j \rho(E_{j,j+i}) v$ has only finitely many nonzero addends. It is clearly enough to prove this in the case when v is an elementary semiinfinite wedge. So let us WLOG assume that v is an elementary semiinfinite wedge. In other words, WLOG assume that $v = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ for some m-degression $(i_0, i_1, i_2, ...)$. Consider this m-degression. By the definition of an m-degression, every sufficiently high $k \in \mathbb{N}$ satisfies $i_k + k = m$. In other words, there exists a $K \in \mathbb{N}$ such that every integer $k \geq K$ satisfies $i_k + k = m$. Consider this K. Then, every integer $j \leq i_K$ appears in the m-degression $(i_0, i_1, i_2, ...)$.

Now, we have the following two observations:

- Every integer $j > i_0 - i$ satisfies $\rho(E_{j,j+i}) v = 0$ (because for every integer $j > i_0 - i$, we have $j + i > i_0$, so that the integer $j+i$ does not appear in the m-degression $(i_0, i_1, i_2, ...)$).
- Every integer $j \leq i_K$ satisfies $\rho(E_{j,j+i}) v = 0$ (because every integer $j \leq i_K$ appears in the m-degression $(i_0, i_1, i_2, ...)$, and because $i \neq 0$).

Combining these two observations, we conclude that every sufficiently large integer j satisfies $\rho(E_{j,j+i}) v = 0$ and that every sufficiently small integer j satisfies $\rho(E_{j,j+i}) v = 0$. Hence, only finitely many integers j satisfy $\rho(E_{j,j+i}) v \neq 0$. Thus, the sum $\sum\limits_{j\in\mathbb{Z}} z_j \rho(E_{j,j+i}) v$ has only finitely many nonzero addends, qed.

addends, and thus makes no sense.

To fix this problem, we define a map $\widehat{\rho}$ which will be a "small" modification of ρ:

Definition 3.7.2. Define a linear map $\widehat{\rho} : \overline{\mathfrak{a}_\infty} \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ by

$$\widehat{\rho}\left((a_{i,j})_{(i,j)\in\mathbb{Z}^2}\right) = \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j} \begin{cases} \rho(E_{i,j}), & \text{unless } i=j \text{ and } i\leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i=j \text{ and } i\leq 0 \end{cases} \tag{131}$$

$$\text{for every } (a_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \overline{\mathfrak{a}_\infty}$$

(where 1 means the endomorphism id of $\wedge^{\frac{\infty}{2},m} V$). Here, the infinite sum $\sum\limits_{(i,j)\in\mathbb{Z}^2} a_{i,j} \begin{cases} \rho(E_{i,j}), & \text{unless } i=j \text{ and } i\leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i=j \text{ and } i\leq 0 \end{cases}$ is well-defined as an endomorphism of $\wedge^{\frac{\infty}{2},m} V$, because for every $v \in \wedge^{\frac{\infty}{2},m} V$, the sum $\sum\limits_{(i,j)\in\mathbb{Z}^2} a_{i,j} \begin{cases} \rho(E_{i,j}), & \text{unless } i=j \text{ and } i\leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i=j \text{ and } i\leq 0 \end{cases} v$ has only finitely many nonzero addends (as Proposition 3.7.4 shows).

The map $\widehat{\rho}$ just defined does not extend the map ρ, but is the unique continuous (in the sense explained above) extension of the map $\widehat{\rho}\mid_{\mathfrak{gl}_\infty}$ to $\overline{\mathfrak{a}_\infty}$ as a linear map. The map $\widehat{\rho}\mid_{\mathfrak{gl}_\infty}$ is, in a certain sense, a "very close approximation to ρ", as can be seen from the following remark:

Remark 3.7.3. From Definition 3.7.2, it follows that

$$\widehat{\rho}(E_{i,j}) = \begin{cases} \rho(E_{i,j}), & \text{unless } i=j \text{ and } i\leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i=j \text{ and } i\leq 0 \end{cases} \qquad \text{for every } (i,j)\in\mathbb{Z}^2. \tag{132}$$

We are not done yet: This map $\widehat{\rho}$ is not a representation of $\overline{\mathfrak{a}_\infty}$. We will circumvent this by defining a central extension $\mathfrak{a}_\infty$ of $\overline{\mathfrak{a}_\infty}$ for which the map $\widehat{\rho}$ (once suitably extended) will be a representation. But first, let us show a lemma that we owe for the definition of $\widehat{\rho}$:

Proposition 3.7.4. Let $(a_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \overline{\mathfrak{a}_\infty}$ and $v \in \wedge^{\frac{\infty}{2},m} V$. Then, the sum

$$\sum_{(i,j)\in\mathbb{Z}^2} a_{i,j} \begin{cases} \rho(E_{i,j}), & \text{unless } i=j \text{ and } i\leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i=j \text{ and } i\leq 0 \end{cases} v$$

has only finitely many nonzero addends.

Proof of Proposition 3.7.4. We know that v is an element of $\wedge^{\frac{\infty}{2},m} V$. Hence, v is a $\mathbb{C}$-linear combination of elements of the form $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ with $(i_0, i_1, i_2, ...)$ being an

m-degression (since $(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)_{(i_0,i_1,i_2,...) \text{ is an } m\text{-degression}}$ is a basis of $\wedge^{\frac{\infty}{2},m} V$). Hence, we can WLOG assume that v is an element of the form $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ with $(i_0, i_1, i_2, ...)$ being an m-degression (because the claim of Proposition 3.7.4 is clearly linear in v). Assume this. Then, $v = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ for some m-degression $(i_0, i_1, i_2, ...)$. Consider this m-degression $(i_0, i_1, i_2, ...)$. By the definition of an m-degression, every sufficiently high $k \in \mathbb{N}$ satisfies $i_k + k = m$. In other words, there exists a $K \in \mathbb{N}$ such that every integer $k \geq K$ satisfies $i_k + k = m$. Consider this K. Then, every integer which is less or equal to i_K appears in the m-degression $(i_0, i_1, i_2, ...)$.

For every $(i,j) \in \mathbb{Z}^2$, let $r_{i,j}$ be the map $\begin{cases} \rho(E_{i,j}), & \text{unless } i = j \text{ and } i \leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i = j \text{ and } i \leq 0 \end{cases} \in \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$. Then, the sum

$$\sum_{(i,j)\in\mathbb{Z}^2} a_{i,j} \begin{cases} \rho(E_{i,j}), & \text{unless } i = j \text{ and } i \leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i = j \text{ and } i \leq 0 \end{cases} v$$

clearly rewrites as $\sum\limits_{(i,j)\in\mathbb{Z}^2} a_{i,j} r_{i,j} v$. Hence, in order to prove Proposition 3.7.4, we only need to prove that the sum $\sum\limits_{(i,j)\in\mathbb{Z}^2} a_{i,j} r_{i,j} v$ has only finitely many nonzero addends.

Since $(a_{i,j})_{(i,j)\in\mathbb{Z}^2} \in \overline{\mathfrak{a}_\infty}$, only finitely many diagonals of the matrix $(a_{i,j})_{(i,j)\in\mathbb{Z}^2}$ are nonzero. In other words, there exists an $M \in \mathbb{N}$ such that

$$\left(\text{the } m\text{-th diagonal of the matrix } (a_{i,j})_{(i,j)\in\mathbb{Z}^2} \text{ is zero for every } m \in \mathbb{Z} \text{ such that } |m| \geq M\right). \tag{133}$$

Consider this M.

Now, we have the following three observations:

- Every $(i,j) \in \mathbb{Z}^2$ such that $j > \max\{i_0, 0\}$ satisfies $r_{i,j} v = 0$ [123] and thus $a_{i,j} \underbrace{r_{i,j} v}_{=0} = 0$.
- Every $(i,j) \in \mathbb{Z}^2$ such that $i \leq \min\{i_K, 0\}$ satisfies $r_{i,j} v = 0$ [124] and thus $a_{i,j} \underbrace{r_{i,j} v}_{=0} = 0$.

[123] *Proof.* Let $(i,j) \in \mathbb{Z}^2$ be such that $j > \max\{i_0, 0\}$. Then, $j > i_0$ and $j > 0$.

Since $j > i_0$, the integer j does not appear in the m-degression $(i_0, i_1, i_2, ...)$. Hence, $\rho(E_{i,j})(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = 0$. Since $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... = v$, this rewrites as $\rho(E_{i,j}) v = 0$.

Since $j > 0$, we cannot have $i = j$ and $i \leq 0$. Now, $r_{i,j} = \begin{cases} \rho(E_{i,j}), & \text{unless } i = j \text{ and } i \leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i = j \text{ and } i \leq 0 \end{cases} = \rho(E_{i,j})$ (since we cannot have $i = j$ and $i \leq 0$), so that $r_{i,j} v = \rho(E_{i,j}) v = 0$, qed.

[124] *Proof.* Let $(i,j) \in \mathbb{Z}^2$ be such that $i \leq \min\{i_K, 0\}$. Then, $i \leq i_K$ and $i \leq 0$.

Since $i \leq i_K$, the integer i appears in the m-degression $(i_0, i_1, i_2, ...)$ (because every integer which is less or equal to i_K appears in the m-degression $(i_0, i_1, i_2, ...)$). We now must be in one of the following two cases:

Case 1: We have $i \neq j$.

Case 2: We have $i = j$.

Let us first consider Case 1. In this case, $i \neq j$. Thus, $\rho(E_{i,j}) v = 0$ (because the integer i appears in the m-degression $(i_0, i_1, i_2, ...)$, so that after applying $\rho(E_{i,j})$ to $v = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$,

- Every $(i,j) \in \mathbb{Z}^2$ such that $|i-j| \geq M$ satisfies $a_{i,j} = 0$ [125] and thus $\underbrace{a_{i,j}}_{=0} r_{i,j} v = 0$.

Now, for any $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Z}$, let $[\alpha, \beta]_{\mathbb{Z}}$ denote the set $\{x \in \mathbb{Z} \mid \alpha \leq x \leq \beta\}$ (this set is finite). It is easy to see that

$$\left(\begin{array}{c} \text{every } (i,j) \in \mathbb{Z}^2 \text{ such that } a_{i,j} r_{i,j} v \neq 0 \text{ satisfies} \\ (i,j) \in [\min\{i_K, 0\}+1, \max\{i_0, 0\}+M-1]_{\mathbb{Z}} \times [\min\{i_K, 0\}-M+2, \max\{i_0, 0\}]_{\mathbb{Z}} \end{array}\right) \qquad (134)$$

[126]. Since $[\min\{i_K, 0\}+1, \max\{i_0, 0\}+M-1]_{\mathbb{Z}} \times [\min\{i_K, 0\}-M+2, \max\{i_0, 0\}]_{\mathbb{Z}}$ is a finite set, this shows that only finitely many $(i,j) \in \mathbb{Z}^2$ satisfy $a_{i,j} r_{i,j} v \neq 0$. In other words, the sum $\sum\limits_{(i,j) \in \mathbb{Z}^2} a_{i,j} r_{i,j} v$ has only finitely many nonzero addends. This proves Proposition 3.7.4.

Our definition of $\widehat{\rho}$ is somewhat unwieldy, since computing $\widehat{\rho}(a) v$ for a matrix $a \in \overline{\mathfrak{a}_\infty}$ and a $v \in \wedge^{\frac{\infty}{2}, m} V$ using it requires writing v as a linear combination of elementary

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we obtain a wedge in which v_i appears twice). On the other hand, $i \neq j$, so that we cannot have $i = j$ and $i \leq 0$. Now, $r_{i,j} = \begin{cases} \rho(E_{i,j}), & \text{unless } i = j \text{ and } i \leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i = j \text{ and } i \leq 0 \end{cases} = \rho(E_{i,j})$ (since we cannot have $i = j$ and $i \leq 0$), and thus $r_{i,j} v = \rho(E_{i,j}) v = 0$.

Now, let us consider Case 2. In this case, $i = j$. Thus, $r_{i,j} = \begin{cases} \rho(E_{i,j}), & \text{unless } i = j \text{ and } i \leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i = j \text{ and } i \leq 0 \end{cases} = \rho(E_{i,j}) - 1$ (since $i = j$ and $i \leq 0$). Since $E_{i,j} = E_{i,i}$ (because $j = i$), this rewrites as $r_{i,j} = \rho(E_{i,i}) - 1$. On the other hand, the integer i appears in the m-degression $(i_0, i_1, i_2, ...)$, so that $\rho(E_{i,i}) v = v$. Hence, from $r_{i,j} = \rho(E_{i,i}) - 1$, we get $r_{i,j} v = (\rho(E_{i,i}) - 1) v = \underbrace{\rho(E_{i,i}) v}_{=v} - v = v - v = 0$.

Thus, in each of the cases 1 and 2, we have proven that $r_{i,j} v = 0$. Hence, $r_{i,j} v = 0$ always holds, qed.

[125] *Proof.* Let $(u,v) \in \mathbb{Z}^2$ be such that $|u-v| \geq M$. Then, since $|v-u| = |u-v| \geq M$, the $(v-u)$-th diagonal of the matrix $(a_{i,j})_{(i,j) \in \mathbb{Z}^2}$ is zero (by (133), applied to $m = v-u$), and thus $a_{u,v} = 0$ (since $a_{u,v}$ is an entry on the $(v-u)$-th diagonal of the matrix $(a_{i,j})_{(i,j) \in \mathbb{Z}^2}$). We thus have shown that every $(u,v) \in \mathbb{Z}^2$ such that $|u-v| \geq M$ satisfies $a_{u,v} = 0$. Renaming (u,v) as (i,j) in this fact, we obtain: Every $(i,j) \in \mathbb{Z}^2$ such that $|i-j| \geq M$ satisfies $a_{i,j} = 0$, qed.

[126] *Proof of (134):* Let $(i,j) \in \mathbb{Z}^2$ be such that $a_{i,j} r_{i,j} v \neq 0$. Then, we cannot have $j > \max\{i_0, 0\}$ (since every $(i,j) \in \mathbb{Z}^2$ such that $j > \max\{i_0, 0\}$ satisfies $a_{i,j} r_{i,j} v = 0$, whereas we have $a_{i,j} r_{i,j} v \neq 0$). In other words, $j \leq \max\{i_0, 0\}$. Also, we cannot have $i \leq \min\{i_K, 0\}$ (since every $(i,j) \in \mathbb{Z}^2$ such that $i \leq \min\{i_K, 0\}$ satisfies $a_{i,j} r_{i,j} v = 0$, whereas we have $a_{i,j} r_{i,j} v \neq 0$). Thus, we have $i > \min\{i_K, 0\}$, so that $i \geq \min\{i_K, 0\} + 1$ (since i and $\min\{i_K, 0\}$ are integers). Finally, we cannot have $|i-j| \geq M$ (since every $(i,j) \in \mathbb{Z}^2$ such that $|i-j| \geq M$ satisfies $a_{i,j} r_{i,j} v = 0$, whereas we have $a_{i,j} r_{i,j} v \neq 0$). Thus, we have $|i-j| < M$, so that $|i-j| \leq M-1$ (since $|i-j|$ and M are integers). Thus, $i-j \leq |i-j| \leq M-1$. Hence, $i \leq \underbrace{j}_{\leq \max\{i_0, 0\}} + M - 1 \leq \max\{i_0, 0\} + M - 1$. Combined with $i \geq \min\{i_K, 0\} + 1$, this yields $i \in [\min\{i_K, 0\}+1, \max\{i_0, 0\}+M-1]_{\mathbb{Z}}$. From $i - j \leq M-1$, we also obtain $j \geq \underbrace{i}_{\geq \min\{i_K, 0\}+1} - (M-1) \geq \min\{i_K, 0\} + 1 - (M-1) = \min\{i_K, 0\} - M + 2$. Combined with $j \leq \max\{i_0, 0\}$, this yields $j \in [\min\{i_K, 0\} - M + 2, \max\{i_0, 0\}]_{\mathbb{Z}}$. Combined with $i \in [\min\{i_K, 0\}+1, \max\{i_0, 0\}+M-1]_{\mathbb{Z}}$, this yields $(i,j) \in [\min\{i_K, 0\}+1, \max\{i_0, 0\}+M-1]_{\mathbb{Z}} \times [\min\{i_K, 0\}-M+2, \max\{i_0, 0\}]_{\mathbb{Z}}$. This proves (134).

semiinfinite wedges. However, since our $\widehat{\rho}$ only slightly differs from ρ, there are many matrices a for which $\widehat{\rho}(a)$ behaves exactly as $\rho(a)$ would if we could extend ρ to $\overline{\mathfrak{a}_\infty}$:

Proposition 3.7.5. Let $m \in \mathbb{Z}$. Let $b_0, b_1, b_2, \ldots$ be vectors in V which satisfy

$$b_i = v_{m-i} \qquad \text{for sufficiently large } i.$$

Let $a \in \overline{\mathfrak{a}_\infty}$. Assume that, for every integer $i \leq 0$, the (i,i)-th entry of a is 0. Then,

$$(\widehat{\rho}(a))(b_0 \wedge b_1 \wedge b_2 \wedge \ldots) = \sum_{k\geq 0} b_0 \wedge b_1 \wedge \ldots \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge \ldots.$$

In particular, the infinite sum $\sum\limits_{k\geq 0} b_0 \wedge b_1 \wedge \ldots \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge \ldots$ is well-defined (i. e., all but finitely many integers $k \geq 0$ satisfy $b_0 \wedge b_1 \wedge \ldots \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge \ldots = 0$).

Proof of Proposition 3.7.5. For every $(i,j) \in \mathbb{Z}^2$, let $a_{i,j}$ be the (i,j)-th entry of the matrix a. Then, $a = (a_{i,j})_{(i,j)\in\mathbb{Z}^2} = \sum\limits_{(i,j)\in\mathbb{Z}^2} a_{i,j} E_{i,j}$. But every $(i,j) \in \mathbb{Z}^2$ such that $i = j$ and $i \leq 0$ satisfies $a_{i,j} = a_{i,i} = 0$ (because we assumed that, for every integer $i \leq 0$, the (i,i)-th entry of a is 0). Thus, $\sum\limits_{\substack{(i,j)\in\mathbb{Z}^2;\\ i=j \text{ and } i\leq 0}} \underbrace{a_{i,j}}_{=0} E_{i,j} = \sum\limits_{\substack{(i,j)\in\mathbb{Z}^2;\\ i=j \text{ and } i\leq 0}} 0E_{i,j} = 0$, so that

$$a = \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}E_{i,j} = \underbrace{\sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ i=j \text{ and } i\leq 0}} 0E_{i,j}}_{=0} + \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j}E_{i,j} = \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j}E_{i,j}.$$

But from $a = (a_{i,j})_{(i,j)\in\mathbb{Z}^2}$, we have

$$\begin{aligned}
\widehat{\rho}(a) &= \widehat{\rho}\left((a_{i,j})_{(i,j)\in\mathbb{Z}^2}\right) = \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j} \begin{cases} \rho(E_{i,j}), & \text{unless } i=j \text{ and } i\leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i=j \text{ and } i\leq 0 \end{cases} \qquad \text{(by (131))} \\
&= \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ i=j \text{ and } i\leq 0}} \underbrace{a_{i,j}}_{=0} \begin{cases} \rho(E_{i,j}), & \text{unless } i=j \text{ and } i\leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i=j \text{ and } i\leq 0 \end{cases} \\
&\qquad + \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j} \underbrace{\begin{cases} \rho(E_{i,j}), & \text{unless } i=j \text{ and } i\leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i=j \text{ and } i\leq 0 \end{cases}}_{\substack{=\rho(E_{i,j}) \\ \text{(since we do not have } (i=j \text{ and } i\leq 0))}} \\
&= \underbrace{\sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ i=j \text{ and } i\leq 0}} 0 \begin{cases} \rho(E_{i,j}), & \text{unless } i=j \text{ and } i\leq 0; \\ \rho(E_{i,j}) - 1, & \text{if } i=j \text{ and } i\leq 0 \end{cases}}_{=0} + \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j}\rho(E_{i,j}) \\
&= \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j}\rho(E_{i,j}),
\end{aligned}$$

so that

$$\begin{aligned}
&\left(\widehat{\rho}\left(a\right)\right)\left(b_0 \wedge b_1 \wedge b_2 \wedge ...\right) \\
&= \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j} \underbrace{\rho\left(E_{i,j}\right)\left(b_0 \wedge b_1 \wedge b_2 \wedge ...\right)}_{\substack{=E_{i,j}\rightharpoonup(b_0\wedge b_1\wedge b_2\wedge ...)\\ =\sum\limits_{k\geq 0} b_0\wedge b_1\wedge ...\wedge b_{k-1}\wedge (E_{i,j}\rightharpoonup b_k)\wedge b_{k+1}\wedge b_{k+2}\wedge ... \\ \text{(by Proposition 3.5.22, applied to } E_{i,j} \text{ instead of } a)}} \\
&= \sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j} \sum_{k\geq 0} b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge \left(E_{i,j} \rightharpoonup b_k\right) \wedge b_{k+1} \wedge b_{k+2} \wedge ... \\
&= \sum_{k\geq 0} b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge \underbrace{\left(\sum_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j}\left(E_{i,j} \rightharpoonup b_k\right)\right)}_{\substack{=\left(\sum\limits_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j}E_{i,j}\right)\rightharpoonup b_k = a\rightharpoonup b_k \\ \left(\text{since } \sum\limits_{\substack{(i,j)\in\mathbb{Z}^2;\\ \text{not } (i=j \text{ and } i\leq 0)}} a_{i,j}E_{i,j}=a\right)}} \wedge b_{k+1} \wedge b_{k+2} \wedge ... \\
&\qquad \left(\begin{array}{c}\text{here, we interchanged the summation signs; this is allowed because (as the reader} \\ \text{can check) all but finitely many } ((i,j),k) \in \mathbb{Z}^2 \times \mathbb{Z} \text{ satisfying } k \geq 0 \text{ and not} \\ (i=j \text{ and } i \leq 0) \text{ satisfy } a_{i,j} \cdot b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (E_{i,j} \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge ... = 0\end{array}\right) \\
&= \sum_{k\geq 0} b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge \left(a \rightharpoonup b_k\right) \wedge b_{k+1} \wedge b_{k+2} \wedge ...
\end{aligned}$$

(and en passant, this argument has shown that the infinite sum $\sum\limits_{k\geq 0} b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge ...$ is well-defined). This proves Proposition 3.7.5.

The issue that remains is that $\widehat{\rho}$ is not a representation of $\overline{\mathfrak{a}_\infty}$. To mitigate this, we will define a central extension of $\overline{\mathfrak{a}_\infty}$ by the so-called *Japanese cocycle*. Let us define this cocycle first:

Theorem 3.7.6. For any $A \in \overline{\mathfrak{a}_\infty}$ and $B \in \overline{\mathfrak{a}_\infty}$, we have $\widehat{\rho}\left([A,B]\right) - \left[\widehat{\rho}\left(A\right), \widehat{\rho}\left(B\right)\right] = \alpha\left(A,B\right)$ where $\alpha\left(A,B\right)$ is a scalar depending on A and B (and where we identify any scalar $\lambda \in \mathbb{C}$ with the matrix $\lambda \cdot \mathrm{id} \in \overline{\mathfrak{a}_\infty}$). This $\alpha\left(A,B\right)$ can be computed as follows: Write A and B as block matrices $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$, where the blocks are separated as follows:

- The left blocks contain the j-th columns for all $j \leq 0$; the right blocks contain the j-th columns for all $j > 0$.

- The upper blocks contain the i-th rows for all $i \leq 0$; the lower blocks contain the i-th rows for all $i > 0$.

Then, $\alpha\left(A,B\right) = \mathrm{Tr}\left(-B_{12}A_{21} + A_{12}B_{21}\right)$. (This trace makes sense because the matrices A_{12}, B_{21}, A_{21}, B_{12} have only finitely many nonzero entries.)

Corollary 3.7.7. The bilinear map $\alpha : \overline{\mathfrak{a}_\infty} \times \overline{\mathfrak{a}_\infty} \to \mathbb{C}$ defined in Theorem 3.7.6 is a 2-cocycle on $\overline{\mathfrak{a}_\infty}$.

We define $\mathfrak{a}_\infty$ as the 1-dimensional central extension $\widehat{\overline{\mathfrak{a}_\infty}}_\alpha$ of $\overline{\mathfrak{a}_\infty}$ by $\mathbb{C}$ using this cocycle α (see Definition 1.5.1 for what this means).

Definition 3.7.8. The 2-cocycle $\alpha : \overline{\mathfrak{a}_\infty} \times \overline{\mathfrak{a}_\infty} \to \mathbb{C}$ introduced in Corollary 3.7.7 is called the *Japanese cocycle*.

The proofs of Theorem 3.7.6 and Corollary 3.7.7 are a homework problem. A few remarks on the Japanese cocycle are in order. It can be explicitly computed by the formula

$$\begin{aligned}
&\alpha\left(\left(a_{i,j}\right)_{(i,j)\in\mathbb{Z}^2}, \left(b_{i,j}\right)_{(i,j)\in\mathbb{Z}^2}\right) \\
&= -\sum_{\substack{i\leq 0;\\ j>0}} b_{i,j}a_{j,i} + \sum_{\substack{i\leq 0;\\ j>0}} a_{i,j}b_{j,i} = -\sum_{\substack{i>0;\\ j\leq 0}} a_{i,j}b_{j,i} + \sum_{\substack{i\leq 0;\\ j>0}} a_{i,j}b_{j,i} \\
&= \sum_{(i,j)\in\mathbb{Z}^2} a_{i,j}b_{j,i}\left([j>0]-[i>0]\right) \qquad \text{for every } \left(a_{i,j}\right)_{(i,j)\in\mathbb{Z}^2}, \left(b_{i,j}\right)_{(i,j)\in\mathbb{Z}^2} \in \overline{\mathfrak{a}_\infty}
\end{aligned}$$

where we are using the *Iverson bracket notation*[127]. The cocycle α owes its name "Japanese cocycle" to the fact that it (first?) appeared in the work of the Tokyo mathematical physicists Date, Jimbo, Kashiwara and Miwa[128].

We are going to prove soon (Proposition 3.7.13 and Corollary 3.7.12) that α is a nontrivial 2-cocycle, but its restriction to $\mathfrak{gl}_\infty$ is trivial. This is a strange situation (given that $\mathfrak{gl}_\infty$ is a dense Lie subalgebra of $\overline{\mathfrak{a}_\infty}$ with respect to a reasonably defined topology), but we will later see the reason for this behavior.

Theorem 3.7.9. Let us extend the linear map $\widehat{\rho} : \overline{\mathfrak{a}_\infty} \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ (introduced in Definition 3.7.2) to a linear map $\widehat{\rho} : \mathfrak{a}_\infty \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ by setting $\widehat{\rho}(K) = \mathrm{id}$. (This makes sense since $\mathfrak{a}_\infty = \overline{\mathfrak{a}_\infty} \oplus \mathbb{C}K$ as vector spaces.) Then, this map $\widehat{\rho} : \mathfrak{a}_\infty \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ is a representation of $\mathfrak{a}_\infty$.

Thus, $\wedge^{\frac{\infty}{2},m} V$ becomes an $\mathfrak{a}_\infty$-module.

[127] This is the notation $[\mathcal{S}]$ for the truth value of any logical statement $\mathcal{S}$ (that is, $[\mathcal{S}]$ denotes the integer $\begin{cases} 1, & \text{if } \mathcal{S} \text{ is true;} \\ 0, & \text{if } \mathcal{S} \text{ is false} \end{cases}$).

[128] More precisely, it is the skew-symmetric bilinear form c in the following paper:

- Etsuro Date, Michio Jimbo, Masaki Kashiwara, Tetuji Miwa, *Transformation Groups for Soliton Equations – Euclidean Lie Algebras and Reduction of the KP Hierarchy*, Publ. RIMS, Kyoto Univ. 18 (1982), pp. 1077–1110.

In this paper, the Lie algebras that we are denoting by $\overline{\mathfrak{a}_\infty}$ and $\mathfrak{a}_\infty$ are called $\mathfrak{pgl}(\infty)$ and $\mathfrak{gl}(\infty)$, respectively.

Definition 3.7.10. Since $\mathfrak{a}_\infty = \overline{\mathfrak{a}_\infty} \oplus \mathbb{C}K$ as vector space, we can define a grading on $\mathfrak{a}_\infty$ as the direct sum of the grading on $\overline{\mathfrak{a}_\infty}$ (which was defined in Definition 3.6.5) and the trivial grading on $\mathbb{C}K$ (that is the grading which puts K in degree 0). This is easily seen to make $\mathfrak{a}_\infty$ a $\mathbb{Z}$-graded Lie algebra. We will consider $\mathfrak{a}_\infty$ to be $\mathbb{Z}$-graded in this way.

Proposition 3.7.11. Let $m \in \mathbb{Z}$. With the grading defined in Definition 3.7.10, the $\mathfrak{a}_\infty$-module $\wedge^{\frac{\infty}{2},m} V$ is graded.

Corollary 3.7.12. The restriction of α to $\mathfrak{gl}_\infty \times \mathfrak{gl}_\infty$ is a 2-coboundary.

Proof of Corollary 3.7.12. Let J be the block matrix $\begin{pmatrix} 0 & 0 \\ 0 & -I_\infty \end{pmatrix} \in \overline{\mathfrak{a}_\infty}$, where the blocks are separated in the same way as in Theorem 3.7.6. Define a linear map $f : \mathfrak{gl}_\infty \to \mathbb{C}$ by

$$(f(A) = \operatorname{Tr}(JA) \qquad \text{for any } A \in \mathfrak{gl}_\infty)$$

[129]. Then, any $A \in \mathfrak{gl}_\infty$ and $B \in \mathfrak{gl}_\infty$ satisfy $\alpha(A,B) = f([A,B])$. This is because (for any $A \in \mathfrak{gl}_\infty$ and $B \in \mathfrak{gl}_\infty$) we can write the matrix $[A,B]$ in the form $[A,B] = \begin{pmatrix} * & * \\ * & [A_{22},B_{22}] + A_{21}B_{12} - B_{21}A_{12} \end{pmatrix}$ (where asterisks mean blocks which we don't care about), so that $J[A,B] = \begin{pmatrix} 0 & 0 \\ * & -([A_{22},B_{22}] + A_{21}B_{12} - B_{21}A_{12}) \end{pmatrix}$ and thus

$$\begin{aligned} &\operatorname{Tr}(J[A,B]) \\ &= -\operatorname{Tr}([A_{22},B_{22}] + A_{21}B_{12} - B_{21}A_{12}) = -\underbrace{\operatorname{Tr}[A_{22},B_{22}]}_{=0} - \underbrace{\operatorname{Tr}(A_{21}B_{12})}_{=\operatorname{Tr}(B_{12}A_{21})} + \underbrace{\operatorname{Tr}(B_{21}A_{12})}_{=\operatorname{Tr}(A_{12}B_{21})} \\ &= -\operatorname{Tr}(B_{12}A_{21}) + \operatorname{Tr}(A_{12}B_{21}) = \operatorname{Tr}(-B_{12}A_{21} + A_{12}B_{21}) = \alpha(A,B). \end{aligned}$$

The proof of Corollary 3.7.12 is thus finished.

But note that this proof does not extend to $\overline{\mathfrak{a}_\infty}$, because f does not continuously extend to $\overline{\mathfrak{a}_\infty}$ (for any reasonable notion of continuity).

Proposition 3.7.13. The 2-cocycle α itself is not a 2-coboundary.

Proof of Proposition 3.7.13. Let T be the shift operator defined above. The span $\langle T^j \mid j \in \mathbb{Z} \rangle$ is an abelian Lie subalgebra of $\overline{\mathfrak{a}_\infty}$ (isomorphic to the abelian Lie algebra $\mathbb{C}[t,t^{-1}]$, and to the quotient $\overline{\mathcal{A}}$ of the Heisenberg algebra $\mathcal{A}$ by its central subalgebra $\langle K \rangle$). Any 2-coboundary must become zero when restricted onto an abelian Lie subalgebra. But the 2-cocycle α, restricted onto the span $\langle T^j \mid j \in \mathbb{Z} \rangle$, does not become 0, since

$$\alpha(T^i, T^j) = \begin{cases} 0, & \text{if } i \neq -j; \\ i, & \text{if } i = -j \end{cases} \qquad \text{for all } i,j \in \mathbb{Z}.$$

Proposition 3.7.13 is thus proven.

In this proof, we have constructed an embedding $\overline{\mathcal{A}} \to \overline{\mathfrak{a}_\infty}$ which sends $\overline{a_j}$ to T^j for every $j \in \mathbb{Z}$. This embedding is crucial to what we are going to do, so let us give it a formal definition:

[129] Note that $\operatorname{Tr}(JA)$ is well-defined for every $A \in \mathfrak{gl}_\infty$, since Remark 3.6.4 **(b)** (applied to J and A instead of A and B) yields that $JA \in \mathfrak{gl}_\infty$.

Definition 3.7.14. The map

$$\overline{\mathcal{A}} \to \overline{\mathfrak{a}_\infty}, \qquad a_j \mapsto T^j$$

(where $\overline{\mathcal{A}}$ is the quotient of the Heisenberg algebra $\mathcal{A}$ by its central subalgebra $\langle K \rangle$) is an embedding of Lie algebras. We will regard this embedding as an inclusion, and thus we will regard $\overline{\mathcal{A}}$ as a Lie subalgebra of $\overline{\mathfrak{a}_\infty}$.

This embedding is easily seen to give rise to an embedding $\mathcal{A} \to \mathfrak{a}_\infty$ of Lie algebras which sends K to K and sends a_j to T^j for every $j \in \mathbb{Z}$. This embedding will also be regarded as an inclusion, so that $\mathcal{A}$ will be considered as a Lie subalgebra of $\mathfrak{a}_\infty$.

It is now easy to see:

Proposition 3.7.15. Extend our map $\widehat{\rho} : \overline{\mathfrak{a}_\infty} \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ to a map $\mathfrak{a}_\infty \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$, also denoted by $\widehat{\rho}$, by setting $\widehat{\rho}(K) = \operatorname{id}$. Then, this map $\widehat{\rho} : \mathfrak{a}_\infty \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ is a Lie algebra homomorphism, i. e., it makes $\wedge^{\frac{\infty}{2},m} V$ into an $\mathfrak{a}_\infty$-module. The element K of $\mathfrak{a}_\infty$ acts as id on this module.

By means of the embedding $\mathcal{A} \to \mathfrak{a}_\infty$, this $\mathfrak{a}_\infty$-module gives rise to an $\mathcal{A}$-module $\wedge^{\frac{\infty}{2},m} V$, on which K acts as id.

In Proposition 3.5.27, we identified $\wedge^{\frac{\infty}{2},m} V$ as an irreducible highest-weight $\mathfrak{gl}_\infty$-module; similarly, we can identify it as an irreducible highest-weight $\mathfrak{a}_\infty$-module:

Proposition 3.7.16. Let $m \in \mathbb{Z}$. Let $\overline{\omega}_m$ be the $\mathbb{C}$-linear map $\mathfrak{a}_\infty[0] \to \mathbb{C}$ which sends every infinite diagonal matrix $\operatorname{diag}(..., d_{-2}, d_{-1}, d_0, d_1, d_2, ...) \in \overline{\mathfrak{a}_\infty}$ to $\begin{cases} \sum\limits_{j=1}^{m} d_j, & \text{if } m \geq 0; \\ -\sum\limits_{j=m+1}^{0} d_j, & \text{if } m < 0 \end{cases}$, and sends K to 1. Then, the graded $\mathfrak{a}_\infty$-module $\wedge^{\frac{\infty}{2},m} V$ is the irreducible highest-weight representation $L_{\overline{\omega}_m}$ of $\mathfrak{a}_\infty$ with highest weight $L_{\overline{\omega}_m}$. Moreover, $L_{\overline{\omega}_m}$ is unitary.

Remark 3.7.17. Note the analogy between the weight $\overline{\omega}_m$ in Proposition 3.7.16 and the weight ω_m in Proposition 3.5.27: The weight ω_m in Proposition 3.5.27 sends every diagonal matrix $\operatorname{diag}(..., d_{-2}, d_{-1}, d_0, d_1, d_2, ...) \in \mathfrak{gl}_\infty$ to $\sum\limits_{j=-\infty}^{m} d_j$. Note that this sum $\sum\limits_{j=-\infty}^{m} d_j$ is well-defined (because for a diagonal matrix $\operatorname{diag}(..., d_{-2}, d_{-1}, d_0, d_1, d_2, ...)$ to lie in $\mathfrak{gl}_\infty$, it has to satisfy $d_j = 0$ for all but finitely many $j \in \mathbb{Z}$).

In analogy to Corollary 3.5.29, we can also show:

Corollary 3.7.18. For every finite sum $\sum\limits_{i\in\mathbb{Z}} k_i\overline{\omega}_i$ with $k_i \in \mathbb{N}$, the representation $L_{\sum\limits_{i\in\mathbb{Z}} k_i\overline{\omega}_i}$ of $\mathfrak{a}_\infty$ is unitary.

3.8. Virasoro actions on $\wedge^{\frac{\infty}{2},m} V$

We can also embed the Virasoro algebra Vir into $\mathfrak{a}_\infty$, and not just in one way, but in infinitely many ways depending on two parameters:

Proposition 3.8.1. Let $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{C}$. Let the Vir-module $V_{\alpha,\beta}$ be defined as in Proposition 2.3.2.

For every $k \in \mathbb{Z}$, let $v_k = t^{-k+\alpha} (dt)^\beta \in V_{\alpha,\beta}$. Here, for any $\ell \in \mathbb{Z}$, the term $t^{\ell+\alpha} (dt)^\beta$ denotes $t^\ell t^\alpha (dt)^\beta$.

According to Proposition 2.3.2 **(b)**, every $m \in \mathbb{Z}$ satisfies

$$L_m v_k = (k - \alpha - \beta (m+1)) v_{k-m} \qquad \text{for every } k \in \mathbb{Z}.$$

Thus, if we write L_m as a matrix with respect to the basis $(v_k)_{k\in\mathbb{Z}}$ of $V_{\alpha,\beta}$, then this matrix lies in $\overline{\mathfrak{a}_\infty}$ (in fact, its only nonzero diagonal is the m-th one).

This defines an injective map $\overline{\varphi_{\alpha,\beta}} : W \to \overline{\mathfrak{a}_\infty}$, which sends every $L_m \in W$ to the matrix representing the action of L_m on $V_{\alpha,\beta}$. This map $\overline{\varphi_{\alpha,\beta}}$ is a Lie algebra homomorphism (since the Vir-module $V_{\alpha,\beta}$ has central charge 0, i. e., is an W-module). Hence, this map $\overline{\varphi_{\alpha,\beta}}$ lifts to an injective map $\widehat{W} \to \mathfrak{a}_\infty$, where $\widehat{W}$ is defined as follows: Let $\widetilde{\alpha} : \overline{\mathfrak{a}_\infty} \times \overline{\mathfrak{a}_\infty} \to \mathbb{C}$ be the Japanese cocycle (this cocycle has been called α in Definition 3.7.8, but here we use the letter α for something different), and let $\widetilde{\alpha}' : W \times W \to \mathbb{C}$ be the restriction of this Japanese cocycle $\widetilde{\alpha} : \overline{\mathfrak{a}_\infty} \times \overline{\mathfrak{a}_\infty} \to \mathbb{C}$ to $W \times W$ via the map $\overline{\varphi_{\alpha,\beta}} \times \overline{\varphi_{\alpha,\beta}} : W \times W \to \overline{\mathfrak{a}_\infty} \times \overline{\mathfrak{a}_\infty}$. Then, $\widehat{W}$ denotes the central extension of W defined by the 2-cocycle $\widetilde{\alpha}'$.

But let us now compute $\widetilde{\alpha}'$ and $\widehat{W}$. In fact, from a straightforward calculation (Homework Set 4 exercise 3) it follows that

$$\widetilde{\alpha}' (L_m, L_n) = \delta_{n,-m} \left(\frac{n^3 - n}{12} c_\beta + 2n h_{\alpha,\beta} \right) \qquad \text{for all } n, m \in \mathbb{Z},$$

where

$$c_\beta = -12\beta^2 + 12\beta - 2 \qquad \text{and} \qquad h_{\alpha,\beta} = \frac{1}{2}\alpha (\alpha + 2\beta - 1).$$

Thus, the 2-cocycle $\widetilde{\alpha}'$ differs from the 2-cocycle ω (defined in Theorem 1.5.2) merely by a multiplicative factor ($\frac{c_\beta}{2}$) and a 2-coboundary (which sends every (L_m, L_n) to $\delta_{n,-m} \cdot 2n h_{\alpha,\beta}$). Thus, the central extension $\widehat{W}$ of W defined by the 2-cocycle $\widetilde{\alpha}'$ is isomorphic (as a Lie algebra) to the central extension of W defined by the 2-cocycle ω, that is, to the Virasoro algebra Vir. This turns the Lie algebra homomorphism $\widehat{W} \to \mathfrak{a}_\infty$ into a homomorphism Vir $\to \mathfrak{a}_\infty$. Let us describe this homomorphism explicitly:

Let $\widehat{L_0}$ be the element $\overline{\varphi_{\alpha,\beta}}(L_0) + h_{\alpha,\beta}K \in \mathfrak{a}_\infty$. Then, the linear map

$$\begin{aligned} \mathrm{Vir} &\to \mathfrak{a}_\infty, \\ L_n &\mapsto \overline{\varphi_{\alpha,\beta}}(L_n) \qquad\qquad \text{for } n \neq 0, \\ L_0 &\mapsto \widehat{L_0}, \\ C &\mapsto c_\beta K \end{aligned}$$

is a Lie algebra homomorphism. Denote this map by $\varphi_{\alpha,\beta}$. By means of this homomorphism, we can restrict the $\mathfrak{a}_\infty$-module $\wedge^{\frac{\infty}{2},m} V$ to a Vir-module. Denote this Vir-module by $\wedge^{\frac{\infty}{2},m} V_{\alpha,\beta}$. Note that $\wedge^{\frac{\infty}{2},m} V_{\alpha,\beta}$ is a Virasoro module with central charge $c = c_\beta$. This $\wedge^{\frac{\infty}{2},m} V_{\alpha,\beta}$ is called the *module of semiinfinite forms*. The vector $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ...$ (defined in Definition 3.5.28) has highest degree (namely, 0).

We have $L_i \psi_m = 0$ for $i > 0$, and we have $L_0 \psi_m = \frac{1}{2}(\alpha - m)(\alpha + 2\beta - 1 - m)\psi_m$. (Proof: Homework exercise.)

Corollary 3.8.2. Let $\alpha, \beta \in \mathbb{C}$. We have a homomorphism

$$\begin{aligned} M_\lambda &\to \wedge^{\frac{\infty}{2},m} V_{\alpha,\beta}, \\ v_\lambda &\mapsto \psi_m \end{aligned}$$

of Virasoro modules, where

$$\lambda = \left(\frac{1}{2}(\alpha - m)(\alpha + 2\beta - 1 - m), -12\beta^2 + 12\beta - 2\right).$$

We will see that this is an isomorphism for generic λ. For concrete λ it is not always one, and can have a rather complicated kernel.

3.9. The dimensions of the homogeneous components of $\wedge^{\frac{\infty}{2},m} V$

Fix $m \in \mathbb{Z}$. We already know from Definition 3.5.25 that $\wedge^{\frac{\infty}{2},m} V$ is a graded $\mathbb{C}$-vector space. More concretely,

$$\wedge^{\frac{\infty}{2},m} V = \bigoplus_{d \geq 0} \left(\wedge^{\frac{\infty}{2},m} V\right)[-d],$$

where every $d \geq 0$ satisfies

$$\left(\wedge^{\frac{\infty}{2},m} V\right)[-d] = \left\langle v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \mid \sum_{k \geq 0} (i_k + k - m) = d \right\rangle.$$

We also know that the m-degressions are in a 1-to-1 correspondence with the partitions. This correspondence maps any m-degression $(i_0, i_1, i_2, ...)$ to the partition $(i_k + k - m)_{k\geq 0}$; this is a partition of the integer $\sum\limits_{k\geq 0} (i_k + k - m)$. As a consequence, for every integer $d \geq 0$, the m-degressions $(i_0, i_1, i_2, ...)$ satisfying $\sum\limits_{k\geq 0} (i_k + k - m) = d$ are in 1-to-1 correspondence with the partitions of d. Hence, for every integer $d \geq 0$, the number of all m-degressions $(i_0, i_1, i_2, ...)$ satisfying $\sum\limits_{k\geq 0} (i_k + k - m) = d$ equals the number of the partitions of d. Thus, for every integer $d \geq 0$, we have

$$\begin{aligned}
&\dim\left(\left(\wedge^{\frac{\infty}{2},m} V\right)[-d]\right) \\
&= \left(\text{the number of } m\text{-degressions } (i_0, i_1, i_2, ...) \text{ satisfying} \sum_{k\geq 0} (i_k + k - m) = d\right) \\
&\qquad \left(\begin{array}{c}\text{since } (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)_{(i_0,i_1,i_2,...) \text{ is an } m\text{-degression satisfying } \sum_{k\geq 0} (i_k+k-m)=d} \\ \text{is a basis of } \left(\wedge^{\frac{\infty}{2},m} V\right)[-d]\end{array}\right) \\
&= (\text{the number of partitions of } d) = p(d),
\end{aligned}$$

where p is the partition function. Hence:

Proposition 3.9.1. Let $m \in \mathbb{Z}$. Every integer $d \geq 0$ satisfies $\dim\left(\left(\wedge^{\frac{\infty}{2},m} V\right)[-d]\right) = p(d)$, where p is the partition function. As a consequence, in the ring of formal power series $\mathbb{C}[[q]]$, we have

$$\sum_{d\geq 0} \dim\left(\left(\wedge^{\frac{\infty}{2},m} V\right)[-d]\right) q^d = \sum_{d\geq 0} p(d)\, q^d = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

3.10. The Boson-Fermion correspondence

Proposition 3.10.1. Let $m \in \mathbb{Z}$. Recall the vector ψ_m defined in Definition 3.5.28.

(a) As an $\mathcal{A}$-module, $\wedge^{\frac{\infty}{2},m} V$ is isomorphic to the Fock module F_m. More precisely, there exists a graded $\mathcal{A}$-module isomorphism $\widetilde{\sigma}_m : F_m \to \wedge^{\frac{\infty}{2},m} V$ of $\mathcal{A}$-modules such that $\widetilde{\sigma}_m(1) = \psi_m$.

(b) As an $\mathcal{A}$-module, $\wedge^{\frac{\infty}{2},m} V$ is isomorphic to the Fock module $\widetilde{F}_m$. More precisely, there exists a graded $\mathcal{A}$-module isomorphism $\sigma_m : \widetilde{F}_m \to \wedge^{\frac{\infty}{2},m} V$ of $\mathcal{A}$-modules such that $\sigma_m(1) = \psi_m$.

Proof of Proposition 3.10.1. **(a)** Let us first notice that in the ring $\mathbb{C}[[q]]$, we have

$$\begin{aligned}
\sum_{d\geq 0}\dim\left(\left(\wedge^{\frac{\infty}{2},m}V\right)[-d]\right)q^d &= \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} \qquad \text{(by Proposition 3.9.1)} \\
&= \sum_{n\geq 0}\dim\left(\underbrace{F}_{\substack{=F_m \\ \text{(as vector spaces)}}}[-n]\right)q^n \qquad \text{(by Definition 2.2.7)} \\
&= \sum_{n\geq 0}\dim\left(F_m[-n]\right)q^n = \sum_{d\geq 0}\dim\left(F_m[-d]\right)q^d.
\end{aligned}$$

By comparing coefficients, this yields that every integer $d \geq 0$ satisfies

$$\dim\left(\left(\wedge^{\frac{\infty}{2},m}V\right)[-d]\right) = \dim\left(F_m[-d]\right). \tag{135}$$

We have $a_i\psi_m = 0$ for all $i > 0$ (by degree considerations), and we also have $K\psi_m = \psi_m$. Besides, it is easy to see that $a_0\psi_m = m\psi_m$ [130].

Hence, Lemma 2.5.13 (applied to m and $\wedge^{\frac{\infty}{2},m}V$ instead of μ and V) yields that

[130] *Proof.* The embedding $\mathcal{A} \to \mathfrak{a}_\infty$ sends a_0 to $T^0 = \mathbf{1}$, where $\mathbf{1}$ denotes the identity matrix in $\mathfrak{a}_\infty$. Thus, $a_0\psi_m = \mathbf{1}\psi_m$. (Note that $\mathbf{1}\psi_m$ needs not equal ψ_m in general, since the action of $\mathfrak{a}_\infty$ on $\wedge^{\frac{\infty}{2},m}V$ is not an associative algebra action, but just a Lie algebra action.) Recall that $\wedge^{\frac{\infty}{2},m}V$ became an $\mathfrak{a}_\infty$-module via the map $\widehat{\rho}$, so that $U\psi_m = \widehat{\rho}(U)\psi_m$ for every $U \in \mathfrak{a}_\infty$. Now,

$$\begin{aligned}
a_0\psi_m &= \mathbf{1}\psi_m = \sum_{i\in\mathbb{Z}} E_{i,i}\psi_m \qquad \left(\text{since } \mathbf{1} = \sum_{i\in\mathbb{Z}} E_{i,i}\right) \\
&= \sum_{i\in\mathbb{Z}} \underbrace{\widehat{\rho}(E_{i,i})}_{\substack{=\begin{cases}\rho(E_{i,i}), & \text{unless } i=i \text{ and } i\leq 0; \\ \rho(E_{i,i})-1, & \text{if } i=i \text{ and } i\leq 0\end{cases} \\ \text{(by the definition of } \widehat{\rho})}} \underbrace{\psi_m}_{=v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots} \\
&\qquad \left(\text{since } U\psi_m = \widehat{\rho}(U)\psi_m \text{ for every } U \in \mathfrak{a}_\infty\right) \\
&= \sum_{i\in\mathbb{Z}} \begin{cases}\rho(E_{i,i}), & \text{unless } i=i \text{ and } i\leq 0; \\ \rho(E_{i,i})-1, & \text{if } i=i \text{ and } i\leq 0\end{cases} \cdot v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots \\
&= \sum_{\substack{i\in\mathbb{Z}; \\ i>0}} \underbrace{\begin{cases}\rho(E_{i,i}), & \text{unless } i=i \text{ and } i\leq 0; \\ \rho(E_{i,i})-1, & \text{if } i=i \text{ and } i\leq 0\end{cases}}_{=\rho(E_{i,i})} \cdot v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots \\
&\qquad + \sum_{\substack{i\in\mathbb{Z}; \\ i\leq 0}} \underbrace{\begin{cases}\rho(E_{i,i}), & \text{unless } i=i \text{ and } i\leq 0; \\ \rho(E_{i,i})-1, & \text{if } i=i \text{ and } i\leq 0\end{cases}}_{=\rho(E_{i,i})-1} \cdot v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots \\
&= \sum_{\substack{i\in\mathbb{Z}; \\ i>0}} \rho(E_{i,i})\cdot v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots + \sum_{\substack{i\in\mathbb{Z}; \\ i\leq 0}} \left(\rho(E_{i,i})-1\right)\cdot v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots.
\end{aligned}$$

Now, we distinguish between two cases:
Case 1: We have $m \geq 0$.
Case 2: We have $m < 0$.

there exists a $\mathbb{Z}$-graded homomorphism $\widetilde{\sigma}_m : F_m \to \wedge^{\frac{\infty}{2},m} V$ of $\mathcal{A}$-modules such that $\widetilde{\sigma}_m(1) = \psi_m$. (An alternative way to prove the existence of this $\widetilde{\sigma}_m$ would be to apply Lemma 2.7.8, making use of the fact (Proposition 2.5.17) that F_m is a Verma module for $\mathcal{A}$.)

This $\widetilde{\sigma}_m$ is injective (since F_m is irreducible) and $\mathbb{Z}$-graded. Hence, for every integer $d \geq 0$, it induces a homomorphism from $F_m[-d]$ to $\left(\wedge^{\frac{\infty}{2},m} V\right)[-d]$. This induced homomorphism must be injective (since $\widetilde{\sigma}_m$ was injective), and thus is an isomorphism (since the vector spaces $F_m[-d]$ and $\left(\wedge^{\frac{\infty}{2},m} V\right)[-d]$ have the same dimension (by (135)) and are both finite-dimensional). Since this holds for every integer $d \geq 0$, this yields that $\widetilde{\sigma}_m$ itself must be an isomorphism. This proves Proposition 3.10.1 **(a)**.

Proposition 3.10.1 **(b)** follows from Proposition 3.10.1 **(a)** due to Proposition 2.2.21 **(b)**.

Note that Proposition 3.10.1 is surprising: It gives an isomorphism between a space of polynomials (the Fock space F_m, also called a *bosonic space*) and a space of wedge products (the space $\wedge^{\frac{\infty}{2},m} V$, also called a *fermionic space*); isomorphisms like this are unheard of in finite-dimensional contexts.

In Case 1, we have

$$\begin{aligned}
a_0\psi_m &= \sum_{\substack{i\in\mathbb{Z};\\ i>0}} \rho(E_{i,i}) \cdot v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots + \sum_{\substack{i\in\mathbb{Z};\\ i\leq 0}} (\rho(E_{i,i}) - 1) \cdot v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots \\
&= \sum_{\substack{i\in\mathbb{Z};\\ i>0;\ i>m}} \underbrace{\rho(E_{i,i}) \cdot v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots}_{\substack{=0\\ \text{(since } i \text{ does not appear in the } m\text{-degression } (m,m-1,m-2,\ldots))}} \\
&\quad + \sum_{\substack{i\in\mathbb{Z};\\ i>0;\ i\leq m}} \underbrace{\rho(E_{i,i}) \cdot v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots}_{\substack{=v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots\\ \text{(since } i \text{ appears in the } m\text{-degression } (m,m-1,m-2,\ldots))}} \\
&\quad + \sum_{\substack{i\in\mathbb{Z};\\ i\leq 0}} \underbrace{(\rho(E_{i,i}) - 1) \cdot v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots}_{\substack{=0\\ \text{(since } i \text{ appears in the } m\text{-degression } (m,m-1,m-2,\ldots)\\ \text{and thus we have } \rho(E_{i,i})\cdot v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots=v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots)}} \\
&\qquad \text{(since we are in Case 1, so that } m \geq 0) \\
&= \underbrace{\sum_{\substack{i\in\mathbb{Z};\\ i>0;\ i>m}} 0}_{=0} + \sum_{\substack{i\in\mathbb{Z};\\ i>0;\ i\leq m}} \underbrace{v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots}_{=\psi_m} + \underbrace{\sum_{\substack{i\in\mathbb{Z};\\ i\leq 0}} 0}_{=0} = \sum_{\substack{i\in\mathbb{Z};\\ i>0;\ i\leq m}} \psi_m = m\psi_m.
\end{aligned}$$

Hence, $a_0\psi_m = m\psi_m$ is proven in Case 1. In Case 2, the proof of $a_0\psi_m = m\psi_m$ is similar (but instead of splitting the $\sum\limits_{\substack{i\in\mathbb{Z};\\ i>0}}$ sum into a $\sum\limits_{\substack{i\in\mathbb{Z};\\ i>0;\ i>m}}$ and a $\sum\limits_{\substack{i\in\mathbb{Z};\\ i>0;\ i\leq m}}$ sum, we must now split the $\sum\limits_{\substack{i\in\mathbb{Z};\\ i\leq 0}}$ sum into a $\sum\limits_{\substack{i\in\mathbb{Z};\\ i\leq 0;\ i>m}}$ and a $\sum\limits_{\substack{i\in\mathbb{Z};\\ i\leq 0;\ i\leq m}}$ sum). Thus, $a_0\psi_m = m\psi_m$ holds in both cases 1 and 2. In other words, the proof of $a_0\psi_m = m\psi_m$ is complete.

Definition 3.10.2. We write $\mathcal{B}^{(m)}$ for the $\mathcal{A}$-module $\widetilde{F}_m$. We write $\mathcal{B}$ for the $\mathcal{A}$-module $\bigoplus\limits_m \mathcal{B}^{(m)} = \bigoplus\limits_m \widetilde{F}_m$. We write $\mathcal{F}^{(m)}$ for the $\mathcal{A}$-module $\wedge^{\frac{\infty}{2},m} V$. We write $\mathcal{F}$ for the $\mathcal{A}$-module $\bigoplus\limits_m \mathcal{F}^{(m)}$.

The isomorphism σ_m (constructed in Proposition 3.10.1 **(b)**) is thus an isomorphism $\mathcal{B}^{(m)} \to \mathcal{F}^{(m)}$. We write σ for the $\mathcal{A}$-module isomorphism $\bigoplus\limits_m \sigma_m : \mathcal{B} \to \mathcal{F}$. This σ is called the *Boson-Fermion Correspondence*.

Note that we can do the same for the Virasoro algebra: If M_λ is irreducible, then the homomorphism $M_\lambda \to \wedge^{\frac{\infty}{2},m} V_{\alpha,\beta}$ is an isomorphism. And we know that Vir is nondegenerate, so M_λ is irreducible for Weil-generic λ.

Corollary 3.10.3. For generic α and β, the Vir-module $\wedge^{\frac{\infty}{2},m} V_{\alpha,\beta}$ is irreducible.

But now, back to the Boson-Fermion Correspondence:

Both $\mathcal{B}$ and $\mathcal{F}$ are $\mathcal{A}$-modules, and Proposition 3.10.1 **(b)** showed us that they are isomorphic as such through the isomorphism $\sigma : \mathcal{B} \to \mathcal{F}$. However, $\mathcal{F}$ is also an $\mathfrak{a}_\infty$-module, whereas $\mathcal{B}$ is not. But of course, with the isomorphism σ being given, we can transfer the $\mathfrak{a}_\infty$-module structure from $\mathcal{F}$ to $\mathcal{B}$. The same can be done with the $\mathfrak{gl}_\infty$-module structure. Let us explicitly define these:

Definition 3.10.4. **(a)** We make $\mathcal{B}$ into an $\mathfrak{a}_\infty$-module by transferring the $\mathfrak{a}_\infty$-module structure on $\mathcal{F}$ (given by the map $\widehat{\rho} : \mathfrak{a}_\infty \to \operatorname{End} \mathcal{F}$) to $\mathcal{B}$ via the isomorphism $\sigma : \mathcal{B} \to \mathcal{F}$. Note that the $\mathcal{A}$-module $\mathcal{B}$ is a restriction of the $\mathfrak{a}_\infty$-module $\mathcal{B}$ (since the $\mathcal{A}$-module $\mathcal{F}$ is the restriction of the $\mathfrak{a}_\infty$-module $\mathcal{F}$). We denote the $\mathfrak{a}_\infty$-module structure on $\mathcal{B}$ by $\widehat{\rho} : \mathfrak{a}_\infty \to \operatorname{End} \mathcal{B}$.

(b) We make $\mathcal{B}$ into a $\mathfrak{gl}_\infty$-module by transferring the $\mathfrak{gl}_\infty$-module structure on $\mathcal{F}$ (given by the map $\rho : \mathfrak{gl}_\infty \to \operatorname{End} \mathcal{F}$) to $\mathcal{B}$ via the isomorphism $\sigma : \mathcal{B} \to \mathcal{F}$. We denote the $\mathfrak{gl}_\infty$-module structure on $\mathcal{B}$ by $\rho : \mathfrak{gl}_\infty \to \operatorname{End} \mathcal{B}$.

How do we describe these module structures on $\mathcal{B}$ explicitly (i. e., in formulas?) This question is answered using the so-called *vertex operator construction*.

But first, some easier things:

Definition 3.10.5. Let $m \in \mathbb{Z}$. Let $i \in \mathbb{Z}$.

(a) We define the so-called *i-th wedging operator* $\widehat{v_i} : \mathcal{F}^{(m)} \to \mathcal{F}^{(m+1)}$ by

$$\widehat{v_i} \cdot \psi = v_i \wedge \psi \qquad \text{for all } \psi \in \mathcal{F}^{(m)}.$$

Here, $v_i \wedge \psi$ is formally defined as follows: Write ψ as a $\mathbb{C}$-linear combination of (well-defined) semiinfinite wedge products $b_0 \wedge b_1 \wedge b_2 \wedge ...$ (for instance, elementary semiinfinite wedges); then, $v_i \wedge \psi$ is obtained by replacing each such product $b_0 \wedge b_1 \wedge b_2 \wedge ...$ by $v_i \wedge b_0 \wedge b_1 \wedge b_2 \wedge ....$

(b) We define the so-called *i-th contraction operator* $\overset{\vee}{v_i} : \mathcal{F}^{(m)} \to \mathcal{F}^{(m-1)}$ as follows:

For every m-degression $(i_0, i_1, i_2, ...)$, we let $\overset{\vee}{v_i}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$ be

$$\begin{cases} 0, & \text{if } i \notin \{i_0, i_1, i_2, ...\}; \\ (-1)^j v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_{j-1}} \wedge v_{i_{j+1}} \wedge v_{i_{j+2}} \wedge ..., & \text{if } i \in \{i_0, i_1, i_2, ...\} \end{cases},$$

where, in the case $i \in \{i_0, i_1, i_2, ...\}$, we denote by j the integer k satisfying $i_k = i$. Thus, the map $\overset{\vee}{v_i}$ is defined on all elementary semiinfinite wedges; we extend this to a map $\mathcal{F}^{(m)} \to \mathcal{F}^{(m-1)}$ by linearity.

Note that the somewhat unwieldy definition of $\overset{\vee}{v_i}$ can be slightly improved: While it only gave a formula for m-degressions, it is easy to see that the same formula holds for straying m-degressions:

Proposition 3.10.6. Let $m \in \mathbb{Z}$ and $i \in \mathbb{Z}$. Let $(i_0, i_1, i_2, ...)$ be a straying m-degression which has no two equal elements. Then,

$$\overset{\vee}{v_i}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \begin{cases} 0, & \text{if } i \notin \{i_0, i_1, i_2, ...\}; \\ (-1)^j v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_{j-1}} \wedge v_{i_{j+1}} \wedge v_{i_{j+2}} \wedge ..., & \text{if } i \in \{i_0, i_1, i_2, ...\} \end{cases},$$

where, in the case $i \in \{i_0, i_1, i_2, ...\}$, we denote by j the integer k satisfying $i_k = i$.

These operators satisfy the relations

$$\widehat{v_i}\widehat{v_j} + \widehat{v_j}\widehat{v_i} = 0, \qquad \overset{\vee}{v_i}\overset{\vee}{v_j} + \overset{\vee}{v_j}\overset{\vee}{v_i} = 0,$$
$$\overset{\vee}{v_i}\widehat{v_j} + \widehat{v_j}\overset{\vee}{v_i} = \delta_{i,j}$$

for all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$.

Definition 3.10.7. For every $i \in \mathbb{Z}$, define $\xi_i = \widehat{v_i}$ and $\xi_i^* = \overset{\vee}{v_i}$.

Then, all $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$ satisfy $\rho(E_{i,j}) = \xi_i \xi_j^*$ and

$$\widehat{\rho}(E_{i,j}) = \begin{cases} \xi_i \xi_j^* - 1, & \text{if } i = j \text{ and } i \leq 0, \\ \xi_i \xi_j^*, & \text{unless } i = j \text{ and } i \leq 0 \end{cases}.$$

The ξ_i and ξ_i^* are called *fermionic operators.*

So what are the ξ_i in terms of a_j ?

3.11. The vertex operator construction

We identify the space $\mathbb{C}[z, z^{-1}, x_1, x_2, ...] = \bigoplus_m z^m \mathbb{C}[x_1, x_2, ...]$ with $\mathcal{B} = \bigoplus_m \mathcal{B}^{(m)}$ by means of identifying $z^m \mathbb{C}[x_1, x_2, ...]$ with $\mathcal{B}^{(m)}$ for every $m \in \mathbb{Z}$ (the identification being made through the map

$$\mathcal{B}^{(m)} \to z^m \mathbb{C}[x_1, x_2, ...],$$
$$p \mapsto z^m \cdot p$$

).

Note also that z (that is, multiplication by z) is an isomorphism of $\mathcal{A}_0$-modules, but not of $\mathcal{A}$-modules.

The Boson-Fermion correspondence goes like this:

$$\mathcal{F} = \bigoplus_m \mathcal{F}^{(m)} \overset{\sigma = \bigoplus_m \sigma_m}{\leftarrow} \mathcal{B} = \bigoplus_m \mathcal{B}^{(m)}.$$

On $\mathcal{F}$ there are operators $\widehat{v_i} = \xi_i$, $\overset{\vee}{v_i} = \xi_i^*$, $\rho(E_{i,j}) = \xi_i \xi_j^*$, $\widehat{\rho}(E_{i,j}) = \begin{cases} \xi_i \xi_j^* - 1, & \text{if } i = j \text{ and } i \leq 0, \\ \xi_i \xi_j^*, & \text{unless } i = j \text{ and } i \leq 0 \end{cases}$. By conjugating with the Boson-Fermion correspondence σ, these operators give rise to operators on $\mathcal{B}$. How do the latter operators look like?

Definition 3.11.1. Introduce the quantum fields

$$X(u) = \sum_{n \in \mathbb{Z}} \xi_n u^n \in (\operatorname{End} \mathcal{F}) \left[\left[u, u^{-1}\right]\right],$$
$$X^*(u) = \sum_{n \in \mathbb{Z}} \xi_n^* u^{-n} \in (\operatorname{End} \mathcal{F}) \left[\left[u, u^{-1}\right]\right],$$
$$\Gamma(u) = \sigma^{-1} \circ X(u) \circ \sigma \in (\operatorname{End} \mathcal{B}) \left[\left[u, u^{-1}\right]\right],$$
$$\Gamma^*(u) = \sigma^{-1} \circ X^*(u) \circ \sigma \in (\operatorname{End} \mathcal{B}) \left[\left[u, u^{-1}\right]\right].$$

Note that $\sigma^{-1} \circ X(u) \circ \sigma$ is to be read as "conjugate every term of the power series $X(u)$ by σ"; in other words, $\sigma^{-1} \circ X(u) \circ \sigma$ means $\sum\limits_{n \in \mathbb{Z}} \left(\sigma^{-1} \circ \xi_n \circ \sigma\right) u^n$.

Recall that $\xi_n = \widehat{v_n}$ sends $\mathcal{F}^{(m)}$ to $\mathcal{F}^{(m+1)}$ for any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$. Thus, every term of the power series $X(u) = \sum\limits_{n \in \mathbb{Z}} \xi_n u^n$ sends $\mathcal{F}^{(m)}$ to $\mathcal{F}^{(m+1)}$ for any $m \in \mathbb{Z}$. Abusing notation, we will abbreviate this fact by saying that $X(u) : \mathcal{F}^{(m)} \to \mathcal{F}^{(m+1)}$ for any $m \in \mathbb{Z}$. Similarly, $X^*(u) : \mathcal{F}^{(m)} \to \mathcal{F}^{(m-1)}$ for any $m \in \mathbb{Z}$ (since $\xi_n^* = \overset{\vee}{v_n}$ sends $\mathcal{F}^{(m)}$ to $\mathcal{F}^{(m-1)}$ for any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$). As a consequence, $\Gamma(u) : \mathcal{B}^{(m)} \to \mathcal{B}^{(m+1)}$ and $\Gamma^*(u) : \mathcal{B}^{(m)} \to \mathcal{B}^{(m-1)}$ for any $m \in \mathbb{Z}$.

Now, here is how we can describe $\Gamma(u)$ and $\Gamma^*(u)$ (and therefore the operators $\sigma^{-1} \circ \xi_n \circ \sigma$ and $\sigma^{-1} \circ \xi_n^* \circ \sigma$) in terms of $\mathcal{B}$:

Theorem 3.11.2. Let $m \in \mathbb{Z}$. On $\mathcal{B}^{(m)}$, we have

$$\Gamma(u) = u^{m+1} z \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right);$$
$$\Gamma^*(u) = u^{-m} z^{-1} \exp\left(-\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(\sum_{j>0} \frac{a_j}{j} u^{-j}\right).$$

Here, $\exp A$ means $1 + A + \dfrac{A^2}{2!} + \dfrac{A^3}{3!} + ...$ for any A for which this series makes any sense.

Let us explain what we mean by the products $\exp\left(\sum\limits_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum\limits_{j>0} \frac{a_j}{j} u^{-j}\right)$ and $\exp\left(-\sum\limits_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(\sum\limits_{j>0} \frac{a_j}{j} u^{-j}\right)$ in Theorem 3.11.2. Why do these products (which are products of exponentials of infinite sums) make any sense? This is easily answered:

- For any $v \in \mathcal{B}^{(m)}$, the term $\exp\left(-\sum\limits_{j>0} \frac{a_j}{j} u^{-j}\right)(v)$ is well-defined and is valued in $\mathcal{B}^{(m)}\left[u^{-1}\right]$. (In fact, if we blindly expand

$$\exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)^{\ell}$$
$$= \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (-1)^{\ell} \sum_{j_1, j_2, \ldots, j_\ell \text{ positive integers}} \frac{a_{j_1} a_{j_2} \ldots a_{j_\ell}}{j_1 j_2 \ldots j_\ell} u^{-(j_1+j_2+\ldots+j_\ell)},$$

  and apply every term of the resulting power series to v, then (for fixed v) only finitely many of these terms yield a nonzero result, since v is a polynomial and thus has finite degree, whereas each a_j lowers degree by j.)

- For any $v \in \mathcal{B}^{(m)}$, the term $\exp\left(\sum\limits_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum\limits_{j>0} \frac{a_j}{j} u^{-j}\right) v$ is well-defined and is valued in $\mathcal{B}^{(m)}((u))$. (In fact, we have just shown that $\exp\left(-\sum\limits_{j>0} \frac{a_j}{j} u^{-j}\right)(v) \in \mathcal{B}^{(m)}\left[u^{-1}\right]$; therefore, applying $\exp\left(\sum\limits_{j>0} \frac{a_{-j}}{j} u^j\right) \in \left(\operatorname{End}\left(\mathcal{B}^{(m)}\right)\right)[[u]]$ to this gives a well-defined power series in $\mathcal{B}^{(m)}((u))$ (because if $\mathfrak{A}$ is an algebra and $\mathfrak{M}$ is an $\mathfrak{A}$-module, then the application of a power series in $\mathfrak{A}[[u]]$ to an element of $\mathfrak{M}\left[u^{-1}\right]$ gives a well-defined element of $\mathfrak{M}((u))$).)

- For any $v \in \mathcal{B}^{(m)}$, the term $\exp\left(-\sum\limits_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(\sum\limits_{j>0} \frac{a_j}{j} u^{-j}\right) v$ is well-defined and is valued in $\mathcal{B}^{(m)}((u))$. (This is proven similarly.)

Thus, the formulas of Theorem 3.11.2 make sense.

Remark 3.11.3. Here is some of physicists' intuition for the right hand sides of the equations in Theorem 3.11.2. [Note: I (=Darij) don't fully understand it, so don't expect me to explain it well.]

Consider the quantum field $a(u) = \sum\limits_{j\in\mathbb{Z}} a_j u^{-j-1} \in U(\mathcal{A})\left[\left[u, u^{-1}\right]\right]$ defined in Section 3.3. Let us work on an informal level, and pretend that integration of series in $U(\mathcal{A})\left[\left[u, u^{-1}\right]\right]$ is well-defined and behaves similar to that of functions on $\mathbb{R}$. Then,

$\int a(u)\,du = -\sum\limits_{j\neq 0} \frac{a_j}{j} u^{-j} + a_0 \log u$. Exponentiating this **"in the normal ordering"** (this means we expand the series $\exp\left(-\sum\limits_{j\neq 0} \frac{a_j}{j} u^{-j} + a_0 \log u\right)$ and replace all products by their normal ordered versions, i. e., shovel all a_m with $m < 0$ to the left and all a_m with $m > 0$ to the right), we obtain

$$\begin{aligned}
: \exp\left(\int a(u)\,du\right) : &= : \exp\left(-\sum_{j\neq 0} \frac{a_j}{j} u^{-j} + a_0 \log u\right) : \\
&= \exp\left(\underbrace{-\sum_{j<0} \frac{a_j}{j} u^{-j}}_{=\sum\limits_{j>0} \frac{a_{-j}}{j} u^j}\right) \cdot \exp\left(a_0 \log u\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \\
&= \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(a_0 \log u\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right).
\end{aligned}$$

But for every $m \in \mathbb{Z}$, we have

$$\begin{aligned}
&\Gamma(u) \\
&= u^{m+1} z \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \qquad \text{(by Theorem 3.11.2)} \\
&= uz \cdot \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \underbrace{u^m}_{\substack{=\exp(m \log u) = \exp(a_0 \log u) \\ \text{(since } a_0 \text{ acts by } m \text{ on } \mathcal{B}^{(m)}, \\ \text{and thus } \exp(a_0 \log u) = \exp(m \log u) \text{ on } \mathcal{B}^{(m)})}} \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \\
&= uz \cdot \underbrace{\exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(a_0 \log u\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)}_{=:\exp\left(\int a(u)du\right):} \\
&= uz \cdot : \exp\left(\int a(u)\,du\right) : .
\end{aligned}$$

Since the right hand side of this equality does not depend on m, we thus have $\Gamma(u) = uz : \exp\left(\int a(u)\,du\right) :$.

Hence, we have rewritten half of the statement of Theorem 3.11.2 as the identity $\Gamma(u) = uz : \exp\left(\int a(u)\,du\right) :$ (which holds on all of $\mathcal{B}$). Similarly, the other half of Theorem 3.11.2 rewrites as the identity $\Gamma^*(u) = z^{-1} : \exp\left(-\int a(u)\,du\right) :$.

This is reminiscent of Euler's formula $y = c \exp\left(\int a(u)\,du\right)$ for the solution y of the differential equation $y' = ay$.

Before we can show Theorem 3.11.2, we state a lemma about the action of $\mathcal{A}$ on $\mathcal{B}$:

Lemma 3.11.4. For every $j \in \mathbb{Z}$, we have $[a_j, \Gamma(u)] = u^j \Gamma(u)$ and $[a_j, \Gamma^*(u)] = -u^j \Gamma^*(u)$.

Proof of Lemma 3.11.4. Let us prove the first formula. Let $j \in \mathbb{Z}$.
On the fermionic space $\mathcal{F}$, the element $a_j \in \mathcal{A}$ acts as

$$\begin{aligned}
\widehat{\rho}\left(T^j\right) &= \sum_i \widehat{\rho}\left(E_{i,i+j}\right) \qquad \left(\text{since } T^j = \sum_{i\in\mathbb{Z}} E_{i,i+j}\right) \\
&= \sum_i \begin{cases} \xi_i \xi_{i+j}^* - 1, & \text{if } i = i+j \text{ and } i \leq 0, \\ \xi_i \xi_{i+j}^*, & \text{unless } i = i+j \text{ and } i \leq 0 \end{cases}
\end{aligned}$$

(since $\widehat{\rho}\left(E_{i,i+j}\right) = \begin{cases} \xi_i \xi_{i+j}^* - 1, & \text{if } i = i+j \text{ and } i \leq 0, \\ \xi_i \xi_{i+j}^*, & \text{unless } i = i+j \text{ and } i \leq 0 \end{cases}$ for every $i \in \mathbb{Z}$). Hence, on $\mathcal{F}$, we have

$$\begin{aligned}
[a_j, X(u)] &= \left[\sum_i \begin{cases} \xi_i \xi_{i+j}^* - 1, & \text{if } i = i+j \text{ and } i \leq 0, \\ \xi_i \xi_{i+j}^*, & \text{unless } i = i+j \text{ and } i \leq 0 \end{cases}, X(u)\right] \\
&= \sum_i \begin{cases} \left[\xi_i \xi_{i+j}^* - 1, X(u)\right], & \text{if } i = i+j \text{ and } i \leq 0, \\ \left[\xi_i \xi_{i+j}^*, X(u)\right], & \text{unless } i = i+j \text{ and } i \leq 0 \end{cases} \\
&= \sum_i \begin{cases} \left[\xi_i \xi_{i+j}^*, X(u)\right], & \text{if } i = i+j \text{ and } i \leq 0, \\ \left[\xi_i \xi_{i+j}^*, X(u)\right], & \text{unless } i = i+j \text{ and } i \leq 0 \end{cases} \\
&\qquad \left(\text{since } \left[\xi_i \xi_{i+j}^* - 1, X(u)\right] = \left[\xi_i \xi_{i+j}^*, X(u)\right]\right) \\
&= \sum_i \left[\xi_i \xi_{i+j}^*, X(u)\right] = \sum_i \left[\xi_i \xi_{i+j}^*, \sum_m \xi_m u^m\right] \qquad \left(\text{since } X(u) = \sum_m \xi_m u^m\right) \\
&= \sum_i \sum_m \underbrace{\left[\xi_i \xi_{i+j}^*, \xi_m\right]}_{\substack{=\delta_{m,i+j}\xi_i \\ \text{(this is easy to check)}}} u^m = \sum_i \sum_m \delta_{m,i+j} \xi_i u^m \\
&= \sum_m \xi_{m-j} u^m = u^j \underbrace{\sum_m \xi_{m-j} u^{m-j}}_{=X(u)} = u^j X(u).
\end{aligned}$$

Conjugating this equation by σ, we obtain $[a_j, \Gamma(u)] = u^j \Gamma(u)$. Similarly, we can prove $[a_j, \Gamma^*(u)] = -u^j \Gamma^*(u)$. Lemma 3.11.4 is proven.

Proof of Theorem 3.11.2. Define an element $\Gamma_+(u)$ of the $\mathbb{C}$-algebra $(\operatorname{End} \mathcal{B})\left[\left[u^{-1}\right]\right]$ by $\Gamma_+(u) = \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)$. Then,

$$[a_i, \Gamma_+(u)] = 0 \qquad \text{if } i \geq 0; \tag{136}$$

$$[a_i, \Gamma_+(u)] = u^i \Gamma_+(u) \qquad \text{if } i < 0. \tag{137}$$

In fact, (136) is trivial (because when $i \geq 0$, the element a_i commutes with a_j for every $j > 0$, and thus also commutes with $\exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)$). To prove (137), it is enough to show that $\left[a_i, \exp\left(-\frac{a_{-i}}{-i}u^i\right)\right] = u^i \exp\left(-\frac{a_{-i}}{-i}u^i\right)$ (since we can write $\Gamma_+(u)$ in the form

$$\Gamma_+(u) = \exp\left(-\sum_{j>0} \frac{a_j}{j}u^{-j}\right) = \prod_{j>0} \exp\left(-\frac{a_j}{j}u^{-j}\right),$$

and it is clear that a_i commutes with all terms $-\frac{a_j}{j}u^{-j}$ for $j \neq -i$). But this is easily checked using the fact that $[a_i, a_{-i}] = i$ and Lemma 3.1.1 (applied to $K = \mathbb{Q}$, $R = (\operatorname{End}\mathcal{B})[[u^{-1}]]$, $\alpha = a_i$, $\beta = a_{-i}$ and $P = \exp\left(-\frac{X}{-i}u^i\right)$). This completes the proof of (137).

Since $\Gamma_+(u)$ is an invertible power series in $(\operatorname{End}\mathcal{B})[[u^{-1}]]$ (because the constant term of $\Gamma_+(u)$ is 1), it makes sense to speak of the power series $\Gamma_+(u)^{-1} \in (\operatorname{End}\mathcal{B})[[u^{-1}]]$. From (136) and (137), we can derive the formulas

$$\left[a_i, \Gamma_+(u)^{-1}\right] = 0 \qquad \text{if } i \geq 0; \tag{138}$$

$$\left[a_i, \Gamma_+(u)^{-1}\right] = -u^i\Gamma_+(u)^{-1} \qquad \text{if } i < 0 \tag{139}$$

(using the standard fact that $[\alpha, \beta^{-1}] = -\beta^{-1}[\alpha, \beta]\beta^{-1}$ for any two elements α and β of a ring such that β is invertible).

Now define a map $\Delta(u) : \mathcal{B}^{(m)} \to \mathcal{B}^{(m)}((u))$ by $\Delta(u) = \Gamma(u)\Gamma_+(u)^{-1}z^{-1}$. Let us check why this definition makes sense:

- For any $v \in \mathcal{B}^{(m)}$, we have $z^{-1}v \in \mathcal{B}^{(m-1)}$, and the term $\Gamma_+(u)^{-1}z^{-1}v$ is well-defined and is valued in $\mathcal{B}^{(m-1)}[u^{-1}]$. [131]

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[131] *Proof.* Recall that $\mathcal{A}$ is a $\mathbb{Z}$-graded Lie algebra, and that $\mathcal{B}$ is a $\mathbb{Z}$-graded $\mathcal{A}$-module concentrated in nonpositive degrees. Let us (for this single proof!) change the $\mathbb{Z}$-gradings on both $\mathcal{A}$ and $\mathcal{B}$ to their inverses (i. e., switch $\mathcal{A}[N]$ with $\mathcal{A}[-N]$ for every $N \in \mathbb{Z}$, and switch $\mathcal{B}[N]$ with $\mathcal{B}[-N]$ for every $N \in \mathbb{Z}$); then, $\mathcal{A}$ remains still a $\mathbb{Z}$-graded Lie algebra, but $\mathcal{B}$ is now a $\mathbb{Z}$-graded $\mathcal{A}$-module concentrated in nonnegative degrees. Moreover, $\mathcal{B}$ is actually a $\mathbb{Z}$-graded $\operatorname{End}_{\mathrm{hg}}\mathcal{B}$-module concentrated in nonnegative degrees.

The power series $\sum\limits_{j>0} \frac{a_j}{j}u^{-j} \in (\operatorname{End}_{\mathrm{hg}}\mathcal{B})\left[\left[u^{-1}\right]\right]$ is now equigraded (since our modified grading on $\mathcal{A}$ has the property that $\deg(a_j) = -j$), so that the power series $\exp\left(\sum\limits_{j>0} \frac{a_j}{j}u^{-j}\right) \in (\operatorname{End}_{\mathrm{hg}}\mathcal{B})\left[\left[u^{-1}\right]\right]$ is equigraded as well (because a consequence of Proposition 3.3.10 **(b)** is that whenever the exponential of an equigraded power series is well-defined, this exponential is also equigraded). Since

$$\Gamma_+(u)^{-1} = \left(\exp\left(-\sum_{j>0} \frac{a_j}{j}u^{-j}\right)\right)^{-1} = \exp\left(\sum_{j>0} \frac{a_j}{j}u^{-j}\right)$$

(since Corollary 3.1.5 (applied to $R = (\operatorname{End}\mathcal{B})\left[\left[u^{-1}\right]\right]$, $I =$ (the ideal of R consisting of all power series with constant term 1), and $\gamma = -\sum\limits_{j>0} \frac{a_j}{j}u^{-j}$) yields $\left(\exp\left(\sum\limits_{j>0} \frac{a_j}{j}u^{-j}\right)\right) \cdot \left(\exp\left(-\sum\limits_{j>0} \frac{a_j}{j}u^{-j}\right)\right) = 1$), this rewrites as follows: The power series $\Gamma_+(u)^{-1} \in (\operatorname{End}_{\mathrm{hg}}\mathcal{B})\left[\left[u^{-1}\right]\right]$ is equigraded.

Therefore, Proposition 3.3.11 **(c)** (applied to $\operatorname{End}_{\mathrm{hg}}\mathcal{B}$, $\mathcal{B}$, $\Gamma_+(u)^{-1}$ and $z^{-1}v$ instead of A, M, f and x) yields that $\Gamma_+(u)^{-1}z^{-1}v$ is a well-defined element of $\mathcal{B}^{(m-1)}\left[u^{-1}\right]$, qed.

* For any $v \in \mathcal{B}^{(m)}$, the term $\Gamma(u)\Gamma_+(u)^{-1} z^{-1}$ is well-defined and is valued in $\mathcal{B}^{(m)}((u))$. [132]

Since $[a_0, z] = z$ and $[a_i, z] = 0$ for all $i \neq 0$, we have

$$[a_i, \Delta(u)] = \begin{cases} 0, & \text{if } i \leq 0; \\ u^i \Delta(u), & \text{if } i > 0 \end{cases} \tag{140}$$

(due to (138), (139) and Lemma 3.11.4). In particular, $[a_i, \Delta(u)] = 0$ if $i \leq 0$. Thus, $\Delta(u)$ is a homomorphism of $\mathcal{A}_-$-modules, where $\mathcal{A}_-$ is the Lie subalgebra $\langle a_{-1}, a_{-2}, a_{-3}, ...\rangle$ of $\mathcal{A}$. (Of course, this formulation means that every term of the formal power series $\Delta(u)$ is a homomorphism of $\mathcal{A}_-$-modules.)

Consider now the element z^m of $z^m \mathbb{C}[x_1, x_2, ...] = \mathcal{B}^{(m)} = \widetilde{F}_m$. Also, consider the element $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ...$ of $\wedge^{\frac{\infty}{2},m} V = \mathcal{F}^{(m)}$ defined in Definition 3.5.28. By the definition of σ_m, we have $\sigma_m(z^m) = \psi_m$. (In fact, z^m is what was denoted by 1 in Proposition 3.10.1.)

From Lemma 2.2.10, it is clear that the Fock module F is generated by 1 as an $\mathcal{A}_-$-module (since $\mathcal{A}_- = \langle a_{-1}, a_{-2}, a_{-3}, ...\rangle$). Since there exists an $\mathcal{A}_-$-module isomorphism $F \to \widetilde{F}$ which sends 1 to 1 (in fact, the map resc of Proposition 2.2.21 is such an isomorphism), this yields that $\widetilde{F}$ is generated by 1 as an $\mathcal{A}_-$-module. Since there exists an $\mathcal{A}_-$-module isomorphism $\widetilde{F} \to \widetilde{F}_m$ which sends 1 to z^m (in fact, multiplication by z^m is such an isomorphism), this yields that $\widetilde{F}_m$ is generated by z^m as an $\mathcal{A}_-$-module. Consequently, the m-th term of the power series $\Delta(u)$ is completely determined by $(\Delta(u))(z^m)$ (because we know that $\Delta(u)$ is a homomorphism of $\mathcal{A}_-$-modules). So let us compute $(\Delta(u))(z^m)$. Since $\Delta(u) : \mathcal{B}^{(m)} \to \mathcal{B}^{(m)}((u))$, we know that $(\Delta(u))(z^m)$ is an element of $\underbrace{\mathcal{B}^{(m)}}_{=z^m \widetilde{F}}((u)) = z^m \widetilde{F}((u))$. In other words, $(\Delta(u))(z^m)$ is z^m times a Laurent series in u whose coefficients are polynomials in $x_1, x_2, x_3, ...$. Denote this Laurent series by Q. Thus, $(\Delta(u))(z^m) = z^m Q$.

For every $i > 0$, we have

$$a_i \Delta(u) = \Delta(u) a_i + \underbrace{[a_i, \Delta(u)]}_{\substack{=u^i \Delta(u) \\ \text{(by (140))}}} = \Delta(u) a_i + u^i \Delta(u),$$

[132] *Proof.* We have just shown that $\Gamma_+(u)^{-1} z^{-1} v \in \mathcal{B}^{(m-1)}[u^{-1}]$. Thus, $\Gamma_+(u)^{-1} z^{-1} v \in \mathcal{B}^{(m-1)}[u^{-1}] \subseteq \mathcal{B}[u^{-1}] \subseteq \mathcal{B}[u, u^{-1}]$.

Recall that $\mathcal{A}$ is a $\mathbb{Z}$-graded Lie algebra, and that $\mathcal{B}$ and $\mathcal{F}$ are $\mathbb{Z}$-graded $\mathcal{A}$-modules concentrated in nonpositive degrees. Let us (for this single proof!) change the $\mathbb{Z}$-gradings on all of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{F}$ to their inverses (i. e., switch $\mathcal{A}[N]$ with $\mathcal{A}[-N]$ for every $N \in \mathbb{Z}$, and switch $\mathcal{B}[N]$ with $\mathcal{B}[-N]$ for every $N \in \mathbb{Z}$, and switch $\mathcal{F}[N]$ with $\mathcal{F}[-N]$ for every $N \in \mathbb{Z}$); then, $\mathcal{A}$ remains still a $\mathbb{Z}$-graded Lie algebra, but $\mathcal{B}$ and $\mathcal{F}$ now are $\mathbb{Z}$-graded $\mathcal{A}$-modules concentrated in nonnegative degrees. Moreover, $\mathcal{B}$ is actually a $\mathbb{Z}$-graded $\operatorname{End}_{\mathrm{hg}} \mathcal{B}$-module concentrated in nonnegative degrees, and $\mathcal{F}$ is a $\mathbb{Z}$-graded $\operatorname{End}_{\mathrm{hg}} \mathcal{F}$-module concentrated in nonnegative degrees.

It is easy to see (from the definition of $X(u)$) that $X(u) \in (\operatorname{End}_{\mathrm{hg}} \mathcal{F})[[u, u^{-1}]]$ is equigraded. As a consequence, $\Gamma(u) \in (\operatorname{End}_{\mathrm{hg}} \mathcal{B})[[u, u^{-1}]]$ is equigraded (since $\Gamma(u) = \sigma^{-1} \circ X(u) \circ \sigma$). Therefore, Proposition 3.3.11 **(b)** (applied to $\operatorname{End}_{\mathrm{hg}} \mathcal{B}$, $\mathcal{B}$, $\Gamma(u)$ and $\Gamma_+(u)^{-1} z^{-1} v$ instead of A, M, f and x) yields that $\Gamma(u)\Gamma_+(u)^{-1} z^{-1}$ is a well-defined element of $\mathcal{B}((u))$. This element actually lies in $\mathcal{B}^{(m)}((u))$ (since $\Gamma(u) : \mathcal{B}^{(m-1)} \to \mathcal{B}^{(m)}$), qed.

so that

$$\left(a_i\Delta\left(u\right)\right)\left(z^m\right)=\left(\Delta\left(u\right)a_i+u^i\Delta\left(u\right)\right)\left(z^m\right)=\Delta\left(u\right)\underbrace{a_iz^m}_{\substack{=0\\ \left(\text{since } a_i=\frac{\partial}{\partial x_i}\right)}}+u^i\underbrace{\left(\Delta\left(u\right)\right)\left(z^m\right)}_{=z^mQ}$$
$$=u^iz^mQ=z^mu^iQ.$$

Since $\left(a_i\Delta\left(u\right)\right)\left(z^m\right)=a_i\underbrace{\left(\left(\Delta\left(u\right)\right)\left(z^m\right)\right)}_{=z^mQ}=z^m\underbrace{a_i}_{=\frac{\partial}{\partial x_i}}Q=z^m\frac{\partial Q}{\partial x_i}$, this rewrites as $z^m\frac{\partial Q}{\partial x_i}=z^mu^iQ$. Hence, for every $i>0$, we have $\frac{\partial Q}{\partial x_i}=u^iQ$. Thus, we can write the formal Laurent series Q in the form $Q=f\left(u\right)\exp\left(\sum\limits_{j>0}x_ju^j\right)$ for some Laurent series $f\left(u\right)\in\mathbb{C}\left(\left(u\right)\right)$.[133] Thus,

$$\begin{aligned}&\left(\Delta\left(u\right)\right)\left(z^m\right)\\&=z^mQ=z^mf\left(u\right)\exp\left(\sum_{j>0}x_ju^j\right)\qquad\left(\text{since } Q=f\left(u\right)\exp\left(\sum_{j>0}x_ju^j\right)\right)\\&=f\left(u\right)\exp\left(\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\left(z^m\right)\qquad\left(\text{since each } \frac{a_{-j}}{j} \text{ acts as multiplication by } x_j \text{ on } \widetilde{F}\right).\end{aligned}$$

In other words, the two maps $\Delta\left(u\right)$ and $f\left(u\right)\exp\left(\sum\limits_{j>0}\frac{a_{-j}}{j}u^j\right)$ are equal on z^m. Since each of these two maps is an $\mathcal{A}_-$-module homomorphism[134], this yields that these two maps must be identical (because $\widetilde{F}_m$ is generated by z^m as an $\mathcal{A}_-$-module). In other words, $\Delta\left(u\right)=f\left(u\right)\exp\left(\sum\limits_{j>0}\frac{a_{-j}}{j}u^j\right)$. Since $\Delta\left(u\right)=\Gamma\left(u\right)\Gamma_+\left(u\right)^{-1}z^{-1}$, this becomes $\Gamma\left(u\right)\Gamma_+\left(u\right)^{-1}z^{-1}=f\left(u\right)\exp\left(\sum\limits_{j>0}\frac{a_{-j}}{j}u^j\right)$, so that

$$\begin{aligned}\Gamma\left(u\right)&=f\left(u\right)\exp\left(\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\cdot z\cdot\Gamma_+\left(u\right)=f\left(u\right)\exp\left(\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\cdot z\cdot\exp\left(-\sum_{j>0}\frac{a_j}{j}u^{-j}\right)\\&\qquad\left(\text{since } \Gamma_+\left(u\right)=\exp\left(-\sum_{j>0}\frac{a_j}{j}u^{-j}\right)\right)\\&=f\left(u\right)z\exp\left(\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\cdot\exp\left(-\sum_{j>0}\frac{a_j}{j}u^{-j}\right)\qquad(141)\end{aligned}$$

[133] This follows from Proposition 3.3.7, applied to $R=\mathbb{C}\left[u\right]$, $U=\mathbb{C}\left(\left(u\right)\right)$, $\left(\alpha_1,\alpha_2,\alpha_3,...\right)=\left(u^1,u^2,u^3,...\right)$ and $P=Q$.

[134] In fact, we know that $\Delta\left(u\right)$ is an $\mathcal{A}_-$-module homomorphism, and it is clear that $f\left(u\right)\exp\left(\sum\limits_{j>0}\frac{a_{-j}}{j}u^j\right)$ is an $\mathcal{A}_-$-module homomorphism because $\mathcal{A}_-$ is an abelian Lie algebra.

on $\mathcal{B}^{(m)}$. It remains to show that $f(u) = u^{m+1}$.

In order to do this, we recall that

$$(\Gamma(u))(z^m) = f(u)\, z \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \underbrace{\exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)(z^m)}_{\substack{=z^m \\ \text{(because } a_j(z^m)=0 \text{ for every } j>0)}} \qquad \text{(by (141))}$$

$$= f(u)\, z \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right)(z^m) = f(u)\, z \exp\left(\sum_{j>0} x_j u^j\right) z^m$$

$$\left(\text{since each } \frac{a_{-j}}{j} \text{ acts as multiplication by } x_j \text{ on } \widetilde{F}\right)$$

$$= f(u) \exp\left(\sum_{j>0} x_j u^j\right) z^{m+1}.$$

On the other hand, back on the fermionic side, for the vector $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge \ldots$, we have

$$(X(u))\,\psi_m = \sum_{n\in\mathbb{Z}} \widehat{v_n}(\psi_m)\, u^n \qquad \left(\text{since } X(u) = \sum_{n\in\mathbb{Z}} \underbrace{\xi_n}_{=\widehat{v_n}} u^n = \sum_{n\in\mathbb{Z}} \widehat{v_n} u^n\right)$$

$$= \sum_{\substack{n\in\mathbb{Z};\\ n\leq m}} \underbrace{\widehat{v_n}(\psi_m)}_{\substack{=0 \\ \text{(since } n\leq m\text{, so that } v_n \\ \text{appears in } v_m\wedge v_{m-1}\wedge v_{m-2}\wedge\ldots=\psi_m)}} u^n + \sum_{\substack{n\in\mathbb{Z};\\ n\geq m+1}} \widehat{v_n}(\psi_m)\, u^n = \sum_{\substack{n\in\mathbb{Z};\\ n\geq m+1}} \widehat{v_n}(\psi_m)\, u^n.$$

Thus, $\sigma^{-1}((X(u))\,\psi_m) = \sigma^{-1}\left(\sum\limits_{\substack{n\in\mathbb{Z};\\ n\geq m+1}} \widehat{v_n}(\psi_m)\, u^n\right)$. Compared with

$$\sigma^{-1}\left((X(u)) \underbrace{\psi_m}_{=\sigma(z^m)}\right) = \sigma^{-1}((X(u))(\sigma(z^m))) = \underbrace{\left(\sigma^{-1}\circ X(u)\circ\sigma\right)}_{=\Gamma(u)}(z^m) = (\Gamma(u))(z^m)$$

$$= f(u) \exp\left(\sum_{j>0} x_j u^j\right) z^{m+1},$$

this yields $\sigma^{-1}\left(\sum\limits_{\substack{n\in\mathbb{Z};\\ n\geq m+1}} \widehat{v_n}(\psi_m)\, u^n\right) = f(u) \exp\left(\sum\limits_{j>0} x_j u^j\right) z^{m+1}$, so that

$$\sigma\left(f(u) \exp\left(\sum_{j>0} x_j u^j\right) z^{m+1}\right) = \sum_{\substack{n\in\mathbb{Z};\\ n\geq m+1}} \widehat{v_n}(\psi_m)\, u^n. \qquad (142)$$

We want to find $f(u)$ by comparing the sides of this equation. In order to do this, we recall that each space $\mathcal{B}^{(i)}$ is graded; hence, $\mathcal{B}$ (being the direct sum of the $\mathcal{B}^{(i)}$) is also graded (by taking the direct sum of all the gradings). Also, each space $\mathcal{F}^{(i)}$ is graded; hence, $\mathcal{F}$ (being the direct sum of the $\mathcal{F}^{(i)}$) is also graded (by taking the direct sum of all the gradings). Since each σ_m is a graded map, the direct sum $\sigma = \bigoplus_{m\in\mathbb{Z}} \sigma_m$ is also graded. Therefore,

$$\begin{aligned} &\sigma\left(\text{0-th homogeneous component of } f(u)\exp\left(\sum_{j>0} x_j u^j\right) z^{m+1}\right) \\ &= \left(\text{0-th homogeneous component of } \sigma\left(f(u)\exp\left(\sum_{j>0} x_j u^j\right) z^{m+1}\right)\right) \\ &= \left(\text{0-th homogeneous component of } \sum_{\substack{n\in\mathbb{Z};\\ n\geq m+1}} \widehat{v_n}(\psi_m) u^n\right) \end{aligned} \tag{143}$$

(by (142)). Now, for every $n \in \mathbb{Z}$ satisfying $n \geq m+1$, the element $\widehat{v_n}(\psi_m)$ equals $v_n \wedge v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ...$, and thus has degree $-(n-m-1)$. Hence, for every non-positive $i \in \mathbb{Z}$, the i-th homogeneous component of the sum $\sum_{\substack{n\in\mathbb{Z};\\ n\geq m+1}} \widehat{v_n}(\psi_m) u^n \in \mathcal{F}$ is $\widehat{v_{m+1-i}}(\psi_m) u^{m+1-i}$. In particular, the 0-th homogeneous component of $\sum_{\substack{n\in\mathbb{Z};\\ n\geq m+1}} \widehat{v_n}(\psi_m) u^n$ is $\widehat{v_{m+1}}(\psi_m) u^{m+1} = \psi_{m+1} u^{m+1}$ (since $\widehat{v_{m+1}}(\psi_m) = v_{m+1} \wedge v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ... = \psi_{m+1}$). Therefore, (143) becomes

$$\sigma\left(\text{0-th homogeneous component of } f(u)\exp\left(\sum_{j>0} x_j u^j\right) z^{m+1}\right) = \psi_{m+1} u^{m+1}. \tag{144}$$

On the other hand, the 0-th homogeneous component of the element $f(u)\exp\left(\sum_{j>0} x_j u^j\right) z^{m+1} \in \mathcal{B}$ is clearly $f(u) z^{m+1}$ (because $\exp\left(\sum_{j>0} x_j u^j\right) = 1 + (\text{terms involving at least one } x_j)$, and every x_j lowers the degree). Thus, (144) becomes $\sigma\left(f(u) z^{m+1}\right) = \psi_{m+1} u^{m+1}$. Since $\sigma\left(f(u) z^{m+1}\right) = f(u) \underbrace{\sigma\left(z^{m+1}\right)}_{=\psi_{m+1}} = f(u) \psi_{m+1}$, this rewrites as $f(u) \psi_{m+1} = \psi_{m+1} u^{m+1}$, so that $f(u) = u^{m+1}$. Hence, (141) becomes

$$\begin{aligned} \Gamma(u) &= \underbrace{f(u)}_{=u^{m+1}} z \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \\ &= u^{m+1} z \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \qquad \text{on } \mathcal{B}^{(m)}. \end{aligned}$$

This proves one of the equalities of Theorem 3.11.2. The other is proven similarly.

Theorem 3.11.2 is proven.

Corollary 3.11.5. Let $m \in \mathbb{Z}$. On $\mathcal{B}^{(m)}$, we have

$$\rho\left(\sum_{(i,j)\in\mathbb{Z}^2} u^i v^{-j} E_{i,j}\right) = \sum_{(i,j)\in\mathbb{Z}^2} u^i v^{-j} \xi_i \xi_j^* = X(u) X^*(v),$$

thus

$$\begin{aligned}
&\sigma^{-1} \circ \rho\left(\sum_{(i,j)\in\mathbb{Z}^2} u^i v^{-j} E_{i,j}\right) \circ \sigma \\
&= \sigma^{-1} \circ X(u) X^*(v) \circ \sigma = \Gamma(u) \Gamma^*(v) \\
&= \frac{1}{1-\dfrac{v}{u}} \cdot \left(\frac{u}{v}\right)^m \exp\left(\sum_{j>0} \frac{u^j - v^j}{j} a_{-j}\right) \exp\left(-\sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_j\right)
\end{aligned}$$

as linear maps from $\mathcal{B}^{(m)}$ to $\mathcal{B}^{(m)}((u,v))$.

Remark 3.11.6. It must be pointed out that the term

$$\frac{1}{1-\dfrac{v}{u}} \cdot \left(\frac{u}{v}\right)^m \exp\left(\sum_{j>0} \frac{u^j - v^j}{j} a_{-j}\right) \exp\left(-\sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_j\right)$$

only makes sense as a map from $\mathcal{B}^{(m)}$ to $\mathcal{B}^{(m)}((u,v))$, but not (for example) as a map from $\mathcal{B}^{(m)}$ to $\mathcal{B}^{(m)}[[u,u^{-1},v,v^{-1}]]$ or as an element of $\left(\operatorname{End}\left(\mathcal{B}^{(m)}\right)\right)[[u,u^{-1},v,v^{-1}]]$. Indeed, $1-\dfrac{v}{u}$ is a zero-divisor in $\mathbb{C}[[u,u^{-1},v,v^{-1}]]$ (since $\left(1-\dfrac{v}{u}\right)\sum\limits_{k\in\mathbb{Z}}\left(\dfrac{v}{u}\right)^k = 0$), so it does not make sense, for example, to multiply a generic element of $\mathcal{B}^{(m)}[[u,u^{-1},v,v^{-1}]]$ by $\dfrac{1}{1-\dfrac{v}{u}}$. An element of $\mathcal{B}^{(m)}((u,v))$ needs not always be a multiple of $1-\dfrac{v}{u}$, but at least when it is, the quotient is unique.

The importance of Corollary 3.11.5 lies in the fact that it gives an easy way to compute the ρ-action of $\mathfrak{gl}_\infty$ on $\mathcal{B}^{(m)}$: In fact, for any $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$, the coefficient of $\sigma^{-1} \circ \rho\left(\sum\limits_{i,j} u^i v^{-j} E_{i,j}\right) \circ \sigma \in \left(\operatorname{End}\left(\mathcal{B}^{(m)}\right)\right)[[u,u^{-1},v,v^{-1}]]$ before $u^p v^{-q}$ is $\sigma^{-1} \circ \rho(E_{p,q}) \circ \sigma$, and this is exactly the action of $E_{p,q}$ on $\mathcal{B}^{(m)}$ obtained by transferring the action ρ of $\mathfrak{gl}_\infty$ on $\mathcal{F}^{(m)}$ to $\mathcal{B}^{(m)}$.

Proof of Corollary 3.11.5. First of all, we clearly have

$$\rho\left(\sum_{(i,j)\in\mathbb{Z}^2} u^i v^{-j} E_{i,j}\right) = \sum_{(i,j)\in\mathbb{Z}^2} u^i v^{-j} \underbrace{\rho\left(E_{i,j}\right)}_{=\xi_i \xi_j^*} = \sum_{(i,j)\in\mathbb{Z}^2} u^i v^{-j} \xi_i \xi_j^*$$
$$= \underbrace{\left(\sum_{i\in\mathbb{Z}} \xi_i u^i\right)}_{=\sum\limits_{n\in\mathbb{Z}} \xi_n u^n = X(u)} \underbrace{\sum_{j\in\mathbb{Z}} \xi_j^* v^{-j}}_{=\sum\limits_{n\in\mathbb{Z}} \xi_n^* v^{-n} = X^*(v)} = X\left(u\right) X^*\left(v\right),$$

so that

$$\sigma^{-1}\circ\rho\left(\sum_{(i,j)\in\mathbb{Z}^2} u^i v^{-j} E_{i,j}\right)\circ\sigma$$
$$= \sigma^{-1}\circ X\left(u\right) X^*\left(v\right)\circ\sigma = \Gamma\left(u\right)\Gamma^*\left(v\right).$$

It thus only remains to prove that

$$\Gamma\left(u\right)\Gamma^*\left(v\right) = \frac{1}{1-\dfrac{v}{u}}\cdot\left(\frac{u}{v}\right)^m \exp\left(\sum_{j>0}\frac{u^j - v^j}{j} a_{-j}\right)\exp\left(-\sum_{j>0}\frac{u^{-j}-v^{-j}}{j} a_j\right).$$

By Theorem 3.11.2 (applied to $m-1$ instead of m), we have

$$\Gamma\left(u\right) = u^m z \exp\left(\sum_{j>0}\frac{a_{-j}}{j} u^j\right)\cdot\exp\left(-\sum_{j>0}\frac{a_j}{j} u^{-j}\right) \qquad \text{on } \mathcal{B}^{(m-1)}.$$

By Theorem 3.11.2, we have

$$\Gamma^*\left(v\right) = v^{-m} z^{-1} \exp\left(-\sum_{j>0}\frac{a_{-j}}{j} v^j\right)\cdot\exp\left(\sum_{j>0}\frac{a_j}{j} v^{-j}\right) \qquad \text{on } \mathcal{B}^{(m)}.$$

Multiplying these two equalities, we obtain

$$\Gamma\left(u\right)\Gamma^*\left(v\right) = u^m v^{-m}\cdot\exp\left(\sum_{j>0}\frac{u^j}{j} a_{-j}\right)\exp\left(-\sum_{j>0}\frac{u^{-j}}{j} a_j\right)$$
$$\cdot\exp\left(-\sum_{j>0}\frac{v^j}{j} a_{-j}\right)\exp\left(\sum_{j>0}\frac{v^{-j}}{j} a_j\right) \qquad \text{on } \mathcal{B}^{(m)}$$

(since multiplication by z commutes with each of $\exp\left(\sum\limits_{j>0}\frac{a_{-j}}{j} u^j\right)$ and $\exp\left(-\sum\limits_{j>0}\frac{a_j}{j} u^{-j}\right)$). We wish to "switch" the second and the third exponential on the right hand side of this equation (although they don't commute). To do so, we notice that each of $-\sum\limits_{j>0}\frac{u^{-j}}{j} a_j$

and $-\sum\limits_{j>0} \dfrac{v^j}{j} a_{-j}$ lies in the ring $\left(\operatorname{End}\left(\mathcal{B}^{(m)}\right)\right)[[u^{-1}, v]]$ [135]. Let I be the ideal of the ring $\left(\operatorname{End}\left(\mathcal{B}^{(m)}\right)\right)[[u^{-1}, v]]$ consisting of all power series with constant term 0. This ring $\left(\operatorname{End}\left(\mathcal{B}^{(m)}\right)\right)[[u^{-1}, v]]$ is a $\mathbb{Q}$-algebra and is complete and Hausdorff with respect to the I-adic topology. Let $\alpha = -\sum\limits_{j>0} \dfrac{u^{-j}}{j} a_j$ and $\beta = -\sum\limits_{j>0} \dfrac{v^j}{j} a_{-j}$. Clearly, both α and β lie in I. Also,

$$\begin{aligned}
[\alpha, \beta] &= \left[-\sum_{j>0} \frac{u^{-j}}{j} a_j, -\sum_{j>0} \frac{v^j}{j} a_{-j} \right] = \sum_{j>0} \sum_{k>0} \frac{u^{-j} v^k}{jk} \underbrace{[a_j, a_{-k}]}_{=\delta_{j,k} j} \\
&= \sum_{j>0} \sum_{k>0} \frac{u^{-j} v^k}{jk} \delta_{j,k} j = \sum_{j>0} \frac{u^{-j} v^j}{jj} j = \sum_{j>0} \frac{1}{j} \left(\frac{v}{u}\right)^j = -\log\left(1 - \frac{v}{u}\right)
\end{aligned}$$

is a power series with coefficients in $\mathbb{Q}$, and thus lies in the center of $\left(\operatorname{End}\left(\mathcal{B}^{(m)}\right)\right)[[u^{-1}, v]]$, and hence commutes with each of α and β. Thus, we can apply Lemma 3.1.9 to $K = \mathbb{Q}$ and $R = \left(\operatorname{End}\left(\mathcal{B}^{(m)}\right)\right)[[u^{-1}, v]]$, and obtain $(\exp \alpha) \cdot (\exp \beta) = (\exp \beta) \cdot (\exp \alpha) \cdot$

[135] This is the ring of formal power series in the indeterminates u^{-1} and v over the ring $\operatorname{End}\left(\mathcal{B}^{(m)}\right)$. Note that $\operatorname{End}\left(\mathcal{B}^{(m)}\right)$ is non-commutative, but the ring of formal power series is still defined in the same way as over commutative rings. The indeterminates u^{-1} and v themselves commute with each other and with each element of $\operatorname{End}\left(\mathcal{B}^{(m)}\right)$.

$(\exp[\alpha,\beta])$. Hence,

$$\begin{aligned}
&\Gamma(u)\Gamma^*(v) \\
&= u^m v^{-m} \cdot \exp\left(\sum_{j>0} \frac{u^j}{j} a_{-j}\right) \underbrace{\exp\left(-\sum_{j>0} \frac{u^{-j}}{j} a_j\right)}_{=\alpha} \\
&\qquad \cdot \underbrace{\exp\left(-\sum_{j>0} \frac{v^j}{j} a_{-j}\right)}_{=\beta} \exp\left(\sum_{j>0} \frac{v^{-j}}{j} a_j\right) \\
&= u^m v^{-m} \cdot \exp\left(\sum_{j>0} \frac{u^j}{j} a_{-j}\right) \cdot \underbrace{(\exp\alpha)\cdot(\exp\beta)}_{=(\exp\beta)\cdot(\exp\alpha)\cdot(\exp[\alpha,\beta])} \cdot \exp\left(\sum_{j>0} \frac{v^{-j}}{j} a_j\right) \\
&= \underbrace{u^m v^{-m}}_{=\left(\frac{u}{v}\right)^m} \cdot \exp\left(\sum_{j>0} \frac{u^j}{j} a_{-j}\right) \cdot \exp \underbrace{\beta}_{=-\sum\limits_{j>0} \frac{v^j}{j} a_{-j}} \\
&\qquad \cdot \exp \underbrace{\alpha}_{=-\sum\limits_{j>0} \frac{u^{-j}}{j} a_j} \cdot \underbrace{\exp[\alpha,\beta]}_{\substack{=\frac{1}{1-\frac{v}{u}} \\ \left(\text{since } [\alpha,\beta] = -\log\left(1-\frac{v}{u}\right)\right)}} \cdot \exp\left(\sum_{j>0} \frac{v^{-j}}{j} a_j\right) \\
&= \left(\frac{u}{v}\right)^m \cdot \exp\left(\sum_{j>0} \frac{u^j}{j} a_{-j}\right) \cdot \exp\left(-\sum_{j>0} \frac{v^j}{j} a_{-j}\right) \\
&\qquad \cdot \exp\left(-\sum_{j>0} \frac{u^{-j}}{j} a_j\right) \cdot \frac{1}{1-\frac{v}{u}} \cdot \exp\left(\sum_{j>0} \frac{v^{-j}}{j} a_j\right)
\end{aligned}$$

$$= \frac{1}{1 - \frac{v}{u}} \cdot \left(\frac{u}{v}\right)^m \cdot \underbrace{\exp\left(\sum_{j>0} \frac{u^j}{j} a_{-j}\right) \cdot \exp\left(-\sum_{j>0} \frac{v^j}{j} a_{-j}\right)}_{\substack{= \exp\left(\sum_{j>0} \frac{u^j - v^j}{j} a_{-j}\right) \\ \text{(by Theorem 3.1.4, applied to } R = \operatorname{End}\left(\mathcal{B}^{(m)}\right)[[u,v]], \\ I = \text{(the ideal of } R \text{ consisting of all power series with constant term 0)}, \\ \alpha = \sum_{j>0} \frac{u^j}{j} a_{-j} \text{ and } \beta = -\sum_{j>0} \frac{v^j}{j} a_{-j})}}$$

$$\cdot \underbrace{\exp\left(-\sum_{j>0} \frac{u^{-j}}{j} a_j\right) \cdot \exp\left(\sum_{j>0} \frac{v^{-j}}{j} a_j\right)}_{\substack{= \exp\left(-\sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_j\right) \\ \text{(by Theorem 3.1.4, applied to } R = \operatorname{End}\left(\mathcal{B}^{(m)}\right)\left[\left[u^{-1},v^{-1}\right]\right], \\ I = \text{(the ideal of } R \text{ consisting of all power series with constant term 0)}, \\ \alpha = -\sum_{j>0} \frac{u^{-j}}{j} a_j \text{ and } \beta = \sum_{j>0} \frac{v^{-j}}{j} a_j)}}$$

$$= \frac{1}{1 - \frac{v}{u}} \cdot \left(\frac{u}{v}\right)^m \exp\left(\sum_{j>0} \frac{u^j - v^j}{j} a_{-j}\right) \exp\left(-\sum_{j>0} \frac{u^{-j} - v^{-j}}{j} a_j\right).$$

This proves Corollary 3.11.5.

3.12. Expliciting σ^{-1} using Schur polynomials

Next we are going to give an explicit (in as far as one can do) formula for $\sigma^{-1}\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right)$ for an elementary semiinfinite wedge $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ....$ Before we do so, we need to introduce the notion of *Schur polynomials*. We first define *elementary Schur polynomials*:

3.12.1. Schur polynomials

Convention 3.12.1. In the following, we let x denote the countable family of indeterminates $(x_1, x_2, x_3, ...)$. Thus, for any polynomial P in countably many indeterminates, we write $P(x)$ for $P(x_1, x_2, x_3, ...)$.

Definition 3.12.2. For every $k \in \mathbb{N}$, let $S_k \in \mathbb{Q}[x_1, x_2, x_3, ...]$ be the coefficient of the power series $\exp\left(\sum_{i \geq 1} x_i z^i\right) \in \mathbb{Q}[x_1, x_2, x_3, ...][[z]]$ before z^k. Then, obviously,

$$\sum_{k \geq 0} S_k(x) z^k = \exp\left(\sum_{i \geq 1} x_i z^i\right). \qquad (145)$$

For example, $S_0(x) = 1$, $S_1(x) = x_1$, $S_2(x) = \frac{x_1^2}{2} + x_2$, $S_3(x) = \frac{x_1^3}{6} + x_1 x_2 + x_3$.

Note that the polynomials S_k that we just defined are **not** symmetric polynomials. Instead, they "represent" the complete symmetric functions in terms of the $\frac{p_i}{i}$ (where p_i are the power sums). Here is what exactly we mean by this:

Definition 3.12.3. Let $N \in \mathbb{N}$, and let y denote a family of N indeterminates $(y_1, y_2, ..., y_N)$. Thus, for any polynomial P in N indeterminates, we write $P(y)$ for $P(y_1, y_2, ..., y_N)$.

Definition 3.12.4. For every $k \in \mathbb{N}$, define the *k-th complete symmetric function* h_k in the variables $y_1, y_2, ..., y_N$ by $h_k(y_1, y_2, ..., y_N) = \sum\limits_{\substack{p_1, p_2, ..., p_N \in \mathbb{N}; \\ p_1+p_2+...+p_N=k}} y_1^{p_1} y_2^{p_2} ... y_N^{p_N}$.

Proposition 3.12.5. In the ring $\mathbb{Q}[y_1, y_2, ..., y_N][[z]]$, we have

$$\sum_{k \geq 0} z^k h_k(y) = \prod_{j=1}^{N} \frac{1}{1 - zy_j}.$$

Proof of Proposition 3.12.5. For every $j \in \{1, 2, ..., N\}$, the sum formula for the geometric series yields $\frac{1}{1 - zy_j} = \sum\limits_{p \in \mathbb{N}} (zy_j)^p = \sum\limits_{p \in \mathbb{N}} y_j^p z^p$. Hence,

$$\begin{aligned}
\prod_{j=1}^{N} \frac{1}{1 - zy_j} &= \prod_{j=1}^{N} \left(\sum_{p \in \mathbb{N}} y_j^p z^p \right) = \sum_{p_1, p_2, ..., p_N \in \mathbb{N}} \underbrace{\left(y_1^{p_1} z^{p_1}\right)\left(y_2^{p_2} z^{p_2}\right) ... \left(y_N^{p_N} z^{p_N}\right)}_{= y_1^{p_1} y_2^{p_2} ... y_N^{p_N} z^{p_1+p_2+...+p_N}} \\
&= \sum_{p_1, p_2, ..., p_N \in \mathbb{N}} y_1^{p_1} y_2^{p_2} ... y_N^{p_N} z^{p_1+p_2+...+p_N} = \sum_{k \geq 0} \underbrace{\sum_{\substack{p_1, p_2, ..., p_N \in \mathbb{N}; \\ p_1+p_2+...+p_N=k}} y_1^{p_1} y_2^{p_2} ... y_N^{p_N}}_{= h_k(y_1, y_2, ..., y_N) = h_k(y)} z^k \\
&= \sum_{k \geq 0} h_k(y) z^k = \sum_{k \geq 0} z^k h_k(y).
\end{aligned}$$

This proves Proposition 3.12.5.

Definition 3.12.6. Let $N \in \mathbb{N}$. We define a map $\operatorname{PSE}_N : \mathbb{C}[x_1, x_2, x_3, ...] \to \mathbb{C}[y_1, y_2, ..., y_N]$ as follows: For every polynomial $P \in [x_1, x_2, x_3, ...]$, let $\operatorname{PSE}_N(P)$ be the result of substituting $x_j = \frac{y_1^j + y_2^j + ... + y_N^j}{j}$ for all positive integers j into the polynomial P.

Clearly, this map PSE_N is a $\mathbb{C}$-algebra homomorphism.

(The notation PSE_N is mine and has been chosen as an abbreviation for "Power Sum Evaluation in N variables".)

Proposition 3.12.7. For every $N \in \mathbb{N}$, we have $h_k(y) = \operatorname{PSE}_N(S_k(x))$ for each $k \in \mathbb{N}$.

Proof of Proposition 3.12.7. Fix $N \in \mathbb{N}$. We know that $\sum\limits_{k\geq 0} S_k(x) z^k = \exp\left(\sum\limits_{i\geq 1} x_i z^i\right)$. Since PSE_N is a $\mathbb{C}$-algebra homomorphism, this yields

$$\begin{aligned}
\sum_{k\geq 0} \operatorname{PSE}_N(S_k(x)) z^k &= \exp\left(\sum_{i\geq 1} \operatorname{PSE}_N(x_i) z^i\right) = \exp\left(\sum_{i\geq 1}\sum_{j=1}^{N} \frac{y_j^i}{i} z^i\right) \\
&\qquad \left(\text{since } \operatorname{PSE}_N(x_i) = \frac{y_1^i + y_2^i + ... + y_N^i}{i} = \sum_{j=1}^{N} \frac{y_j^i}{i}\right) \\
&= \exp\left(\sum_{j=1}^{N}\sum_{i\geq 1} \frac{y_j^i}{i} z^i\right) = \prod_{j=1}^{N} \exp\left(\sum_{i\geq 1} \frac{y_j^i}{i} z^i\right) \\
&= \prod_{j=1}^{N} \exp\underbrace{\left(\sum_{i\geq 1} \frac{y_j^i z^i}{i}\right)}_{=-\log(1-y_j z)} = \prod_{j=1}^{N} \underbrace{\exp\left(-\log\left(1-y_j z\right)\right)}_{=\frac{1}{1-y_j z}=\frac{1}{1-zy_j}} \\
&= \prod_{j=1}^{N} \frac{1}{1-zy_j} = \sum_{k\geq 0} z^k h_k(y) \qquad \text{(by Proposition 3.12.5)}.
\end{aligned}$$

By comparing coefficients in this equality, we conclude that $\operatorname{PSE}_N(S_k(x)) = h_k(y)$ for each $k \in \mathbb{N}$. Proposition 3.12.7 is proven.

Definition 3.12.8. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ be a partition, so that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 0$ are integers.

We define $S_\lambda(x) \in \mathbb{Q}[x_1, x_2, x_3, ...]$ to be the polynomial

$$\begin{aligned}
&\det\begin{pmatrix}
S_{\lambda_1}(x) & S_{\lambda_1+1}(x) & S_{\lambda_1+2}(x) & ... & S_{\lambda_1+m-1}(x) \\
S_{\lambda_2-1}(x) & S_{\lambda_2}(x) & S_{\lambda_2+1}(x) & ... & S_{\lambda_2+m-2}(x) \\
S_{\lambda_3-2}(x) & S_{\lambda_3-1}(x) & S_{\lambda_3}(x) & ... & S_{\lambda_3+m-3}(x) \\
... & ... & ... & ... & ... \\
S_{\lambda_m-m+1}(x) & S_{\lambda_m-m+2}(x) & S_{\lambda_m-m+3}(x) & ... & S_{\lambda_m}(x)
\end{pmatrix} \\
&= \det\left(\left(S_{\lambda_i+j-i}(x)\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right),
\end{aligned}$$

where S_j denotes 0 if $j < 0$. (Note that this does not depend on trailing zeroes in the partition; in other words, $S_{(\lambda_1,\lambda_2,...,\lambda_m)}(x) = S_{(\lambda_1,\lambda_2,...,\lambda_m,0,0,...,0)}(x)$ for any number of zeroes. This is because any nonnegative integers m and ℓ, any $m \times m$-matrix A, any $m \times \ell$-matrix B and any upper unitriangular $\ell \times \ell$-matrix C satisfy $\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A$.)

We refer to $S_\lambda(x)$ as the *bosonic Schur polynomial corresponding to the partition* λ.

To a reader acquainted with the Schur polynomials of combinatorics (and representation theory of symmetric groups), this definition may look familiar, but it should be reminded that our polynomial $S_\lambda(x)$ is **not a symmetric function per se** (this is why we call it "bosonic Schur polynomial" and not just simply "Schur polynomial");

instead, it can be made into a symmetric function – and this will, indeed, be the λ-Schur polynomial known from combinatorics – by substituting for each x_j the term $\dfrac{(j\text{-th power sum symmetric function})}{j}$. We will prove this in Proposition 3.12.10 (albeit only for finitely many variables). Let us first formulate one of the many definitions of Schur polynomials from combinatorics:

Definition 3.12.9. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ be a partition, so that $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_m \geq 0$ are integers. We define λ_ℓ to mean 0 for all integers $\ell > m$; thus, we obtain a nonincreasing sequence $(\lambda_1, \lambda_2, \lambda_3, ...)$ of nonnegative integers.

Let $N \in \mathbb{N}$.

The so-called λ-*Schur module* V_λ *over* $\mathrm{GL}(N)$ is defined to be the $\mathrm{GL}(N)$-module $\mathrm{Hom}_{S_n}\left(S^\lambda, \left(\mathbb{C}^N\right)^{\otimes n}\right)$, where n denotes the number $\lambda_1 + \lambda_2 + ... + \lambda_m$ and S^λ denotes the Specht module over the symmetric group S_n corresponding to the partition λ. (The $\mathrm{GL}(N)$-module structure on $\mathrm{Hom}_{S_n}\left(S^\lambda, \left(\mathbb{C}^N\right)^{\otimes n}\right)$ is obtained from the $\mathrm{GL}(N)$-module structure on $\mathbb{C}^N$.) This λ-Schur module V_λ is not only a $\mathrm{GL}(N)$-module, but also a $\mathfrak{gl}(N)$-module. If $\lambda_{N+1} = 0$, then V_λ is irreducible both as a representation of $\mathrm{GL}(N)$ and as a representation of $\mathfrak{gl}(N)$. If $\lambda_{N+1} \neq 0$, then $V_\lambda = 0$.

It is known that there exists a unique polynomial $\chi_\lambda \in \mathbb{Q}[y_1, y_2, ..., y_N]$ (depending both on λ and on N) such that every diagonal matrix $A = \mathrm{diag}(a_1, a_2, ..., a_N) \in \mathrm{GL}(N)$ satisfies $\chi_\lambda(a_1, a_2, ..., a_N) = (\mathrm{Tr}\mid_{V_\lambda})(A)$ (where $(\mathrm{Tr}\mid_{V_\lambda})(A)$ means the trace of the action of $A \in \mathrm{GL}(N)$ on V_λ by means of the $\mathrm{GL}(N)$-module structure on V_λ). In the language of representation theory, χ_λ is thus the character of the $\mathrm{GL}(N)$-module V_λ. This polynomial χ_λ is called the λ-*th Schur polynomial in* N *variables.*

Now, the relation between the S_λ and the Schur polynomials looks like this:

Proposition 3.12.10. Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ be a partition. Then, $\chi_\lambda(y_1, y_2, ..., y_N) = \mathrm{PSE}_N(S_\lambda(x))$.

This generalizes Proposition 3.12.7 (in fact, set $\lambda = (k)$ and notice that $V_\lambda = S^k\mathbb{C}^N$).

Proof of Proposition 3.12.10. Define h_k to mean 0 for every $k < 0$.

Proposition 3.12.7 yields $h_k(y) = \mathrm{PSE}_N(S_k(x))$ for each $k \in \mathbb{N}$. Since $h_k(y) = \mathrm{PSE}_N(S_k(x))$ also holds for every negative integer k (since every negative integer k satisfies $h_k = 0$ and $S_k = 0$), we thus conclude that

$$h_k(y) = \mathrm{PSE}_N(S_k(x)) \qquad \text{for every } k \in \mathbb{Z}. \tag{146}$$

We know that χ_λ is the λ-th Schur polynomial in N variables. By the first Giambelli formula, this yields that

$$\chi_\lambda(y_1, y_2, ..., y_N)$$
$$= \det \underbrace{\begin{pmatrix} h_{\lambda_1}(y) & h_{\lambda_1+1}(y) & h_{\lambda_1+2}(y) & ... & h_{\lambda_1+m-1}(y) \\ h_{\lambda_2-1}(y) & h_{\lambda_2}(y) & h_{\lambda_2+1}(y) & ... & h_{\lambda_2+m-2}(y) \\ h_{\lambda_3-2}(y) & h_{\lambda_3-1}(y) & h_{\lambda_3}(y) & ... & h_{\lambda_3+m-3}(y) \\ ... & ... & ... & ... & ... \\ h_{\lambda_m-m+1}(y) & h_{\lambda_m-m+2}(y) & h_{\lambda_m-m+3}(y) & ... & h_{\lambda_m}(y) \end{pmatrix}}_{=\left(h_{\lambda_i+j-i}(y)\right)_{1\leq i\leq m,\ 1\leq j\leq m}}$$
$$= \det\left(\left(h_{\lambda_i+j-i}(y)\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right) = \det\left(\left(\mathrm{PSE}_N\left(S_{\lambda_i+j-i}(x)\right)\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right)$$
$$\qquad \text{(by (146))}$$
$$= \mathrm{PSE}_N \underbrace{\left(\det\left(\left(S_{\lambda_i+j-i}(x)\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right)\right)}_{=S_\lambda(x)}$$
$$\qquad \left(\begin{array}{c}\text{since } \mathrm{PSE}_N \text{ is a } \mathbb{C}\text{-algebra homomorphism, whereas } \det \text{ is a polynomial} \\ \text{(and any } \mathbb{C}\text{-algebra homomorphism commutes with any polynomial)}\end{array}\right)$$
$$= \mathrm{PSE}_N\left(S_\lambda(x)\right).$$

Proposition 3.12.10 is proven.

3.12.2. The statement of the fact

Theorem 3.12.11. Whenever $(i_0, i_1, i_2, ...)$ is a 0-degression (see Definition 3.5.12 for what this means), we have $\sigma^{-1}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = S_\lambda(x)$ where $\lambda = (i_0 + 0, i_1 + 1, i_2 + 2, ...)$. (Note that this λ is indeed a partition since $(i_0, i_1, i_2, ...)$ is a 0-degression.)

We are going to give two proofs of this theorem. The first proof will be covered in Section 3.13, whereas the second proof will encompass Section 3.14.

3.13. Expliciting σ^{-1} using Schur polynomials: first proof

3.13.1. The power sums are algebraically independent

Our first proof of Theorem 3.12.11 will require some lemmata from algebraic combinatorics. First of all:

Lemma 3.13.1. Let $N \in \mathbb{N}$. For every positive integer j, let p_j denote the polynomial $y_1^j + y_2^j + ... + y_N^j \in \mathbb{C}[y_1, y_2, ..., y_N]$. Then, the polynomials p_1, p_2, ..., p_N are algebraically independent.

In order to prove this fact, we need the following known facts (which we won't prove):

Lemma 3.13.2. Let $N \in \mathbb{N}$. For every $j \in \mathbb{N}$, let e_j denote the j-th elementary symmetric polynomial $\sum\limits_{1\leq i_1<i_2<...<i_j\leq N} y_{i_1}y_{i_2}...y_{i_j}$ in $\mathbb{C}[y_1, y_2, ..., y_N]$. Then, the elements e_1, e_2, ..., e_N are algebraically independent.

Lemma 3.13.2 is one half of a known theorem. The other half says that the elements e_1, e_2, ..., e_N generate the $\mathbb{C}$-algebra of symmetric polynomials in $\mathbb{C}[y_1, y_2, ..., y_N]$. We will prove neither of these halves; they are both classical and well-known (under the name "fundamental theorem of symmetric polynomials", which is usually formulated in a more general setting when $\mathbb{C}$ is replaced by any commutative ring).

Lemma 3.13.3. Let $N \in \mathbb{N}$. For every positive integer j, define p_j as in Lemma 3.13.1. For every $j \in \mathbb{N}$, define e_j as in Lemma 3.13.2. Then, every $k \in \mathbb{N}$ satisfies $ke_k = \sum\limits_{i=1}^{k} (-1)^{i-1} e_{k-i} p_i$.

This lemma is known as the *Newton identity* (or identities), and won't be proven due to being well-known. But we will use it to derive the following corollary:

Corollary 3.13.4. Let $N \in \mathbb{N}$. For every positive integer j, define p_j as in Lemma 3.13.1. For every $j \in \mathbb{N}$, define e_j as in Lemma 3.13.2. Then, for every positive $k \in \mathbb{N}$, there exists a polynomial $P_k \in \mathbb{Q}[T_1, T_2, ..., T_k]$ such that $p_k = P_k(e_1, e_2, ..., e_k)$ and $P_k - (-1)^{k-1} kT_k \in \mathbb{Q}[T_1, T_2, ..., T_{k-1}]$. (Here, of course, $\mathbb{Q}[T_1, T_2, ..., T_{k-1}]$ is identified with a subalgebra of $\mathbb{Q}[T_1, T_2, ..., T_k]$.)

Proof of Corollary 3.13.4. We will prove Corollary 3.13.4 by strong induction over k:

Induction step: Let ℓ be a positive integer. Assume that Corollary 3.13.4 holds for every positive integer $k < \ell$. We must then prove that Corollary 3.13.4 holds for $k = \ell$.

Corollary 3.13.4 holds for every positive integer $k < \ell$ (by the induction hypothesis). In other words, for every $k < \ell$, there exists a polynomial $P_k \in \mathbb{Q}[T_1, T_2, ..., T_k]$ such that $p_k = P_k(e_1, e_2, ..., e_k)$ and $P_k - (-1)^{k-1} kT_k \in \mathbb{Q}[T_1, T_2, ..., T_{k-1}]$. Consider these polynomials P_1, P_2, ..., $P_{\ell-1}$.

Applying Lemma 3.13.3 to $k = \ell$, we obtain

$$\begin{aligned} \ell e_\ell &= \sum_{i=1}^{\ell} (-1)^{i-1} e_{\ell-i} p_i = \sum_{k=1}^{\ell} (-1)^{k-1} e_{\ell-k} p_k \qquad \text{(here, we renamed } i \text{ as } k \text{ in the sum)} \\ &= \sum_{k=1}^{\ell-1} (-1)^{k-1} e_{\ell-k} p_k + (-1)^{\ell-1} \underbrace{e_{\ell-\ell}}_{=e_0=1} p_\ell = \sum_{k=1}^{\ell-1} (-1)^{k-1} e_{\ell-k} p_k + (-1)^{\ell-1} p_\ell, \end{aligned}$$

so that $(-1)^{\ell-1} p_\ell = \ell e_\ell - \sum\limits_{k=1}^{\ell-1} (-1)^{k-1} e_{\ell-k} p_k$ and thus

$$p_\ell = (-1)^{\ell-1} \left(\ell e_\ell - \sum_{k=1}^{\ell-1} (-1)^{k-1} e_{\ell-k} p_k \right) = (-1)^{\ell-1} \ell e_\ell - (-1)^{\ell-1} \sum_{k=1}^{\ell-1} (-1)^{k-1} e_{\ell-k} p_k.$$

Now, define a polynomial $P_\ell \in \mathbb{Q}[T_1, T_2, ..., T_\ell]$ by

$$P_\ell = (-1)^{\ell-1} \ell T_\ell - (-1)^{\ell-1} \sum_{k=1}^{\ell-1} (-1)^{k-1} T_{\ell-k} P_k (T_1, T_2, ..., T_k).$$

Then,

$$P_\ell - (-1)^{\ell-1} \ell T_\ell = -(-1)^{\ell-1} \sum_{k=1}^{\ell-1} (-1)^{k-1} \underbrace{T_{\ell-k}}_{\substack{\in \mathbb{Q}[T_1,T_2,...,T_{\ell-1}] \\ (\text{since } \ell-k \leq \ell-1)}} \underbrace{P_k(T_1,T_2,...,T_k)}_{\substack{\in \mathbb{Q}[T_1,T_2,...,T_{\ell-1}] \\ (\text{since } k \leq \ell-1)}} \in \mathbb{Q}[T_1,T_2,...,T_{\ell-1}].$$

Moreover, $P_\ell = (-1)^{\ell-1} \ell T_\ell - (-1)^{\ell-1} \sum\limits_{k=1}^{\ell-1} (-1)^{k-1} T_{\ell-k} P_k (T_1, T_2, ..., T_k)$ yields

$$\begin{aligned} P_\ell(e_1,e_2,...,e_\ell) &= (-1)^{\ell-1} \ell e_\ell - (-1)^{\ell-1} \sum_{k=1}^{\ell-1} (-1)^{k-1} e_{\ell-k} \underbrace{P_k(e_1,e_2,...,e_k)}_{\substack{=p_k \\ (\text{by the definition of } P_k)}} \\ &= (-1)^{\ell-1} \ell e_\ell - (-1)^{\ell-1} \sum_{k=1}^{\ell-1} (-1)^{k-1} e_{\ell-k} p_k = p_\ell. \end{aligned}$$

We thus have shown that $p_\ell = P_\ell(e_1, e_2, ..., e_\ell)$ and $P_\ell - (-1)^{\ell-1} \ell T_\ell \in \mathbb{Q}[T_1, T_2, ..., T_{\ell-1}]$. Thus, there exists a polynomial $P_\ell \in \mathbb{Q}[T_1, T_2, ..., T_\ell]$ such that $p_\ell = P_\ell(e_1, e_2, ..., e_\ell)$ and $P_\ell - (-1)^{\ell-1} \ell T_\ell \in \mathbb{Q}[T_1, T_2, ..., T_{\ell-1}]$. In other words, Corollary 3.13.4 holds for $k = \ell$. This completes the induction step. The induction proof of Corollary 3.13.4 is thus complete.

Proof of Lemma 3.13.1. Assume the contrary. Thus, the polynomials p_1, p_2, ..., p_N are algebraically dependent. Hence, there exists a nonzero polynomial $Q \in \mathbb{C}[U_1, U_2, ..., U_N]$ such that $Q(p_1, p_2, ..., p_N) = 0$. Consider this Q.

Consider the lexicographic order on the monomials in $\mathbb{C}[T_1, T_2, ..., T_N]$ given by $T_1 < T_2 < ... < T_N$.

For every $j \in \mathbb{N}$, define e_j as in Lemma 3.13.2. For every positive $k \in \mathbb{N}$, Corollary 3.13.4 guarantees the existence of a polynomial $P_k \in \mathbb{Q}[T_1, T_2, ..., T_k]$ such that $p_k = P_k(e_1, e_2, ..., e_k)$ and $P_k - (-1)^{k-1} kT_k \in \mathbb{Q}[T_1, T_2, ..., T_{k-1}]$. Consider such a polynomial P_k.

For every $k \in \{1, 2, ..., N\}$, there exists a polynomial $Q_k \in \mathbb{Q}[T_1, T_2, ..., T_{k-1}]$ such that $P_k - (-1)^{k-1} kT_k = Q_k(T_1, T_2, ..., T_{k-1})$ (since $P_k - (-1)^{k-1} kT_k \in \mathbb{Q}[T_1, T_2, ..., T_{k-1}]$). Consider such a polynomial Q_k.

For every $k \in \{1, 2, ..., N\}$, let $\widetilde{P}_k$ be the polynomial $P_k(T_1, T_2, ..., T_k) \in \mathbb{C}[T_1, T_2, ..., T_N]$. (This is the same polynomial as P_k, but now considered as a polynomial in N variables over $\mathbb{C}$ rather than in k variables over $\mathbb{Q}$.)

Then, for every $k \in \{1, 2, ..., N\}$, we have

$$\begin{aligned} \widetilde{P}_k(e_1,e_2,...,e_N) &= P_k(e_1,e_2,...,e_k) \qquad \left(\text{since } \widetilde{P}_k = P_k(T_1,T_2,...,T_k)\right) \\ &= p_k. \end{aligned}$$

Also, for every $k \in \{1, 2, ..., N\}$, the leading monomial[136] of $\widetilde{P}_k$ (with respect to the

[136] Here, "monomial" means "monomial without coefficient", and the "leading monomial" of a polynomial means the highest monomial (with nonzero coefficient) of the polynomial.

lexicographic order defined above) is T_k [137]. Since the leading monomial of a product of polynomials equals the product of their leading monomials, this yields that for every $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$,

$$\text{the leading monomial of } \widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N} \text{ is } T_1^{\alpha_1} T_2^{\alpha_2} ... T_N^{\alpha_N}. \tag{147}$$

Since every $k \in \{1, 2, ..., N\}$ satisfies $p_k = \widetilde{P}_k(e_1, e_2, ..., e_N)$, we have

$$\begin{aligned} Q(p_1, p_2, ..., p_N) &= Q\left(\widetilde{P}_1(e_1, e_2, ..., e_N), \widetilde{P}_2(e_1, e_2, ..., e_N), ..., \widetilde{P}_N(e_1, e_2, ..., e_N)\right) \\ &= \left(Q\left(\widetilde{P}_1, \widetilde{P}_2, ..., \widetilde{P}_N\right)\right)(e_1, e_2, ..., e_N). \end{aligned}$$

Hence, $Q(p_1, p_2, ..., p_N) = 0$ rewrites as $\left(Q\left(\widetilde{P}_1, \widetilde{P}_2, ..., \widetilde{P}_N\right)\right)(e_1, e_2, ..., e_N) = 0$. Since e_1, e_2, ..., e_N are algebraically independent (by Lemma 3.13.2), this yields $Q\left(\widetilde{P}_1, \widetilde{P}_2, ..., \widetilde{P}_N\right) = 0$. Since $Q \neq 0$, this shows that the elements $\widetilde{P}_1$, $\widetilde{P}_2$, ..., $\widetilde{P}_N$ are algebraically dependent. In other words, the family $\left(\widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N}\right)_{(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N}$ is linearly dependent. Thus, there exists a family $(\lambda_{\alpha_1, \alpha_2, ..., \alpha_N})_{(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N}$ of elements of $\mathbb{C}$ such that:

- all but finitely many $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ satisfy $\lambda_{\alpha_1, \alpha_2, ..., \alpha_N} = 0$;
- not all $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ satisfy $\lambda_{\alpha_1, \alpha_2, ..., \alpha_N} = 0$;
- we have $\sum\limits_{(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N} \lambda_{\alpha_1, \alpha_2, ..., \alpha_N} \widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N} = 0$.

Consider this family. By identifying every N-tuple $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ with the monomial $T_1^{\alpha_1} T_2^{\alpha_2} ... T_N^{\alpha_N} \in \mathbb{C}[T_1, T_2, ..., T_N]$, we obtain a lexicographic order on the N-tuples $(\alpha_1, \alpha_2, ..., \alpha_N)$ (from the lexicographic order on the monomials in $\mathbb{C}[T_1, T_2, ..., T_N]$).

Since all but finitely many $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ satisfy $\lambda_{\alpha_1, \alpha_2, ..., \alpha_N} = 0$, but not all $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ satisfy $\lambda_{\alpha_1, \alpha_2, ..., \alpha_N} = 0$, there exists a highest (with respect to the above-defined order) $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ satisfying $\lambda_{\alpha_1, \alpha_2, ..., \alpha_N} \neq 0$.

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[137] *Proof.* Let $k \in \{1, 2, ..., N\}$. Then,

$$\begin{aligned} \underbrace{\widetilde{P}_k}_{= P_k(T_1, T_2, ..., T_k)} - (-1)^{k-1} k T_k &= P_k(T_1, T_2, ..., T_k) - (-1)^{k-1} k T_k \\ &= \underbrace{\left(P_k - (-1)^{k-1} k T_k\right)}_{= Q_k(T_1, T_2, ..., T_{k-1})} (T_1, T_2, ..., T_k) \\ &= (Q_k(T_1, T_2, ..., T_{k-1}))(T_1, T_2, ..., T_k) = Q_k(T_1, T_2, ..., T_{k-1}), \end{aligned}$$

so that $\widetilde{P}_k = (-1)^{k-1} k T_k + Q_k(T_1, T_2, ..., T_{k-1})$. Hence, the only monomials which occur with nonzero coefficient in the polynomial $\widetilde{P}_k$ are the monomial T_k (occurring with coefficient $(-1)^{k-1} k$) and the monomials of the polynomial $Q_k(T_1, T_2, ..., T_{k-1})$. But the latter monomials don't contain any variable other than T_1, T_2, ..., T_{k-1} (because they are monomials of the polynomial $Q_k(T_1, T_2, ..., T_{k-1})$), and thus are smaller than the monomial T_k (because any monomial which doesn't contain any variable other than T_1, T_2, ..., T_{k-1} is smaller than any monomial which contains T_k (since we have a lexicographic order given by $T_1 < T_2 < ... < T_N$)). Hence, the leading monomial of $\widetilde{P}_k$ must be T_k, qed.

Let this $(\alpha_1, \alpha_2, ..., \alpha_N)$ be called $(\beta_1, \beta_2, ..., \beta_N)$. Then, $(\beta_1, \beta_2, ..., \beta_N)$ is the highest $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ satisfying $\lambda_{\alpha_1,\alpha_2,...,\alpha_N} \neq 0$. Thus, $\lambda_{\beta_1,\beta_2,...,\beta_N} \neq 0$, but

$$\text{every } (\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N \text{ higher than } (\beta_1, \beta_2, ..., \beta_N) \text{ satisfies } \lambda_{\alpha_1,\alpha_2,...,\alpha_N} = 0. \tag{148}$$

Now it is easy to see that for every $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ satisfying $(\alpha_1, \alpha_2, ..., \alpha_N) \neq (\beta_1, \beta_2, ..., \beta_N)$, the term

$$\lambda_{\alpha_1,\alpha_2,...,\alpha_N} \widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N} \text{ is a } \mathbb{C}\text{-linear combination of monomials smaller than } T_1^{\beta_1} T_2^{\beta_2} ... T_N^{\beta_N}. \tag{149}$$

[138] As a consequence,

$$\sum_{\substack{(\alpha_1,\alpha_2,...,\alpha_N) \in \mathbb{N}^N; \\ (\alpha_1,\alpha_2,...,\alpha_N) \neq (\beta_1,\beta_2,...,\beta_N)}} \lambda_{\alpha_1,\alpha_2,...,\alpha_N} \widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N}$$

is a sum of $\mathbb{C}$-linear combinations of monomials smaller than $T_1^{\beta_1} T_2^{\beta_2} ... T_N^{\beta_N}$, and thus itself a $\mathbb{C}$-linear combination of monomials smaller than $T_1^{\beta_1} T_2^{\beta_2} ... T_N^{\beta_N}$.

Now,

$$\begin{aligned} 0 &= \sum_{(\alpha_1,\alpha_2,...,\alpha_N) \in \mathbb{N}^N} \lambda_{\alpha_1,\alpha_2,...,\alpha_N} \widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N} \\ &= \lambda_{\beta_1,\beta_2,...,\beta_N} \widetilde{P}_1^{\beta_1} \widetilde{P}_2^{\beta_2} ... \widetilde{P}_N^{\beta_N} + \sum_{\substack{(\alpha_1,\alpha_2,...,\alpha_N) \in \mathbb{N}^N; \\ (\alpha_1,\alpha_2,...,\alpha_N) \neq (\beta_1,\beta_2,...,\beta_N)}} \lambda_{\alpha_1,\alpha_2,...,\alpha_N} \widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N}, \end{aligned}$$

so that

$$\sum_{\substack{(\alpha_1,\alpha_2,...,\alpha_N) \in \mathbb{N}^N; \\ (\alpha_1,\alpha_2,...,\alpha_N) \neq (\beta_1,\beta_2,...,\beta_N)}} \lambda_{\alpha_1,\alpha_2,...,\alpha_N} \widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N} = -\lambda_{\beta_1,\beta_2,...,\beta_N} \widetilde{P}_1^{\beta_1} \widetilde{P}_2^{\beta_2} ... \widetilde{P}_N^{\beta_N}.$$

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[138] *Proof of (149).* Let $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ satisfy $(\alpha_1, \alpha_2, ..., \alpha_N) \neq (\beta_1, \beta_2, ..., \beta_N)$. Since the lexicographic order is a total order, we must be in one of the following two cases:

Case 1: We have $(\alpha_1, \alpha_2, ..., \alpha_N) \geq (\beta_1, \beta_2, ..., \beta_N)$.

Case 2: We have $(\alpha_1, \alpha_2, ..., \alpha_N) < (\beta_1, \beta_2, ..., \beta_N)$.

First, consider Case 1. In this case, $(\alpha_1, \alpha_2, ..., \alpha_N) \geq (\beta_1, \beta_2, ..., \beta_N)$, so that $(\alpha_1, \alpha_2, ..., \alpha_N) > (\beta_1, \beta_2, ..., \beta_N)$ (since $(\alpha_1, \alpha_2, ..., \alpha_N) \neq (\beta_1, \beta_2, ..., \beta_N)$). Thus, $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ is higher than $(\beta_1, \beta_2, ..., \beta_N)$. Hence, $\lambda_{\alpha_1,\alpha_2,...,\alpha_N} = 0$ (by (148)), so that $\lambda_{\alpha_1,\alpha_2,...,\alpha_N} \widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N} = 0$ is clearly a $\mathbb{C}$-linear combination of monomials smaller than $T_1^{\alpha_1} T_2^{\alpha_2} ... T_N^{\alpha_N}$. Thus, (149) holds in Case 1.

Now, let us consider Case 2. In this case, $(\alpha_1, \alpha_2, ..., \alpha_N) < (\beta_1, \beta_2, ..., \beta_N)$, so that $T_1^{\alpha_1} T_2^{\alpha_2} ... T_N^{\alpha_N} < T_1^{\beta_1} T_2^{\beta_2} ... T_N^{\beta_N}$ (because the order on N-tuples is obtained from the order on monomials by identifying every N-tuple $(\alpha_1, \alpha_2, ..., \alpha_N) \in \mathbb{N}^N$ with the monomial $T_1^{\alpha_1} T_2^{\alpha_2} ... T_N^{\alpha_N} \in \mathbb{C}[T_1, T_2, ..., T_N]$).

Due to (147), every monomial which occurs with nonzero coefficient in $\widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N}$ is smaller or equal to $T_1^{\alpha_1} T_2^{\alpha_2} ... T_N^{\alpha_N}$. Combined with $T_1^{\alpha_1} T_2^{\alpha_2} ... T_N^{\alpha_N} < T_1^{\beta_1} T_2^{\beta_2} ... T_N^{\beta_N}$, this yields that every monomial which occurs with nonzero coefficient in $\widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N}$ is smaller than $T_1^{\beta_1} T_2^{\beta_2} ... T_N^{\beta_N}$. Hence, $\widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N}$ is a $\mathbb{C}$-linear combination of monomials smaller than $T_1^{\beta_1} T_2^{\beta_2} ... T_N^{\beta_N}$. Thus, $\lambda_{\alpha_1,\alpha_2,...,\alpha_N} \widetilde{P}_1^{\alpha_1} \widetilde{P}_2^{\alpha_2} ... \widetilde{P}_N^{\alpha_N}$ is a $\mathbb{C}$-linear combination of monomials smaller than $T_1^{\beta_1} T_2^{\beta_2} ... T_N^{\beta_N}$. We have thus proven that (149) holds in Case 2.

Hence, (149) holds in each of cases 1 and 2. Since no other cases are possible, this yields that (149) always holds.

Since we know that $\sum\limits_{\substack{(\alpha_1,\alpha_2,...,\alpha_N)\in\mathbb{N}^N;\\ (\alpha_1,\alpha_2,...,\alpha_N)\neq(\beta_1,\beta_2,...,\beta_N)}} \lambda_{\alpha_1,\alpha_2,...,\alpha_N} \widetilde{P}_1^{\alpha_1}\widetilde{P}_2^{\alpha_2}...\widetilde{P}_N^{\alpha_N}$ is a $\mathbb{C}$-linear combination of monomials smaller than $T_1^{\beta_1}T_2^{\beta_2}...T_N^{\beta_N}$, we thus conclude that $-\lambda_{\beta_1,\beta_2,...,\beta_N}\widetilde{P}_1^{\beta_1}\widetilde{P}_2^{\beta_2}...\widetilde{P}_N^{\beta_N}$ is a $\mathbb{C}$-linear combination of monomials smaller than $T_1^{\beta_1}T_2^{\beta_2}...T_N^{\beta_N}$. Since $-\lambda_{\beta_1,\beta_2,...,\beta_N}$ is invertible (because $\lambda_{\beta_1,\beta_2,...,\beta_N}\neq 0$), this yields that $\widetilde{P}_1^{\beta_1}\widetilde{P}_2^{\beta_2}...\widetilde{P}_N^{\beta_N}$ is a $\mathbb{C}$-linear combination of monomials smaller than $T_1^{\beta_1}T_2^{\beta_2}...T_N^{\beta_N}$. In other words, every monomial which occurs with nonzero coefficient in $\widetilde{P}_1^{\beta_1}\widetilde{P}_2^{\beta_2}...\widetilde{P}_N^{\beta_N}$ is less than $T_1^{\beta_1}T_2^{\beta_2}...T_N^{\beta_N}$. In particular, the leading monomial of $\widetilde{P}_1^{\beta_1}\widetilde{P}_2^{\beta_2}...\widetilde{P}_N^{\beta_N}$ is less than $T_1^{\beta_1}T_2^{\beta_2}...T_N^{\beta_N}$. But this contradicts the fact that (due to (147), applied to $(\alpha_1,\alpha_2,...,\alpha_N)=(\beta_1,\beta_2,...,\beta_N)$) the leading monomial of $\widetilde{P}_1^{\beta_1}\widetilde{P}_2^{\beta_2}...\widetilde{P}_N^{\beta_N}$ is $T_1^{\beta_1}T_2^{\beta_2}...T_N^{\beta_N}$.

This contradiction shows that our assumption was wrong. Hence, Lemma 3.13.1 is proven.

(I have learned the above proof from:

Julia Pevtsova and Nate Bottman, *504A Fall 2009 Homework Set 3*, `http://www.math.washington.edu/~julia/teaching/504_Fall2009/HW7_sol.pdf` .)

We will apply Lemma 3.13.1 not directly, but through the following corollary:

Corollary 3.13.5. Let P and Q be polynomials in $\mathbb{C}[x_1,x_2,x_3,...]$. Assume that $\operatorname{PSE}_N(P)=\operatorname{PSE}_N(Q)$ for every sufficiently high $N\in\mathbb{N}$. Then, $P=Q$.

Proof of Corollary 3.13.5. Any polynomial (even if it is a polynomial in infinitely many indeterminates) has only finitely many indeterminates actually appear in it. Hence, only finitely many indeterminates appear in $P-Q$. Thus, there exists an $M\in\mathbb{N}$ such that no indeterminates other than x_1, x_2, ..., x_M appear in $P-Q$. Consider this M.

Recall that $\operatorname{PSE}_N(P)=\operatorname{PSE}_N(Q)$ for every sufficiently high $N\in\mathbb{N}$. Thus, there exists an $N\in\mathbb{N}$ such that $N\geq M$ and $\operatorname{PSE}_N(P)=\operatorname{PSE}_N(Q)$. Pick such an N.

No indeterminates other than x_1, x_2, ..., x_M appear in $P-Q$. Since $N\geq M$, this clearly yields that no indeterminates other than x_1, x_2, ..., x_N appear in $P-Q$. Hence, there exists a polynomial $R\in\mathbb{C}[x_1,x_2,...,x_N]$ such that $P-Q=R(x_1,x_2,...,x_N)$. Consider this R.

Now, let us use the notations of Lemma 3.13.1.

We defined $\operatorname{PSE}_N(P-Q)$ as the result of substituting $x_j=\dfrac{y_1^j+y_2^j+...+y_N^j}{j}$ for all positive integers j into the polynomial $P-Q$. Since $y_1^j+y_2^j+...+y_N^j=p_j$ for all positive integers j, this rewrites as follows: $\operatorname{PSE}_N(P-Q)$ is the result of substituting $x_j=\dfrac{p_j}{j}$ for all positive integers j into the polynomial $P-Q$. In other words,

$$\begin{aligned}\operatorname{PSE}_N(P-Q)&=\underbrace{(P-Q)}_{=R(x_1,x_2,...,x_N)}\left(\frac{p_1}{1},\frac{p_2}{2},\frac{p_3}{3},...\right)=(R(x_1,x_2,...,x_N))\left(\frac{p_1}{1},\frac{p_2}{2},\frac{p_3}{3},...\right)\\&=R\left(\frac{p_1}{1},\frac{p_2}{2},...,\frac{p_N}{N}\right).\end{aligned}$$

But since PSE_N is a $\mathbb{C}$-algebra homomorphism, we have $\operatorname{PSE}_N(P-Q)=\operatorname{PSE}_N(P)-\operatorname{PSE}_N(Q)=0$ (since $\operatorname{PSE}_N(P)=\operatorname{PSE}_N(Q)$). Thus,

$$R\left(\frac{p_1}{1},\frac{p_2}{2},...,\frac{p_N}{N}\right)=\operatorname{PSE}_N(P-Q)=0.$$

Since $\frac{p_1}{1}, \frac{p_2}{2}, ..., \frac{p_N}{N}$ are algebraically independent (because Lemma 3.13.1 yields that $p_1, p_2, ..., p_N$ are algebraically independent), this yields $R = 0$, so that $P - Q = \underbrace{R}_{=0}(x_1, x_2, ..., x_N) = 0$, thus $P = Q$. Corollary 3.13.5 is proven.

Corollary 3.13.5 allows us to prove equality of polynomials in $\mathbb{C}[x_1, x_2, x_3, ...]$ by means of evaluating them at power sums. Now, let us show what such evaluations look like for the Schur functions:

3.13.2. First proof of Theorem 3.12.11

Theorem 3.13.6. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...)$ be a partition, so that $\lambda_1 \geq \lambda_2 \geq ...$ are nonnegative integers.

Let N be a nonnegative integer such that $\lambda_{N+1} = 0$. Then,

$$\mathrm{PSE}_N(S_\lambda(x)) = \frac{\det\left(\left(y_i^{\lambda_j+N-j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}.$$

Proof of Theorem 3.13.6. We will not really prove this theorem; we will just reduce it to a known fact about Schur functions.

In fact, let m be an integer such that $\lambda_{m+1} = 0$ (such an integer clearly exists). Then, the partition λ can also be written in the form $(\lambda_1, \lambda_2, ..., \lambda_m)$. Hence, by the first Giambelli formula, the λ-th Schur polynomial evaluated at $(y_1, y_2, ..., y_N)$ equals

$$\det\begin{pmatrix} h_{\lambda_1}(y) & h_{\lambda_1+1}(y) & h_{\lambda_1+2}(y) & ... & h_{\lambda_1+m-1}(y) \\ h_{\lambda_2-1}(y) & h_{\lambda_2}(y) & h_{\lambda_2+1}(y) & ... & h_{\lambda_2+m-2}(y) \\ h_{\lambda_3-2}(y) & h_{\lambda_3-1}(y) & h_{\lambda_3}(y) & ... & h_{\lambda_3+m-3}(y) \\ ... & ... & ... & ... & ... \\ h_{\lambda_m-m+1}(y) & h_{\lambda_m-m+2}(y) & h_{\lambda_m-m+3}(y) & ... & h_{\lambda_m}(y) \end{pmatrix}$$
$$= \det\left(\left(h_{\lambda_i+j-i}(y)\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right).$$

But since the λ-th Schur polynomial evaluated at $(y_1, y_2, ..., y_N)$ also equals $\frac{\det\left(\left(y_i^{\lambda_j+N-j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}$ (by the "Vandermonde-determinant" definition of Schur polynomials), this yields that

$$\det\left(\left(h_{\lambda_i+j-i}(y)\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right) = \frac{\det\left(\left(y_i^{\lambda_j+N-j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}.$$

Comparing this with the equality $\det\left(\left(h_{\lambda_i+j-i}(y)\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right) = \mathrm{PSE}_N(S_\lambda(x))$

(which was verified during the proof of Proposition 3.12.10), we obtain

$$\frac{\det\left(\left(y_i^{\lambda_{j-1}+N-j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)} = \operatorname{PSE}_N\left(S_\lambda\left(x\right)\right).$$

Theorem 3.13.6 is thus proven.

We will now use a harmless-looking result about determinants:

Proposition 3.13.7. Let $N \in \mathbb{N}$. Let $(a_{i,j})_{1\leq i\leq N,\ 1\leq j\leq N}$ be an $N \times N$-matrix of elements of a commutative ring R. Let $b_1, b_2, ..., b_N$ be N elements of R. Then,

$$\sum_{k=1}^{N} \det\left(\left(a_{i,j} b_i^{\delta_{j,k}}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right) = \left(b_1 + b_2 + ... + b_N\right) \det\left(\left(a_{i,j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right). \tag{150}$$

Equivalently (in more reader-friendly terms):

$$\begin{aligned} &\det\begin{pmatrix} b_1 a_{1,1} & a_{1,2} & ... & a_{1,N} \\ b_2 a_{2,1} & a_{2,2} & ... & a_{2,N} \\ ... & ... & ... & ... \\ b_N a_{N,1} & a_{N,2} & ... & a_{N,N} \end{pmatrix} + \det\begin{pmatrix} a_{1,1} & b_1 a_{1,2} & ... & a_{1,N} \\ a_{2,1} & b_2 a_{2,2} & ... & a_{2,N} \\ ... & ... & ... & ... \\ a_{N,1} & b_N a_{N,2} & ... & a_{N,N} \end{pmatrix} \\ &\quad + ... + \det\begin{pmatrix} a_{1,1} & a_{1,2} & ... & b_1 a_{1,N} \\ a_{2,1} & a_{2,2} & ... & b_2 a_{2,N} \\ ... & ... & ... & ... \\ a_{N,1} & a_{N,2} & ... & b_N a_{N,N} \end{pmatrix} \\ &= \left(b_1 + b_2 + ... + b_N\right) \det\begin{pmatrix} a_{1,1} & a_{1,2} & ... & a_{1,N} \\ a_{2,1} & a_{2,2} & ... & a_{2,N} \\ ... & ... & ... & ... \\ a_{N,1} & a_{N,2} & ... & a_{N,N} \end{pmatrix}. \end{aligned} \tag{151}$$

Proof of Proposition 3.13.7. Recall the explicit formula for a determinant of a matrix as a sum over permutations: For every $N \times N$-matrix $(c_{i,j})_{1\leq i\leq N,\ 1\leq j\leq N}$, we have

$$\det\left(\left(c_{i,j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right) = \sum_{\sigma\in S_N} (-1)^\sigma \prod_{j=1}^{N} c_{\sigma(j),j}. \tag{152}$$

Applied to $(c_{i,j})_{1\leq i\leq N,\ 1\leq j\leq N} = (a_{i,j})_{1\leq i\leq N,\ 1\leq j\leq N}$, this yields

$$\det\left(\left(a_{i,j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right) = \sum_{\sigma\in S_N} (-1)^\sigma \prod_{j=1}^{N} a_{\sigma(j),j}. \tag{153}$$

For every $k \in \{1, 2, ..., N\}$, we can apply (152) to $(c_{i,j})_{1\leq i\leq N,\ 1\leq j\leq N} = \left(a_{i,j} b_i^{\delta_{j,k}}\right)_{1\leq i\leq N,\ 1\leq j\leq N}$, and obtain

$$\det\left(\left(a_{i,j}b_i^{\delta_{j,k}}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right) = \sum_{\sigma\in S_N}(-1)^\sigma \underbrace{\prod_{j=1}^N\left(a_{\sigma(j),j}b_{\sigma(j)}^{\delta_{j,k}}\right)}_{=\prod_{j=1}^N a_{\sigma(j),j}\prod_{j=1}^N b_{\sigma(j)}^{\delta_{j,k}}}$$
$$= \sum_{\sigma\in S_N}(-1)^\sigma \prod_{j=1}^N a_{\sigma(j),j} \underbrace{\prod_{j=1}^N b_{\sigma(j)}^{\delta_{j,k}}}_{=b_{\sigma(k)}^{\delta_{k,k}}\prod_{\substack{j\in\{1,2,\ldots,N\};\\ j\neq k}} b_{\sigma(j)}^{\delta_{j,k}}}$$
$$= \sum_{\sigma\in S_N}(-1)^\sigma \prod_{j=1}^N a_{\sigma(j),j} \underbrace{b_{\sigma(k)}^{\delta_{k,k}}}_{\substack{=b_{\sigma(k)}\\ (\text{since } \delta_{k,k}=1)}} \prod_{\substack{j\in\{1,2,\ldots,N\};\\ j\neq k}} \underbrace{b_{\sigma(j)}^{\delta_{j,k}}}_{\substack{=1\\ (\text{since } j\neq k \text{ and thus } \delta_{j,k}=0)}}$$
$$= \sum_{\sigma\in S_N}(-1)^\sigma \prod_{j=1}^N a_{\sigma(j),j}b_{\sigma(k)} \underbrace{\prod_{\substack{j\in\{1,2,\ldots,N\};\\ j\neq k}} 1}_{=1} = \sum_{\sigma\in S_N}(-1)^\sigma \prod_{j=1}^N a_{\sigma(j),j}b_{\sigma(k)}.$$

Hence,

$$\sum_{k=1}^N \det\left(\left(a_{i,j}b_i^{\delta_{j,k}}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)$$
$$= \sum_{k=1}^N\sum_{\sigma\in S_N}(-1)^\sigma\prod_{j=1}^N a_{\sigma(j),j}b_{\sigma(k)} = \sum_{\sigma\in S_N}(-1)^\sigma\prod_{j=1}^N a_{\sigma(j),j} \underbrace{\sum_{k=1}^N b_{\sigma(k)}}_{\substack{=\sum_{k=1}^N b_k\\ (\text{since } \sigma \text{ is a permutation})}}$$
$$= \sum_{\sigma\in S_N}(-1)^\sigma\prod_{j=1}^N a_{\sigma(j),j}\sum_{k=1}^N b_k = \underbrace{\left(\sum_{k=1}^N b_k\right)}_{=b_1+b_2+\ldots+b_N} \underbrace{\sum_{\sigma\in S_N}(-1)^\sigma\prod_{j=1}^N a_{\sigma(j),j}}_{\substack{=\det\left((a_{i,j})_{1\leq i\leq N,\ 1\leq j\leq N}\right)\\ (\text{by (153)})}}$$
$$= (b_1+b_2+\ldots+b_N)\det\left((a_{i,j})_{1\leq i\leq N,\ 1\leq j\leq N}\right).$$

This proves Proposition 3.13.7.

Corollary 3.13.8. Let $N\in\mathbb{N}$. Let $(i_0,i_1,\ldots,i_{N-1})\in\mathbb{Z}^N$ be such that $i_{j-1}+N>0$ for every $j\in\{1,2,\ldots,N\}$. Let $m\in\mathbb{N}$. Then,

$$\sum_{k=1}^N \det\left(\left(y_i^{i_{j-1}+\delta_{j,k}m+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)$$
$$= (y_1^m+y_2^m+\ldots+y_N^m)\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right).$$

Proof of Corollary 3.13.8. Applying Proposition 3.13.7 to $R = \mathbb{C}[y_1, y_2, ..., y_N]$, $(a_{i,j})_{1\leq i\leq N,\ 1\leq j\leq N} = \left(y_i^{i_{j-1}+N}\right)_{1\leq i\leq N,\ 1\leq j\leq N}$ and $b_i = y_i^m$, we obtain

$$\sum_{k=1}^{N} \det\left(\left(y_i^{i_{j-1}+N-1}\left(y_i^m\right)^{\delta_{j,k}}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)$$
$$= \left(y_1^m + y_2^m + ... + y_N^m\right)\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right).$$

Since any $i \in \{1, 2, ..., N\}$, $j \in \{1, 2, ..., N\}$ and $k \in \{1, 2, ..., N\}$ satisfy $y_i^{i_{j-1}+N-1}\left(y_i^m\right)^{\delta_{j,k}} = y_i^{i_{j-1}+N+\delta_{j,k}m} = y_i^{i_{j-1}+\delta_{j,k}m+N-1}$, this rewrites as

$$\sum_{k=1}^{N} \det\left(\left(y_i^{i_{j-1}+\delta_{j,k}m+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)$$
$$= \left(y_1^m + y_2^m + ... + y_N^m\right)\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right).$$

Corollary 3.13.8 is proven.

Now, to the main proof.

Proof of Theorem 3.12.11. Define a $\mathbb{C}$-linear map $\tau : \mathcal{F}^{(0)} \to \mathbb{C}[x_1, x_2, x_3, ...]$ by

$$\tau\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right) = S_{(i_0+0, i_1+1, i_2+2, ...)}(x) \qquad \text{for every 0-degression } (i_0, i_1, i_2, ...).$$

(This definition makes sense, because we know that $\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right)_{(i_0,i_1,i_2,...) \text{ is a 0-degression}}$ is a basis of $\wedge^{\frac{\infty}{2},0} V = \mathcal{F}^{(0)}$.)

Our aim is to prove that $\tau = \sigma^{-1}$.

1st step: First of all, the definition of τ (applied to the 0-degression $(0, -1, -2, ...)$) yields

$$\tau\left(v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...\right) = S_{(0+0,-1+1,-2+2,...)}(x) = S_{(0,0,0,...)}(x) = 1.$$

2nd step: If $N \in \mathbb{N}$, and $(i_0, i_1, i_2, ...)$ is a straying 0-degression, then we say that $(i_0, i_1, i_2, ...)$ is *N-finished* if the following two conditions (154) and (155) hold:

$$\left(\text{every integer } k \geq N \text{ satisfies } i_k + k = 0\right); \tag{154}$$
$$\left(\text{each of the integers } i_0,\ i_1,\ ...,\ i_{N-1} \text{ is } > -N\right). \tag{155}$$

Now, we claim the following:

For any $N \in \mathbb{N}$, and any N-finished straying 0-degression $(i_0, i_1, i_2, ...)$, we have

$$\mathrm{PSE}_N\left(\tau\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right)\right) = \frac{\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}. \tag{156}$$

Proof of (156): Let $N \in \mathbb{N}$, and let $(i_0, i_1, i_2, ...)$ be an N-finished straying 0-degression.

Since $(i_0, i_1, i_2, ...)$ is N-finished, we conclude (by the definition of "N-finished") that it satisfies the conditions (154) and (155).

If some two of the integers i_0, i_1, ..., i_{N-1} are equal, then (156) is true.[139] Hence, for the rest of this proof, we assume that no two of the integers i_0, i_1, ..., i_{N-1} are equal. Then, there exists a permutation ϕ of the set $\{0, 1, ..., N-1\}$ such that $i_{\phi^{-1}(0)} > i_{\phi^{-1}(1)} > ... > i_{\phi^{-1}(N-1)}$. Consider this ϕ.

It is easy to see that $i_{\phi^{-1}(0)} > i_{\phi^{-1}(1)} > ... > i_{\phi^{-1}(N-1)} > -N$. [140]

Let π be the finitary permutation of $\mathbb{N}$ which sends every $k \in \mathbb{N}$ to $\begin{cases} \phi(k), & \text{if } k \in \{0, 1, ..., N-1\}; \\ k, & \text{if } k \notin \{0, 1, ..., N-1\} \end{cases}$. Then, $(-1)^\pi = (-1)^\phi$; moreover, every $k \in \mathbb{N}$ satisfies

$$\pi^{-1}(k) = \begin{cases} \phi^{-1}(k), & \text{if } k \in \{0, 1, ..., N-1\}; \\ k, & \text{if } k \notin \{0, 1, ..., N-1\} \end{cases}. \tag{157}$$

In particular, every integer $k \geq N$ satisfies $\pi^{-1}(k) = k$.

From (157), it is clear that

$$\text{every } k \in \{0, 1, ..., N-1\} \text{ satisfies } \pi^{-1}(k) = \phi^{-1}(k). \tag{158}$$

Hence, $i_{\pi^{-1}(0)} > i_{\pi^{-1}(1)} > ... > i_{\pi^{-1}(N-1)} > -N$ (since $i_{\phi^{-1}(0)} > i_{\phi^{-1}(1)} > ... > i_{\phi^{-1}(N-1)} > -N$).

Now, every integer $k \geq N$ satisfies $\pi^{-1}(k) = k$, thus $i_{\pi^{-1}(k)} = i_k = -k$ (since (154) yields $i_k + k = 0$). Hence, $-N = i_{\pi^{-1}(N)} > i_{\pi^{-1}(N+1)} > i_{\pi^{-1}(N+2)} > ...$ (because $-N = -N > -(N+1) > -(N+2) > ...$). Combined with $i_{\pi^{-1}(0)} > i_{\pi^{-1}(1)} > ... > i_{\pi^{-1}(N-1)} > -N$, this becomes

$$i_{\pi^{-1}(0)} > i_{\pi^{-1}(1)} > ... > i_{\pi^{-1}(N-1)} > -N = i_{\pi^{-1}(N)} > i_{\pi^{-1}(N+1)} > i_{\pi^{-1}(N+2)} > ....$$

Thus,

$$i_{\pi^{-1}(0)} > i_{\pi^{-1}(1)} > ... > i_{\pi^{-1}(N-1)} > i_{\pi^{-1}(N)} > i_{\pi^{-1}(N+1)} > i_{\pi^{-1}(N+2)} > ....$$

In other words, the sequence $\left(i_{\pi^{-1}(0)}, i_{\pi^{-1}(1)}, i_{\pi^{-1}(2)}, ...\right)$ is strictly decreasing. Since every sufficiently high $k \in \mathbb{N}$ satisfies $i_{\pi^{-1}(k)} + k = 0$ (in fact, every $k \geq N$ satisfies $i_{\pi^{-1}(k)} = -k$ and thus $i_{\pi^{-1}(k)} + k = 0$), this sequence $\left(i_{\pi^{-1}(0)}, i_{\pi^{-1}(1)}, i_{\pi^{-1}(2)}, ...\right)$ must thus be a 0-degression. Hence, by the definition of τ, we have

$$\tau\left(v_{i_{\pi^{-1}(0)}} \wedge v_{i_{\pi^{-1}(1)}} \wedge v_{i_{\pi^{-1}(2)}} \wedge ...\right) = S_{\left(i_{\pi^{-1}(0)}+0, i_{\pi^{-1}(1)}+1, i_{\pi^{-1}(2)}+2, ...\right)}(x).$$

[139] *Proof.* Assume that some two of the integers i_0, i_1, ..., i_{N-1} are equal. Then, some two elements of the sequence $(i_0, i_1, i_2, ...)$ are equal, so that $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... = 0$ (by the definition of $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$) and thus $\mathrm{PSE}_N(\tau(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)) = \mathrm{PSE}_N(0) = 0$. Thus, the left hand side of (156) is 0. On the other hand, the matrix $\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}$ has two equal columns (since two of the integers i_0, i_1, ..., i_{N-1} are equal) and thus its determinant vanishes, i. e., we have $\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right) = 0$, so that the right hand side of (156) is 0.

Thus, both the left hand side and the right hand side of (156) are 0. Hence, (156) is true, qed.

[140] *Proof.* Every $j \in \{0, 1, ..., N-1\}$ satisfies $\phi^{-1}(j) \in \phi^{-1}(\{0, 1, ..., N-1\}) = \{0, 1, ..., N-1\}$. Hence, for every $j \in \{0, 1, ..., N-1\}$, the integer $i_{\phi^{-1}(j)}$ is one of the integers i_0, i_1, ..., i_{N-1}, and therefore $> -N$ (due to (155)). That is, $i_{\phi^{-1}(j)} > -N$ for every $j \in \{0, 1, ..., N-1\}$. Combining this with $i_{\phi^{-1}(0)} > i_{\phi^{-1}(1)} > ... > i_{\phi^{-1}(N-1)}$, we get $i_{\phi^{-1}(0)} > i_{\phi^{-1}(1)} > ... > i_{\phi^{-1}(N-1)} > -N$.

Since π is a finitary permutation of $\mathbb{N}$ such that $\left(i_{\pi^{-1}(0)}, i_{\pi^{-1}(1)}, i_{\pi^{-1}(2)}, ...\right)$ is a 0-degression, it is clear that π is the straightening permutation of $(i_0, i_1, i_2, ...)$. Thus, by the definition of $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$, we have

$$\begin{aligned} v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... &= \underbrace{(-1)^{\pi}}_{=(-1)^{\phi}} v_{i_{\pi^{-1}(0)}} \wedge v_{i_{\pi^{-1}(1)}} \wedge v_{i_{\pi^{-1}(2)}} \wedge ... \\ &= (-1)^{\phi} v_{i_{\pi^{-1}(0)}} \wedge v_{i_{\pi^{-1}(1)}} \wedge v_{i_{\pi^{-1}(2)}} \wedge ..., \end{aligned}$$

so that

$$\begin{aligned} &\operatorname{PSE}_N \left(\tau \left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right)\right) \\ &= \operatorname{PSE}_N \left(\tau \left((-1)^{\phi} v_{i_{\pi^{-1}(0)}} \wedge v_{i_{\pi^{-1}(1)}} \wedge v_{i_{\pi^{-1}(2)}} \wedge ...\right)\right) \\ &= (-1)^{\phi} \operatorname{PSE}_N \underbrace{\left(\tau \left(v_{i_{\pi^{-1}(0)}} \wedge v_{i_{\pi^{-1}(1)}} \wedge v_{i_{\pi^{-1}(2)}} \wedge ...\right)\right)}_{\substack{=S_{\left(i_{\pi^{-1}(0)}+0, i_{\pi^{-1}(1)}+1, i_{\pi^{-1}(2)}+2, ...\right)}(x) \\ \text{(by the definition of } \tau \text{, since } \left(i_{\pi^{-1}(0)}, i_{\pi^{-1}(1)}, i_{\pi^{-1}(2)}, ...\right) \text{ is a 0-degression)}}} \\ &= (-1)^{\phi} \operatorname{PSE}_N \left(S_{\left(i_{\pi^{-1}(0)}+0, i_{\pi^{-1}(1)}+1, i_{\pi^{-1}(2)}+2, ...\right)}(x)\right). \end{aligned} \tag{159}$$

Let μ be the partition $\left(i_{\pi^{-1}(0)} + 0, i_{\pi^{-1}(1)} + 1, i_{\pi^{-1}(2)} + 2, ...\right)$. For every positive integer α, let μ_α denote the α-th part of the partition μ, so that $\mu = (\mu_1, \mu_2, \mu_3, ...)$. Then, every $j \in \{1, 2, ..., N\}$ satisfies

$$\begin{aligned} \mu_j &= i_{\pi^{-1}(j-1)} + (j-1) && \text{(by the definition of } \mu\text{)} \\ &= i_{\phi^{-1}(j-1)} + (j-1) && \left(\text{since (158) (applied to } k = j-1\text{) yields } \pi^{-1}(j-1) = \phi^{-1}(j-1)\right), \end{aligned}$$

so that $\mu_j + N - j = i_{\phi^{-1}(j-1)} + (j-1) + N - j = i_{\phi^{-1}(j-1)} + N - 1$. Hence,

$$\det\left(\left(y_i^{\mu_j+N-j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right) = \det\left(\left(y_i^{i_{\phi^{-1}(j-1)}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right). \tag{160}$$

But the matrix $\left(y_i^{i_{\phi^{-1}(j-1)}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}$ is obtained from the matrix $\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}$ by permuting the columns using the permutation ϕ. Hence,

$$\det\left(\left(y_i^{i_{\phi^{-1}(j-1)}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right) = (-1)^{\phi} \det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)$$

(since permuting the columns of a matrix changes the determinant by the sign of the permutation). Combining this with (160), we obtain

$$\det\left(\left(y_i^{\mu_j+N-j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right) = (-1)^{\phi} \det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right). \tag{161}$$

Also, by the definition of μ, we have $\mu_{N+1} = i_{\pi^{-1}(N)} + N = 0$ (because $-N = i_{\pi^{-1}(N)}$), and thus we can apply Theorem 3.13.6 to μ instead of λ. This results in

$$\begin{aligned}
\operatorname{PSE}_N\left(S_\mu\left(x\right)\right) &= \frac{\det\left(\left(y_i^{\mu_j+N-j}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)} \\
&= \frac{\left(-1\right)^\phi \det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)} \qquad (162)
\end{aligned}$$

(by (161)). But (159) becomes

$$\begin{aligned}
&\operatorname{PSE}_N\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right) \\
&= \left(-1\right)^\phi \operatorname{PSE}_N\left(S_{\left(i_{\pi^{-1}(0)}+0, i_{\pi^{-1}(1)}+1, i_{\pi^{-1}(2)}+2,...\right)}\left(x\right)\right) \\
&= \left(-1\right)^\phi \operatorname{PSE}_N\left(S_\mu\left(x\right)\right) \qquad \left(\text{since } \left(i_{\pi^{-1}(0)}+0, i_{\pi^{-1}(1)}+1, i_{\pi^{-1}(2)}+2,...\right)=\mu\right) \\
&= \left(-1\right)^\phi \frac{\left(-1\right)^\phi \det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)} \qquad \left(\text{by (162)}\right) \\
&= \frac{\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}.
\end{aligned}$$

This proves (156). The proof of the 2nd step is thus complete.

3rd step: Consider the action of the Heisenberg algebra $\mathcal{A}$ on $\widetilde{F}=\mathcal{B}^{(0)}$ and $\wedge^{\frac{\infty}{2},0}V=\mathcal{F}^{(0)}$. We will now prove that the map $\tau:\wedge^{\frac{\infty}{2},0}V\to\widetilde{F}$ satisfies

$$\tau\circ a_{-m}=a_{-m}\circ\tau \qquad \text{for every positive integer } m. \qquad (163)$$

Proof of (163): Let m be a positive integer.

Let $\left(i_0,i_1,i_2,...\right)$ be a 0-degression. By the definition of a 0-degression, $\left(i_0,i_1,i_2,...\right)$ is a strictly decreasing sequence of integers such that every sufficiently high $k\in\mathbb{N}$ satisfies $i_k+k=0$. In other words, there exists an $\ell\in\mathbb{N}$ such that every integer $k\geq\ell$ satisfies $i_k+k=0$. Consider this ℓ.

Let N be any integer satisfying $N\geq\ell+m$. Then, it is easy to see that, for every integer $k\geq N$, we have $i_k+m=i_{k-m}$.

By the definition of the $\mathcal{A}$-module structure on $\wedge^{\frac{\infty}{2},0}V$, the action of a_{-m} on $\wedge^{\frac{\infty}{2},0}V$ is $\widehat{\rho}\left(T^{-m}\right)$, where T is the shift operator. Thus,

$$a_{-m}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)=\left(\widehat{\rho}\left(T^{-m}\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right). \qquad (164)$$

Since $m\neq 0$, the matrix T^{-m} has the property that, for every integer i, the $\left(i,i\right)$-th entry of T^{-m} is 0. Hence, Proposition 3.7.5 (applied to 0, T^{-m} and v_{i_k} instead of m,

a and b_k) yields

$$\begin{aligned}
&\left(\widehat{\rho}\left(T^{-m}\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\\
&=\sum_{k\geq 0} v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge \underbrace{\left(T^{-m}\rightharpoonup v_{i_k}\right)}_{=v_{i_k+m}}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\\
&=\sum_{k\geq 0} v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{i_k+m}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\\
&=\underbrace{\sum_{\substack{k\geq 0;\\ k<N}}}_{=\sum\limits_{k=0}^{N-1}} \underbrace{v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{i_k+m}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...}_{\substack{=v_{i_0+\delta_{0,k}m}\wedge v_{i_1+\delta_{1,k}m}\wedge ...\wedge v_{i_{k-1}+\delta_{k-1,k}m}\wedge v_{i_k+\delta_{k,k}m}\wedge v_{i_{k+1}+\delta_{k+1,k}m}\wedge v_{i_{k+2}+\delta_{k+2,k}m}\wedge ...\\ \text{(here we are simply making use of the fact that every } j\in\mathbb{N} \text{ such that } j\neq k \text{ satisfies}\\ i_j=i_j+\delta_{j,k}m \text{ (since } \delta_{j,k}=0\text{), whereas } i_k+m=i_k+\delta_{k,k}m \text{ (since } \delta_{k,k}=1\text{))}}}\\
&\qquad +\sum_{k\geq N} \underbrace{v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{i_k+m}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...}_{\substack{=0 \text{ (because the sequence } (i_0,i_1,...,i_{k-1},i_k+m,i_{k+1},i_{k+2},...)\\ \text{has two equal elements (since } i_k+m=i_{k-m}\text{))}}}\\
&=\sum_{k=0}^{N-1} v_{i_0+\delta_{0,k}m}\wedge v_{i_1+\delta_{1,k}m}\wedge ...\wedge v_{i_{k-1}+\delta_{k-1,k}m}\wedge v_{i_k+\delta_{k,k}m}\wedge v_{i_{k+1}+\delta_{k+1,k}m}\wedge v_{i_{k+2}+\delta_{k+2,k}m}\wedge ...\\
&=\sum_{k=0}^{N-1} v_{i_0+\delta_{0,k}m}\wedge v_{i_1+\delta_{1,k}m}\wedge v_{i_2+\delta_{2,k}m}\wedge ....
\end{aligned}$$

Combined with (164), this yields

$$a_{-m}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)=\sum_{k=0}^{N-1} v_{i_0+\delta_{0,k}m}\wedge v_{i_1+\delta_{1,k}m}\wedge v_{i_2+\delta_{2,k}m}\wedge ...,$$

so that

$$\begin{aligned}
&\mathrm{PSE}_N\left(\tau\left(a_{-m}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge...\right)\right)\right)\\
&=\mathrm{PSE}_N\left(\tau\left(\sum_{k=0}^{N-1} v_{i_0+\delta_{0,k}m}\wedge v_{i_1+\delta_{1,k}m}\wedge v_{i_2+\delta_{2,k}m}\wedge...\right)\right)\\
&=\sum_{k=0}^{N-1}\underbrace{\mathrm{PSE}_N\left(\tau\left(v_{i_0+\delta_{0,k}m}\wedge v_{i_1+\delta_{1,k}m}\wedge v_{i_2+\delta_{2,k}m}\wedge...\right)\right)}_{\substack{=\dfrac{\det\left(\left(y_i^{i_{j-1}+\delta_{j-1,k}m+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}\\ \text{(by (156), applied to }\left(i_0+\delta_{0,k}m,i_1+\delta_{1,k}m,i_2+\delta_{2,k}m,...\right)\\ \text{instead of }(i_0,i_1,i_2,...)\text{ (since }\left(i_0+\delta_{0,k}m,i_1+\delta_{1,k}m,i_2+\delta_{2,k}m,...\right)\\ \text{is easily seen to be an }N\text{-finished straying 0-degression))}}}\\
&\qquad\left(\text{since }\mathrm{PSE}_N\text{ and }\tau\text{ are both linear}\right)\\
&=\sum_{k=0}^{N-1}\frac{\det\left(\left(y_i^{i_{j-1}+\delta_{j-1,k}m+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}\\
&=\sum_{k=1}^{N}\frac{\det\left(\left(y_i^{i_{j-1}+\delta_{j-1,k-1}m+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}\\
&\qquad\left(\text{here, we substituted }k-1\text{ for }k\text{ in the sum}\right)\\
&=\sum_{k=1}^{N}\frac{\det\left(\left(y_i^{i_{j-1}+\delta_{j,k}m+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}\qquad\left(\text{since }\delta_{j-1,k-1}=\delta_{j,k}\text{ for all }j\text{ and }k\right)\\
&=\frac{1}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}\underbrace{\sum_{k=1}^{N}\det\left(\left(y_i^{i_{j-1}+\delta_{j,k}m+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}_{\substack{=\left(y_1^m+y_2^m+...+y_N^m\right)\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)\\ \text{(by Corollary 3.13.8)}}}\\
&=\frac{1}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}\left(y_1^m+y_2^m+...+y_N^m\right)\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)\\
&=\left(y_1^m+y_2^m+...+y_N^m\right)\cdot\frac{\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}.
\end{aligned}\tag{165}$$

On the other hand, since $(i_0,i_1,i_2,...)$ is strictly decreasing, $(i_0,i_1,i_2,...)$ is N-finished. Thus, (156) yields

$$\mathrm{PSE}_N\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge...\right)\right)=\frac{\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}.\tag{166}$$

Now, (165) becomes

$$\begin{aligned}
&\operatorname{PSE}_N\left(\tau\left(a_{-m}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\right)\\
&=\underbrace{\left(y_1^m+y_2^m+...+y_N^m\right)}_{\substack{=m\operatorname{PSE}_N(x_m)\\ \text{(since the definition of }\operatorname{PSE}_N\text{ yields}\\ \operatorname{PSE}_N(x_m)=\frac{y_1^m+y_2^m+...+y_N^m}{m})}}\cdot\underbrace{\frac{\det\left(\left(y_i^{i_{j-1}+N-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}{\det\left(\left(y_i^{j-1}\right)_{1\leq i\leq N,\ 1\leq j\leq N}\right)}}_{\substack{=\operatorname{PSE}_N\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\\ \text{(by (166))}}}\\
&=m\operatorname{PSE}_N\left(x_m\right)\cdot\operatorname{PSE}_N\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\\
&=\operatorname{PSE}_N\underbrace{\left(mx_m\cdot\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)}_{\substack{=a_{-m}\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\\ \text{(since }a_{-m}\text{ acts on }\widetilde{F}\text{ as multiplication by }mx_m)}}\\
&\qquad\text{(since }\operatorname{PSE}_N\text{ is a }\mathbb{C}\text{-algebra homomorphism)}\\
&=\operatorname{PSE}_N\left(a_{-m}\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\right).
\end{aligned}$$

Now forget that we fixed N. We thus have shown that every integer $N\geq\ell+m$ satisfies

$$\operatorname{PSE}_N\left(\tau\left(a_{-m}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\right)=\operatorname{PSE}_N\left(a_{-m}\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\right).$$

Hence,

$$\operatorname{PSE}_N\left(\tau\left(a_{-m}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\right)=\operatorname{PSE}_N\left(a_{-m}\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)\right)$$

for every sufficiently high $N\in\mathbb{N}$. Thus, Corollary 3.13.5 (applied to $P=\tau\left(a_{-m}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)$ and $Q=a_{-m}\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)$) yields that

$$\tau\left(a_{-m}\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)=a_{-m}\left(\tau\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right).$$

In other words,

$$\left(\tau\circ a_{-m}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)=\left(a_{-m}\circ\tau\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right).$$

Now forget that we fixed $(i_0,i_1,i_2,...)$. We have thus shown that $\left(\tau\circ a_{-m}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)=\left(a_{-m}\circ\tau\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)$ for every 0-degression $(i_0,i_1,i_2,...)$. Hence, the maps $\tau\circ a_{-m}$ and $a_{-m}\circ\tau$ are equal to each other on a basis of $\wedge^{\frac{\infty}{2},0}V$ (namely, on the basis $\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)_{(i_0,i_1,i_2,...)\text{ is a 0-degression}}$). Since these two maps are linear, this yields that these two maps must be identical, i. e., we have $\tau\circ a_{-m}=a_{-m}\circ\tau$. This proves (163). The proof of the 3rd step is thus complete.

4th step: We can now easily conclude Theorem 3.12.11.

Let $\mathcal{A}_-$ be the Lie subalgebra $\langle a_{-1},a_{-2},a_{-3},...\rangle$ of $\mathcal{A}$. Then, τ is an $\mathcal{A}_-$-module homomorphism $\wedge^{\frac{\infty}{2},0}V\to\widetilde{F}$ (according to (163)).

Consider the element $\psi_0=v_0\wedge v_{-1}\wedge v_{-2}\wedge ...$ of $\wedge^{\frac{\infty}{2},0}V=\mathcal{F}^{(0)}$. By the definition of σ_0, we have $\sigma_0(1)=\psi_0$, so that $\sigma_0^{-1}(\psi_0)=1$. Compared with

$$\begin{aligned}
\tau\left(\psi_0\right)&=\tau\left(v_0\wedge v_{-1}\wedge v_{-2}\wedge ...\right)\qquad\text{(since }\psi_0=v_0\wedge v_{-1}\wedge v_{-2}\wedge ...)\\
&=1,
\end{aligned}$$

this yields $\tau(\psi_0) = \sigma_0^{-1}(\psi_0)$.

From Lemma 2.2.10, it is clear that the Fock module F is generated by 1 as an $\mathcal{A}_-$-module (since $\mathcal{A}_- = \langle a_{-1}, a_{-2}, a_{-3}, ...\rangle$). Since there exists an $\mathcal{A}_-$-module isomorphism $F \to \widetilde{F}$ which sends 1 to 1 (in fact, the map resc of Proposition 2.2.21 is such an isomorphism), this yields that $\widetilde{F}$ is generated by 1 as an $\mathcal{A}_-$-module. Since there exists an $\mathcal{A}_-$-module isomorphism $\widetilde{F} \to \wedge^{\frac{\infty}{2},0} V$ which sends 1 to ψ_0 (in fact, the map σ_0 is such an isomorphism, since $\sigma_0(1) = \psi_0$), this yields that $\wedge^{\frac{\infty}{2},0} V$ is generated by ψ_0 as an $\mathcal{A}_-$-module. Hence, if two $\mathcal{A}_-$-module homomorphisms from $\wedge^{\frac{\infty}{2},0} V$ to another $\mathcal{A}_-$-module are equal to each other on ψ_0, then they must be identical. We can apply this observation to the two $\mathcal{A}_-$-module homomorphisms $\tau : \wedge^{\frac{\infty}{2},0} V \to \widetilde{F}$ and $\sigma_0^{-1} : \wedge^{\frac{\infty}{2},0} V \to \widetilde{F}$ (which are equal to each other on ψ_0, since $\tau(\psi_0) = \sigma_0^{-1}(\psi_0)$), and conclude that these homomorphisms are identical, i. e., we have $\tau = \sigma_0^{-1}$. Now, every 0-degression $(i_0, i_1, i_2, ...)$ satisfies

$$\begin{aligned}
\sigma^{-1}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) &= \underbrace{\sigma_0^{-1}}_{=\tau}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \tau(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) \\
&= S_{(i_0+0, i_1+1, i_2+2, ...)}(x) \qquad \text{(by the definition of } \tau) \\
&= S_\lambda(x),
\end{aligned}$$

where $\lambda = (i_0 + 0, i_1 + 1, i_2 + 2, ...)$. This proves Theorem 3.12.11.

3.14. Expliciting σ^{-1} using Schur polynomials: second proof

We are next going to give a second proof of Theorem 3.12.11. We will give this proof in two versions: The first version (Subsection 3.14.7) will proceed by manipulations with infinite matrices, using various properties of infinite matrices acting on $\wedge^{\frac{\infty}{2},m} V$. Since we are not going to prove all these properties, this first version is not completely self-contained (although the missing proofs are easy to fill in). The second version (Subsection 3.14.8) will be a rewriting of the first version without the use of all these properties of infinite matrices; it is self-contained. Both versions of the proof require lengthy preparations, some of which (like the definition of $\mathrm{GL}(\infty)$) will also turn out useful to us later.

3.14.1. The multivariate Taylor formula

Before we step to the second proof of Theorem 3.12.11, we show a lemma about polynomials over $\mathbb{Q}$-algebras:

Lemma 3.14.1. Let K be a commutative $\mathbb{Q}$-algebra, let $(y_1, y_2, y_3, ...)$ be a sequence of elements of K, and let $(z_1, z_2, z_3, ...)$ be a sequence of new symbols. Denote the sequence $(y_1, y_2, y_3, ...)$ by y. Denote the sequence $(z_1, z_2, z_3, ...)$ by z. Then, every

$P \in K[z_1, z_2, z_3, ...]$ satisfies

$$\exp\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right) P(z) = P(y+z).$$

Here, $y+z$ means the componentwise sum of the sequences y and z (so that $y+z = (y_1+z_1, y_2+z_2, y_3+z_3, ...)$).

Lemma 3.14.1 is actually a multivariate generalization of the famous Taylor formula

$$\exp\left(\alpha \frac{\partial}{\partial \xi}\right) P(\xi) = P(\alpha+\xi)$$

which holds for any polynomial $P \in K[\xi]$ and any $\alpha \in K$.

Proof of Lemma 3.14.1. Let A be the map

$$\exp\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right) : K[z_1, z_2, z_3, ...] \to K[z_1, z_2, z_3, ...]$$

(this is easily seen to be well-defined). Let B be the map

$$K[z_1, z_2, z_3, ...] \to K[z_1, z_2, z_3, ...], \qquad P \mapsto P(y+z).$$

We have $A = \exp\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)$, so that A is the exponential of a derivation (since $\sum_{s>0} y_s \frac{\partial}{\partial z_s}$ is a derivation). Thus, A is a K-algebra homomorphism (since there is a known fact that the exponential of a derivation is a K-algebra homomorphism). Combined with the fact that B is a K-algebra homomorphism (in fact, B is an evaluation homomorphism), this yields that both A and B are K-algebra homomorphisms.

Now, let k be a positive integer. We will prove that $Az_k = Bz_k$.

We have

$$\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right) z_k = \sum_{s>0} y_s \frac{\partial}{\partial z_s} z_k = y_k \underbrace{\frac{\partial}{\partial z_k} z_k}_{=1} + \sum_{\substack{s>0;\\ s\neq k}} y_s \underbrace{\frac{\partial}{\partial z_s} z_k}_{\substack{=0\\ (\text{since } s\neq k)}} = y_k + \underbrace{\sum_{\substack{s>0;\\ s\neq k}} y_s 0}_{=0} = y_k,$$

so that

$$\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)^2 z_k = \left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right) \underbrace{\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right) z_k}_{=y_k} = \left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right) y_k = \sum_{s>0} y_s \underbrace{\frac{\partial}{\partial z_s} y_k}_{=0} = 0.$$

As a consequence,

$$\text{every integer } i \geq 2 \text{ satisfies } \left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)^i z_k = 0. \tag{167}$$

Now, since $A = \exp\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right) = \sum_{i\in\mathbb{N}} \frac{1}{i!}\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)^i$, we have

$$\begin{aligned}
Az_k &= \sum_{i\in\mathbb{N}} \frac{1}{i!}\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)^i z_k \\
&= \underbrace{\frac{1}{0!}}_{=1} \underbrace{\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)^0}_{=\mathrm{id}} z_k + \underbrace{\frac{1}{1!}}_{=1} \underbrace{\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)^1}_{=\sum_{s>0} y_s \frac{\partial}{\partial z_s}} z_k + \sum_{i\geq 2} \frac{1}{i!} \underbrace{\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)^i z_k}_{\substack{=0 \\ \text{(by (167))}}} \\
&= \underbrace{\mathrm{id}\, z_k}_{=z_k} + \underbrace{\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right) z_k}_{=y_k} + \underbrace{\sum_{i\geq 2} \frac{1}{i!} 0}_{=0} = z_k + y_k = y_k + z_k.
\end{aligned}$$

Compared to

$$\begin{aligned}
Bz_k &= z_k(y+z) \qquad \text{(by the definition of } B\text{)} \\
&= y_k + z_k,
\end{aligned}$$

this yields $Az_k = Bz_k$.

Now, forget that we fixed k. We thus have shown that $Az_k = Bz_k$ for every positive integer k. In other words, the maps A and B coincide on the set $\{z_1, z_2, z_3, ...\}$. Since the set $\{z_1, z_2, z_3, ...\}$ generates $K[z_1, z_2, z_3, ...]$ as a K-algebra, this yields that the maps A and B coincide on a generating set of the K-algebra $K[z_1, z_2, z_3, ...]$. Since A and B are K-algebra homomorphisms, this yields that $A = B$ (because if two K-algebra homomorphisms coincide on a K-algebra generating set of their domain, then they must be equal). Hence, every $P \in K[z_1, z_2, z_3, ...]$ satisfies

$$\underbrace{\exp\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)}_{=A=B} \underbrace{P(z)}_{=P} = BP = P(y+z)$$

(by the definition of B). This proves Lemma 3.14.1.

3.14.2. $\mathrm{GL}(\infty)$ and $\mathrm{M}(\infty)$

We now introduce the groups $\mathrm{GL}(\infty)$ and $\mathrm{M}(\infty)$ and their actions on $\wedge^{\frac{\infty}{2},m} V$. On the one hand, this will prepare us to the second proof of Theorem 3.12.11; on the other hand, these group actions are of autonomous interest, and we will meet them again in Subsection 3.15.2.

Definition 3.14.2. We let $\mathrm{M}(\infty)$ denote the set $\mathrm{id} + \mathfrak{gl}_\infty$. In other words, we let $\mathrm{M}(\infty)$ denote the set of all infinite matrices (infinite in both directions) which are equal to the infinite identity matrix id in all but finitely many entries.

Clearly, $\mathrm{M}(\infty) \subseteq \overline{\mathfrak{a}_\infty}$ as sets. We notice that:

Proposition 3.14.3. **(a)** For every $A \in \mathrm{M}(\infty)$ and $B \in \mathrm{M}(\infty)$, the matrix AB is well-defined and lies in $\mathrm{M}(\infty)$.

(b) We have $\mathrm{id} \in \mathrm{M}(\infty)$ (where id denotes the infinite identity matrix).

(c) The set $\mathrm{M}(\infty)$ becomes a monoid under multiplication of matrices.

(d) If a matrix $A \in \mathrm{M}(\infty)$ is invertible, then its inverse also lies in $\mathrm{M}(\infty)$.

(e) Denote by $\mathrm{GL}(\infty)$ the subset $\{A \in \mathrm{M}(\infty) \mid A \text{ is invertible}\}$ of $\mathrm{M}(\infty)$. Then, $\mathrm{GL}(\infty)$ becomes a group under multiplication of matrices.

Remark 3.14.4. In Proposition 3.14.3, a matrix $A \in \mathrm{M}(\infty)$ is said to be *invertible* if there exists an infinite matrix B (with rows and columns indexed by integers) satisfying $AB = BA = \mathrm{id}$. The matrix B is then called the *inverse* of A. Note that we don't a-priori require that B lie in $\mathrm{M}(\infty)$, or any other "finiteness conditions" for B; Proposition 3.14.3 **(d)** shows that these conditions are automatically satisfied.

Definition 3.14.5. Let $\mathrm{GL}(\infty)$ denote the group $\mathrm{GL}(\infty)$ defined in Proposition 3.14.3 **(e)**.

Proof of Proposition 3.14.3. **(a)** Let $A \in \mathrm{M}(\infty)$ and $B \in \mathrm{M}(\infty)$. Since $A \in \mathrm{M}(\infty) = \mathrm{id} + \mathfrak{gl}_\infty$, there exists an $a \in \mathfrak{gl}_\infty$ such that $A = \mathrm{id} + a$. Consider this a.

Since $B \in \mathrm{M}(\infty) = \mathrm{id} + \mathfrak{gl}_\infty$, there exists a $b \in \mathfrak{gl}_\infty$ such that $B = \mathrm{id} + b$. Consider this b.

Since $A = \mathrm{id} + a$ and $B = \mathrm{id} + b$, we have $AB = (\mathrm{id} + a)(\mathrm{id} + b) = \mathrm{id} + a + b + ab$, which is clearly well-defined (because $a \in \mathfrak{gl}_\infty$ and $b \in \mathfrak{gl}_\infty$ lead to ab being well-defined) and lies in $\mathrm{M}(\infty)$ (since $\underbrace{a}_{\in \mathfrak{gl}_\infty} + \underbrace{b}_{\in \mathfrak{gl}_\infty} + \underbrace{ab}_{\substack{\in \mathfrak{gl}_\infty \\ (\text{since } a \in \mathfrak{gl}_\infty \text{ and } b \in \mathfrak{gl}_\infty)}} \in \mathfrak{gl}_\infty + \mathfrak{gl}_\infty + \mathfrak{gl}_\infty \subseteq \mathfrak{gl}_\infty$ and thus $\mathrm{id} + a + b + ab \in \mathrm{id} + \mathfrak{gl}_\infty = \mathrm{M}(\infty)$). This proves Proposition 3.14.3 **(a)**.

(b) Trivial.

(c) Follows from **(a)** and **(b)**.

(d) Let $A \in \mathrm{M}(\infty)$ be invertible.

Since $A \in \mathrm{M}(\infty) = \mathrm{id} + \mathfrak{gl}_\infty$, there exists an $a \in \mathfrak{gl}_\infty$ such that $A = \mathrm{id} + a$. Consider this a.

Since A is invertible, there exists an infinite matrix B (with rows and columns indexed by integers) satisfying $AB = BA = \mathrm{id}$ (according to how we defined "invertible" in Remark 3.14.4). Consider this B. This B is the inverse of A. Let $b = B - \mathrm{id}$. Then, $B = \mathrm{id} + b$. Since $A = \mathrm{id} + a$ and $B = \mathrm{id} + b$, we have $AB = (\mathrm{id} + a)(\mathrm{id} + b) = \mathrm{id} + a + b + ab$, which is clearly well-defined (because $a \in \mathfrak{gl}_\infty$ leads to ab being well-defined). Since $\mathrm{id} = AB = \mathrm{id} + ab + a + b$, we have $0 = ab + a + b$.

Let us introduce two notations that we will use during this proof:

- For any infinite matrix M and any pair (i, j) of integers, let us denote by $M_{i,j}$ the (i, j)-th entry of the matrix M. (In particular, for any pair (i, j) of integers, we denote by $a_{i,j}$ the (i, j)-th entry of the matrix a (not of the matrix A !), and we denote by $b_{i,j}$ the (i, j)-th entry of the matrix b (not of the matrix B !).)

- For any assertion $\mathcal{A}$, let $[\mathcal{A}]$ denote the integer $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is wrong} \end{cases}$.

Since $a \in \mathfrak{gl}_\infty$, only finitely many entries of the matrix a are nonzero. In particular, this yields that only finitely many columns of the matrix a are nonzero. Hence, there exists a nonnegative integer N such that

$$\left(\text{for every integer } j \text{ with } |j| > N, \text{ the } j\text{-th column of } a \text{ is zero}\right). \tag{168}$$

Consider this N. Clearly,

$$\left(\text{for every } (i,j) \in \mathbb{Z}^2 \text{ such that } |j| > N, \text{ we have } a_{i,j} = 0\right) \tag{169}$$

(because for every $(i,j) \in \mathbb{Z}^2$ such that $|j| > N$, the j-th column of a is zero (by (168)), so that every entry on the j-th column of a is zero, so that $a_{i,j}$ is zero (because the element $a_{i,j}$ is the (i,j)-th entry of a, hence an entry on the j-th column of a)).

Recall that only finitely many entries of the matrix a are nonzero. In particular, this yields that only finitely many rows of the matrix a are nonzero. Hence, there exists a nonnegative integer M such that

$$\left(\text{for every integer } i \text{ with } |i| > M, \text{ the } i\text{-th row of } a \text{ is zero}\right). \tag{170}$$

Consider this M. Clearly,

$$\left(\text{for every } (i,j) \in \mathbb{Z}^2 \text{ such that } |i| > M, \text{ we have } a_{i,j} = 0\right) \tag{171}$$

(because for every $(i,j) \in \mathbb{Z}^2$ such that $|i| > M$, the i-th row of a is zero (by (170)), so that every entry on the i-th row of a is zero, so that $a_{i,j}$ is zero (because the element $a_{i,j}$ is the (i,j)-th entry of a, hence an entry on the i-th row of a)).

Let $P = \max\{M, N\}$. Clearly, $P \geq M$ and $P \geq N$. It is now easy to see that

$$\text{any } (i,j) \in \mathbb{Z}^2 \text{ satisfies } a_{i,j} = [|i| \leq P] \cdot a_{i,j}. \tag{172}$$

[141] Similarly,

$$\text{any } (i,j) \in \mathbb{Z}^2 \text{ satisfies } a_{i,j} = [|j| \leq P] \cdot a_{i,j}. \tag{173}$$

Let b' be the infinite matrix (with rows and columns indexed by integers) defined by

$$\left(b'_{i,j} = [|i| \leq P] \cdot [|j| \leq P] \cdot b_{i,j} \qquad \text{for all } (i,j) \in \mathbb{Z}^2\right). \tag{174}$$

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[141] *Proof of (172):* Let $(i,j) \in \mathbb{Z}^2$. Then, we must be in one of the following three cases:

Case 1: We don't have $|i| \leq P$.

Case 2: We have $|i| \leq P$.

Let us consider Case 1 first. In this case, we don't have $|i| \leq P$. Thus, $[|i| \leq P] = 0$ and $|i| > P$. From $|i| > P \geq M$, we conclude that $a_{i,j} = 0$ (by (171)). Compared with $\underbrace{[|i| \leq P]}_{=0} \cdot a_{i,j} = 0$, this yields $a_{i,j} = [|i| \leq P] \cdot a_{i,j}$. Hence, (172) is proven in Case 1.

Finally, let us consider Case 2. In this case, we have $|i| \leq P$. Hence, $[|i| \leq P] = 1$. Thus, $\underbrace{[|i| \leq P]}_{=1} \cdot a_{i,j} = a_{i,j}$. Hence, (172) is proven in Case 2.

Altogether, we have thus proven (172) in each of the two cases 1 and 2. Since these two cases cover all possibilities, this shows that (172) always holds. Thus, (172) is proven.

It is clear that only finitely many entries of b' are nonzero[142]. In other words, $b' \in \mathfrak{gl}_\infty$, so that $\operatorname{id} + b' \in \operatorname{id} + \mathfrak{gl}_\infty = \mathrm{M}(\infty)$.

We will now prove that $A(\operatorname{id} + b') = \operatorname{id}$.

For every $(i, j) \in \mathbb{Z}^2$, we have

$$\begin{aligned}
&(ab' + a + b')_{i,j} \\
&= \underbrace{(ab')_{i,j}}_{\substack{= \sum_{k\in\mathbb{Z}} a_{i,k} b'_{k,j} \\ \text{(by the definition of the} \\ \text{product of two matrices)}}} + a_{i,j} + \underbrace{b'_{i,j}}_{\substack{=[|i|\leq P]\cdot[|j|\leq P]\cdot b_{i,j} \\ \text{(by (174))}}} \\
&= \sum_{k\in\mathbb{Z}} a_{i,k} \underbrace{b'_{k,j}}_{\substack{=[|k|\leq P]\cdot[|j|\leq P]\cdot b_{k,j} \\ \text{(by (174), applied to} \\ k \text{ instead of } i)}} + a_{i,j} + [|i| \leq P] \cdot [|j| \leq P] \cdot b_{i,j} \\
&= \sum_{k\in\mathbb{Z}} \underbrace{a_{i,k} [|k| \leq P]}_{\substack{=[|k|\leq P]\cdot a_{i,k} = a_{i,k} \\ \text{(since (173) (applied to} \\ k \text{ instead of } j) \text{ yields } a_{i,k} = [|k|\leq P]\cdot a_{i,k})}} \cdot [|j| \leq P] \cdot b_{k,j} + \underbrace{a_{i,j}}_{\substack{=[|i|\leq P]\cdot a_{i,j} \\ \text{(by (172))}}} + [|i| \leq P] \cdot [|j| \leq P] \cdot b_{i,j} \\
&= \sum_{k\in\mathbb{Z}} \underbrace{a_{i,k}}_{\substack{=[|i|\leq P]\cdot a_{i,k} \\ \text{(by (172), applied to} \\ k \text{ instead of } j)}} \cdot [|j| \leq P] \cdot b_{k,j} + [|i| \leq P] \cdot \underbrace{a_{i,j}}_{\substack{=[|j|\leq P]\cdot a_{i,j} \\ \text{(by (173))}}} + [|i| \leq P] \cdot [|j| \leq P] \cdot b_{i,j} \\
&= \sum_{k\in\mathbb{Z}} [|i| \leq P] \cdot a_{i,k} \cdot [|j| \leq P] \cdot b_{k,j} + [|i| \leq P] \cdot [|j| \leq P] \cdot a_{i,j} + [|i| \leq P] \cdot [|j| \leq P] \cdot b_{i,j} \\
&= [|i| \leq P] \cdot [|j| \leq P] \cdot \left(\sum_{k\in\mathbb{Z}} a_{i,k} b_{k,j} + a_{i,j} + b_{i,j} \right) = [|i| \leq P] \cdot [|j| \leq P] \cdot \underbrace{\left((ab)_{i,j} + a_{i,j} + b_{i,j} \right)}_{\substack{=(ab+a+b)_{i,j} = 0 \\ \text{(since } ab+a+b=0)}} \\
&\qquad \left(\begin{array}{c} \text{since } (ab)_{i,j} = \sum_{k\in\mathbb{Z}} a_{i,k} b_{k,j} \text{ (by the definition of the product of two matrices),} \\ \text{so that } \sum_{k\in\mathbb{Z}} a_{i,k} b_{k,j} = (ab)_{i,j} \end{array} \right) \\
&= 0.
\end{aligned}$$

Thus, $ab' + a + b' = 0$. Since $A = \operatorname{id} + a$, we have $A(\operatorname{id} + b') = (\operatorname{id} + a)(\operatorname{id} + b') = \operatorname{id} + \underbrace{ab' + a + b'}_{=0} = \operatorname{id}$.

We thus have shown that $A(\operatorname{id} + b') = \operatorname{id}$.

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[142] *Proof.* Let $(i, j) \in \mathbb{Z}^2$ such that $b'_{i,j} \neq 0$. Then, $|i| \leq P$ (because otherwise, we would have $[|i| \leq P] = 0$, so that $b'_{i,j} = \underbrace{[|i| \leq P]}_{=0} \cdot [|j| \leq P] \cdot b_{i,j} = 0$, contradicting to $b'_{i,j} \neq 0$), so that $i \in \{-P, -P+1, ..., P\}$, and similarly $j \in \{-P, -P+1, ..., P\}$. Hence, $(i, j) \in \{-P, -P+1, ..., P\}^2$ (since $i \in \{-P, -P+1, ..., P\}$ and $j \in \{-P, -P+1, ..., P\}$).

Now forget that we fixed (i, j). We thus have showed that every $(i, j) \in \mathbb{Z}^2$ such that $b'_{i,j} \neq 0$ satisfies $(i, j) \in \{-P, -P+1, ..., P\}^2$. Since there are only finitely many $(i, j) \in \{-P, -P+1, ..., P\}^2$, this yields that there are only finitely many $(i, j) \in \mathbb{Z}^2$ such that $b'_{i,j} \neq 0$. In other words, there are only finitely many $(i, j) \in \mathbb{Z}^2$ such that the (i, j)-th entry of b' is nonzero. In other words, only finitely many entries of b' are nonzero, qed.

Now, it is easy to see that the products $B(A(\operatorname{id}+b'))$ and $(BA)(\operatorname{id}+b')$ are well-defined and satisfy associativity, i. e., we have $B(A(\operatorname{id}+b')) = (BA)(\operatorname{id}+b')$. Now,

$$B = B \cdot \underbrace{\operatorname{id}}_{=A(\operatorname{id}+b')} = B(A(\operatorname{id}+b')) = \underbrace{(BA)}_{=\operatorname{id}}(\operatorname{id}+b') = \operatorname{id}+b' \in \mathrm{M}(\infty).$$

Since B is the inverse of A, this yields that the inverse of A lies in $\mathrm{M}(\infty)$. This proves Proposition 3.14.3 **(d)**.

(e) Follows from **(c)** and **(d)**.

The proof of Proposition 3.14.3 is complete.

We now construct a group action of $\mathrm{GL}(\infty)$ on $\mathcal{F}^{(m)}$ that is related to the Lie algebra action ρ of $\mathfrak{gl}_\infty$ on $\mathcal{F}^{(m)}$ in the same way as the action of a Lie group on a representation is usually related to its "derivative" action of the corresponding Lie algebra:

Definition 3.14.6. Let $m \in \mathbb{Z}$. We define an action $\varrho : \mathrm{M}(\infty) \to \operatorname{End}\left(\mathcal{F}^{(m)}\right)$ of the monoid $\mathrm{M}(\infty)$ on the vector space $\mathcal{F}^{(m)} = \wedge^{\frac{\infty}{2},m} V$ as follows: For every $A \in \mathrm{M}(\infty)$ and every m-degression $(i_0, i_1, i_2, ...)$, we set

$$(\varrho(A))(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = Av_{i_0} \wedge Av_{i_1} \wedge Av_{i_2} \wedge ....$$

(This is then extended to the whole $\mathcal{F}^{(m)}$ by linearity.) It is very easy to see that this is well-defined (because $Av_k = v_k$ for all sufficiently small k) and indeed gives a monoid action.

The restriction $\varrho\mid_{\mathrm{GL}(\infty)}: \mathrm{GL}(\infty) \to \operatorname{End}\left(\mathcal{F}^{(m)}\right)$ to $\mathrm{GL}(\infty)$ is thus a group action of $\mathrm{GL}(\infty)$ on $\mathcal{F}^{(m)}$.

Since we have defined an action of $\mathrm{M}(\infty)$ on $\mathcal{F}^{(m)}$ for every $m \in \mathbb{Z}$, we thus obtain an action of $\mathrm{M}(\infty)$ on $\mathcal{F} = \bigoplus_{m\in\mathbb{Z}} \mathcal{F}^{(m)}$ (namely, the direct sum of the previous actions). This latter action will also be denoted by ϱ.

Note that the letter ϱ is a capital rho, as opposed to ρ which is the lowercase rho.

When A is a matrix in $\mathrm{M}(\infty)$, the endomorphism $\varrho(A)$ of $\mathcal{F}^{(m)}$ can be seen as an infinite analogue of the endomorphisms $\wedge^\ell A$ of $\wedge^\ell V$ defined for all $\ell \in \mathbb{N}$.

We are next going to give an explicit formula for the action of $\varrho(A)$ on $\mathcal{F}^{(m)}$ in terms of (infinite) minors of A. The formula will be an infinite analogue of the following well-known formula:

Proposition 3.14.7. Let P be a finite-dimensional $\mathbb{C}$-vector space with basis $(e_1, e_2, ..., e_n)$, and let Q be a finite-dimensional $\mathbb{C}$-vector space with basis $(f_1, f_2, ..., f_m)$. Let $\ell \in \mathbb{N}$.

Let $f : P \to Q$ be a linear map, and let A be the $m \times n$-matrix which represents this map f with respect to the bases $(e_1, e_2, ..., e_n)$ and $(f_1, f_2, ..., f_m)$ of P and Q.

Let i_1, i_2, ..., i_ℓ be integers such that $1 \leq i_1 < i_2 < ... < i_\ell \leq n$. For any ℓ integers j_1, j_2, ..., j_ℓ satisfying $1 \leq j_1 < j_2 < ... < j_\ell \leq m$, let $A^{i_1,i_2,...,i_\ell}_{j_1,j_2,...,j_\ell}$ denote the matrix which is obtained from A by removing all columns except for the i_1-th, the i_2-th, ..., the i_ℓ-th ones and removing all rows except for the j_1-th, the j_2-th, ..., the j_ℓ-th ones. Then,

$$\left(\wedge^\ell f\right)(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_\ell}) = \sum_{\substack{j_1,\ j_2,\ ...,\ j_\ell \text{ are } \ell \text{ integers;} \\ 1\leq j_1<j_2<...<j_\ell\leq m}} \det\left(A^{i_1,i_2,...,i_\ell}_{j_1,j_2,...,j_\ell}\right) e_{j_1} \wedge e_{j_2} \wedge ... \wedge e_{j_\ell}.$$

Note that Proposition 3.14.7 is the main link between exterior powers and minors of matrices. It is commonly used both to prove results involving exterior powers and to give slick proofs of identities involving minors.

In order to obtain an infinite analogue of this result, we need to first define determinants of infinite matrices. This cannot be done for arbitrary infinite matrices, but there exist classes of infinite matrices for which a notion of determinant can be made sense of. Let us define it for so-called "upper almost-unitriangular" matrices:

Definition 3.14.8. **(a)** In the following, when S and T are two sets of integers (not necessarily finite), an *$S \times T$-matrix* will mean a matrix whose rows are indexed by the elements of S and whose columns are indexed by the elements of T. (Hence, the elements of $\mathfrak{gl}_\infty$, as well as those of $\overline{\mathfrak{a}_\infty}$ and those of $\mathrm{M}(\infty)$, are $\mathbb{Z} \times \mathbb{Z}$-matrices.)

(b) If S is a set of integer, then an $S \times S$-matrix B over $\mathbb{C}$ is said to be *upper unitriangular* if it satisfies the following two assertions:

– All entries on the main diagonal of B are $= 1$.

– All entries of B below the main diagonal are $= 0$.

(c) An $\mathbb{N} \times \mathbb{N}$-matrix B over $\mathbb{C}$ is said to be *upper almost-unitriangular* if it satisfies the following two assertions:

– All but finitely many of the entries on the main diagonal of B are $= 1$.

– All but finitely many of the entries of B below the main diagonal are $= 0$.

(d) Let B be an upper almost-unitriangular $\mathbb{N} \times \mathbb{N}$-matrix over $\mathbb{C}$. Then, we can write the matrix B in the form $\begin{pmatrix} C & D \\ 0 & E \end{pmatrix}$ for some $n \in \mathbb{N}$, some $\{0, 1, ..., n-1\} \times \{0, 1, ..., n-1\}$-matrix C, some $\{0, 1, ..., n-1\} \times \{n, n+1, n+2, ...\}$-matrix D, and some upper unitriangular $\{n, n+1, n+2, ...\} \times \{n, n+1, n+2, ...\}$-matrix E. The matrix C in such a representation of B will be called a *faithful block-triangular truncation* of B.

(e) Let B be an upper almost-unitriangular $\mathbb{N} \times \mathbb{N}$-matrix over $\mathbb{C}$. We define the *determinant* $\det B$ of the matrix B to be $\det C$, where C is a faithful block-triangular truncation of B. This is well-defined, because a faithful block-triangular truncation of B exists and because the determinant $\det C$ does not depend on the choice of the faithful block-triangular truncation C. (The latter assertion follows from the fact that $\det \begin{pmatrix} F & G \\ 0 & H \end{pmatrix} = \det F$ for any $n \in \mathbb{N}$, any $k \in \mathbb{N}$, any $n \times n$-matrix F, any $n \times k$-matrix G, and any upper unitriangular $k \times k$-matrix H.)

Now, the following fact (an analogue of Proposition 3.14.7) gives an explicit formula for the action of $\varrho(A)$:

Remark 3.14.9. Let $(i_0, i_1, i_2, ...)$ be an m-degression. Let $A \in \mathrm{M}(\infty)$. For any m-degression $(j_0, j_1, j_2, ...)$, let $A^{i_0, i_1, i_2, ...}_{j_0, j_1, j_2, ...}$ denote the $\mathbb{N} \times \mathbb{N}$-matrix defined by

$$\left(\left(\text{the } (u,v)\text{-th entry of } A^{i_0, i_1, i_2, ...}_{j_0, j_1, j_2, ...}\right) = \left(\text{the } (j_u, i_v)\text{-th entry of } A\right) \qquad \text{for every } (u,v) \in \mathbb{N}^2\right).$$

(In other words, let $A^{i_0, i_1, i_2, ...}_{j_0, j_1, j_2, ...}$ denote the matrix which is obtained from A by removing all columns except for the i_0-th, the i_1-th, the i_2-th, etc. ones and removing all rows except for the j_0-th, the j_1-th, the j_2-th, etc. ones, and then inverting the order of the rows, and inverting the order of the columns.) Then, for any m-degression $(j_0, j_1, j_2, ...)$, the matrix $A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}$ is upper almost-unitriangular (in fact, one can easily check that more is true: all but finitely many entries of $A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}$ are equal to the corresponding entries of the identity $\mathbb{N} \times \mathbb{N}$ matrix), and thus the determinant $\det\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right)$ makes sense (according to Definition 3.14.8 **(e)**). We have

$$(\varrho(A))(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{(j_0,j_1,j_2,...) \text{ is an } m\text{-degression}} \det\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right) v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ....$$

The analogy between Remark 3.14.9 and Proposition 3.14.7 is slightly obscured by technicalities (such as the fact that Remark 3.14.9 only concerns itself with certain endomorphisms of V and not with homomorphisms between different vector spaces, and the fact that the m-degressions in Remark 3.14.9 are decreasing, while the ℓ-tuples $(i_1, i_2, ..., i_\ell)$ and $(j_1, j_2, ..., j_\ell)$ in Proposition 3.14.7 are increasing). Still, it should be rather evident why Remark 3.14.9 is (informally speaking) a consequence of "the $\ell = \infty$ case" of Proposition 3.14.7.

3.14.3. Semiinfinite vectors and actions of $\mathfrak{u}_\infty$ and $\mathrm{U}(\infty)$ on $\wedge^{\frac{\infty}{2},m} V$

The actions of $\mathfrak{gl}_\infty$, $\mathfrak{a}_\infty$, $\mathrm{M}(\infty)$ and $\mathrm{GL}(\infty)$ on $\wedge^{\frac{\infty}{2},m} V$ have many good properties, but for what we want to do with them, they are in some sense "too small" (even $\mathfrak{a}_\infty$). Of course, we cannot let the space of **all** infinite matrices act on $\wedge^{\frac{\infty}{2},m} V$ (this space is not even a Lie algebra), but it turns out that we can get away with restricting ourselves to strictly upper-triangular infinite matrices. First, let us define a kind of completion of V:

Definition 3.14.10. **(a)** A family $(x_i)_{i\in\mathbb{Z}}$ of elements of some additive group indexed by integers is said to be *semiinfinite* if every sufficiently high $i \in \mathbb{Z}$ satisfies $x_i = 0$.

(b) Let $\widehat{V}$ be the vector subspace $\left\{v \in \mathbb{C}^{\mathbb{Z}} \mid v \text{ is semiinfinite}\right\}$ of $\mathbb{C}^{\mathbb{Z}}$. Let $\mathfrak{u}_\infty$ denote the Lie algebra of all **strictly** upper-triangular infinite matrices (with rows and columns indexed by integers). It is easy to see that the Lie algebra $\mathfrak{u}_\infty$ acts on the vector space $\widehat{V}$ in the obvious way: namely, for any $a \in \mathfrak{u}_\infty$ and $v \in \widehat{V}$, we let $a \rightharpoonup v$ be the product of the matrix a with the column vector v. Here, every element $(x_i)_{i\in\mathbb{Z}}$ of $\widehat{V}$ is identified with the column vector $\begin{pmatrix} ... \\ x_{-2} \\ x_{-1} \\ x_0 \\ x_1 \\ x_2 \\ ... \end{pmatrix}$.

The vector space V defined in Definition 3.5.2 clearly is a subspace of $\widehat{V}$. Restricting the $\mathfrak{u}_\infty$-action on $\widehat{V}$ to an $(\mathfrak{u}_\infty \cap \mathfrak{gl}_\infty)$-action on V yields the same $(\mathfrak{u}_\infty \cap \mathfrak{gl}_\infty)$-module as restricting the $\mathfrak{gl}_\infty$-action on V to an $(\mathfrak{u}_\infty \cap \mathfrak{gl}_\infty)$-action on V.

We thus have obtained an $\mathfrak{u}_\infty$-module $\widehat{V}$, which is a kind of completion of V. One could now hope that this allows us to construct an $\mathfrak{u}_\infty$-module structure on some kind of completion of $\wedge^{\frac{\infty}{2},m} V$. A quick observation shows that this works better than one would expect, because we don't have to take any completion of $\wedge^{\frac{\infty}{2},m} V$ (although we can if we want to). We can make $\wedge^{\frac{\infty}{2},m} V$ itself an $\mathfrak{u}_\infty$-module:

Definition 3.14.11. Let $\ell \in \mathbb{Z}$. Let $\pi_\ell : \widehat{V} \to V$ be the linear map which sends every $(x_i)_{i\in\mathbb{Z}} \in \widehat{V}$ to $\left(\begin{cases} x_i, & \text{if } i \geq \ell; \\ 0, & \text{if } i < \ell \end{cases} \right)_{i\in\mathbb{Z}} \in V$. (It is very easy to see that this map π_ℓ is well-defined.)

Definition 3.14.12. Let $m \in \mathbb{Z}$. Let $b_0, b_1, b_2, ...$ be vectors in $\widehat{V}$ which satisfy

$$\pi_{m-i}(b_i) = v_{m-i} \qquad \text{for all sufficiently large } i.$$

Define an element $b_0 \wedge b_1 \wedge b_2 \wedge ...$ of $\wedge^{\frac{\infty}{2},m} V$ as follows: Pick some $N \in \mathbb{N}$ such that every $i > N$ satisfies $\pi_{m-i}(b_i) = v_{m-i}$. (Such an N exists, since we know that $\pi_{m-i}(b_i) = v_{m-i}$ for all sufficiently large i.) Then, we define $b_0 \wedge b_1 \wedge b_2 \wedge ...$ to be the element

$$\pi_{m-N}(b_0) \wedge \pi_{m-N}(b_1) \wedge ... \wedge \pi_{m-N}(b_N) \wedge v_{m-N-1} \wedge v_{m-N-2} \wedge v_{m-N-3} \wedge ... \in \wedge^{\frac{\infty}{2},m} V.$$

This element does not depend on the choice of N (according to Proposition 3.14.13 below). Hence, $b_0 \wedge b_1 \wedge b_2 \wedge ...$ is well-defined.

The next few propositions state some properties of wedge products of elements of $\widehat{V}$ similar to some properties of wedge products of elements of V stated above. We will not prove them; neither of them is actually difficult to verify.

Proposition 3.14.13. Let $m \in \mathbb{Z}$. Let $b_0, b_1, b_2, ...$ be vectors in $\widehat{V}$ which satisfy

$$\pi_{m-i}(b_i) = v_{m-i} \qquad \text{for all sufficiently large } i.$$

If we pick some $N \in \mathbb{N}$ such that every $i > N$ satisfies $\pi_{m-i}(b_i) = v_{m-i}$, then the element

$$\pi_{m-N}(b_0) \wedge \pi_{m-N}(b_1) \wedge ... \wedge \pi_{m-N}(b_N) \wedge v_{m-N-1} \wedge v_{m-N-2} \wedge v_{m-N-3} \wedge ... \in \wedge^{\frac{\infty}{2},m} V$$

does not depend on the choice of N.

Proposition 3.14.14. The wedge product defined in Definition 3.14.12 is antisymmetric and multilinear (in the appropriate sense).

Definition 3.14.15. Let $m \in \mathbb{Z}$. Define an action of the Lie algebra $\mathfrak{u}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$ by the equation

$$a \rightharpoonup (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{k \geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge (a \rightharpoonup v_{i_k}) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...$$

for all $a \in \mathfrak{u}_\infty$ and all elementary semiinfinite wedges $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ (and by linear extension).

Proposition 3.14.16. Let $m \in \mathbb{Z}$. Then, Definition 3.14.15 really defines a representation of the Lie algebra $\mathfrak{u}_\infty$ on the vector space $\wedge^{\frac{\infty}{2},m} V$.

Proposition 3.14.17. Let $m \in \mathbb{Z}$. Let $b_0, b_1, b_2, ...$ be vectors in $\widehat{V}$ which satisfy

$$\pi_{m-i}(b_i) = v_{m-i} \qquad \text{for all sufficiently large } i.$$

Let $a \in \mathfrak{u}_\infty$. Then,

$$a \rightharpoonup (b_0 \wedge b_1 \wedge b_2 \wedge ...) = \sum_{k \geq 0} b_0 \wedge b_1 \wedge ... \wedge b_{k-1} \wedge (a \rightharpoonup b_k) \wedge b_{k+1} \wedge b_{k+2} \wedge ....$$

Definition 3.14.18. Let $m \in \mathbb{Z}$. Let $\rho : \mathfrak{u}_\infty \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ be the representation of $\mathfrak{u}_\infty$ on $\wedge^{\frac{\infty}{2},m} V$ defined in Definition 3.14.15. (We denote this representation by the same letter ρ as the representation $\mathfrak{gl}_\infty \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ from Definition 3.7.1. This is intentional and unproblematic, because both of these representations have the same restriction onto $\mathfrak{u}_\infty \cap \mathfrak{gl}_\infty$.)

Remark 3.14.19. Let $m \in \mathbb{Z}$. Let $a \in \mathfrak{u}_\infty \cap \overline{\mathfrak{a}_\infty}$. Then, $\rho(a) = \widehat{\rho}(a)$ (where $\rho(a)$ is defined according to Definition 3.14.18, and $\widehat{\rho}(a)$ is defined according to Definition 3.7.2).

Definition 3.14.20. We let $\mathrm{U}(\infty)$ denote the set $\mathrm{id} + \mathfrak{u}_\infty$. In other words, $\mathrm{U}(\infty)$ is the set of all upper-triangular infinite matrices (with rows and columns indexed by integers) whose all diagonal entries are $= 1$. This set $\mathrm{U}(\infty)$ is easily seen to be a group (with respect to matrix multiplication). Inverses in this group can be computed by means of the formula $(I_\infty + a)^{-1} = \sum_{k=0}^{\infty} a^k$ for all $a \in \mathfrak{u}_\infty$ [143]

[143]Here, we are using the fact that, for every $a \in \mathfrak{u}_\infty$, the sum $\sum_{k=0}^{\infty} a^k$ converges entrywise (i. e., for

Definition 3.14.21. Let $m \in \mathbb{Z}$. We define an action $\varrho : \mathrm{U}(\infty) \to \mathrm{End}\left(\mathcal{F}^{(m)}\right)$ of the group $\mathrm{U}(\infty)$ on the vector space $\mathcal{F}^{(m)} = \wedge^{\frac{\infty}{2},m} V$ as follows: For every $A \in \mathrm{U}(\infty)$ and every m-degression $(i_0, i_1, i_2, ...)$, we set

$$(\varrho(A))(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = Av_{i_0} \wedge Av_{i_1} \wedge Av_{i_2} \wedge ....$$

(This is then extended to the whole $\mathcal{F}^{(m)}$ by linearity.) It is very easy to see that this is well-defined (because $\pi_{v_k}(Av_k) = v_k$ for all sufficiently small k) and indeed gives a group action. (We denote this action by the same letter ϱ as the action $\mathrm{M}(\infty) \to \mathrm{End}\left(\mathcal{F}^{(m)}\right)$ from Definition 3.14.6. This is intentional and unproblematic, because both of these actions have the same restriction onto $\mathrm{U}(\infty) \cap \mathrm{M}(\infty)$.)

In analogy to Remark 3.14.9, we have:

Remark 3.14.22. Let $(i_0, i_1, i_2, ...)$ be an m-degression. Let $A \in \mathrm{U}(\infty)$. For any m-degression $(j_0, j_1, j_2, ...)$, let $A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}$ denote the $\mathbb{N} \times \mathbb{N}$-matrix defined by

$$\left(\left(\text{the } (u,v)\text{-th entry of } A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right) = \left(\text{the } (j_u, i_v)\text{-th entry of } A\right) \qquad \text{for every } (u,v) \in \mathbb{N}^2\right).$$

(In other words, let $A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}$ denote the matrix which is obtained from A by removing all columns except for the i_0-th, the i_1-th, the i_2-th, etc. ones and removing all rows except for the j_0-th, the j_1-th, the j_2-th, etc. ones, and then inverting the order of the rows, and inverting the order of the columns.) Then, for any m-degression $(j_0, j_1, j_2, ...)$, the matrix $\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right)^T$ is upper almost-unitriangular, and thus the determinant $\det\left(\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right)^T\right)$ makes sense (according to Definition 3.14.8 **(e)**). We have

$$(\varrho(A))(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{(j_0,j_1,j_2,...) \text{ is an } m\text{-degression}} \det\left(\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right)^T\right) v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ....$$

The analogy between Remark 3.14.9 and Remark 3.14.22 is somewhat marred by the fact that the transposed matrix $\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right)^T$ is used in Remark 3.14.22 instead of the

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every $(i,j) \in \mathbb{Z}^2$, the sum $\sum_{k=0}^{\infty} \left(\text{the } (i,j)\text{-th entry of } a^k\right)$ converges in the discrete topology). Here is why this holds:

Since $a \in \mathfrak{u}_\infty$, we know that the (i,j)-th entry of a is 0 for all $(i,j) \in \mathbb{Z}^2$ satisfying $i > j - 1$. From this, it is easy to conclude (by induction over k) that for every $k \in \mathbb{N}$, the (i,j)-th entry of a^k is 0 for all $(i,j) \in \mathbb{Z}^2$ satisfying $i > j - k$. Hence, for every $(i,j) \in \mathbb{Z}^2$, the (i,j)-th entry of a^k is 0 for all nonnegative integers k satisfying $k > j - i$. As a consequence, for every $(i,j) \in \mathbb{Z}^2$, all but finitely many addends of the sum

$$\sum_{k=0}^{\infty} \left(\text{the } (i,j)\text{-th entry of } a^k\right)$$

are 0. In other words, for every $(i,j) \in \mathbb{Z}^2$, the sum $\sum_{k=0}^{\infty} \left(\text{the } (i,j)\text{-th entry of } a^k\right)$ converges in the discrete topology. Hence, the sum $\sum_{k=0}^{\infty} a^k$ converges entrywise, qed.

matrix $A^{i_0,i_1,i_2,\ldots}_{j_0,j_1,j_2,\ldots}$. This is merely a technical difference, and if we would have defined the determinant of a **lower** almost-unitriangular matrix, we could have avoided using the transpose in Remark 3.14.22.

Remark 3.14.23. There is a way to "merge" $\mathrm{GL}(\infty)$ and $\mathrm{U}(\infty)$ into a bigger group of infinite matrices. Indeed, let $\mathrm{M}^{\mathrm{U}}(\infty)$ the set of all matrices $A \in \mathrm{U}(\infty)$ such that all but finitely many among the $(i,j) \in \mathbb{Z}^2$ satisfying $i \geq j$ satisfy (the (i,j)-th entry of A) $= \delta_{i,j}$. (Note that this condition does not restrict the (i,j)-th entry of A for any $(i,j) \in \mathbb{Z}^2$ satisfying $i < j$. That is, the entries of A above the main diagonal can be arbitrary, but the entries of A below and on the main diagonal have to coincide with the respective entries of the identity matrix save for finitely many exceptions, if A is to lie in $\mathrm{M}^{\mathrm{U}}(\infty)$.) Then, it is easy to see that $\mathrm{M}^{\mathrm{U}}(\infty)$ is a monoid. The group of all invertible elements of this monoid (where "invertible" means "having an inverse in the monoid $\mathrm{M}^{\mathrm{U}}(\infty)$") is a group which has both $\mathrm{GL}(\infty)$ and $\mathrm{U}(\infty)$ as subgroups. Actually, this group is $\mathrm{GL}(\infty) \cdot \mathrm{U}(\infty)$, as the reader can easily check.

We will need neither the monoid $\mathrm{M}^{\mathrm{U}}(\infty)$ nor this group in the following.

3.14.4. The exponential relation between ρ and ϱ

We now come to a relation which connects the actions ρ and ϱ. It comes in a $\mathrm{GL}(\infty)$ version, a $\mathrm{U}(\infty)$ version, and a finitary version; we will formulate all three, but only prove the latter. First, the $\mathrm{GL}(\infty)$ version:

Theorem 3.14.24. Let $a \in \mathfrak{gl}_\infty$. Let $m \in \mathbb{Z}$. Then, the exponential $\exp a$ is a well-defined element of $\mathrm{GL}(\infty)$ and satisfies $\varrho(\exp a) = \exp(\rho(a))$ in $\mathrm{End}\left(\mathcal{F}^{(m)}\right)$.

It should be noticed that Theorem 3.14.24, unlike most of the other results we have been stating, does rely on the ground field being $\mathbb{C}$; otherwise, there would be no guarantee that $\exp a$ is well-defined. However, if we assume, for example, that a is strictly upper-triangular, or that the entries of a belong to some ideal I of the ground ring such that the ground ring is complete and Hausdorff in the I-adic topology, then the statement of Theorem 3.14.24 would be guaranteed over any ground ring which is a commutative $\mathbb{Q}$-algebra.

The $\mathrm{U}(\infty)$ version does not depend on the ground ring at all (as long as the ground ring is a $\mathbb{Q}$-algebra):

Theorem 3.14.25. Let $a \in \mathfrak{u}_\infty$. Let $m \in \mathbb{Z}$. Then, the exponential $\exp a$ is a well-defined element of $\mathrm{U}(\infty)$ and satisfies $\varrho(\exp a) = \exp(\rho(a))$ in $\mathrm{End}\left(\mathcal{F}^{(m)}\right)$.

We have now stated the $\mathrm{GL}(\infty)$ and the $\mathrm{U}(\infty)$ versions of the relation between ρ and ϱ. Before we state the finitary version, we define a finite analogue of the map ρ:

Definition 3.14.26. Let P be a vector space, and let $\ell \in \mathbb{N}$. Let $\rho_{P,\ell} : \mathfrak{gl}(P) \to \mathrm{End}\left(\wedge^\ell P\right)$ denote the representation of the Lie algebra $\mathfrak{gl}(P)$ on the ℓ-th exterior power of the defining representation P of $\mathfrak{gl}(P)$. By the definition of the ℓ-th exterior power of a representation of a Lie algebra, this representation $\rho_{P,\ell}$ satisfies

$$(\rho_{P,\ell}(a))(p_1 \wedge p_2 \wedge ... \wedge p_\ell) = \sum_{k=1}^{\ell} p_1 \wedge p_2 \wedge ... \wedge p_{k-1} \wedge (a \rightharpoonup p_k) \wedge p_{k+1} \wedge p_{k+2} \wedge ... \wedge p_\ell \tag{175}$$

for every $a \in \mathfrak{gl}(P)$ and any $p_1, p_2, ..., p_\ell \in P$. (Recall that $a \rightharpoonup p = ap$ for every $a \in \mathfrak{gl}(P)$ and $p \in P$.)

Finally, let us state the finitary version of Theorem 3.14.24 and Theorem 3.14.25. To see why it is analogous to the two aforementioned theorems, one should keep in mind that $\rho_{P,\ell}$ is an analogue of ρ in the finite case, while $\wedge^\ell A$ is an analogue of $\varrho(A)$.

Theorem 3.14.27. Let P be a vector space. Let $a \in \mathfrak{gl}(P)$ be a nilpotent linear map. Then, the exponential $\exp a$ is a well-defined element of $\mathrm{GL}(P)$ and satisfies $\wedge^\ell(\exp a) = \exp(\rho_{P,\ell}(a))$ in $\mathrm{End}\left(\wedge^\ell P\right)$ for every $\ell \in \mathbb{N}$.

Note that we have formulated Theorem 3.14.27 only for nilpotent $a \in \mathfrak{gl}(P)$. We could have also formulated it for arbitrary $a \in \mathfrak{gl}(P)$ under some mild conditions on P (such as P being finite-dimensional), but then it would depend on the ground field being $\mathbb{C}$, which is something we would like to avoid (as we are going to apply this theorem to a different ground field).

First proof of Theorem 3.14.27 (sketched). Since a is nilpotent, it is known that the exponential $\exp a$ is a well-defined element of $\mathrm{GL}(P)$.

Let $\ell \in \mathbb{N}$. Now define an endomorphism $\rho'_{P,\ell}(a) : P^{\otimes \ell} \to P^{\otimes \ell}$ by

$$\rho'_{P,\ell}(a) = \sum_{k=1}^{\ell} \mathrm{id}_P^{\otimes(k-1)} \otimes a \otimes \mathrm{id}_P^{\otimes(\ell-k)}.$$

Let also $\pi : P^{\otimes \ell} \to \wedge^\ell P$ be the canonical projection (since $\wedge^\ell P$ is defined as a quotient vector space of $P^{\otimes \ell}$). Clearly, π is surjective.

It is easy to see that $\pi \circ \left(\rho'_{P,\ell}(a)\right) = (\rho_{P,\ell}(a)) \circ \pi$. From this, one can conclude that

$$\pi \circ \left(\rho'_{P,\ell}(a)\right)^m = (\rho_{P,\ell}(a))^m \circ \pi \qquad \text{for every } m \in \mathbb{N}. \tag{176}$$

On the other hand, a routine induction proves that every $m \in \mathbb{N}$ satisfies

$$\left(\rho'_{P,\ell}(a)\right)^m = \sum_{\substack{(i_1,i_2,...,i_\ell) \in \mathbb{N}^\ell; \\ i_1+i_2+...+i_\ell = m}} \frac{m!}{i_1! i_2! ... i_\ell!} a^{i_1} \otimes a^{i_2} \otimes ... \otimes a^{i_\ell}. \tag{177}$$

Now, $\exp a = \sum_{i\in\mathbb{N}} \frac{1}{i!} a^i$, whence

$$\begin{aligned}
(\exp a)^{\otimes \ell} &= \left(\sum_{i\in\mathbb{N}} \frac{1}{i!} a^i\right)^{\otimes \ell} = \sum_{(i_1,i_2,...,i_\ell)\in\mathbb{N}^\ell} \left(\frac{1}{i_1!} a^{i_1}\right) \otimes \left(\frac{1}{i_2!} a^{i_2}\right) \otimes ... \otimes \left(\frac{1}{i_\ell!} a^{i_\ell}\right) \\
&\qquad \text{(by the product rule)} \\
&= \sum_{(i_1,i_2,...,i_\ell)\in\mathbb{N}^\ell} \frac{1}{i_1! i_2! ... i_\ell!} a^{i_1} \otimes a^{i_2} \otimes ... \otimes a^{i_\ell} \\
&= \sum_{m\in\mathbb{N}} \sum_{\substack{(i_1,i_2,...,i_\ell)\in\mathbb{N}^\ell; \\ i_1+i_2+...+i_\ell=m}} \frac{1}{i_1! i_2! ... i_\ell!} a^{i_1} \otimes a^{i_2} \otimes ... \otimes a^{i_\ell} \\
&= \sum_{m\in\mathbb{N}} \frac{1}{m!} \underbrace{\sum_{\substack{(i_1,i_2,...,i_\ell)\in\mathbb{N}^\ell; \\ i_1+i_2+...+i_\ell=m}} \frac{m!}{i_1! i_2! ... i_\ell!} a^{i_1} \otimes a^{i_2} \otimes ... \otimes a^{i_\ell}}_{\substack{=\left(\rho'_{P,\ell}(a)\right)^m \\ \text{(by (177))}}} = \sum_{m\in\mathbb{N}} \frac{1}{m!} \left(\rho'_{P,\ell}(a)\right)^m \\
&= \exp\left(\rho'_{P,\ell}(a)\right).
\end{aligned}$$

Note that this shows that $\exp\left(\rho'_{P,\ell}(a)\right)$ is well-defined. But since $\exp\left(\rho'_{P,\ell}(a)\right) = \sum_{m\in\mathbb{N}} \frac{1}{m!} \left(\rho'_{P,\ell}(a)\right)^m$, we have

$$\begin{aligned}
\pi \circ \left(\exp\left(\rho'_{P,\ell}(a)\right)\right) &= \pi \circ \left(\sum_{m\in\mathbb{N}} \frac{1}{m!} \left(\rho'_{P,\ell}(a)\right)^m\right) = \sum_{m\in\mathbb{N}} \frac{1}{m!} \underbrace{\pi \circ \left(\rho'_{P,\ell}(a)\right)^m}_{\substack{=\left(\rho_{P,\ell}(a)\right)^m \circ \pi \\ \text{(by (176))}}} \\
&\qquad \text{(since composition of linear maps is bilinear)} \\
&= \sum_{m\in\mathbb{N}} \frac{1}{m!} \left(\rho_{P,\ell}(a)\right)^m \circ \pi = \underbrace{\left(\sum_{m\in\mathbb{N}} \frac{1}{m!} \left(\rho_{P,\ell}(a)\right)^m\right)}_{=\exp\left(\rho_{P,\ell}(a)\right)} \circ \pi = \left(\exp\left(\rho_{P,\ell}(a)\right)\right) \circ \pi,
\end{aligned}$$

and this also shows that $\exp\left(\rho_{P,\ell}(a)\right)$ is well-defined (since π is surjective).

Since we have proven earlier that $(\exp a)^{\otimes \ell} = \exp\left(\rho'_{P,\ell}(a)\right)$, the equality $\pi \circ \left(\exp\left(\rho'_{P,\ell}(a)\right)\right) = \left(\exp\left(\rho_{P,\ell}(a)\right)\right) \circ \pi$ rewrites as $\pi \circ (\exp a)^{\otimes \ell} = \left(\exp\left(\rho_{P,\ell}(a)\right)\right) \circ \pi$.

On the other hand, since the projection $\pi : P^{\otimes \ell} \to \wedge^\ell P$ is functorial in P, we have $\pi \circ (\exp a)^{\otimes \ell} = \left(\wedge^\ell (\exp a)\right) \circ \pi$. Thus,

$$\left(\wedge^\ell (\exp a)\right) \circ \pi = \pi \circ (\exp a)^{\otimes \ell} = \left(\exp\left(\rho_{P,\ell}(a)\right)\right) \circ \pi.$$

Since the morphism π is right-cancellable (since it is surjective), this yields $\wedge^\ell (\exp a) = \exp\left(\rho_{P,\ell}(a)\right)$. This proves Theorem 3.14.27.

Second proof of Theorem 3.14.27 (sketched). Since a is nilpotent, it is known that the exponential $\exp a$ is a well-defined unipotent element of $\mathrm{GL}(P)$. But for every $\ell \in \mathbb{N}$, the ℓ-th exterior power of any unipotent element of $\mathrm{GL}(P)$ is a unipotent element of

$\mathrm{GL}\left(\wedge^{\ell} P\right)$. Since $\exp a$ is a unipotent element of $\mathrm{GL}(P)$, this yields that $\wedge^{\ell}(\exp a)$ is a unipotent element of $\mathrm{GL}\left(\wedge^{\ell} P\right)$ for every $\ell \in \mathbb{N}$. Hence, the logarithm $\log\left(\wedge^{\ell}(\exp a)\right)$ is well-defined for every $\ell \in \mathbb{N}$.

On the other hand, consider the map $\wedge(\exp a) : \wedge P \to \wedge P$. This map is an algebra homomorphism (because generally, if Q and R are two vector spaces, and $f : Q \to R$ is a linear map, then $\wedge f : \wedge Q \to \wedge R$ is an algebra homomorphism) and identical with the direct sum $\bigoplus\limits_{\ell \in \mathbb{N}} \wedge^{\ell}(\exp a) : \bigoplus\limits_{\ell \in \mathbb{N}} \wedge^{\ell} P \to \bigoplus\limits_{\ell \in \mathbb{N}} \wedge^{\ell} P$ of the linear maps $\wedge^{\ell}(\exp a) : \wedge^{\ell} P \to \wedge^{\ell} P$.

Since $\wedge(\exp a) = \bigoplus\limits_{\ell \in \mathbb{N}} \wedge^{\ell}(\exp a)$, we have $\log(\wedge(\exp a)) = \log\left(\bigoplus\limits_{\ell \in \mathbb{N}} \wedge^{\ell}(\exp a)\right) = \bigoplus\limits_{\ell \in \mathbb{N}} \log\left(\wedge^{\ell}(\exp a)\right)$ (because logarithms on direct sums are componentwise).[144] As a consequence, every $\ell \in \mathbb{N}$ and every $p_1, p_2, ..., p_\ell \in P$ satisfy $p_1 \wedge p_2 \wedge ... \wedge p_\ell \in \wedge^{\ell} P$ and thus $(\log(\wedge(\exp a)))(p_1 \wedge p_2 \wedge ... \wedge p_\ell) = \left(\log\left(\wedge^{\ell}(\exp a)\right)\right)(p_1 \wedge p_2 \wedge ... \wedge p_\ell)$.

But it is well-known that if A is an algebra and $f : A \to A$ is an algebra endomorphism such that $\log f$ is well-defined, then $\log f : A \to A$ is a derivation. Applied to $A = \wedge P$ and $f = \wedge(\exp a)$, this yields that $\log(\wedge(\exp a)) : \wedge P \to \wedge P$ is a derivation.

But every $p \in P$ satisfies

$$(\log(\wedge(\exp a)))(p) = a \rightharpoonup p, \qquad (178)$$

where p is viewed as an element of $\wedge^1 P \subseteq \wedge P$. [145]

Now recall the Leibniz identity for derivations. In its general form, it says that if A is an algebra, M is an A-bimodule, and $d : A \to M$ is a derivation, then every $\ell \in \mathbb{N}$ and every $p_1, p_2, ..., p_\ell \in A$ satisfy

$$d(p_1 p_2 ... p_\ell) = \sum_{k=1}^{\ell} p_1 p_2 ... p_{k-1} d(p_k) p_{k+1} p_{k+2} ... p_\ell.$$

Applying this to $A = \wedge P$, $M = \wedge P$ and $d = \log(\wedge(\exp a))$, we conclude that every $\ell \in \mathbb{N}$ and every $p_1, p_2, ..., p_\ell \in \wedge P$ satisfy

$$(\log(\wedge(\exp a)))(p_1 p_2 ... p_\ell) = \sum_{k=1}^{\ell} p_1 p_2 ... p_{k-1} (\log(\wedge(\exp a)))(p_k) p_{k+1} p_{k+2} ... p_\ell$$

(since $\log(\wedge(\exp a)) : \wedge P \to \wedge P$ is a derivation). Thus, every $\ell \in \mathbb{N}$ and every

[144] Note that the map $\wedge(\exp a)$ needs not be unipotent, but the logarithm $\log(\wedge(\exp a))$ nevertheless makes sense because the map $\wedge(\exp a)$ is a direct sum of unipotent maps (and thus is locally unipotent).

[145] *Proof of (178):* Let $p \in P$. Since $\log(\wedge(\exp a)) = \bigoplus\limits_{\ell \in \mathbb{N}} \log\left(\wedge^{\ell}(\exp a)\right)$ and $p \in P = \wedge^1 P$, we have

$$(\log(\wedge(\exp a)))(p) = \left(\log \underbrace{\left(\wedge^1(\exp a)\right)}_{=\exp a}\right)(p) = \underbrace{(\log(\exp a))}_{=a}(p) = ap = a \rightharpoonup p.$$

This proves (178).

$p_1, p_2, ..., p_\ell \in P$ satisfy

$$\left(\log\left(\wedge\left(\exp a\right)\right)\right)\left(p_1 p_2 ... p_\ell\right) = \sum_{k=1}^{\ell} p_1 p_2 ... p_{k-1} \underbrace{\left(\log\left(\wedge\left(\exp a\right)\right)\right)\left(p_k\right)}_{\substack{=a \rightharpoonup p_k \\ \text{(by (178), applied to } p=p_k\text{)}}} p_{k+1} p_{k+2} ... p_\ell$$
$$= \sum_{k=1}^{\ell} \underbrace{p_1 p_2 ... p_{k-1} \left(a \rightharpoonup p_k\right) p_{k+1} p_{k+2} ... p_\ell}_{\substack{=p_1 \wedge p_2 \wedge ... \wedge p_{k-1} \wedge (a \rightharpoonup p_k) \wedge p_{k+1} \wedge p_{k+2} \wedge ... \wedge p_\ell \\ \text{(since the multiplication in } \wedge P \text{ is given by the wedge product)}}}$$
$$= \sum_{k=1}^{\ell} p_1 \wedge p_2 \wedge ... \wedge p_{k-1} \wedge \left(a \rightharpoonup p_k\right) \wedge p_{k+1} \wedge p_{k+2} \wedge ... \wedge p_\ell$$
$$= \left(\rho_{P,\ell}\left(a\right)\right)\left(p_1 \wedge p_2 \wedge ... \wedge p_\ell\right) \qquad \text{(by (175))}. \qquad (179)$$

On the other hand, every $\ell \in \mathbb{N}$ and every $p_1, p_2, ..., p_\ell \in P$ satisfy

$$\left(\log\left(\wedge\left(\exp a\right)\right)\right) \underbrace{\left(p_1 p_2 ... p_\ell\right)}_{\substack{=p_1 \wedge p_2 \wedge ... \wedge p_\ell \\ \text{(since the multiplication in } \wedge P \text{ is given by the wedge product)}}}$$
$$= \left(\log\left(\wedge\left(\exp a\right)\right)\right)\left(p_1 \wedge p_2 \wedge ... \wedge p_\ell\right) = \left(\log\left(\wedge^\ell\left(\exp a\right)\right)\right)\left(p_1 \wedge p_2 \wedge ... \wedge p_\ell\right).$$

Compared with (179), this yields

$$\left(\rho_{P,\ell}\left(a\right)\right)\left(p_1 \wedge p_2 \wedge ... \wedge p_\ell\right) = \left(\log\left(\wedge^\ell\left(\exp a\right)\right)\right)\left(p_1 \wedge p_2 \wedge ... \wedge p_\ell\right)$$

for every $\ell \in \mathbb{N}$ and every $p_1, p_2, ..., p_\ell \in P$.

Now fix $\ell \in \mathbb{N}$. We know that

$$\left(\rho_{P,\ell}\left(a\right)\right)\left(p_1 \wedge p_2 \wedge ... \wedge p_\ell\right) = \left(\log\left(\wedge^\ell\left(\exp a\right)\right)\right)\left(p_1 \wedge p_2 \wedge ... \wedge p_\ell\right)$$

for every $p_1, p_2, ..., p_\ell \in P$. Since the vector space $\wedge^\ell P$ is spanned by elements of the form $p_1 \wedge p_2 \wedge ... \wedge p_\ell$ with $p_1, p_2, ..., p_\ell \in P$, this yields that the two linear maps $\rho_{P,\ell}(a)$ and $\log\left(\wedge^\ell\left(\exp a\right)\right)$ are equal to each other on a spanning set of the vector space $\wedge^\ell P$. Therefore, these two maps must be identical (because if two linear maps are equal to each other on a spanning set of their domain, then they must always be identical). In other words, $\rho_{P,\ell}(a) = \log\left(\wedge^\ell\left(\exp a\right)\right)$. Exponentiating this equality, we obtain $\exp\left(\rho_{P,\ell}(a)\right) = \wedge^\ell\left(\exp a\right)$. This proves Theorem 3.14.27.

3.14.5. Reduction to fermions

We are now going to reduce Theorem 3.12.11 to a "purely fermionic" statement – a statement (Theorem 3.14.32) not involving the bosonic space $\mathcal{B}$ or the Boson-Fermion correspondence σ in any way. We will later (Subsection 3.14.6) generalize this statement, and yet later prove the generalization.

First, a definition:

Definition 3.14.28. Let $\mathbf{R}$ (not to be confused with the field $\mathbb{R}$) be a commutative $\mathbb{Q}$-algebra. We denote by $\mathcal{A}_{\mathbf{R}}$ the Heisenberg algebra defined over the ground ring $\mathbf{R}$ in lieu of $\mathbb{C}$. We denote by $\mathcal{B}^{(0)}_{\mathbf{R}}$ the $\mathcal{A}_{\mathbf{R}}$-module $\mathcal{B}^{(0)}$ defined over the ground ring

$\mathbf{R}$ in lieu of $\mathbb{C}$. We denote by $\mathcal{F}^{(0)}_{\mathbf{R}}$ the $\mathcal{A}_{\mathbf{R}}$-module $\mathcal{F}^{(0)}$ defined over the ground ring $\mathbf{R}$ in lieu of $\mathbb{C}$. We denote by $\sigma_{\mathbf{R}}$ the map σ defined over the ground ring $\mathbf{R}$ in lieu of $\mathbb{C}$. (This $\sigma_{\mathbf{R}}$ is thus a graded $\mathcal{A}_{\mathbf{R}}$-module homomorphism $\mathcal{B}_{\mathbf{R}} \to \mathcal{F}_{\mathbf{R}}$, where $\mathcal{B}_{\mathbf{R}}$ and $\mathcal{F}_{\mathbf{R}}$ are the $\mathcal{A}_{\mathbf{R}}$-modules $\mathcal{B}$ and $\mathcal{F}$ defined over the ground ring $\mathbf{R}$ in lieu of $\mathbb{C}$.)

Next, some preparations:

Proposition 3.14.29. Let $\mathbf{R}$ be a commutative $\mathbb{Q}$-algebra. Let $y_1, y_2, y_3, ...$ be some elements of $\mathbf{R}$.

(a) Let M be a $\mathbb{Z}$-graded $\mathcal{A}_{\mathbf{R}}$-module concentrated in nonpositive degrees (i. e., satisfying $M[n] = 0$ for all positive integers n). The map $\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...) : M \to M$ is well-defined, in the following sense: For every $m \in M$, expanding the expression $\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...) m$ yields an infinite sum with only finitely many nonzero addends.

(b) Let M and N be two $\mathbb{Z}$-graded $\mathcal{A}_{\mathbf{R}}$-modules concentrated in nonpositive degrees. Let $\eta : M \to N$ be an $\mathcal{A}_{\mathbf{R}}$-module homomorphism. Then,

$$(\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...)) \circ \eta = \eta \circ (\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...))$$

as maps from M to N.

(c) Consider the $\mathbb{Z}$-graded $\mathcal{A}_{\mathbf{R}}$-module $\mathcal{F}^{(0)}_{\mathbf{R}}$. This $\mathbb{Z}$-graded $\mathcal{A}_{\mathbf{R}}$-module $\mathcal{F}^{(0)}_{\mathbf{R}}$ is concentrated in nonpositive degrees. Hence, by Theorem 3.14.32, the map $\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...) : \mathcal{F}^{(0)}_{\mathbf{R}} \to \mathcal{F}^{(0)}_{\mathbf{R}}$ is well-defined. Thus, $\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...) \cdot (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$ is well-defined for every 0-degression $(i_0, i_1, i_2, ...)$.

Proof of Proposition 3.14.29. **(a)** Let $m \in M$. We will prove that expanding the expression $\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...) m$ yields an infinite sum with only finitely many nonzero terms.

Since M is $\mathbb{Z}$-graded, we can write m in the form $m = \sum\limits_{n\in\mathbb{Z}} m_n$ for a family $(m_n)_{n\in\mathbb{Z}}$ of elements of M which satisfy ($m_n \in M[n]$ for every $n \in \mathbb{Z}$) and ($m_n = 0$ for all but finitely many $n \in \mathbb{Z}$). Consider this family $(m_n)_{n\in\mathbb{Z}}$. We know that $m_n = 0$ for all but finitely many $n \in \mathbb{Z}$. In other words, there exists a finite subset I of $\mathbb{Z}$ such that every $n \in \mathbb{Z} \setminus I$ satisfies $m_n = 0$. Consider this I. Let s be an integer which is smaller than every element of I. (Such an s exists since I is finite.) Then,

$$fm = 0 \qquad \text{for every integer } q \geq -s \text{ and every } f \in U_{\mathbf{R}}(\mathcal{A}_{\mathbf{R}})[q] \tag{180}$$

(where $U_{\mathbf{R}}$ means "enveloping algebra over the ground ring $\mathbf{R}$"). [146]

[146] *Proof of (180):* Let $q \geq -s$ be an integer, and let $f \in U_{\mathbf{R}}(\mathcal{A}_{\mathbf{R}})[q]$. Since s is smaller than every element of I, we have $s < n$ for every $n \in I$. Thus, $q \geq -\underbrace{s}_{<n} > -n$ for every $n \in I$, so that $q + n > 0$ for every $n \in I$ and thus $M[q+n] = 0$ for every $n \in I$ (since M is concentrated in nonpositive degrees).

Expanding the expression $\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right) m$, we obtain

$$\begin{aligned}
&\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right) m \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \left(\underbrace{y_1 a_1 + y_2 a_2 + y_3 a_3 + ...}_{= \sum\limits_{j\in\{1,2,3,...\}} y_j a_j} \right)^i m = \sum_{i=0}^{\infty} \frac{1}{i!} \underbrace{\left(\sum_{j\in\{1,2,3,...\}} y_j a_j \right)^i}_{= \sum\limits_{(j_1,j_2,...,j_i)\in\{1,2,3,...\}^i} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i}} m \\
&= \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{(j_1,j_2,...,j_i)\in\{1,2,3,...\}^i} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i} m \\
&= \sum_{\substack{i\in\mathbb{N};\\ (j_1,j_2,...,j_i)\in\{1,2,3,...\}^i}} \frac{1}{i!} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i} m.
\end{aligned}$$

But this infinite sum has only finitely many nonzero addends[147]. Thus, we have shown that for every $m \in M$, expanding the expression $\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right) m$ yields

---

Notice that M is a graded $\mathcal{A}_{\mathbf{R}}$-module, thus a graded $U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)$-module. But

$$m = \sum_{n\in\mathbb{Z}} m_n = \sum_{n\in I} m_n + \sum_{n\in\mathbb{Z}\setminus I} \underbrace{m_n}_{\substack{=0\\ (\text{since } n\in\mathbb{Z}\setminus I)}} = \sum_{n\in I} m_n + \underbrace{\sum_{n\in\mathbb{Z}\setminus I} 0}_{=0} = \sum_{n\in I} m_n,$$

so that

$$fm = f\sum_{n\in I} m_n = \sum_{n\in I} \underbrace{fm_n}_{\substack{\in M[q+n]\\ (\text{since } f\in U_{\mathbf{R}}(\mathcal{A}_{\mathbf{R}})[q] \text{ and } m_n\in M[n],\\ \text{and since } M \text{ is a graded } U_{\mathbf{R}}(\mathcal{A}_{\mathbf{R}})\text{-module})}} \in \sum_{n\in I} \underbrace{M\left[q+n\right]}_{\substack{=0\\ (\text{since } n\in I)}} = \sum_{n\in I} 0 = 0,$$

so that $fm = 0$, qed.

[147] *Proof.* Let $i \in \mathbb{N}$ and $(j_1, j_2, ..., j_i) \in \{1, 2, 3, ...\}^i$ be such that $\frac{1}{i!} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i} m \neq 0$. Since $a_{j_k} \in U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)[j_k]$ for every $k \in \{1, 2, ..., i\}$, we have

$$\begin{aligned}
a_{j_1} a_{j_2} ... a_{j_i} &\in \left(U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)[j_1]\right)\left(U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)[j_2]\right) ... \left(U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)[j_i]\right) \\
&\subseteq U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)\left[j_1 + j_2 + ... + j_i\right],
\end{aligned}$$

so that

$$\frac{1}{i!} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i} \in \frac{1}{i!} y_{j_1} y_{j_2} ... y_{j_i} U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)\left[j_1 + j_2 + ... + j_i\right] \subseteq U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)\left[j_1 + j_2 + ... + j_i\right].$$

Hence, if $j_1 + j_2 + ... + j_i \geq -s$, then $\frac{1}{i!} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i} m = 0$ (by (180), applied to $f = \frac{1}{i!} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i}$ and $q = j_1 + j_2 + ... + j_i$), contradicting $\frac{1}{i!} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i} m \neq 0$. As a consequence, we cannot have $j_1 + j_2 + ... + j_i \geq -s$. We must thus have $j_1 + j_2 + ... + j_i < -s$.

Now forget that we fixed i and $(j_1, j_2, ..., j_i)$. We thus have shown that every $i \in \mathbb{N}$ and $(j_1, j_2, ..., j_i) \in \{1, 2, 3, ...\}^i$ such that $\frac{1}{i!} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i} m \neq 0$ must satisfy $j_1 + j_2 + ... + j_i < -s$. Since there are only finitely many pairs $(i, (j_1, j_2, ..., j_i))$ of $i \in \mathbb{N}$ and $(j_1, j_2, ..., j_i) \in \{1, 2, 3, ...\}^i$ satisfying $j_1 + j_2 + ... + j_i < -s$, this yields that there are only finitely many pairs $(i, (j_1, j_2, ..., j_i))$ of $i \in \mathbb{N}$ and $(j_1, j_2, ..., j_i) \in \{1, 2, 3, ...\}^i$ satisfying $\frac{1}{i!} y_{j_1} y_{j_2} ... y_{j_i} a_{j_1} a_{j_2} ... a_{j_i} m \neq 0$.

an infinite sum with only finitely many nonzero addends. This proves Proposition 3.14.29 **(a)**.

(b) In order to prove Proposition 3.14.29 **(b)**, we must clearly show that

$$\eta\left(\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)m\right)=\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)\cdot\eta\left(m\right) \qquad (181)$$

for every $m\in M$.

Fix $m\in M$. Since η is an $\mathcal{A}_{\mathbf{R}}$-module homomorphism, η must also be an $U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)$-module homomorphism (since every $\mathcal{A}_{\mathbf{R}}$-module homomorphism is an $U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)$-module homomorphism). Thus,

$$\eta\left(gm\right)=g\cdot\eta\left(m\right) \qquad \text{for every } g\in U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right).$$

If $\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)$ was an element of $U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)$, then we could apply this to $g=\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)$ and conclude (181) immediately. Unfortunately, $\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)$ is not an element of $U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)$, but this problem is easy to amend: By Proposition 3.14.29 **(a)**, we can find a finite partial sum g of the expanded power series $\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)$ satisfying

$$\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)m=gm \qquad \text{and}$$
$$\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)\cdot\eta\left(m\right)=g\cdot\eta\left(m\right).$$

Consider such a g. Since g is only a finite partial sum, we have $g\in U_{\mathbf{R}}\left(\mathcal{A}_{\mathbf{R}}\right)$, and thus $\eta\left(gm\right)=g\cdot\eta\left(m\right)$. Hence,

$$\eta\left(\underbrace{\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)m}_{=gm}\right)=\eta\left(gm\right)=g\cdot\eta\left(m\right)$$
$$=\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)\cdot\eta\left(m\right),$$

so that (181) is proven. Thus, Proposition 3.14.29 **(b)** is proven.

(c) This is obvious.

Let us make a remark which we will only use in the "finitary" version of our proof of Theorem 3.14.32. First, a definition:

Definition 3.14.30. For every commutative ring $\mathbf{R}$, let $\mathcal{A}_{+\mathbf{R}}$ be the Lie algebra $\mathcal{A}_+$ defined for the ground ring $\mathbf{R}$ instead of $\mathbb{C}$.

Now, it is easy to see that Proposition 3.14.29 holds with $\mathcal{A}_{\mathbf{R}}$ replaced by $\mathcal{A}_{+\mathbf{R}}$. We will only use the analogues of parts **(a)** and **(b)**:

Proposition 3.14.31. Let $\mathbf{R}$ be a commutative $\mathbb{Q}$-algebra. Let $y_1, y_2, y_3, ...$ be some elements of $\mathbf{R}$.

(a) Let M be a $\mathbb{Z}$-graded $\mathcal{A}_{+\mathbf{R}}$-module concentrated in nonpositive degrees (i. e., satisfying $M\left[n\right]=0$ for all positive integers n). The map

In other words, the infinite sum $\sum\limits_{\substack{i\in\mathbb{N};\\ (j_1,j_2,...,j_i)\in\{1,2,3,...\}^i}}\frac{1}{i!}y_{j_1}y_{j_2}...y_{j_i}a_{j_1}a_{j_2}...a_{j_i}m$ has only finitely many nonzero addends, qed.

$\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...) : M \to M$ is well-defined, in the following sense: For every $m \in M$, expanding the expression $\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...)\, m$ yields an infinite sum with only finitely many nonzero addends.

(b) Let M and N be two $\mathbb{Z}$-graded $\mathcal{A}_{+\mathbf{R}}$-modules concentrated in nonpositive degrees. Let $\eta : M \to N$ be an $\mathcal{A}_{+\mathbf{R}}$-module homomorphism. Then,

$$(\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...)) \circ \eta = \eta \circ (\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...))$$

as maps from M to N.

Proof of Proposition 3.14.31. In order to obtain proofs of Proposition 3.14.31, it is enough to simply replace $\mathcal{A}_{\mathbf{R}}$ by $\mathcal{A}_{+\mathbf{R}}$ throughout the proof of parts **(a)** and **(b)** of Proposition 3.14.29.

Now, let us state the "fermionic" version of Theorem 3.12.11:

Theorem 3.14.32. Let $\mathbf{R}$ be a commutative $\mathbb{Q}$-algebra. Let $y_1, y_2, y_3, ...$ be some elements of $\mathbf{R}$. Denote by y the family $(y_1, y_2, y_3, ...)$. Let $(i_0, i_1, i_2, ...)$ be a 0-degression.

The $(v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...)$-coordinate of $\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...) \cdot (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$ (this is a well-defined element of $\mathcal{F}^{(0)}_{\mathbf{R}}$ due to Proposition 3.14.29 **(c)**) with respect to the basis[148] $(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...)_{(j_0, j_1, j_2, ...) \text{ a 0-degression}}$ of $\mathcal{F}^{(0)}_{\mathbf{R}}$ equals $S_{(i_k+k)_{k\geq 0}}(y)$. (Here, we are using the fact that $(i_k + k)_{k\geq 0}$ is a partition for every 0-degression $(i_0, i_1, i_2, ...)$. This follows from Proposition 3.5.24, applied to $m = 0$.)

Let us see how this yields Theorem 3.12.11:

Proof of Theorem 3.12.11 using Theorem 3.14.32. Fix a 0-degression $(i_0, i_1, i_2, ...)$; then, $i_0 > i_1 > i_2 > ...$ and $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \in \mathcal{F}^{(0)}$. Let λ be the partition $(i_0 + 0, i_1 + 1, i_2 + 2, ...)$.

Denote the element $\sigma^{-1}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) \in \mathcal{B}^{(0)}$ by $P(x)$. We need to show that $P(x) = S_\lambda(x)$.

From now on, we let y denote another countable family of indeterminates $(y_1, y_2, y_3, ...)$ (rather than a finite family like the $(y_1, y_2, ..., y_N)$ of Definition 3.12.3). Thus, whenever Q is a polynomial in countably many indeterminates, $Q(y)$ will mean $Q(y_1, y_2, y_3, ...)$.

Let $\mathbf{R}$ be the polynomial ring $\mathbb{C}[y_1, y_2, y_3, ...]$. Then, y is a family of elements of $\mathbf{R}$.

By the definition of $\mathcal{B}^{(0)}_{\mathbf{R}}$, we have $\mathcal{B}^{(0)}_{\mathbf{R}} = \mathbf{R}[x_1, x_2, x_3, ...]$ as a vector space, so that $\mathcal{B}^{(0)}_{\mathbf{R}} = (\mathbb{C}[y_1, y_2, y_3, ...])[x_1, x_2, x_3, ...]$ as a vector space. Let us denote by $1 \in \mathcal{B}^{(0)}$ the unity of the algebra $\mathbb{C}[x_1, x_2, x_3, ...]$. Clearly, $\mathcal{B}^{(0)} \subseteq \mathcal{B}^{(0)}_{\mathbf{R}}$, and thus $1 \in \mathcal{B}^{(0)} \subseteq \mathcal{B}^{(0)}_{\mathbf{R}}$.

We still let x denote the whole collection of variables $(x_1, x_2, x_3, ...)$. Also, let $x + y$ denote the family $(x_1 + y_1, x_2 + y_2, x_3 + y_3, ...)$ of elements of $\mathcal{B}^{(0)}_{\mathbf{R}}$.

Recall the $\mathbb{C}$-bilinear form $(\cdot, \cdot) : F \times F \to \mathbb{C}$ defined in Proposition 2.2.24. Since $F = \widetilde{F} = \mathcal{B}^{(0)}$ (as vector spaces), this form $(\cdot, \cdot)$ is a $\mathbb{C}$-bilinear form $\mathcal{B}^{(0)} \times \mathcal{B}^{(0)} \to \mathbb{C}$. Since the definition of the form did not depend of the ground ring, we can analogously define an $\mathbf{R}$-bilinear form $(\cdot, \cdot) : \mathcal{B}^{(0)}_{\mathbf{R}} \times \mathcal{B}^{(0)}_{\mathbf{R}} \to \mathbf{R}$. The restriction of this latter $\mathbf{R}$-bilinear form $(\cdot, \cdot) : \mathcal{B}^{(0)}_{\mathbf{R}} \times \mathcal{B}^{(0)}_{\mathbf{R}} \to \mathbf{R}$ to $\mathcal{B}^{(0)} \times \mathcal{B}^{(0)}$ is clearly the former $\mathbb{C}$-bilinear form $(\cdot, \cdot) : \mathcal{B}^{(0)} \times \mathcal{B}^{(0)} \to \mathbb{C}$; therefore we will use the same notation for these two forms.

[148] Here, "basis" means "$\mathbf{R}$-module basis", not "$\mathbb{C}$-vector space basis".

In the following, elements of $\mathcal{B}^{(0)}_{\mathbf{R}} = \mathbf{R}[x_1, x_2, x_3, ...]$ will be considered as polynomials in the variables $x_1, x_2, x_3, ...$ over the ring $\mathbf{R}$, and not as polynomials in the variables $x_1, x_2, x_3, ..., y_1, y_2, y_3, ...$ over the field $\mathbb{C}$. Hence, for an $R \in \mathcal{B}^{(0)}_{\mathbf{R}}$, the notation $R(0,0,0,...)$ will mean the result of substituting 0 for the variables $x_1, x_2, x_3, ...$ in R (but the variables $y_1, y_2, y_3, ...$ will stay unchanged!). We will abbreviate $R(0,0,0,...)$ by $R(0)$.

Every polynomial $R \in \mathcal{B}^{(0)}$ satisfies:

$$R(0) = \left(\begin{array}{c} \text{the } (v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...) \text{-coordinate of } \sigma(R) \\ \text{with respect to the basis } (v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...)_{(j_0,j_1,j_2,...) \text{ a 0-degression}} \text{ of } \mathcal{F}^{(0)} \end{array} \right) \tag{182}$$

[149]. Since the proof of (182) clearly does not depend on the ground ring, an analogous result holds over the ring $\mathbf{R}$: Every polynomial $R \in \mathcal{B}^{(0)}_{\mathbf{R}}$ satisfies

$$R(0) = \left(\begin{array}{c} \text{the } (v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...) \text{-coordinate of } \sigma_{\mathbf{R}}(R) \\ \text{with respect to the basis } (v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...)_{(j_0,j_1,j_2,...) \text{ a 0-degression}} \text{ of } \mathcal{F}^{(0)}_{\mathbf{R}} \end{array} \right) \tag{183}$$

[150].

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[149] *Proof of (182).* Let $R \in \mathcal{B}^{(0)}$. Thus, $R \in \mathcal{B}^{(0)} = \widetilde{F}$.

Let $p_{0,\mathcal{B}}$ be the canonical projection of the graded space $\mathcal{B}^{(0)}$ onto its 0-th homogeneous component $\mathcal{B}^{(0)}[0] = \mathbb{C} \cdot 1$, and let $p_{0,\mathcal{F}}$ be the canonical projection of the graded space $\mathcal{F}^{(0)}$ onto its 0-th homogeneous component $\mathcal{F}^{(0)}[0] = \mathbb{C}\psi_0$. Since $\sigma_0 : \mathcal{B}^{(0)} \to \mathcal{F}^{(0)}$ is a graded homomorphism, σ_0 commutes with the projections on the 0-th graded components; in other words, $\sigma_0 \circ p_{0,\mathcal{B}} = p_{0,\mathcal{F}} \circ \sigma_0$. Now, we know that $p_{0,\mathcal{B}}(R) = R(0) \cdot 1$ (since $\mathcal{B} = \widetilde{F} = \mathbb{C}[x_1, x_2, x_3, ...]$), and thus $(\sigma_0 \circ p_{0,\mathcal{B}})(R) = \sigma_0 \left(\underbrace{p_{0,\mathcal{B}}(R)}_{=R(1)\cdot 1} \right) = \sigma_0 (R(0) \cdot 1) = R(0) \cdot \underbrace{\sigma_0(1)}_{=\psi_0} = R(0)\psi_0$.

On the other hand, let κ denote the $(v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...)$-coordinate of $\sigma(R)$ with respect to the basis $(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...)_{(j_0,j_1,j_2,...) \text{ a 0-degression}}$ of $\mathcal{F}^{(0)}$. Then, the projection of $\sigma(R)$ onto the 0-th graded component $\mathcal{F}^{(0)}[0]$ of $\mathcal{F}^{(0)}$ is $\kappa \cdot v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...$ (because the basis $(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...)_{(j_0,j_1,j_2,...) \text{ a 0-degression}}$ of $\mathcal{F}^{(0)}$ is a graded basis, and the 0-th graded component $\mathcal{F}^{(0)}[0]$ of $\mathcal{F}^{(0)}$ is spanned by $(v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...)$). In other words, $p_{0,\mathcal{F}}(\sigma(R)) = \kappa \cdot \underbrace{v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...}_{=\psi_0} = \kappa\psi_0$. Hence,

$$R(0)\psi_0 = \underbrace{(\sigma_0 \circ p_{0,\mathcal{B}})}_{=p_{0,\mathcal{F}} \circ \sigma_0}(R) = (p_{0,\mathcal{F}} \circ \sigma_0)(R) = p_{0,\mathcal{F}}(\sigma(R)) = \kappa\psi_0.$$

Thus, $(R(0) - \kappa)\psi_0 = \underbrace{R(0)\psi_0}_{=\kappa\psi_0} - \kappa\psi_0 = \kappa\psi_0 - \kappa\psi_0 = 0$.

But ψ_0 is an element of a basis of $\mathcal{F}^{(0)}$ (namely, of the basis $(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...)_{(j_0,j_1,j_2,...) \text{ a 0-degression}}$). Thus, every scalar $\mu \in \mathbb{C}$ satisfying $\mu\psi_0 = 0$ must satisfy $\mu = 0$. Applying this to $\mu = R(0) - \kappa$, we obtain $R(0) - \kappa = 0$ (since $(R(0) - \kappa)\psi_0 = 0$). Thus,

$$R(0) = \kappa = \left(\begin{array}{c} \text{the } (v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...) \text{-coordinate of } \sigma(R) \\ \text{with respect to the basis } (v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...)_{(j_0,j_1,j_2,...) \text{ a 0-degression}} \text{ of } \mathcal{F}^{(0)} \end{array} \right).$$

This proves (182).

[150] Of course, "basis" means "$\mathbf{R}$-module basis" and no longer "$\mathbb{C}$-vector space basis" in this statement.

On the other hand, for every polynomial $R \in \mathcal{B}^{(0)}$, we can view $R = R(x)$ as an element of $\mathcal{B}^{(0)}_{\mathbf{R}}$ (since $\mathcal{B}^{(0)} \subseteq \mathcal{B}^{(0)}_{\mathbf{R}}$), and this way we obtain

$$\begin{aligned}
&\left(1, \exp\left(\underbrace{y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots}_{=\sum\limits_{s>0} y_s a_s}\right) R(x)\right) = \left(1, \exp\left(\sum_{s>0} y_s a_s\right) R(x)\right) \\
&= \left(1, \exp\left(\sum_{s>0} y_s \frac{\partial}{\partial x_s}\right) R(x)\right) \qquad \left(\text{since } a_s \text{ acts as } \frac{\partial}{\partial x_s} \text{ on } \mathcal{B}^{(0)} \text{ for every } s \geq 1\right) \\
&= (1, R(x+y)) \\
&\qquad \left(\begin{array}{c} \text{since } \exp\left(\sum\limits_{s>0} y_s \dfrac{\partial}{\partial x_s}\right) R(x) = R(x+y) \\ \text{by Lemma 3.14.1 (applied to } R,\ (x_1, x_2, x_3, \ldots) \text{ and } \mathbf{R} \\ \text{instead of } P,\ (z_1, z_2, z_3, \ldots) \text{ and } K) \end{array}\right) \\
&= (R(x+y))(0) \\
&\qquad \left(\begin{array}{c} \text{because the analogue of Proposition 2.2.24 } \textbf{(b)} \text{ for} \\ \text{the ground ring } \mathbf{R} \text{ yields } (1, Q) = Q(0) \text{ for every } Q \in \mathcal{B}^{(0)}_{\mathbf{R}} \end{array}\right) \\
&= R(y)
\end{aligned} \tag{184}$$

in $\mathbf{R}$.

Recall that the map $\sigma_{\mathbf{R}}$ is defined analogously to σ but for the ground ring $\mathbf{R}$ instead of $\mathbb{C}$. Thus, $\sigma_{\mathbf{R}}(Q) = \sigma(Q)$ for every $Q \in \mathcal{B}^{(0)}$. Applied to $Q = P(x)$, this yields $\sigma_{\mathbf{R}}(P(x)) = \sigma(P(x)) = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots$ (since $P(x) = \sigma^{-1}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots)$).

On the other hand, since $\sigma_{\mathbf{R}} : \mathcal{B}^{(0)}_{\mathbf{R}} \to \mathcal{F}^{(0)}_{\mathbf{R}}$ is an $\mathcal{A}_{\mathbf{R}}$-module homomorphism, and since $\mathcal{B}^{(0)}_{\mathbf{R}}$ and $\mathcal{F}^{(0)}_{\mathbf{R}}$ are two $\mathcal{A}_{\mathbf{R}}$-modules concentrated in nonpositive degrees, we can apply Proposition 3.14.29 **(b)** to $\sigma_{\mathbf{R}}$, $\mathcal{B}^{(0)}_{\mathbf{R}}$ and $\mathcal{F}^{(0)}_{\mathbf{R}}$ instead of η, M and N. As a result, we obtain

$$\sigma_{\mathbf{R}} \circ (\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots)) = (\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots)) \circ \sigma_{\mathbf{R}}$$

as maps from $\mathcal{B}^{(0)}_{\mathbf{R}}$ to $\mathcal{F}^{(0)}_{\mathbf{R}}$. This easily yields

$$\begin{aligned}
\sigma_{\mathbf{R}}(\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots) P(x)) &= \underbrace{(\sigma_{\mathbf{R}} \circ (\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots)))}_{=(\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots)) \circ \sigma_{\mathbf{R}}} (P(x)) \\
&= ((\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots)) \circ \sigma_{\mathbf{R}}) (P(x)) \\
&= \exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots) \cdot \underbrace{\sigma_{\mathbf{R}}(P(x))}_{=v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots} \\
&= \exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots) \cdot (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots).
\end{aligned} \tag{185}$$

But (184) (applied to $R = P$) yields

$$(1, \exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots) P(x)) = P(y),$$

so that

$$\begin{aligned}
& P(y) \\
& = (1, \exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots) P(x)) \\
& = (\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots) P(x))(0) \\
& \qquad \left(\begin{array}{c} \text{because the analogue of Proposition 2.2.24 } \textbf{(b)} \text{ for} \\ \text{the ground ring } \mathbf{R} \text{ yields } (1, Q) = Q(0) \text{ for every } Q \in \mathcal{B}_{\mathbf{R}}^{(0)} \end{array} \right) \\
& = \left(\begin{array}{c} \text{the } (v_0 \wedge v_{-1} \wedge v_{-2} \wedge \ldots) \text{-coordinate of } \sigma_{\mathbf{R}} (\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots) P(x)) \\ \text{with respect to the basis } (v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge \ldots)_{(j_0, j_1, j_2, \ldots) \text{ a 0-degression}} \text{ of } \mathcal{F}_{\mathbf{R}}^{(0)} \end{array} \right) \\
& \qquad \text{(by (183), applied to } R = \exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots) P(x)) \\
& = \left(\begin{array}{c} \text{the } (v_0 \wedge v_{-1} \wedge v_{-2} \wedge \ldots) \text{-coordinate of} \\ \exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + \ldots) \cdot (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots) \\ \text{with respect to the basis } (v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge \ldots)_{(j_0, j_1, j_2, \ldots) \text{ a 0-degression}} \text{ of } \mathcal{F}_{\mathbf{R}}^{(0)} \end{array} \right) \\
& \qquad \text{(by (185))} \\
& = S_{(i_k + k)_{k \geq 0}} (y) \qquad \text{(by Theorem 3.14.32)} \\
& = S_\lambda (y) \qquad \left(\text{since } (i_k + k)_{k \geq 0} = (i_0 + 0, i_1 + 1, i_2 + 2, \ldots) = \lambda \right).
\end{aligned}$$

Substituting x_i for y_i in this equation, we obtain $P(x) = S_\lambda(x)$ (since both P and S_λ are polynomials in $\mathbb{C}[x_1, x_2, x_3, \ldots]$). Thus,

$$S_\lambda(x) = P(x) = \sigma^{-1}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge \ldots).$$

This proves Theorem 3.12.11.

3.14.6. Skew Schur polynomials

Rather than prove Theorem 3.14.32 directly, let us formulate and verify a stronger statement which will be in no way harder to prove. First, we need a definition:

Definition 3.14.33. Let λ and μ be two partitions.

(a) We write $\mu \subseteq \lambda$ if every $i \in \{1, 2, 3, \ldots\}$ satisfies $\lambda_i \geq \mu_i$, where the partitions λ and μ have been written in the forms $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ and $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$.

(b) We define a polynomial $S_{\lambda/\mu}(x) \in \mathbb{Q}[x_1, x_2, x_3, \ldots]$ as follows: Write λ and μ in the forms $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ for some $m \in \mathbb{N}$. Then, let $S_{\lambda/\mu}(x)$ be the polynomial

$$\begin{aligned}
& \det \begin{pmatrix} S_{\lambda_1 - \mu_1}(x) & S_{\lambda_1 - \mu_2 + 1}(x) & S_{\lambda_1 - \mu_3 + 2}(x) & \ldots & S_{\lambda_1 - \mu_m + m - 1}(x) \\ S_{\lambda_2 - \mu_1 - 1}(x) & S_{\lambda_2 - \mu_2}(x) & S_{\lambda_2 - \mu_3 + 1}(x) & \ldots & S_{\lambda_2 - \mu_m + m - 2}(x) \\ S_{\lambda_3 - \mu_1 - 2}(x) & S_{\lambda_3 - \mu_2 - 1}(x) & S_{\lambda_3 - \mu_3}(x) & \ldots & S_{\lambda_3 - \mu_m + m - 3}(x) \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ S_{\lambda_m - \mu_1 - m + 1}(x) & S_{\lambda_m - \mu_2 - m + 2}(x) & S_{\lambda_m - \mu_3 - m + 3}(x) & \ldots & S_{\lambda_m - \mu_m}(x) \end{pmatrix} \\
& = \det \left(\left(S_{\lambda_i - \mu_j + j - i}(x) \right)_{1 \leq i \leq m, \ 1 \leq j \leq m} \right),
\end{aligned}$$

where S_j denotes 0 if $j < 0$. (Note that this does not depend on the choice of m (that is, increasing m at the cost of padding the partitions λ and μ with trailing zeroes does not change the value of $\det\left(\left(S_{\lambda_i-\mu_j+j-i}(x)\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right)$. This is because any nonnegative integers m and ℓ, any $m\times m$-matrix A, any $m\times\ell$-matrix B and any upper unitriangular $\ell\times\ell$-matrix C satisfy $\det\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \det A$.)

We refer to $S_{\lambda/\mu}(x)$ as the *bosonic Schur polynomial corresponding to the skew partition* λ/μ.

Before we formulate the strengthening of Theorem 3.14.32, three remarks:

Remark 3.14.34. Let $\varnothing$ denote the partition $(0,0,0,...)$. For every partition λ, we have $\varnothing\subseteq\lambda$ and $S_{\lambda/\varnothing}(x)=S_\lambda(x)$.

Remark 3.14.35. Let λ and μ be two partitions. Then, $S_{\lambda/\mu}(x)=0$ unless $\mu\subseteq\lambda$.

Remark 3.14.36. Recall that in Definition 3.14.8 **(c)**, we defined the notion of an "upper almost-unitriangular" $\mathbb{N}\times\mathbb{N}$-matrix. In the same way, we can define the notion of an "upper almost-unitriangular" $\{1,2,3,...\}\times\{1,2,3,...\}$-matrix.

In Definition 3.14.8 **(e)**, we defined the determinant of an upper almost-unitriangular $\mathbb{N}\times\mathbb{N}$-matrix. Analogously, we can define the determinant of an upper almost-unitriangular $\{1,2,3,...\}\times\{1,2,3,...\}$-matrix.

Let $\lambda=(\lambda_1,\lambda_2,\lambda_3,...)$ and $\mu=(\mu_1,\mu_2,\mu_3,...)$ be two partitions. Then, the $\{1,2,3,...\}\times\{1,2,3,...\}$-matrix $\left(S_{\lambda_i-\mu_j+j-i}(x)\right)_{(i,j)\in\{1,2,3,...\}^2}$ is upper almost-unitriangular, and we have

$$\begin{aligned} S_{\lambda/\mu}(x) &= \det\left(\left(S_{\lambda_i-\mu_j+j-i}(x)\right)_{(i,j)\in\{1,2,3,...\}^2}\right) \\ &= \det\begin{pmatrix} S_{\lambda_1-\mu_1}(x) & S_{\lambda_1-\mu_2+1}(x) & S_{\lambda_1-\mu_3+2}(x) & ... \\ S_{\lambda_2-\mu_1-1}(x) & S_{\lambda_2-\mu_2}(x) & S_{\lambda_2-\mu_3+1}(x) & ... \\ S_{\lambda_3-\mu_1-2}(x) & S_{\lambda_3-\mu_2-1}(x) & S_{\lambda_3-\mu_3}(x) & ... \\ ... & ... & ... & ... \end{pmatrix}. \end{aligned} \tag{186}$$

All of the above three remarks follow easily from Definition 3.14.33.

Now, let us finally give the promised strengthening of Theorem 3.14.32:

Theorem 3.14.37. Let $\mathbf{R}$ be a commutative $\mathbb{Q}$-algebra. Let $y_1,y_2,y_3,...$ be some elements of $\mathbf{R}$. Denote by y the family $(y_1,y_2,y_3,...)$. Let $(i_0,i_1,i_2,...)$ be a 0-degression. Recall that $\exp(y_1a_1+y_2a_2+y_3a_3+...)\cdot(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge...)$ is a well-defined element of $\mathcal{F}_{\mathbf{R}}^{(0)}$ due to Proposition 3.14.29 **(c)**. Recall also that $(j_k+k)_{k\geq 0}$ is a partition for every 0-degression $(j_0,j_1,j_2,...)$ (this follows from Proposition 3.5.24, applied to 0 and $(j_0,j_1,j_2,...)$ instead of m and $(i_0,i_1,i_2,...)$). In particular, $(i_k+k)_{k\geq 0}$ is a partition.

We have

$$\begin{aligned} &\exp(y_1a_1+y_2a_2+y_3a_3+...)\cdot(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge...) \\ &= \sum_{\substack{(j_0,j_1,j_2,...)\text{ a 0-degression;} \\ (j_k+k)_{k\geq 0}\subseteq(i_k+k)_{k\geq 0}}} S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y)\cdot v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge.... \end{aligned} \tag{187}$$

(Note that the sum on the right hand side of (187) is a finite sum, since only finitely many 0-degressions $(j_0, j_1, j_2, ...)$ satisfy $(j_k + k)_{k\geq 0} \subseteq (i_k + k)_{k\geq 0}$.)

Before we prove this, let us see how this yields Theorem 3.14.32:

Proof of Theorem 3.14.32 using Theorem 3.14.37. Remark 3.14.34 (applied to $\lambda = (i_k + k)_{k\geq 0}$) yields $\varnothing \subseteq (i_k + k)_{k\geq 0}$ and $S_{(i_k+k)_{k\geq 0}/\varnothing}(x) = S_{(i_k+k)_{k\geq 0}}(x)$. By substituting y for x in the equality $S_{(i_k+k)_{k\geq 0}/\varnothing}(x) = S_{(i_k+k)_{k\geq 0}}(x)$, we conclude $S_{(i_k+k)_{k\geq 0}/\varnothing}(y) = S_{(i_k+k)_{k\geq 0}}(y)$.

Theorem 3.14.37 yields that (187) holds.

On the other hand, every 0-degression $(j_0, j_1, j_2, ...)$ satisfying $(j_k + k)_{k\geq 0} \not\subseteq (i_k + k)_{k\geq 0}$ must satisfy

$$S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... = 0 \tag{188}$$

[151]. Hence, each of the addends of the infinite sum $\sum\limits_{\substack{(j_0,j_1,j_2,...) \text{ a 0-degression;} \\ (j_k+k)_{k\geq 0} \not\subseteq (i_k+k)_{k\geq 0}}} S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...$ equals 0. Thus, the infinite sum $\sum\limits_{\substack{(j_0,j_1,j_2,...) \text{ a 0-degression;} \\ (j_k+k)_{k\geq 0} \not\subseteq (i_k+k)_{k\geq 0}}} S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...$ is well-defined and equals 0. We thus have

$$0 = \sum_{\substack{(j_0,j_1,j_2,...) \text{ a 0-degression;} \\ (j_k+k)_{k\geq 0} \not\subseteq (i_k+k)_{k\geq 0}}} S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge .... \tag{189}$$

Adding this equality to (187), we obtain

$$\begin{aligned}
&\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...) \cdot (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) \\
&= \sum_{\substack{(j_0,j_1,j_2,...) \text{ a 0-degression;} \\ (j_k+k)_{k\geq 0} \subseteq (i_k+k)_{k\geq 0}}} S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \\
&\qquad + \sum_{\substack{(j_0,j_1,j_2,...) \text{ a 0-degression;} \\ (j_k+k)_{k\geq 0} \not\subseteq (i_k+k)_{k\geq 0}}} S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \\
&= \sum_{(j_0,j_1,j_2,...) \text{ a 0-degression}} S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ....
\end{aligned}$$

Hence, the $(v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...)$-coordinate of $\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...) \cdot (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$ with respect to the basis $(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...)_{(j_0,j_1,j_2,...) \text{ a 0-degression}}$ of $\mathcal{F}_{\mathbf{R}}^{(0)}$ equals

$$\begin{aligned}
S_{(i_k+k)_{k\geq 0}/(-k+k)_{k\geq 0}}(y) &= S_{(i_k+k)_{k\geq 0}/\varnothing}(y) \qquad \left(\text{since } (-k+k)_{k\geq 0} = (0)_{k\geq 0} = (0,0,0,...) = \varnothing\right) \\
&= S_{(i_k+k)_{k\geq 0}}(y).
\end{aligned}$$

This proves Theorem 3.14.32 using Theorem 3.14.37.

[151] *Proof.* Let $(j_0, j_1, j_2, ...)$ be a 0-degression satisfying $(j_k + k)_{k\geq 0} \not\subseteq (i_k + k)_{k\geq 0}$. We know that $(i_k + k)_{k\geq 0}$ and $(j_k + k)_{k\geq 0}$ are partitions. Thus, Remark 3.14.35 (applied to $\lambda = (i_k + k)_{k\geq 0}$ and $\mu = (j_k + k)_{k\geq 0}$) yields that $S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(x) = 0$ unless $(j_k + k)_{k\geq 0} \subseteq (i_k + k)_{k\geq 0}$. Since we don't have $(j_k + k)_{k\geq 0} \subseteq (i_k + k)_{k\geq 0}$ (because by assumption, we have $(j_k + k)_{k\geq 0} \not\subseteq (i_k + k)_{k\geq 0}$), we thus know that $S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(x) = 0$. Substituting y for x in this equation, we obtain $S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) = 0$, so that $S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... = 0$, qed.

3.14.7. Proof of Theorem 3.14.37 using $\mathrm{U}(\infty)$

One final easy lemma:

Lemma 3.14.38. For every $n \in \mathbb{Z}$, let c_n be an element of $\mathbb{C}$. Assume that $c_n = 0$ for every negative $n \in \mathbb{Z}$. Consider the shift operator $T : V \to V$ of Definition 3.6.2. Then, $\sum\limits_{k \geq 0} c_k T^k = (c_{j-i})_{(i,j)\in\mathbb{Z}^2}$.

The proof of this lemma is immediate from the definition of T.

We now give a proof of Theorem 3.14.37 using the actions $\rho : \mathfrak{u}_\infty \to \operatorname{End}\left(\wedge^{\frac{\infty}{2},m} V\right)$ and $\varrho : \mathrm{U}(\infty) \to \mathrm{GL}\left(\wedge^{\frac{\infty}{2},m} V\right)$ introduced in Subsection 3.14.3 and their properties.

First proof of Theorem 3.14.37. In order to simplify notation, we assume that $\mathbf{R} = \mathbb{C}$. (All the arguments that we will make in the following are independent of the ground ring, as long as the ground ring is a commutative $\mathbb{Q}$-algebra. Therefore, we are actually allowed to assume that $\mathbf{R} = \mathbb{C}$.) Since we assumed that $\mathbf{R} = \mathbb{C}$, we have $\mathcal{A}_{\mathbf{R}} = \mathcal{A}$ and $\mathcal{F}^{(0)}_{\mathbf{R}} = \mathcal{F}^{(0)}$.

Now consider the shift operator $T : V \to V$ of Definition 3.6.2. As a matrix in $\overline{\mathfrak{a}_\infty}$, this T is the matrix which has 1's on the diagonal right above the main one, and 0's everywhere else. The embedding $\mathcal{A} \to \mathfrak{a}_\infty$ that we are using to define the action of $\mathcal{A}$ on $\mathcal{F}^{(0)}$ sends a_j to T^j for every $j \in \mathbb{Z}$. Thus, every positive integer j satisfies

$$a_j \mid_{\mathcal{F}^{(0)}} = T^j \mid_{\mathcal{F}^{(0)}} = \widehat{\rho}\left(T^j\right) = \rho\left(T^j\right)$$
$$\left(\text{by Remark 3.14.19, applied to } m = 0 \text{ and } a = T^j \text{ (since } T^j \in \mathfrak{u}_\infty \cap \overline{\mathfrak{a}_\infty})\right).$$

Since $y_1 a_1 + y_2 a_2 + y_3 a_3 + ... = \sum\limits_{j \geq 1} y_j a_j$, we have

$$\begin{aligned}(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...) \mid_{\mathcal{F}^{(0)}} &= \left(\sum_{j\geq 1} y_j a_j\right) \mid_{\mathcal{F}^{(0)}} = \sum_{j \geq 1} y_j \underbrace{\left(a_j \mid_{\mathcal{F}^{(0)}}\right)}_{=\rho(T^j)} = \sum_{j\geq 1} y_j \rho\left(T^j\right) \\ &= \rho\left(\sum_{j\geq 1} y_j T^j\right).\end{aligned}$$

Here, we have used the fact that $\sum\limits_{j\geq 1} y_j T^j \in \mathfrak{u}_\infty$ (this ensures that $\rho\left(\sum\limits_{j\geq 1} y_j T^j\right)$ is well-defined).

On the other hand, substituting y for x in (145), we obtain

$$\sum_{k\geq 0} S_k(y) z^k = \exp\left(\sum_{i\geq 1} y_i z^i\right) \qquad \text{in } \mathbb{C}[[z]].$$

Substituting T for z in this equality, we obtain $\sum\limits_{k\geq 0} S_k(y) T^k = \exp\left(\sum\limits_{i\geq 1} y_i T^i\right)$. Thus,

$$\exp\left(\sum_{j\geq 1} y_j T^j\right) = \exp\left(\sum_{i\geq 1} y_i T^i\right) = \sum_{k\geq 0} S_k(y) T^k = \left(S_{j-i}(y)\right)_{(i,j)\in\mathbb{Z}^2} \qquad (190)$$

(by Lemma 3.14.38, applied to $c_n = S_n(y)$ (since $S_n(y) = 0$ for every negative $n \in \mathbb{Z}$)).

Now,

$$\begin{aligned}
&(\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...)) \mid_{\mathcal{F}^{(0)}} \\
&= \exp\underbrace{((y_1a_1 + y_2a_2 + y_3a_3 + ...) \mid_{\mathcal{F}^{(0)}})}_{=\rho\left(\sum\limits_{j\geq 1} y_jT^j\right)} \\
&= \exp\left(\rho\left(\sum_{j\geq 1} y_jT^j\right)\right) = \varrho\left(\exp\left(\sum_{j\geq 1} y_jT^j\right)\right) \\
&\qquad \left(\begin{array}{c}\text{since Theorem 3.14.25 (applied to } a = \sum\limits_{j\geq 1} y_jT^j\text{) yields} \\ \varrho\left(\exp\left(\sum\limits_{j\geq 1} y_jT^j\right)\right) = \exp\left(\rho\left(\sum\limits_{j\geq 1} y_jT^j\right)\right)\end{array}\right) \\
&= \varrho\left((S_{j-i}(y))_{(i,j)\in\mathbb{Z}^2}\right) \qquad \text{(by (190))}.
\end{aligned}$$

Denote the matrix $(S_{j-i}(y))_{(i,j)\in\mathbb{Z}^2} \in \mathrm{U}(\infty)$ by A. Thus, we have

$$(\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...)) \mid_{\mathcal{F}^{(0)}} = \varrho\left(\underbrace{(S_{j-i}(y))_{(i,j)\in\mathbb{Z}^2}}_{=A}\right) = \varrho(A).$$

Hence,

$$\begin{aligned}
&\exp(y_1a_1 + y_2a_2 + y_3a_3 + ...) \cdot (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) \\
&= (\varrho(A))(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = \sum_{(j_0,j_1,j_2,...) \text{ is a 0-degression}} \det\left(\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right)^T\right) v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...
\end{aligned} \tag{191}$$

$$\text{(by Remark 3.14.22, applied to } m = 0).$$

But a close look at the matrix $\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right)^T$ proves that

$$\det\left(\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right)^T\right) = S_{(i_k+k)_{k\geq 0}/(j_k+k)_{k\geq 0}}(y) \qquad \text{for every 0-degression } (j_0, j_1, j_2, ...) \tag{192}$$

[152].

[152] *Proof of (192):* Let $(j_0, j_1, j_2, ...)$ be a 0-degression. Since $A = (S_{j-i}(y))_{(i,j)\in\mathbb{Z}^2}$, we have $A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...} = (S_{i_v-j_u}(y))_{(u,v)\in\mathbb{N}^2}$, so that $\left(A^{i_0,i_1,i_2,...}_{j_0,j_1,j_2,...}\right)^T = (S_{i_v-j_u}(y))_{(v,u)\in\mathbb{N}^2}$. But define two partitions λ and μ by $\lambda = (i_k + k)_{k\geq 0}$ and $\mu = (j_k + k)_{k\geq 0}$. Write the partitions λ and μ in the forms $\lambda = (\lambda_1, \lambda_2, \lambda_3, ...)$ and $\mu = (\mu_1, \mu_2, \mu_3, ...)$. Then, $\lambda_v = i_{v-1} + (v-1)$ for every $v \in \{1, 2, 3, ...\}$, and $\mu_u = j_{u-1} + (u-1)$ for every $u \in \{1, 2, 3, ...\}$. Thus, for every $(u, v) \in \{1, 2, 3, ...\}^2$, we have

$$\underbrace{\lambda_v}_{=i_{v-1}+(v-1)} - \underbrace{\mu_u}_{=j_{u-1}+(u-1)} + u - v = (i_{v-1} + (v-1)) - (j_{u-1} + (u-1)) + u - v = i_{v-1} - j_{u-1}. \tag{193}$$

But (186) yields $S_{\lambda/\mu}(x) = \det\left((S_{\lambda_i-\mu_j+j-i}(x))_{(i,j)\in\{1,2,3,...\}^2}\right)$. Substituting y for x in this

Now, (191) becomes

$$\begin{aligned}
&\exp\left(y_1a_1+y_2a_2+y_3a_3+\ldots\right)\cdot\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge\ldots\right)\\
&=\sum_{(j_0,j_1,j_2,\ldots)\text{ is a 0-degression}}\underbrace{\det\left(\left(A^{i_0,i_1,i_2,\ldots}_{j_0,j_1,j_2,\ldots}\right)^T\right)}_{\substack{=S_{(i_k+k)_{k\geq0}\diagup(j_k+k)_{k\geq0}}(y)\\ \text{(by (192))}}} v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge\ldots\\
&=\sum_{(j_0,j_1,j_2,\ldots)\text{ is a 0-degression}} S_{(i_k+k)_{k\geq0}\diagup(j_k+k)_{k\geq0}}(y)\cdot v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge\ldots\\
&=\sum_{\substack{(j_0,j_1,j_2,\ldots)\text{ a 0-degression;}\\ (j_k+k)_{k\geq0}\subseteq(i_k+k)_{k\geq0}}} S_{(i_k+k)_{k\geq0}\diagup(j_k+k)_{k\geq0}}(y)\cdot v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge\ldots\\
&\qquad+\underbrace{\sum_{\substack{(j_0,j_1,j_2,\ldots)\text{ a 0-degression;}\\ (j_k+k)_{k\geq0}\not\subseteq(i_k+k)_{k\geq0}}} S_{(i_k+k)_{k\geq0}\diagup(j_k+k)_{k\geq0}}(y)\cdot v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge\ldots}_{\substack{=0\\ \text{(by (189))}}}\\
&=\sum_{\substack{(j_0,j_1,j_2,\ldots)\text{ a 0-degression;}\\ (j_k+k)_{k\geq0}\subseteq(i_k+k)_{k\geq0}}} S_{(i_k+k)_{k\geq0}\diagup(j_k+k)_{k\geq0}}(y)\cdot v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge\ldots.
\end{aligned}$$

This proves Theorem 3.14.37.

We can now combine the above to obtain a proof of Theorem 3.12.11:

Second proof of Theorem 3.12.11. We have proven Theorem 3.14.32 using Theorem 3.14.37. Since we know that Theorem 3.14.37 holds, this yields that Theorem 3.14.32 holds. This, in turn, entails that Theorem 3.12.11 holds (since we have proven Theorem 3.12.11 using Theorem 3.14.32).

3.14.8. "Finitary" proof of Theorem 3.14.37

The above second proof of Theorem 3.12.11 had the drawback of requiring a slew of new notions (those of $\mathfrak{u}_\infty$, of $\mathrm{U}(\infty)$, of the determinant of an almost upper-triangular

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equality, we obtain

$$\begin{aligned}
S_{\lambda\diagup\mu}(y)&=\det\left(\left(S_{\lambda_i-\mu_j+j-i}(y)\right)_{(i,j)\in\{1,2,3,\ldots\}^2}\right)=\det\left(\left(S_{\lambda_v-\mu_u+u-v}(y)\right)_{(v,u)\in\{1,2,3,\ldots\}^2}\right)\\
&\qquad\text{(here, we substituted }(v,u)\text{ for }(i,j)\text{)}\\
&=\det\left(\left(S_{i_{v-1}-j_{u-1}}(y)\right)_{(v,u)\in\{1,2,3,\ldots\}^2}\right)\qquad\text{(by (193))}\\
&=\det\left(\underbrace{\left(S_{i_v-j_u}(y)\right)_{(v,u)\in\mathbb{N}^2}}_{=\left(A^{i_0,i_1,i_2,\ldots}_{j_0,j_1,j_2,\ldots}\right)^T}\right)\qquad\text{(here, we substituted }(v,u)\text{ for }(v-1,u-1)\text{)}\\
&=\det\left(\left(A^{i_0,i_1,i_2,\ldots}_{j_0,j_1,j_2,\ldots}\right)^T\right).
\end{aligned}$$

Since $\lambda=(i_k+k)_{k\geq0}$ and $\mu=(j_k+k)_{k\geq0}$, this rewrites as $S_{(i_k+k)_{k\geq0}\diagup(j_k+k)_{k\geq0}}(y)=\det\left(\left(A^{i_0,i_1,i_2,\ldots}_{j_0,j_1,j_2,\ldots}\right)^T\right)$. This proves (192).

matrix etc.) and of their properties (Proposition 3.14.13, Remark 3.14.22, Theorem 3.14.25 and others). We will now give a proof of Theorem 3.12.11 which is more or less equivalent to the second proof of Theorem 3.12.11 shown above, but avoiding these new notions. It will eschew using infinite matrices other than those in $\overline{\mathfrak{a}_\infty}$, and instead work with finite objects most of the time.

Since we already know how to derive Theorem 3.12.11 from Theorem 3.14.37, we only need to verify Theorem 3.14.37.

Let us first introduce some finite-dimensional subspaces of the vector space V:

Definition 3.14.39. Let α and β be integers such that $\alpha - 1 \leq \beta$.

(a) Then, $V_{]\alpha,\beta]}$ will denote the vector subspace of V spanned by the vectors $v_{\alpha+1}$, $v_{\alpha+2}$, ..., v_β. It is clear that $(v_{\alpha+1}, v_{\alpha+2}, ..., v_\beta)$ is a basis of this vector space $V_{]\alpha,\beta]}$, so that $\dim\left(V_{]\alpha,\beta]}\right) = \beta - \alpha$.

(b) Let $T_{]\alpha,\beta]}$ be the endomorphism of the vector space $V_{]\alpha,\beta]}$ defined by

$$\left(T_{]\alpha,\beta]}(v_i) = \begin{cases} v_{i-1}, & \text{if } i > \alpha+1; \\ 0, & \text{if } i = \alpha+1 \end{cases} \qquad \text{for all } i \in \{\alpha+1, \alpha+2, ..., \beta\}\right).$$

(c) We let $\mathcal{A}_+$ be the Lie subalgebra $\langle a_1, a_2, a_3, ...\rangle$ of $\mathcal{A}$. This Lie subalgebra $\mathcal{A}_+$ is abelian. We define an $\mathcal{A}_+$-module structure on the vector space $V_{]\alpha,\beta]}$ by letting a_i act as $T^i_{]\alpha,\beta]}$ for every positive integer i. (This is well-defined, since the powers of $T_{]\alpha,\beta]}$ commute, just as the elements of $\mathcal{A}_+$.) Thus, for every $\ell \in \mathbb{N}$, the ℓ-th exterior power $\wedge^\ell\left(V_{]\alpha,\beta]}\right)$ is canonically equipped with an $\mathcal{A}_+$-module structure.

(d) For every $\ell \in \mathbb{N}$, let $R_{\ell,]\alpha,\beta]} : \wedge^\ell\left(V_{]\alpha,\beta]}\right) \to \wedge^{\frac{\infty}{2},\alpha+\ell} V$ be the linear map defined by

$$\left(\begin{array}{r} R_{\ell,]\alpha,\beta]}(b_1 \wedge b_2 \wedge ... \wedge b_\ell) = b_1 \wedge b_2 \wedge ... \wedge b_\ell \wedge v_\alpha \wedge v_{\alpha-1} \wedge v_{\alpha-2} \wedge ... \\ \text{for any } b_1, b_2, ..., b_\ell \in V_{]\alpha,\beta]} \end{array}\right).$$

Remark 3.14.40. Let α and β be integers such that $\alpha - 1 \leq \beta$.

(a) The $(\beta-\alpha)$-tuple $(v_\beta, v_{\beta-1}, ..., v_{\alpha+1})$ is a basis of this vector space $V_{]\alpha,\beta]}$. With respect to this basis, the endomorphism $T_{]\alpha,\beta]}$ of $V_{]\alpha,\beta]}$ is represented by the $(\beta-\alpha) \times (\beta-\alpha)$ matrix $\begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ 1 & 0 & 0 & ... & 0 & 0 \\ 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & ... & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$.

(b) We have $T^{\beta-\alpha}_{]\alpha,\beta]} = 0$.

(c) For every sequence $(y_1, y_2, y_3, ...)$ of elements of $\mathbb{C}$, the endomorphism $\sum\limits_{i=1}^{\infty} y_i T^i_{]\alpha,\beta]}$ of $V_{]\alpha,\beta]}$ is well-defined and nilpotent.

(d) For every sequence $(y_1, y_2, y_3, ...)$ of elements of $\mathbb{C}$, the endomorphism $\exp\left(\sum\limits_{i=1}^{\infty} y_i T^i_{]\alpha,\beta]}\right)$ of $V_{]\alpha,\beta]}$ is well-defined.

(e) For every sequence $(y_1, y_2, y_3, ...)$ of elements of $\mathbb{C}$, the endomorphism $\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...)$ of $V_{]\alpha,\beta]}$ is well-defined.

(f) Every $j \in \mathbb{N}$ satisfies

$$T^j_{]\alpha,\beta]} v_u = \begin{cases} v_{u-j}, & \text{if } u - j > \alpha; \\ 0, & \text{if } u - j \leq \alpha \end{cases} \qquad \text{for every } u \in \{\alpha+1, \alpha+2, ..., \beta\}. \tag{194}$$

(g) For every $n \in \mathbb{Z}$, let c_n be an element of $\mathbb{C}$. Assume that $c_n = 0$ for every negative $n \in \mathbb{Z}$. Then, the sum $\sum_{k \geq 0} c_k T^k_{]\alpha,\beta]}$ is a well-defined endomorphism of $V_{]\alpha,\beta]}$, and the matrix representing this endomorphism with respect to the basis $(v_\beta, v_{\beta-1}, ..., v_{\alpha+1})$ of $V_{]\alpha,\beta]}$ is $(c_{i-j})_{(i,j)\in\{1,2,...,\beta-\alpha\}^2}$.

Proof of Remark 3.14.40. Parts **(a)** through **(f)** of Remark 3.14.40 are trivial, and part **(g)** is just the finitary analogue of Lemma 3.14.38 and proven in the same way.

This completes the proof of Remark 3.14.40.

A less trivial observation is the following:

Proposition 3.14.41. Let α and β be integers such that $\alpha - 1 \leq \beta$. Let $\ell \in \mathbb{N}$. Then, $R_{\ell,]\alpha,\beta]} : \wedge^\ell \left(V_{]\alpha,\beta]}\right) \to \mathcal{F}^{(\alpha+\ell)}$ is an $\mathcal{A}_+$-module homomorphism (where the $\mathcal{A}_+$-module structure on $\mathcal{F}^{(\alpha+\ell)}$ is obtained by restricting the $\mathcal{A}$-module structure on $\mathcal{F}^{(\alpha+\ell)}$).

Proof of Proposition 3.14.41. Let T be the shift operator defined in Definition 3.6.2.

Let j be a positive integer. Then, for every integer $i \leq 0$, the (i,i)-th entry of the matrix T^j is 0 (in fact, the matrix T^j has only zeroes on its main diagonal). Moreover, from the definition of T, it follows quickly that

$$T^j v_u = v_{u-j} \qquad \text{for every } u \in \mathbb{Z}. \tag{195}$$

Let $(i_0, i_1, ..., i_{\ell-1})$ be an ℓ-tuple of elements of $\{\alpha+1, \alpha+2, ..., \beta\}$ such that $i_0 > i_1 > ... > i_{\ell-1}$. We will prove that

$$a_j \rightharpoonup \left(R_{\ell,]\alpha,\beta]}\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\ell-1}}\right)\right) = R_{\ell,]\alpha,\beta]}\left(a_j \rightharpoonup \left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\ell-1}}\right)\right). \tag{196}$$

Indeed, let us extend the ℓ-tuple $(i_0, i_1, ..., i_{\ell-1})$ to a sequence $(i_0, i_1, i_2, ...)$ of integers by setting ($i_k = \ell + \alpha - k$ for every $k \in \{\ell, \ell+1, \ell+2, ...\}$). Then, $(i_0, i_1, i_2, ...) = (i_0, i_1, ..., i_{\ell-1}, \alpha, \alpha-1, \alpha-2, ...)$. As a consequence, the sequence $(i_0, i_1, i_2, ...)$ is strictly decreasing (since $i_0 > i_1 > ... > i_{\ell-1} > \alpha > \alpha - 1 > \alpha - 2 > ...$) and hence an $(\alpha+\ell)$-degression. Note that

$$v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge v_{i_k - j} \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... = 0 \qquad \text{for every } k \in \mathbb{N} \text{ satisfying } k \geq \ell \tag{197}$$

[153]. Also,

$$v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge v_{i_k - j} \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... = 0 \qquad \text{for every } k \in \mathbb{N} \text{ satisfying } k < \ell \text{ and } i_k - j \leq \alpha. \tag{198}$$

[153] *Proof of (197):* Let $k \in \mathbb{N}$ satisfy $k \geq \ell$. Then, $k + j \geq \ell$ as well (since j is positive), so that $i_{k+j} = \ell + \alpha - (k+j) = \underbrace{(\ell + \alpha - k)}_{\substack{=i_k \\ (\text{since } k \geq \ell)}} - j = i_k - j$. Thus, the sequence $(i_0, i_1, ..., i_{k-1}, i_k - j, i_{k+1}, i_{k+2}, ...)$ has two equal terms (since $k + j \neq k$ (due to j being positive)). Thus, $v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge v_{i_k - j} \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... = 0$. This proves (197).

[154]

The definition of $R_{\ell,]\alpha,\beta]}$ yields

$$\begin{aligned}
&R_{\ell,]\alpha,\beta]}\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{\ell-1}}\right)\\
&=v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{\ell-1}}\wedge v_\alpha\wedge v_{\alpha-1}\wedge v_{\alpha-2}\wedge ...\\
&=v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ... \qquad \left(\text{since } \left(i_0,i_1,...,i_{\ell-1},\alpha,\alpha-1,\alpha-2,...\right)=\left(i_0,i_1,i_2,...\right)\right),
\end{aligned}$$

so that

$$\begin{aligned}
&a_j\rightharpoonup\left(R_{\ell,]\alpha,\beta]}\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{\ell-1}}\right)\right)\\
&=a_j\rightharpoonup\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)=\left(\widehat{\rho}\left(T^j\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\\
&\qquad\left(\text{since } a_j\mid_{\mathcal{F}^{(\alpha+\ell)}}=T^j\mid_{\mathcal{F}^{(\alpha+\ell)}}=\widehat{\rho}\left(T^j\right)\right)\\
&=\sum_{k\geq 0}v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge\underbrace{\left(T^j\rightharpoonup v_{i_k}\right)}_{\substack{=T^jv_{i_k}=v_{i_k-j}\\ \text{(by (195), applied to } u=i_k)}}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\\
&\qquad\left(\begin{array}{c}\text{by Proposition 3.7.5, applied to } \left(b_0,b_1,b_2,...\right)=\left(v_{i_0},v_{i_1},v_{i_2},...\right) \text{ and } a=T^j\\ \text{(since for every integer } i\leq 0\text{, the } (i,i)\text{-th entry of } T^j \text{ is } 0).\end{array}\right)\\
&=\sum_{k\geq 0}v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{i_k-j}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ....
\end{aligned}$$

The sum on the right hand side of this equation is infinite, but lots of its terms vanish: Namely, all its terms with $k\geq\ell$ vanish (because of (197)), and all its terms with $k<\ell$ and $i_k-j\leq\alpha$ vanish (because of (198)). We can thus replace the $\sum\limits_{k\geq 0}$ sign by a $\sum\limits_{\substack{k\geq 0;\\ k<\ell;\\ i_k-j>\alpha}}$ sign, and obtain

$$\begin{aligned}
&a_j\rightharpoonup\left(R_{\ell,]\alpha,\beta]}\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{\ell-1}}\right)\right)\\
&=\sum_{\substack{k\geq 0;\\ k<\ell;\\ i_k-j>\alpha}}v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{i_k-j}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge .... \qquad (199)
\end{aligned}$$

[154] *Proof of (198):* Let $k\in\mathbb{N}$ satisfy $k<\ell$ and $i_k-j\leq\alpha$. Every integer $\leq\alpha$ is contained in the sequence $\left(i_0,i_1,i_2,...\right)$ (since $\left(i_0,i_1,i_2,...\right)=\left(i_0,i_1,...,i_{\ell-1},\alpha,\alpha-1,\alpha-2,...\right)$). Since $i_k-j\leq\alpha$, this yields that the integer i_k-j is contained in the sequence $\left(i_0,i_1,i_2,...\right)$. Hence, there exists a $p\in\mathbb{N}$ such that $i_p=i_k-j$. Consider this p.

Since $k<\ell$, we have $i_k\in\{\alpha+1,\alpha+2,...,\beta\}$, so that $i_k>\alpha\geq i_k-j=i_p$, and hence $i_k\neq i_p$. Thus, $k\neq p$. Since $i_k-j=i_p$ and $k\neq p$, the sequence $\left(i_0,i_1,...,i_{k-1},i_k-j,i_{k+1},i_{k+2},...\right)$ has two equal terms. Thus, $v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{i_k-j}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...=0$, and this proves (198).

On the other hand, by the definition of the $\mathcal{A}_+$-module $\wedge^\ell \left(V_{]\alpha,\beta]}\right)$, we have

$$\begin{aligned}
& a_j \rightharpoonup \left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\ell-1}}\right) \\
&= \sum_{k=0}^{\ell-1} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \underbrace{\left(a_j \rightharpoonup v_{i_k}\right)}_{\substack{=T^j_{]\alpha,\beta]} v_{i_k} \\ \text{(since } a_j \text{ acts as } T^j_{]\alpha,\beta]} \\ \text{on } V_{]\alpha,\beta]})}} \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \wedge v_{i_{\ell-1}} \\
&= \sum_{k=0}^{\ell-1} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(T^j_{]\alpha,\beta]} v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \wedge v_{i_{\ell-1}}.
\end{aligned}$$

Applying the linear map $R_{\ell,]\alpha,\beta]}$ to this equation, we obtain

$$\begin{aligned}
& R_{\ell,]\alpha,\beta]} \left(a_j \rightharpoonup \left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\ell-1}}\right)\right) \\
&= \sum_{k=0}^{\ell-1} \underbrace{R_{\ell,]\alpha,\beta]} \left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(T^j_{]\alpha,\beta]} v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \wedge v_{i_{\ell-1}}\right)}_{\substack{= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(T^j_{]\alpha,\beta]} v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \wedge v_{i_{\ell-1}} \wedge v_\alpha \wedge v_{\alpha-1} \wedge v_{\alpha-2} \wedge ... \\ \text{(by the definition of } R_{\ell,]\alpha,\beta]})}} \\
&= \underbrace{\sum_{k=0}^{\ell-1}}_{= \sum_{\substack{k \geq 0; \\ k < \ell}}} \underbrace{v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(T^j_{]\alpha,\beta]} v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \wedge v_{i_{\ell-1}} \wedge v_\alpha \wedge v_{\alpha-1} \wedge v_{\alpha-2} \wedge ...}_{\substack{= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(T^j_{]\alpha,\beta]} v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \\ \text{(since } (i_0, i_1, ..., i_{\ell-1}, \alpha, \alpha-1, \alpha-2, ...) = (i_0, i_1, i_2, ...))}} \\
&= \sum_{\substack{k \geq 0; \\ k < \ell}} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \underbrace{\left(T^j_{]\alpha,\beta]} v_{i_k}\right)}_{\substack{= \begin{cases} v_{i_k - j}, & \text{if } i_k - j > \alpha; \\ 0, & \text{if } i_k - j \leq \alpha \end{cases} \\ \text{(by (194), applied to } u = i_k)}} \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \\
&= \sum_{\substack{k \geq 0; \\ k < \ell}} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \begin{cases} v_{i_k - j}, & \text{if } i_k - j > \alpha; \\ 0, & \text{if } i_k - j \leq \alpha \end{cases} \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ... \\
&= \sum_{\substack{k \geq 0; \\ k < \ell; \\ i_k - j > \alpha}} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge v_{i_k - j} \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ....
\end{aligned}$$

Compared with (199), this yields

$$a_j \rightharpoonup \left(R_{\ell,]\alpha,\beta]} \left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\ell-1}}\right)\right) = R_{\ell,]\alpha,\beta]} \left(a_j \rightharpoonup \left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\ell-1}}\right)\right).$$

We have thus proven (196).

Now, forget that we fixed j and $(i_0, i_1, ..., i_{\ell-1})$. We have thus proven the equality (196) for every positive integer j and every ℓ-tuple $(i_0, i_1, ..., i_{\ell-1})$ of elements of $\{\alpha+1, \alpha+2, ..., \beta\}$ such that $i_0 > i_1 > ... > i_{\ell-1}$.

Since $\mathcal{A}_+ = \langle a_1, a_2, a_3, ... \rangle$ and since $\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\ell-1}}\right)_{\beta \geq i_0 > i_1 > ... > i_{\ell-1} \geq \alpha+1}$ is a basis of the vector space $\wedge^\ell \left(V_{]\alpha,\beta]}\right)$, this yields (by linearity) that $x \rightharpoonup \left(R_{\ell,]\alpha,\beta]} (w)\right) =$

$R_{\ell,]\alpha,\beta]}(x \rightharpoonup w)$ holds for every $x \in \mathcal{A}_+$ and $w \in \wedge^\ell \left(V_{]\alpha,\beta]}\right)$. Thus, $R_{\ell,]\alpha,\beta]}$ is an $\mathcal{A}_+$-module homomorphism. This proves Proposition 3.14.41.

Now, we can turn to the promised proof:

Second proof of Theorem 3.14.37. In order to simplify notation, we assume that $\mathbf{R} = \mathbb{C}$. (All the arguments that we will make in the following are independent of the ground ring, as long as the ground ring is a commutative $\mathbb{Q}$-algebra. Therefore, we are actually allowed to assume that $\mathbf{R} = \mathbb{C}$.) Since we assumed that $\mathbf{R} = \mathbb{C}$, we have $\mathcal{A}_\mathbf{R} = \mathcal{A}$ and $\mathcal{F}^{(0)}_\mathbf{R} = \mathcal{F}^{(0)}$.

Since $(i_0, i_1, i_2, ...)$ is a 0-degression, every sufficiently high $k \in \mathbb{N}$ satisfies $i_k + k = 0$. In other words, there exists some $K \in \mathbb{N}$ such that every $k \in \mathbb{N}$ satisfying $k \geq K$ satisfies $i_k + k = 0$. Consider this K. WLOG assume that $K > 0$ (else, replace K by $K + 1$). Since every $k \in \mathbb{N}$ satisfying $k \geq K$ satisfies $i_k + k = 0$ and thus $i_k = -k$, we have $(i_0, i_1, i_2, ...) = (i_0, i_1, i_2, ..., i_{K-1}, -K, -(K+1), -(K+2), ...) = (i_0, i_1, i_2, ..., i_{K-1}, -K, -K-1, -K-2, ...)$. In particular, $i_K = -K$.

Let $\alpha = i_K$ and $\beta = i_0$. Since $(i_0, i_1, i_2, ...)$ is a 0-degression, we have $i_0 > i_1 > i_2 > ....$ Thus, $i_0 > i_1 > i_2 > ... > i_{K-1} > i_K$. In other words, $i_0 \geq i_0 > i_1 > i_2 > ... > i_{K-1} > i_K$. Since $i_0 = \beta$ and $i_K = \alpha$, this rewrites as $\beta \geq i_0 > i_1 > i_2 > ... > i_{K-1} > \alpha$. Thus, the integers i_0, i_1, i_2, ..., i_{K-1} lie in the set $\{\alpha+1, \alpha+2, ..., \beta\}$. Hence, the vectors v_{i_0}, v_{i_1}, ..., $v_{i_{K-1}}$ lie in the vector space $V_{]\alpha,\beta]}$. Thus, the definition of the map $R_{K,]\alpha,\beta]}$ (defined according to Definition 3.14.39 **(d)**) yields

$$\begin{aligned} R_{K,]\alpha,\beta]}\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}}\right) &= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}} \wedge v_\alpha \wedge v_{\alpha-1} \wedge v_{\alpha-2} \wedge ... \\ &= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}} \wedge v_{-K} \wedge v_{-K-1} \wedge v_{-K-2} \wedge ... \\ &\qquad (\text{since } \alpha = i_K = -K) \\ &= v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \end{aligned} \tag{200}$$

(since $(i_0, i_1, i_2, ..., i_{K-1}, -K, -K-1, -K-2, ...) = (i_0, i_1, i_2, ...)$).

For every $p \in \{1, 2, ..., K\}$, define an integer $\widetilde{i}_p$ by $\widetilde{i}_p = \beta + 1 - i_{p-1}$.

Subtracting the chain of inequalities $\beta \geq i_0 > i_1 > i_2 > ... > i_{K-1} > \alpha$ from $\beta + 1$, we obtain $\beta+1-\beta \leq \beta+1-i_0 < \beta+1-i_1 < \beta+1-i_2 < ... < \beta+1-i_{K-1} < \beta+1-\alpha$. Since $\beta + 1 - i_{p-1} = \widetilde{i}_p$ for every $p \in \{1, 2, ..., K\}$, this rewrites as $\beta + 1 - \beta \leq \widetilde{i}_1 < \widetilde{i}_2 < \widetilde{i}_3 < ... < \widetilde{i}_K < \beta + 1 - \alpha$.

This simplifies to $1 \leq \widetilde{i}_1 < \widetilde{i}_2 < \widetilde{i}_3 < ... < \widetilde{i}_K < \beta + 1 - \alpha$. Since $\widetilde{i}_K$ and $\beta + 1 - \alpha$ are integers, we obtain $\widetilde{i}_K \leq \beta - \alpha$ from $\widetilde{i}_K < \beta + 1 - \alpha$. Thus, $1 \leq \widetilde{i}_1 < \widetilde{i}_2 < \widetilde{i}_3 < ... < \widetilde{i}_K \leq \beta - \alpha$.

On the other hand, substituting y for x in (145), we obtain

$$\sum_{k\geq 0} S_k(y) z^k = \exp\left(\sum_{i\geq 1} y_i z^i\right) \qquad \text{in } \mathbb{C}[[z]].$$

Substituting $T_{]\alpha,\beta]}$ for z in this equality, we obtain

$$\sum_{k\geq 0} S_k(y) T^k_{]\alpha,\beta]} = \exp\left(\sum_{i\geq 1} y_i T^i_{]\alpha,\beta]}\right). \tag{201}$$

From Remark 3.14.40, we know that the endomorphisms $\exp\left(\sum\limits_{i=1}^{\infty} y_i T^i_{]\alpha,\beta]}\right)$ and $\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...)$ of $V_{]\alpha,\beta]}$ are well-defined. Denote the endomorphism

$\exp(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...)$ of $V_{]\alpha,\beta]}$ by f. Then,

$$f = \exp\left(\underbrace{y_1 a_1 + y_2 a_2 + y_3 a_3 + ...}_{=\sum_{i=1}^{\infty} y_i a_i}\right) = \exp\left(\sum_{i=1}^{\infty} y_i \underbrace{a_i}_{\substack{=T^i_{]\alpha,\beta]} \\ \text{(since the action of } a_i \text{ on } V_{]\alpha,\beta]} \\ \text{was defined to be } T^i_{]\alpha,\beta]})}}\right) = \exp\left(\sum_{i=1}^{\infty} y_i T^i_{]\alpha,\beta]}\right)$$
$$= \exp\left(\sum_{i\geq 1} y_i T^i_{]\alpha,\beta]}\right) = \sum_{k\geq 0} S_k(y) T^k_{]\alpha,\beta]} \qquad \text{(by (201))}.$$

Note that $S_n(y) \in \mathbb{C}$ for every $n \in \mathbb{Z}$ (since we assumed that $\mathbf{R} = \mathbb{C}$). Also note that $S_n(y) = 0$ for every negative $n \in \mathbb{Z}$ (since $S_n = 0$ for every negative n). Hence, according to Remark 3.14.40 **(g)** (applied to $c_n = S_n(y)$), the sum $\sum_{k\geq 0} S_k(y) T^k_{]\alpha,\beta]}$ is a well-defined endomorphism of $V_{]\alpha,\beta]}$, and the matrix representing this endomorphism with respect to the basis $(v_\beta, v_{\beta-1}, ..., v_{\alpha+1})$ of $V_{]\alpha,\beta]}$ is $(S_{i-j}(y))_{(i,j)\in\{1,2,...,\beta-\alpha\}^2}$. Denote this matrix $(S_{i-j}(y))_{(i,j)\in\{1,2,...,\beta-\alpha\}^2}$ by A. Let $n = \beta - \alpha$, and denote the basis $(v_\beta, v_{\beta-1}, ..., v_{\alpha+1})$ of $V_{]\alpha,\beta]}$ by $(e_1, e_2, ..., e_n)$. Then,

$$e_k = v_{\beta+1-k} \qquad \text{for every } k \in \{1, 2, ..., n\}. \tag{202}$$

As a consequence,

$$e_{\beta+1-k} = v_k \qquad \text{for every } k \in \{\alpha+1, \alpha+2, ..., \beta\} \tag{203}$$

(because for every $k \in \{\alpha+1, \alpha+2, ..., \beta\}$, we can apply (202) to $\beta+1-k$ instead of k, and thus obtain $e_{\beta+1-k} = v_{\beta+1-(\beta+1-k)} = v_k$).

We have shown that the matrix representing the endomorphism $\sum_{k\geq 0} S_k(y) T^k_{]\alpha,\beta]}$ of $V_{]\alpha,\beta]}$ with respect to the basis $(v_\beta, v_{\beta-1}, ..., v_{\alpha+1})$ of $V_{]\alpha,\beta]}$ is $(S_{i-j}(y))_{(i,j)\in\{1,2,...,\beta-\alpha\}^2}$. Since $\sum_{k\geq 0} S_k(y) T^k_{]\alpha,\beta]} = f$, $(v_\beta, v_{\beta-1}, ..., v_{\alpha+1}) = (e_1, e_2, ..., e_n)$, and $(S_{i-j}(y))_{(i,j)\in\{1,2,...,\beta-\alpha\}^2} = A$, this rewrites as follows: The matrix representing the endomorphism f of $V_{]\alpha,\beta]}$ with respect to the basis $(e_1, e_2, ..., e_n)$ of $V_{]\alpha,\beta]}$ is A. In other words, A is the $n \times n$-matrix which represents the map f with respect to the bases $(e_1, e_2, ..., e_n)$ and $(e_1, e_2, ..., e_n)$ of $V_{]\alpha,\beta]}$ and $V_{]\alpha,\beta]}$. Therefore, we can apply Proposition 3.14.7 to n, $V_{]\alpha,\beta]}$, $V_{]\alpha,\beta]}$, $(e_1, e_2, ..., e_n)$, $(e_1, e_2, ..., e_n)$, K, f, A and $\left(\widetilde{i}_1, \widetilde{i}_2, \widetilde{i}_3, ..., \widetilde{i}_K\right)$ instead of m, P, Q, $(e_1, e_2, ..., e_n)$, $(f_1, f_2, ..., f_m)$, ℓ, f, A and $(i_1, i_2, ..., i_\ell)$. As a result, we obtain

$$\left(\wedge^K(f)\right)\left(e_{\widetilde{i}_1} \wedge e_{\widetilde{i}_2} \wedge ... \wedge e_{\widetilde{i}_K}\right) = \sum_{\substack{j_1, j_2, ..., j_K \text{ are } K \text{ integers;} \\ 1\leq j_1<j_2<...<j_K\leq\beta-\alpha}} \det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{j_1,j_2,...,j_K}\right) e_{j_1}\wedge e_{j_2}\wedge ...\wedge e_{j_K}. \tag{204}$$

But every $p \in \{1, 2, ..., K\}$ satisfies $e_{\widetilde{i}_p} = v_{i_{p-1}}$ [155]. Hence, $\left(e_{\widetilde{i}_1}, e_{\widetilde{i}_2}, ..., e_{\widetilde{i}_K}\right) =$

[155] This is because $\widetilde{i}_p = \beta + 1 - i_{p-1}$, so that $\beta + 1 - \widetilde{i}_p = i_{p-1}$ and now

$$e_{\widetilde{i}_p} = v_{\beta+1-\widetilde{i}_p} \qquad \left(\text{by (202), applied to } \widetilde{i}_p \text{ instead of } k\right)$$
$$= v_{i_{p-1}} \qquad \left(\text{since } \beta + 1 - \widetilde{i}_p = i_{p-1}\right).$$

$\left(v_{i_0}, v_{i_1}, ..., v_{i_{K-1}}\right)$. Consequently, $e_{\widetilde{i_1}} \wedge e_{\widetilde{i_2}} \wedge ... \wedge e_{\widetilde{i_K}} = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}}$. Thus, (204) rewrites as

$$\left(\wedge^K (f)\right)\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}}\right) = \sum_{\substack{j_1, j_2, ..., j_K \text{ are } K \text{ integers;} \\ 1 \leq j_1 < j_2 < ... < j_K \leq \beta - \alpha}} \det\left(A^{\widetilde{i_1}, \widetilde{i_2}, ..., \widetilde{i_K}}_{j_1, j_2, ..., j_K}\right) e_{j_1} \wedge e_{j_2} \wedge ... \wedge e_{j_K}. \tag{205}$$

But $\sum\limits_{i \geq 1} y_i T^i_{]\alpha,\beta]} \in \mathfrak{gl}\left(V_{]\alpha,\beta]}\right)$ is a nilpotent linear map (by Remark 3.14.40 **(c)**). Hence, Theorem 3.14.27 (applied to $P = V_{]\alpha,\beta]}$, $a = \sum\limits_{i \geq 1} y_i T^i_{]\alpha,\beta]}$ and $\ell = K$) yields that the exponential $\exp\left(\sum\limits_{i \geq 1} y_i T^i_{]\alpha,\beta]}\right)$ is a well-defined element of $\mathrm{U}\left(V_{]\alpha,\beta]}\right)$ and satisfies $\wedge^K\left(\exp\left(\sum\limits_{i \geq 1} y_i T^i_{]\alpha,\beta]}\right)\right) = \exp\left(\rho_{V_{]\alpha,\beta]},K}\left(\sum\limits_{i \geq 1} y_i T^i_{]\alpha,\beta]}\right)\right)$. Since $\exp\left(\sum\limits_{i \geq 1} y_i T^i_{]\alpha,\beta]}\right) = f$, this rewrites as

$$\wedge^K (f) = \exp\left(\rho_{V_{]\alpha,\beta]},K}\left(\sum_{i \geq 1} y_i T^i_{]\alpha,\beta]}\right)\right). \tag{206}$$

But it is easy to see that

$$\rho_{V_{]\alpha,\beta]},K}\left(\sum_{i \geq 1} y_i T^i_{]\alpha,\beta]}\right) = y_1 a_1 + y_2 a_2 + y_3 a_3 + ... \tag{207}$$

as endomorphisms of $\wedge^K\left(V_{]\alpha,\beta]}\right)$ [156]. Hence, (206) rewrites as

$$\wedge^K (f) = \exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right).$$

[156] *Proof of (207).* By the definition of $\rho_{V_{]\alpha,\beta]},K}$, we know that $\rho_{V_{]\alpha,\beta]},K} : \mathfrak{gl}\left(V_{]\alpha,\beta]}\right) \to \operatorname{End}\left(\wedge^K\left(V_{]\alpha,\beta]}\right)\right)$ denotes the representation of the Lie algebra $\mathfrak{gl}\left(V_{]\alpha,\beta]}\right)$ on the K-th exterior power of the defining representation $V_{]\alpha,\beta]}$ of $\mathfrak{gl}\left(V_{]\alpha,\beta]}\right)$. Hence,

$$\rho_{V_{]\alpha,\beta]},K}\left(\sum_{i \geq 1} y_i T^i_{]\alpha,\beta]}\right) = \left(\sum_{i \geq 1} y_i T^i_{]\alpha,\beta]}\right) \mid_{\wedge^K\left(V_{]\alpha,\beta]}\right)}.$$

Hence, every $\xi_1, \xi_2, ..., \xi_K \in V_{]\alpha,\beta]}$ satisfy

$$\left(\rho_{V_{]\alpha,\beta]},K}\left(\sum_{i\geq 1} y_i T^i_{]\alpha,\beta]}\right)\right)(\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_K)$$
$$= \left(\sum_{i\geq 1} y_i T^i_{]\alpha,\beta]}\right) \rightharpoonup (\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_K)$$
$$= \sum_{i\geq 1} y_i \underbrace{T^i_{]\alpha,\beta]} \rightharpoonup (\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_K)}_{\substack{=\sum\limits_{k=1}^{K} \xi_1\wedge\xi_2\wedge...\wedge\xi_{k-1}\wedge\left(T^i_{]\alpha,\beta]}\rightharpoonup\xi_k\right)\wedge\xi_{k+1}\wedge\xi_{k+2}\wedge...\wedge\xi_K \\ \text{(by the definition of the } \mathfrak{gl}\left(V_{]\alpha,\beta]}\right)\text{-module } \wedge^K\left(V_{]\alpha,\beta]}\right))}}$$
$$= \sum_{i\geq 1} y_i \sum_{k=1}^{K} \xi_1 \wedge \xi_2 \wedge ... \wedge \xi_{k-1} \wedge \underbrace{\left(T^i_{]\alpha,\beta]} \rightharpoonup \xi_k\right)}_{=T^i_{]\alpha,\beta]}\xi_k} \wedge \xi_{k+1} \wedge \xi_{k+2} \wedge ... \wedge \xi_K$$
$$= \sum_{i\geq 1} y_i \sum_{k=1}^{K} \xi_1 \wedge \xi_2 \wedge ... \wedge \xi_{k-1} \wedge T^i_{]\alpha,\beta]}\xi_k \wedge \xi_{k+1} \wedge \xi_{k+2} \wedge ... \wedge \xi_K.$$

On the other hand, every $\xi_1, \xi_2, ..., \xi_K \in V_{]\alpha,\beta]}$ satisfy

$$\underbrace{(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...)}_{=\sum\limits_{i\geq 1} y_i a_i}(\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_K)$$
$$= \sum_{i\geq 1} y_i \underbrace{a_i (\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_K)}_{\substack{=\sum\limits_{k=1}^{K} \xi_1\wedge\xi_2\wedge...\wedge\xi_{k-1}\wedge a_i\xi_k\wedge\xi_{k+1}\wedge\xi_{k+2}\wedge...\wedge\xi_K \\ \text{(by the definition of the } \mathcal{A}_+\text{-module } \wedge^K\left(V_{]\alpha,\beta]}\right))}}$$
$$= \sum_{i\geq 1} y_i \sum_{k=1}^{K} \xi_1 \wedge \xi_2 \wedge ... \wedge \xi_{k-1} \wedge \underbrace{a_i\xi_k}_{\substack{=T^i_{]\alpha,\beta]}\xi_k \\ \text{(since the element } a_i \text{ of } \mathcal{A}_+ \text{ acts} \\ \text{on } V_{]\alpha,\beta]} \text{ by } T^i_{]\alpha,\beta]})}} \wedge \xi_{k+1} \wedge \xi_{k+2} \wedge ... \wedge \xi_K$$
$$= \sum_{i\geq 1} y_i \sum_{k=1}^{K} \xi_1 \wedge \xi_2 \wedge ... \wedge \xi_{k-1} \wedge T^i_{]\alpha,\beta]}\xi_k \wedge \xi_{k+1} \wedge \xi_{k+2} \wedge ... \wedge \xi_K$$
$$= \left(\rho_{V_{]\alpha,\beta]},K}\left(\sum_{i\geq 1} y_i T^i_{]\alpha,\beta]}\right)\right)(\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_K).$$

In other words, the two endomorphisms $y_1 a_1 + y_2 a_2 + y_3 a_3 + ...$ and $\rho_{V_{]\alpha,\beta]},K}\left(\sum\limits_{i\geq 1} y_i T^i_{]\alpha,\beta]}\right)$ of $\wedge^K\left(V_{]\alpha,\beta]}\right)$ are equal to each other on the set $\left\{\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_K \mid \xi_1, \xi_2, ..., \xi_K \in V_{]\alpha,\beta]}\right\}$. Since the set $\left\{\xi_1 \wedge \xi_2 \wedge ... \wedge \xi_K \mid \xi_1, \xi_2, ..., \xi_K \in V_{]\alpha,\beta]}\right\}$ is a spanning set of the vector space $\wedge^K\left(V_{]\alpha,\beta]}\right)$, Hence,

$$\begin{aligned}
&\underbrace{\left(\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)\right)}_{=\wedge^K(f)}\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{K-1}}\right)\\
&=\left(\wedge^K(f)\right)\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{K-1}}\right)\\
&=\sum_{\substack{j_1,\ j_2,\ ...,\ j_K \text{ are } K \text{ integers;}\\ 1\leq j_1<j_2<...<j_K\leq\beta-\alpha}}\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{j_1,j_2,...,j_K}\right)e_{j_1}\wedge e_{j_2}\wedge ...\wedge e_{j_K}\qquad\text{(by (205))}\\
&=\sum_{\substack{j_0,\ j_1,\ ...,\ j_{K-1} \text{ are } K \text{ integers;}\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}}\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right)\\
&\qquad\qquad\underbrace{e_{\beta+1-j_0}\wedge e_{\beta+1-j_1}\wedge ...\wedge e_{\beta+1-j_{K-1}}}_{\substack{=v_{j_0}\wedge v_{j_1}\wedge...\wedge v_{j_{K-1}}\\ \text{(due to (203))}}}\\
&\qquad\text{(here, we substituted } \left(\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}\right) \text{ for } \left(j_1,j_2,...,j_K\right))\\
&=\sum_{\substack{j_0,\ j_1,\ ...,\ j_{K-1} \text{ are } K \text{ integers;}\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}}\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right)v_{j_0}\wedge v_{j_1}\wedge ...\wedge v_{j_{K-1}}.
\end{aligned}\tag{208}$$

But every K-tuple $(j_0,j_1,...,j_{K-1})$ of integers such that $1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha$ satisfies

$$\sum_{\substack{j_K,\ j_{K+1},\ j_{K+2},\ ... \text{ are integers;}\\ j_k=-k \text{ for every } k\geq K}}\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right)=\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right)$$

(since the sum $\sum\limits_{\substack{j_K,\ j_{K+1},\ j_{K+2},\ ... \text{ are integers;}\\ j_k=-k \text{ for every } k\geq K}}\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right)$ has only one ad-

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this entails that the two endomorphisms $y_1a_1+y_2a_2+y_3a_3+...$ and $\rho_{V_{]\alpha,\beta]},K}\left(\sum\limits_{i\geq 1}y_iT^i_{]\alpha,\beta]}\right)$ of $\wedge^K\left(V_{]\alpha,\beta]}\right)$ are identical. This proves (207).

dend). Thus, (208) becomes

$$\left(\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge...\wedge v_{i_{K-1}}\right)$$
$$=\sum_{\substack{j_0,\ j_1,\ ...,\ j_{K-1}\text{ are }K\text{ integers;}\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}}\underbrace{\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right)}_{=\sum\limits_{\substack{j_K,\ j_{K+1},\ j_{K+2},\ ...\text{ are integers;}\\ j_k=-k\text{ for every }k\geq K}}\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right)}v_{j_0}\wedge v_{j_1}\wedge...\wedge v_{j_{K-1}}$$
$$=\underbrace{\sum_{\substack{j_0,\ j_1,\ ...,\ j_{K-1}\text{ are }K\text{ integers;}\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}}\ \sum_{\substack{j_K,\ j_{K+1},\ j_{K+2},\ ...\text{ are integers;}\\ j_k=-k\text{ for every }k\geq K}}}_{=\sum\limits_{\substack{(j_0,j_1,j_2,...)\in\mathbb{Z}^{\mathbb{N}};\\ j_k=-k\text{ for every }k\geq K;\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}}}\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right)v_{j_0}\wedge v_{j_1}\wedge...\wedge v_{j_{K-1}}$$
$$=\sum_{\substack{(j_0,j_1,j_2,...)\in\mathbb{Z}^{\mathbb{N}};\\ j_k=-k\text{ for every }k\geq K;\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}}\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right)v_{j_0}\wedge v_{j_1}\wedge...\wedge v_{j_{K-1}}.$$

Applying the linear map $R_{K,]\alpha,\beta]}$ to this equality, we obtain

$$\begin{aligned}
& R_{K,]\alpha,\beta]}\left(\left(\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{K-1}}\right)\right) \\
&= \sum_{\substack{(j_0,j_1,j_2,...)\in\mathbb{Z}^{\mathbb{N}};\\ j_k=-k \text{ for every } k\geq K;\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}} \det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right) \\
&\qquad \underbrace{R_{K,]\alpha,\beta]}\left(v_{j_0}\wedge v_{j_1}\wedge ...\wedge v_{j_{K-1}}\right)}_{\substack{=v_{j_0}\wedge v_{j_1}\wedge...\wedge v_{j_{K-1}}\wedge v_\alpha\wedge v_{\alpha-1}\wedge v_{\alpha-2}\wedge...\\ \text{(by the definition of } R_{K,]\alpha,\beta]})}} \\
&= \sum_{\substack{(j_0,j_1,j_2,...)\in\mathbb{Z}^{\mathbb{N}};\\ j_k=-k \text{ for every } k\geq K;\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}} \det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right) \\
&\qquad v_{j_0}\wedge v_{j_1}\wedge ...\wedge v_{j_{K-1}}\wedge v_\alpha\wedge v_{\alpha-1}\wedge v_{\alpha-2}\wedge ... \\
&= \sum_{\substack{(j_0,j_1,j_2,...)\in\mathbb{Z}^{\mathbb{N}};\\ j_k=-k \text{ for every } k\geq K;\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}} \det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right) \\
&\qquad \underbrace{v_{j_0}\wedge v_{j_1}\wedge ...\wedge v_{j_{K-1}}\wedge v_{-K}\wedge v_{-K-1}\wedge v_{-K-2}\wedge ...}_{\substack{=v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge...\\ \text{(since every } k\geq K \text{ satisfies } -k=j_k\text{, and therefore}\\ (j_0,j_1,...,j_{K-1},-K,-K-1,-K-2,...)=(j_0,j_1,...,j_{K-1},j_K,j_{K+1},j_{K+2},...)=(j_0,j_1,j_2,...))}} \\
&\qquad \left(\text{since } \alpha=i_K=-K\right) \\
&= \sum_{\substack{(j_0,j_1,j_2,...)\in\mathbb{Z}^{\mathbb{N}};\\ j_k=-k \text{ for every } k\geq K;\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}} \det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right) v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge ....
\end{aligned} \tag{209}$$

Let us now notice that when $(j_0,j_1,j_2,...)\in\mathbb{Z}^{\mathbb{N}}$ is a sequence of integers satisfying ($j_k=-k$ for every $k\geq K$), then a very straightforward argument shows that $1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha$ holds if and only if $(j_0,j_1,j_2,...)$ is a 0-degression satisfying $j_0\leq\beta$. Hence, we can replace the sum sign $\sum\limits_{\substack{(j_0,j_1,j_2,...)\in\mathbb{Z}^{\mathbb{N}};\\ j_k=-k \text{ for every } k\geq K;\\ 1\leq\beta+1-j_0<\beta+1-j_1<...<\beta+1-j_{K-1}\leq\beta-\alpha}}$ by $\sum\limits_{\substack{(j_0,j_1,j_2,...) \text{ is a 0-degression};\\ j_k=-k \text{ for every } k\geq K;\\ j_0\leq\beta}}$ in (209). Hence, (209) becomes

$$\begin{aligned}
& R_{K,]\alpha,\beta]}\left(\left(\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{K-1}}\right)\right) \\
&= \sum_{\substack{(j_0,j_1,j_2,...) \text{ is a 0-degression};\\ j_k=-k \text{ for every } k\geq K;\\ j_0\leq\beta}} \det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right) v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge ....
\end{aligned} \tag{210}$$

But it is easily revealed that

$$\det\left(A^{\widetilde{i}_1,\widetilde{i}_2,...,\widetilde{i}_K}_{\beta+1-j_0,\beta+1-j_1,...,\beta+1-j_{K-1}}\right) = S_{(i_k+k)_{k\geq 0}\diagup (j_k+k)_{k\geq 0}}(y) \tag{211}$$

for any 0-degression $(j_0, j_1, j_2, ...)$ satisfying $(j_k = -k$ for every $k \geq K)$ and $j_0 \leq \beta$ [157]. Therefore, (210) simplifies to

$$R_{K,]\alpha,\beta]}\left(\left(\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right)\right)\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}}\right)\right)$$
$$= \sum_{\substack{(j_0,j_1,j_2,...) \text{ is a 0-degression;}\\ j_k=-k \text{ for every } k\geq K;\\ j_0\leq\beta}} S_{(i_k+k)_{k\geq 0}\diagup(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge .... \tag{212}$$

But Proposition 3.14.41 (applied to K instead of ℓ) yields that $R_{K,]\alpha,\beta]} : \wedge^K \left(V_{]\alpha,\beta]}\right) \to \mathcal{F}^{(\alpha+K)}$ is an $\mathcal{A}_+$-module homomorphism. Since $\underbrace{\alpha}_{=i_K=-K} + K = -K + K = 0$, this rewrites as follows: $R_{K,]\alpha,\beta]} : \wedge^K \left(V_{]\alpha,\beta]}\right) \to \mathcal{F}^{(0)}$ is an $\mathcal{A}_+$-module homomorphism.

Now, $\mathcal{A}_+$ is a graded Lie subalgebra of $\mathcal{A}$, and it is easy to define a grading on the $\mathcal{A}_+$-module $\wedge^K \left(V_{]\alpha,\beta]}\right)$ such that $\wedge^K \left(V_{]\alpha,\beta]}\right)$ is concentrated in nonpositive degrees.[158] Thus, applying Proposition 3.14.31 **(b)** to $\mathbb{C}$, $\wedge^K \left(V_{]\alpha,\beta]}\right)$, $\mathcal{F}^{(0)}$ and $R_{K,]\alpha,\beta]}$ instead of $\mathbf{R}$, M, N and η, we obtain

$$\left(\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right)\right) \circ R_{K,]\alpha,\beta]} = R_{K,]\alpha,\beta]} \circ \left(\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right)\right)$$

as maps from $\wedge^K \left(V_{]\alpha,\beta]}\right)$ to $\mathcal{F}^{(0)}$. Hence,

$$\left(\left(\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right)\right) \circ R_{K,]\alpha,\beta]}\right)\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}}\right)$$
$$= \left(R_{K,]\alpha,\beta]} \circ \left(\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right)\right)\right)\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}}\right)$$
$$= R_{K,]\alpha,\beta]}\left(\left(\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right)\right)\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}}\right)\right)$$
$$= \sum_{\substack{(j_0,j_1,j_2,...) \text{ is a 0-degression;}\\ j_k=-k \text{ for every } k\geq K;\\ j_0\leq\beta}} S_{(i_k+k)_{k\geq 0}\diagup(j_k+k)_{k\geq 0}}(y) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \qquad \text{(by (212))}.$$

Compared with

$$\left(\left(\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right)\right) \circ R_{K,]\alpha,\beta]}\right)\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}}\right)$$
$$= \left(\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right)\right)\underbrace{\left(R_{K,]\alpha,\beta]}\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{K-1}}\right)\right)}_{\substack{=v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\\ \text{(by (200))}}}$$
$$= \left(\exp\left(y_1 a_1 + y_2 a_2 + y_3 a_3 + ...\right)\right)\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right),$$

[157] The proof of (211) is completely straightforward and left to the reader. The ingredients of the proof are the equality $A = \left(S_{i-j}(y)\right)_{(i,j)\in\{1,2,...,\beta-\alpha\}^2}$ (which we used to define A), the definition of $\widetilde{i}_v$ (namely, $\widetilde{i}_v = \beta + 1 - i_{v-1}$ for every $v \in \{1, 2, ..., K\}$), and the definition of the skew Schur function $S_{\lambda\diagup\mu}(x)$ as a determinant of a (finite!) matrix.

[158] Indeed, let us define a grading on the vector space $V_{]\alpha,\beta]}$ by setting the degree of v_i to be $\alpha + 1 - i$ for every $i \in \{\alpha+1, \alpha+2, ..., \beta\}$. Then, the vector space $V_{]\alpha,\beta]}$ is concentrated in nonpositive degrees, so that its K-th exterior power $\wedge^K \left(V_{]\alpha,\beta]}\right)$ is also concentrated in nonpositive degrees. On the other hand, $V_{]\alpha,\beta]}$ is a graded $\mathcal{A}_+$-module (this is very easy to check), so that its K-th exterior power $\wedge^K \left(V_{]\alpha,\beta]}\right)$ is also a graded $\mathcal{A}_+$-module.

this becomes

$$\begin{aligned}
&\left(\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\\
&=\sum_{\substack{(j_0,j_1,j_2,...)\text{ is a 0-degression;}\\ j_k=-k\text{ for every }k\geq K;\\ j_0\leq\beta}} S_{(i_k+k)_{k\geq 0}\diagup (j_k+k)_{k\geq 0}}\left(y\right)\cdot v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge ...\\
&=\sum_{\substack{(j_0,j_1,j_2,...)\text{ is a 0-degression;}\\ (j_k+k)_{k\geq 0}\subseteq (i_k+k)_{k\geq 0}\\ j_k=-k\text{ for every }k\geq K;\\ j_0\leq\beta}} S_{(i_k+k)_{k\geq 0}\diagup (j_k+k)_{k\geq 0}}\left(y\right)\cdot v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge ... \qquad (213)
\end{aligned}$$

(here, we deprived the sum of all addends for which $(j_k+k)_{k\geq 0}\not\subseteq (i_k+k)_{k\geq 0}$, because (188) shows that all such addends are 0).

But for any 0-degression $(j_0, j_1, j_2, ...)$ satisfying $(j_k+k)_{k\geq 0}\subseteq (i_k+k)_{k\geq 0}$, we automatically have ($j_k=-k$ for every $k\geq K$) and $j_0\leq\beta$ (this is very easy to see). Hence, we can replace the summation sign $\sum\limits_{\substack{(j_0,j_1,j_2,...)\text{ is a 0-degression;}\\ (j_k+k)_{k\geq 0}\subseteq (i_k+k)_{k\geq 0}\\ j_k=-k\text{ for every }k\geq K;\\ j_0\leq\beta}}$ on the right hand side of (213) by a $\sum\limits_{\substack{(j_0,j_1,j_2,...)\text{ is a 0-degression;}\\ (j_k+k)_{k\geq 0}\subseteq (i_k+k)_{k\geq 0}}}$ sign. Thus, (213) simplifies to

$$\begin{aligned}
&\left(\exp\left(y_1a_1+y_2a_2+y_3a_3+...\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\\
&=\sum_{\substack{(j_0,j_1,j_2,...)\text{ is a 0-degression;}\\ (j_k+k)_{k\geq 0}\subseteq (i_k+k)_{k\geq 0}}} S_{(i_k+k)_{k\geq 0}\diagup (j_k+k)_{k\geq 0}}\left(y\right)\cdot v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge ....
\end{aligned}$$

This proves Theorem 3.14.37.

3.15. Applications to integrable systems

Let us show how these things can be applied to partial differential equations.

Convention 3.15.1. If v is a function in several variables x_1, x_2, ..., x_k, then, for every $i\in\{1,2,...,k\}$, the derivative of v by the variable x_i will be denoted by $\partial_{x_i}v$ and by v_{x_i}. In other words, $\partial_{x_i}v=v_{x_i}=\dfrac{\partial}{\partial x_i}v$. (For example, if v is a function in two variables x and t, then v_t will mean the derivative of v by t.)

The PDE (partial differential equation) we will be concerned with is the **Korteweg-de Vries equation** (abbreviated as **KdV equation**): This is the equation $u_t=\dfrac{3}{2}uu_x+\dfrac{1}{4}u_{xxx}$ for a function $u(t,x)$. [159]

[159] There seems to be no consistent definition of the KdV equation across literature. We defined the KdV equation as $u_t=\dfrac{3}{2}uu_x+\dfrac{1}{4}u_{xxx}$ because this is the form most suited to our approach. Some other authors, instead, define the KdV equation as $v_t=v_{xxx}+6vv_x$ for a function $v(t,x)$. Others define it as $w_t+ww_x+w_{xxx}=0$ for a function $w(t,x)$. Yet others define it as $q_t+q_{xxx}+6qq_x=0$

We will discuss several interesting solutions of this equation. Here is the most basic family of solutions:

$$u(t) = \frac{2a^2}{\cosh^2\left(a\left(x+a^2t\right)\right)} \qquad \text{(for } a \text{ being arbitrary but fixed)}.$$

These are so-called "traveling wave solutions". It is a peculiar kind of wave: it has only one bump; it is therefore called a *soliton* (or *solitary wave*). Such waves never occur in linear systems. Note that when we speak of "wave", we are imagining a time-dependent 2-dimensional graph with the x-axis showing t, the y-axis showing $u(t)$, and the time parameter being x. So when we speak of "traveling wave", we mean that it is a wave for any fixed time x and "travels" when x moves.

The first to study this kind of waves was J. S. Russell in 1834, describing the motion of water in a shallow canal (tsunami waves are similar). The first models for these waves were found by Korteweg-de Vries in 1895.

The term $\frac{1}{4}u_{xxx}$ in the Korteweg-de Vries equation $u_t = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}$ is called the *dispersion term.*

Exercise: Solve the equation $u_t = \frac{3}{2}uu_x$. (Note that the waves solving this equation develop shocks, in contrast to those solving the Korteweg-de Vries equation.)

The Korteweg-de Vries equation is famous for having lots of explicit solutions (unexpectedly for a nonlinear partial differential equation). We will construct some of them using infinite-dimensional Lie algebras. (There are many other ways to construct solutions. In some sense, every field of mathematics is related to some of its solutions.)

We will also study the **Kadomtsev-Petviashvili equation** (abbreviated as **KP equation**)

$$u_{yy} = \left(u_t - \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}\right)_x$$

(or, after some rescaling, $\frac{3}{4}\partial_y^2 u = \partial_x\left(\partial_t u - \frac{3}{2}u\partial_x u - \frac{1}{4}\partial_x^3 u\right)$) on a function $u(t,x,y)$. We will obtain functions which solve this equation (among others).

We are going to use the *infinite Grassmannian* for this. First, recall what the *finite Grassmannian* is:

3.15.1. The finite Grassmannian

for a function $q(t,x)$. These equations are not literally equivalent, but can be transformed into each other by very simple substitutions. In fact, for a function $u(t,x)$, we have the following equivalence of assertions:

$$\begin{aligned}
&\left(\text{the function } u(t,x) \text{ satisfies the equation } u_t = \frac{3}{2}uu_x + \frac{1}{4}u_{xxx}\right)\\
\Longleftrightarrow\ &\left(\text{the function } v(t,x) := u(4t,x) \text{ satisfies the equation } v_t = v_{xxx} + 6vv_x\right)\\
\Longleftrightarrow\ &\left(\text{the function } w(t,x) := 6u(-4t,x) \text{ satisfies the equation } w_t + ww_x + w_{xxx} = 0\right)\\
\Longleftrightarrow\ &\left(\text{the function } q(t,x) := u(-4t,x) \text{ satisfies the equation } q_t + q_{xxx} + 6qq_x = 0\right).
\end{aligned}$$

Definition 3.15.2. Let k and n be integers satisfying $0 \leq k \leq n$. Let V be the $\mathbb{C}$-vector space $\mathbb{C}^n$. Let $(v_1, v_2, ..., v_n)$ be the standard basis of $\mathbb{C}^n$. Recall that $\wedge^k V$ is a representation of $\operatorname{GL}(V)$ with a highest-weight vector $v_1 \wedge v_2 \wedge ... \wedge v_k$. Denote by Ω the orbit of $v_1 \wedge v_2 \wedge ... \wedge v_k$ under $\operatorname{GL}(V)$.

Proposition 3.15.3. Let k and n be integers satisfying $0 \leq k \leq n$. We have $\Omega = \left\{x \in \wedge^k V \text{ nonzero } \mid x = x_1 \wedge x_2 \wedge ... \wedge x_k \text{ for some } x_i \in V\right\}$. Also, $x_1 \wedge x_2 \wedge ... \wedge x_k \neq 0$ if and only if $x_1, x_2, ..., x_k$ are linearly independent.

Proof. Very easy.

Definition 3.15.4. Let V be a $\mathbb{C}$-vector space. Let k be a nonnegative integer. The *k-Grassmannian of V* is defined to be the set of all k-dimensional vector subspaces of V. This set is denoted by $\operatorname{Gr}(k, V)$.

When V is a finite-dimensional $\mathbb{C}$-vector space, there is a way to define the structure of a projective variety on the Grassmannian $\operatorname{Gr}(k, V)$. While we won't ever need the existence of this structure, we will need the so-called Plücker embedding which is the main ingredient in defining this structure:[160]

Definition 3.15.5. Let k and n be integers satisfying $0 \leq k \leq n$. Let V be the $\mathbb{C}$-vector space $\mathbb{C}^n$. The *Plücker embedding* (corresponding to n and k) is defined as the map

$$\operatorname{Pl} : \operatorname{Gr}(k, V) \to \mathbb{P}\left(\wedge^k V\right),$$
$$\left(\begin{array}{c} k\text{-dimensional subspace of } V \\ \text{with basis } x_1, x_2, ..., x_k \end{array}\right) \mapsto \left(\begin{array}{c} \text{projection of} \\ x_1 \wedge x_2 \wedge ... \wedge x_k \in \wedge^k V \diagdown \{0\} \\ \text{on } \mathbb{P}\left(\wedge^k V\right) \end{array}\right).$$

It is easy to see that this is well-defined (i. e., that the projection of $x_1 \wedge x_2 \wedge ... \wedge x_k \in \wedge^k V \diagdown \{0\}$ on $\mathbb{P}\left(\wedge^k V\right)$ does not depend on the choice of basis $x_1, x_2, ..., x_k$). The image of this map is $\operatorname{Im}\operatorname{Pl} = \Omega \diagup (\text{scalars})$.

Proposition 3.15.6. This map Pl is injective.

Proof of Proposition 3.15.6. Proving Proposition 3.15.6 boils down to showing that if λ is a complex number and v_1, v_2, ..., v_k, w_1, w_2, ..., w_k are any vectors in a vector space U satisfying $v_1 \wedge v_2 \wedge ... \wedge v_k = \lambda \cdot w_1 \wedge w_2 \wedge ... \wedge w_k \neq 0$, then the vector subspace of U spanned by the vectors v_1, v_2, ..., v_k is identical with the vector subspace of U spanned by the vectors w_1, w_2, ..., w_k. This is a well-known fact. The details are left to the reader.

Thus, $\operatorname{Gr}(k, V) \cong \Omega \diagup (\text{scalars})$. (For algebraic geometers: Ω is the total space of the determinant bundle on $\operatorname{Gr}(k, V)$ (but only the nonzero elements).)

[160] In the following definition (and further below), we use the notation $\mathbb{P}(W)$ for the projective space of a $\mathbb{C}$-vector space W. This projective space is defined to be the quotient set $(W \setminus \{0\}) / \sim$, where $\sim$ is the proportionality relation (i.e., two vectors w_1 and w_2 in $W \setminus \{0\}$ satisfy $w_1 \sim w_2$ if and only if they are linearly dependent).

We are now going to describe the image $\operatorname{Im}\operatorname{Pl}$ by algebraic equations. These equations go under the name *Plücker relations.*

First, we define (in analogy to Definition 3.10.5) "wedging" and "contraction" operators on the exterior algebra of V:

Definition 3.15.7. Let $n \in \mathbb{N}$. Let $k \in \mathbb{Z}$. Let V be the vector space $\mathbb{C}^n$. Let $(v_1, v_2, ..., v_n)$ be the standard basis of V. Let $i \in \{1, 2, ..., n\}$.

(a) We define the so-called *i-th wedging operator* $\widehat{v_i} : \wedge^k V \to \wedge^{k+1} V$ by

$$\widehat{v_i} \cdot \psi = v_i \wedge \psi \qquad \text{for all } \psi \in \wedge^k V.$$

(b) We define the so-called *i-th contraction operator* $\overset{\vee}{v_i} : \wedge^k V \to \wedge^{k-1} V$ as follows:

For every k-tuple $(i_1, i_2, ..., i_k)$ of integers satisfying $1 \leq i_1 < i_2 < ... < i_k \leq n$, we let $\overset{\vee}{v_i}(v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_k})$ be

$$\begin{cases} 0, & \text{if } i \notin \{i_1, i_2, ..., i_k\}; \\ (-1)^{j-1} v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_{j-1}} \wedge v_{i_{j+1}} \wedge v_{i_{j+2}} \wedge ... \wedge v_{i_k}, & \text{if } i \in \{i_1, i_2, ..., i_k\} \end{cases},$$

where, in the case $i \in \{i_1, i_2, ..., i_k\}$, we denote by j the integer ℓ satisfying $i_\ell = i$. Thus, the map $\overset{\vee}{v_i}$ is defined on a basis of the vector space $\wedge^k V$; we extend this to a map $\wedge^k V \to \wedge^{k-1} V$ by linearity.

Note that, for every negative $\ell \in \mathbb{Z}$, we understand $\wedge^\ell V$ to mean the zero space.

Now we can formulate the Plücker relations as follows:

Theorem 3.15.8. Let $n \in \mathbb{N}$. Let $k \in \mathbb{Z}$. We consider the vector space $V = \mathbb{C}^n$ with its standard basis $(v_1, v_2, ..., v_n)$. Let $S = \sum\limits_{i=1}^{n} \widehat{v_i} \otimes \overset{\vee}{v_i} : \wedge^k V \otimes \wedge^k V \to \wedge^{k+1} V \otimes \wedge^{k-1} V$.

(a) This map S does not depend on the choice of the basis and is $\operatorname{GL}(V)$-invariant[161]. In other words, for **any** basis $(w_1, w_2, ..., w_n)$ of V, we have $S = \sum\limits_{i=1}^{n} \widehat{w_i} \otimes \overset{\vee}{w_i}$ (where the maps $\widehat{w_i}$ and $\overset{\vee}{w_i}$ are defined just as $\widehat{v_i}$ and $\overset{\vee}{v_i}$, but with respect to the basis $(w_1, w_2, ..., w_n)$).

(b) Let $k \in \{1, 2, ..., n\}$. A nonzero element $\tau \in \wedge^k V$ belongs to Ω if and only if $S(\tau \otimes \tau) = 0$.

(c) The map S is $\operatorname{M}(V)$-invariant. (Here, $\operatorname{M}(V)$ denotes the multiplicative monoid of all endomorphisms of V.)

Part **(b)** of this theorem is what is actually called the Plücker relations, although it is not how these relations are usually formulated in literature. For a more classical formulation, see Theorem 3.15.9. Of course, Theorem 3.15.8 **(b)** not only shows when an element of $\wedge^k V$ belongs to Ω, but also shows when an element of $\mathbb{P}(\wedge^k V)$ lies in $\operatorname{Im}\operatorname{Pl}$ (because an element of $\mathbb{P}(\wedge^k V)$ is an equivalence class of elements of $\wedge^k V \setminus \{0\}$, and lies in $\operatorname{Im}\operatorname{Pl}$ if and only if its representatives lie in Ω).

Proof of Theorem 3.15.8. Before we start proving the theorem, let us introduce some notations.

[161] The word "$\operatorname{GL}(V)$-invariant" here means "invariant under the action of $\operatorname{GL}(V)$ on the space of all linear operators $\wedge^k V \otimes \wedge^k V \to \wedge^{k+1} V \otimes \wedge^{k-1} V$". So, for an operator from $\wedge^k V \otimes \wedge^k V$ to $\wedge^{k+1} V \otimes \wedge^{k-1} V$ to be $\operatorname{GL}(V)$-invariant means the same as for it to be $\operatorname{GL}(V)$-equivariant.

First of all, for every basis $(e_1, e_2, ..., e_n)$ of V, let $(e_1^*, e_2^*, ..., e_n^*)$ denote its dual basis (this is a basis of V^*).

Next, for any element $v \in V$ we define the so called *v-wedging operator* $\widehat{v} : \wedge^k V \to \wedge^{k+1} V$ by

$$\widehat{v} \cdot \psi = v \wedge \psi \qquad \text{for all } \psi \in \wedge^k V.$$

Of course, this definition does not conflict with Definition 3.15.7 **(a)**. (In fact, for every $i \in \{1, 2, ..., n\}$, the v_i-wedging operator that we just defined is exactly identical with the i-th wedging operator defined in Definition 3.15.7 **(a)**, and hence there is no harm from denoting both of them by $\widehat{v_i}$.)

Further, for any $f \in V^*$, we define the so called *f-contraction operator* $\overset{\vee}{f} : \wedge^k V \to \wedge^{k-1} V$ by

$$\overset{\vee}{f} \cdot (u_1 \wedge u_2 \wedge ... \wedge u_k) = \sum_{i=1}^{k} (-1)^{i-1} f(u_i) \cdot u_1 \wedge u_2 \wedge ... \wedge u_{i-1} \wedge u_{i+1} \wedge u_{i+2} \wedge ... \wedge u_k$$
$$\text{for all } u_1, u_2, ..., u_k \in V.$$

[162] These contraction operators are connected to the contraction operators defined in Definition 3.15.7 **(b)**: Namely, $\overset{\vee}{v_i} = \overset{\vee}{v_i^*}$ for every $i \in \{1, 2, ..., n\}$. More generally, $\overset{\vee}{e_i} = \overset{\vee}{e_i^*}$ for every basis $(e_1, e_2, ..., e_n)$ of V (where the maps $\widehat{e_i}$ and $\overset{\vee}{e_i}$ are defined just as $\widehat{v_i}$ and $\overset{\vee}{v_i}$, but with respect to the basis $(e_1, e_2, ..., e_n)$).

The f-contraction operators, however, have a major advantage against the contraction operators defined in Definition 3.15.7 **(b)**: In fact, the former are canonical (i. e., they can be defined in the same way for every vector space instead of V, and then they are canonical maps that don't depend on any choice of basis), while the latter have the basis $(v_1, v_2, ..., v_n)$ "hard-coded" into them.

Note that many sources denote the f-contraction operator by i_f and call it the *interior product operator* with f.

It is easy to see that

$$\overset{\vee}{f}\widehat{v} + \widehat{v}\overset{\vee}{f} = f(v) \cdot \mathrm{id} \qquad \text{for all } f \in V^* \text{ and } v \in V \tag{214}$$

(where, in the case $k = 0$, we interpret $\widehat{v}\overset{\vee}{f}$ as 0).

(a) We will give a basis-free definition of S. This will prove the basis independence.

There is a unique vector space isomorphism $\Phi : V^* \otimes V \to \operatorname{End} V$ which satisfies

$$\Phi(f \otimes v) = (\text{the map } V \to V \text{ sending each } w \text{ to } f(w)v) \qquad \text{for all } f \in V^* \text{ and } v \in V.$$

This Φ and its inverse isomorphism Φ^{-1} are actually basis-independent.

Now, define a map

$$T : V^* \otimes V \otimes \wedge^k V \otimes \wedge^k V \to \wedge^{k+1} V \otimes \wedge^{k-1} V$$

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[162] In order to prove that this is well-defined, we need to check that the term $\sum_{i=1}^{k} (-1)^{i-1} f(u_i) \cdot u_1 \wedge u_2 \wedge ... \wedge u_{i-1} \wedge u_{i+1} \wedge u_{i+2} \wedge ... \wedge u_k$ depends multilinearly and antisymmetrically on $u_1, u_2, ..., u_k$. This is easy and left to the reader.

by

$$T\left(f\otimes v\otimes\psi\otimes\phi\right)=\left(\widehat{v}\cdot\psi\right)\otimes\left(\overset{\vee}{f}\cdot\phi\right)\qquad\text{for all } f\in V^{*},\ v\in V,\ \psi\in\wedge^{k}V\text{ and }\phi\in\wedge^{k}V.$$

This map T is clearly well-defined (because $\widehat{v}\cdot\psi$ depends bilinearly on v and ψ, and because $\overset{\vee}{f}\cdot\phi$ depends bilinearly on f and ϕ).

It is now easy to show that S is the map $\wedge^{k}V\otimes\wedge^{k}V\to\wedge^{k+1}V\otimes\wedge^{k-1}V$ which sends $\psi\otimes\phi$ to $T\left(\Phi^{-1}\left(\mathrm{id}_V\right)\otimes\psi\otimes\phi\right)$ for all $\psi\in\wedge^{k}V$ and $\phi\in\wedge^{k}V$. [163] This shows immediately that S is basis-independent (since T and Φ^{-1} are basis-independent).

Since S is basis-independent, it is clear that S is $\mathrm{GL}\left(V\right)$-invariant (because the action of $\mathrm{GL}\left(V\right)$ transforms S into the same operator S but constructed for a different basis; but since S is basis-independent, this other S must be the S that we started with). This proves Theorem 3.15.8 **(a)**.

(b) Let $\tau\in\Omega$ be nonzero.

1) First let us show that if $\tau\in\Omega$, then $S\left(\tau\otimes\tau\right)=0$.

In order to show this, it is enough to prove that $S\left(\tau\otimes\tau\right)=0$ holds in the case $\tau=v_1\wedge v_2\wedge...\wedge v_k$ (since S is $\mathrm{GL}\left(V\right)$-invariant, and Ω is the $\mathrm{GL}\left(V\right)$-orbit of $v_1\wedge v_2\wedge...\wedge v_k$).

But this is obvious, because for every $i\in\{1,2,...,n\}$, either $\widehat{v_i}$ or $\overset{\vee}{v_i}$ annihilates $v_1\wedge v_2\wedge...\wedge v_k$.

2) Let us now (conversely) prove that if $S\left(\tau\otimes\tau\right)=0$, then $\tau\in\Omega$.

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[163] *Proof.* Consider the map $\wedge^{k}V\otimes\wedge^{k}V\to\wedge^{k+1}V\otimes\wedge^{k-1}V$ which sends $\psi\otimes\phi$ to $T\left(\Phi^{-1}\left(\mathrm{id}_V\right)\otimes\psi\otimes\phi\right)$ for all $\psi\in\wedge^{k}V$ and $\phi\in\wedge^{k}V$. This map is clearly well-defined. Now, since $\Phi^{-1}\left(\mathrm{id}_V\right)=\sum\limits_{i=1}^{n}v_i^{*}\otimes v_i$ (because every $w\in V$ satisfies

$$\left(\Phi\left(\sum_{i=1}^{n}v_i^{*}\otimes v_i\right)\right)(w)=\sum_{i=1}^{n}\underbrace{\left(\Phi\left(v_i^{*}\otimes v_i\right)\right)(w)}_{\substack{=v_i^{*}(w)v_i\\ \text{(by the definition of }\Phi)}}=\sum_{i=1}^{n}v_i^{*}(w)\,v_i=w$$

$$\text{(since }\left(v_1^{*},v_2^{*},...,v_n^{*}\right)\text{ is the dual basis of }\left(v_1,v_2,...,v_n\right))$$

$$=\mathrm{id}_V(w),$$

so that $\Phi\left(\sum\limits_{i=1}^{n}v_i^{*}\otimes v_i\right)=\mathrm{id}_V$), this map sends $\psi\otimes\phi$ to

$$T\left(\underbrace{\Phi^{-1}\left(\mathrm{id}_V\right)}_{=\sum\limits_{i=1}^{n}v_i^{*}\otimes v_i}\otimes\psi\otimes\phi\right)=T\left(\sum_{i=1}^{n}v_i^{*}\otimes v_i\otimes\psi\otimes\phi\right)=\sum_{i=1}^{n}\underbrace{T\left(v_i^{*}\otimes v_i\otimes\psi\otimes\phi\right)}_{\substack{=\left(\widehat{v_i}\cdot\psi\right)\otimes\left(\overset{\vee}{v_i^{*}}\cdot\phi\right)\\ \text{(by the definition of }T)}}$$

$$=\sum_{i=1}^{n}\left(\widehat{v_i}\cdot\psi\right)\otimes\left(\underbrace{\overset{\vee}{v_i^{*}}}_{=\overset{\vee}{v_i}}\cdot\phi\right)=\sum_{i=1}^{n}\left(\widehat{v_i}\cdot\psi\right)\otimes\left(\overset{\vee}{v_i}\cdot\phi\right)$$

for all $\psi\in\wedge^{k}V$ and $\phi\in\wedge^{k}V$. In other words, this map is the map $\sum\limits_{i=1}^{n}\widehat{v_i}\otimes\overset{\vee}{v_i}=S$. So we have shown that S is the map $\wedge^{k}V\otimes\wedge^{k}V\to\wedge^{k+1}V\otimes\wedge^{k-1}V$ which sends $\psi\otimes\phi$ to $T\left(\Phi^{-1}\left(\mathrm{id}_V\right)\otimes\psi\otimes\phi\right)$ for all $\psi\in\wedge^{k}V$ and $\phi\in\wedge^{k}V$, qed.

(There is a combinatorial proof of this in the infinite setting in the Kac-Raina book, but we will make a different proof here.)

Define $E \subseteq V$ to be the set $\{v \in V \mid \widehat{v}\tau = 0\}$. Define $E' \subseteq V^*$ to be the set $\left\{f \in V^* \mid \overset{\vee}{f}\tau = 0\right\}$. Clearly, E is a subspace of V, and E' is a subspace of V^*.

We know that all $v \in E$ and $f \in E'$ satisfy $\left(\overset{\vee}{f}\widehat{v} + \widehat{v}\overset{\vee}{f}\right)\tau = 0$ (since the definition of E yields $\widehat{v}\tau = 0$, and the definition of E' yields $\overset{\vee}{f}\tau = 0$). But $\underbrace{\left(\overset{\vee}{f}\widehat{v} + \widehat{v}\overset{\vee}{f}\right)}_{\substack{=f(v)\,\mathrm{id} \\ (\text{by } (214))}}\tau = f(v)\tau$, so this yields $f(v)\tau = 0$, and thus $f(v) = 0$ (since $\tau \neq 0$). Thus, $E \subseteq E'^{\perp}$.

Let $m = \dim E$ and $r = \dim\left(E'^{\perp}\right)$. Pick a basis $(e_1, e_2, ..., e_n)$ of V such that $(e_1, e_2, ..., e_m)$ is a basis of E and such that $(e_1, e_2, ..., e_r)$ is a basis of $E'^{\perp}$. (Such a basis clearly exists.)

Clearly, for every $i \in \{1, 2, ..., m\}$, we have $e_i \in E$ and thus $\widehat{e_i}\tau = 0$ (by the definition of E).

Also, for every $i \in \{r+1, r+2, ..., n\}$, we have $\overset{\vee}{e_i^*}\tau = 0$ (because $i > r$, so that $e_i^*(e_j) = 0$ for all $j \in \{1, 2, ..., r\}$, so that $e_i^*(E') = 0$ (since $(e_1, e_2, ..., e_r)$ is a basis of $E'^{\perp}$), so that $e_i^* \in \left(E'^{\perp}\right)^{\perp} = E'$).

The vectors $\widehat{e_i}\tau$ for $i \in \{m+1, m+2, ..., n\}$ are linearly independent (because if some linear combination of them was zero, then some linear combination of the e_i with $i \in \{m+1, m+2, ..., n\}$ would lie in $\{v \in V \mid \widehat{v}\tau = 0\} = E$, but this contradicts the fact that $(e_1, e_2, ..., e_m)$ is a basis of E). Hence, the vectors $\widehat{e_i}\tau$ for $i \in \{m+1, m+2, ..., r\}$ are linearly independent.

We defined S using the basis $(v_1, v_2, ..., v_n)$ of V by the formula $S = \sum\limits_{i=1}^{n} \widehat{v_i} \otimes \overset{\vee}{v_i}$. Since S did not depend on the basis, we get the same S if we define it using the basis $(e_1, e_2, ..., e_n)$. Thus, we have $S = \sum\limits_{i=1}^{n} \widehat{e_i} \otimes \overset{\vee}{e_i}$. Hence,

$$S(\tau \otimes \tau) = \sum_{i=1}^{m} \underbrace{\widehat{e_i}\tau}_{\substack{=0 \\ (\text{since } i \in \{1,2,...,m\})}} \otimes \overset{\vee}{e_i^*}\tau + \sum_{i=m+1}^{r} \widehat{e_i}\tau \otimes \overset{\vee}{e_i^*}\tau + \sum_{i=r+1}^{n} \widehat{e_i}\tau \otimes \underbrace{\overset{\vee}{e_i^*}\tau}_{\substack{=0 \\ (\text{since } i \in \{r+1,r+2,...,n\})}}$$
$$= \sum_{i=m+1}^{r} \widehat{e_i}\tau \otimes \overset{\vee}{e_i^*}\tau.$$

Thus, $S(\tau \otimes \tau) = 0$ rewrites as $\sum\limits_{i=m+1}^{r} \widehat{e_i}\tau \otimes \overset{\vee}{e_i^*}\tau = 0$. But since the vectors $\widehat{e_i}\tau$ for $i \in \{m+1, m+2, ..., r\}$ are linearly independent, this yields that $\overset{\vee}{e_i^*}\tau = 0$ for any $i \in \{m+1, m+2, ..., r\}$. Thus, for every $i \in \{m+1, m+2, ..., r\}$, we have $e_i^* \in \left\{f \in V^* \mid \overset{\vee}{f}\tau = 0\right\} = E'$, so that $e_i^*\left(E'^{\perp}\right) = 0$. But on the other hand, for every $i \in \{m+1, m+2, ..., r\}$, we have $e_i \in E'^{\perp}$ (since $(e_1, e_2, ..., e_r)$ is a basis of $E'^{\perp}$, and since $i \leq r$). Thus, for every $i \in \{m+1, m+2, ..., r\}$, we have $1 = e_i^* \left(\underbrace{e_i}_{\in E'^{\perp}} \right) \in e_i^* \left(E'^{\perp} \right) = 0$. This is a contradiction unless there are no $i \in \{m+1, m+2, ..., r\}$ at all.

So we conclude that there are no $i \in \{m+1, m+2, ..., r\}$ at all. In other words, $m = r$. Thus, $\dim E = m = r = \dim \left(E'^{\perp} \right)$. Combined with $E \subseteq E'^{\perp}$, this yields $E = E'^{\perp}$.

Now, recall that $(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k})_{1 \leq i_1 < i_2 < ... < i_k \leq n}$ is a basis of $\wedge^k V$. Hence, we can write τ in the form $\tau = \sum\limits_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1, i_2, ..., i_k} e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}$ for some scalars $\lambda_{i_1, i_2, ..., i_k} \in \mathbb{C}$.

Now, we will prove:

Observation 1: For every k-tuple $(j_1, j_2, ..., j_k)$ of integers satisfying $1 \leq j_1 < j_2 < ... < j_k \leq n$ and $\{1, 2, ..., m\} \not\subseteq \{j_1, j_2, ..., j_k\}$, we have $\lambda_{j_1, j_2, ..., j_k} = 0$.

Proof of Observation 1: Let $(j_1, j_2, ..., j_k)$ be a k-tuple of integers satisfying $1 \leq j_1 < j_2 < ... < j_k \leq n$ and $\{1, 2, ..., m\} \not\subseteq \{j_1, j_2, ..., j_k\}$. Then, there exists an $i \in \{1, 2, ..., m\}$ such that $i \notin \{j_1, j_2, ..., j_k\}$. Consider this i. As we saw above, this yields $\widehat{e_i} \tau = 0$. Thus,

$$\begin{aligned} 0 = \widehat{e_i} \tau = e_i \wedge \tau &= \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1, i_2, ..., i_k} e_i \wedge e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k} \\ &\qquad \left(\text{since } \tau = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1, i_2, ..., i_k} e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k} \right) \\ &= \sum_{\substack{1 \leq i_1 < i_2 < ... < i_k \leq n; \\ i \notin \{i_1, i_2, ..., i_k\}}} \lambda_{i_1, i_2, ..., i_k} e_i \wedge e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k} \\ &\qquad \left(\text{since all terms of the sum with } i \in \{i_1, i_2, ..., i_k\} \text{ are } 0 \right). \end{aligned}$$

Thus, for every k-tuple $(i_1, i_2, ..., i_k)$ of integers satisfying $1 \leq i_1 < i_2 < ... < i_k \leq n$ and $i \notin \{i_1, i_2, ..., i_k\}$, we must have $\lambda_{i_1, i_2, ..., i_k} = 0$ (because the wedge products $e_i \wedge e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}$ for all such k-tuples are linearly independent elements of $\wedge^{k+1} V$). Applied to $(i_1, i_2, ..., i_k) = (j_1, j_2, ..., j_k)$, this yields that $\lambda_{j_1, j_2, ..., j_k} = 0$. Observation 1 is proven.

Observation 2: For every k-tuple $(j_1, j_2, ..., j_k)$ of integers satisfying $1 \leq j_1 < j_2 < ... < j_k \leq n$ and $\{j_1, j_2, ..., j_k\} \not\subseteq \{1, 2, ..., m\}$, we have $\lambda_{j_1, j_2, ..., j_k} = 0$.

Proof of Observation 2: Let $(j_1, j_2, ..., j_k)$ be a k-tuple of integers satisfying $1 \leq j_1 < j_2 < ... < j_k \leq n$ and $\{j_1, j_2, ..., j_k\} \not\subseteq \{1, 2, ..., m\}$. Then, there exists an $i \in \{j_1, j_2, ..., j_k\}$ such that $i \notin \{1, 2, ..., m\}$. Consider this i. Then, $i \notin \{1, 2, ..., m\}$, so that $i > m = r$, so that $i \in \{r+1, r+2, ..., n\}$. As we saw above, this yields $\overset{\vee}{e_i^*} \tau = 0$.

Thus,

$$0 = \underbrace{\overset{\vee}{e_i^*}}_{=\overset{\vee}{e_i}} \tau = \overset{\vee}{e_i}\tau = \sum_{1\leq i_1<i_2<...<i_k\leq n} \lambda_{i_1,i_2,...,i_k} \overset{\vee}{e_i} \cdot (e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k})$$

$$\left(\text{since } \tau = \sum_{1\leq i_1<i_2<...<i_k\leq n} \lambda_{i_1,i_2,...,i_k} e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k} \right)$$

$$= \sum_{\substack{1\leq i_1<i_2<...<i_k\leq n;\\ i\in\{i_1,i_2,...,i_k\}}} \lambda_{i_1,i_2,...,i_k} \overset{\vee}{e_i} \cdot (e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k})$$

$$\left(\text{since all terms of the sum with } i \notin \{i_1, i_2, ..., i_k\} \text{ are } 0\right).$$

Thus, for every k-tuple $(i_1, i_2, ..., i_k)$ of integers satisfying $1 \leq i_1 < i_2 < ... < i_k \leq n$ and $i \in \{i_1, i_2, ..., i_k\}$, we must have $\lambda_{i_1,i_2,...,i_k} = 0$ (because the wedge products $\overset{\vee}{e_i} \cdot (e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k})$ for all such k-tuples are linearly independent elements of $\wedge^{k-1}V$ [164]). Applied to $(i_1, i_2, ..., i_k) = (j_1, j_2, ..., j_k)$, this yields that $\lambda_{j_1,j_2,...,j_k} = 0$. Observation 2 is proven.

Now, every k-tuple $(j_1, j_2, ..., j_k)$ of integers satisfying $1 \leq j_1 < j_2 < ... < j_k \leq n$ must satisfy either $\{1, 2, ..., m\} \not\subseteq \{j_1, j_2, ..., j_k\}$, or $\{j_1, j_2, ..., j_k\} \not\subseteq \{1, 2, ..., m\}$, or $(1, 2, ..., m) = (j_1, j_2, ..., j_k)$. In the first of these three cases, we have $\lambda_{j_1,j_2,...,j_k} = 0$ by Observation 1; in the second case, we have $\lambda_{j_1,j_2,...,j_k} = 0$ by Observation 2. Hence, the only case where $\lambda_{j_1,j_2,...,j_k}$ can be nonzero is the third case, i. e., the case when $(1, 2, ..., m) = (j_1, j_2, ..., j_k)$. Hence, the only nonzero addend that the sum $\sum\limits_{1\leq i_1<i_2<...<i_k\leq n} \lambda_{i_1,i_2,...,i_k} e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}$ can have is the addend for $(i_1, i_2, ..., i_k) = (1, 2, ..., m)$. Thus, all other addends of this sum can be removed, and therefore $\tau = \sum\limits_{1\leq i_1<i_2<...<i_k\leq n} \lambda_{i_1,i_2,...,i_k} e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k}$ rewrites as $\tau = \lambda_{1,2,...,m} e_1 \wedge e_2 \wedge ... \wedge e_m$. Since $\tau \neq 0$, we thus have $\lambda_{1,2,...,m} \neq 0$. Hence, $m = k$ (because $\lambda_{1,2,...,m} e_1 \wedge e_2 \wedge ... \wedge e_m = \tau \in \wedge^k V$). Hence,

$$\tau = \lambda_{1,2,...,m} e_1 \wedge e_2 \wedge ... \wedge e_m = \lambda_{1,2,...,m} e_1 \wedge e_2 \wedge ... \wedge e_k = (\lambda_{1,2,...,m} e_1) \wedge e_2 \wedge e_3 \wedge ... \wedge e_k.$$

Now, since $\lambda_{1,2,...,m} \neq 0$, the n-tuple $(\lambda_{1,2,...,m} e_1, e_2, e_3, ..., e_n)$ is a basis of V. Thus, there exists an element of $\mathrm{GL}(V)$ which sends $(v_1, v_2, ..., v_n)$ to $(\lambda_{1,2,...,m} e_1, e_2, e_3, ..., e_n)$. This element therefore sends $v_1 \wedge v_2 \wedge ... \wedge v_k$ to $(\lambda_{1,2,...,m} e_1) \wedge e_2 \wedge e_3 \wedge ... \wedge e_k = \tau$. Hence, τ lies in the $\mathrm{GL}(V)$-orbit of $v_1 \wedge v_2 \wedge ... \wedge v_k$. Since this orbit was called Ω, this becomes $\tau \in \Omega$.

We thus have shown that if $S(\tau \otimes \tau) = 0$, then $\tau \in \Omega$. This completes the proof of Theorem 3.15.8 **(b)**.

(c) We know from Theorem 3.15.8 **(a)** that S is $\mathrm{GL}(V)$-invariant. Since $\mathrm{GL}(V)$ is Zariski-dense in $\mathrm{M}(V)$, this yields that S is $\mathrm{M}(V)$-invariant (because the $\mathrm{M}(V)$-invariance of S can be written as a collection of polynomial identities). This proves Theorem 3.15.8 **(c)**.

We can rewrite Theorem 3.15.8 **(b)** in coordinates:

[164] To check this, it is enough to recall how $\overset{\vee}{e_i} \cdot (e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_k})$ was defined: It was defined to be $(-1)^{j-1} e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_{j-1}} \wedge e_{i_{j+1}} \wedge e_{i_{j+2}} \wedge ... \wedge e_{i_k}$, where j is the integer ℓ satisfying $i_\ell = i$.

Theorem 3.15.9. Let $n \in \mathbb{N}$. Let $k \in \{1, 2, ..., n\}$. We consider the vector space $V = \mathbb{C}^n$ with its standard basis $(v_1, v_2, ..., v_n)$.

Let $\tau \in \wedge^k V$ be nonzero.

For every subset K of $\{1, 2, ..., n\}$, let v_K denote the element of $\wedge^{|K|} V$ defined by $v_K = v_{k_1} \wedge v_{k_2} \wedge ... \wedge v_{k_\ell}$ where k_1, k_2, ..., k_ℓ are the elements of K in increasing order. We know that $(v_K)_{K \subseteq \{1,2,...,n\},\ |K|=k}$ is a basis of the vector space $\wedge^k V$. For every subset K of $\{1, 2, ..., n\}$ satisfying $|K| = k$, let P_K be the K-coordinate of τ with respect to this basis.

Then, $\tau \in \Omega$ if and only if

$$\left(\begin{array}{c} \text{for all } I \subseteq \{1, 2, ..., n\} \text{ with } |I| = k-1 \text{ and all } J \subseteq \{1, 2, ..., n\} \\ \text{with } |J| = k+1, \text{ we have } \sum\limits_{j \in J;\ j \notin I} (-1)^{\mu(j)} (-1)^{\nu(j)-1} P_{I \cup \{j\}} P_{J \setminus \{j\}} = 0 \end{array} \right), \tag{215}$$

where $\nu(j)$ is the integer ℓ for which j is the ℓ-th smallest element of the set J, and where $\mu(j)$ is the number of elements of the set I which are smaller than j.

Proof of Theorem 3.15.9 (sketched). We know that $(v_K)_{K \subseteq \{1,2,...,n\},\ |K|=k+1}$ is a basis of $\wedge^{k+1} V$, and $(v_K)_{K \subseteq \{1,2,...,n\},\ |K|=k-1}$ is a basis of $\wedge^{k-1} V$. Hence, $(v_K \otimes v_L)_{K \subseteq \{1,2,...,n\},\ |K|=k+1,\ L \subseteq \{1,2,...,n\},\ |L|=k-1}$ is a basis of $\wedge^{k+1} V \otimes \wedge^{k-1} V$. It is not hard to check that the $v_J \otimes v_I$-coordinate (with respect to this basis) of $S(\tau \otimes \tau)$ is precisely $\sum\limits_{j \in J;\ j \notin I} (-1)^{\mu(j)} (-1)^{\nu(j)-1} P_{I \cup \{j\}} P_{J \setminus \{j\}}$ for all $I \subseteq \{1, 2, ..., n\}$ with $|I| = k-1$ and all $J \subseteq \{1, 2, ..., n\}$ with $|J| = k+1$. Hence, (215) holds if and only if every coordinate of $S(\tau \otimes \tau)$ is zero, i. e., if $S(\tau \otimes \tau) = 0$, but the latter condition is equivalent to $\tau \in \Omega$ (because of Theorem 3.15.8 **(b)**). This proves Theorem 3.15.9.

Note that the $\Longrightarrow$ direction of Theorem 3.15.9 can be formulated as a determinantal identity:

Corollary 3.15.10. Let $n \in \mathbb{N}$. Let $k \in \{1, 2, ..., n\}$. Let $\begin{pmatrix} x_{11} & x_{12} & ... & x_{1k} \\ x_{21} & x_{22} & ... & x_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & ... & x_{nk} \end{pmatrix}$ be any matrix with n rows and k columns.

For every $I \subseteq \{1, 2, ..., n\}$ with $|I| = k$, let P_I be the minor of this matrix obtained by only keeping the rows whose indices lie in I (and throwing all other rows away).

Then, for all $I \subseteq \{1, 2, ..., n\}$ with $|I| = k-1$ and all $J \subseteq \{1, 2, ..., n\}$ with $|J| = k+1$, we have $\sum\limits_{j \in J;\ j \notin I} (-1)^{\mu(j)} (-1)^{\nu(j)-1} P_{I \cup \{j\}} P_{J \setminus \{j\}} = 0$ (where $\mu(j)$ and $\nu(j)$ are defined as in Theorem 3.15.9).

Example: If $n = 4$ and $k = 2$, then the claim of Corollary 3.15.10 is easily simplified to the single equation $P_{12} P_{34} + P_{14} P_{23} - P_{13} P_{24} = 0$ (where we abbreviate two-element sets $\{i, j\}$ by ij).

Proof of Corollary 3.15.10 (sketched). WLOG assume $k \leq n$ (else, everything is vacuously true).

For every $i \in \{1, 2, ..., k\}$, let $x_i \in V$ be the vector $\begin{pmatrix} x_{1i} \\ x_{2i} \\ \vdots \\ x_{ni} \end{pmatrix}$, where V is as in Theorem 3.15.9. Since Corollary 3.15.10 is a collection of polynomial identities, we can WLOG assume that the vectors x_1, x_2, ..., x_k are linearly independent (since the set of linearly independent k-tuples $(x_1, x_2, ..., x_k)$ of vectors in V is Zariski-dense in V^k). Then, there exists an element of $\mathrm{GL}(V)$ which maps v_1, v_2, ..., v_k to x_1, x_2, ..., x_k. Thus, $x_1 \wedge x_2 \wedge ... \wedge x_k \in \Omega$ (since Ω is the orbit of $v_1 \wedge v_2 \wedge ... \wedge v_k$ under $\mathrm{GL}(V)$). Now, apply Theorem 3.15.9 to $\tau = x_1 \wedge x_2 \wedge ... \wedge x_k$, and Corollary 3.15.10 follows.

Of course, this was not the easiest way to prove Corollary 3.15.10. We could just as well have derived Corollary 3.15.10 from the Cauchy-Binet identity, and thus given a new proof for the $\Longrightarrow$ direction of Theorem 3.15.9; but the $\Longleftarrow$ direction is not that easy.

3.15.2. The semiinfinite Grassmannian: preliminary work

Now we prepare for the semiinfinite Grassmannian:

Let ψ_0 denote the elementary semiinfinite wedge $v_0 \wedge v_{-1} \wedge v_{-2} \wedge ... \in \mathcal{F}^{(0)}$. We recall the action $\varrho : \mathrm{M}(\infty) \to \mathrm{End}\left(\mathcal{F}^{(m)}\right)$ of the monoid $\mathrm{M}(\infty)$ on $\mathcal{F}^{(m)}$ for every $m \in \mathbb{Z}$. This action was defined in Definition 3.14.6.

Definition 3.15.11. From now on, Ω denotes the subset $\mathrm{GL}(\infty) \cdot \psi_0$ of $\mathcal{F}^{(0)}$. (Here and in the following, we abbreviate $(\varrho(A))\, v$ by Av for every $A \in \mathrm{M}(\infty)$ and $v \in \mathcal{F}^{(m)}$ and every $m \in \mathbb{Z}$. In particular, $\mathrm{GL}(\infty)\, \psi_0$ means $(\varrho(\mathrm{GL}(\infty)))\, \psi_0$.)

Proposition 3.15.12. For all 0-degressions $(i_0, i_1, i_2, ...)$, we have $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \in \Omega$.

Proof of Proposition 3.15.12. Let $(i_0, i_1, i_2, ...)$ be a 0-degression. Then, there exists a permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$ which fixes all but finitely many integers (i. e., is a finitary permutation of $\mathbb{Z}$), and satisfies $i_k = \sigma(-k)$ for every $k \in \mathbb{N}$. Since σ fixes all but finitely many integers, we can represent σ by a matrix in $\mathrm{GL}(\infty)$. Let us (by abuse of notation) denote this matrix by σ again. Then, every $k \in \mathbb{N}$ satisfies $v_{i_k} = v_{\sigma(-k)} = \sigma v_{-k}$. Thus,

$$v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... = \sigma v_0 \wedge \sigma v_{-1} \wedge \sigma v_{-2} \wedge ... = \sigma \underbrace{(v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...)}_{=\psi_0} = \sigma \psi_0 \in \mathrm{GL}(\infty)\, \psi_0 = \Omega.$$

This proves Proposition 3.15.12.

Next, an "infinite" analogue of Theorem 3.15.8:

Theorem 3.15.13. For every $m \in \mathbb{Z}$, define a map $S : \mathcal{F}^{(m)} \otimes \mathcal{F}^{(m)} \to \mathcal{F}^{(m+1)} \otimes \mathcal{F}^{(m-1)}$ by $S = \sum\limits_{i \in \mathbb{Z}} \widehat{v_i} \otimes \overset{\vee}{v_i}$. (Note that the map S is well-defined because, for every $T \in \mathcal{F}^{(m)} \otimes \mathcal{F}^{(m)}$, only finitely many terms of the infinite sum $\sum\limits_{i \in \mathbb{Z}} \left(\widehat{v_i} \otimes \overset{\vee}{v_i} \right)(T)$ are nonzero.)

(a) For every $m \in \mathbb{Z}$, this map S is $\mathrm{GL}(\infty)$-invariant.
(b) Let $\tau \in \mathcal{F}^{(0)}$ be nonzero. Then, $\tau \in \Omega$ if and only if $S(\tau \otimes \tau) = 0$.
(c) For every $m \in \mathbb{Z}$, the map S is $\mathrm{M}(\infty)$-invariant.

We are going to prove this theorem by reducing it to its "finite-dimensional version" (i. e., Theorem 3.15.8). This reduction requires us to link the set Ω with its finite-dimensional analoga. To do this, we set up some definitions:

3.15.3. Proof of Theorem 3.15.13

While the following definitions and results are, superficially seen, auxiliary to the proof of Theorem 3.15.13, their use is not confined to this proof. They can be used to derive various results about semiinfinite wedges (elements of $\mathcal{F}^{(m)}$ for integer m) from similar statements about finite wedges (elements of $\wedge^k W$ for integer k and finite-dimensional W). Our proof of Theorem 3.15.13 below will be just one example of such a derivation.

Note that most of the proofs in this subsection are straightforward and boring and are easier to do by the reader than to understand from these notes.

Definition 3.15.14. Let V be the vector space $\mathbb{C}^{(\mathbb{Z})} = \left\{ (x_i)_{i\in\mathbb{Z}} \mid x_i \in \mathbb{C}; \text{ only finitely many } x_i \text{ are nonzero} \right\}$ as defined in Definition 3.5.2. Let $(v_j)_{j\in\mathbb{Z}}$ be the basis of V introduced in Definition 3.5.2.

For every $N \in \mathbb{N}$, let V_N denote the $(2N+1)$-dimensional vector subspace $\langle v_{-N}, v_{-N+1}, ..., v_N \rangle$ of V. It is clear that $V_0 \subseteq V_1 \subseteq V_2 \subseteq ...$ and $V = \bigcup\limits_{N\in\mathbb{N}} V_N$.

It should be noticed that this vector subspace V_N is what has been called $V_{]-N-1,N]}$ in Definition 3.14.39.

Definition 3.15.15. Let $N \in \mathbb{N}$. Let $\mathrm{M}(V_N)$ denote the set of all $(2N+1) \times (2N+1)$-matrices over $\mathbb{C}$ whose rows are indexed by elements of $\{-N, -N+1, ..., N\}$ and whose columns are also indexed by elements of $\{-N, -N+1, ..., N\}$. Define a map $i_N : \mathrm{M}(V_N) \to \mathrm{M}(\infty)$ as follows: For every matrix $A \in \mathrm{M}(V_N)$, let $i_N(A)$ be the infinite matrix (with rows and columns indexed by integers) such that

$$\left(\begin{array}{l} \text{(the } (i,j)\text{-th entry of } i_N(A)) \\ = \begin{cases} \text{(the } (i,j)\text{-th entry of } A), & \text{if } (i,j) \in \{-N, -N+1, ..., N\}^2; \\ \delta_{i,j}, & \text{if } (i,j) \in \mathbb{Z}^2 \diagdown \{-N, -N+1, ..., N\}^2 \end{cases} \\ \qquad\qquad\qquad\qquad\qquad \text{for every } (i,j) \in \mathbb{Z}^2 \end{array} \right).$$

It is easy to see that this map i_N is well-defined (i. e., for every $A \in \mathrm{M}(V_N)$, the matrix $i_N(A)$ that we just defined really lies in $\mathrm{M}(\infty)$), injective and a monoid homomorphism.

The vector space V_N has a basis $(v_{-N}, v_{-N+1}, ..., v_N)$ which is indexed by the set $\{-N, -N+1, ..., N\}$. Thus, we can identify matrices in $\mathrm{M}(V_N)$ with endomorphisms of the vector space V_N in the obvious way. Hence, the invertible elements of $\mathrm{M}(V_N)$ are identified with the invertible endomorphisms of the vector space V_N, i. e., with the elements of $\mathrm{GL}(V_N)$. The injective map $i_N : \mathrm{M}(V_N) \to \mathrm{M}(\infty)$ restricts to an injective map $i_N \mid_{\mathrm{GL}(V_N)} : \mathrm{GL}(V_N) \to \mathrm{GL}(\infty)$.

Remark 3.15.16. Here is a more lucid way to describe the map i_N we just defined:

Let $I_{-\infty}$ be the infinite identity matrix whose rows are indexed by all negative integers, and whose columns are indexed by all negative integers.

Let I_{∞} be the infinite identity matrix whose rows are indexed by all positive integers, and whose columns are indexed by all positive integers.

For any matrix $A \in \mathrm{M}(V_N)$, we define $i_N(A)$ to be the block-diagonal matrix $\begin{pmatrix} I_{-\infty} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{\infty} \end{pmatrix}$ whose diagonal blocks are $I_{-\infty}$, A and I_{∞}, where the first block covers the rows with indices smaller than $-N$ (and therefore also the columns with indices smaller than $-N$), the second block covers the rows with indices in $\{-N, -N+1, ..., N\}$ (and therefore also the columns with indices in $\{-N, -N+1, ..., N\}$), and the third block covers the rows with indices larger than N (and therefore also the columns with indices larger than N). From this definition, it becomes clear why i_N is a monoid homomorphism. (In fact, it is clear that the block-diagonal matrix $\begin{pmatrix} I_{-\infty} & 0 & 0 \\ 0 & I_{2N+1} & 0 \\ 0 & 0 & I_{\infty} \end{pmatrix}$ is the identity matrix, and using the rules for computing with block matrices it is also easy to see that $\begin{pmatrix} I_{-\infty} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{\infty} \end{pmatrix} \begin{pmatrix} I_{-\infty} & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I_{\infty} \end{pmatrix} = \begin{pmatrix} I_{-\infty} & 0 & 0 \\ 0 & AB & 0 \\ 0 & 0 & I_{\infty} \end{pmatrix}$ for all $A \in \mathrm{M}(V_N)$ and $B \in \mathrm{M}(V_N)$.)

Remark 3.15.17. **(a)** Every $N \in \mathbb{N}$ satisfies

$$i_N(\mathrm{M}(V_N)) = \left\{ A \in \mathrm{M}(\infty) \mid \left(\begin{array}{c} (\text{the } (i,j)\text{-th entry of } A) = \delta_{i,j} \text{ for every} \\ (i,j) \in \mathbb{Z}^2 \diagdown \{-N, -N+1, ..., N\}^2 \end{array} \right) \right\}.$$

(b) We have $i_0(\mathrm{M}(V_0)) \subseteq i_1(\mathrm{M}(V_1)) \subseteq i_2(\mathrm{M}(V_2)) \subseteq ....$

(c) We have $\mathrm{M}(\infty) = \bigcup\limits_{N \in \mathbb{N}} i_N(\mathrm{M}(V_N))$.

Proof of Remark 3.15.17. **(a)** Let $N \in \mathbb{N}$. Then,

$$\left\{ A \in \mathrm{M}(\infty) \mid \left(\begin{array}{c} (\text{the } (i,j)\text{-th entry of } A) = \delta_{i,j} \text{ for every} \\ (i,j) \in \mathbb{Z}^2 \diagdown \{-N, -N+1, ..., N\}^2 \end{array} \right) \right\} \subseteq i_N(\mathrm{M}(V_N))$$

[165] and

$$i_N\left(\mathrm{M}\left(V_N\right)\right) \subseteq \left\{A \in \mathrm{M}\left(\infty\right) \ \mid \ \left(\begin{array}{c}\left(\text{the } (i,j)\text{-th entry of } A\right) = \delta_{i,j} \text{ for every} \\ (i,j) \in \mathbb{Z}^2 \diagdown \left\{-N, -N+1, ..., N\right\}^2\end{array}\right)\right\}$$

(by the definition of i_N). Combining these two relations, we obtain

$$i_N\left(\mathrm{M}\left(V_N\right)\right) = \left\{A \in \mathrm{M}\left(\infty\right) \ \mid \ \left(\begin{array}{c}\left(\text{the } (i,j)\text{-th entry of } A\right) = \delta_{i,j} \text{ for every} \\ (i,j) \in \mathbb{Z}^2 \diagdown \left\{-N, -N+1, ..., N\right\}^2\end{array}\right)\right\}.$$

This proves Remark 3.15.17 **(a)**.

(b) By Remark 3.15.17 **(a)**, for any $N \in \mathbb{N}$, the set $i_N\left(\mathrm{M}\left(V_N\right)\right)$ is the set of all matrices $A \in \mathrm{M}\left(\infty\right)$ satisfying the condition

$$\left(\left(\text{the } (i,j)\text{-th entry of } A\right) = \delta_{i,j} \text{ for every } (i,j) \in \mathbb{Z}^2 \diagdown \left\{-N, -N+1, ..., N\right\}^2\right).$$

If this condition is satisfied for some N, then it is (all the more) satisfied for $N+1$ instead of N. Hence, $i_N\left(\mathrm{M}\left(V_N\right)\right) \subseteq i_{N+1}\left(\mathrm{M}\left(V_{N+1}\right)\right)$ for any $N \in \mathbb{N}$. Thus, $i_0\left(\mathrm{M}\left(V_0\right)\right) \subseteq i_1\left(\mathrm{M}\left(V_1\right)\right) \subseteq i_2\left(\mathrm{M}\left(V_2\right)\right) \subseteq ....$ This proves Remark 3.15.17 **(b)**.

(c) Let $B \in \mathrm{M}\left(\infty\right)$ be arbitrary. We will now construct an $N \in \mathbb{N}$ such that $B \in i_N\left(\mathrm{M}\left(V_N\right)\right)$.

Since $B \in \mathrm{M}\left(\infty\right) = \mathrm{id} + \mathfrak{gl}_\infty$, there exists a $b \in \mathfrak{gl}_\infty$ such that $B = \mathrm{id} + b$. Consider this b.

For any $(i,j) \in \mathbb{Z}^2$, let $b_{i,j}$ denote the (i,j)-th entry of the matrix b.

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[165] *Proof.* To prove this, it is clearly enough to show that every matrix $A \in \mathrm{M}\left(\infty\right)$ which satisfies

$$\left(\left(\text{the } (i,j)\text{-th entry of } A\right) = \delta_{i,j} \text{ for every } (i,j) \in \mathbb{Z}^2 \diagdown \left\{-N, -N+1, ..., N\right\}^2\right) \tag{216}$$

lies in $i_N\left(\mathrm{M}\left(V_N\right)\right)$. So let $A \in \mathrm{M}\left(\infty\right)$ be a matrix which satisfies (216). We must prove that $A \in i_N\left(\mathrm{M}\left(V_N\right)\right)$.

Indeed, let $B \in \mathrm{M}\left(V_N\right)$ be the matrix defined by

$$\left(\left(\text{the } (i,j)\text{-th entry of } B\right) = \left(\text{the } (i,j)\text{-th entry of } A\right) \text{ for every } (i,j) \in \left\{-N, -N+1, ..., N\right\}^2\right). \tag{217}$$

Then, $i_N\left(B\right) = A$ (because for every $(i,j) \in \mathbb{Z}^2$, we have

$$\begin{aligned}
&\left(\text{the } (i,j)\text{-th entry of } i_N\left(B\right)\right) \\
&= \begin{cases} \left(\text{the } (i,j)\text{-th entry of } B\right), & \text{if } (i,j) \in \left\{-N, -N+1, ..., N\right\}^2; \\ \delta_{i,j}, & \text{if } (i,j) \in \mathbb{Z}^2 \diagdown \left\{-N, -N+1, ..., N\right\}^2 \end{cases} \\
&\qquad \left(\text{by the definition of } i_N\left(B\right)\right) \\
&= \begin{cases} \left(\text{the } (i,j)\text{-th entry of } A\right), & \text{if } (i,j) \in \left\{-N, -N+1, ..., N\right\}^2; \\ \delta_{i,j}, & \text{if } (i,j) \in \mathbb{Z}^2 \diagdown \left\{-N, -N+1, ..., N\right\}^2 \end{cases} \\
&\qquad \left(\text{by (217)}\right) \\
&= \begin{cases} \left(\text{the } (i,j)\text{-th entry of } A\right), & \text{if } (i,j) \in \left\{-N, -N+1, ..., N\right\}^2; \\ \left(\text{the } (i,j)\text{-th entry of } A\right), & \text{if } (i,j) \in \mathbb{Z}^2 \diagdown \left\{-N, -N+1, ..., N\right\}^2 \end{cases} \\
&\qquad \left(\text{since } \delta_{i,j} = \left(\text{the } (i,j)\text{-th entry of } A\right) \text{ for every } (i,j) \in \mathbb{Z}^2 \diagdown \left\{-N, -N+1, ..., N\right\}^2 \text{ (by (216))}\right) \\
&= \left(\text{the } (i,j)\text{-th entry of } A\right)
\end{aligned}$$

). Thus, $A = i_N\left(B\right) \in i_N\left(\mathrm{M}\left(V_N\right)\right)$ (since $B \in \mathrm{M}\left(V_N\right)$), qed.

Since $b \in \mathfrak{gl}_\infty$, only finitely many entries of the matrix b are nonzero. In other words, only finitely many $(u,v) \in \mathbb{Z}^2$ satisfy $((u,v)\text{-th entry of } b) \neq 0$. In other words, only finitely many $(u,v) \in \mathbb{Z}^2$ satisfy $b_{u,v} \neq 0$ (since $((u,v)\text{-th entry of } b) = b_{u,v}$). In other words, the set $\{\max\{|u|,|v|\} \mid (u,v) \in \mathbb{Z}^2;\ b_{u,v} \neq 0\}$ is finite.

Let

$$N = \max\left\{\max\{|u|,|v|\} \mid (u,v) \in \mathbb{Z}^2;\ b_{u,v} \neq 0\right\}.$$

[166] This N is a well-defined nonnegative integer (since the set $\{\max\{|u|,|v|\} \mid (u,v) \in \mathbb{Z}^2;\ b_{u,v} \neq 0\}$ is finite).

Let $(i,j) \in \mathbb{Z}^2 \diagdown \{-N,-N+1,...,N\}^2$. Then, $(i,j) \notin \{-N,-N+1,...,N\}^2$. We are now going to show that $b_{i,j} = 0$.

In fact, assume (for the sake of contradiction) that $b_{i,j} \neq 0$. Thus, $(i,j) \in \{(u,v) \in \mathbb{Z}^2 \mid b_{u,v} \neq 0\}$. Hence,

$$\max\{|i|,|j|\} \in \left\{\max\{|u|,|v|\} \mid (u,v) \in \mathbb{Z}^2;\ b_{u,v} \neq 0\right\}.$$

Since any element of a finite set is less or equal to the maximum of the set, this yields

$$\max\{|i|,|j|\} \leq \max\left\{\max\{|u|,|v|\} \mid (u,v) \in \mathbb{Z}^2;\ b_{u,v} \neq 0\right\} = N.$$

Thus, $|i| \leq \max\{|i|,|j|\} \leq N$, so that $i \in \{-N,-N+1,...,N\}$ and similarly $j \in \{-N,-N+1,...,N\}$. Hence, $(i,j) \in \{-N,-N+1,...,N\}^2$ (because $i \in \{-N,-N+1,...,N\}$ and $j \in \{-N,-N+1,...,N\}$), which contradicts $(i,j) \notin \{-N,-N+1,...,N\}^2$. This contradiction shows that our assumption (that $b_{i,j} \neq 0$) was wrong. We thus have $b_{i,j} = 0$.

Since $B = \operatorname{id} + b$, we have:

$$(\text{the } (i,j)\text{-th entry of } B) = \underbrace{(\text{the } (i,j)\text{-th entry of } \operatorname{id})}_{=\delta_{i,j}} + \underbrace{(\text{the } (i,j)\text{-th entry of } b)}_{=b_{i,j}=0} = \delta_{i,j}.$$

Now, forget that we fixed (i,j). We thus have shown that $(\text{the } (i,j)\text{-th entry of } B) = \delta_{i,j}$ for every $(i,j) \in \mathbb{Z}^2 \diagdown \{-N,-N+1,...,N\}^2$. In other words,

$$\begin{aligned} B &\in \left\{A \in \mathrm{M}(\infty) \mid \left(\begin{array}{c} (\text{the } (i,j)\text{-th entry of } A) = \delta_{i,j} \text{ for every} \\ (i,j) \in \mathbb{Z}^2 \diagdown \{-N,-N+1,...,N\}^2 \end{array}\right)\right\} = i_N(\mathrm{M}(V_N)) \\ &\qquad (\text{by Remark 3.15.17 } \textbf{(a)}) \\ &\subseteq \bigcup_{P \in \mathbb{N}} i_P(\mathrm{M}(V_P)). \end{aligned}$$

Now forget that we fixed B. We thus have proven that every $B \in \mathrm{M}(\infty)$ satisfies $B \in \bigcup_{P\in\mathbb{N}} i_P(\mathrm{M}(V_P))$. In other words, $\mathrm{M}(\infty) \subseteq \bigcup_{P\in\mathbb{N}} i_P(\mathrm{M}(V_P)) = \bigcup_{N\in\mathbb{N}} i_N(\mathrm{M}(V_N))$ (here, we renamed the index P as N). Combined with the obvious inclusion $\bigcup_{N\in\mathbb{N}} i_N(\mathrm{M}(V_N)) \subseteq \mathrm{M}(\infty)$, this yields $\mathrm{M}(\infty) = \bigcup_{N\in\mathbb{N}} i_N(\mathrm{M}(V_N))$. Remark 3.15.17 **(c)** is therefore proven.

[166] Here, we set $\max\left\{\max\{|u|,|v|\} \mid (u,v) \in \mathbb{Z}^2;\ b_{u,v} \neq 0\right\}$ to be 0 if the set $\left\{\max\{|u|,|v|\} \mid (u,v) \in \mathbb{Z}^2;\ b_{u,v} \neq 0\right\}$ is empty.

Definition 3.15.18. Let $N \in \mathbb{N}$ and $m \in \mathbb{Z}$. We define a linear map $j_N^{(m)} : \wedge^{N+m+1}(V_N) \to \mathcal{F}^{(m)}$ by setting

$$\left(\begin{array}{c} j_N^{(m)} (b_0 \wedge b_1 \wedge ... \wedge b_{N+m}) = b_0 \wedge b_1 \wedge ... \wedge b_{N+m} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ... \\ \text{for any } b_0, b_1, ..., b_{N+m} \in V_N \end{array} \right).$$

This map $j_N^{(m)}$ is well-defined (because $b_0 \wedge b_1 \wedge ... \wedge b_{N+m} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ...$ is easily seen to lie in $\mathcal{F}^{(m)}$ and depend multilinearly and anti-symmetrically on $b_0, b_1, ..., b_{N+m}$) and injective (because the elements of the basis $\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}\right)_{N \geq i_0 > i_1 > ... > i_{N+m} \geq -N}$ of $\wedge^{N+m+1}(V_N)$ are sent by $j_N^{(m)}$ to pairwise distinct elements of the basis $(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)_{(i_0, i_1, i_2, ...) \text{ is an } m\text{-degression}}$ of $\mathcal{F}^{(m)}$).

In the terminology of Definition 3.14.39, the map $j_N^{(m)}$ that we have just defined is the map $R_{N+m+1,]-N-1,N]}$.

Our definitions of $j_N^{(m)}$ and of i_N satisfy reasonable compatibilities:

Proposition 3.15.19. Let $N \in \mathbb{N}$ and $m \in \mathbb{Z}$. For any $u \in \wedge^{N+m+1}(V_N)$ and $A \in \mathrm{M}(V_N)$, we have

$$i_N(A) \cdot j_N^{(m)}(u) = j_N^{(m)}(Au).$$

(Here, of course, $i_N(A) \cdot j_N^{(m)}(u)$ stands for $(\varrho(i_N(A)))\left(j_N^{(m)}(u)\right)$.)

Proof of Proposition 3.15.19. Let $A \in \mathrm{M}(V_N)$ and $u \in \wedge^{N+m+1}(V_N)$. We must prove the equality $i_N(A) \cdot j_N^{(m)}(u) = j_N^{(m)}(Au)$. Since this equality is linear in u, we can WLOG assume that u is an element of the basis $\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}\right)_{N \geq i_0 > i_1 > ... > i_{N+m} \geq -N}$ of $\wedge^{N+m+1}(V_N)$. Assume this. Then, there exists an $N+m+1$-tuple $(i_0, i_1, ..., i_{N+m})$ of integers such that $N \geq i_0 > i_1 > ... > i_{N+m} \geq -N$ and $u = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}$. Consider this $N+m+1$-tuple.

By the definition of $i_N(A)$, we have

$$(i_N(A) \cdot v_k = Av_k \qquad \text{for every } k \in \{-N, -N+1, ..., N\}) \tag{218}$$

and

$$(i_N(A) \cdot v_k = v_k \qquad \text{for every } k \in \mathbb{Z} \diagdown \{-N, -N+1, ..., N\}). \tag{219}$$

Note that every $\ell \in \{0, 1, ..., N+m\}$ satisfies $i_\ell \in \{-N, -N+1, ..., N\}$ (since $N \geq i_0 > i_1 > ... > i_{N+m} \geq -N$ and thus $N \geq i_\ell \geq -N$) and thus

$$i_N(A) \cdot v_{i_\ell} = Av_{i_\ell} \tag{220}$$

(by (218), applied to $k = i_\ell$). Also, every positive integer r satisfies $-N-r \in \mathbb{Z} \diagdown \{-N, -N+1, ..., N\}$ and thus

$$i_N(A) \cdot v_{-N-r} = v_{-N-r} \tag{221}$$

(by (219), applied to $k = -N-r$).

Now, since $u = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}$, we have

$$\begin{aligned} j_N^{(m)} (u) &= j_N^{(m)} \left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \right) \\ &= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ... \end{aligned}$$

(by the definition of $j_N^{(m)}$), so that

$$\begin{aligned} & i_N (A) \cdot j_N^{(m)} (u) \\ &= i_N (A) \cdot \left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ... \right) \\ &= \underbrace{i_N (A) \cdot v_{i_0} \wedge i_N (A) \cdot v_{i_1} \wedge ... \wedge i_N (A) \cdot v_{i_{N+m}}}_{\substack{= Av_{i_0} \wedge Av_{i_1} \wedge ... \wedge Av_{i_{N+m}} \\ \text{(because every } \ell \in \{0,1,...,N+m\} \text{ satisfies } i_N(A) \cdot v_{i_\ell} = Av_{i_\ell} \text{ (by (220)))}}} \\ &\qquad \wedge \underbrace{i_N (A) \cdot v_{-N-1} \wedge i_N (A) \cdot v_{-N-2} \wedge i_N (A) \cdot v_{-N-3} \wedge ...}_{\substack{= v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ... \\ \text{(because every positive integer } r \text{ satisfies } i_N(A) \cdot v_{-N-r} = v_{-N-r} \text{ (by (221)))}}} \\ &\qquad \left(\text{by the definition of the action } \varrho : \mathrm{M} (\infty) \to \operatorname{End} \left(\mathcal{F}^{(m)} \right) \right) \\ &= Av_{i_0} \wedge Av_{i_1} \wedge ... \wedge Av_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge .... \end{aligned} \tag{222}$$

On the other hand, since $u = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}$, we have $Au = Av_{i_0} \wedge Av_{i_1} \wedge ... \wedge Av_{i_{N+m}}$, so that

$$\begin{aligned} j_N^{(m)} (Au) &= j_N^{(m)} \left(Av_{i_0} \wedge Av_{i_1} \wedge ... \wedge Av_{i_{N+m}} \right) \\ &= Av_{i_0} \wedge Av_{i_1} \wedge ... \wedge Av_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ... \end{aligned}$$

(by the definition of $j_N^{(m)}$). Compared with (222), this yields $i_N (A) \cdot j_N^{(m)} (u) = j_N^{(m)} (Au)$. This proves Proposition 3.15.19.

An important property of the maps $j_N^{(m)}$ is that their images (for fixed m and varying N) cover (not just span, but actually cover) all of $\mathcal{F}^{(m)}$:

Proposition 3.15.20. Let $m \in \mathbb{Z}$.

(a) We have

$$j_0^{(m)} \left(\wedge^{0+m+1} (V_0) \right) \subseteq j_1^{(m)} \left(\wedge^{1+m+1} (V_1) \right) \subseteq j_2^{(m)} \left(\wedge^{2+m+1} (V_2) \right) \subseteq ....$$

(b) For every $Q \in \mathbb{N}$, we have $\mathcal{F}^{(m)} = \bigcup\limits_{\substack{N \in \mathbb{N}; \\ N \geq Q}} j_N^{(m)} \left(\wedge^{N+m+1} (V_N) \right)$.

Actually, the "$N \geq Q$" in Proposition 3.15.20 **(b)** doesn't have much effect since Proposition 3.15.20 **(a)** yields $\bigcup\limits_{\substack{N \in \mathbb{N}; \\ N \geq Q}} j_N^{(m)} \left(\wedge^{N+m+1} (V_N) \right) = \bigcup\limits_{N \in \mathbb{N}} j_N^{(m)} \left(\wedge^{N+m+1} (V_N) \right)$; but we prefer to put it in because it is needed in our application.

Proof of Proposition 3.15.20. **(a)** Let $N \in \mathbb{N}$. From the definitions of j_N and j_{N+1}, it is easy to see that

$$j_N^{(m)} (b_0 \wedge b_1 \wedge ... \wedge b_{N+m}) = j_{N+1}^{(m)} (b_0 \wedge b_1 \wedge ... \wedge b_{N+m} \wedge v_{-N-1})$$

for any $b_0, b_1, ..., b_{N+m} \in V_N$. Due to linearity, this yields that $j_N^{(m)}(a) = j_{N+1}^{(m)}(a \wedge v_{-N-1})$ for any $a \in \wedge^{N+m+1}(V_N)$. Hence, $j_N^{(m)}(a) = j_{N+1}^{(m)}(a \wedge v_{-N-1}) \in j_{N+1}^{(m)}\left(\wedge^{(N+1)+m+1}(V_{N+1})\right)$ for any $a \in \wedge^{N+m+1}(V_N)$. In other words, $j_N^{(m)}\left(\wedge^{N+m+1}(V_N)\right) \subseteq j_{N+1}^{(m)}\left(\wedge^{(N+1)+m+1}(V_{N+1})\right)$.

We thus have proven that every $N \in \mathbb{N}$ satisfies

$$j_N^{(m)}\left(\wedge^{N+m+1}(V_N)\right) \subseteq j_{N+1}^{(m)}\left(\wedge^{(N+1)+m+1}(V_{N+1})\right).$$

In other words,

$$j_0^{(m)}\left(\wedge^{0+m+1}(V_0)\right) \subseteq j_1^{(m)}\left(\wedge^{1+m+1}(V_1)\right) \subseteq j_2^{(m)}\left(\wedge^{2+m+1}(V_2)\right) \subseteq ....$$

Proposition 3.15.20 **(a)** is proven.

(b) We need three notations:

- For any m-degression $\mathbf{i}$, define a nonnegative integer $\operatorname{exting}(\mathbf{i})$ as the largest $k \in \mathbb{N}$ satisfying $i_k + k \neq m$ [167], where $\mathbf{i}$ is written in the form $(i_0, i_1, i_2, ...)$. (Such a largest k indeed exists, because (by the definition of an m-degression) every sufficiently high $k \in \mathbb{N}$ satisfies $i_k + k = m$.)

- For any m-degression $\mathbf{i}$, define an integer $\operatorname{head}(\mathbf{i})$ by $\operatorname{head}(\mathbf{i}) = i_0$, where $\mathbf{i}$ is written in the form $(i_0, i_1, i_2, ...)$.

- For any m-degression $\mathbf{i}$, define an element $v_{\mathbf{i}}$ of $\mathcal{F}^{(m)}$ by $v_{\mathbf{i}} = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$, where $\mathbf{i}$ is written in the form $(i_0, i_1, i_2, ...)$.

Thus, $(v_{\mathbf{i}})_{\mathbf{i}\text{ is an }m\text{-degression}} = (v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)_{(i_0,i_1,i_2,...)\text{ is an }m\text{-degression}}$. Since $(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)_{(i_0,i_1,i_2,...)\text{ is an }m\text{-degression}}$ is a basis of the vector space $\mathcal{F}^{(m)}$, we thus conclude that $(v_{\mathbf{i}})_{\mathbf{i}\text{ is an }m\text{-degression}}$ is a basis of the vector space $\mathcal{F}^{(m)}$.

Now we prove a simple fact:

$$\left(\begin{array}{c}\text{If } \mathbf{i} \text{ is an } m\text{-degression, and } P \text{ is an integer such that} \\ P \geq \max\{0, \operatorname{exting}(\mathbf{i}) - m, \operatorname{head}(\mathbf{i})\}, \text{ then } v_{\mathbf{i}} \in j_P^{(m)}\left(\wedge^{P+m+1}(V_P)\right)\end{array}\right). \tag{223}$$

Proof of (223): Let $\mathbf{i}$ be an m-degression, and P be an integer such that $P \geq \max\{0, \operatorname{exting}(\mathbf{i}) - m, \operatorname{head}(\mathbf{i})\}$. Write $\mathbf{i}$ in the form $(i_0, i_1, i_2, ...)$. Then, $\operatorname{exting}(\mathbf{i})$ is the largest $k \in \mathbb{N}$ satisfying $i_k + k \neq m$ (by the definition of $\operatorname{exting}(\mathbf{i})$). Hence,

$$\text{every } k \in \mathbb{N} \text{ such that } k > \operatorname{exting}(\mathbf{i}) \text{ satisfies } i_k + k = m. \tag{224}$$

Since $P \geq \max\{0, \operatorname{exting}(\mathbf{i}) - m, \operatorname{head}(\mathbf{i})\} \geq 0$, the map $j_P^{(m)}$ and the space V_P are well-defined.

Since $P \geq \max\{0, \operatorname{exting}(\mathbf{i}) - m, \operatorname{head}(\mathbf{i})\} \geq \operatorname{exting}(\mathbf{i}) - m$, we have $P + m \geq \operatorname{exting}(\mathbf{i}) \geq 0$. Now,

$$\text{every positive integer } \ell \text{ satisfies } i_{P+m+\ell} = -P - \ell \tag{225}$$

[168]. Applied to $\ell = 1$, this yields $i_{P+m+1} = -P - 1$.

[167] If no such k exists, then we set $\operatorname{exting}(\mathbf{i})$ to be 0.

[168] *Proof of (225):* Let $\ell \in \mathbb{N}$ be a positive integer. Then, $P + m + \underbrace{\ell}_{>0} > P + m \geq \operatorname{exting}(\mathbf{i})$. Hence, (224) (applied to $k = P+m+\ell$) yields $i_{P+m+\ell} + P + m + \ell = m$. In other words, $i_{P+m+\ell} = -P - \ell$. This proves (225).

Notice also that $P \geq \max\{0, \operatorname{exting}(\mathbf{i}) - m, \operatorname{head}(\mathbf{i})\} \geq \operatorname{head}(\mathbf{i}) = i_0$ (by the definition of $\operatorname{head}(\mathbf{i})$). Now it is easy to see that

$$\text{every } k \in \mathbb{N} \text{ such that } k \leq P + m \text{ satisfies } v_{i_k} \in V_P. \tag{226}$$

[169] Hence, $v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{P+m}} \in \wedge^{P+m+1}(V_P)$. Now, by the definition of $j_P^{(m)}$, we have

$$\begin{aligned}
& j_P^{(m)}\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{P+m}}\right) \\
&= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{P+m}} \wedge \underbrace{v_{-P-1} \wedge v_{-P-2} \wedge v_{-P-3} \wedge ...}_{\substack{= v_{i_{P+m+1}} \wedge v_{i_{P+m+2}} \wedge v_{i_{P+m+3}} \wedge ... \\ \text{(because every positive integer } \ell \\ \text{satisfies } -P-\ell = i_{P+m+\ell} \text{ (by (225)))}}} \\
&= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{P+m}} \wedge v_{i_{P+m+1}} \wedge v_{i_{P+m+2}} \wedge v_{i_{P+m+3}} \wedge ... = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... = v_{\mathbf{i}}
\end{aligned}$$

(since $v_{\mathbf{i}}$ was defined as $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$). Thus, $v_{\mathbf{i}} = j_P^{(m)}\left(\underbrace{v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{P+m}}}_{\in \wedge^{P+m+1}(V_P)}\right) \in j_P^{(m)}\left(\wedge^{P+m+1}(V_P)\right)$. This proves (223).

Now, fix an arbitrary $Q \in \mathbb{N}$.

Let w be any element of $\mathcal{F}^{(m)}$. Since $(v_{\mathbf{i}})_{\mathbf{i} \text{ is an } m\text{-degression}}$ is a basis of $\mathcal{F}^{(m)}$, we can write w as a linear combination of elements of the family $(v_{\mathbf{i}})_{\mathbf{i} \text{ is an } m\text{-degression}}$. Since every linear combination contains only finitely many vectors, this yields that we can write w as a linear combination of **finitely many** elements of the family $(v_{\mathbf{i}})_{\mathbf{i} \text{ is an } m\text{-degression}}$. In other words, there exists a finite set S of m-degressions such that w is a linear combination of the family $(v_{\mathbf{i}})_{\mathbf{i} \in S}$. Consider this S. Since w is a linear combination of the family $(v_{\mathbf{i}})_{\mathbf{i} \in S}$, we can find a scalar $\lambda_{\mathbf{i}} \in \mathbb{C}$ for every $\mathbf{i} \in S$ such that $w = \sum\limits_{\mathbf{i} \in S} \lambda_{\mathbf{i}} v_{\mathbf{i}}$. Consider these scalars $\lambda_{\mathbf{i}}$. Let

$$P = \max\{Q, \max\{\max\{0, \operatorname{exting}(\mathbf{j}) - m, \operatorname{head}(\mathbf{j})\} \mid \mathbf{j} \in S\}\}$$

(where the maximum of the empty set is to be understood as 0). Then, first of all, $P \geq Q$. Second, every $\mathbf{i} \in S$ satisfies

$$\begin{aligned}
P &= \max\{Q, \max\{\max\{0, \operatorname{exting}(\mathbf{j}) - m, \operatorname{head}(\mathbf{j})\} \mid \mathbf{j} \in S\}\} \\
&\geq \max\{\max\{0, \operatorname{exting}(\mathbf{j}) - m, \operatorname{head}(\mathbf{j})\} \mid \mathbf{j} \in S\} \\
&\geq \max\{0, \operatorname{exting}(\mathbf{i}) - m, \operatorname{head}(\mathbf{i})\} \\
&\qquad \left(\begin{array}{c} \text{since } \max\{0, \operatorname{exting}(\mathbf{i}) - m, \operatorname{head}(\mathbf{i})\} \text{ is an element of the set} \\ \{\max\{0, \operatorname{exting}(\mathbf{j}) - m, \operatorname{head}(\mathbf{j})\} \mid \mathbf{j} \in S\} \text{ (because } \mathbf{i} \in S\text{),} \\ \text{and the maximum of a set is } \geq \text{ to any element of this set} \end{array}\right)
\end{aligned}$$

[169] *Proof of (226):* Let $k \in \mathbb{N}$ be such that $k \leq P + m$. Thus, $k < P + m + 1$.

Since $(i_0, i_1, i_2, ...) = \mathbf{i}$ is an m-degression, the sequence $(i_0, i_1, i_2, ...)$ is strictly decreasing, i. e., we have $i_0 > i_1 > i_2 > ....$ As a consequence, $i_0 \geq i_k$ (since $0 \leq k$) and $i_k > i_{P+m+1}$ (since $k < P + m + 1$). Since $i_k > i_{P+m+1} = -P - 1$, we have $i_k \geq -P$ (since both i_k and $-P$ are integers). Combining $P \geq i_0 \geq i_k$ with $i_k \geq -P$, we obtain $P \geq i_k \geq -P$. Hence, $v_{i_k} \in \langle v_{-P}, v_{-P+1}, ..., v_P \rangle = V_P$ (because V_P is defined as $\langle v_{-P}, v_{-P+1}, ..., v_P \rangle$). This proves (226).

and thus $v_{\mathbf{i}} \in j_P^{(m)} \left(\wedge^{P+m+1} (V_P) \right)$ (by (223)). Hence,

$$w = \sum_{\mathbf{i} \in S} \lambda_{\mathbf{i}} \underbrace{v_{\mathbf{i}}}_{\in j_P^{(m)} \left(\wedge^{P+m+1} (V_P) \right)} \in \sum_{\mathbf{i} \in S} \lambda_{\mathbf{i}} j_P^{(m)} \left(\wedge^{P+m+1} (V_P) \right) \subseteq j_P^{(m)} \left(\wedge^{P+m+1} (V_P) \right)$$

$$\left(\text{since } j_P^{(m)} \left(\wedge^{P+m+1} (V_P) \right) \text{ is a vector space} \right)$$

$$\subseteq \bigcup_{\substack{N \in \mathbb{N}; \\ N \geq Q}} j_N^{(m)} \left(\wedge^{N+m+1} (V_N) \right) \qquad (\text{since } P \geq Q).$$

Now, forget that we fixed w. We thus have proven that every $w \in \mathcal{F}^{(m)}$ satisfies $w \in \bigcup_{\substack{N \in \mathbb{N}; \\ N \geq Q}} j_N^{(m)} \left(\wedge^{N+m+1} (V_N) \right)$. Thus, $\mathcal{F}^{(m)} \subseteq \bigcup_{\substack{N \in \mathbb{N}; \\ N \geq Q}} j_N^{(m)} \left(\wedge^{N+m+1} (V_N) \right)$. Combined with the obvious inclusion $\bigcup_{\substack{N \in \mathbb{N}; \\ N \geq Q}} j_N^{(m)} \left(\wedge^{N+m+1} (V_N) \right) \subseteq \mathcal{F}^{(m)}$, this yields $\mathcal{F}^{(m)} = \bigcup_{\substack{N \in \mathbb{N}; \\ N \geq Q}} j_N^{(m)} \left(\wedge^{N+m+1} (V_N) \right)$. Proposition 3.15.20 **(b)** is thus proven.

What comes next is almost a carbon copy of Definition 3.15.7:

Definition 3.15.21. Let $N \in \mathbb{N}$. Let $k \in \mathbb{Z}$. Let $i \in \{-N, -N+1, ..., N\}$.

(a) We define the so-called *i-th wedging operator* $\widehat{v_i^{(N)}} : \wedge^k (V_N) \to \wedge^{k+1} (V_N)$ by

$$\widehat{v_i^{(N)}} \cdot \psi = v_i \wedge \psi \qquad \text{for all } \psi \in \wedge^k (V_N).$$

(b) We define the so-called *i-th contraction operator* $\overset{\vee}{v_i^{(N)}} : \wedge^k (V_N) \to \wedge^{k-1} (V_N)$ as follows:

For every k-tuple $(i_1, i_2, ..., i_k)$ of integers satisfying $N \geq i_1 > i_2 > ... > i_k \geq -N$, we let $\overset{\vee}{v_i^{(N)}} (v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_k})$ be

$$\begin{cases} 0, & \text{if } i \notin \{i_1, i_2, ..., i_k\}; \\ (-1)^{j-1} v_{i_1} \wedge v_{i_2} \wedge ... \wedge v_{i_{j-1}} \wedge v_{i_{j+1}} \wedge v_{i_{j+2}} \wedge ... \wedge v_{i_k}, & \text{if } i \in \{i_1, i_2, ..., i_k\} \end{cases},$$

where, in the case $i \in \{i_1, i_2, ..., i_k\}$, we denote by j the integer ℓ satisfying $i_\ell = i$. Thus, the map $\overset{\vee}{v_i^{(N)}}$ is defined on a basis of the vector space $\wedge^k (V_N)$; we extend this to a map $\wedge^k (V_N) \to \wedge^{k-1} (V_N)$ by linearity.

Note that, for every negative $\ell \in \mathbb{Z}$, we understand $\wedge^\ell (V_N)$ to mean the zero space.

Also:

Definition 3.15.22. For every $N \in \mathbb{N}$ and $k \in \{1, 2, ..., 2N+1\}$, let $\Omega_N^{(k)}$ denote the orbit of $v_N \wedge v_{N-1} \wedge ... \wedge v_{N-k+1}$ under the action of $\mathrm{GL}(V_N)$.

The following lemma, then, is an easy corollary of Theorem 3.15.8:

Lemma 3.15.23. Let $N \in \mathbb{N}$ and $k \in \mathbb{Z}$. Let $S_N^{(k)} = \sum\limits_{i=-N}^{N} \widehat{v_i^{(N)}} \otimes \overset{\vee}{v_i^{(N)}} : \wedge^k (V_N) \otimes \wedge^k (V_N) \to \wedge^{k+1} (V_N) \otimes \wedge^{k-1} (V_N)$.

(a) This map $S_N^{(k)}$ does not depend on the choice of the basis of V_N, and is $\operatorname{GL}(V_N)$-invariant. In other words, for **any** basis $(w_N, w_{N-1}, ..., w_{-N})$ of V_N, we have $S_N^{(k)} = \sum\limits_{i=-N}^{N} \widehat{w_i^{(N)}} \otimes \overset{\vee}{w_i^{(N)}}$ (where the maps $\widehat{w_i^{(N)}}$ and $\overset{\vee}{w_i^{(N)}}$ are defined just as $\widehat{v_i^{(N)}}$ and $\overset{\vee}{v_i^{(N)}}$, but with respect to the basis $(w_N, w_{N-1}, ..., w_{-N})$).

(b) Let $k \in \{1, 2, ..., 2N+1\}$. A nonzero element $\tau \in \wedge^k (V_N)$ belongs to $\Omega_N^{(k)}$ if and only if $S_N^{(k)} (\tau \otimes \tau) = 0$.

(c) The map $S_N^{(k)}$ is $\operatorname{M}(V_N)$-invariant.

Proof of Lemma 3.15.23. If we set $n = 2N + 1$ in Theorem 3.15.8, and do the following renaming operations:

- rename the standard basis $(v_1, v_2, ..., v_n)$ as $(v_N, v_{N-1}, ..., v_{-N})$;
- rename the vector space V as V_N;
- rename the map S as $S_N^{(k)}$;
- rename the basis $(w_1, w_2, ..., w_n)$ as $(w_N, w_{N-1}, ..., w_{-N})$;
- rename the maps $\widehat{v_i}$ as $\widehat{v_i^{(N)}}$;
- rename the maps $\overset{\vee}{v_i}$ as $\overset{\vee}{v_i^{(N)}}$;
- rename the maps $\widehat{w_i}$ as $\widehat{w_i^{(N)}}$;
- rename the maps $\overset{\vee}{w_i}$ as $\overset{\vee}{w_i^{(N)}}$;
- rename the set Ω as $\Omega_N^{(k)}$;

then what we obtain is exactly the statement of Lemma 3.15.23. Thus, Lemma 3.15.23 is proven.

The maps $S_N^{(k)}$ have their own compatibility relation with the $j_N^{(m)}$:

Lemma 3.15.24. Let $N \in \mathbb{N}$ and $m \in \mathbb{Z}$. Define the notation $S_N^{(N+m+1)}$ as in Lemma 3.15.23. Then,

$$\left(j_N^{(m+1)} \otimes j_N^{(m-1)}\right) \circ S_N^{(N+m+1)} = S \circ \left(j_N^{(m)} \otimes j_N^{(m)}\right).$$

Proof of Lemma 3.15.24. Define the maps $\widehat{v_i^{(N)}}$ and $\overset{\vee}{v}_i^{(N)}$ (for all $i \in \{-N, -N+1, ..., N\}$) as in Definition 3.15.21. Define the maps $\widehat{v_i}$ and $\overset{\vee}{v_i}$ (for all $i \in \mathbb{Z}$) as in Definition 3.10.5.

a) Let us first show that

$$j_N^{(m+1)} \circ \widehat{v_i^{(N)}} = \widehat{v_i} \circ j_N^{(m)} \qquad \text{for every } i \in \{N, N-1, ..., -N\}. \tag{227}$$

Proof of (227): Let $i \in \{N, N-1, ..., -N\}$. In order to prove (227), it is clearly enough to show that $\left(j_N^{(m+1)} \circ \widehat{v_i^{(N)}}\right)(u) = \left(\widehat{v_i} \circ j_N^{(m)}\right)(u)$ for every $u \in \wedge^{N+m+1}(V_N)$.

So let u be any element of $\wedge^{N+m+1}(V_N)$. We must prove the equality $\left(j_N^{(m+1)} \circ \widehat{v_i^{(N)}}\right)(u) = \left(\widehat{v_i} \circ j_N^{(m)}\right)(u)$. Since this equality is linear in u, we can WLOG assume that u is an element of the basis $\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}\right)_{N \geq i_0 > i_1 > ... > i_{N+m} \geq -N}$ of $\wedge^{N+m+1}(V_N)$. Assume this. Then, there exists an $N+m+1$-tuple $(i_0, i_1, ..., i_{N+m})$ of integers such that $N \geq i_0 > i_1 > ... > i_{N+m} \geq -N$ and $u = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}$. Consider this $N+m+1$-tuple.

Comparing

$$\begin{aligned}
\left(j_N^{(m+1)} \circ \widehat{v_i^{(N)}}\right)(u) &= j_N^{(m+1)} \underbrace{\left(\widehat{v_i^{(N)}}(u)\right)}_{\substack{= v_i \wedge u \\ \text{(by the definition of } \widehat{v_i^{(N)}})}} = j_N^{(m+1)} \left(v_i \wedge \underbrace{u}_{= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}}\right) \\
&= j_N^{(m+1)} \left(v_i \wedge v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}\right) \\
&= v_i \wedge v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ... \\
&\qquad \left(\text{by the definition of } j_N^{(m+1)}\right)
\end{aligned}$$

with

$$\begin{aligned}
\left(\widehat{v_i} \circ j_N^{(m)}\right)(u) &= \widehat{v_i}\left(j_N^{(m)}(u)\right) = v_i \wedge j_N^{(m)} \left(\underbrace{u}_{= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}}\right) \qquad (\text{by the definition of } \widehat{v_i}) \\
&= v_i \wedge \underbrace{j_N^{(m)}\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}\right)}_{\substack{= v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ... \\ \text{(by the definition of } j_N^{(m)})}} \\
&= v_i \wedge v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ...,
\end{aligned}$$

we obtain $\left(j_N^{(m+1)} \circ \widehat{v_i^{(N)}}\right)(u) = \left(\widehat{v_i} \circ j_N^{(m)}\right)(u)$. This is exactly what we needed to prove in order to complete the proof of (227). The proof of (227) is thus finished.

b) Let us next show that

$$j_N^{(m+1)} \circ \overset{\vee}{v}_i^{(N)} = \overset{\vee}{v_i} \circ j_N^{(m)} \qquad \text{for every } i \in \{N, N-1, ..., -N\}. \tag{228}$$

Proof of (228): Let $i \in \{N, N-1, ..., -N\}$. In order to prove (228), it is clearly enough to show that $\left(j_N^{(m+1)} \circ \overset{\vee}{v}_i^{(N)} \right)(u) = \left(\overset{\vee}{v}_i \circ j_N^{(m)} \right)(u)$ for every $u \in \wedge^{N+m+1}(V_N)$.

So let u be any element of $\wedge^{N+m+1}(V_N)$. We must prove the equality $\left(j_N^{(m+1)} \circ \overset{\vee}{v}_i^{(N)} \right)(u) = \left(\overset{\vee}{v}_i \circ j_N^{(m)} \right)(u)$. Since this equality is linear in u, we can WLOG assume that u is an element of the basis $\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \right)_{N \geq i_0 > i_1 > ... > i_{N+m} \geq -N}$ of $\wedge^{N+m+1}(V_N)$. Assume this. Then, there exists an $N+m+1$-tuple $(i_0, i_1, ..., i_{N+m})$ of integers such that $N \geq i_0 > i_1 > ... > i_{N+m} \geq -N$ and $u = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}$. Consider this $N+m+1$-tuple.

Let $(j_0, j_1, j_2, ...)$ be the sequence $(i_0, i_1, ..., i_{N+m}, -N-1, -N-2, -N-3, ...)$. From $u = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}$, we obtain

$$\begin{aligned} j_N^{(m)}(u) &= j_N^{(m)}\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \right) = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ... \\ &\qquad \left(\text{by the definition of } j_N^{(m)} \right) \\ &= v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \\ &\qquad \left(\text{since } (i_0, i_1, ..., i_{N+m}, -N-1, -N-2, -N-3, ...) = (j_0, j_1, j_2, ...) \right). \end{aligned} \tag{229}$$

We distinguish between two cases:

Case 1: We have $i \notin \{i_0, i_1, ..., i_{N+m}\}$.

Case 2: We have $i \in \{i_0, i_1, ..., i_{N+m}\}$.

Let us first consider Case 1. In this case, from $u = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}$, we obtain

$$\begin{aligned} &\overset{\vee}{v}_i^{(N)}(u) \\ &= \overset{\vee}{v}_i^{(N)}\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \right) \\ &= \begin{cases} 0, & \text{if } i \notin \{i_0, i_1, ..., i_{N+m}\}; \\ (-1)^{j-1} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{(j-1)-1}} \wedge v_{i_{(j-1)+1}} \wedge v_{i_{(j-1)+2}} \wedge ... \wedge v_{i_{N+m}}, & \text{if } i \in \{i_0, i_1, ..., i_{N+m}\} \end{cases} \\ &\qquad \left(\text{by the definition of } \overset{\vee}{v}_i^{(N)} \right), \end{aligned}$$

where, in the case $i \in \{i_0, i_1, ..., i_{N+m}\}$, we denote by j the integer ℓ satisfying $i_{\ell-1} = i$. [170] Since $i \notin \{i_0, i_1, ..., i_{N+m}\}$ (because we are in Case 1), this simplifies to

$$\overset{\vee}{v}_i^{(N)}(u) = 0.$$

On the other hand, combining $i \notin \{-N-1, -N-2, -N-3, ...\}$ (which is because $i \in \{N, N-1, ..., -N\}$) with $i \notin \{i_0, i_1, ..., i_{N+m}\}$ (which is because we are in Case 1),

[170] If you are wondering where the -1 (for example, in $i_{\ell-1}$ and in $i_{(j-1)-1}$) comes from: It comes from the fact that the indexing of our $N+m+1$-tuple $\left(v_{i_0}, v_{i_1}, ..., v_{i_{N+m}} \right)$ begins with 0, and not with 1 as in Definition 3.15.7.

we obtain

$$\begin{aligned}
& i \notin \{i_0, i_1, ..., i_{N+m}\} \cup \{-N-1, -N-2, -N-3, ...\} \\
& \quad = \{i_0, i_1, ..., i_{N+m}, -N-1, -N-2, -N-3, ...\} = \{j_0, j_1, j_2, ...\} \\
& \qquad \qquad \left(\text{since } (i_0, i_1, ..., i_{N+m}, -N-1, -N-2, -N-3, ...) = (j_0, j_1, j_2, ...)\right).
\end{aligned}$$

Now,

$$\begin{aligned}
\left(\overset{\vee}{v}_i \circ j_N^{(m)}\right)(u) &= \overset{\vee}{v}_i \left(j_N^{(m)}(u)\right) = \overset{\vee}{v}_i \left(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...\right) \\
& \qquad \left(\text{since } j_N^{(m)}(u) = v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \text{ by (229)}\right) \\
&= \begin{cases} 0, & \text{if } i \notin \{j_0, j_1, j_2, ...\}; \\ (-1)^j v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \wedge v_{j_{j-1}} \wedge v_{j_{j+1}} \wedge v_{j_{j+2}} \wedge ..., & \text{if } i \in \{j_0, j_1, j_2, ...\} \end{cases} \\
& \qquad \left(\text{by the definition of } \overset{\vee}{v}_i\right),
\end{aligned}$$

where, in the case $i \in \{j_0, j_1, j_2, ...\}$, we denote by j the integer k satisfying $j_k = i$. Since $i \notin \{j_0, j_1, j_2, ...\}$, this simplifies to

$$\left(\overset{\vee}{v}_i \circ j_N^{(m)}\right)(u) = 0.$$

Compared with

$$\left(j_N^{(m+1)} \circ \overset{\vee}{v_i^{(N)}}\right)(u) = j_N^{(m+1)} \underbrace{\left(\overset{\vee}{v_i^{(N)}}(u)\right)}_{=0} = 0,$$

this yields $\left(j_N^{(m+1)} \circ \overset{\vee}{v_i^{(N)}}\right)(u) = \left(\overset{\vee}{v}_i \circ j_N^{(m)}\right)(u)$. We have thus proven $\left(j_N^{(m+1)} \circ \overset{\vee}{v_i^{(N)}}\right)(u) = \left(\overset{\vee}{v}_i \circ j_N^{(m)}\right)(u)$ in Case 1.

Next, let us consider Case 2. In this case, $i \in \{i_0, i_1, ..., i_{N+m}\}$, so there exists an $\ell \in \{0, 1, ..., N+m\}$ such that $i_\ell = i$. Denote this ℓ by κ. Then, $i_\kappa = i$. Clearly,

$$\begin{aligned}
& (i_0, i_1, ..., i_{\kappa-1}, i_{\kappa+1}, i_{\kappa+2}, ..., i_{N+m}, -N-1, -N-2, -N-3, ...) \\
& = \left(\text{result of removing the } \kappa+1\text{-th term from the sequence } \underbrace{(i_0, i_1, ..., i_{N+m}, -N-1, -N-2, -N-3, ...)}_{=(j_0, j_1, j_2, ...)}\right) \\
& = \left(\text{result of removing the } \kappa+1\text{-th term from the sequence } (j_0, j_1, j_2, ...)\right) \\
& = (j_0, j_1, ..., j_{\kappa-1}, j_{\kappa+1}, j_{\kappa+2}, ...). \qquad (230)
\end{aligned}$$

From $u = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}$, we obtain

$$\begin{aligned}
&\overset{\vee}{v_i^{(N)}}(u) \\
&= \overset{\vee}{v_i^{(N)}}\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}\right) \\
&= \begin{cases} 0, & \text{if } i \notin \{i_0, i_1, ..., i_{N+m}\}; \\ (-1)^{j-1} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{(j-1)-1}} \wedge v_{i_{(j-1)+1}} \wedge v_{i_{(j-1)+2}} \wedge ... \wedge v_{i_{N+m}}, & \text{if } i \in \{i_0, i_1, ..., i_{N+m}\} \end{cases} \\
&\qquad \left(\text{by the definition of } \overset{\vee}{v_i^{(N)}}\right),
\end{aligned}$$

where, in the case $i \in \{i_0, i_1, ..., i_{N+m}\}$, we denote by j the integer ℓ satisfying $i_{\ell-1} = i$. [171] Since $i \in \{i_0, i_1, ..., i_{N+m}\}$, this simplifies to

$$\overset{\vee}{v_i^{(N)}}(u) = (-1)^{j-1} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{(j-1)-1}} \wedge v_{i_{(j-1)+1}} \wedge v_{i_{(j-1)+2}} \wedge ... \wedge v_{i_{N+m}},$$

where we denote by j the integer ℓ satisfying $i_{\ell-1} = i$. Since the integer ℓ satisfying $i_{\ell-1} = i$ is $\kappa+1$ (because $i_{(\kappa+1)-1} = i_\kappa = i$), this rewrites as

$$\begin{aligned}
\overset{\vee}{v_i^{(N)}}(u) &= (-1)^{(\kappa+1)-1} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{((\kappa+1)-1)-1}} \wedge v_{i_{((\kappa+1)-1)+1}} \wedge v_{i_{((\kappa+1)-1)+2}} \wedge ... \wedge v_{i_{N+m}} \\
&= (-1)^\kappa v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\kappa-1}} \wedge v_{i_{\kappa+1}} \wedge v_{i_{\kappa+2}} \wedge ... \wedge v_{i_{N+m}}
\end{aligned}$$

(since $(\kappa+1)-1 = \kappa$). Thus,

$$\begin{aligned}
&\left(j_N^{(m+1)} \circ \overset{\vee}{v_i^{(N)}}\right)(u) \\
&= j_N^{(m+1)}\left(\underbrace{\overset{\vee}{v_i^{(N)}}(u)}_{=(-1)^\kappa v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\kappa-1}} \wedge v_{i_{\kappa+1}} \wedge v_{i_{\kappa+2}} \wedge ... \wedge v_{i_{N+m}}}\right) \\
&= j_N^{(m+1)}\left((-1)^\kappa v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\kappa-1}} \wedge v_{i_{\kappa+1}} \wedge v_{i_{\kappa+2}} \wedge ... \wedge v_{i_{N+m}}\right) \\
&= (-1)^\kappa v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{\kappa-1}} \wedge v_{i_{\kappa+1}} \wedge v_{i_{\kappa+2}} \wedge ... \wedge v_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ... \\
&\qquad \left(\text{by the definition of } j_N^{(m+1)}\right) \\
&= (-1)^\kappa v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \wedge v_{j_{\kappa-1}} \wedge v_{j_{\kappa+1}} \wedge v_{j_{\kappa+2}} \wedge ... \qquad (231) \\
&\qquad \left(\begin{array}{c} \text{since (230) yields} \\ (i_0, i_1, ..., i_{\kappa-1}, i_{\kappa+1}, i_{\kappa+2}, ..., i_{N+m}, -N-1, -N-2, -N-3, ...) \\ = (j_0, j_1, ..., j_{\kappa-1}, j_{\kappa+1}, j_{\kappa+2}, ...) \end{array}\right).
\end{aligned}$$

On the other hand,

$$i \in \{i_0, i_1, ..., i_{N+m}\} \subseteq \{i_0, i_1, ..., i_{N+m}, -N-1, -N-2, -N-3, ...\} = \{j_0, j_1, j_2, ...\}$$
$$\left(\text{since } (i_0, i_1, ..., i_{N+m}, -N-1, -N-2, -N-3, ...) = (j_0, j_1, j_2, ...)\right).$$

[171] If you are wondering where the -1 (for example, in $i_{\ell-1}$ and in $i_{(j-1)-1}$) comes from: It comes from the fact that the indexing of our $N+m+1$-tuple $\left(v_{i_0}, v_{i_1}, ..., v_{i_{N+m}}\right)$ begins with 0, and not with 1 as in Definition 3.15.7.

Moreover, the integer k satisfying $j_k = i$ is κ [172]. Now,

$$\left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)(u) = \overset{\vee}{v_i}\left(j_N^{(m)}(u)\right) = \overset{\vee}{v_i}\left(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...\right)$$

$$\left(\text{since } j_N^{(m)}(u) = v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \text{ by (229)}\right)$$

$$= \begin{cases} 0, & \text{if } i \notin \{j_0, j_1, j_2, ...\}; \\ (-1)^j v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \wedge v_{j_{j-1}} \wedge v_{j_{j+1}} \wedge v_{j_{j+2}} \wedge ..., & \text{if } i \in \{j_0, j_1, j_2, ...\} \end{cases}$$

$$\left(\text{by the definition of } \overset{\vee}{v_i}\right),$$

where, in the case $i \in \{j_0, j_1, j_2, ...\}$, we denote by j the integer k satisfying $j_k = i$. Since $i \in \{j_0, j_1, j_2, ...\}$, this simplifies to

$$\left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)(u) = (-1)^j v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \wedge v_{j_{j-1}} \wedge v_{j_{j+1}} \wedge v_{j_{j+2}} \wedge ...,$$

where we denote by j the integer k satisfying $j_k = i$. Since the integer k satisfying $j_k = i$ is κ, this rewrites as

$$\left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)(u) = (-1)^\kappa v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \wedge v_{j_{\kappa-1}} \wedge v_{j_{\kappa+1}} \wedge v_{j_{\kappa+2}} \wedge ....$$

Compared with (231), this yields $\left(j_N^{(m+1)} \circ \widehat{v_i^{(N)}}\right)(u) = \left(\widehat{v_i} \circ j_N^{(m)}\right)(u)$. This is exactly what we needed to prove in order to complete the proof of (228). The proof of (228) is thus finished.

c) Let us next show that

$$\widehat{v_i} \circ j_N^{(m)} = 0 \qquad \text{for every } i \in \{-N-1, -N-2, -N-3, ...\}. \tag{232}$$

Proof of (232): Let $i \in \{-N-1, -N-2, -N-3, ...\}$. In order to prove (232), it is clearly enough to show that $\left(\widehat{v_i} \circ j_N^{(m)}\right)(u) = 0$ for every $u \in \wedge^{N+m+1}(V_N)$.

So let u be any element of $\wedge^{N+m+1}(V_N)$. We must prove the equality $\left(\widehat{v_i} \circ j_N^{(m)}\right)(u) = 0$. Since this equality is linear in u, we can WLOG assume that u is an element of the basis $\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}\right)_{N \geq i_0 > i_1 > ... > i_{N+m} \geq -N}$ of $\wedge^{N+m+1}(V_N)$. Assume this. Then, there exists an $N+m+1$-tuple $(i_0, i_1, ..., i_{N+m})$ of integers such that $N \geq i_0 > i_1 > ... > i_{N+m} \geq -N$ and $u = v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}}$. Consider this $N+m+1$-tuple.

The vector v_i occurs twice in the semiinfinite wedge $v_i \wedge v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ...$ (namely, it occurs once in the very beginning of this wedge, and then it occurs again in the $v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ...$ part (because $i \in \{-N-1, -N-2, -N-3, ...\}$)). Hence, the semiinfinite wedge $v_i \wedge v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{N+m}} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ...$ equals 0 (since a semiinfinite wedge in which a vector occurs more than once must always be equal to 0).

[172] because

$$\begin{aligned} j_\kappa &= i_\kappa \qquad \left(\text{since } (i_0, i_1, ..., i_{N+m}, -N-1, -N-2, -N-3, ...) = (j_0, j_1, j_2, ...) \text{ and } \kappa \in \{0, 1, ..., N+m\}\right) \\ &= i \end{aligned}$$

Now,

$$\left(\widehat{v_i}\circ j_N^{(m)}\right)(u)=\widehat{v_i}\left(j_N^{(m)}(u)\right)=v_i\wedge j_N^{(m)}\left(\underbrace{u}_{=v_{i_0}\wedge v_{i_1}\wedge\ldots\wedge v_{i_{N+m}}}\right)\qquad\left(\text{by the definition of }\widehat{v_i}\right)$$
$$=v_i\wedge\underbrace{j_N^{(m)}\left(v_{i_0}\wedge v_{i_1}\wedge\ldots\wedge v_{i_{N+m}}\right)}_{\substack{=v_{i_0}\wedge v_{i_1}\wedge\ldots\wedge v_{i_{N+m}}\wedge v_{-N-1}\wedge v_{-N-2}\wedge v_{-N-3}\wedge\ldots\\ \left(\text{by the definition of }j_N^{(m)}\right)}}$$
$$=v_i\wedge v_{i_0}\wedge v_{i_1}\wedge\ldots\wedge v_{i_{N+m}}\wedge v_{-N-1}\wedge v_{-N-2}\wedge v_{-N-3}\wedge\ldots$$
$$=0\qquad(\text{as we proved above}).$$

This is exactly what we needed to prove in order to complete the proof of (232). The proof of (232) is thus finished.

d) Let us now show that

$$\overset{\vee}{v_i}\circ j_N^{(m)}=0\qquad\text{for every }i\in\{N+1,N+2,N+3,\ldots\}.\qquad(233)$$

Proof of (233): Let $i\in\{N+1,N+2,N+3,\ldots\}$. In order to prove (228), it is clearly enough to show that $\left(\overset{\vee}{v_i}\circ j_N^{(m)}\right)(u)=0$ for every $u\in\wedge^{N+m+1}(V_N)$.

So let u be any element of $\wedge^{N+m+1}(V_N)$. We must prove the equality $\left(\overset{\vee}{v_i}\circ j_N^{(m)}\right)(u)=0$. Since this equality is linear in u, we can WLOG assume that u is an element of the basis $\left(v_{i_0}\wedge v_{i_1}\wedge\ldots\wedge v_{i_{N+m}}\right)_{N\geq i_0>i_1>\ldots>i_{N+m}\geq -N}$ of $\wedge^{N+m+1}(V_N)$. Assume this. Then, there exists an $N+m+1$-tuple $(i_0,i_1,\ldots,i_{N+m})$ of integers such that $N\geq i_0>i_1>\ldots>i_{N+m}\geq -N$ and $u=v_{i_0}\wedge v_{i_1}\wedge\ldots\wedge v_{i_{N+m}}$. Consider this $N+m+1$-tuple.

Notice that $i\in\{N+1,N+2,N+3,\ldots\}$, so that $i\notin\{N,N-1,\ldots,-N\}$ and $i\notin\{N,N-1,N-2,\ldots\}$.

Since $N\geq i_0>i_1>\ldots>i_{N+m}\geq -N$, we have $\{i_0,i_1,\ldots,i_{N+m}\}\subseteq\{N,N-1,\ldots,-N\}$ and thus $i\notin\{i_0,i_1,\ldots,i_{N+m}\}$ (because $i\notin\{N,N-1,\ldots,-N\}$).

Let $(j_0,j_1,j_2,\ldots)$ be the sequence $(i_0,i_1,\ldots,i_{N+m},-N-1,-N-2,-N-3,\ldots)$. Then,

$$\{j_0,j_1,j_2,\ldots\}=\{i_0,i_1,\ldots,i_{N+m},-N-1,-N-2,-N-3,\ldots\}$$
$$=\underbrace{\{i_0,i_1,\ldots,i_{N+m}\}}_{\subseteq\{N,N-1,\ldots,-N\}}\cup\{-N-1,-N-2,-N-3,\ldots\}$$
$$\subseteq\{N,N-1,\ldots,-N\}\cup\{-N-1,-N-2,-N-3,\ldots\}=\{N,N-1,N-2,\ldots\}.$$

Thus, $i\notin\{j_0,j_1,j_2,\ldots\}$ (since $i\notin\{N,N-1,N-2,\ldots\}$).

From $u=v_{i_0}\wedge v_{i_1}\wedge\ldots\wedge v_{i_{N+m}}$, we obtain

$$j_N^{(m)}(u)=j_N^{(m)}\left(v_{i_0}\wedge v_{i_1}\wedge\ldots\wedge v_{i_{N+m}}\right)=v_{i_0}\wedge v_{i_1}\wedge\ldots\wedge v_{i_{N+m}}\wedge v_{-N-1}\wedge v_{-N-2}\wedge v_{-N-3}\wedge\ldots$$
$$\left(\text{by the definition of }j_N^{(m)}\right)$$
$$=v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge\ldots$$
$$(\text{since }(i_0,i_1,\ldots,i_{N+m},-N-1,-N-2,-N-3,\ldots)=(j_0,j_1,j_2,\ldots)),\qquad(234)$$

so that

$$\left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)(u) = \overset{\vee}{v_i}\left(j_N^{(m)}(u)\right) = \overset{\vee}{v_i}\left(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...\right)$$
$$\left(\text{since } j_N^{(m)}(u) = v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \text{ by (234)}\right)$$
$$= \begin{cases} 0, & \text{if } i \notin \{j_0, j_1, j_2, ...\}; \\ (-1)^j v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \wedge v_{j_{j-1}} \wedge v_{j_{j+1}} \wedge v_{j_{j+2}} \wedge ..., & \text{if } i \in \{j_0, j_1, j_2, ...\} \end{cases}$$
$$\left(\text{by the definition of } \overset{\vee}{v_i}\right),$$

where, in the case $i \in \{j_0, j_1, j_2, ...\}$, we denote by j the integer k satisfying $j_k = i$. Since $i \notin \{j_0, j_1, j_2, ...\}$, this simplifies to $\left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)(u) = 0$.

This is exactly what we needed to prove in order to complete the proof of (233). The proof of (233) is thus finished.

e) Now it is the time to draw conclusions.

We have $S = \sum\limits_{i \in \mathbb{Z}} \widehat{v_i} \otimes \overset{\vee}{v_i}$ (by the definition of S). Thus,

$$S \circ \left(j_N^{(m)} \otimes j_N^{(m)}\right) = \left(\sum_{i \in \mathbb{Z}} \widehat{v_i} \otimes \overset{\vee}{v_i}\right) \circ \left(j_N^{(m)} \otimes j_N^{(m)}\right) = \sum_{i \in \mathbb{Z}} \underbrace{\left(\widehat{v_i} \otimes \overset{\vee}{v_i}\right) \circ \left(j_N^{(m)} \otimes j_N^{(m)}\right)}_{=\left(\widehat{v_i} \circ j_N^{(m)}\right) \otimes \left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)}$$
$$= \sum_{i \in \mathbb{Z}} \left(\widehat{v_i} \circ j_N^{(m)}\right) \otimes \left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)$$
$$= \sum_{i=-\infty}^{-N-1} \underbrace{\left(\widehat{v_i} \circ j_N^{(m)}\right)}_{\substack{=0 \\ \text{(by (232))}}} \otimes \left(\overset{\vee}{v_i} \circ j_N^{(m)}\right) + \sum_{i=-N}^{N} \underbrace{\left(\widehat{v_i} \circ j_N^{(m)}\right)}_{\substack{=j_N^{(m+1)} \circ \widehat{v_i^{(N)}} \\ \text{(by (227))}}} \otimes \underbrace{\left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)}_{\substack{=j_N^{(m+1)} \circ \overset{\vee}{v_i^{(N)}} \\ \text{(by (228))}}} + \sum_{i=N+1}^{\infty} \left(\widehat{v_i} \circ j_N^{(m)}\right) \otimes \underbrace{\left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)}_{\substack{=0 \\ \text{(by (233))}}}$$
$$= \underbrace{\sum_{i=-\infty}^{-N-1} 0 \otimes \left(\overset{\vee}{v_i} \circ j_N^{(m)}\right)}_{=0} + \sum_{i=-N}^{N} \underbrace{\left(j_N^{(m+1)} \circ \widehat{v_i^{(N)}}\right) \otimes \left(j_N^{(m+1)} \circ \overset{\vee}{v_i^{(N)}}\right)}_{=\left(j_N^{(m+1)} \otimes j_N^{(m-1)}\right) \circ \left(\widehat{v_i^{(N)}} \otimes \overset{\vee}{v_i^{(N)}}\right)} + \underbrace{\sum_{i=N+1}^{\infty} \left(\widehat{v_i} \circ j_N^{(m)}\right) \otimes 0}_{=0}$$
$$= \sum_{i=-N}^{N} \left(j_N^{(m+1)} \otimes j_N^{(m-1)}\right) \circ \left(\widehat{v_i^{(N)}} \otimes \overset{\vee}{v_i^{(N)}}\right) = \left(j_N^{(m+1)} \otimes j_N^{(m-1)}\right) \circ \left(\sum_{i=-N}^{N} \widehat{v_i^{(N)}} \otimes \overset{\vee}{v_i^{(N)}}\right).$$

But since $S_N^{(N+m+1)} = \sum\limits_{i=-N}^{N} \widehat{v_i^{(N)}} \otimes \overset{\vee}{v_i^{(N)}}$ (by the definition of $S_N^{(N+m+1)}$), this rewrites as

$$S \circ \left(j_N^{(m)} \otimes j_N^{(m)}\right) = \left(j_N^{(m+1)} \otimes j_N^{(m-1)}\right) \circ \underbrace{\left(\sum_{i=-N}^{N} \widehat{v_i^{(N)}} \otimes \overset{\vee}{v_i^{(N)}}\right)}_{=S_N^{(N+m+1)}} = \left(j_N^{(m+1)} \otimes j_N^{(m-1)}\right) \circ S_N^{(N+m+1)}.$$

This proves Lemma 3.15.24.

Now we can finally come to proving Theorem 3.15.13:

Proof of Theorem 3.15.13. Let $\varrho' : \mathrm{M}(\infty) \to \mathrm{End}(\mathcal{F} \otimes \mathcal{F})$ be the action of the monoid $\mathrm{M}(\infty)$ on the tensor product of the $\mathrm{M}(\infty)$-module $\mathcal{F}$ with itself. Clearly,

$$\varrho'(M) = \varrho(M) \otimes \varrho(M) \qquad \text{for every } M \in \mathrm{M}(\infty)$$

(because this is how one defines the tensor product of two modules over a monoid).

(c) Let $m \in \mathbb{Z}$. Let $M \in \mathrm{M}(\infty)$. Let $v \in \mathcal{F}^{(m)}$ and $w \in \mathcal{F}^{(m)}$. We are going to prove that $(S \circ \varrho'(M))(v \otimes w) = (\varrho'(M) \circ S)(v \otimes w)$.

Since $M \in \mathrm{M}(\infty) = \bigcup\limits_{N \in \mathbb{N}} i_N(\mathrm{M}(V_N))$ (by Remark 3.15.17 **(c)**), there exists an $R \in \mathbb{N}$ such that $M \in i_R(\mathrm{M}(V_R))$. Consider this R.

Since $v \in \mathcal{F}^{(m)} = \bigcup\limits_{\substack{N \in \mathbb{N};\\ N \geq R}} j_N^{(m)}\left(\wedge^{N+m+1}(V_N)\right)$ (by Proposition 3.15.20 **(b)**, applied to $Q = R$), there exists some $T \in \mathbb{N}$ such that $T \geq R$ and $v \in j_T^{(m)}\left(\wedge^{T+m+1}(V_T)\right)$. Consider this T.

Since $w \in \mathcal{F}^{(m)} = \bigcup\limits_{\substack{N \in \mathbb{N};\\ N \geq T}} j_N^{(m)}\left(\wedge^{N+m+1}(V_N)\right)$ (by Proposition 3.15.20 **(b)**, applied to $Q = T$), there exists some $P \in \mathbb{N}$ such that $P \geq T$ and $w \in j_P^{(m)}\left(\wedge^{P+m+1}(V_P)\right)$. Consider this P. There exists a $w' \in \wedge^{P+m+1}(V_P)$ such that $w = j_P^{(m)}(w')$ (because $w \in j_P^{(m)}\left(\wedge^{P+m+1}(V_P)\right)$). Consider this w'.

Applying Proposition 3.15.20 **(a)**, we get $j_0^{(m)}(\wedge^{0+m+1}(V_0)) \subseteq j_1^{(m)}(\wedge^{1+m+1}(V_1)) \subseteq j_2^{(m)}(\wedge^{2+m+1}(V_2)) \subseteq ....$ Thus, $j_T^{(m)}\left(\wedge^{T+m+1}(V_T)\right) \subseteq j_P^{(m)}\left(\wedge^{P+m+1}(V_P)\right)$ (since $T \leq P$), so that $v \in j_T^{(m)}\left(\wedge^{T+m+1}(V_T)\right) \subseteq j_P^{(m)}\left(\wedge^{P+m+1}(V_P)\right)$. Hence, there exists a $v' \in \wedge^{P+m+1}(V_P)$ such that $v = j_P^{(m)}(v')$. Consider this v'. Since $v = j_P^{(m)}(v')$ and $w = j_P^{(m)}(w')$, we have

$$v \otimes w = j_P^{(m)}(v') \otimes j_P^{(m)}(w') = \left(j_P^{(m)} \otimes j_P^{(m)}\right)(v' \otimes w'). \tag{235}$$

Since $R \leq T \leq P$, we have $i_R(\mathrm{M}(V_R)) \subseteq i_P(\mathrm{M}(V_P))$ (since Remark 3.15.17 **(b)** yields $i_0(\mathrm{M}(V_0)) \subseteq i_1(\mathrm{M}(V_1)) \subseteq i_2(\mathrm{M}(V_2)) \subseteq ...$). Thus, $M \in i_R(\mathrm{M}(V_R)) \subseteq i_P(\mathrm{M}(V_P))$. In other words, there exists an $A \in \mathrm{M}(V_P)$ such that $M = i_P(A)$. Consider this A.

In the following, we will write the action of $\mathrm{M}(\infty)$ on $\mathcal{F}$ as a left action. In other words, we will abbreviate $(\varrho(N))u$ by Nu, wherever $N \in \mathrm{M}(\infty)$ and $u \in \mathcal{F}$. Similarly, we will write the action of $\mathrm{M}(\infty)$ on $\mathcal{F} \otimes \mathcal{F}$ (this action is obtained by tensoring the $\mathrm{M}(\infty)$-module $\mathcal{F}$ with itself); this action satisfies $\varrho'(A) = \varrho(A) \otimes \varrho(A)$.

Let us also denote by ϱ the action of the monoid $\mathrm{M}(V_N)$ on $\wedge(V_N)$. Moreover, let us denote by ϱ' the action of the monoid $\mathrm{M}(V_N)$ on $\wedge(V_N) \otimes \wedge(V_N)$ (this action is obtained by tensoring the $\mathrm{M}(V_N)$-module $\wedge(V_N)$ with itself).

We notice that every $\ell \in \mathbb{Z}$ satisfies

$$(\varrho(M)) \circ j_P^{(\ell)} = j_P^{(\ell)} \circ (\varrho(A)). \tag{236}$$

[173]

Applying Lemma 3.15.24 to $N = P$, we obtain

$$\left(j_P^{(m+1)} \otimes j_P^{(m-1)}\right) \circ S_P^{(P+m+1)} = S \circ \left(j_P^{(m)} \otimes j_P^{(m)}\right). \tag{237}$$

On the other hand, the map $S_P^{(P+m+1)}$ is $\mathrm{M}(\infty)$-invariant (by Lemma 3.15.23 **(c)**, applied to $N = P$ and $k = P + m + 1$), so that

$$S_P^{(P+m+1)} \circ (\varrho' (A)) = (\varrho' (A)) \circ S_P^{(P+m+1)}.$$

Since $\varrho' (A) = \varrho (A) \otimes \varrho (A)$, this rewrites as

$$S_P^{(P+m+1)} \circ (\varrho (A) \otimes \varrho (A)) = (\varrho (A) \otimes \varrho (A)) \circ S_P^{(P+m+1)}. \tag{238}$$

Comparing

$$\begin{aligned}
& S \circ \underbrace{(\varrho' (M))}_{=\varrho(M)\otimes\varrho(M)} \circ \left(j_P^{(m)} \otimes j_P^{(m)}\right) \\
&= S \circ \underbrace{(\varrho (M) \otimes \varrho (M)) \circ \left(j_P^{(m)} \otimes j_P^{(m)}\right)}_{=\left((\varrho(M))\circ j_P^{(m)}\right)\otimes\left((\varrho(M))\circ j_P^{(m)}\right)} \\
&= S \circ \left(\underbrace{\left((\varrho (M)) \circ j_P^{(m)}\right)}_{\substack{=j_P^{(m)}\circ(\varrho(A)) \\ \text{(by (236), applied to } \ell=m)}} \otimes \underbrace{\left((\varrho (M)) \circ j_P^{(m)}\right)}_{\substack{=j_P^{(m)}\circ(\varrho(A)) \\ \text{(by (236), applied to } \ell=m)}} \right) \\
&= S \circ \underbrace{\left(\left(j_P^{(m)} \circ (\varrho (A))\right) \otimes \left(j_P^{(m)} \circ (\varrho (A))\right)\right)}_{=\left(j_P^{(m)}\otimes j_P^{(m)}\right)\circ(\varrho(A)\otimes\varrho(A))} = \underbrace{S \circ \left(j_P^{(m)} \otimes j_P^{(m)}\right)}_{\substack{=\left(j_P^{(m+1)}\otimes j_P^{(m-1)}\right)\circ S_P^{(P+m+1)} \\ \text{(by (237))}}} \circ (\varrho (A) \otimes \varrho (A)) \\
&= \left(j_P^{(m+1)} \otimes j_P^{(m-1)}\right) \circ \underbrace{S_P^{(P+m+1)} \circ (\varrho (A) \otimes \varrho (A))}_{\substack{=(\varrho(A)\otimes\varrho(A))\circ S_P^{(P+m+1)} \\ \text{(by (238))}}} = \left(j_P^{(m+1)} \otimes j_P^{(m-1)}\right) \circ (\varrho (A) \otimes \varrho (A)) \circ S_P^{(P+m+1)}
\end{aligned}$$

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[173] *Proof of (236):* Let $\ell \in \mathbb{Z}$. Every $u \in \mathcal{F}^{(\ell)}$ satisfies

$$\begin{aligned}
\left((\varrho (M)) \circ j_P^{(\ell)}\right) (u) &= (\varrho (M)) \left(j_P^{(\ell)} (u)\right) = \underbrace{M}_{=i_P(A)} \cdot j_P^{(\ell)} (u) = i_P (A) \cdot j_P^{(\ell)} u = j_P^{(\ell)} \underbrace{(Au)}_{=(\varrho(A))u} \\
&\qquad \text{(by Proposition 3.15.19, applied to } P \text{ and } \ell \text{ instead of } N \text{ and } m) \\
&= j_P^{(\ell)} ((\varrho (A)) u) = \left(j_P^{(\ell)} \circ (\varrho (A))\right) (u).
\end{aligned}$$

Thus, $(\varrho (M)) \circ j_P^{(\ell)} = j_P^{(\ell)} \circ (\varrho (A))$, so that (236) is proven.

with

$$\begin{aligned}
&\underbrace{\left(\varrho'\left(M\right)\right)}_{=\varrho(M)\otimes\varrho(M)} \circ \underbrace{S\circ\left(j_P^{(m)}\otimes j_P^{(m)}\right)}_{\substack{=\left(j_P^{(m+1)}\otimes j_P^{(m-1)}\right)\circ S_P^{(P+m+1)}\\ \text{(by (237))}}}\\
&=\underbrace{\left(\varrho\left(M\right)\otimes\varrho\left(M\right)\right)\circ\left(j_P^{(m+1)}\otimes j_P^{(m-1)}\right)}_{=\left((\varrho(M))\circ j_P^{(m+1)}\right)\otimes\left((\varrho(M))\circ j_P^{(m-1)}\right)}\circ S_P^{(P+m+1)}\\
&=\left(\underbrace{\left(\left(\varrho\left(M\right)\right)\circ j_P^{(m+1)}\right)}_{\substack{=j_P^{(m+1)}\circ(\varrho(A))\\ \text{(by (236), applied to }\ell=m+1)}}\otimes\underbrace{\left(\left(\varrho\left(M\right)\right)\circ j_P^{(m-1)}\right)}_{\substack{=j_P^{(m-1)}\circ(\varrho(A))\\ \text{(by (236), applied to }\ell=m-1)}}\right)\circ S_P^{(P+m+1)}\\
&=\underbrace{\left(\left(j_P^{(m+1)}\circ\left(\varrho\left(A\right)\right)\right)\otimes\left(j_P^{(m-1)}\circ\left(\varrho\left(A\right)\right)\right)\right)}_{=\left(j_P^{(m+1)}\otimes j_P^{(m-1)}\right)\circ(\varrho(A)\otimes\varrho(A))}\circ S_P^{(P+m+1)}\\
&=\left(j_P^{(m+1)}\otimes j_P^{(m-1)}\right)\circ\left(\varrho\left(A\right)\otimes\varrho\left(A\right)\right)\circ S_P^{(P+m+1)},
\end{aligned}$$

we obtain

$$S\circ\left(\varrho'\left(M\right)\right)\circ\left(j_P^{(m)}\otimes j_P^{(m)}\right)=\left(\varrho'\left(M\right)\right)\circ S\circ\left(j_P^{(m)}\otimes j_P^{(m)}\right). \tag{239}$$

Now,

$$\begin{aligned}
&\left(S\circ\left(\varrho'\left(M\right)\right)\right)\underbrace{\left(v\otimes w\right)}_{\substack{=\left(j_P^{(m)}\otimes j_P^{(m)}\right)(v'\otimes w')\\ \text{(by (235))}}}\\
&=\left(S\circ\left(\varrho'\left(M\right)\right)\right)\left(\left(j_P^{(m)}\otimes j_P^{(m)}\right)\left(v'\otimes w'\right)\right)=\underbrace{\left(S\circ\left(\varrho'\left(M\right)\right)\circ\left(j_P^{(m)}\otimes j_P^{(m)}\right)\right)}_{\substack{=(\varrho'(M))\circ S\circ\left(j_P^{(m)}\otimes j_P^{(m)}\right)\\ \text{(by (239))}}}\left(v'\otimes w'\right)\\
&=\left(\left(\varrho'\left(M\right)\right)\circ S\circ\left(j_P^{(m)}\otimes j_P^{(m)}\right)\right)\left(v'\otimes w'\right)=\left(\left(\varrho'\left(M\right)\right)\circ S\right)\underbrace{\left(\left(j_P^{(m)}\otimes j_P^{(m)}\right)\left(v'\otimes w'\right)\right)}_{\substack{=v\otimes w\\ \text{(by (235))}}}\\
&=\left(\left(\varrho'\left(M\right)\right)\circ S\right)\left(v\otimes w\right).
\end{aligned}$$

Now forget that we fixed v and w. We thus have proven that $\left(S\circ\varrho'\left(M\right)\right)\left(v\otimes w\right)=\left(\varrho'\left(M\right)\circ S\right)\left(v\otimes w\right)$ for every $v\in\mathcal{F}^{(m)}$ and $w\in\mathcal{F}^{(m)}$. In other words, the two maps $S\circ\varrho'\left(M\right)$ and $\varrho'\left(M\right)\circ S$ are equal to each other on every pure tensor in $\mathcal{F}^{(m)}\otimes\mathcal{F}^{(m)}$. Thus, these two maps must be identical (on $\mathcal{F}^{(m)}\otimes\mathcal{F}^{(m)}$). In other words, $S\circ\varrho'\left(M\right)=\varrho'\left(M\right)\circ S$.

Now forget that we fixed M. We have proven that $S\circ\varrho'\left(M\right)=\varrho'\left(M\right)\circ S$ for every $M\in\mathrm{M}\left(\infty\right)$. In other words, S is $\mathrm{M}\left(\infty\right)$-invariant. This proves Theorem 3.15.13 **(c)**.

(a) Theorem 3.15.13 **(a)** follows from Theorem 3.15.13 **(c)** since $\mathrm{GL}\left(\infty\right)\subseteq\mathrm{M}\left(\infty\right)$.

(b) $\Longrightarrow$: Assume that $\tau \in \Omega$. We want to prove that $S(\tau \otimes \tau) = 0$.

Since $\Omega = \mathrm{GL}(\infty) \cdot \psi_0$, we have $\tau \in \Omega = \mathrm{GL}(\infty) \cdot \psi_0$. In other words, there exists $A \in \mathrm{GL}(\infty)$ such that $\tau = A\psi_0$. Consider this A.

It is easy to see that

$$\overset{\vee}{v_i}(\psi_0) = 0 \qquad \text{for every integer } i > 0. \tag{240}$$

[174] Also,

$$\widehat{v_i}(\psi_0) = 0 \qquad \text{for every integer } i \leq 0. \tag{241}$$

[175]

Since $S = \sum\limits_{i \in \mathbb{Z}} \widehat{v_i} \otimes \overset{\vee}{v_i}$, we have

$$\begin{aligned} S(\psi_0 \otimes \psi_0) &= \sum_{i \in \mathbb{Z}} \underbrace{\left(\widehat{v_i} \otimes \overset{\vee}{v_i}\right)(\psi_0 \otimes \psi_0)}_{=\widehat{v_i}(\psi_0) \otimes \overset{\vee}{v_i}(\psi_0)} = \sum_{i \in \mathbb{Z}} \widehat{v_i}(\psi_0) \otimes \overset{\vee}{v_i}(\psi_0) \\ &= \sum_{\substack{i \in \mathbb{Z};\\ i \leq 0}} \underbrace{\widehat{v_i}(\psi_0)}_{\substack{=0\\ \text{(by (241))}}} \otimes \overset{\vee}{v_i}(\psi_0) + \sum_{\substack{i \in \mathbb{Z};\\ i > 0}} \widehat{v_i}(\psi_0) \otimes \underbrace{\overset{\vee}{v_i}(\psi_0)}_{\substack{=0\\ \text{(by (240))}}} \\ &= \underbrace{\sum_{\substack{i \in \mathbb{Z};\\ i \leq 0}} 0 \otimes \overset{\vee}{v_i}(\psi_0)}_{=0} + \underbrace{\sum_{\substack{i \in \mathbb{Z};\\ i > 0}} \widehat{v_i}(\psi_0) \otimes 0}_{=0} = 0. \end{aligned}$$

Now, since $\tau = A\psi_0$, we have $\tau \otimes \tau = A\psi_0 \otimes A\psi_0 = A(\psi_0 \otimes \psi_0)$, so that

$$\begin{aligned} S(\tau \otimes \tau) &= S(A(\psi_0 \otimes \psi_0)) \\ &= A \cdot \underbrace{S(\psi_0 \otimes \psi_0)}_{=0} \qquad \text{(since } S \text{ is } \mathrm{M}(\infty)\text{-linear (by Theorem 3.15.13 } \textbf{(c)}\text{))} \\ &= A \cdot 0 = 0. \end{aligned}$$

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[174] *Proof of (240):* Let $i > 0$ be an integer. Then,

$$\begin{aligned} \overset{\vee}{v_i}(\psi_0) &= \overset{\vee}{v_i}(v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...) \qquad (\text{since } \psi_0 = v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...) \\ &= \begin{cases} 0, & \text{if } i \notin \{0, -1, -2, ...\}; \\ (-1)^j v_0 \wedge v_{-1} \wedge v_{-2} \wedge ... \wedge v_{-(j-1)} \wedge v_{-(j+1)} \wedge v_{-(j+2)} \wedge ..., & \text{if } i \in \{0, -1, -2, ...\} \end{cases} \\ &\qquad \left(\text{by the definition of } \overset{\vee}{v_i}\right), \end{aligned}$$

where, in the case $i \in \{0, -1, -2, ...\}$, we denote by j the integer k satisfying $-k = i$. Since $i \notin \{0, -1, -2, ...\}$ (because $i > 0$), this simplifies to $\overset{\vee}{v_i}(\psi_0) = 0$. This proves (240).

[175] *Proof of (241):* Let $i \leq 0$ be an integer. Since $\psi_0 = v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...$, we have

$$\widehat{v_i}(\psi_0) = \widehat{v_i}(v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...) = v_i \wedge v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...$$

(by the definition of $\widehat{v_i}$). But the semiinfinite wedge $v_i \wedge v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...$ contains the vector v_i twice (in fact, it contains the vector v_i once in its very beginning, and once again in its $v_0 \wedge v_{-1} \wedge v_{-2} \wedge ...$ part (since $i \leq 0$)), and thus must equal 0 (since any semiinfinite wedge which contains a vector more than once must equal 0). We thus have

$$\widehat{v_i}(\psi_0) = v_i \wedge v_0 \wedge v_{-1} \wedge v_{-2} \wedge ... = 0.$$

This proves (241).

This proves the $\Longrightarrow$ direction of Theorem 3.15.13 **(b)**.

$\Longleftarrow$: Let $\tau \in \mathcal{F}^{(0)}$ be such that $S(\tau \otimes \tau) = 0$. We want to prove that $\tau \in \Omega$.

Since $\tau \in \mathcal{F}^{(0)} = \bigcup\limits_{\substack{N\in\mathbb{N};\\ N\geq 0}} j_N^{(0)}\left(\wedge^{N+0+1}(V_N)\right)$ (by Proposition 3.15.20 **(b)**, applied to $m = 0$ and $Q = 0$), there exists some $N \in \mathbb{N}$ such that $N \geq 0$ and $\tau \in j_N^{(0)}\left(\wedge^{N+0+1}(V_N)\right)$. Consider this N.

Lemma 3.15.24 (applied to $m = 0$) yields

$$\left(j_N^{(1)} \otimes j_N^{(-1)}\right) \circ S_N^{(N+1)} = S \circ \left(j_N^{(0)} \otimes j_N^{(0)}\right). \tag{242}$$

Recall that the map $j_N^{(m)}$ is injective for every $m \in \mathbb{Z}$. In particular, the maps $j_N^{(1)}$ and $j_N^{(-1)}$ are injective, so that the map $j_N^{(1)} \otimes j_N^{(-1)}$ is also injective.

But $\tau \in j_N^{(0)}\left(\wedge^{N+0+1}(V_N)\right) = j_N^{(0)}\left(\wedge^{N+1}(V_N)\right)$. In other words, there exists some $\tau' \in \wedge^{N+1}(V_N)$ such that $\tau = j_N^{(0)}(\tau')$. Consider this τ'.

Since $\tau = j_N^{(0)}(\tau')$, we have $\tau \otimes \tau = j_N^{(0)}(\tau') \otimes j_N^{(0)}(\tau') = \left(j_N^{(0)} \otimes j_N^{(0)}\right)(\tau' \otimes \tau')$, so that

$$\begin{aligned} S(\tau\otimes\tau) &= S\left(\left(j_N^{(0)} \otimes j_N^{(0)}\right)(\tau' \otimes \tau')\right) = \underbrace{\left(S \circ \left(j_N^{(0)} \otimes j_N^{(0)}\right)\right)}_{\substack{=\left(j_N^{(1)}\otimes j_N^{(-1)}\right)\circ S_N^{(N+1)} \\ \text{(by (242))}}}(\tau' \otimes \tau') \\ &= \left(\left(j_N^{(1)} \otimes j_N^{(-1)}\right) \circ S_N^{(N+1)}\right)(\tau' \otimes \tau') = \left(j_N^{(1)} \otimes j_N^{(-1)}\right)\left(S_N^{(N+1)}(\tau' \otimes \tau')\right). \end{aligned}$$

Compared with $S(\tau \otimes \tau) = 0$, this yields $\left(j_N^{(1)} \otimes j_N^{(-1)}\right)\left(S_N^{(N+1)}(\tau' \otimes \tau')\right) = 0$. Since $j_N^{(1)} \otimes j_N^{(-1)}$ is injective, this yields $S_N^{(N+1)}(\tau' \otimes \tau') = 0$. But Lemma 3.15.23 **(b)** (applied to $N+1$ and τ' instead of k and τ) yields that τ' belongs to $\Omega_N^{(N+1)}$ if and only if $S_N^{(N+1)}(\tau' \otimes \tau') = 0$. Since we know that $S_N^{(N+1)}(\tau' \otimes \tau') = 0$, we can thus conclude that τ' belongs to $\Omega_N^{(N+1)}$. Since $\Omega_N^{(N+1)}$ is the orbit of $v_N \wedge v_{N-1} \wedge ... \wedge v_{N-(N+1)+1}$ under the action of $\mathrm{GL}(V_N)$ (this is how $\Omega_N^{(N+1)}$ was defined), this yields that τ' belongs to the orbit of $v_N \wedge v_{N-1} \wedge ... \wedge v_{N-(N+1)+1}$ under the action of $\mathrm{GL}(V_N)$. In other words, there exists some $A \in \mathrm{GL}(V_N)$ such that $\tau' = A \cdot \left(v_N \wedge v_{N-1} \wedge ... \wedge v_{N-(N+1)+1}\right)$. Consider this A.

We have $\tau' = A \cdot \left(v_N \wedge v_{N-1} \wedge ... \wedge \underbrace{v_{N-(N+1)+1}}_{=v_0}\right) = A \cdot (v_N \wedge v_{N-1} \wedge ... \wedge v_0)$.

There clearly exists an invertible linear map $B \in \mathrm{GL}(V_N)$ which sends v_N, v_{N-1}, ..., v_0 to v_0, v_{-1}, ..., v_{-N}, respectively[176]. Pick such a B. Then, $B \cdot (v_N \wedge v_{N-1} \wedge ... \wedge v_0) = v_0 \wedge v_{-1} \wedge ... \wedge v_{-N}$ (since B sends v_N, v_{N-1}, ..., v_0 to v_0, v_{-1}, ..., v_{-N}, respectively), so that $B^{-1} \cdot (v_0 \wedge v_{-1} \wedge ... \wedge v_{-N}) = v_N \wedge v_{N-1} \wedge ... \wedge v_0$ and thus

$$A \underbrace{B^{-1} \cdot (v_0 \wedge v_{-1} \wedge ... \wedge v_{-N})}_{=v_N\wedge v_{N-1}\wedge ...\wedge v_0} = A \cdot (v_N \wedge v_{N-1} \wedge ... \wedge v_0) = \tau'.$$

[176] *Proof.* Since $(v_N, v_{N-1}, ..., v_{-N})$ is a basis of V_N, there exists a linear map $B \in \mathrm{End}(V_N)$ which sends v_i to $\begin{cases} v_{i-N}, & \text{if } i \geq 0; \\ v_{-i}, & \text{if } i < 0 \end{cases}$ for every $i \in \{N, N-1, ..., -N\}$. This linear map B is invertible (since it permutes the elements of the basis $(v_N, v_{N-1}, ..., v_{-N})$ of V_N), and thus lies in $\mathrm{GL}(V_N)$, and it clearly sends v_N, v_{N-1}, ..., v_0 to v_0, v_{-1}, ..., v_{-N}, respectively. Qed.

Let $M = i_N\left(AB^{-1}\right)$. Then, $M = i_N \underbrace{\left(AB^{-1}\right)}_{\in \mathrm{GL}(V_N)} \in i_N\left(\mathrm{GL}\left(V_N\right)\right) \subseteq \mathrm{GL}\left(\infty\right)$. Also,

$$j_N^{(0)}\left(v_0 \wedge v_{-1} \wedge ... \wedge v_{-N}\right) = v_0 \wedge v_{-1} \wedge ... \wedge v_{-N} \wedge v_{-N-1} \wedge v_{-N-2} \wedge v_{-N-3} \wedge ...$$
$$\left(\text{by the definition of } j_N^{(0)}\right)$$
$$= v_0 \wedge v_{-1} \wedge v_{-2} \wedge ... = \psi_0.$$

Now,

$$\underbrace{M}_{=i_N(AB^{-1})} \cdot \underbrace{\psi_0}_{=j_N^{(0)}(v_0\wedge v_{-1}\wedge ...\wedge v_{-N})}$$
$$= i_N\left(AB^{-1}\right) \cdot j_N^{(0)}\left(v_0 \wedge v_{-1} \wedge ... \wedge v_{-N}\right) = j_N^{(0)}\left(\underbrace{AB^{-1} \cdot \left(v_0 \wedge v_{-1} \wedge ... \wedge v_{-N}\right)}_{=\tau'}\right)$$
$$\left(\text{by Proposition 3.15.19, applied to } 0,\ AB^{-1} \text{ and } v_0 \wedge v_{-1} \wedge ... \wedge v_{-N} \text{ instead of } m,\ A \text{ and } u\right)$$
$$= j_N^{(0)}\left(\tau'\right) = \tau.$$

Thus, $\tau = \underbrace{M}_{\in \mathrm{GL}(\infty)} \cdot \psi_0 \in \mathrm{GL}\left(\infty\right) \cdot \psi_0 = \Omega$. This proves the $\Longleftarrow$ direction of Theorem 3.15.13 **(b)**.

3.15.4. The semiinfinite Grassmannian

Denote $\Omega / \mathbb{C}^\times$ by Gr; this is called the *semiinfinite Grassmannian.*

Think of the space V as $\mathbb{C}\left[t, t^{-1}\right]$ (by identifying v_i with t^{-i}). Then, $\left\langle v_0, v_{-1}, v_{-2}, ...\right\rangle = \mathbb{C}\left[t\right]$.

Exercise: Then, Gr is the set

$$\left\{E \subseteq V \text{ subspace } \mid \left(\begin{array}{c} E \supseteq t^N \mathbb{C}\left[t\right] \text{ for sufficiently large } N \text{, and} \\ \dim\left(E / t^N \mathbb{C}\left[t\right]\right) = N \text{ for sufficiently large } N \end{array}\right)\right\}.$$

[177] (Note that when the relations $E \supseteq t^N \mathbb{C}\left[t\right]$ and $\dim\left(E / t^N \mathbb{C}\left[t\right]\right) = N$ hold for *some* N, it is easy to see that they also hold for all greater N.)

We can also replace $\mathbb{C}\left[t, t^{-1}\right]$ with $\mathbb{C}\left((t)\right)$ (the formal Laurent series), and then

$$\mathrm{Gr} = \left\{E \subseteq V \text{ subspace } \mid \left(\begin{array}{c} E \supseteq t^N \mathbb{C}\left[[t]\right] \text{ for sufficiently large } N \text{, and} \\ \dim\left(E / t^N \mathbb{C}\left[[t]\right]\right) = N \text{ for sufficiently large } N \end{array}\right)\right\}.$$

For any $E \in \mathrm{Gr}$, there exists some $N \in \mathbb{N}$ such that $t^N \mathbb{C}\left[t\right] \subseteq E \subseteq t^{-N} \mathbb{C}\left[t\right]$, so that the quotient $E / t^N \mathbb{C}\left[t\right] \subseteq t^{-N} \mathbb{C}\left[t\right] / t^N \mathbb{C}\left[t\right] \cong \mathbb{C}^{2N}$.

Thus, $\mathrm{Gr} = \bigcup\limits_{N \geq 1} \mathrm{Gr}\left(N, 2N\right)$ (a nested union). (By a variation of this construction, $\mathrm{Gr} = \bigcup\limits_{N \geq 1} \bigcup\limits_{M \geq 1} \mathrm{Gr}\left(N, N+M\right)$.)

[177] Here, "subspace" means "$\mathbb{C}$-vector subspace".

3.15.5. The preimage of the Grassmannian under the Boson-Fermion correspondence: the Hirota bilinear relations

Now, how do we actually use these things to find solutions to the Kadomtsev-Petviashvili equations and other integrable systems?

By Theorem 3.15.13 **(b)**, the elements of Ω are exactly the nonzero elements τ of $\mathcal{F}^{(0)}$ satisfying $S(\tau \otimes \tau) = 0$. We might wonder what happens to these elements under the Boson-Fermion correspondence σ: how can their preimages under σ be described? In other words, can we find a necessary and sufficient condition for a polynomial $\tau \in \mathcal{B}^{(0)}$ to satisfy $\sigma(\tau) \in \Omega$ (without using σ in this very condition)?

Recall the power series $X(u) = \sum\limits_{i\in\mathbb{Z}} \xi_i u^i$ and $X^*(u) = \sum\limits_{i\in\mathbb{Z}} \xi_i^* u^{-i}$ defined in Definition 3.11.1. These power series "act" on the fermionic space $\mathcal{F}$. The word "act" has been put in inverted commas here because it is not the power series but their coefficients which really act on $\mathcal{F}$, whereas the power series themselves only map elements of $\mathcal{F}$ to elements of $\mathcal{F}((u))$. This, actually, is an important observation:

$$\text{every } \omega \in \mathcal{F} \text{ satisfies } X(u)\,\omega \in \mathcal{F}((u)) \text{ and } X^*(u)\,\omega \in \mathcal{F}((u)). \tag{243}$$

[178]

Let $\tau \in \mathcal{B}^{(0)}$ be arbitrary. We want to find an equivalent form for the equation $S(\sigma(\tau) \otimes \sigma(\tau)) = 0$ which does not refer to σ.

Let us give two definitions first:

Definition 3.15.25. Let A and B be two $\mathbb{C}$-vector spaces, and let u be a symbol. Then, the map

$$A((u)) \times B((u)) \to (A \otimes B)((u)),$$
$$\left(\sum_{i\in\mathbb{Z}} a_i u^i, \sum_{i\in\mathbb{Z}} b_i u^i\right) \mapsto \sum_{i\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_j \otimes b_{i-j}\right) u^i$$
$$\text{(where all } a_i \text{ lie in } A \text{ and all } b_i \text{ lie in } B\text{)}$$

is well-defined (in fact, it is easy to see that for any Laurent series $\sum\limits_{i\in\mathbb{Z}} a_i u^i \in A((u))$ with all a_i lying in A, any Laurent series $\sum\limits_{i\in\mathbb{Z}} b_i u^i \in B((u))$ with all b_i lying in B, and any integer $i \in \mathbb{Z}$, the sum $\sum\limits_{j\in\mathbb{Z}} a_j \otimes b_{i-j}$ has only finitely many addends and vanishes if i is small enough) and $\mathbb{C}$-bilinear. Hence, it induces a $\mathbb{C}$-linear map

$$A((u)) \otimes B((u)) \to (A \otimes B)((u)),$$
$$\left(\sum_{i\in\mathbb{Z}} a_i u^i\right) \otimes \left(\sum_{i\in\mathbb{Z}} b_i u^i\right) \mapsto \sum_{i\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} a_j \otimes b_{i-j}\right) u^i$$
$$\text{(where all } a_i \text{ lie in } A \text{ and all } b_i \text{ lie in } B\text{)}.$$

This map will be denoted by $\Omega_{A,B,u}$.

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[178] *Proof of (243):* Let $\omega \in \mathcal{F}$. Since $X(u) = \sum\limits_{i\in\mathbb{Z}} \xi_i u^i$, we have $X(u)\,\omega = \sum\limits_{i\in\mathbb{Z}} \xi_i(\omega)\, u^i \in \mathcal{F}((u))$, because every sufficiently small $i \in \mathbb{Z}$ satisfies $\xi_i(\omega) = 0$ (this is easy to see). On the other hand, since $X^*(u) = \sum\limits_{i\in\mathbb{Z}} \xi_i^* u^{-i}$, we have $X^*(u) = \sum\limits_{i\in\mathbb{Z}} \xi_i^*(\omega)\, u^{-i} \in \mathcal{F}((u))$, since every sufficiently high $i \in \mathbb{Z}$ satisfies $\xi_i^*(\omega) = 0$ (this, again, is easy to see). This proves (243).

More can be said about the map $\Omega_{A,B,u}$: It factors as a composition of the canonical projection $A((u)) \otimes B((u)) \to A((u)) \otimes_{\mathbb{C}((u))} B((u))$ with a $\mathbb{C}((u))$-linear map $A((u)) \otimes_{\mathbb{C}((u))} B((u)) \to (A \otimes B)((u))$. We won't need this in the following. What we will need is the following observation:

Remark 3.15.26. Let A and B be two $\mathbb{C}$-algebras, and let u be a symbol. Then, the map $\Omega_{A,B,u}$ is $A \otimes B$-linear.

Definition 3.15.27. Let A be a $\mathbb{C}$-vector space, and let u be a symbol. Then, $\mathrm{CT}_u : A((u)) \to A$ will denote the map which sends every Laurent series $\sum\limits_{i\in\mathbb{Z}} a_i u^i \in A((u))$ (where all a_i lie in A) to $a_0 \in A$. The image of a Laurent series α under CT_u will be called the **constant term** of α. The map CT_u is clearly A-linear.

This notion of "constant term" we have thus defined for Laurent series is, of course, completely analogous to the one used for polynomials and formal power series. The label CT_u is an abbreviation for "constant term with respect to the variable u".

Now, for every $\omega \in \mathcal{F}^{(0)}$ and $\rho \in \mathcal{F}^{(0)}$, we have

$$S(\omega \otimes \rho) = \mathrm{CT}_u \left(\Omega_{\mathcal{F},\mathcal{F},u}\left(X(u)\,\omega \otimes X^*(u)\,\rho\right)\right). \tag{244}$$

[179]

Now, let $\tau \in \mathcal{B}^{(0)}$. Due to (243) (applied to $\omega = \sigma(\tau)$), we have $X(u)\,\sigma(\tau) \in \mathcal{F}((u))$ and $X^*(u)\,\sigma(\tau) \in \mathcal{F}((u))$.

[179] *Proof of (244):* Let $\omega \in \mathcal{F}^{(0)}$ and $\rho \in \mathcal{F}^{(0)}$. Since $X(u) = \sum\limits_{i\in\mathbb{Z}} \xi_i u^i$ and $X^*(u) = \sum\limits_{i\in\mathbb{Z}} \xi_i^* u^{-i} = \sum\limits_{i\in\mathbb{Z}} \xi_{-i}^* u^i$ (here, we substituted $-i$ for i in the sum), we have

$$X(u)\,\omega \otimes X^*(u)\,\rho = \left(\sum_{i\in\mathbb{Z}} \xi_i u^i\right)\omega \otimes \left(\sum_{i\in\mathbb{Z}} \xi_{-i}^* u^i\right)\rho = \left(\sum_{i\in\mathbb{Z}} \xi_i(\omega)\,u^i\right) \otimes \left(\sum_{i\in\mathbb{Z}} \xi_{-i}^*(\rho)\,u^i\right),$$

so that

$$\begin{aligned}
&\Omega_{\mathcal{F},\mathcal{F},u}\left(X(u)\,\omega \otimes X^*(u)\,\rho\right) \\
&= \Omega_{\mathcal{F},\mathcal{F},u}\left(\left(\sum_{i\in\mathbb{Z}} \xi_i(\omega)\,u^i\right) \otimes \left(\sum_{i\in\mathbb{Z}} \xi_{-i}^*(\rho)\,u^i\right)\right) = \sum_{i\in\mathbb{Z}}\left(\sum_{j\in\mathbb{Z}} \xi_j(\omega) \otimes \xi_{-(i-j)}^*(\rho)\right)u^i
\end{aligned}$$

(by the definition of $\Omega_{\mathcal{F},\mathcal{F},u}$). Thus (by the definition of CT_u) we have

$$\begin{aligned}
&\mathrm{CT}_u\left(\Omega_{\mathcal{F},\mathcal{F},u}\left(X(u)\,\omega \otimes X^*(u)\,\rho\right)\right) \\
&= \sum_{j\in\mathbb{Z}} \xi_j(\omega) \otimes \xi_{-(0-j)}^*(\rho) = \sum_{j\in\mathbb{Z}} \xi_j(\omega) \otimes \xi_j^*(\rho) = \sum_{i\in\mathbb{Z}} \underbrace{\xi_i}_{=\widehat{v_i}}(\omega) \otimes \underbrace{\xi_i^*}_{=\overset{\vee}{v_i}}(\rho) \\
&\qquad\qquad \text{(here, we substituted } i \text{ for } j \text{ in the sum)} \\
&= \sum_{i\in\mathbb{Z}} \widehat{v_i}(\omega) \otimes \overset{\vee}{v_i}(\rho) = \underbrace{\left(\sum_{i\in\mathbb{Z}} \widehat{v_i} \otimes \overset{\vee}{v_i}\right)}_{\substack{=S \\ \text{(because this is how } S \text{ was defined)}}}(\omega \otimes \rho) = S(\omega \otimes \rho),
\end{aligned}$$

so that (244) is proven.

Now, let us abuse notation and denote by σ the map from $\mathcal{B}((u))$ to $\mathcal{F}((u))$ which is canonically induced by the Boson-Fermion correspondence $\sigma : \mathcal{B} \to \mathcal{F}$. Then, of course, this new map $\sigma : \mathcal{B}((u)) \to \mathcal{F}((u))$ is also an isomorphism. Then, the equalities $\Gamma(u) = \sigma^{-1} \circ X(u) \circ \sigma$ and $\Gamma^*(u) = \sigma^{-1} \circ X^*(u) \circ \sigma$ (from Definition 3.11.1) are not just abbreviations for termwise equalities (as we explained them back in Definition 3.11.1), but also hold literally (if we interpret σ to mean our isomorphism $\sigma : \mathcal{B}((u)) \to \mathcal{F}((u))$ rather than the original Boson-Fermion correspondence $\sigma : \mathcal{B} \to \mathcal{F}$). As a consequence, $\sigma \circ \Gamma(u) = X(u) \circ \sigma$ and $\sigma \circ \Gamma^*(u) = X^*(u) \circ \sigma$. Thus,

$$\sigma(\Gamma(u)\tau) = \underbrace{(\sigma \circ \Gamma(u))}_{=X(u)\circ\sigma} \tau = (X(u) \circ \sigma)\tau = X(u)\sigma(\tau)$$

and

$$\sigma(\Gamma^*(u)\tau) = \underbrace{(\sigma \circ \Gamma^*(u))}_{=X^*(u)\circ\sigma} \tau = (X^*(u) \circ \sigma)\tau = X^*(u)\sigma(\tau),$$

so that

$$\underbrace{X(u)\sigma(\tau)}_{=\sigma(\Gamma(u)\tau)} \otimes \underbrace{X^*(u)\sigma(\tau)}_{=\sigma(\Gamma^*(u)\tau)} = \sigma(\Gamma(u)\tau) \otimes \sigma(\Gamma^*(u)\tau) = (\sigma \otimes \sigma)(\Gamma(u)\tau \otimes \Gamma^*(u)\tau).$$

Now,

$$\begin{aligned}
S(\sigma(\tau) \otimes \sigma(\tau)) &= \mathrm{CT}_u \left(\Omega_{\mathcal{F},\mathcal{F},u} \underbrace{(X(u)\sigma(\tau) \otimes X^*(u)\sigma(\tau))}_{=(\sigma\otimes\sigma)(\Gamma(u)\tau\otimes\Gamma^*(u)\tau)} \right) \\
&\qquad \text{(by (244), applied to } \omega = \sigma(\tau) \text{ and } \rho = \sigma(\tau)) \\
&= \mathrm{CT}_u \left(\Omega_{\mathcal{F},\mathcal{F},u} ((\sigma \otimes \sigma)(\Gamma(u)\tau \otimes \Gamma^*(u)\tau)) \right) \\
&= \underbrace{(\mathrm{CT}_u \circ \Omega_{\mathcal{F},\mathcal{F},u} \circ (\sigma \otimes \sigma))}_{\substack{=(\sigma\otimes\sigma)\circ\mathrm{CT}_u \circ\Omega_{\mathcal{B},\mathcal{B},u} \\ \text{(since } \mathrm{CT}_u \text{ and } \Omega_{A,B,u} \text{ are functorial)}}} (\Gamma(u)\tau \otimes \Gamma^*(u)\tau) \\
&= ((\sigma \otimes \sigma) \circ \mathrm{CT}_u \circ \Omega_{\mathcal{B},\mathcal{B},u})(\Gamma(u)\tau \otimes \Gamma^*(u)\tau) \\
&= (\sigma \otimes \sigma)(\mathrm{CT}_u(\Omega_{\mathcal{B},\mathcal{B},u}(\Gamma(u)\tau \otimes \Gamma^*(u)\tau))).
\end{aligned}$$

Therefore, the equation $S(\sigma(\tau) \otimes \sigma(\tau)) = 0$ is equivalent to $(\sigma \otimes \sigma)(\mathrm{CT}_u(\Omega_{\mathcal{B},\mathcal{B},u}(\Gamma(u)\tau \otimes \Gamma^*(u)\tau))) = 0$. This latter equation, in turn, is equivalent to $\mathrm{CT}_u(\Omega_{\mathcal{B},\mathcal{B},u}(\Gamma(u)\tau \otimes \Gamma^*(u)\tau)) = 0$ (since $\sigma \otimes \sigma$ is an isomorphism[180]). This, in turn, is equivalent to $(z^{-1} \otimes z) \cdot \mathrm{CT}_u(\Omega_{\mathcal{B},\mathcal{B},u}(\Gamma(u)\tau \otimes \Gamma^*(u)\tau)) = 0$ (because $z^{-1} \otimes z$

[180]because σ is an isomorphism

is an invertible element of $\mathcal{B}\otimes\mathcal{B}$). Since

$$\begin{aligned}
&\left(z^{-1}\otimes z\right)\cdot\operatorname{CT}_u\left(\Omega_{\mathcal{B},\mathcal{B},u}\left(\Gamma\left(u\right)\tau\otimes\Gamma^*\left(u\right)\tau\right)\right)\\
&=\operatorname{CT}_u\left(\underbrace{\left(z^{-1}\otimes z\right)\cdot\Omega_{\mathcal{B},\mathcal{B},u}\left(\Gamma\left(u\right)\tau\otimes\Gamma^*\left(u\right)\tau\right)}_{\substack{=\Omega_{\mathcal{B},\mathcal{B},u}\left(\left(z^{-1}\otimes z\right)\left(\Gamma(u)\tau\otimes\Gamma^*(u)\tau\right)\right)\\ \text{(since }\Omega_{\mathcal{B},\mathcal{B},u}\text{ is }\mathcal{B}\otimes\mathcal{B}\text{-linear)}}}\right)\\
&\qquad\left(\text{since }\operatorname{CT}_u\text{ is }\mathcal{B}\otimes\mathcal{B}\text{-linear (by Remark 3.15.26)}\right)\\
&=\operatorname{CT}_u\left(\Omega_{\mathcal{B},\mathcal{B},u}\underbrace{\left(\left(z^{-1}\otimes z\right)\left(\Gamma\left(u\right)\tau\otimes\Gamma^*\left(u\right)\tau\right)\right)}_{=z^{-1}\Gamma(u)\tau\otimes z\Gamma^*(u)\tau}\right)\\
&=\operatorname{CT}_u\left(\Omega_{\mathcal{B},\mathcal{B},u}\left(z^{-1}\Gamma\left(u\right)\tau\otimes z\Gamma^*\left(u\right)\tau\right)\right),
\end{aligned}$$

this is equivalent to $\operatorname{CT}_u\left(\Omega_{\mathcal{B},\mathcal{B},u}\left(z^{-1}\Gamma\left(u\right)\tau\otimes z\Gamma^*\left(u\right)\tau\right)\right)=0$. Let us combine what we have proven: We have proven the equivalence of assertions

$$\left(S\left(\sigma\left(\tau\right)\otimes\sigma\left(\tau\right)\right)=0\right)\iff\left(\operatorname{CT}_u\left(\Omega_{\mathcal{B},\mathcal{B},u}\left(z^{-1}\Gamma\left(u\right)\tau\otimes z\Gamma^*\left(u\right)\tau\right)\right)=0\right).\qquad(245)$$

Now, let us simplify $\operatorname{CT}_u\left(\Omega_{\mathcal{B},\mathcal{B},u}\left(z^{-1}\Gamma\left(u\right)\tau\otimes z\Gamma^*\left(u\right)\tau\right)\right)$.

For this, we recall that $\mathcal{B}^{(0)}=\widetilde{F}=\mathbb{C}\left[x_1,x_2,x_3,...\right]$. Thus, the elements of $\mathcal{B}^{(0)}$ are polynomials in the countably many indeterminates x_1, x_2, x_3, ... We are going to interpret the elements of $\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)}$ as polynomials in "twice as many" indeterminates; by this we mean the following:

Convention 3.15.28. Let $(x_1',x_2',x_3',...)$ and $(x_1'',x_2'',x_3'',...)$ be two countable families of new symbols. We denote the family $(x_1',x_2',x_3',...)$ by x', and we denote the family $(x_1'',x_2'',x_3'',...)$ by x''. Thus, if $P\in\mathbb{C}\left[x_1,x_2,x_3,...\right]$, we will denote by $P\left(x'\right)$ the polynomial $P\left(x_1',x_2',x_3',...\right)$, and we will denote by $P\left(x''\right)$ the polynomial $P\left(x_1'',x_2'',x_3'',...\right)$.

The $\mathbb{C}$-linear map

$$\begin{aligned}\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)}&\to\mathbb{C}\left[x_1',x_1'',x_2',x_2'',x_3',x_3'',...\right],\\ P\otimes Q&\mapsto P\left(x'\right)Q\left(x''\right)\end{aligned}$$

is a $\mathbb{C}$-algebra isomorphism. By means of this isomorphism, we are going to identify $\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)}$ with $\mathbb{C}\left[x_1',x_1'',x_2',x_2'',x_3',x_3'',...\right]$.

Another convention:

Convention 3.15.29. For any $P\in\mathcal{B}^{(0)}\left(\left(u\right)\right)$ and any family $(y_1,y_2,y_3,...)$ of pairwise commuting elements of a $\mathbb{C}$-algebra A, we define an element $P\left(y_1,y_2,y_3,...\right)$ of $A\left(\left(u\right)\right)$ as follows: Write P in the form $P=\sum\limits_{i\in\mathbb{Z}}P_i\cdot u^i$ for some $P_i\in\mathcal{B}^{(0)}$, and set $P\left(y_1,y_2,y_3,...\right)=\sum\limits_{i\in\mathbb{Z}}P_i\left(y_1,y_2,y_3,...\right)\cdot u^i$. (In words, $P\left(y_1,y_2,y_3,...\right)$ is defined by substituting $y_1,y_2,y_3,...$ for the variables $x_1,x_2,x_3,...$ in P while keeping the variable u unchanged).

Now, let us notice that:

Lemma 3.15.30. For any $P \in \mathcal{B}^{(0)}((u))$ and $Q \in \mathcal{B}^{(0)}((u))$, we have

$$\Omega_{\mathcal{B},\mathcal{B},u}(P \otimes Q) = P(x') \cdot Q(x'')$$

(where $P(x')$ and $Q(x'')$ are to be understood according to Convention 3.15.29 and Convention 3.15.28, and where $\mathcal{B}^{(0)} \otimes \mathcal{B}^{(0)}$ is identified with $\mathbb{C}[x'_1, x''_1, x'_2, x''_2, x'_3, x''_3, ...]$ according to Convention 3.15.28).

Proof of Lemma 3.15.30. Let $P \in \mathcal{B}^{(0)}((u))$ and $Q \in \mathcal{B}^{(0)}((u))$. Write P in the form $P = \sum\limits_{i \in \mathbb{Z}} P_i \cdot u^i$ for some $P_i \in \mathcal{B}^{(0)}$. Write Q in the form $Q = \sum\limits_{i \in \mathbb{Z}} Q_i \cdot u^i$ for some $Q_i \in \mathcal{B}^{(0)}$. Since $P = \sum\limits_{i \in \mathbb{Z}} P_i \cdot u^i$ and $Q = \sum\limits_{i \in \mathbb{Z}} Q_i \cdot u^i$, we have

$$\begin{aligned}
\Omega_{\mathcal{B},\mathcal{B},u}(P \otimes Q) &= \Omega_{\mathcal{B},\mathcal{B},u}\left(\left(\sum_{i \in \mathbb{Z}} P_i \cdot u^i\right) \otimes \left(\sum_{i \in \mathbb{Z}} Q_i \cdot u^i\right)\right) \\
&= \sum_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} \underbrace{P_j \otimes Q_{i-j}}_{\substack{=P_j(x') \cdot Q_{i-j}(x'') \\ \text{(due to our identification of} \\ \mathcal{B}^{(0)} \otimes \mathcal{B}^{(0)} \text{ with} \\ \mathbb{C}[x'_1, x''_1, x'_2, x''_2, x'_3, x''_3, ...])}}\right) u^i \qquad \text{(by the definition of } \Omega_{\mathcal{B},\mathcal{B},u}) \\
&= \sum_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} P_j(x') \cdot Q_{i-j}(x'')\right) u^i
\end{aligned}$$

and

$$\begin{aligned}
P(x') \cdot Q(x'') &= \underbrace{\left(\sum_{i \in \mathbb{Z}} P_i \cdot u^i\right)(x')}_{\substack{=\sum\limits_{i \in \mathbb{Z}} P_i(x') \cdot u^i = \sum\limits_{j \in \mathbb{Z}} P_j(x') \cdot u^j \\ \text{(here, we renamed } i \text{ as } j)}} \cdot \underbrace{\left(\sum_{i \in \mathbb{Z}} Q_i \cdot u^i\right)(x'')}_{=\sum\limits_{i \in \mathbb{Z}} Q_i(x'') \cdot u^i} \\
&= \left(\sum_{j \in \mathbb{Z}} P_j(x') \cdot u^j\right) \cdot \left(\sum_{i \in \mathbb{Z}} Q_i(x'') \cdot u^i\right) = \sum_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} P_j(x') \cdot u^j \cdot Q_i(x'') \cdot u^i \\
&= \sum_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} P_j(x') \cdot \underbrace{u^j \cdot Q_{i-j}(x'') \cdot u^{i-j}}_{=Q_{i-j}(x'') \cdot u^i} \\
&\qquad \text{(here, we substituted } i-j \text{ for } i \text{ in the second sum)} \\
&= \sum_{j \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} P_j(x') \cdot Q_{i-j}(x'') \cdot u^i = \sum_{i \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} P_j(x') \cdot Q_{i-j}(x'')\right) u^i = \Omega_{\mathcal{B},\mathcal{B},u}(P \otimes Q).
\end{aligned}$$

This proves Lemma 3.15.30.

Now, Theorem 3.11.2 (applied to $m = 0$) yields

$$\Gamma(u) = uz \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \qquad \text{and} \tag{246}$$

$$\Gamma^*(u) = z^{-1} \exp\left(-\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(\sum_{j>0} \frac{a_j}{j} u^{-j}\right) \tag{247}$$

on $\mathcal{B}^{(0)}$. Thus,

$$\begin{aligned}
z^{-1}\Gamma(u)\tau &= z^{-1}uz \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)\tau \qquad \text{(by (246))} \\
&= u \exp\left(\sum_{j>0} \frac{a_{-j}}{j} u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{a_j}{j} u^{-j}\right)\tau \\
&= u \exp\left(\sum_{j>0} \frac{jx_j}{j} u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{\left(\frac{\partial}{\partial x_j}\right)}{j} u^{-j}\right)\tau \\
&\qquad \left(\begin{array}{c} \text{since } a_j \text{ acts as } \dfrac{\partial}{\partial x_j} \text{ on } \widetilde{F} \text{ for every } j > 0, \\ \text{and since } a_{-j} \text{ acts as } jx_j \text{ on } \widetilde{F} \text{ for every } j > 0 \end{array}\right) \\
&= u \exp\left(\sum_{j>0} x_j u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{1}{j} \frac{\partial}{\partial x_j} u^{-j}\right)\tau,
\end{aligned}$$

so that

$$\begin{aligned}
\left(z^{-1}\Gamma(u)\tau\right)(x') &= \left(u \exp\left(\sum_{j>0} x_j u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{1}{j} \frac{\partial}{\partial x_j} u^{-j}\right)\tau\right)(x') \\
&= u \exp\left(\sum_{j>0} x'_j u^j\right) \cdot \exp\left(-\sum_{j>0} \frac{1}{j} \frac{\partial}{\partial x'_j} u^{-j}\right)(\tau(x')) . \qquad (248)
\end{aligned}$$

Also,

$$\begin{aligned}
z\Gamma^*(u)\tau &= zz^{-1}\exp\left(-\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\cdot\exp\left(\sum_{j>0}\frac{a_j}{j}u^{-j}\right)\tau \qquad \text{(by (247))}\\
&= \exp\left(-\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\cdot\exp\left(\sum_{j>0}\frac{a_j}{j}u^{-j}\right)\tau\\
&= \exp\left(-\sum_{j>0}\frac{jx_j}{j}u^j\right)\cdot\exp\left(\sum_{j>0}\frac{\left(\frac{\partial}{\partial x_j}\right)}{j}u^{-j}\right)\tau\\
&\qquad \left(\begin{array}{c}\text{since } a_j \text{ acts as } \dfrac{\partial}{\partial x_j} \text{ on } \widetilde{F} \text{ for every } j>0, \text{ and}\\ \text{since } a_{-j} \text{ acts as } jx_j \text{ on } \widetilde{F} \text{ for every } j>0\end{array}\right)\\
&= \exp\left(-\sum_{j>0}x_ju^j\right)\cdot\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x_j}u^{-j}\right)\tau,
\end{aligned}$$

so that

$$\begin{aligned}
\left(z\Gamma^*(u)\tau\right)(x'') &= \left(\exp\left(-\sum_{j>0}x_ju^j\right)\cdot\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x_j}u^{-j}\right)\tau\right)(x'')\\
&= \exp\left(-\sum_{j>0}x''_ju^j\right)\cdot\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)(\tau(x'')). \qquad (249)
\end{aligned}$$

Now,

$$\begin{aligned}
&\Omega_{\mathcal{B},\mathcal{B},u}\left(z^{-1}\Gamma(u)\tau\otimes z\Gamma^*(u)\tau\right)\\
&= \left(z^{-1}\Gamma(u)\tau\right)(x')\cdot\left(z\Gamma^*(u)\tau\right)(x'')\\
&\qquad \left(\text{by Lemma 3.15.30, applied to } P=z^{-1}\Gamma(u)\tau \text{ and } Q=z\Gamma^*(u)\tau\right)\\
&= u\exp\left(\sum_{j>0}x'_ju^j\right)\cdot\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right)(\tau(x'))\\
&\qquad \cdot\exp\left(-\sum_{j>0}x''_ju^j\right)\cdot\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)(\tau(x''))\\
&\qquad \text{(by (248) and (249))}\\
&= u\exp\left(\sum_{j>0}x'_ju^j\right)\cdot\exp\left(-\sum_{j>0}x''_ju^j\right)\\
&\qquad \cdot\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right)(\tau(x'))\cdot\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)(\tau(x'')). \qquad (250)
\end{aligned}$$

We are going to rewrite the right hand side of this equality. First of all, notice that Theorem 3.1.4 (applied to $R=\left(\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)}\right)((u))$,

$I = \left(\text{closure of the ideal of } R \text{ generated by } x'_j \text{ and } x''_j \text{ with } j \text{ ranging over all positive integers}\right)$, $\alpha = \sum\limits_{j>0} x'_j u^j$ and $\beta = -\sum\limits_{j>0} x''_j u^j$) yields

$$\exp\left(\sum_{j>0} x'_j u^j + \left(-\sum_{j>0} x''_j u^j\right)\right) = \exp\left(\sum_{j>0} x'_j u^j\right) \cdot \exp\left(-\sum_{j>0} x''_j u^j\right).$$

Thus,

$$\exp\left(\sum_{j>0} x'_j u^j\right) \cdot \exp\left(-\sum_{j>0} x''_j u^j\right) = \exp \underbrace{\left(\sum_{j>0} x'_j u^j + \left(-\sum_{j>0} x''_j u^j\right)\right)}_{=\sum\limits_{j>0} u^j\left(x'_j - x''_j\right)}$$
$$= \exp\left(\sum_{j>0} u^j \left(x'_j - x''_j\right)\right). \tag{251}$$

Now, let us recall a very easy fact: If ϕ is an endomorphism of a vector space V, and v is a vector in V such that $\phi v = 0$, then $(\exp \phi) v$ is well-defined (in the sense that the power series $\sum\limits_{n \geq 0} \dfrac{1}{n!} \phi^n v$ converges) and satisfies $(\exp \phi) v = v$. Applying this fact to $V = \left(\mathcal{B}^{(0)} \otimes \mathcal{B}^{(0)}\right)[u, u^{-1}]$, $\phi = \sum\limits_{j>0} \dfrac{1}{j} \dfrac{\partial}{\partial x''_j} u^{-j}$ and $v = \tau(x')$, we see that $\exp\left(\sum\limits_{j>0} \dfrac{1}{j} \dfrac{\partial}{\partial x''_j} u^{-j}\right)(\tau(x'))$ is well-defined and satisfies

$$\exp\left(\sum_{j>0} \frac{1}{j} \frac{\partial}{\partial x''_j} u^{-j}\right)(\tau(x')) = \tau(x') \tag{252}$$

(since $\left(\sum\limits_{j>0} \dfrac{1}{j} \dfrac{\partial}{\partial x''_j} u^{-j}\right)(\tau(x')) = \sum\limits_{j>0} \dfrac{1}{j} \underbrace{\dfrac{\partial}{\partial x''_j}(\tau(x'))}_{=0} u^{-j} = 0$). The same argument (with x'_j and x''_j switching places) shows that $\exp\left(\sum\limits_{j>0} \dfrac{1}{j} \dfrac{\partial}{\partial x'_j} u^{-j}\right)(\tau(x''))$ is well-defined and satisfies

$$\exp\left(\sum_{j>0} \frac{1}{j} \frac{\partial}{\partial x'_j} u^{-j}\right)(\tau(x'')) = \tau(x''). \tag{253}$$

Now,

$$\exp\left(-\sum_{j>0} \frac{u^{-j}}{j}\left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j}\right)\right) = \exp\left(\left(-\sum_{j>0} \frac{1}{j} \frac{\partial}{\partial x'_j} u^{-j}\right) + \sum_{j>0} \frac{1}{j} \frac{\partial}{\partial x''_j} u^{-j}\right)$$
$$= \exp\left(-\sum_{j>0} \frac{1}{j} \frac{\partial}{\partial x'_j} u^{-j}\right) \circ \exp\left(\sum_{j>0} \frac{1}{j} \frac{\partial}{\partial x''_j} u^{-j}\right) \tag{254}$$

[181] and similarly

$$\exp\left(-\sum_{j>0}\frac{u^{-j}}{j}\left(\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}\right)\right)=\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)\circ\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right). \tag{255}$$

But since $-\sum\limits_{j>0}\frac{u^{-j}}{j}\left(\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}\right)$ is a derivation (from $\left(\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)}\right)[u,u^{-1}]$ to $\left(\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)}\right)[u,u^{-1}]$), its exponential $\exp\left(-\sum\limits_{j>0}\frac{u^{-j}}{j}\left(\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}\right)\right)$ is a $\mathbb{C}$-algebra homomorphism (since exponentials of derivations are $\mathbb{C}$-algebra homomorphisms), so

[181] Here, the last equality sign follows from Theorem 3.1.4, applied to

$$R=\left(\begin{array}{c}\text{closure of the }\mathbb{C}\left[u,u^{-1}\right]\text{-subalgebra of }\operatorname{End}_{\mathbb{C}[u,u^{-1}]}\left(\left(\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)}\right)\left[u,u^{-1}\right]\right)\\ \text{generated by }\frac{\partial}{\partial x'_j}\text{ and }\frac{\partial}{\partial x''_j}\text{ with }j\text{ ranging over all positive integers}\end{array}\right),$$

$$I=\left(\begin{array}{c}\text{closure of the ideal of }R\text{ generated by }\frac{\partial}{\partial x'_j}\text{ and }\frac{\partial}{\partial x''_j}\text{ with}\\ j\text{ ranging over all positive integers}\end{array}\right),$$

$$\alpha=-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j},\qquad\text{and}\qquad\beta=\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}.$$

that

$$\begin{aligned}
&\exp\left(-\sum_{j>0}\frac{u^{-j}}{j}\left(\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}\right)\right)\left(\tau\left(x'\right)\tau\left(x''\right)\right)\\
&= \underbrace{\exp\left(-\sum_{j>0}\frac{u^{-j}}{j}\left(\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}\right)\right)}_{\substack{=\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right)\circ\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)\\ \text{(by (254))}}}\left(\tau\left(x'\right)\right)\\
&\qquad\cdot\underbrace{\exp\left(-\sum_{j>0}\frac{u^{-j}}{j}\left(\frac{\partial}{\partial x'_j}-\frac{\partial}{\partial x''_j}\right)\right)}_{\substack{=\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)\circ\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right)\\ \text{(by (255))}}}\left(\tau\left(x''\right)\right)\\
&=\left(\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right)\circ\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)\right)\left(\tau\left(x'\right)\right)\\
&\qquad\cdot\left(\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)\circ\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right)\right)\left(\tau\left(x''\right)\right)\\
&=\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right)\underbrace{\left(\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)\left(\tau\left(x'\right)\right)\right)}_{\substack{=\tau(x')\\ \text{(by (252))}}}\\
&\qquad\cdot\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)\underbrace{\left(\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right)\left(\tau\left(x''\right)\right)\right)}_{\substack{=\tau(x'')\\ \text{(by (253))}}}\\
&=\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x'_j}u^{-j}\right)\left(\tau\left(x'\right)\right)\cdot\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x''_j}u^{-j}\right)\left(\tau\left(x''\right)\right).
\end{aligned}$$

Hence, (250) becomes

$$\Omega_{\mathcal{B},\mathcal{B},u}\left(z^{-1}\Gamma\left(u\right)\tau\otimes z\Gamma^{*}\left(u\right)\tau\right)$$
$$=\underbrace{u\exp\left(\sum_{j>0}x_j'u^j\right)\cdot\exp\left(-\sum_{j>0}x_j''u^j\right)}_{\substack{=\exp\left(\sum\limits_{j>0}u^j\left(x_j'-x_j''\right)\right)\\ \text{(by (251))}}}$$
$$\cdot\underbrace{\exp\left(-\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x_j'}u^{-j}\right)\left(\tau\left(x'\right)\right)\cdot\exp\left(\sum_{j>0}\frac{1}{j}\frac{\partial}{\partial x_j''}u^{-j}\right)\left(\tau\left(x''\right)\right)}_{=\exp\left(-\sum\limits_{j>0}\frac{u^{-j}}{j}\left(\frac{\partial}{\partial x_j'}-\frac{\partial}{\partial x_j''}\right)\right)\left(\tau\left(x'\right)\tau\left(x''\right)\right)}$$
$$=u\exp\left(\sum_{j>0}u^j\left(x_j'-x_j''\right)\right)\cdot\exp\left(-\sum_{j>0}\frac{u^{-j}}{j}\left(\frac{\partial}{\partial x_j'}-\frac{\partial}{\partial x_j''}\right)\right)\left(\tau\left(x'\right)\tau\left(x''\right)\right). \quad (256)$$

Thus, (245) rewrites as

$$\left(S\left(\sigma\left(\tau\right)\otimes\sigma\left(\tau\right)\right)=0\right)$$
$$\Longleftrightarrow\left(\mathrm{CT}_u\left(u\exp\left(\sum_{j>0}u^j\left(x_j'-x_j''\right)\right)\cdot\exp\left(-\sum_{j>0}\frac{u^{-j}}{j}\left(\frac{\partial}{\partial x_j'}-\frac{\partial}{\partial x_j''}\right)\right)\left(\tau\left(x'\right)\tau\left(x''\right)\right)\right)=0\right). \quad (257)$$

This already gives a criterion for a $\tau\in\mathcal{B}^{(0)}$ to satisfy $\sigma\left(\tau\right)\in\Omega$, but it is yet a rather messy one. We are going to simplify it in the following. First, we do a substitution of variables:

Convention 3.15.31. Let $\left(y_1,y_2,y_3,...\right)$ be a sequence of new symbols. We identify the $\mathbb{C}$-algebra $\mathbb{C}\left[x_1,y_1,x_2,y_2,x_3,y_3,...\right]$ with the $\mathbb{C}$-algebra $\mathbb{C}\left[x_1',x_1'',x_2',x_2'',x_3',x_3'',...\right]=\mathcal{B}^{(0)}\otimes\mathcal{B}^{(0)}$ by the following substitution:

$$x_j'=x_j-y_j \qquad \text{for every } j>0;$$
$$x_j''=x_j+y_j \qquad \text{for every } j>0.$$

If we define the sum and the difference of two sequences by componentwise addition resp. subtraction, then this rewrites as follows:

$$x'=x-y;$$
$$x''=x+y.$$

It is now easy to see that

$$x_j'-x_j''=-2y_j \qquad \text{for every } j>0,$$

and
$$\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j} = -\frac{\partial}{\partial y_j} \qquad \text{for every } j > 0$$
(where $\dfrac{\partial}{\partial x'_j}$ and $\dfrac{\partial}{\partial x''_j}$ mean differentiation over the variables x'_j and x''_j in the polynomial ring $\mathbb{C}[x'_1, x''_1, x'_2, x''_2, x'_3, x''_3, ...]$, whereas $\dfrac{\partial}{\partial y_j}$ means differentiation over the variable y_j in the polynomial ring $\mathbb{C}[x_1, y_1, x_2, y_2, x_3, y_3, ...]$). As a consequence,

$$u \exp\left(\sum_{j>0} u^j \underbrace{\left(x'_j - x''_j\right)}_{=-2y_j}\right) \cdot \exp\left(-\sum_{j>0} \frac{u^{-j}}{j} \underbrace{\left(\frac{\partial}{\partial x'_j} - \frac{\partial}{\partial x''_j}\right)}_{=-\frac{\partial}{\partial y_j}}\right) \left(\tau\left(\underbrace{x'}_{=x-y}\right) \tau\left(\underbrace{x''}_{=x+y}\right)\right)$$
$$= u \exp\left(-2\sum_{j>0} u^j y_j\right) \cdot \exp\left(\sum_{j>0} \frac{u^{-j}}{j} \frac{\partial}{\partial y_j}\right) \left(\tau(x-y)\,\tau(x+y)\right).$$

Hence, (257) rewrites as
$$(S(\sigma(\tau) \otimes \sigma(\tau)) = 0)$$
$$\Longleftrightarrow \left(\mathrm{CT}_u\left(u \exp\left(-2\sum_{j>0} u^j y_j\right) \cdot \exp\left(\sum_{j>0} \frac{u^{-j}}{j} \frac{\partial}{\partial y_j}\right) \left(\tau(x-y)\,\tau(x+y)\right)\right) = 0\right). \tag{258}$$

To simplify this even further, a new notation is needed:

Definition 3.15.32. Let K be a commutative ring. Let $(x_1, x_2, x_3, ...)$, $(z_1, z_2, z_3, ...)$, and $(w_1, w_2, w_3, ...)$ be three disjoint families of indeterminates. Denote by x the family $(x_1, x_2, x_3, ...)$, and denote by z the family $(z_1, z_2, z_3, ...)$.

(a) For any polynomial $r \in K[x_1, x_2, x_3, ..., z_1, z_2, z_3, ...]$, let $r \mid_{z=0}$ denote the polynomial in $K[x_1, x_2, x_3, ...]$ obtained by substituting $(0, 0, 0, ...)$ for $(z_1, z_2, z_3, ...)$ in P.

(b) Consider the differential operators $\dfrac{\partial}{\partial z_1}, \dfrac{\partial}{\partial z_2}, \dfrac{\partial}{\partial z_3}, ...$ on $K[x_1, x_2, x_3, ..., z_1, z_2, z_3, ...]$. For any power series $P \in K[[w_1, w_2, w_3, ...]]$, let $P(\partial_z)$ mean the value of P when applied to the family $\left(\dfrac{\partial}{\partial z_1}, \dfrac{\partial}{\partial z_2}, \dfrac{\partial}{\partial z_3}, ...\right)$ (that is, the result of substituting $\dfrac{\partial}{\partial z_j}$ for each w_j in P). This value is a well-defined differential operator on $K[x_1, x_2, x_3, ..., z_1, z_2, z_3, ...]$ (due to Remark 3.15.33 below).

(c) For any power series $P \in K[[w_1, w_2, w_3, ...]]$ and any two polynomials $f \in K[x_1, x_2, x_3, ...]$ and $g \in K[x_1, x_2, x_3, ...]$, define a polynomial $A(P, f, g) \in K[x_1, x_2, x_3, ...]$ by
$$A(P, f, g) = \left(P(\partial_z)\left(f(x-z)\,g(x+z)\right)\right)\mid_{z=0}.$$

Remark 3.15.33. Let K be a commutative ring. Let $(x_1, x_2, x_3, ...)$, $(z_1, z_2, z_3, ...)$, and $(w_1, w_2, w_3, ...)$ be three disjoint families of indeterminates. Let $P \in K[[w_1, w_2, w_3, ...]]$ be a power series. Then, if we apply the power series P to the family $\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, ...\right)$, we obtain a well-defined endomorphism of $K[x_1, x_2, x_3, ..., z_1, z_2, z_3, ...]$.

Proof of Remark 3.15.33. Let $\mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ be defined as in Convention 2.2.23. Write the power series P in the form

$$P = \sum_{(i_1,i_2,i_3,...)\in\mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}} \lambda_{(i_1,i_2,i_3,...)} w_1^{i_1} w_2^{i_2} w_3^{i_3} ...$$

for $\lambda_{(i_1,i_2,i_3,...)} \in K$. Then, if we apply the power series P to the family $\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, ...\right)$, we obtain

$$\sum_{(i_1,i_2,i_3,...)\in\mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}} \lambda_{(i_1,i_2,i_3,...)} \left(\frac{\partial}{\partial z_1}\right)^{i_1} \left(\frac{\partial}{\partial z_2}\right)^{i_2} \left(\frac{\partial}{\partial z_3}\right)^{i_3} ....$$

In order to prove that this is a well-defined endomorphism of $K[x_1, x_2, x_3, ..., z_1, z_2, z_3, ...]$, we must prove that for every $r \in K[x_1, x_2, x_3, ..., z_1, z_2, z_3, ...]$, the sum

$$\sum_{(i_1,i_2,i_3,...)\in\mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}} \lambda_{(i_1,i_2,i_3,...)} \left(\left(\frac{\partial}{\partial z_1}\right)^{i_1} \left(\frac{\partial}{\partial z_2}\right)^{i_2} \left(\frac{\partial}{\partial z_3}\right)^{i_3} ...\right) r$$

is well-defined, i. e., has only finitely many nonzero addends. But this is clear, because only finitely many $(i_1, i_2, i_3, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ satisfy $\left(\left(\frac{\partial}{\partial z_1}\right)^{i_1} \left(\frac{\partial}{\partial z_2}\right)^{i_2} \left(\frac{\partial}{\partial z_3}\right)^{i_3} ...\right) r \neq 0$ [182]. Hence, we have proven that the sum $\sum_{(i_1,i_2,i_3,...)\in\mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}} \lambda_{(i_1,i_2,i_3,...)} \left(\frac{\partial}{\partial z_1}\right)^{i_1} \left(\frac{\partial}{\partial z_2}\right)^{i_2} \left(\frac{\partial}{\partial z_3}\right)^{i_3} ...$ is a well-defined endomorphism of $K[x_1, x_2, x_3, ..., z_1, z_2, z_3, ...]$. Since this sum is the result of applying the power series P to the family $\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, ...\right)$, we thus conclude that applying the power series P to the family $\left(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, ...\right)$ yields a well-defined endomorphism of $K[x_1, x_2, x_3, ..., z_1, z_2, z_3, ...]$. Remark 3.15.33 is proven.

Example: If $P(w) = w_1$ (the first variable), then

$$A(P, f, g) = \left(\frac{\partial}{\partial z_1}(f(x - z) g(x + z))\right) |_{z=0} = -\frac{\partial f}{\partial x_1} g + \frac{\partial g}{\partial x_1} f.$$

Lemma 3.15.34. For any three polynomials P, f, g, we have $A(P, f, g) = A(P_-, g, f)$, where $P_-(w) = P(-w)$.

[182] This is because r is a polynomial, so that only finitely many variables occur in r, and the degrees of the monomials of r are bounded from above.

Corollary 3.15.35. For any two polynomials P and f, we have $A(P, f, f) = 0$ if P is odd.

This is clear from the definition.

We now state the so-called *Hirota bilinear relations*, which are a simplified version of (258):

Theorem 3.15.36 (Hirota bilinear relations)**.** Let $\tau \in \mathcal{B}^{(0)}$ be a nonzero vector. Let $(y_1, y_2, y_3, ...)$ and $(w_1, w_2, w_3, ...)$ be two families of new symbols. Let $\widetilde{w}$ denote the sequence $\left(\frac{w_1}{1}, \frac{w_2}{2}, \frac{w_3}{3}, ...\right)$. Define the elementary Schur polynomials S_k as in Definition 3.12.2.

Then, $\sigma(\tau) \in \Omega$ if and only if

$$A\left(\sum_{j=0}^{\infty} S_j(-2y) S_{j+1}(\widetilde{w}) \exp\left(\sum_{s>0} y_s w_s\right), \tau, \tau\right) = 0, \tag{259}$$

where the term $A\left(\sum\limits_{j=0}^{\infty} S_j(-2y) S_{j+1}(\widetilde{w}) \exp\left(\sum\limits_{s>0} y_s w_s\right), \tau, \tau\right)$ is to be interpreted by applying Definition 3.15.32 **(c)** to $K = \mathbb{C}[[y_1, y_2, y_3, ...]]$ (since $\sum\limits_{j=0}^{\infty} S_j(-2y) S_{j+1}(\widetilde{w}) \exp\left(\sum\limits_{s>0} y_s w_s\right) \in (\mathbb{C}[[y_1, y_2, y_3, ...]])[[w_1, w_2, w_3, ...]]$ and $\tau \in \mathcal{B}^{(0)} = \mathbb{C}[x_1, x_2, x_3, ...] \subseteq (\mathbb{C}[[y_1, y_2, y_3, ...]])[x_1, x_2, x_3, ...]$).

Before we prove this, we need a simple lemma about polynomials:

Lemma 3.15.37. Let K be a commutative $\mathbb{Q}$-algebra. Let $(y_1, y_2, y_3, ...)$ and $(z_1, z_2, z_3, ...)$ be two sequences of new symbols. Denote the sequence $(y_1, y_2, y_3, ...)$ by y. Denote the sequence $(z_1, z_2, z_3, ...)$ by z. Denote by $\widetilde{\partial_y}$ the sequence $\left(\frac{1}{1}\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \frac{1}{3}\frac{\partial}{\partial y_3}, ...\right)$ of endomorphisms of $(K[[y_1, y_2, y_3, ...]])[z_1, z_2, z_3, ...]$. Denote by $\widetilde{\partial_z}$ the sequence $\left(\frac{1}{1}\frac{\partial}{\partial z_1}, \frac{1}{2}\frac{\partial}{\partial z_2}, \frac{1}{3}\frac{\partial}{\partial z_3}, ...\right)$ of endomorphisms of $(K[[y_1, y_2, y_3, ...]])[z_1, z_2, z_3, ...]$. Let P and Q be two elements of $K[w_1, w_2, w_3, ...]$ (where $(w_1, w_2, w_3, ...)$ is a further sequence of new symbols). Then,

$$Q\left(\widetilde{\partial_y}\right)(P(y+z)) = Q\left(\widetilde{\partial_z}\right)(P(y+z)).$$

Proof of Lemma 3.15.37. Let D be the K-subalgebra of $\operatorname{End}((K[[y_1, y_2, y_3, ...]])[z_1, z_2, z_3, ...])$ generated by $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_3}, ..., \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \frac{\partial}{\partial z_3}, ...$ Then, clearly, D is a commutative K-algebra (since its generators commute), and all elements of the sequences $\widetilde{\partial_y}$ and $\widetilde{\partial_z}$ lie in D (since $\widetilde{\partial_y} = \left(\frac{1}{1}\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \frac{1}{3}\frac{\partial}{\partial y_3}, ...\right)$ and $\widetilde{\partial_z} = \left(\frac{1}{1}\frac{\partial}{\partial z_1}, \frac{1}{2}\frac{\partial}{\partial z_2}, \frac{1}{3}\frac{\partial}{\partial z_3}, ...\right)$).

Let I be the ideal of D generated by $\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}$ with i ranging over the positive integers. Then, $\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i} \in I$ for every positive integer i. Hence, every positive integer

i satisfies $\frac{1}{i}\frac{\partial}{\partial y_i} \equiv \frac{1}{i}\frac{\partial}{\partial z_i} \operatorname{mod} I$ (since $\frac{1}{i}\frac{\partial}{\partial y_i} - \frac{1}{i}\frac{\partial}{\partial z_i} = \frac{1}{i}\underbrace{\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}\right)}_{\in I} \in I$). In other words, for every positive integer i, the i-th element of the sequence $\widetilde{\partial}_y$ is congruent to the i-th element of the sequence $\widetilde{\partial}_z$ modulo I (since the i-th element of the sequence $\widetilde{\partial}_y$ is $\frac{1}{i}\frac{\partial}{\partial y_i}$, while the i-th element of the sequence $\widetilde{\partial}_z$ is $\frac{1}{i}\frac{\partial}{\partial z_i}$). Thus, each element of the sequence $\widetilde{\partial}_y$ is congruent to the corresponding element of the sequence $\widetilde{\partial}_z$ modulo I. Hence, $Q\left(\widetilde{\partial}_y\right) \equiv Q\left(\widetilde{\partial}_z\right) \operatorname{mod} I$ (since Q is a polynomial, and I is an ideal). Hence,

$$\begin{aligned} & Q\left(\widetilde{\partial}_y\right) - Q\left(\widetilde{\partial}_z\right) \in I \\ & = \left(\text{ideal of } D \text{ generated by } \frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i} \text{ with } i \text{ ranging over the positive integers}\right). \end{aligned}$$

In other words, $Q\left(\widetilde{\partial}_y\right) - Q\left(\widetilde{\partial}_z\right)$ is a D-linear combinations of terms of the form $\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}$ with i ranging over the positive integers. Thus, we can write $Q\left(\widetilde{\partial}_y\right) - Q\left(\widetilde{\partial}_z\right)$ in the form $Q\left(\widetilde{\partial}_y\right) - Q\left(\widetilde{\partial}_z\right) = \sum\limits_{i>0} d_i \circ \left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}\right)$, where each d_i is an element of D, and all but finitely many $i > 0$ satisfy $d_i = 0$. Consider these d_i.

But it is easy to see that

$$\text{every positive integer } i \text{ satisfies } \left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}\right)\left(P\left(y+z\right)\right) = 0. \tag{260}$$

[183] Thus,

$$\begin{aligned}
&Q\left(\widetilde{\partial}_y\right)(P(y+z)) - Q\left(\widetilde{\partial}_z\right)(P(y+z)) \\
&= \underbrace{\left(Q\left(\widetilde{\partial}_y\right) - Q\left(\widetilde{\partial}_z\right)\right)}_{=\sum\limits_{i>0} d_i \circ \left(\dfrac{\partial}{\partial y_i} - \dfrac{\partial}{\partial z_i}\right)} (P(y+z)) = \sum_{i>0} \left(d_i \circ \left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}\right)\right)(P(y+z)) \\
&= \sum_{i>0} d_i \underbrace{\left(\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}\right)(P(y+z))\right)}_{\substack{=0 \\ \text{(by (260))}}} = \sum_{i>0} \underbrace{d_i(0)}_{=0} = 0.
\end{aligned}$$

In other words, $Q\left(\widetilde{\partial}_y\right)(P(y+z)) = Q\left(\widetilde{\partial}_z\right)(P(y+z))$. This proves Lemma 3.15.37.

Proof of Theorem 3.15.36. We introduce a new family of indeterminates $(z_1, z_2, z_3, ...)$.

---

[183] *Proof of (260):* Let i be a positive integer. Let us identify $\mathbb{C}[w_1, w_2, w_3, ...]$ with $(\mathbb{C}[w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ...])[w_i]$. Then, $P \in \mathbb{C}[w_1, w_2, w_3, ...] = (\mathbb{C}[w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ...])[w_i]$, so that we can write P as a polynomial in the variable w_i over the ring $\mathbb{C}[w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ...]$. In other words, we can write P in the form $P = \sum\limits_{n\in\mathbb{N}} p_n w_i^n$, where every $n \in \mathbb{N}$ satisfies $p_n \in \mathbb{C}[w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ...]$ and all but finitely many $n \in \mathbb{N}$ satisfy $p_n = 0$. Consider these p_n.

Let $n \in \mathbb{N}$ be arbitrary. Consider $p_n \in \mathbb{C}[w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ...]$ as an element of $\mathbb{C}[w_1, w_2, w_3, ...]$ (by means of the canonical embedding $\mathbb{C}[w_1, w_2, ..., w_{i-1}, w_{i+1}, w_{i+2}, ...] \subseteq \mathbb{C}[w_1, w_2, w_3, ...]$). Then, p_n is a polynomial in which the variable w_i does not occur. Hence, $p_n(y+z)$ is a polynomial in which neither of the variables y_i and z_i occur. Thus, $\dfrac{\partial}{\partial y_i}(p_n(y+z)) = 0$ and $\dfrac{\partial}{\partial z_i}(p_n(y+z)) = 0$.

On the other hand, it is very easy to check that $\dfrac{\partial}{\partial y_i}(y_i+z_i)^n = \dfrac{\partial}{\partial z_i}(y_i+z_i)^n$ (in fact, this is obvious in the case when $n = 0$, and in every other case follows from $\dfrac{\partial}{\partial y_i}(y_i+z_i)^n = n(y_i+z_i)^{n-1}$ and $\dfrac{\partial}{\partial z_i}(y_i+z_i)^n = n(y_i+z_i)^{n-1}$). Now, by the Leibniz rule,

$$\begin{aligned}
\frac{\partial}{\partial y_i}\left(p_n(y+z)\cdot(y_i+z_i)^n\right) &= \underbrace{\left(\frac{\partial}{\partial y_i}(p_n(y+z))\right)}_{=0=\frac{\partial}{\partial z_i}(p_n(y+z))} \cdot (y_i+z_i)^n + p_n(y+z)\cdot \underbrace{\frac{\partial}{\partial y_i}(y_i+z_i)^n}_{=\frac{\partial}{\partial z_i}(y_i+z_i)^n} \\
&= \left(\frac{\partial}{\partial z_i}(p_n(y+z))\right)\cdot(y_i+z_i)^n + p_n(y+z)\cdot\frac{\partial}{\partial z_i}(y_i+z_i)^n.
\end{aligned}$$

Compared with

$$\frac{\partial}{\partial z_i}\left(p_n(y+z)\cdot(y_i+z_i)^n\right) = \left(\frac{\partial}{\partial z_i}(p_n(y+z))\right)\cdot(y_i+z_i)^n + p_n(y+z)\cdot\frac{\partial}{\partial z_i}(y_i+z_i)^n$$

(this follows from the Leibniz rule), this yields

$$\frac{\partial}{\partial y_i}\left(p_n(y+z)\cdot(y_i+z_i)^n\right) = \frac{\partial}{\partial z_i}\left(p_n(y+z)\cdot(y_i+z_i)^n\right). \tag{261}$$

Now, forget that we fixed $n \in \mathbb{N}$. We have shown that every $n \in \mathbb{N}$ satisfies (261). Now, since

Denote this family by z. (This z has nothing to do with the element z of $\mathcal{B}$. It is best to forget about $\mathcal{B}$ here, and only think about $\mathcal{B}^{(0)} = \mathbb{C}[x_1, x_2, x_3, ...]$.) Denote by $\widetilde{\partial_z}$ the sequence $\left(\frac{1}{1}\frac{\partial}{\partial z_1}, \frac{1}{2}\frac{\partial}{\partial z_2}, \frac{1}{3}\frac{\partial}{\partial z_3}, ...\right)$.

Denote by $\widetilde{\partial_y}$ the sequence $\left(\frac{1}{1}\frac{\partial}{\partial y_1}, \frac{1}{2}\frac{\partial}{\partial y_2}, \frac{1}{3}\frac{\partial}{\partial y_3}, ...\right)$. Also, let $-2y$ be the sequence $(-2y_1, -2y_2, -2y_3, ...)$. Then,

$$\sum_{k=0}^{\infty} S_k(-2y) u^k = \sum_{k\geq 0} S_k(-2y) u^k = \exp\left(\sum_{i\geq 1} -2y_i u^i\right)$$
$$\text{(by (145), with } -2y \text{ substituted for } x \text{ and } u \text{ substituted for } z\text{)}$$
$$= \exp\left(\sum_{j\geq 1} -2y_j u^j\right) = \exp\left(-2\sum_{j>0} u^j y_j\right) \qquad (262)$$

and

$$\sum_{k=0}^{\infty} S_k\left(\widetilde{\partial_y}\right) u^{-k} = \sum_{k\geq 0} S_k\left(\widetilde{\partial_y}\right) u^{-k} = \exp\left(\sum_{i\geq 1} \frac{1}{i}\frac{\partial}{\partial y_i} u^{-i}\right)$$
$$\left(\text{by (145), with } \widetilde{\partial_y} \text{ substituted for } x \text{ and } u^{-1} \text{ substituted for } z\right)$$
$$= \exp\left(\sum_{j\geq 1} \frac{1}{j}\frac{\partial}{\partial y_j} u^{-j}\right) = \exp\left(\sum_{j>0} \frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\right). \qquad (263)$$

Applying Lemma 3.14.1 to $K = (\mathbb{C}[[y_1, y_2, y_3, ...]])[x_1, x_2, x_3, ...]$ and $P = \tau(x+z)\tau(x-z)$, we obtain

$$\exp\left(\sum_{s>0} y_s \frac{\partial}{\partial z_s}\right)(\tau(x+z)\tau(x-z)) = \tau(x+y+z)\tau(x-y-z). \qquad (264)$$

$P = \sum_{n\in\mathbb{N}} p_n w_i^n$, we have $P(y+z) = \sum_{n\in\mathbb{N}} p_n(y+z)\cdot(y_i+z_i)^n$, so that

$$\begin{aligned}
&\left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}\right)(P(y+z)) \\
&= \left(\frac{\partial}{\partial y_i} - \frac{\partial}{\partial z_i}\right)\left(\sum_{n\in\mathbb{N}} p_n(y+z)\cdot(y_i+z_i)^n\right) \\
&= \sum_{n\in\mathbb{N}} \underbrace{\frac{\partial}{\partial y_i}\left(p_n(y+z)\cdot(y_i+z_i)^n\right)}_{\substack{=\frac{\partial}{\partial z_i}(p_n(y+z)\cdot(y_i+z_i)^n) \\ \text{(by (261))}}} - \sum_{n\in\mathbb{N}} \frac{\partial}{\partial z_i}\left(p_n(y+z)\cdot(y_i+z_i)^n\right) \\
&= \sum_{n\in\mathbb{N}} \frac{\partial}{\partial z_i}\left(p_n(y+z)\cdot(y_i+z_i)^n\right) - \sum_{n\in\mathbb{N}} \frac{\partial}{\partial z_i}\left(p_n(y+z)\cdot(y_i+z_i)^n\right) = 0.
\end{aligned}$$

This proves (260).

Now,

$$\begin{aligned}
&\mathrm{CT}_u\left(u\exp\left(-2\sum_{j>0}u^jy_j\right)\exp\left(\sum_{j>0}\frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\right)\left(\tau\left(x-y\right)\tau\left(x+y\right)\right)\right)\\
&=\mathrm{CT}_u\left(\underbrace{u\exp\left(-2\sum_{j>0}u^jy_j\right)}_{\substack{=\sum\limits_{k=0}^{\infty}S_k(-2y)u^k\\ \text{(by (262))}}}\underbrace{\exp\left(\sum_{j>0}\frac{u^{-j}}{j}\frac{\partial}{\partial y_j}\right)}_{\substack{=\sum\limits_{k=0}^{\infty}S_k\left(\widetilde{\partial_y}\right)u^{-k}\\ \text{(by (263))}}}\left(\tau\left(x+y+z\right)\tau\left(x-y-z\right)\right)\right)|_{z=0}\\
&=\mathrm{CT}_u\left(u\left(\sum_{k=0}^{\infty}S_k\left(-2y\right)u^k\right)\left(\sum_{k=0}^{\infty}S_k\left(\widetilde{\partial_y}\right)u^{-k}\right)\left(\tau\left(x+y+z\right)\tau\left(x-y-z\right)\right)\right)|_{z=0}\\
&=\sum_{j=0}^{\infty}S_j\left(-2y\right)\underbrace{S_{j+1}\left(\widetilde{\partial_y}\right)\left(\tau\left(x+y+z\right)\tau\left(x-y-z\right)\right)}_{\substack{=S_{j+1}\left(\widetilde{\partial_z}\right)\left(\tau(x+y+z)\tau(x-y-z)\right)\\ \text{(by Lemma 3.15.37, applied to}\\ K=\mathbb{C}[x_1,x_2,x_3,\ldots],\ P=\tau(x+w)\tau(x-w)\text{ and }Q=S_{j+1}(w))}}|_{z=0}\\
&=\sum_{j=0}^{\infty}S_j\left(-2y\right)S_{j+1}\left(\widetilde{\partial_z}\right)\underbrace{\left(\tau\left(x+y+z\right)\tau\left(x-y-z\right)\right)}_{\substack{=\exp\left(\sum\limits_{s>0}y_s\frac{\partial}{\partial z_s}\right)\left(\tau(x+z)\tau(x-z)\right)\\ \text{(by (264))}}}|_{z=0}\\
&=\sum_{j=0}^{\infty}S_j\left(-2y\right)S_{j+1}\left(\widetilde{\partial_z}\right)\exp\left(\sum_{s>0}y_s\frac{\partial}{\partial z_s}\right)\left(\tau\left(x+z\right)\tau\left(x-z\right)\right)|_{z=0}\ .
\end{aligned}$$

Compared with the fact that (by the definition of $A\left(\sum\limits_{j=0}^{\infty}S_j\left(-2y\right)S_{j+1}\left(\widetilde{w}\right)\exp\left(\sum\limits_{s>0}y_sw_s\right),\tau,\tau\right)$) we have

$$\begin{aligned}
&A\left(\sum_{j=0}^{\infty}S_j\left(-2y\right)S_{j+1}\left(\widetilde{w}\right)\exp\left(\sum_{s>0}y_sw_s\right),\tau,\tau\right)\\
&=\underbrace{\left(\sum_{j=0}^{\infty}S_j\left(-2y\right)S_{j+1}\left(\widetilde{w}\right)\exp\left(\sum_{s>0}y_sw_s\right)\right)\left(\partial_z\right)}_{=\sum\limits_{j=0}^{\infty}S_j(-2y)S_{j+1}\left(\widetilde{\partial_z}\right)\exp\left(\sum\limits_{s>0}y_s\frac{\partial}{\partial z_s}\right)}\left(\tau\left(x+z\right)\tau\left(x-z\right)\right)|_{z=0}\\
&=\sum_{j=0}^{\infty}S_j\left(-2y\right)S_{j+1}\left(\widetilde{\partial_z}\right)\exp\left(\sum_{s>0}y_s\frac{\partial}{\partial z_s}\right)\left(\tau\left(x+z\right)\tau\left(x-z\right)\right)|_{z=0},
\end{aligned}$$

this yields

$$\begin{aligned}
&\operatorname{CT}_u \left(u \exp\left(-2 \sum_{j>0} u^j y_j \right) \exp\left(\sum_{j>0} \frac{u^{-j}}{j} \frac{\partial}{\partial y_j} \right) \left(\tau\left(x-y\right) \tau\left(x+y\right) \right) \right) \\
&= A\left(\sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s \right), \tau, \tau \right).
\end{aligned}$$

Hence, (258) rewrites as follows:

$$\left(S\left(\sigma\left(\tau\right) \otimes \sigma\left(\tau\right)\right) = 0 \right) \iff \left(A\left(\sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s \right), \tau, \tau \right) = 0 \right).$$

Since $S\left(\sigma\left(\tau\right) \otimes \sigma\left(\tau\right)\right) = 0$ is equivalent to $\sigma\left(\tau\right) \in \Omega$ (by Theorem 3.15.13 **(b)**, applied to $\sigma\left(\tau\right)$ instead of τ), this rewrites as follows:

$$\left(\sigma\left(\tau\right) \in \Omega \right) \iff \left(A\left(\sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s \right), \tau, \tau \right) = 0 \right).$$

This proves Theorem 3.15.36.

Theorem 3.15.36 tells us that a nonzero $\tau \in \mathcal{B}^{(0)}$ satisfies $\sigma\left(\tau\right) \in \Omega$ if and only if it satisfies the equation (259). The left hand side of this equation is a power series with respect to the variables $y_1, y_2, y_3, \ldots$. A power series is 0 if and only if each of its coefficients is 0. Hence, the equation (259) holds if and only if for each monomial in $y_1, y_2, y_3, \ldots$, the coefficient of the left hand side of (259) in front of this monomial is 0. Thus, the equation (259) is equivalent to **a system of infinitely many equations**, one for each monomial in $y_1, y_2, y_3, \ldots$. We don't know of a good way to describe these equations (without using the variables $y_1, y_2, y_3, \ldots$), but we can describe the equations corresponding to the simplest among our monomials: the monomials of degree 0 and those of degree 1.

In the following, we consider $\left(\mathbb{C}\left[\left[y_1, y_2, y_3, \ldots\right]\right]\right)\left[x_1, x_2, x_3, \ldots\right]$ as a subring of $\left(\mathbb{C}\left[x_1, x_2, x_3, \ldots\right]\right)\left[\left[y_1, y_2, y_3, \ldots\right]\right]$. For every commutative ring K, every element T of $K\left[\left[y_1, y_2, y_3, \ldots\right]\right]$ and any monomial[184] $\mathfrak{m}$ in the variables $y_1, y_2, y_3, \ldots$, we denote by $T\left[\mathfrak{m}\right]$ the coefficient of the monomial $\mathfrak{m}$ in the power series T. (For example, $\left(\exp\left(x_2 y_2\right)\right)\left[y_2^3\right] = \frac{x_2^3}{6}$; note that $K = \mathbb{C}\left[x_1, x_2, x_3, \ldots\right]$ in this example, so that x_2 counts as a constant!)

For every $P \in \left(\mathbb{C}\left[\left[y_1, y_2, y_3, \ldots\right]\right]\right)\left[\left[w_1, w_2, w_3, \ldots\right]\right]$ and every monomial $\mathfrak{m}$ in the variables $y_1, y_2, y_3, \ldots$, we have

$$\left(A\left(P, \tau, \tau\right)\right)\left[\mathfrak{m}\right] = A\left(P\left[\mathfrak{m}\right], \tau, \tau\right). \tag{265}$$

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[184] When we say "monomial", we mean a monomial without coefficient.

[185] *Proof.* We have $P = \sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} P\left[\mathfrak{n}\right] \cdot \mathfrak{n}$. Since the map

$$\left(\mathbb{C}\left[\left[y_1, y_2, y_3, \ldots\right]\right]\right)\left[\left[w_1, w_2, w_3, \ldots\right]\right] \to \left(\mathbb{C}\left[\left[y_1, y_2, y_3, \ldots\right]\right]\right)\left[x_1, x_2, x_3, \ldots\right],$$
$$Q \mapsto A\left(Q, \tau, \tau\right)$$

Now, let us describe the equations that are obtained from (259) by taking coefficients before monomials of degree 0 and 1:

Monomials of degree 0: The only monomial of degree 0 in $y_1, y_2, y_3, ...$ is 1. We have

$$\left(A\left(\sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s\right), \tau, \tau\right)\right)[1]$$
$$= A\left(\underbrace{\left(\sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s\right)\right)[1]}_{=S_1(\widetilde{w})=w_1}, \tau, \tau\right)$$
$$\left(\text{by (265), applied to } P = \sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s\right) \text{ and } \mathfrak{m} = 1\right)$$
$$= A\left(w_1, \tau, \tau\right) = 0 \qquad \left(\text{by Corollary 3.15.35, since } w_1 \text{ is odd}\right).$$

Therefore, if we take coefficients with respect to the monomial 1 in the equation (265), we obtain a tautology.

Monomials of degree 1: This will be more interesting. The monomials of degree

is $\mathbb{C}\left[\left[y_1, y_2, y_3, ...\right]\right]$-linear, we have

$$A\left(\sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} P\left[\mathfrak{n}\right] \cdot \mathfrak{n}, \tau, \tau\right) = \sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} A\left(P\left[\mathfrak{n}\right], \tau, \tau\right) \cdot \mathfrak{n}.$$

But $P = \sum\limits_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} P\left[\mathfrak{n}\right] \cdot \mathfrak{n}$ shows that

$$A\left(P, \tau, \tau\right) = A\left(\sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} P\left[\mathfrak{n}\right] \cdot \mathfrak{n}, \tau, \tau\right) = \sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} A\left(P\left[\mathfrak{n}\right], \tau, \tau\right) \cdot \mathfrak{n},$$

so that the coefficient of $A\left(P, \tau, \tau\right)$ before $\mathfrak{m}$ equals $A\left(P\left[\mathfrak{m}\right], \tau, \tau\right)$. Since we denoted the coefficient of $A\left(P, \tau, \tau\right)$ before $\mathfrak{m}$ by $\left(A\left(P, \tau, \tau\right)\right)\left[\mathfrak{m}\right]$, this rewrites as $\left(A\left(P, \tau, \tau\right)\right)\left[\mathfrak{m}\right] = A\left(P\left[\mathfrak{m}\right], \tau, \tau\right)$, qed.

1 in $y_1, y_2, y_3, ...$ are $y_1, y_2, y_3, ...$ Let r be a positive integer. We have

$$\left(A\left(\sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s\right), \tau, \tau\right)\right)[y_r]$$
$$= A\left(\underbrace{\left(\sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s\right)\right)[y_r]}_{\substack{=-2S_{r+1}(\widetilde{w})+w_1w_r \\ \text{(by easy computations)}}}, \tau, \tau\right)$$
$$\left(\text{by (265), applied to } P = \sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s\right) \text{ and } \mathfrak{m} = y_r\right)$$
$$= A\left(-2S_{r+1}\left(\widetilde{w}\right) + w_1 w_r, \tau, \tau\right). \tag{266}$$

Denote the polynomial $-2S_{r+1}\left(\widetilde{w}\right) + w_1 w_r$ by $T_r\left(w\right)$. Then, (266) rewrites as

$$\left(A\left(\sum_{j=0}^{\infty} S_j\left(-2y\right) S_{j+1}\left(\widetilde{w}\right) \exp\left(\sum_{s>0} y_s w_s\right), \tau, \tau\right)\right)[y_r] = A\left(T_r\left(w\right), \tau, \tau\right). \tag{267}$$

We have $T_1\left(w\right) = w_2$, $T_2\left(w\right) = -\frac{w_1^3}{3} - \frac{2w_3}{3}$ and $T_3\left(w\right) = \frac{w_1 w_3}{3} - \frac{w_4}{2} - \frac{w_2^2}{4} - \frac{w_1^4}{12} - \frac{w_1^2 w_2}{2}$. Since $T_1\left(w\right)$ and $T_2\left(w\right)$ are odd, we have $A\left(T_1\left(w\right), \tau, \tau\right) = 0$ and $A\left(T_2\left(w\right), \tau, \tau\right) = 0$ (by Corollary 3.15.35). Therefore, taking coefficients with respect to the monomials y_1 and y_2 in the equation (265) yields tautologies. However, $T_3\left(w\right)$ is **not odd**. Applying

(267) to $r = 3$, we obtain

$$\left(A \left(\sum_{j=0}^{\infty} S_j(-2y) S_{j+1}(\widetilde{w}) \exp\left(\sum_{s>0} y_s w_s \right), \tau, \tau \right) \right) [y_3]$$
$$= A(T_3(w), \tau, \tau) = A\left(\frac{w_1 w_3}{3} - \frac{w_4}{2} - \frac{w_2^2}{4} - \frac{w_1^4}{12} - \frac{w_1^2 w_2}{2}, \tau, \tau \right)$$
$$= A\left(\frac{w_1 w_3}{3} - \frac{w_2^2}{4} - \frac{w_1^4}{12}, \tau, \tau \right) + \underbrace{A\left(-\frac{w_4}{2} - \frac{w_1^2 w_2}{2}, \tau, \tau \right)}_{\substack{=0 \\ \text{(by Corollary 3.15.35, since} \\ -\frac{w_4}{2} - \frac{w_1^2 w_2}{2} \text{ is odd)}}}$$
$$= A\left(\frac{w_1 w_3}{3} - \frac{w_2^2}{4} - \frac{w_1^4}{12}, \tau, \tau \right) = \left(\left(\frac{\frac{\partial}{\partial z_1}\frac{\partial}{\partial z_3}}{3} - \frac{\left(\frac{\partial}{\partial z_2}\right)^2}{4} - \frac{\left(\frac{\partial}{\partial z_1}\right)^4}{12} \right) (\tau(x-z)\tau(x+z)) \right) |_{z=0}$$
$$\left(\text{by the definition of } A\left(\frac{w_1 w_3}{3} - \frac{w_2^2}{4} - \frac{w_1^4}{12}, \tau, \tau \right) \right)$$
$$= \frac{1}{12} \left(\left(4 \frac{\partial}{\partial z_1}\frac{\partial}{\partial z_3} - 3\left(\frac{\partial}{\partial z_2}\right)^2 - \left(\frac{\partial}{\partial z_1}\right)^4 \right) (\tau(x-z)\tau(x+z)) \right) |_{z=0}$$
$$= \frac{1}{12} \left(\left(4 \frac{\partial}{\partial w_1}\frac{\partial}{\partial w_3} - 3\left(\frac{\partial}{\partial w_2}\right)^2 - \left(\frac{\partial}{\partial w_1}\right)^4 \right) (\tau(x-w)\tau(x+w)) \right) |_{w=0} .$$

Since $\frac{\partial}{\partial w_j} = \partial_{w_j}$ for every j, we rewrite this as

$$\left(A \left(\sum_{j=0}^{\infty} S_j(-2y) S_{j+1}(\widetilde{w}) \exp\left(\sum_{s>0} y_s w_s \right), \tau, \tau \right) \right) [y_3]$$
$$= \frac{1}{12} \left(\left(4\partial_{w_1}\partial_{w_3} - 3\partial_{w_2}^2 - \partial_{w_1}^4 \right) (\tau(x-w)\tau(x+w)) \right) |_{w=0} .$$

Hence, taking coefficients with respect to the monomial y_3 in the equation (259), we obtain
$$\frac{1}{12} \left(\left(4\partial_{w_1}\partial_{w_3} - 3\partial_{w_2}^2 - \partial_{w_1}^4 \right) (\tau(x-w)\tau(x+w)) \right) |_{w=0} = 0.$$
In other words,

$$\left(\partial_{w_1}^4 + 3\partial_{w_2}^2 - 4\partial_{w_1}\partial_{w_3} \right) (\tau(x-w)\tau(x+w)) |_{w=0} = 0. \qquad (268)$$

This does not yet look like a PDE in any usual form. We will now transform it into one.

We make the substitution $x_1 = x$, $x_2 = y$, $x_3 = t$, $x_m = c_m$ for $m \geq 4$. Here, x, y, t and c_m (for $m \geq 4$) are new symbols (in particularly, x and y no longer denote the sequences $(x_1, x_2, x_3, ...)$ and $(y_1, y_2, y_3, ...)$). Let $u = 2\partial_x^2 \log \tau$.

Proposition 3.15.38. The polynomial $\tau(x, y, t, c_4, c_5, ...)$ satisfies (268) if and only if the function u satisfies the KP equation

$$\frac{3}{4}\partial_y^2 u = \partial_x \left(\partial_t u - \frac{3}{2} u \partial_x u - \frac{1}{4}\partial_x^3 u\right)$$

(where c_4, c_5, c_6, ... are considered as constants).

Proof of Proposition 3.15.38. Optional homework exercise.

Thus, we know that any element τ of Ω gives rise to a solution of the KP equation (namely, the solution is $2\partial_x^2 \log(\sigma^{-1}(\tau))$). Two elements of Ω differing from each other by a scalar factor yield one and the same solution of the KP equation. Hence, any element of Gr gives rise to a solution of the KP equation. Since we know how to produce elements of Gr, we thus know how to produce solutions of the KP equation!

This does not give **all** solutions, and in fact we cannot even hope to find all solutions explicitly (since they depend on boundary conditions, and these can be arbitrarily nonexplicit), but we will use this to find a dense subset of them (in an appropriate sense).

The KP equation is not the KdV (Korteweg-de Vries) equation; but if we have a solution of the KP equation which does not depend on y, then this solution satisfies $\partial_t u - \frac{3}{2} u \partial_x u - \frac{1}{4}\partial_x^3 u = \text{const}$, and with some work it gives rise to a solution of the KdV equation (under appropriate decay-at-infinity conditions).

The equations corresponding to the coefficients of the monomials y_4, y_5, ... in (265) correspond to the *KP hierarchy* of higher-order PDEs. There is no point in writing them up explicitly; they become more and more complicated.

Corollary 3.15.39. Let λ be a partition. Then, $2\partial_x^2 \log(S_\lambda(x, y, t, c_4, c_5, ...))$ is a solution of the KP equation (and of the whole KP hierarchy), where c_4, c_5, c_6, ... are considered as constants.

Proof of Corollary 3.15.39. Write λ in the form $\lambda = (\lambda_0, \lambda_1, \lambda_2, ...)$. Let $(i_0, i_1, i_2, ...)$ be the sequence defined by $i_k = \lambda_k - k$ for every $k \in \mathbb{N}$. Then, $(i_0, i_1, i_2, ...)$ is a 0-degression, and we know that the elementary semiinfinite wedge $v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$ is in Ω. But Theorem 3.12.11 yields $\sigma^{-1}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...) = S_\lambda(x)$ (since $\lambda = (i_0 + 0, i_1 + 1, i_2 + 2, ...)$), so that $\sigma(S_\lambda(x)) = v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \in \Omega$. Hence, the function $2\partial_x^2 \log(S_\lambda(x, y, t, c_4, c_5, ...))$ satisfies the KP equation (and the whole KP hierarchy). This proves Corollary 3.15.39.

3.15.6. [unfinished] n-soliton solutions of KdV

Now we will construct other solutions of the KdV equations (which are called multi-soliton solutions).

We will identify the $\mathcal{A}$-modules $\mathcal{B}^{(0)}$ and $\mathcal{F}^{(0)}$ along the Boson-Fermion correspondence σ.

Definition 3.15.40. Define a quantum field $\Gamma(u, v) \in \left(\operatorname{End}\left(\mathcal{B}^{(0)}\right)\right)[[u, u^{-1}, v, v^{-1}]]$ by

$$\Gamma(u, v) = \exp\left(\sum_{j \geq 1} \frac{u^j - v^j}{j} a_{-j}\right) \exp\left(-\sum_{j \geq 1} \frac{u^{-j} - v^{-j}}{j} a_j\right). \qquad (269)$$

It is possible to rewrite the equality (269) in the following form:

$$\Gamma(u,v) = u : \Gamma(u)\Gamma^*(v) : . \tag{270}$$

However, before we can make sense of this equality (270), we need to explain what we mean by $: \Gamma(u)\Gamma^*(v) :$. Theorem 3.11.2 (applied to $m=-1$ and to $m=0$) yields that

$$\Gamma(u) = z\exp\left(\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\cdot\exp\left(-\sum_{j>0}\frac{a_j}{j}u^{-j}\right) \qquad \text{on } \mathcal{B}^{(-1)} \tag{271}$$

and

$$\Gamma^*(u) = z^{-1}\exp\left(-\sum_{j>0}\frac{a_{-j}}{j}u^j\right)\cdot\exp\left(\sum_{j>0}\frac{a_j}{j}u^{-j}\right) \qquad \text{on } \mathcal{B}^{(0)}. \tag{272}$$

Renaming u as v in (272), we obtain

$$\Gamma^*(v) = z^{-1}\exp\left(-\sum_{j>0}\frac{a_{-j}}{j}v^j\right)\cdot\exp\left(\sum_{j>0}\frac{a_j}{j}v^{-j}\right) \qquad \text{on } \mathcal{B}^{(0)}. \tag{273}$$

If we now extend the "normal ordered product" which we have defined on $U(\mathcal{A})$ to a "normal ordered multiplication map" $U(\mathcal{A})[z][[u,u^{-1}]] \times U(\mathcal{A})[z][[v,v^{-1}]] \to U(\mathcal{A})[z][[u,u^{-1},v,v^{-1}]]$

[...] [This isn't really that easy to formalize, and this formalization is wrong.]

[According to Etingof, one can put these power series on a firm footing by defining a series $\gamma \in (\operatorname{Hom}(A,B))[[u,u^{-1}]]$ (where A and B are two **graded** vector spaces) to be "sampled-rational" if every homogeneous $w \in A$ and every homogeneous $f \in B^*$ satisfy $\langle f, \gamma w\rangle \in \mathbb{C}(u)$. Sampled-rational power series form a torsion-free $\mathbb{C}(u)$-module[186]. And limits are defined sample-wise (see below). But it probably needs some explanations how $\mathbb{C}(u)$ is embedded in $\mathbb{C}[[u,u^{-1}]]$ (or what it means for an element of $\mathbb{C}[[u,u^{-1}]]$ to be a rational function).]

We will use the following notation, generalizing Definition 3.15.25:

Definition 3.15.41. Let A and B be two $\mathbb{C}$-vector spaces, and let $(u_1,u_2,...,u_\ell)$ be a sequence of distinct symbols. For every ℓ-tuple $\mathbf{i} \in \mathbb{Z}^\ell$, define a monomial $\mathbf{u}^{\mathbf{i}} \in \mathbb{C}((u_1,u_2,...,u_\ell))$ by $\mathbf{u}^{\mathbf{i}} = u_1^{i_1}u_2^{i_2}...u_\ell^{i_\ell}$, where the ℓ-tuple $\mathbf{i}$ is written in the form $\mathbf{i} = (i_1,i_2,...,i_\ell)$. Then, the map

$$A((u_1,u_2,...,u_\ell)) \times B((u_1,u_2,...,u_\ell)) \to (A\otimes B)((u_1,u_2,...,u_\ell)),$$

$$\left(\sum_{\mathbf{i}\in\mathbb{Z}^\ell} a_{\mathbf{i}}\mathbf{u}^{\mathbf{i}}, \sum_{\mathbf{i}\in\mathbb{Z}^\ell} b_{\mathbf{i}}\mathbf{u}^{\mathbf{i}}\right) \mapsto \sum_{\mathbf{i}\in\mathbb{Z}^\ell}\left(\sum_{\mathbf{j}\in\mathbb{Z}^\ell} a_{\mathbf{j}}\otimes b_{\mathbf{i}-\mathbf{j}}\right)\mathbf{u}^{\mathbf{i}}$$

(where all $a_{\mathbf{i}}$ lie in A and all $b_{\mathbf{i}}$ lie in B)

[186]But I don't think the composition of any two sampled-rational power series is sampled-rational. Ideas?

is well-defined (in fact, it is easy to see that for any Laurent series $\sum_{\mathbf{i}\in\mathbb{Z}^\ell} a_{\mathbf{i}}\mathbf{u}^{\mathbf{i}} \in A((u_1, u_2, ..., u_\ell))$ with all $a_{\mathbf{i}}$ lying in A, any Laurent series $\sum_{\mathbf{i}\in\mathbb{Z}^\ell} b_{\mathbf{i}}\mathbf{u}^{\mathbf{i}} \in B((u_1, u_2, ..., u_\ell))$ with all $b_{\mathbf{i}}$ lying in B, and any ℓ-tuple $\mathbf{i} \in \mathbb{Z}^\ell$, the sum $\sum_{\mathbf{j}\in\mathbb{Z}^\ell} a_{\mathbf{j}} \otimes b_{\mathbf{i}-\mathbf{j}}$ has only finitely many addends and vanishes if any coordinate of $\mathbf{i}$ is small enough) and $\mathbb{C}$-bilinear. Hence, it induces a $\mathbb{C}$-linear map

$$A((u_1, u_2, ..., u_\ell)) \otimes B((u_1, u_2, ..., u_\ell)) \to (A \otimes B)((u_1, u_2, ..., u_\ell)),$$
$$\left(\sum_{\mathbf{i}\in\mathbb{Z}^\ell} a_{\mathbf{i}}\mathbf{u}^{\mathbf{i}}\right) \otimes \left(\sum_{\mathbf{i}\in\mathbb{Z}^\ell} b_{\mathbf{i}}\mathbf{u}^{\mathbf{i}}\right) \mapsto \sum_{\mathbf{i}\in\mathbb{Z}^\ell} \left(\sum_{\mathbf{j}\in\mathbb{Z}^\ell} a_{\mathbf{j}} \otimes b_{\mathbf{i}-\mathbf{j}}\right) \mathbf{u}^{\mathbf{i}}$$
$$\text{(where all } a_{\mathbf{i}} \text{ lie in } A \text{ and all } b_{\mathbf{i}} \text{ lie in } B\text{)}.$$

This map will be denoted by $\Omega_{A,B,(u_1,u_2,...,u_\ell)}$. Clearly, when $\ell = 1$, this map $\Omega_{A,B,(u_1)}$ is identical with the map Ω_{A,B,u_1} defined in Definition 3.15.25.

Proposition 3.15.42. If $\tau \in \Omega$ and $a \in \mathbb{C}$, then

$$(1 + a\Gamma(u, v))\tau \in \Omega_{u,v},$$

where

$$\Omega_{u,v} = \left\{\tau \in \mathcal{B}^{(0)}((u, v)) \mid S(\tau \otimes \tau) = 0\right\}.$$

(Here, the S really means not the map $S : \mathcal{B}^{(0)} \otimes \mathcal{B}^{(0)} \to \mathcal{B}^{(1)} \otimes \mathcal{B}^{(-1)}$ itself, but rather the map $\left(\mathcal{B}^{(0)} \otimes \mathcal{B}^{(0)}\right)((u, v)) \to \left(\mathcal{B}^{(1)} \otimes \mathcal{B}^{(-1)}\right)((u, v))$ it induces. And $\tau \otimes \tau$ means not $\tau \otimes \tau \in \mathcal{B}^{(0)} \otimes \mathcal{B}^{(0)}$ but rather $\Omega_{\mathcal{B}^{(0)},\mathcal{B}^{(0)},(u,v)}(\tau \otimes \tau) \in \left(\mathcal{B}^{(0)} \otimes \mathcal{B}^{(0)}\right)((u, v))$.)

Corollary 3.15.43. For any $a^{(1)}, a^{(2)}, ..., a^{(n)} \in \mathbb{C}$, we have

$$\left(1 + a^{(1)}\Gamma(u_1, v_1)\right)\left(1 + a^{(2)}\Gamma(u_2, v_2)\right) ... \left(1 + a^{(n)}\Gamma(u_n, v_n)\right)\mathbf{1}$$
$$\in \Omega$$

(in fact, in an appropriate $\Omega_{u_1,v_1,u_2,v_2,...}$ rather than in Ω itself).

Idea of proof of Proposition. We will prove $\Gamma(u, v)^2 = 0$, but we will have to make sense of a term like $\Gamma(u, v)^2$ in order to define this. Thus, $1 + a\Gamma(u, v)$ will become $\exp(a\Gamma(u, v))$.

We will formalize this proof later.

But first, here is the punchline of this:

Proposition 3.15.44. Let $a^{(1)}, a^{(2)}, ..., a^{(n)} \in \mathbb{C}$. If $\tau = \left(1 + a^{(1)}\Gamma(u_1, v_1)\right)\left(1 + a^{(2)}\Gamma(u_2, v_2)\right) ... \left(1 + a^{(n)}\Gamma(u_n, v_n)\right)\mathbf{1}$, then $2\partial_x^2 \log \tau$ is given by a convergent series and defines a solution of KP depending on the parameters $a^{(i)}$, u_i and v_i.

This solution is called an *n-soliton solution.*

For $n = 1$, we have

$$\tau = (1 + a\Gamma(u,v))\,\mathbf{1} = 1 + a\exp\left((u-v)\,x + \left(u^2 - v^2\right)y + \left(u^3 - v^3\right)t + \left(u^4 - v^4\right)c_4 + ...\right).$$

Absorb the c_i parameters into a single constant c, which can be absorbed into a. So we get

$$\tau = 1 + a\exp\left((u-v)\,x + \left(u^2 - v^2\right)y + \left(u^3 - v^3\right)t\right).$$

This τ satisfies

$$2\partial_x^2 \log\tau = \frac{(u-v)^2}{2}\frac{1}{\cosh^2\left(\dfrac{1}{2}\left((u-v)\,x + \left(u^2-v^2\right)y + \left(u^3-v^3\right)\tau\right)\right)}.$$

Call this function U. To make it independent of y (so we get a solution of KdV equation), we set $v = -u$, and this becomes

$$U = \frac{2u^2}{\cosh^2\left(ux + u^3 t\right)}.$$

This is exactly the soliton solution of KdV.

But let us now give the promised proof of Proposition 3.15.42.

Proof of Proposition 3.15.42. Recall that $\Gamma(u,v) = u : \Gamma(u)\,\Gamma^*(v) :$. We can show:

Lemma 3.15.45. We have

$$\Gamma(u)\,\Gamma(v) = (u-v)\cdot : \Gamma(u)\,\Gamma(v) :$$

and

$$\Gamma(u)\,\Gamma^*(v) = \frac{1}{u-v} : \Gamma(u)\,\Gamma^*(v) :$$

and

$$\Gamma^*(u)\,\Gamma(v) = \frac{1}{u-v} : \Gamma^*(u)\,\Gamma(v) :$$

and

$$\Gamma^*(u)\,\Gamma^*(v) = (u-v)\cdot : \Gamma^*(u)\,\Gamma^*(v) : .$$

Proof of Lemma 3.15.45. We have

$$...\exp\left(\sum_{j>0}\frac{a_j}{j}u^{-j}\right)\exp\left(\sum_{k>0}\frac{a_{-k}}{k}v^k\right)...$$

and we have to switch these two terms. We get something like

$$\exp\left(-\log\left(1 - \frac{v}{u}\right)\right) = \frac{1}{1 - \dfrac{v}{u}} = \frac{u}{u-v}.$$

Etc.

We can generalize this: If $\varepsilon = 1$ or $\varepsilon = -1$, we can define Γ_ε by $\Gamma_{+1} = \Gamma$ and $\Gamma_{-1} = \Gamma^*$. Then,

Proposition 3.15.46. We have

$$\Gamma_{\varepsilon_1}(u_1)\Gamma_{\varepsilon_2}(u_2)...\Gamma_{\varepsilon_n}(u_n) = \prod_{i<j}(u_i - u_j)^{\varepsilon_i\varepsilon_j} : \Gamma_{\varepsilon_1}(u_1)\Gamma_{\varepsilon_2}(u_2)...\Gamma_{\varepsilon_n}(u_n) : .$$

Here, series are being expanded in the region where $|u_1| > |u_2| > ... > |u_n|$.

Corollary 3.15.47. The matrix elements of $\Gamma_{\varepsilon_1}(u_1)\Gamma_{\varepsilon_2}(u_2)...\Gamma_{\varepsilon_n}(u_n)$ (this means expressions of the form $(w^*, \Gamma_{\varepsilon_1}(u_1)\Gamma_{\varepsilon_2}(u_2)...\Gamma_{\varepsilon_n}(u_n)w)$ with $w \in \mathcal{B}^{(0)}$ and $w^* \in \mathcal{B}^{(0)*}$ (where * means restricted dual); a priori, these are only series) are series which converge to rational functions of the form

$$P(u) \cdot \prod_{i<j}(u_i - u_j)^{\varepsilon_i\varepsilon_j}, \qquad \text{where } P \in \mathbb{C}\left[u_1^{\pm 1}, u_2^{\pm 1}, ..., u_n^{\pm 1}\right].$$

This follows from the Proposition since matrix elements of normal ordered products are Laurent polynomials.

Corollary 3.15.48. We have $\Gamma(u', v')\Gamma(u, v) = \dfrac{(u'-u)(v'-v)}{(v'-u)(u'-v)} : \Gamma(u', v')\Gamma(u, v) :$.

Here, we cancelled $u - v$ and $u' - v'$ which is okay because our rational functions lie in an integral domain.

As a corollary of this corollary, we have:

Corollary 3.15.49. If $u \neq v$, then $\lim\limits_{\substack{u' \to u; \\ v' \to v}} \Gamma(u', v')\Gamma(u, v) = 0$. By which we mean that for any $w \in \mathcal{B}^{(0)}$ and $w^* \in \mathcal{B}^{(0)*}$, we have $\lim\limits_{\substack{u' \to u; \\ v' \to v}} (w^*, \Gamma(u', v')\Gamma(u, v)w) = 0$ as a rational function.

Informally, this can be written $(\Gamma(u, v))^2 = 0$. But this does not really make sense in a formal sense since we are not supposed to take squares of such power series.

Proof of Proposition 3.15.42. Recall that our idea was to use $1 + a\Gamma = \exp(a\Gamma)$ since $\Gamma^2 = 0$. But this is not rigorous since we cannot speak of Γ^2. So here is the actual proof:

We have (abbreviating $\Gamma(u, v)$ by Γ occasionally)

$$\begin{aligned} & S((1 + a\Gamma(u, v))\tau \otimes (1 + a\Gamma(u, v))\tau) \\ &= \underbrace{S(\tau \otimes \tau)}_{\substack{=0 \\ (\text{since } \tau \in \Omega)}} + a \underbrace{S(\Gamma \otimes 1 + 1 \otimes \Gamma)(\tau \otimes \tau)}_{\substack{=0 \\ (\text{since } S \text{ commutes with } \mathfrak{gl}_\infty, \\ \text{and coefficients of } \Gamma \text{ are in } \mathfrak{gl}_\infty, \\ \text{and } S(\tau\otimes\tau)=0)}} + a^2 S(\Gamma \otimes \Gamma)(\tau \otimes \tau) \\ &= a^2 S(\Gamma \otimes \Gamma)(\tau \otimes \tau). \end{aligned}$$

Remains to prove that $S(\Gamma \otimes \Gamma)(\tau \otimes \tau) = 0$.

We have

$$\begin{aligned}
&S\left(\Gamma\otimes\Gamma\right)\left(\tau\otimes\tau\right)\\
&=\lim_{\substack{u'\to u;\\ v'\to v}}\frac{1}{2}S\left(\Gamma\left(u,v\right)\tau\otimes\Gamma\left(u',v'\right)\tau+\Gamma\left(u',v'\right)\tau\otimes\Gamma\left(u,v\right)\tau\right)\\
&=\frac{1}{2}\lim_{\substack{u'\to u;\\ v'\to v}}\underbrace{S\left(\Gamma\left(u',v'\right)\otimes 1+1\otimes\Gamma\left(u',v'\right)\right)\left(\Gamma\left(u,v\right)\otimes 1+1\otimes\Gamma\left(u,v\right)\right)\left(\tau\otimes\tau\right)}_{\substack{=0\\ \text{(since } S \text{ commutes with these things)}}}\\
&\qquad -\frac{1}{2}\lim_{\substack{u'\to u;\\ v'\to v}}S\left(\underbrace{\Gamma\left(u',v'\right)\Gamma\left(u,v\right)}_{\to 0}\otimes 1+1\otimes\underbrace{\Gamma\left(u',v'\right)\Gamma\left(u,v\right)}_{\to 0}\right)\left(\tau\otimes\tau\right)\\
&=0.
\end{aligned}$$

This proves Proposition 3.15.42.

3.16. [unfinished] Representations of Vir revisited

We now come back to the representation theory of the Virasoro algebra Vir.

Recall that to every pair $\lambda=(c,h)$, we can attach a Verma module $M_\lambda^+=M_{c,h}^+$ over Vir. We will denote this module by $M_\lambda=M_{c,h}$, and its v_λ^+ by v_λ.

This module M_λ has a symmetric bilinear form $(\cdot,\cdot):M_\lambda^+\times M_\lambda^+\to\mathbb{C}$ such that $(v_\lambda,v_\lambda)=1$ and $(L_nv,w)=(v,L_{-n}w)$ for all $n\in\mathbb{Z}$, $v\in M_\lambda$ and $w\in M_\lambda$. This form is called the *Shapovalov form*, and is obtained from the invariant bilinear form $M_\lambda^+\times M_{-\lambda}^-\to\mathbb{C}$ by means of the involution on Vir.

Also, if $\lambda\in\mathbb{R}^2$, the module M_λ^+ has a Hermitian form $\langle\cdot,\cdot\rangle$ satisfying the same conditions.

We recall that M_λ has a unique irreducible quotient L_λ. We have asked questions about when it is unitary, etc.. We will try to answer some of these questions today.

Convention 3.16.1. Let us change the grading of the Virasoro algebra Vir to $\deg L_i=-i$. Correspondingly, M_λ becomes $M_\lambda=\bigoplus\limits_{n\geq 0}M_\lambda[n]$.

For any $n\geq 0$, we have the polynomial $\det_n(c,h)$ which is the determinant of the contravariant form $(\cdot,\cdot)$ in degree n. This polynomial is defined up to a constant scalar. Let us recall how it is defined:

Let (w_j) be a basis of $U(\mathrm{Vir}_-)[n]$ (where Vir_- is $\langle L_{-1},L_{-2},L_{-3},...\rangle$; this is now the **positive** part of Vir). Then, $\det_n(c,h)=\det\left(\left(w_Iv_\lambda,w_Jv_\lambda\right)_{I,J}\right)$. If we change the basis by a matrix S, the determinant multiplies by $(\det S)^2$.

For a Hermitian form, we can do the same when (c,h) is real, but then $\det_n(c,h)$ is defined up to a **positive** scalar, because now the determinant multiplies by $|\det S|^2$. Hence it makes sense to say that $\det_n(c,h)>0$.

Proposition 3.16.2. We have $\det_n(c,h)=0$ if and only if there exists a singular vector $w\neq 0$ in $M_{c,h}$ of degree $\leq n$ and >0.

In particular, if $\det_n(c,h)=0$, then $\det_{n+1}(c,h)=0$.

In fact, we will see that $\det_{n+1}$ is divisible by $\det_n$.

Proof of Proposition. Apparently this is supposed to follow from something we did.

We recall examples:

$$\det_1 = 2h,$$
$$\det_2 = 2h\left(16h^2 + 2hc - 10h + c\right).$$

Also recall that $M_{c,h}$ is irreducible if and only if every positive n satisfies $\det_n(c,h) \neq 0$.

Proposition 3.16.3. Let $(c,h) \in \mathbb{R}^2$. If $M_{c,h}$ is unitary, then $\det_n(c,h) > 0$ for all positive n.

More generally, if $L_{c,h}[n] \cong M_{c,h}[n]$ for some positive n, and $L_{c,h}$ is unitary, then $\det_n(c,h) > 0$.

Proof of Proposition. A positive-definite Hermitian matrix has positive determinant.

Theorem 3.16.4. Fix c. Regard $\det_m(c,h)$ as a polynomial in h. Then,

$$\det_m(c,h) = K \cdot h^{\sum\limits_{\substack{r,s\geq 1;\\ rs\leq m}} p(m-rs)} + (\text{lower terms})$$

for some nonzero constant K (which depends on the choice of the basis).

Proof. We computed before the leading term of $\det_m$ for any graded Lie algebra.

$\left(L_{-k}^{m_k}...L_{-1}^{m_1}v_\lambda, L_{-k}^{n_k}...L_{-1}^{n_1}v_\lambda\right)$: the main contribution to the leading term comes from diagonal.

What degree in h do we get?

If μ is a partition of m, we can write $m = 1k_1(\mu) + 2k_2(\mu) + ...$, where $k_i(\mu)$ is the number of times i occurs in μ.

$\left(L_{-\ell}^{k_\ell}...L_{-1}^{k_1}v, L_{-\ell}^{k_\ell}...L_{-1}^{k_1}v\right) = \left(v, L_1^{k_1}...L_\ell^{k_\ell}L_{-\ell}^{k_\ell}...L_{-1}^{k_1}v\right)$.

So μ contributes $k_1 + ... + k_\ell$ to the exponent of h.

So we conclude that the total exponent of h is $\sum\limits_{\mu\vdash m}\sum\limits_i k_i(\mu)$.

The rest is easy combinatorics:

Let $m(r,s)$ denote the number of partitions of m in which r occurs exactly s times. Then, $m(r,s) = p(m-rs) - p(m-r(s+1))$. Thus, with m and r fixed,

$$\begin{aligned}
\sum_s sm(r,s) &= \sum_s s\left(p(m-rs) - p(m-r(s+1))\right) \\
&= \sum_s sp(m-rs) - \sum_s sp(m-r(s+1)) \\
&= \sum_s sp(m-rs) - \sum_s (s-1)p(m-rs) \\
&\qquad \text{(here, we substituted } s-1 \text{ for } s \text{ in the second sum)} \\
&= \sum_s \underbrace{(s-(s-1))}_{=1} p(m-rs) = \sum_s p(m-rs).
\end{aligned}$$

So our job is to show that $\sum\limits_{\mu \vdash m} \sum\limits_{i} k_i(\mu) = \sum\limits_{\substack{r,s \geq 1; \\ rs \leq m}} sm(r,s)$. But $\sum\limits_{\substack{s \geq 1; \\ s \leq m}} sm(r,s)$ is the total number of occurrences of r in all partitions of m. Summed over r, it yields the total number of parts of all partitions of m. But this is also $\sum\limits_{\mu \vdash m} \sum\limits_{i} k_i(\mu)$, qed.

We now quote a theorem which was proved independently by Kac and Feigin-Fuchs:

Theorem 3.16.5. Suppose $rs \leq m$. Then, if

$$h = h_{r,s}(c) := \frac{1}{48}\left((13-c)\left(r^2+s^2\right) + \sqrt{(c-1)(c-25)}\left(r^2-s^2\right) - 24rs - 2 + 2c\right),$$

then $\det_m(c,h) = 0$. (This is true for any of the branches of the square root.)

Theorem 3.16.6. If $h = h_{r,s}(c)$, then $M_{c,h}$ has a nonzero singular vector in degree $1 \leq d \leq rs$.

Theorem 3.16.7 (Kac, also proved by Feigin-Fuchs)**.** We have

$$\det_m(c,h) = K_m \cdot \prod_{\substack{r,s \geq 1; \\ rs \leq m}} \left(h - h_{r,s}(c)\right)^{p(m-rs)},$$

where K_m is some constant. Note that we should choose the same branch of the square root in $\sqrt{(c-1)(c-25)}$ for $h_{r,s}$ and $h_{s,r}$. The square roots "cancel out" and give way to a polynomial in h and c.

To prove these, we will use the following lemma:

Lemma 3.16.8. Let $A(t)$ be a polynomial in one variable t with values in $\operatorname{End} V$, where V is a finite-dimensional vector space. Suppose that $\dim \operatorname{Ker}(A(0)) \geq n$. Then, $\det(A(t))$ is divisible by t^n.

Proof of Lemma 3.16.8. Pick a basis $e_1, e_2, ..., e_m$ of V such that the first n vectors $e_1, e_2, ..., e_n$ are in $\operatorname{Ker}(A(0))$. Then, the matrix of $A(t)$ in this basis has first n columns divisible by t, so that its determinant $\det(A(t))$ is divisible by t^n.

Proof of Theorem 3.16.7. Let $A = A(h)$ be the matrix of the contravariant form in degree m, considered as a polynomial in h. If $h = h_{r,s}(c)$, we have a singular vector w in degree $1 \leq d \leq rs$ (by Theorem 3.16.6), which generates a Verma submodule $M_{c,h'} \subseteq M_{c,h}$ (by Homework Set 3 problem 1) (the c is the same since c is central and thus acts by the same number on all vectors).

So $M_{c,h}[m] \supseteq M_{c,h'}[m-d]$. We also have $\dim(M_{c,h'}[m-d]) = p(m-d) \geq p(m-rs)$ (since $d \leq rs$) and $M_{c,h'}[m-d] \subseteq \operatorname{Ker}(\cdot,\cdot)$ (when $h = h_{r,s}$). Hence, $\dim(\operatorname{Ker}(\cdot,\cdot)) \geq p(m-rs)$. By Lemma 3.16.8, this yields that $\det_m(c,h)$ is divisible by $(h - h_{r,s}(c))^{p(m-rs)}$.

But it is easy to see that for Weil-generic c, the $h - h_{r,s}(c)$ are different, so that $\det_m(c,h)$ is divisible by $\prod\limits_{\substack{r,s \geq 1; \\ rs \leq m}} (h - h_{r,s}(c))^{p(m-rs)}$. But by Theorem 3.16.4, the leading term of $\det_m(c,h)$ is $K \cdot h^{\sum\limits_{\substack{r,s \geq 1; \\ rs \leq m}} p(m-rs)}$, which has exactly the same degree. So $\det_m(c,h)$ is a constant multiple of $\prod\limits_{\substack{r,s \geq 1; \\ rs \leq m}} (h - h_{r,s}(c))^{p(m-rs)}$. Theorem 3.16.7 is proven.

We will not prove Theorem 3.16.6, since we do not have the tools for that.

Corollary 3.16.9. The module $M_{c,h}$ is irreducible if and only if (c,h) does not lie on the lines

$$h - h_{r,r}(c) = 0 \iff h + \left(r^2 - 1\right)(c-1)/24 = 0$$

and quadrics (in fact, hyperbolas if we are over $\mathbb{R}$)

$$\begin{aligned} &(h - h_{r,s}(c))(h - h_{s,r}(c)) = 0 \\ \iff &\left(h - \frac{(r-s)^2}{4}\right)^2 + \frac{h}{24}(c-1)\left(r^2+s^2-2\right) + \frac{1}{576}\left(r^2-1\right)\left(s^2-1\right)(c-1)^2 \\ &\quad + \frac{1}{48}(c-1)(r-s)^2(rs+1) = 0. \end{aligned}$$

Corollary 3.16.10. (1) Let $h \geq 0$ and $c \geq 1$. Then, $L_{c,h}$ is unitary.

(2) Let $h > 0$ and $c > 1$. Then, $M_{c,h} \cong L_{c,h}$, so that $M_{c,h}$ is irreducible.

Proof of Corollary 3.16.10. **(2)** Lines and hyperbolas do not pass through the region. For part **(1)** we need a lemma:

Lemma 3.16.11. Let $\mathfrak{g}$ be a graded Lie algebra (with $\dim \mathfrak{g}_i \neq \infty$) with a real structure $\dagger$. Let $U \subseteq \mathfrak{g}_{0\mathbb{R}}^*$ be the set of all λ such that L_λ is unitary. Then, U is closed in the usual metric.

[**Note:** This lemma possibly needs additional assumptions, like the assumption that the map $\dagger$ reverses the degree (i. e., every $j \in \mathbb{Z}$ satisfies $\dagger(\mathfrak{g}_j) \subseteq \mathfrak{g}_{-j}$) and that $\mathfrak{g}_0$ is an abelian Lie algebra.]

Proof of Lemma. It follows from the fact that if (A_n) is a sequence of positive definite Hermitian matrices, and $\lim\limits_{n\to\infty} A_n = A_\infty$, then A_∞ is nonnegative definite.

Okay, sorry, we are not going to use this lemma; we will derive the special case we need.

Now I claim that if $h > 0$ and $c > 1$, then $L_{c,h} = M_{c,h}$ is unitary. We know this is true for some points of this region (namely, the ones "above the zigzag line"). Then everything follows from the fact that if $A(t)$ is a continuous family of nondegenerate Hermitian matrices parametrized by $t \in [0,1]$ such that $A(0) > 0$, then $A(t) > 0$ for every t. (This fact is because the signature of a nondegenerate Hermitian matrix is a continuous map to a discrete set, and thus constant on connected components.)

e. g., consider $M_{1,h}$ as a limit of $M_{1+\frac{1}{n},h}$ (this is irreducible for large n).

So the matrix of the form in $M_{1,h}[m]$ is a limit of the matrices for $M_{1+\frac{1}{n},h}[m]$. So the matrix for $M_{1,h}[m]$ is ≥ 0. But kernel lies in $J_{1,h}[m]$, so the form on $L_{1,h}[m] = \left(M_{1,h} \diagup J_{1,h}\right)[m]$ is strictly positive.

By analyzing the Kac curves, we can show (although *we* will *not* show) that in the region $0 \leq c < 1$, there are only countably many points where we possibly can have unitarity:

$c(m) = 1 - \dfrac{6}{(m+2)(m+3)}$;

$h_{r,s}(m) = \dfrac{((m+3)r - (m+2)s)^2 - 1}{4(m+2)(m+3)}$ with $1 \leq r \leq s \leq m+1$.

for $m \geq 0$.

In fact we will show that at these points we indeed have unitary representations.

Proposition 3.16.12. **(1)** If $c \geq 0$ and $L_{c,h}$ is unitary, then $h = 0$.

(2) We have $L_{0,h} = M_{0,h}$ if and only if $h \neq \dfrac{m^2-1}{24}$ for all $m \geq 0$.

(3) We have $L_{1,h} = M_{1,h}$ if and only if $h \neq \dfrac{m^2}{24}$ for all $m \geq 0$.

Proof. **(2)** and **(3)** follow immediately from the Kac determinant formula. For **(1)**, just compute $\det \begin{pmatrix} (L_{-N}^2 v, L_{-N}^2 v) & (L_{-N}^2 v, L_{-2N} v) \\ (L_{-2N} v, L_{-N}^2 v) & (L_{-2N} v, L_{-2N} v) \end{pmatrix} = 4N^3 h^2 (8h - 5N)$ (this is < 0 for high enough N as long as $h \neq 0$), so that the only possibility for unitarity is $h = 0$.

4. Affine Lie algebras

4.1. Introducing $\widehat{\mathfrak{gl}_n}$

Definition 4.1.1. Let V denote the vector representation of $\mathfrak{gl}_\infty$ defined in Definition 3.5.2.

Let n be a positive integer. Consider $L\mathfrak{gl}_n = \mathfrak{gl}_n [t, t^{-1}]$; this is the loop algebra of the Lie algebra $\mathfrak{gl}_n$. This loop algebra clearly acts on $\mathbb{C}^n [t, t^{-1}]$ (by $(At^i) \rightharpoonup (wt^j) = Awt^{i+j}$ for all $A \in \mathfrak{gl}_n$, $w \in \mathbb{C}^n$, $i \in \mathbb{Z}$ and $j \in \mathbb{Z}$). But we can identify the vector space $\mathbb{C}^n [t, t^{-1}]$ with V as follows: Let $(e_1, e_2, ..., e_n)$ be the standard basis of $\mathbb{C}^n$. Then we identify $e_i t^k \in \mathbb{C}^n [t, t^{-1}]$ with $v_{i-kn} \in V$ for every $i \in \{1, 2, ..., n\}$ and $k \in \mathbb{Z}$. The action of $L\mathfrak{gl}_n$ on $\mathbb{C}^n [t, t^{-1}]$ now becomes an action of $L\mathfrak{gl}_n$ on V. Hence, $L\mathfrak{gl}_n$ maps into $\operatorname{End} V$. More precisely, $L\mathfrak{gl}_n$ maps into $\overline{\mathfrak{a}_\infty} \subseteq \operatorname{End} V$. Here is a direct way to construct this mapping:

Let $a(t) \in L\mathfrak{gl}_n$ be a Laurent polynomial with coefficients in $\mathfrak{gl}_n$. Write $a(t)$ in the form $a(t) = \sum\limits_{k \in \mathbb{Z}} a_k t^k$ with all a_k lying in $\mathfrak{gl}_n$. Then, let $\operatorname{Toep}_n(a(t))$ be the matrix

$$\begin{pmatrix} ... & ... & ... & ... & ... \\ ... & a_0 & a_1 & a_2 & ... \\ ... & a_{-1} & a_0 & a_1 & ... \\ ... & a_{-2} & a_{-1} & a_0 & ... \\ ... & ... & ... & ... & ... \end{pmatrix} \in \overline{\mathfrak{a}_\infty}.$$

Formally speaking, this matrix is defined as the matrix whose $(ni + \alpha, nj + \beta)$-th entry equals the (α, β)-th entry of the $n \times n$ matrix a_{j-i} for all $i \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\alpha \in \{1, 2, ..., n\}$ and $\beta \in \{1, 2, ..., n\}$. In other words, this is the block matrix consisting of infinitely many $n \times n$-blocks such that the "i-th block diagonal" is filled with a_i's for every $i \in \mathbb{Z}$.

We thus have defined a map $\operatorname{Toep}_n : L\mathfrak{gl}_n \to \overline{\mathfrak{a}_\infty}$. This map Toep_n is injective, and is exactly the map $L\mathfrak{gl}_n \to \overline{\mathfrak{a}_\infty}$ we obtain from the above action of $L\mathfrak{gl}_n$ on V. In particular, this map Toep_n is a Lie algebra homomorphism.

In the following, we will often regard the injective map Toep_n as an inclusion, i. e., we will identify any $a(t) \in L\mathfrak{gl}_n$ with its image $\operatorname{Toep}_n(a(t)) \in \overline{\mathfrak{a}_\infty}$.

Note that I chose the notation Toep_n because of the notion of Toeplitz matrices. For any $a(t) \in L\mathfrak{gl}_n$, the matrix $\operatorname{Toep}_n(a(t))$ can be called an infinite "block-Toeplitz" matrix. If $n = 1$, then $\operatorname{Toep}_1(a(t))$ is an actual infinite Toeplitz matrix.

Example 4.1.2. Since $\mathfrak{gl}_1$ is a 1-dimensional abelian Lie algebra, we can identify $L\mathfrak{gl}_1$ with the Lie algebra $\overline{\mathcal{A}}$. The image $\operatorname{Toep}_1(L\mathfrak{gl}_1)$ is the abelian Lie subalgebra $\langle T^j \mid j \in \mathbb{Z}\rangle$ of $\overline{\mathfrak{a}_\infty}$ (where T is the shift operator) and is isomorphic to $\overline{\mathcal{A}}$.

It is easy to see that:

Proposition 4.1.3. Let n be a positive integer. Define an associative algebra structure on $L\mathfrak{gl}_n = \mathfrak{gl}_n[t, t^{-1}]$ by

$$\left(at^i\right) \cdot \left(bt^j\right) = abt^{i+j} \qquad \text{for all } a \in \mathfrak{gl}_n,\ b \in \mathfrak{gl}_n,\ i \in \mathbb{Z} \text{ and } j \in \mathbb{Z}.$$

Then, Toep_n is not only a Lie algebra homomorphism, but also a homomorphism of associative algebras.

Proof of Proposition 4.1.3. Let $a(t) \in L\mathfrak{gl}_n$ and $b(t) \in L\mathfrak{gl}_n$. Write $a(t)$ in the form $a(t) = \sum\limits_{k\in\mathbb{Z}} a_k t^k$ with all a_k lying in $\mathfrak{gl}_n$. Write $b(t)$ in the form $b(t) = \sum\limits_{k\in\mathbb{Z}} b_k t^k$ with all b_k lying in $\mathfrak{gl}_n$. By the definition of Toep_n, we have

$$\operatorname{Toep}_n(a(t)) = \begin{pmatrix} ... & ... & ... & ... & ... \\ ... & a_0 & a_1 & a_2 & ... \\ ... & a_{-1} & a_0 & a_1 & ... \\ ... & a_{-2} & a_{-1} & a_0 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}$$

$$\text{and} \qquad \operatorname{Toep}_n(b(t)) = \begin{pmatrix} ... & ... & ... & ... & ... \\ ... & b_0 & b_1 & b_2 & ... \\ ... & b_{-1} & b_0 & b_1 & ... \\ ... & b_{-2} & b_{-1} & b_0 & ... \\ ... & ... & ... & ... & ... \end{pmatrix}.$$

Hence,

$$\begin{aligned}
&\operatorname{Toep}_n(a(t)) \cdot \operatorname{Toep}_n(b(t)) \\
&= \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & a_0 & a_1 & a_2 & \ldots \\ \ldots & a_{-1} & a_0 & a_1 & \ldots \\ \ldots & a_{-2} & a_{-1} & a_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \cdot \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & b_0 & b_1 & b_2 & \ldots \\ \ldots & b_{-1} & b_0 & b_1 & \ldots \\ \ldots & b_{-2} & b_{-1} & b_0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \\
&= \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \sum\limits_{k\in\mathbb{Z}} a_{k-(-1)} b_{-1-k} & \sum\limits_{k\in\mathbb{Z}} a_{k-(-1)} b_{0-k} & \sum\limits_{k\in\mathbb{Z}} a_{k-(-1)} b_{1-k} & \ldots \\ \ldots & \sum\limits_{k\in\mathbb{Z}} a_{k-0} b_{-1-k} & \sum\limits_{k\in\mathbb{Z}} a_{k-0} b_{0-k} & \sum\limits_{k\in\mathbb{Z}} a_{k-0} b_{1-k} & \ldots \\ \ldots & \sum\limits_{k\in\mathbb{Z}} a_{k-1} b_{-1-k} & \sum\limits_{k\in\mathbb{Z}} a_{k-1} b_{0-k} & \sum\limits_{k\in\mathbb{Z}} a_{k-1} b_{1-k} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \\
&\qquad \text{(by the rule for multiplying block matrices)} \\
&= \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \sum\limits_{k\in\mathbb{Z}} a_k b_{(-1)+(-1)-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{(-1)+0-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{(-1)+1-k} & \ldots \\ \ldots & \sum\limits_{k\in\mathbb{Z}} a_k b_{0+(-1)-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{0+0-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{0+1-k} & \ldots \\ \ldots & \sum\limits_{k\in\mathbb{Z}} a_k b_{1+(-1)-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{1+0-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{1+1-k} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix} \qquad (274) \\
&\qquad \left(\text{since any } (i,j) \in \mathbb{Z}^2 \text{ satisfies } \sum_{k\in\mathbb{Z}} a_{k-i} b_{j-k} = \sum_{k\in\mathbb{Z}} a_k b_{i+j-k}\right).
\end{aligned}$$

On the other hand, multiplying $a(t) = \sum\limits_{k\in\mathbb{Z}} a_k t^k$ and $b(t) = \sum\limits_{k\in\mathbb{Z}} b_k t^k$, we obtain

$$a(t) \cdot b(t) = \left(\sum_{k\in\mathbb{Z}} a_k t^k\right) \cdot \left(\sum_{k\in\mathbb{Z}} b_k t^k\right) = \sum_{i\in\mathbb{Z}} \left(\sum_{k\in\mathbb{Z}} a_k b_{i-k}\right) t^i$$

(by the definition of the product of two Laurent polynomials), so that

$$\operatorname{Toep}_n(a(t) \cdot b(t)) = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \sum\limits_{k\in\mathbb{Z}} a_k b_{(-1)+(-1)-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{(-1)+0-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{(-1)+1-k} & \ldots \\ \ldots & \sum\limits_{k\in\mathbb{Z}} a_k b_{0+(-1)-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{0+0-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{0+1-k} & \ldots \\ \ldots & \sum\limits_{k\in\mathbb{Z}} a_k b_{1+(-1)-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{1+0-k} & \sum\limits_{k\in\mathbb{Z}} a_k b_{1+1-k} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix}$$

(by the definition of Toep_n). Compared with (274), this yields $\operatorname{Toep}_n(a(t)) \cdot \operatorname{Toep}_n(b(t)) = \operatorname{Toep}_n(a(t) \cdot b(t))$.

Now forget that we fixed $a(t)$ and $b(t)$. We thus have proven that every $a(t) \in L\mathfrak{gl}_n$ and $b(t) \in L\mathfrak{gl}_n$ satisfy $\operatorname{Toep}_n(a(t)) \cdot \operatorname{Toep}_n(b(t)) = \operatorname{Toep}_n(a(t) \cdot b(t))$. Combined with the fact that $\operatorname{Toep}_n(1) = \mathrm{id}$ (this is very easy to prove), this yields that Toep_n is a homomorphism of associative algebras. Hence, Toep_n is also a homomorphism of Lie algebras. Proposition 4.1.3 is proven.

Recall that the Lie algebra $\overline{\mathfrak{a}_\infty}$ has a central extension $\mathfrak{a}_\infty$, which equals $\overline{\mathfrak{a}_\infty} \oplus \mathbb{C}K$ as a vector space but has its Lie bracket defined using the cocycle α.

Proposition 4.1.4. Let $\alpha : \overline{\mathfrak{a}_\infty} \times \overline{\mathfrak{a}_\infty} \to \mathbb{C}$ be the Japanese cocycle.

Let $n \in \mathbb{N}$. Let $\omega : L\mathfrak{gl}_n \times L\mathfrak{gl}_n \to \mathbb{C}$ be the 2-cocycle on $L\mathfrak{gl}_n$ which is defined by

$$\omega\left(a\left(t\right), b\left(t\right)\right) = \sum_{k\in\mathbb{Z}} k \operatorname{Tr}\left(a_k b_{-k}\right) \qquad \text{for all } a\left(t\right) \in L\mathfrak{gl}_n \text{ and } b\left(t\right) \in L\mathfrak{gl}_n \quad (275)$$

(where we write $a\left(t\right)$ in the form $a\left(t\right) = \sum_{i\in\mathbb{Z}} a_i t^i$ with $a_i \in \mathfrak{gl}_n$, and where we write $b\left(t\right)$ in the form $b\left(t\right) = \sum_{i\in\mathbb{Z}} b_i t^i$ with $b_i \in \mathfrak{gl}_n$).

Then, the restriction of the Japanese cocycle $\alpha : \overline{\mathfrak{a}_\infty} \times \overline{\mathfrak{a}_\infty} \to \mathbb{C}$ to $L\mathfrak{gl}_n \times L\mathfrak{gl}_n$ is the 2-cocycle ω.

Remark 4.1.5. The 2-cocycle ω in Proposition 4.1.4 coincides with the cocycle ω defined in Definition 1.7.1 in the case when $\mathfrak{g} = \mathfrak{gl}_n$ and $(\cdot, \cdot)$ is the form $\mathfrak{gl}_n \times \mathfrak{gl}_n \to \mathbb{C}$, $(a, b) \mapsto \operatorname{Tr}(ab)$. The 1-dimensional central extension $\widehat{\mathfrak{gl}_n}_\omega$ induced by this 2-cocycle ω (by the procedure shown in Definition 1.7.1) will be denoted by $\widehat{\mathfrak{gl}_n}$ in the following. Note that $\widehat{\mathfrak{gl}_n} = L\mathfrak{gl}_n \oplus \mathbb{C}K$ as a vector space.

Note that the equality (275) can be rewritten in the suggestive form

$$\omega\left(a\left(t\right), b\left(t\right)\right) = \operatorname{Res}_{t=0} \operatorname{Tr}\left(da\left(t\right) b\left(t\right)\right) \qquad \text{for all } a\left(t\right) \in L\mathfrak{gl}_n \text{ and } b\left(t\right) \in L\mathfrak{gl}_n$$

(as long as the "matrix-valued differential form" $da\left(t\right) b\left(t\right)$ is understood correctly).

Proof of Proposition 4.1.4. We need to prove that $\alpha\left(a\left(t\right), b\left(t\right)\right) = \omega\left(a\left(t\right), b\left(t\right)\right)$ for any $a\left(t\right) \in L\mathfrak{gl}_n$ and $b\left(t\right) \in L\mathfrak{gl}_n$ (where, of course, we consider $a\left(t\right)$ and $b\left(t\right)$ as elements of $\overline{\mathfrak{a}_\infty}$ in the term $\alpha\left(a\left(t\right), b\left(t\right)\right)$).

Write $a\left(t\right)$ in the form $a\left(t\right) = \sum_{k\in\mathbb{Z}} a_k t^k$ with all a_k lying in $\mathfrak{gl}_n$. Write $b\left(t\right)$ in the form $b\left(t\right) = \sum_{k\in\mathbb{Z}} b_k t^k$ with all b_k lying in $\mathfrak{gl}_n$.

In the following, for any integers u and v, the (u, v)-th *block* of a matrix will mean the submatrix obtained by leaving only the rows numbered $un+1$, $un+2$, ..., $(u+1)n$ and the columns numbered $vn+1$, $vn+2$, ..., $(v+1)n$. (This, of course, makes sense only when the matrix has such rows and such columns.)

By the definition of our embedding $\operatorname{Toep}_n(a(t)) : L\mathfrak{gl}_n \to \overline{\mathfrak{a}_\infty}$, we have

$$a(t) = \operatorname{Toep}_n(a(t)) = \left(\begin{array}{ccccc} ... & ... & ... & ... & ... \\ ... & a_0 & a_1 & a_2 & ... \\ ... & a_{-1} & a_0 & a_1 & ... \\ ... & a_{-2} & a_{-1} & a_0 & ... \\ ... & ... & ... & ... & ... \end{array}\right) \qquad \text{and}$$

$$b(t) = \operatorname{Toep}_n(b(t)) = \left(\begin{array}{ccccc} ... & ... & ... & ... & ... \\ ... & b_0 & b_1 & b_2 & ... \\ ... & b_{-1} & b_0 & b_1 & ... \\ ... & b_{-2} & b_{-1} & b_0 & ... \\ ... & ... & ... & ... & ... \end{array}\right),$$

where the matrices $\left(\begin{array}{ccccc} ... & ... & ... & ... & ... \\ ... & a_0 & a_1 & a_2 & ... \\ ... & a_{-1} & a_0 & a_1 & ... \\ ... & a_{-2} & a_{-1} & a_0 & ... \\ ... & ... & ... & ... & ... \end{array}\right)$ and $\left(\begin{array}{ccccc} ... & ... & ... & ... & ... \\ ... & b_0 & b_1 & b_2 & ... \\ ... & b_{-1} & b_0 & b_1 & ... \\ ... & b_{-2} & b_{-1} & b_0 & ... \\ ... & ... & ... & ... & ... \end{array}\right)$ are understood as block matrices made of $n \times n$ blocks.

In order to compute $\alpha(a(t), b(t))$, let us write these two infinite matrices $a(t)$ and $b(t)$ as 2×2 block matrices **made of infinite blocks each**, where the blocks are separated as follows:

- The left blocks contain the j-th columns for all $j \leq 0$; the right blocks contain the j-th columns for all $j > 0$.

- The upper blocks contain the i-th rows for all $i \leq 0$; the lower blocks contain the i-th rows for all $i > 0$.

Written like this, the matrix $a(t)$ takes the form $\left(\begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array}\right)$ with

$$A_{11} = \left(\begin{array}{cccc} ... & ... & ... & ... \\ ... & a_0 & a_1 & a_2 \\ ... & a_{-1} & a_0 & a_1 \\ ... & a_{-2} & a_{-1} & a_0 \end{array}\right), \qquad A_{12} = \left(\begin{array}{cccc} ... & ... & ... & ... \\ a_3 & a_4 & a_5 & ... \\ a_2 & a_3 & a_4 & ... \\ a_1 & a_2 & a_3 & ... \end{array}\right),$$

$$A_{21} = \left(\begin{array}{cccc} ... & a_{-3} & a_{-2} & a_{-1} \\ ... & a_{-4} & a_{-3} & a_{-2} \\ ... & a_{-5} & a_{-4} & a_{-3} \\ ... & ... & ... & ... \end{array}\right), \qquad A_{22} = \left(\begin{array}{cccc} a_0 & a_1 & a_2 & ... \\ a_{-1} & a_0 & a_1 & ... \\ a_{-2} & a_{-1} & a_0 & ... \\ ... & ... & ... & ... \end{array}\right),$$

and the matrix $b(t)$ takes the form $\left(\begin{array}{cc} B_{11} & B_{12} \\ B_{21} & B_{22} \end{array}\right)$ with similarly-defined blocks B_{11}, B_{12}, B_{21} and B_{22}.

By the definition of α given in Theorem 3.7.6, we now have $\alpha(a(t), b(t)) = \operatorname{Tr}(-B_{12}A_{21} + A_{12}B_{21})$. We now need to compute $B_{12}A_{21}$ and $A_{12}B_{21}$ in order to simplify this.

Now, since $B_{12} = \left(\begin{array}{cccc} ... & ... & ... & ... \\ b_3 & b_4 & b_5 & ... \\ b_2 & b_3 & b_4 & ... \\ b_1 & b_2 & b_3 & ... \end{array}\right)$ and $A_{21} = \left(\begin{array}{cccc} ... & a_{-3} & a_{-2} & a_{-1} \\ ... & a_{-4} & a_{-3} & a_{-2} \\ ... & a_{-5} & a_{-4} & a_{-3} \\ ... & ... & ... & ... \end{array}\right)$, the matrix $B_{12}A_{21}$ is a matrix whose rows and columns are indexed by nonpositive integers, and whose (i,j)-th block equals $\sum\limits_{k\in\mathbb{Z};\ k>0} b_{k-(i+1)}a_{-k+(j+1)}$ for any pair of negative integers i and j. Similarly, the matrix $A_{12}B_{21}$ is a matrix whose rows and columns are indexed by nonpositive integers, and whose (i,j)-th block equals $\sum\limits_{k\in\mathbb{Z};\ k>0} a_{k-(i+1)}b_{-k+(j+1)}$ for any pair of negative integers i and j. Thus, the matrix $-B_{12}A_{21}+A_{12}B_{21}$ is a matrix whose rows and columns are indexed by nonpositive integers, and whose (i,j)-th block equals $-\sum\limits_{k\in\mathbb{Z};\ k>0} b_{k-(i+1)}a_{-k+(j+1)}+\sum\limits_{k\in\mathbb{Z};\ k>0} a_{k-(i+1)}b_{-k+(j+1)}$ for any pair of negative integers i and j. But since $\operatorname{Tr}(-B_{12}A_{21}+A_{12}B_{21})$ is clearly the sum of the traces of the (i,i)-th blocks of the matrix $-B_{12}A_{21}+A_{12}B_{21}$ over all negative integers i, we thus have

$$\begin{aligned}
\operatorname{Tr}(-B_{12}A_{21}+A_{12}B_{21}) &= \sum_{i\in\mathbb{Z};\ i<0} \operatorname{Tr}\left(-\sum_{k\in\mathbb{Z};\ k>0} b_{k-(i+1)}a_{-k+(i+1)} + \sum_{k\in\mathbb{Z};\ k>0} a_{k-(i+1)}b_{-k+(i+1)}\right) \\
&= \sum_{i\in\mathbb{Z};\ i\leq 0} \operatorname{Tr}\left(-\sum_{k\in\mathbb{Z};\ k>0} b_{k-i}a_{-k+i} + \sum_{k\in\mathbb{Z};\ k>0} a_{k-i}b_{-k+i}\right) \\
&\qquad \left(\text{here, we substituted } i \text{ for } i+1\right) \\
&= \sum_{i\in\mathbb{Z};\ i\geq 0} \operatorname{Tr}\left(-\sum_{k\in\mathbb{Z};\ k>0} b_{k+i}a_{-k-i} + \sum_{k\in\mathbb{Z};\ k>0} a_{k+i}b_{-k-i}\right) \\
&\qquad \left(\text{here, we substituted } i \text{ for } -i \text{ in the first sum}\right).
\end{aligned}$$

We are now going to split the first sum on the right hand side and get the Tr out of it. To see that this is allowed, we notice that each of the infinite sums $\sum\limits_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}} b_{k+i}a_{-k-i}$ and $\sum\limits_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}} a_{k+i}b_{-k-i}$ converges with respect to the discrete topology[187]. Hence, we can

[187] *Proof.* Since $\sum\limits_{k\in\mathbb{Z}} a_k t^k = a(t) \in L\mathfrak{gl}_n$, only finitely many $k\in\mathbb{Z}$ satisfy $a_k \neq 0$. Hence, there exists some $N\in\mathbb{Z}$ such that every $\nu\in\mathbb{Z}$ satisfying $\nu<N$ satisfies $a_\nu = 0$. Consider this N. Any pair $(i,k)\in\mathbb{Z}^2$ such that $k+i>-N$ satisfies $-k-i = -\underbrace{(k+i)}_{>-N} < N$ and thus $a_{-k-i}=0$ (because we know that every $\nu\in\mathbb{Z}$ satisfying $\nu<N$ satisfies $a_\nu=0$) and thus $b_{k+i}a_{-k-i}=0$. Thus, all but finitely many pairs $(i,k)\in\mathbb{Z}^2$ such that $i\geq 0$ and $k>0$ satisfy $b_{k+i}a_{-k-i}=0$ (because it is clear that all but finitely many pairs $(i,k)\in\mathbb{Z}^2$ such that $i\geq 0$ and $k>0$ satisfy $k+i>-N$). In other words, the sum $\sum\limits_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}} b_{k+i}a_{-k-i}$ converges with respect to the discrete topology. A similar argument shows that the sum $\sum\limits_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}} a_{k+i}b_{-k-i}$ converges with respect to the discrete topology.

transform these sums as we please: For example,

$$\begin{aligned}
&\sum_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}} b_{k+i}a_{-k-i}\\
&=\sum_{\substack{\ell\in\mathbb{Z};\\ \ell>0}} \sum_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0;\\ k+i=\ell}} \underbrace{b_{k+i}}_{\substack{=b_\ell\\ (\text{since } k+i=\ell)}} \underbrace{a_{-k-i}}_{\substack{=a_{-(k+i)}=a_{-\ell}\\ (\text{since } k+i=\ell)}} \qquad (\text{since } k+i>0 \text{ for all } i\geq 0 \text{ and } k>0)\\
&=\sum_{\substack{\ell\in\mathbb{Z};\\ \ell>0}} \underbrace{\sum_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0;\\ k+i=\ell}} b_\ell a_{-\ell}}_{\substack{=\ell b_\ell a_{-\ell}\\ (\text{since there exist exactly } \ell \text{ pairs } (i,k)\in\mathbb{Z}^2\\ \text{satisfying } i\geq 0,\ k>0 \text{ and } k+i=\ell)}} = \sum_{\substack{\ell\in\mathbb{Z};\\ \ell>0}} \ell b_\ell a_{-\ell} = \sum_{\substack{k\in\mathbb{Z};\\ k>0}} k b_k a_{-k} \qquad (276)
\end{aligned}$$

(here, we renamed the summation index ℓ as k) and similarly

$$\sum_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}} a_{k+i}b_{-k-i} = \sum_{\substack{k\in\mathbb{Z};\\ k>0}} k a_k b_{-k}. \qquad (277)$$

The equality (276) becomes

$$\begin{aligned}
\sum_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}} b_{k+i}a_{-k-i} &= \sum_{\substack{k\in\mathbb{Z};\\ k>0}} k b_k a_{-k} = \sum_{\substack{k\in\mathbb{Z};\\ k<0}} (-k)\, b_{-k} a_k\\
&\qquad (\text{here, we substituted } k \text{ for } -k \text{ in the first sum})\\
&= -\sum_{\substack{k\in\mathbb{Z};\\ k<0}} k b_{-k} a_k,
\end{aligned}$$

so that

$$\sum_{\substack{k\in\mathbb{Z};\\ k<0}} k b_{-k} a_k = -\sum_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}} b_{k+i}a_{-k-i}. \qquad (278)$$

But

$$\begin{aligned}
&\omega\left(a\left(t\right),b\left(t\right)\right)\\
&=\sum_{k\in\mathbb{Z}}k\operatorname{Tr}\left(a_kb_{-k}\right)=\sum_{\substack{k\in\mathbb{Z};\\ k<0}}k\underbrace{\operatorname{Tr}\left(a_kb_{-k}\right)}_{=\operatorname{Tr}\left(b_{-k}a_k\right)}+\underbrace{0\operatorname{Tr}\left(a_0b_{-0}\right)}_{=0}+\sum_{\substack{k\in\mathbb{Z};\\ k>0}}k\operatorname{Tr}\left(a_kb_{-k}\right)\\
&=\underbrace{\sum_{\substack{k\in\mathbb{Z};\\ k<0}}k\operatorname{Tr}\left(b_{-k}a_k\right)}_{=\operatorname{Tr}\left(\sum\limits_{\substack{k\in\mathbb{Z};\\ k<0}}kb_{-k}a_k\right)}+\underbrace{\sum_{\substack{k\in\mathbb{Z};\\ k>0}}k\operatorname{Tr}\left(a_kb_{-k}\right)}_{=\operatorname{Tr}\left(\sum\limits_{\substack{k\in\mathbb{Z};\\ k>0}}ka_kb_{-k}\right)}\\
&=\operatorname{Tr}\left(\underbrace{\sum_{\substack{k\in\mathbb{Z};\\ k<0}}kb_{-k}a_k}_{\substack{=-\sum\limits_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}}b_{k+i}a_{-k-i}\\ \text{(by (278))}}}\right)+\operatorname{Tr}\left(\underbrace{\sum_{\substack{k\in\mathbb{Z};\\ k>0}}ka_kb_{-k}}_{\substack{=\sum\limits_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}}a_{k+i}b_{-k-i}\\ \text{(by (277))}}}\right)\\
&=\operatorname{Tr}\left(-\sum_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}}b_{k+i}a_{-k-i}\right)+\operatorname{Tr}\left(\sum_{\substack{(i,k)\in\mathbb{Z}^2;\\ i\geq 0;\ k>0}}a_{k+i}b_{-k-i}\right)\\
&=\operatorname{Tr}\left(-\sum_{i\in\mathbb{Z};\ i\geq 0}\ \sum_{k\in\mathbb{Z};\ k>0}b_{k+i}a_{-k-i}\right)+\operatorname{Tr}\left(\sum_{i\in\mathbb{Z};\ i\geq 0}\ \sum_{k\in\mathbb{Z};\ k>0}a_{k+i}b_{-k-i}\right)\\
&\qquad\qquad\text{(here, we have unfolded our single sums into double sums)}\\
&=\sum_{i\in\mathbb{Z};\ i\geq 0}\operatorname{Tr}\left(-\sum_{k\in\mathbb{Z};\ k>0}b_{k+i}a_{-k-i}\right)+\sum_{i\in\mathbb{Z};\ i\geq 0}\operatorname{Tr}\left(\sum_{k\in\mathbb{Z};\ k>0}a_{k+i}b_{-k-i}\right)\\
&=\sum_{i\in\mathbb{Z};\ i\geq 0}\operatorname{Tr}\left(-\sum_{k\in\mathbb{Z};\ k>0}b_{k+i}a_{-k-i}+\sum_{k\in\mathbb{Z};\ k>0}a_{k+i}b_{-k-i}\right)=\operatorname{Tr}\left(-B_{12}A_{21}+A_{12}B_{21}\right)\\
&=\alpha\left(a\left(t\right),b\left(t\right)\right).
\end{aligned}$$

Thus, $\alpha\left(a\left(t\right),b\left(t\right)\right)=\omega\left(a\left(t\right),b\left(t\right)\right)$ is proven, so we have verified Proposition 4.1.4.

Note that Proposition 4.1.4 gives a new proof of Proposition 3.7.13. This proof (whose details are left to the reader) uses two easy facts:

- If $\sigma:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ is a 2-coboundary on a Lie algebra $\mathfrak{g}$, and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$, then $\sigma\mid_{\mathfrak{h}\times\mathfrak{h}}$ must be a 2-coboundary on $\mathfrak{h}$.

- For any positive integer n, the 2-cocycle ω of Proposition 4.1.4 is not a 2-coboundary.

But if we look closely at this argument, we see that it is not a completely new proof; it is a direct generalization of the proof of Proposition 3.7.13 that we gave above. In fact, in the particular case when $n = 1$, our embedding of $L\mathfrak{gl}_n$ into $\overline{\mathfrak{a}_\infty}$ becomes the canonical injection of the abelian Lie subalgebra $\langle T^j \mid j \in \mathbb{Z}\rangle$ into $\overline{\mathfrak{a}_\infty}$ (where T is as in the proof of Proposition 3.7.13), and we see that what we just did was generalizing that abelian Lie subalgebra.

Definition 4.1.6. Due to Proposition 4.1.4, the restriction of the 2-cocycle α to $L\mathfrak{gl}_n \times L\mathfrak{gl}_n$ is the 2-cocycle ω. Thus, the 1-dimensional central extension of $L\mathfrak{gl}_n$ determined by the 2-cocycle ω canonically injects into the 1-dimensional central extension of $\overline{\mathfrak{a}_\infty}$ determined by the 2-cocycle α. If we recall that the 1-dimensional central extension of $L\mathfrak{gl}_n$ by the 2-cocycle ω is $\widehat{\mathfrak{gl}_n}$ whereas the 1-dimensional central extension of $\overline{\mathfrak{a}_\infty}$ determined by the 2-cocycle α is $\mathfrak{a}_\infty$, we can rewrite this as follows: We have an injection $\widehat{\mathfrak{gl}_n} \to \mathfrak{a}_\infty$ which lifts the inclusion $L\mathfrak{gl}_n \subseteq \overline{\mathfrak{a}_\infty}$ and sends K to K. We denote this inclusion map $\widehat{\mathfrak{gl}_n} \to \mathfrak{a}_\infty$ by $\widehat{\mathrm{Toep}_n}$, but we will often consider it as an inclusion.

Similarly, we can get an inclusion $\widehat{\mathfrak{sl}_n} \subseteq \mathfrak{a}_\infty$ which lifts the inclusion $L\mathfrak{sl}_n \subseteq \overline{\mathfrak{a}_\infty}$.

So $\mathcal{B}^{(m)} \cong \mathcal{F}^{(m)}$ is a module over $\widehat{\mathfrak{gl}_n}$ and $\widehat{\mathfrak{sl}_n}$ at level 1 (this means that K acts as 1).

Corollary 4.1.7. There is a Lie algebra isomorphism $\widehat{\phi} : \mathcal{A} \to \widehat{\mathfrak{gl}_1}$ which sends K to K and sends a_m to $T^m \in \widehat{\mathfrak{gl}_1}$ for every $m \in \mathbb{Z}$. (Here, we are considering the injection $\widehat{\mathfrak{gl}_1} \to \mathfrak{a}_\infty$ as an inclusion, so that $\widehat{\mathfrak{gl}_1}$ is identified with the image of this inclusion.)

Proof of Corollary 4.1.7. There is an obvious Lie algebra isomorphism $\phi : \overline{\mathcal{A}} \to L\mathfrak{gl}_1$ which sends a_m to $t^m \in L\mathfrak{gl}_1$ for every $m \in \mathbb{Z}$. This isomorphism ϕ is easily seen to satisfy

$$\omega\left(\phi\left(x\right), \phi\left(y\right)\right) = \omega'\left(x, y\right) \qquad \text{for all } x \in \overline{\mathcal{A}} \text{ and } y \in \overline{\mathcal{A}}, \tag{279}$$

where $\omega : L\mathfrak{gl}_1 \times L\mathfrak{gl}_1 \to \mathbb{C}$ is the 2-cocycle on $L\mathfrak{gl}_1$ defined in Proposition 4.1.4, and $\omega' : \overline{\mathcal{A}} \times \overline{\mathcal{A}} \to \mathbb{C}$ is the 2-cocycle on $\overline{\mathcal{A}}$ defined by

$$\omega'\left(a_k, a_\ell\right) = k\delta_{k,-\ell} \qquad \text{for all } k \in \mathbb{Z} \text{ and } \ell \in \mathbb{Z}.$$

Thus, the Lie algebra isomorphism $\phi : \overline{\mathcal{A}} \to L\mathfrak{gl}_1$ gives rise to an isomorphism $\widehat{\phi}$ from the extension of $\overline{\mathcal{A}}$ defined by the 2-cocycle ω' to the extension of $L\mathfrak{gl}_1$ defined by the 2-cocycle ω. Since the extension of $\overline{\mathcal{A}}$ defined by the 2-cocycle ω' is $\mathcal{A}$, while the extension of $L\mathfrak{gl}_1$ defined by the 2-cocycle ω is $\widehat{\mathfrak{gl}_1}$, this rewrites as follows: The Lie algebra isomorphism $\phi : \overline{\mathcal{A}} \to L\mathfrak{gl}_1$ gives rise to an isomorphism $\widehat{\phi} : \mathcal{A} \to \widehat{\mathfrak{gl}_1}$. This isomorphism $\widehat{\phi}$ sends K to K, and sends a_m to $t^m \in \widehat{\mathfrak{gl}_1}$ for every $m \in \mathbb{Z}$. Since t^m corresponds to T^m under our inclusion $\widehat{\mathfrak{gl}_1} \to \mathfrak{a}_\infty$ (in fact, $\mathrm{Toep}_1\left(t^m\right) = T^m$), this shows that $\widehat{\phi}$ sends a_m to $T^m \in \widehat{\mathfrak{gl}_1}$ for every $m \in \mathbb{Z}$. Corollary 4.1.7 is thus proven.

Proposition 4.1.8. Let n be a positive integer. Consider the shift operator T. Let us regard the injections $\overline{\mathfrak{a}_\infty} \to \mathfrak{a}_\infty$, $L\mathfrak{gl}_n \to \overline{\mathfrak{a}_\infty}$ and $\widehat{\mathfrak{gl}_n} \to \mathfrak{a}_\infty$ as inclusions, so that $L\mathfrak{gl}_n$, $\widehat{\mathfrak{gl}_n}$ and $\mathfrak{a}_\infty$ all become subspaces of $\mathfrak{a}_\infty$.

(a) For every $m \in \mathbb{Z}$, we have $T^m \in L\mathfrak{gl}_n \subseteq \widehat{\mathfrak{gl}_n}$.

(b) We have $\widehat{\mathfrak{gl}_1} \subseteq \widehat{\mathfrak{gl}_n}$. Hence, the Lie algebra isomorphism $\widehat{\phi} : \mathcal{A} \to \widehat{\mathfrak{gl}_1}$ constructed in Corollary 4.1.7 induces a Lie algebra injection $\mathcal{A} \to \widehat{\mathfrak{gl}_n}$ (which sends every $a \in \mathcal{A}$ to $\widehat{\phi}(a) \in \widehat{\mathfrak{gl}_n}$). The restriction of the $\widehat{\mathfrak{gl}_n}$-module $\mathcal{F}^{(m)}$ by means of this injection is the $\mathcal{A}$-module $\mathcal{F}^{(m)}$ that we know.

First proof of Proposition 4.1.8. **(a)** We recall that $T = \begin{pmatrix} ... & ... & ... & ... & ... & ... \\ ... & 0 & 1 & 0 & 0 & ... \\ ... & 0 & 0 & 1 & 0 & ... \\ ... & 0 & 0 & 0 & 1 & ... \\ ... & 0 & 0 & 0 & 0 & ... \\ ... & ... & ... & ... & ... & ... \end{pmatrix}$ (this is the matrix which has 1's on the 1-st diagonal and 0's everywhere else). Clearly, $T \in \overline{\mathfrak{a}_\infty}$. We want to prove that T lies in $L\mathfrak{gl}_n \subseteq \overline{\mathfrak{a}_\infty}$.

Let $a_0 = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \end{pmatrix}$ (this is the $n \times n$ matrix which has 1's on the 1-st diagonal and 0's everywhere else).

Let $a_1 = \begin{pmatrix} 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ 0 & 0 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & 0 & ... & 0 \\ 1 & 0 & 0 & ... & 0 \end{pmatrix}$ (this is the $n \times n$ matrix which has a 1 in its lowermost leftmost corner, and 0's everywhere else).

Then, $T = \operatorname{Toep}_n(a_0 + ta_1)$. Thus, for every $m \in \mathbb{N}$, we have

$$\begin{aligned} T^m &= (\operatorname{Toep}_n(a_0 + ta_1))^m = \operatorname{Toep}_n((a_0 + ta_1)^m) \qquad \text{(because of Proposition 4.1.3)} \\ &\in \operatorname{Toep}_n(L\mathfrak{gl}_n) = L\mathfrak{gl}_n \qquad \text{(since we regard } \operatorname{Toep}_n \text{ as an inclusion)}. \end{aligned}$$

Since it is easy to see that $T^{-1} \in L\mathfrak{gl}_n$ as well[188], a similar argument yields that $(T^{-1})^m \in L\mathfrak{gl}_n$ for all $m \in \mathbb{N}$. In other words, $T^{-m} \in L\mathfrak{gl}_n$ for all $m \in \mathbb{N}$. In other words, $T^m \in \mathfrak{gl}_n$ for all nonpositive integers m. Combined with the fact that $T^m \in L\mathfrak{gl}_n$ for all $m \in \mathbb{N}$, this yields that $T^m \in L\mathfrak{gl}_n$ for all $m \in \mathbb{Z}$. Since $L\mathfrak{gl}_n \subseteq \widehat{\mathfrak{gl}_n}$, we thus have $T^m \in L\mathfrak{gl}_n \subseteq \widehat{\mathfrak{gl}_n}$ for all $m \in \mathbb{Z}$. This proves Proposition 4.1.8 **(a)**.

(b) For every $a(t) \in L\mathfrak{gl}_1$, we have $\operatorname{Toep}_1(a(t)) \in \langle T^j \mid j \in \mathbb{Z} \rangle$ [189]. Thus, $\operatorname{Toep}_1(L\mathfrak{gl}_1) \subseteq \langle T^j \mid j \in \mathbb{Z} \rangle$. Since we are considering Toep_1 as an inclusion, this becomes $L\mathfrak{gl}_1 \subseteq \langle T^j \mid j \in \mathbb{Z} \rangle$. Combined with $\langle T^j \mid j \in \mathbb{Z} \rangle \subseteq L\mathfrak{gl}_n$ (because every $m \in \mathbb{Z}$ satisfies $T^m \in L\mathfrak{gl}_n$ (according to Proposition 4.1.8 **(a)**)), this yields $L\mathfrak{gl}_1 \subseteq L\mathfrak{gl}_n$. Thus, $\widehat{\mathfrak{gl}_1} \subseteq \widehat{\mathfrak{gl}_n}$.

Hence, the Lie algebra isomorphism $\widehat{\phi} : \mathcal{A} \to \widehat{\mathfrak{gl}_1}$ constructed in Corollary 4.1.7 induces a Lie algebra injection $\mathcal{A} \to \widehat{\mathfrak{gl}_n}$ (which sends every $a \in \mathcal{A}$ to $\widehat{\phi}(a) \in \widehat{\mathfrak{gl}_n}$). This

[188] This is analogous to $T \in L\mathfrak{gl}_n$ (because T^{-1} is the matrix which has 1's on the (-1)-st diagonal and 0's everywhere else).

[189] *Proof.* Let $a(t) \in L\mathfrak{gl}_1$. Write $a(t)$ in the form $\sum\limits_{i \in \mathbb{Z}} a_i t^i$ with $a_i \in \mathfrak{gl}_1$. Then, of course, the a_i are

injection is exactly the embedding $\mathcal{A} \to \mathfrak{a}_\infty$ constructed in Definition 3.7.14 (apart from the fact that its target is $\widehat{\mathfrak{gl}_n}$ rather than $\mathfrak{a}_\infty$). Hence, the restriction of the $\widehat{\mathfrak{gl}_n}$-module $\mathcal{F}^{(m)}$ by means of this injection is the $\mathcal{A}$-module $\mathcal{F}^{(m)}$ that we know[190]. This proves Proposition 4.1.8 **(b)**.

Our inclusions $L\mathfrak{gl}_n \subseteq \overline{\mathfrak{a}_\infty}$ and $\widehat{\mathfrak{gl}_n} \subseteq \mathfrak{a}_\infty$ can be somewhat refined: For any positive integers n and N satisfying $n \mid N$, we have $L\mathfrak{gl}_n \subseteq L\mathfrak{gl}_N$ and $\widehat{\mathfrak{gl}_n} \subseteq \widehat{\mathfrak{gl}_N}$. Let us formulate this more carefully without abuse of notation:

Proposition 4.1.9. Let n and N be positive integers such that $n \mid N$. Then, the inclusion $\operatorname{Toep}_n : L\mathfrak{gl}_n \to \overline{\mathfrak{a}_\infty}$ factors through the inclusion $\operatorname{Toep}_N : L\mathfrak{gl}_N \to \overline{\mathfrak{a}_\infty}$. More precisely:

Let $d = \dfrac{N}{n}$. Let $\operatorname{Toep}_{n,N} : L\mathfrak{gl}_n \to L\mathfrak{gl}_N$ be the map which sends every $a(t) \in L\mathfrak{gl}_n$ to

$$\sum_{\ell \in \mathbb{Z}} \underbrace{\begin{pmatrix} a_{(j-i)d} & a_{(j-i)d+1} & a_{(j-i)d+2} & \cdots & a_{(j-i)d+(d-1)} \\ a_{(j-i)d-1} & a_{(j-i)d} & a_{(j-i)d+1} & \cdots & a_{(j-i)d+(d-2)} \\ a_{(j-i)d-2} & a_{(j-i)d-1} & a_{(j-i)d} & \cdots & a_{(j-i)d+(d-3)} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{(j-i)d-(d-1)} & a_{(j-i)d-(d-2)} & a_{(j-i)d-(d-3)} & \cdots & a_{(j-i)d} \end{pmatrix}}_{\substack{\text{this is an } N\times N\text{-matrix constructed as a } d\times d\text{-block matrix} \\ \text{consisting of } n\times n\text{-blocks; one can formally define this matrix} \\ \text{as the } N\times N\text{-matrix whose } (nI+\alpha, nJ+\beta)\text{-th entry equals} \\ \text{the } (\alpha,\beta)\text{-th entry of } a_{(j-i)d+J-I} \text{ for all } I\in\{0,1,\ldots,d-1\}, \\ J\in\{0,1,\ldots,d-1\},\ \alpha\in\{1,2,\ldots,n\} \text{ and } \beta\in\{1,2,\ldots,n\}}} t^\ell \in L\mathfrak{gl}_N$$

(where we write $a(t)$ in the form $a(t) = \sum_{i\in\mathbb{Z}} a_i t^i$ with $a_i \in \mathfrak{gl}_n$).

(a) We have $\operatorname{Toep}_N \circ \operatorname{Toep}_{n,N} = \operatorname{Toep}_n$. In other words, we can regard $\operatorname{Toep}_{n,N}$ as an inclusion map $L\mathfrak{gl}_n \to L\mathfrak{gl}_N$ which forms a commutative triangle with the inclusion maps $\operatorname{Toep}_n : L\mathfrak{gl}_n \to \overline{\mathfrak{a}_\infty}$ and $\operatorname{Toep}_N : L\mathfrak{gl}_N \to \overline{\mathfrak{a}_\infty}$. In other words, if we consider $L\mathfrak{gl}_n$ and $L\mathfrak{gl}_N$ as Lie subalgebras of $\overline{\mathfrak{a}_\infty}$ (by means of the injections $\operatorname{Toep}_n : L\mathfrak{gl}_n \to \overline{\mathfrak{a}_\infty}$ and $\operatorname{Toep}_N : L\mathfrak{gl}_N \to \overline{\mathfrak{a}_\infty}$), then $L\mathfrak{gl}_n \subseteq L\mathfrak{gl}_N$.

(b) If we consider $\operatorname{Toep}_{n,N}$ as an inclusion map $L\mathfrak{gl}_n \to L\mathfrak{gl}_N$, then the 2-cocycle $\omega : L\mathfrak{gl}_n \times L\mathfrak{gl}_n \to \mathbb{C}$ defined in Proposition 4.1.4 is the restriction of the similarly-defined 2-cocycle $\omega : L\mathfrak{gl}_N \times L\mathfrak{gl}_N \to \mathbb{C}$ (we also call it ω because it is constructed similarly) to $L\mathfrak{gl}_n \times L\mathfrak{gl}_n$. As a consequence, the inclusion map $\operatorname{Toep}_{n,N} : L\mathfrak{gl}_n \to L\mathfrak{gl}_N$ induces a Lie algebra injection $\widehat{\operatorname{Toep}_{n,N}} : \widehat{\mathfrak{gl}_n} \to \widehat{\mathfrak{gl}_N}$ which satisfies $\widehat{\operatorname{Toep}_N} \circ \widehat{\operatorname{Toep}_{n,N}} = \widehat{\operatorname{Toep}_n}$. Thus, this injection $\widehat{\operatorname{Toep}_{n,N}}$ forms a commutative triangle with the inclusion maps $\widehat{\operatorname{Toep}_n} : \widehat{\mathfrak{gl}_n} \to \mathfrak{a}_\infty$ and $\widehat{\operatorname{Toep}_N} : \widehat{\mathfrak{gl}_N} \to \mathfrak{a}_\infty$. In other words,

scalars (since $\mathfrak{gl}_1 = \mathbb{C}$). By the definition of Toep_1, we have

$$\operatorname{Toep}_1(a(t)) = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_0 & a_1 & a_2 & \cdots \\ \cdots & a_{-1} & a_0 & a_1 & \cdots \\ \cdots & a_{-2} & a_{-1} & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} = \sum_{i\in\mathbb{Z}} a_i T^i \in \left\langle T^j \mid j \in \mathbb{Z} \right\rangle,$$

qed.

[190] because both the $\widehat{\mathfrak{gl}_n}$-module $\mathcal{F}^{(m)}$ and the $\mathcal{A}$-module $\mathcal{F}^{(m)}$ were defined as restrictions of the $\mathfrak{a}_\infty$-module $\mathcal{F}^{(m)}$

if we consider $\widehat{\mathfrak{gl}_n}$ and $\widehat{\mathfrak{gl}_N}$ as Lie subalgebras of $\mathfrak{a}_\infty$ (by means of the injections $\widehat{\operatorname{Toep}_n} : \widehat{\mathfrak{gl}_n} \to \mathfrak{a}_\infty$ and $\widehat{\operatorname{Toep}_N} : \widehat{\mathfrak{gl}_N} \to \mathfrak{a}_\infty$), then $\widehat{\mathfrak{gl}_n} \subseteq \widehat{\mathfrak{gl}_N}$.

Proof of Proposition 4.1.9. **(a)** The proof of Proposition 4.1.9 **(a)** is completely straightforward. (One has to show that the $(Ni + nI + \alpha, Nj + nJ + \beta)$-th entry of $\left(\operatorname{Toep}_N \circ \operatorname{Toep}_{n,N}\right)(a(t))$ equals the $(Ni + nI + \alpha, Nj + nJ + \beta)$-th entry of $\operatorname{Toep}_n(a(t))$ for every $a(t) \in L\mathfrak{gl}_n$, every $i \in \mathbb{Z}$, every $j \in \mathbb{Z}$, every $I \in \{0, 1, ..., d-1\}$, $J \in \{0, 1, ..., d-1\}$, $\alpha \in \{1, 2, ..., n\}$ and $\beta \in \{1, 2, ..., n\}$.)

(b) The 2-cocycle $\omega : L\mathfrak{gl}_n \times L\mathfrak{gl}_n \to \mathbb{C}$ defined in Proposition 4.1.4 is the restriction of the similarly-defined 2-cocycle $\omega : L\mathfrak{gl}_N \times L\mathfrak{gl}_N \to \mathbb{C}$ to $L\mathfrak{gl}_n \times L\mathfrak{gl}_n$. (This is because both of these 2-cocycles are restrictions of the Japanese cocycle $\alpha : \overline{\mathfrak{a}_\infty} \times \overline{\mathfrak{a}_\infty} \to \mathbb{C}$, as shown in Proposition 4.1.4.) This proves Proposition 4.1.9.

Note that Proposition 4.1.9 can be used to derive Proposition 4.1.8:

Second proof of Proposition 4.1.8. **(a)** For every $m \in \mathbb{Z}$, we have $T^m \in \widehat{\mathfrak{gl}_1}$ (because the Lie algebra isomorphism $\widehat{\phi}$ constructed in Corollary 4.1.7 satisfies $\phi(a_m) = T^m$, so that $T^m \in \phi(a_m) \in \widehat{\mathfrak{gl}_1}$). Thus, for every $m \in \mathbb{Z}$, we have $T^m \in \widehat{\mathfrak{gl}_1} \cap \overline{\mathfrak{a}_\infty} = L\mathfrak{gl}_1$.

Due to Proposition 4.1.9 **(a)**, we have $L\mathfrak{gl}_1 \subseteq L\mathfrak{gl}_n$ (since $1 \mid n$). Thus, for every $m \in \mathbb{Z}$, we have $T^m \in L\mathfrak{gl}_1 \subseteq L\mathfrak{gl}_n \subseteq \widehat{\mathfrak{gl}_n}$. This proves Proposition 4.1.8 **(a)**.

(b) Due to Proposition 4.1.9 **(b)**, we have $\widehat{\mathfrak{gl}_1} \subseteq \widehat{\mathfrak{gl}_n}$ (since $1 \mid n$). Hence, the Lie algebra isomorphism $\widehat{\phi} : \mathcal{A} \to \widehat{\mathfrak{gl}_1}$ constructed in Corollary 4.1.7 induces a Lie algebra injection $\mathcal{A} \to \widehat{\mathfrak{gl}_n}$ (which sends every $a \in \mathcal{A}$ to $\widehat{\phi}(a) \in \widehat{\mathfrak{gl}_n}$). Formally speaking, this injection is the map $\widehat{\operatorname{Toep}_{1,n}} \circ \widehat{\phi} : \mathcal{A} \to \widehat{\mathfrak{gl}_n}$ (because the injection $\widehat{\mathfrak{gl}_1} \to \widehat{\mathfrak{gl}_n}$ is $\widehat{\operatorname{Toep}_{1,n}}$). Therefore, the restriction of the $\widehat{\mathfrak{gl}_n}$-module $\mathcal{F}^{(m)}$ by means of this injection is

$$\begin{aligned}
&\left(\text{the restriction of the } \widehat{\mathfrak{gl}_n}\text{-module } \mathcal{F}^{(m)} \text{ by means of the injection } \widehat{\operatorname{Toep}_{1,n}} \circ \widehat{\phi} : \mathcal{A} \to \widehat{\mathfrak{gl}_n}\right) \\
&= \left(\text{the restriction of the } \mathfrak{a}_\infty\text{-module } \mathcal{F}^{(m)} \text{ by means of the injection } \widehat{\operatorname{Toep}_n} \circ \widehat{\operatorname{Toep}_{1,n}} \circ \widehat{\phi} : \mathcal{A} \to \mathfrak{a}_\infty\right) \\
&\qquad \left(\begin{array}{c} \text{because the } \widehat{\mathfrak{gl}_n}\text{-module } \mathcal{F}^{(m)} \text{ itself was the restriction of the } \mathfrak{a}_\infty\text{-module } \mathcal{F}^{(m)} \\ \text{by means of the injection } \widehat{\operatorname{Toep}_n} : \widehat{\mathfrak{gl}_n} \to \mathfrak{a}_\infty \end{array}\right) \\
&= \left(\text{the restriction of the } \mathfrak{a}_\infty\text{-module } \mathcal{F}^{(m)} \text{ by means of the injection } \widehat{\operatorname{Toep}_1} \circ \widehat{\phi} : \mathcal{A} \to \mathfrak{a}_\infty\right) \\
&\qquad \left(\begin{array}{c} \text{since } \widehat{\operatorname{Toep}_n} \circ \widehat{\operatorname{Toep}_{1,n}} = \widehat{\operatorname{Toep}_1} \\ \text{(by Proposition 4.1.9 } \textbf{(b)}\text{, applied to } n \text{ and } 1 \text{ instead of } N \text{ and } n) \end{array}\right) \\
&= \left(\begin{array}{c} \text{the restriction of the } \mathfrak{a}_\infty\text{-module } \mathcal{F}^{(m)} \text{ by means of the} \\ \text{embedding } \mathcal{A} \to \mathfrak{a}_\infty \text{ constructed in Definition 3.7.14} \end{array}\right) \\
&\qquad \left(\begin{array}{c} \text{because } \widehat{\operatorname{Toep}_1} \circ \widehat{\phi} : \mathcal{A} \to \mathfrak{a}_\infty \text{ is exactly the} \\ \text{embedding } \mathcal{A} \to \mathfrak{a}_\infty \text{ constructed in Definition 3.7.14} \end{array}\right) \\
&= \left(\text{the } \mathcal{A}\text{-module } \mathcal{F}^{(m)} \text{ that we know}\right).
\end{aligned}$$

This proves Proposition 4.1.8 **(b)**.

4.2. The semidirect product $\widetilde{\mathfrak{gl}_n}$ and its representation theory

4.2.1. Extending affine Lie algebras by derivations

Now we give a definition pertaining to general affine Lie algebras:

Definition 4.2.1. If $\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$ is an affine Lie algebra (the $\oplus$ sign here only means a direct sum of vector spaces, not a direct sum of Lie algebras), then there exists a unique linear map $d : \widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ such that $d(a(t)) = ta'(t)$ for every $a(t) \in L\mathfrak{g}$ (so that $d\left(at^\ell\right) = \ell at^\ell$ for every $a \in \mathfrak{g}$ and $\ell \in \mathbb{N}$) and $d(K) = 0$. This linear map d is a derivation (as can be easily checked). Thus, the abelian Lie algebra $\mathbb{C}d$ (a one-dimensional Lie algebra) acts on the Lie algebra $\widehat{\mathfrak{g}}$ by derivations (in the obvious way, with d acting as d). Thus, a semidirect product $\mathbb{C}d \ltimes \widehat{\mathfrak{g}}$ is well-defined (according to Definition 3.2.1).

Set $\widetilde{\mathfrak{g}} = \mathbb{C}d \ltimes \widehat{\mathfrak{g}}$. Clearly, $\widetilde{\mathfrak{g}} = \mathbb{C}d \oplus \widehat{\mathfrak{g}}$ as vector space. The Lie algebra $\widetilde{\mathfrak{g}}$ is graded by taking the grading of $\widehat{\mathfrak{g}}$ and additionally giving d the degree 0.

One can wonder which $\widehat{\mathfrak{g}}$-modules can be extended to $\widetilde{\mathfrak{g}}$-modules. This can't be generally answered, but here is a partial uniqueness result:

Lemma 4.2.2. Let $\mathfrak{g}$ be a Lie algebra, and d be the unique derivation $\widehat{\mathfrak{g}} \to \widehat{\mathfrak{g}}$ constructed in Definition 4.2.1. Let M be a $\widehat{\mathfrak{g}}$-module, and v an element of M such that M is generated by v as a $\widehat{\mathfrak{g}}$-module. Then, there exists **at most one** extension of the $\widehat{\mathfrak{g}}$-representation on M to $\widetilde{\mathfrak{g}}$ such that $dv = 0$.

Proof of Lemma 4.2.2. Let $\rho_1 : \widetilde{\mathfrak{g}} \to \operatorname{End} M$ and $\rho_2 : \widetilde{\mathfrak{g}} \to \operatorname{End} M$ be two extensions of the $\widehat{\mathfrak{g}}$-representation on M to $\widetilde{\mathfrak{g}}$ such that $\rho_1(d)v = 0$ and $\rho_2(d)v = 0$. If we succeed in showing that $\rho_1 = \rho_2$, then Lemma 4.2.2 will be proven.

Let U be the subset $\{u \in M \mid \rho_1(d)u = \rho_2(d)u\}$ of M. Clearly, U is a vector subspace of M. Also, $v \in U$ (since $\rho_1(d)v = 0 = \rho_2(d)v$). We will now show that U is a $\widehat{\mathfrak{g}}$-submodule of M.

In fact, since ρ_1 is an action of $\widetilde{\mathfrak{g}}$ on M, every $m \in M$ and every $\alpha \in \widehat{\mathfrak{g}}$ satisfy

$$(\rho_1(d))(\rho_1(\alpha)m) - (\rho_1(\alpha))(\rho_1(d)m) = \rho_1([d,\alpha])m.$$

Since $\rho_1(\alpha)m = \alpha \rightharpoonup m$ (because the action ρ_1 extends the $\widehat{\mathfrak{g}}$-representation on M) and $[d,\alpha] = d(\alpha)$ (by the definition of the Lie bracket on the semidirect product $\widetilde{\mathfrak{g}} = \mathbb{C}d \ltimes \widehat{\mathfrak{g}}$), this rewrites as follows: Every $m \in M$ and every $\alpha \in \widehat{\mathfrak{g}}$ satisfy

$$(\rho_1(d))(\alpha \rightharpoonup m) - (\rho_1(\alpha))(\rho_1(d)m) = \rho_1(d(\alpha))m.$$

Since $(\rho_1(\alpha))(\rho_1(d)m) = \alpha \rightharpoonup (\rho_1(d)m)$ (again because the action ρ_1 extends the $\widehat{\mathfrak{g}}$-representation on M) and $\rho_1(d(\alpha))m = (d(\alpha)) \rightharpoonup m$ (for the same reason), this further rewrites as follows: Every $m \in M$ and every $\alpha \in \widehat{\mathfrak{g}}$ satisfy

$$(\rho_1(d))(\alpha \rightharpoonup m) - \alpha \rightharpoonup (\rho_1(d)m) = (d(\alpha)) \rightharpoonup m. \tag{280}$$

Now, let $m \in U$ and $\alpha \in \widehat{\mathfrak{g}}$ be arbitrary. Then, $\rho_1(d)m = \rho_2(d)m$ (by the definition of U, since $m \in U$), but we have

$$(\rho_1(d))(\alpha \rightharpoonup m) = \alpha \rightharpoonup (\rho_1(d)m) + (d(\alpha)) \rightharpoonup m$$

(by (280)) and

$$(\rho_2(d))(\alpha \rightharpoonup m) = \alpha \rightharpoonup (\rho_2(d)m) + (d(\alpha)) \rightharpoonup m$$

(similarly). Hence,

$$\begin{aligned}(\rho_1(d))(\alpha \rightharpoonup m) &= \alpha \rightharpoonup \underbrace{(\rho_1(d)m)}_{=\rho_2(d)m} + (d(\alpha)) \rightharpoonup m \\ &= \alpha \rightharpoonup (\rho_2(d)m) + (d(\alpha)) \rightharpoonup m = (\rho_2(d))(\alpha \rightharpoonup m),\end{aligned}$$

so that $\alpha \rightharpoonup m \in U$ (by the definition of U).

Now forget that we fixed $m \in U$ and $\alpha \in \widehat{\mathfrak{g}}$. We thus have showed that $\alpha \rightharpoonup m \in U$ for every $m \in U$ and $\alpha \in \widehat{\mathfrak{g}}$. In other words, U is a $\widehat{\mathfrak{g}}$-submodule of M. Since $v \in U$, this yields that U is a $\widehat{\mathfrak{g}}$-submodule of M containing v, and thus must be the whole M (since M is generated by v as a $\widehat{\mathfrak{g}}$-module). Thus, $M = U = \{u \in M \mid \rho_1(d)u = \rho_2(d)u\}$. Hence, every $u \in M$ satisfies $\rho_1(d)u = \rho_2(d)u$. Thus, $\rho_1(d) = \rho_2(d)$.

Combining $\rho_1 \mid_{\widehat{\mathfrak{g}}} = \rho_2 \mid_{\widehat{\mathfrak{g}}}$ (because both ρ_1 and ρ_2 are extensions of the $\widehat{\mathfrak{g}}$-representation on M, and thus coincide on $\widehat{\mathfrak{g}}$) and $\rho_1 \mid_{\mathbb{C}d} = \rho_2 \mid_{\mathbb{C}d}$ (because $\rho_1(d) = \rho_2(d)$), we obtain $\rho_1 = \rho_2$ (because the vector space $\widetilde{\mathfrak{g}} = \mathbb{C}d \ltimes \widehat{\mathfrak{g}}$ is generated by $\mathbb{C}d$ and $\widehat{\mathfrak{g}}$, and thus two linear maps which coincide on $\mathbb{C}d$ and on $\widehat{\mathfrak{g}}$ must be identical). Thus, as we said above, Lemma 4.2.2 is proven.

4.2.2. $\widetilde{\mathfrak{gl}_n}$

Applying Definition 4.2.1 to $\mathfrak{g} = \mathfrak{gl}_n$, we obtain a Lie algebra $\widetilde{\mathfrak{gl}_n}$. We want to study its highest weight theory.

Convention 4.2.3. For the sake of disambiguation, let us, in the following, use $E_{i,j}^{\mathfrak{gl}_n}$ to denote the elementary matrices of $\mathfrak{gl}_n$ (these are defined for $(i,j) \in \{1,2,...,n\}^2$), and use $E_{i,j}^{\mathfrak{gl}_\infty}$ to denote the elementary matrices of $\mathfrak{gl}_\infty$ (these are defined for $(i,j) \in \mathbb{Z}^2$).

Definition 4.2.4. We can make $L\mathfrak{gl}_n$ into a graded Lie algebra by setting $\deg E_{i,j}^{\mathfrak{gl}_n} = j - i$ (this, so far, is the standard grading on $\mathfrak{gl}_n$) and $\deg t = n$. Consequently, $\widehat{\mathfrak{gl}_n} = \mathbb{C}K \oplus \mathfrak{gl}_n$ (this is just a direct sum of vector spaces) becomes a graded Lie algebra with $\deg K = 0$, and $\widetilde{\mathfrak{gl}_n} = \mathbb{C}d \oplus \widehat{\mathfrak{gl}_n}$ (again, this is only a direct sum of vector spaces) becomes a graded Lie algebra with $\deg d = 0$.

The triangular decomposition of $\widetilde{\mathfrak{gl}_n}$ is $\widetilde{\mathfrak{gl}_n} = \widetilde{\mathfrak{n}_-} \oplus \widetilde{\mathfrak{h}} \oplus \widetilde{\mathfrak{n}_+}$. Here, $\widetilde{\mathfrak{h}} = \mathbb{C}K \oplus \mathbb{C}d \oplus \mathfrak{h}$ where $\mathfrak{h}$ is the Lie algebra of diagonal $n \times n$ matrices (in other words, $\mathfrak{h} = \left\langle E_{1,1}^{\mathfrak{gl}_n}, E_{2,2}^{\mathfrak{gl}_n}, ..., E_{n,n}^{\mathfrak{gl}_n} \right\rangle$). Further, $\widetilde{\mathfrak{n}_+} = \mathfrak{n}_+ \oplus t\mathfrak{gl}_n[t]$ (where $\mathfrak{n}_+$ is the Lie algebra of strictly upper-triangular matrices) and $\widetilde{\mathfrak{n}_-} = \mathfrak{n}_- \oplus t^{-1}\mathfrak{gl}_n[t^{-1}]$ (where $\mathfrak{n}_-$ is the Lie algebra of strictly lower-triangular matrices).

Definition 4.2.5. For every $m \in \mathbb{Z}$, define the weight $\widetilde{\omega}_m \in \widetilde{\mathfrak{h}}^*$ by

$$\begin{aligned}
\widetilde{\omega}_m \left(E_{i,i}^{\mathfrak{gl}_n} \right) &= \begin{cases} 1, & \text{if } i \leq \overline{m}; \\ 0, & \text{if } i > \overline{m} \end{cases} + \frac{m - \overline{m}}{n} \qquad \text{for all } i \in \{1, 2, ..., n\}; \\
\widetilde{\omega}_m (K) &= 1; \\
\widetilde{\omega}_m (d) &= 0,
\end{aligned}$$

where $\overline{m}$ is the remainder of m modulo n (that is, the element of $\{0, 1, ..., n-1\}$ satisfying $m \equiv \overline{m} \bmod n$).

Note that we can rewrite the definition of $\widetilde{\omega}_m \left(E_{i,i}^{\mathfrak{gl}_n} \right)$ as

$$\begin{aligned}
&\widetilde{\omega}_m \left(E_{i,i}^{\mathfrak{gl}_n} \right) \\
&= \begin{cases} (\text{the number of all } j \in \mathbb{Z} \text{ such that } j \equiv i \bmod n \text{ and } 1 \leq j \leq m), & \text{if } m \geq 0; \\ -(\text{the number of all } j \in \mathbb{Z} \text{ such that } j \equiv i \bmod n \text{ and } m < j \leq 0), & \text{if } m \leq 0 \end{cases}.
\end{aligned}$$

4.2.3. The $\widetilde{\mathfrak{gl}_n}$-module $\mathcal{F}^{(m)}$

A natural question to ask about representations of $\widehat{\mathfrak{g}}$ is when and how they can be extended to representations of $\widetilde{\mathfrak{g}}$. Here is an answer for $\mathfrak{g} = \mathfrak{gl}_n$ and the representation $\mathcal{F}^{(m)}$:

Proposition 4.2.6. Let $m \in \mathbb{Z}$. Let ψ_m be the element $v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ... \in \mathcal{F}^{(m)}$.

There exists a unique extension of the $\widehat{\mathfrak{gl}_n}$-representation on $\mathcal{F}^{(m)}$ to $\widetilde{\mathfrak{gl}_n}$ such that $d\psi_m = 0$. The action of d in this extension is given by

$$d \left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \right) = \left(\sum_{k \geq 0} \left(\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{i_k}{n} \right\rceil \right) \right) \cdot v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$$

for every m-degression $(i_0, i_1, i_2, ...)$.

Note that the infinite sum $\sum\limits_{k \geq 0} \left(\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{i_k}{n} \right\rceil \right)$ in Proposition 4.2.6 is well-defined[191].

Proof of Proposition 4.2.6. Uniqueness: Let us prove that there exists **at most one** extension of the $\widehat{\mathfrak{gl}_n}$-representation on $\mathcal{F}^{(m)}$ to $\widetilde{\mathfrak{gl}_n}$ such that $d\psi_m = 0$.

By Proposition 4.1.8 **(b)**, the $\mathcal{A}$-module $\mathcal{F}^{(m)}$ is a restriction of the $\widehat{\mathfrak{gl}_n}$-module $\mathcal{F}^{(m)}$. As a consequence, $\mathcal{F}^{(m)}$ is generated by ψ_m as a $\widehat{\mathfrak{gl}_n}$-module (since $\mathcal{F}^{(m)}$ is generated by ψ_m as an $\mathcal{A}$-module). Hence, by Lemma 4.2.2 (applied to $\mathfrak{g} = \mathfrak{gl}_n$, $M = \mathcal{F}^{(m)}$ and $v = \psi_m$), there exists **at most one** extension of the $\widehat{\mathfrak{gl}_n}$-representation on $\mathcal{F}^{(m)}$ to $\widetilde{\mathfrak{gl}_n}$ such that $d\psi_m = 0$.

[191] In fact, $(i_0, i_1, i_2, ...)$ is an m-degression. Hence, every sufficiently high $k \geq 0$ satisfies $i_k + k = m$ and thus $m - k = i_k$ and thus $\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{i_k}{n} \right\rceil = 0$. Thus, all but finitely many addends of the infinite sum $\sum\limits_{k \geq 0} \left(\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{i_k}{n} \right\rceil \right)$ are zero, so that this sum is well-defined, qed.

Existence: Let us now show that there exists an extension of the $\widehat{\mathfrak{gl}_n}$-representation on $\mathcal{F}^{(m)}$ to $\widetilde{\mathfrak{gl}_n}$ such that $d\psi_m = 0$.

In fact, let us construct this extension. In order to do so, it is clearly enough to define the action of d on $\mathcal{F}^{(m)}$ (because an action of $\widehat{\mathfrak{gl}_n}$ on $\mathcal{F}^{(m)}$ is already defined), and then show that every $A \in \widehat{\mathfrak{gl}_n}$ satisfies

$$\left[d \mid_{\mathcal{F}^{(m)}}, A \mid_{\mathcal{F}^{(m)}}\right] = [d, A]_{\widetilde{\mathfrak{gl}_n}} \mid_{\mathcal{F}^{(m)}} . \tag{281}$$

[192]

Let us define the action of d on $\mathcal{F}^{(m)}$ by stipulating that

$$d\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right) = \left(\sum_{k\geq 0} \left(\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{i_k}{n} \right\rceil\right)\right) \cdot v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ... \tag{282}$$

for every m-degression $(i_0, i_1, i_2, ...)$. (This is extended by linearity to the whole of $\mathcal{F}^{(m)}$, since $\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right)_{(i_0,i_1,i_2,...) \text{ is an } m\text{-degression}}$ is a basis of $\mathcal{F}^{(m)}$.)

It is rather clear that (282) holds not only for every m-degression $(i_0, i_1, i_2, ...)$, but also for every straying m-degression $(i_0, i_1, i_2, ...)$. [193] Renaming $(i_0, i_1, i_2, ...)$ as $(j_0, j_1, j_2, ...)$ and renaming the summation index k as p, we can rewrite this as follows: We have

$$d\left(v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ...\right) = \left(\sum_{p\geq 0} \left(\left\lceil \frac{m-p}{n} \right\rceil - \left\lceil \frac{j_p}{n} \right\rceil\right)\right) \cdot v_{j_0} \wedge v_{j_1} \wedge v_{j_2} \wedge ... \tag{283}$$

for every straying m-degression $(j_0, j_1, j_2, ...)$.

We now need to prove that every $A \in \widehat{\mathfrak{gl}_n}$ satisfies (281). Since this equation (281) is linear in A, we need to check it only in the case when $A = K$ and in the case when $A = at^\ell$ for some $a \in \mathfrak{gl}_n$ and some $\ell \in \mathbb{Z}$ (because the vector space $\widehat{\mathfrak{gl}_n}$ is generated by K and all elements of the form at^ℓ for some $a \in \mathfrak{gl}_n$ and some $\ell \in \mathbb{Z}$). But checking the equation (281) in the case when $A = K$ is trivial[194]. Hence, it only remains to check the equation (281) in the case when $A = at^\ell$ for some $a \in \mathfrak{gl}_n$ and some $\ell \in \mathbb{Z}$.

So let $a \in \mathfrak{gl}_n$ and $\ell \in \mathbb{Z}$ be arbitrary. We can WLOG assume that if $\ell = 0$, then the diagonal entries of the matrix a are zero[195]. Let us assume this. (The purpose of this assumption is to ensure that we can apply Proposition 3.7.5 to at^ℓ in lieu of a.)

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[192] Here, for every $\xi \in \widetilde{\mathfrak{gl}_n}$, we denote by $\xi \mid_{\mathcal{F}^{(m)}}$ the action of ξ on $\mathcal{F}^{(m)}$. Besides, $[d, A]_{\widetilde{\mathfrak{gl}_n}}$ means the Lie bracket of d and A in the Lie algebra $\widetilde{\mathfrak{gl}_n}$.

[193] In fact, if $(i_0, i_1, i_2, ...)$ is a straying m-degression with no two equal elements, and π is its straightening permutation, then $\sum\limits_{k\geq 0} \left(\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{i_k}{n} \right\rceil\right) = \sum\limits_{k\geq 0} \left(\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{i_{\pi^{-1}(k)}}{n} \right\rceil\right)$, and this readily yields (282). If $(i_0, i_1, i_2, ...)$ is a straying m-degression with two equal elements, then (282) is even more obvious (since both sides of (282) are zero in this case).

[194] In fact, $K \mid_{\mathcal{F}^{(m)}} = \mathrm{id}$, so that $\left[d \mid_{\mathcal{F}^{(m)}}, K \mid_{\mathcal{F}^{(m)}}\right] = \left[d \mid_{\mathcal{F}^{(m)}}, \mathrm{id}\right] = 0$, and by the definition of a semidirect product of Lie algebras we have $[d, K]_{\widetilde{\mathfrak{gl}_n}} = d(K) = 0$, so that both sides of (281) are zero in the case $A = K$, so that (281) trivially holds in the case when $A = K$.

[195] Here is why this assumption is allowed:

We must prove that every $a \in \mathfrak{gl}_n$ and $\ell \in \mathbb{Z}$ satisfy the equation (281) for $A = at^\ell$. In other words, we must prove that every $a \in \mathfrak{gl}_n$ and $\ell \in \mathbb{Z}$ satisfy $\left[d \mid_{\mathcal{F}^{(m)}}, \left(at^\ell\right) \mid_{\mathcal{F}^{(m)}}\right] = \left[d, \left(at^\ell\right)\right]_{\widetilde{\mathfrak{gl}_n}} \mid_{\mathcal{F}^{(m)}}$. If $\ell \neq 0$, then our assumption (that if $\ell = 0$, then the diagonal entries of the

Let $(i_0, i_1, i_2, ...)$ be an m-degression.

We recall that, when we embedded $L\mathfrak{gl}_n$ into $\overline{\mathfrak{a}_\infty}$, we identified the element $at^\ell \in L\mathfrak{gl}_n$ with the matrix $\operatorname{Toep}_n\left(at^\ell\right)$ whose $(ni+\alpha, nj+\beta)$-th entry equals

$$\left\{ \begin{array}{l} \text{the } (\alpha,\beta)\text{-th entry of } a, \qquad \text{if } j-i=\ell; \\ 0, \qquad \text{if } j-i\neq\ell \end{array} \right.$$

for all $i \in \mathbb{Z}$, $j \in \mathbb{Z}$, $\alpha \in \{1, 2, ..., n\}$ and $\beta \in \{1, 2, ..., n\}$. Hence, for every $j \in \mathbb{Z}$ and $\beta \in \{1, 2, ..., n\}$, we have

$$\begin{aligned}
&\left(\operatorname{Toep}_n\left(at^\ell\right)\right) \rightharpoonup v_{nj+\beta} \\
&= \sum_{i\in\mathbb{Z}} \sum_{\alpha\in\{1,2,...,n\}} \left\{ \begin{array}{l} \text{the } (\alpha,\beta)\text{-th entry of } a, \qquad \text{if } j-i=\ell; \\ 0, \qquad \text{if } j-i\neq\ell \end{array} \right. v_{ni+\alpha} \\
&= \sum_{\alpha\in\{1,2,...,n\}} \underbrace{\sum_{i\in\mathbb{Z}} \left\{ \begin{array}{l} \text{the } (\alpha,\beta)\text{-th entry of } a, \qquad \text{if } j-i=\ell; \\ 0, \qquad \text{if } j-i\neq\ell \end{array} \right. v_{ni+\alpha}}_{\substack{=(\text{the } (\alpha,\beta)\text{-th entry of } a) v_{n(j-\ell)+\alpha} \\ (\text{since there is precisely one } i\in\mathbb{Z} \text{ satisfying } j-i=\ell, \text{ namely } i=j-\ell)}} \\
&= \sum_{\alpha\in\{1,2,...,n\}} (\text{the } (\alpha,\beta)\text{-th entry of } a)\, v_{n(j-\ell)+\alpha}. \qquad (284)
\end{aligned}$$

Note that the matrix $\operatorname{Toep}_n\left(at^\ell\right)$ has the property that, for every integer $i \leq 0$, the (i,i)-th entry of $\operatorname{Toep}_n\left(at^\ell\right)$ is 0. (This is due to our assumption that if $\ell = 0$, then the diagonal entries of the matrix a are zero.) As a consequence, we can apply Proposition 3.7.5 to $\operatorname{Toep}_n\left(at^\ell\right)$ and v_{i_k} instead of a and b_k, and obtain

$$\begin{aligned}
&\left(\widehat{\rho}\left(\operatorname{Toep}_n\left(at^\ell\right)\right)\right)\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right) \\
&= \sum_{k\geq 0} v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(\left(\operatorname{Toep}_n\left(at^\ell\right)\right) \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge .... \qquad (285)
\end{aligned}$$

Now, we can check that, for every $k \geq 0$, we have

$$\begin{aligned}
&d\left(v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(\left(\operatorname{Toep}_n\left(at^\ell\right)\right) \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge ...\right) \\
&= \left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil - \left\lceil\frac{i_p}{n}\right\rceil\right) + \ell\right) \\
&\qquad \cdot v_{i_0} \wedge v_{i_1} \wedge ... \wedge v_{i_{k-1}} \wedge \left(\left(\operatorname{Toep}_n\left(at^\ell\right)\right) \rightharpoonup v_{i_k}\right) \wedge v_{i_{k+1}} \wedge v_{i_{k+2}} \wedge .... \qquad (286)
\end{aligned}$$

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matrix a are zero) is clearly allowed (because it only makes a statement about the case $\ell = 0$). So we only need to consider the case $\ell = 0$. In this case, the equation which we must prove (this is the equation $\left[d\mid_{\mathcal{F}^{(m)}}, \left(at^\ell\right)\mid_{\mathcal{F}^{(m)}}\right] = \left[d, \left(at^\ell\right)\right]_{\widetilde{\mathfrak{gl}_n}} \mid_{\mathcal{F}^{(m)}}$) simplifies to $\left[d\mid_{\mathcal{F}^{(m)}}, a\mid_{\mathcal{F}^{(m)}}\right] = [d, a]_{\widetilde{\mathfrak{gl}_n}} \mid_{\mathcal{F}^{(m)}}$. This equation is clearly linear in a. Hence, we can WLOG assume that either the matrix a is diagonal, or all diagonal entries of the matrix a are zero (because every $n \times n$ matrix can be decomposed as a sum of a diagonal matrix with a matrix all of whose diagonal entries are zero). But in the case when the matrix a is diagonal, the equation $\left[d\mid_{\mathcal{F}^{(m)}}, a\mid_{\mathcal{F}^{(m)}}\right] = [d, a]_{\widetilde{\mathfrak{gl}_n}} \mid_{\mathcal{F}^{(m)}}$ is very easy to check (the details of this are left to the reader). Hence, it is enough to only consider the case when the diagonal entries of the matrix a are 0. Of course, our assumption is justified in this case.

Thus, we are allowed to make the assumption that if $\ell = 0$, then the diagonal entries of the matrix a are zero.

Since $A = at^{\ell}$, we have $A\mid_{\mathcal{F}^{(m)}} = \left(at^{\ell}\right)\mid_{\mathcal{F}^{(m)}} = \widehat{\rho}\left(\operatorname{Toep}_n\left(at^{\ell}\right)\right)$ (because the element $at^{\ell} \in L\mathfrak{gl}_n$ was identified with the matrix $\operatorname{Toep}_n\left(at^{\ell}\right)$ and this matrix acts on $\mathcal{F}^{(m)}$ via $\widehat{\rho}$). Thus, we can rewrite (285) as

$$\begin{aligned}&\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\\&=\sum_{k\geq 0} v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge\left(\left(\operatorname{Toep}_n\left(at^{\ell}\right)\right)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge .... \qquad (289)\end{aligned}$$

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[196] *Proof of (286):* Let $k \geq 0$. Write the integer i_k in the form $nj+\beta$ for some $j \in \mathbb{Z}$ and $\beta \in \{1,2,...,n\}$. Then,

$$\left(\operatorname{Toep}_n\left(at^{\ell}\right)\right)\rightharpoonup v_{i_k} = \left(\operatorname{Toep}_n\left(at^{\ell}\right)\right)\rightharpoonup v_{nj+\beta} = \sum_{\alpha\in\{1,2,...,n\}} \left(\text{the } (\alpha,\beta)\text{-th entry of } a\right) v_{n(j-\ell)+\alpha}$$

due to (284). Hence,

$$\begin{aligned}&v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge \underbrace{\left(\left(\operatorname{Toep}_n\left(at^{\ell}\right)\right)\rightharpoonup v_{i_k}\right)}_{=\sum\limits_{\alpha\in\{1,2,...,n\}}(\text{the }(\alpha,\beta)\text{-th entry of }a)v_{n(j-\ell)+\alpha}} \wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\\&=\sum_{\alpha\in\{1,2,...,n\}} \left(\text{the } (\alpha,\beta)\text{-th entry of } a\right)\cdot v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{n(j-\ell)+\alpha}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge .... \qquad (287)\end{aligned}$$

Now, fix $\alpha \in \{1,2,...,n\}$. Let $(j_0, j_1, j_2, ...)$ be the straying m-degression $(i_0, i_1, i_2, ..., i_{k-1}, n(j-\ell)+\alpha, i_{k+1}, i_{k+2}, ...)$. Then, $j_p = i_p$ for every $p \geq 0$ satisfying $p \neq k$.

Comparing $\left\lceil \dfrac{i_k}{n}\right\rceil = j+1$ (since $i_k = nj+\beta$ with $\beta \in \{1,2,...,n\}$) with $\left\lceil \dfrac{j_k}{n}\right\rceil = j-\ell+1$ (since $j_k = n(j-\ell)+\alpha$ with $\alpha \in \{1,2,...,n\}$), we get $\left\lceil \dfrac{j_k}{n}\right\rceil = \left\lceil \dfrac{i_k}{n}\right\rceil - \ell$.

Since $\left\lceil \dfrac{j_p}{n}\right\rceil = \left\lceil \dfrac{i_p}{n}\right\rceil$ for every $p \geq 0$ satisfying $p \neq k$ (because every $p \geq 0$ satisfying $p \neq k$ satisfies $j_p = i_p$), the two sums $\sum\limits_{p\geq 0}\left(\left\lceil \dfrac{m-p}{n}\right\rceil - \left\lceil \dfrac{j_p}{n}\right\rceil\right)$ and $\sum\limits_{p\geq 0}\left(\left\lceil \dfrac{m-p}{n}\right\rceil - \left\lceil \dfrac{i_p}{n}\right\rceil\right)$ differ only in their k-th addends. Since the k-th addends differ in ℓ (because $\left\lceil \dfrac{j_k}{n}\right\rceil = \left\lceil \dfrac{i_k}{n}\right\rceil - \ell$), this yields $\sum\limits_{p\geq 0}\left(\left\lceil \dfrac{m-p}{n}\right\rceil - \left\lceil \dfrac{j_p}{n}\right\rceil\right) = \sum\limits_{p\geq 0}\left(\left\lceil \dfrac{m-p}{n}\right\rceil - \left\lceil \dfrac{i_p}{n}\right\rceil\right) + \ell$.

But since $(i_0, i_1, i_2, ..., i_{k-1}, n(j-\ell)+\alpha, i_{k+1}, i_{k+2}, ...) = (j_0, j_1, j_2, ...)$, we have

$$\begin{aligned}&d\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{n(j-\ell)+\alpha}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\right)\\&= d\left(v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge ...\right) = \underbrace{\left(\sum_{p\geq 0}\left(\left\lceil \frac{m-p}{n}\right\rceil - \left\lceil \frac{j_p}{n}\right\rceil\right)\right)}_{=\sum\limits_{p\geq 0}\left(\left\lceil \frac{m-p}{n}\right\rceil - \left\lceil \frac{i_p}{n}\right\rceil\right)+\ell}\cdot \underbrace{v_{j_0}\wedge v_{j_1}\wedge v_{j_2}\wedge ...}_{\substack{=v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{n(j-\ell)+\alpha}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\\ (\text{since } (j_0,j_1,j_2,...)=(i_0,i_1,i_2,...,i_{k-1},n(j-\ell)+\alpha,i_{k+1},i_{k+2},...))}}\\&=\left(\sum_{p\geq 0}\left(\left\lceil \frac{m-p}{n}\right\rceil - \left\lceil \frac{i_p}{n}\right\rceil\right)+\ell\right)\cdot v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{n(j-\ell)+\alpha}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge .... \qquad (288)\end{aligned}$$

Applying d to this equality, we get

$$\begin{aligned}
& d\left(\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right) \\
&= \sum_{k\geq 0} \underbrace{d\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge\left(\left(\operatorname{Toep}_n\left(at^\ell\right)\right)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\right)}_{\substack{=\left(\sum\limits_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)+\ell\right)\cdot v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge\left(\left(\operatorname{Toep}_n\left(at^\ell\right)\right)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ... \\ \text{(by (286))}}} \\
&= \left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)+\ell\right) \\
&\qquad \cdot \underbrace{\sum_{k\geq 0} v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge\left(\left(\operatorname{Toep}_n\left(at^\ell\right)\right)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...}_{\substack{=\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right) \\ \text{(by (289))}}} \\
&= \left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)+\ell\right)\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right) \\
&= \left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)\right)\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right) \\
&\qquad +\ell\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right).
\end{aligned} \tag{290}$$

Now forget that we fixed α. Now, applying d to the equality (287), we get

$$\begin{aligned}
& d\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge\left(\left(\operatorname{Toep}_n\left(at^\ell\right)\right)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\right) \\
&= \sum_{\alpha\in\{1,2,...,n\}} \left(\text{the }(\alpha,\beta)\text{-th entry of }a\right) \\
&\qquad \cdot \underbrace{d\left(v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{n(j-\ell)+\alpha}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...\right)}_{\substack{=\left(\sum\limits_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)+\ell\right)\cdot v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{n(j-\ell)+\alpha}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ... \\ \text{(by (288))}}} \\
&= \sum_{\alpha\in\{1,2,...,n\}} \left(\text{the }(\alpha,\beta)\text{-th entry of }a\right) \\
&\qquad \cdot \left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)+\ell\right)\cdot v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{n(j-\ell)+\alpha}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ... \\
&= \left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)+\ell\right) \\
&\qquad \cdot \underbrace{\sum_{\alpha\in\{1,2,...,n\}} \left(\text{the }(\alpha,\beta)\text{-th entry of }a\right)\cdot v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge v_{n(j-\ell)+\alpha}\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...}_{\substack{=v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge\left(\left(\operatorname{Toep}_n\left(at^\ell\right)\right)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ... \\ \text{(by (287))}}} \\
&= \left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)+\ell\right)\cdot v_{i_0}\wedge v_{i_1}\wedge ...\wedge v_{i_{k-1}}\wedge\left(\left(\operatorname{Toep}_n\left(at^\ell\right)\right)\rightharpoonup v_{i_k}\right)\wedge v_{i_{k+1}}\wedge v_{i_{k+2}}\wedge ...,
\end{aligned}$$

so that (286) is proven.

Since

$$\left(A\mid_{\mathcal{F}^{(m)}}\right)\underbrace{\left(d\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)}_{\substack{=\left(\sum\limits_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{j_p}{n}\right\rceil\right)\right)\cdot v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\\ \text{(by (283), applied to }(j_0,j_1,j_2,...)=(i_0,i_1,i_2,...))}}$$
$$=\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(\left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{j_p}{n}\right\rceil\right)\right)\cdot v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)$$
$$=\left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)\right)\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)$$

and

$$\left([d,A]_{\widetilde{\mathfrak{gl}_n}}\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)$$
$$=\left(\ell A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)$$
$$\left(\begin{array}{c}\text{since, by the definition of the Lie bracket on the semidirect product}\\ \widetilde{\mathfrak{gl}_n}=\mathbb{C}d\ltimes\widehat{\mathfrak{gl}_n}\text{, we have }[d,A]_{\widetilde{\mathfrak{gl}_n}}=d\underbrace{(A)}_{=at^\ell}=d\left(at^\ell\right)=\ell\underbrace{at^\ell}_{=A}=\ell A\end{array}\right)$$
$$=\ell\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right),$$

we can rewrite (290) as

$$d\left(\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)$$
$$=\underbrace{\left(\sum_{p\geq 0}\left(\left\lceil\frac{m-p}{n}\right\rceil-\left\lceil\frac{i_p}{n}\right\rceil\right)\right)\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)}_{=\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(d\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)}$$
$$+\underbrace{\ell\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)}_{=\left([d,A]_{\widetilde{\mathfrak{gl}_n}}\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)}$$
$$=\left(A\mid_{\mathcal{F}^{(m)}}\right)\left(d\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)\right)+\left([d,A]_{\widetilde{\mathfrak{gl}_n}}\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right).$$

In other words,

$$\left(\left(d\mid_{\mathcal{F}^{(m)}}\right)\circ\left(A\mid_{\mathcal{F}^{(m)}}\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)$$
$$=\left(\left(A\mid_{\mathcal{F}^{(m)}}\right)\circ\left(d\mid_{\mathcal{F}^{(m)}}\right)\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)+\left([d,A]_{\widetilde{\mathfrak{gl}_n}}\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)$$
$$=\left(\left(A\mid_{\mathcal{F}^{(m)}}\right)\circ\left(d\mid_{\mathcal{F}^{(m)}}\right)+[d,A]_{\widetilde{\mathfrak{gl}_n}}\mid_{\mathcal{F}^{(m)}}\right)\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right).$$

Since this holds for every m-degression $(i_0,i_1,i_2,...)$, this yields that $\left(d\mid_{\mathcal{F}^{(m)}}\right)\circ\left(A\mid_{\mathcal{F}^{(m)}}\right)=\left(A\mid_{\mathcal{F}^{(m)}}\right)\circ\left(d\mid_{\mathcal{F}^{(m)}}\right)+[d,A]_{\widetilde{\mathfrak{gl}_n}}\mid_{\mathcal{F}^{(m)}}$ (because $\left(v_{i_0}\wedge v_{i_1}\wedge v_{i_2}\wedge ...\right)_{(i_0,i_1,i_2,...)\text{ is an }m\text{-degression}}$ is a basis of $\mathcal{F}^{(m)}$). In other words,

$$[d,A]_{\widetilde{\mathfrak{gl}_n}}\mid_{\mathcal{F}^{(m)}}=\left(d\mid_{\mathcal{F}^{(m)}}\right)\circ\left(A\mid_{\mathcal{F}^{(m)}}\right)-\left(A\mid_{\mathcal{F}^{(m)}}\right)\circ\left(d\mid_{\mathcal{F}^{(m)}}\right)=\left[d\mid_{\mathcal{F}^{(m)}},A\mid_{\mathcal{F}^{(m)}}\right].$$

In other words, (281) holds.

We have thus checked the equation (281) in the case when $A = at^{\ell}$ for some $a \in \mathfrak{gl}_n$ and some $\ell \in \mathbb{Z}$. As explained above, this completes the proof of the equation (281) for every $A \in \widehat{\mathfrak{gl}_n}$. Hence, we have constructed an action of d on $\mathcal{F}^{(m)}$. This action clearly satisfies $d\psi_m = 0$ (because $\psi_m = v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ...$, so that

$$\begin{aligned} d\psi_m &= d\left(v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ...\right) \\ &= \left(\sum_{k\geq 0} \underbrace{\left(\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{m-k}{n} \right\rceil\right)}_{\substack{=0 \\ \text{(by (282), applied to } i_k = m-k)}}\right) \cdot v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ... \\ &= 0 \end{aligned}$$

). Hence, we have proven the existence of an extension of the $\widehat{\mathfrak{gl}_n}$-representation on $\mathcal{F}^{(m)}$ to $\widetilde{\mathfrak{gl}_n}$ such that $d\psi_m = 0$.

Altogether, we have now proven both the uniqueness and the existence of an extension of the $\widehat{\mathfrak{gl}_n}$-representation on $\mathcal{F}^{(m)}$ to $\widetilde{\mathfrak{gl}_n}$ such that $d\psi_m = 0$. Moreover, in the proof of the existence, we have showed that the action of d in this extension is given by

$$d\left(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...\right) = \left(\sum_{k\geq 0} \left(\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{i_k}{n} \right\rceil\right)\right) \cdot v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...$$

for every m-degression $(i_0, i_1, i_2, ...)$ (because we defined this extension using (282)). This completes the proof of Proposition 4.2.6.

Next, an irreducibility result:

Proposition 4.2.7. Let $m \in \mathbb{Z}$. Let ψ_m be the element $v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ... \in \mathcal{F}^{(m)}$.

(a) The $\widehat{\mathfrak{gl}_n}$-module $\mathcal{F}^{(m)}$ is irreducible.

(b) Let $\widehat{\rho}\mid_{\widetilde{\mathfrak{gl}_n}} : \widetilde{\mathfrak{gl}_n} \to \operatorname{End}\left(\mathcal{F}^{(m)}\right)$ denote the unique extension of the $\widehat{\mathfrak{gl}_n}$-representation on $\mathcal{F}^{(m)}$ to $\widetilde{\mathfrak{gl}_n}$ such that $d\psi_m = 0$. (This is well-defined due to Proposition 4.2.6.)

The $\widetilde{\mathfrak{gl}_n}$-module $\left(\mathcal{F}^{(m)}, \widehat{\rho}\mid_{\widetilde{\mathfrak{gl}_n}}\right)$ is irreducible with highest weight $\widetilde{\omega}_m$.

Proof of Proposition 4.2.7. **(a)** By Proposition 2.2.9, we know that F is an irreducible $\mathcal{A}_0$-module. In other words, $\mathcal{B}^{(m)}$ is an irreducible $\mathcal{A}_0$-module (since $\mathcal{B}^{(m)} = F_m = F$ as $\mathcal{A}_0$-modules). Hence, $\mathcal{B}^{(m)}$ is also an irreducible $\mathcal{A}$-module (since the $\mathcal{A}_0$-module $\mathcal{B}^{(m)}$ is a restriction of the $\mathcal{A}$-module $\mathcal{B}^{(m)}$).

Since the Boson-Fermion correspondence $\sigma_m : \mathcal{B}^{(m)} \to \mathcal{F}^{(m)}$ is an $\mathcal{A}$-module isomorphism, we have $\mathcal{B}^{(m)} \cong \mathcal{F}^{(m)}$ as $\mathcal{A}$-modules. Since $\mathcal{B}^{(m)}$ is an irreducible $\mathcal{A}$-module, this yields that $\mathcal{F}^{(m)}$ is an irreducible $\mathcal{A}$-module.

By Proposition 4.1.8 **(b)**, the $\mathcal{A}$-module $\mathcal{F}^{(m)}$ is a restriction of the $\widehat{\mathfrak{gl}_n}$-module $\mathcal{F}^{(m)}$. Since the $\mathcal{A}$-module $\mathcal{F}^{(m)}$ is irreducible, this yields that the $\widehat{\mathfrak{gl}_n}$-module $\mathcal{F}^{(m)}$ is irreducible. Proposition 4.2.7 **(a)** is proven.

(b) It is easy to check that $\widetilde{\mathfrak{n}_+}\psi_m = 0$ and $x\psi_m = \widetilde{\omega}_m(x)\psi_m$ for every $x \in \widetilde{\mathfrak{h}}$.

Proof. Proving that $\widetilde{\mathfrak{n}_+}\psi_m = 0$ is easy, since $\widetilde{\mathfrak{n}_+}$ embeds into $\mathfrak{a}_\infty$ as strictly upper-triangular matrices (and $\mathcal{F}^{(m)}$ is a graded $\mathfrak{a}_\infty$-module).

In order to prove that $x\psi_m = \widetilde{\omega}_m(x)\psi_m$ for every $x \in \widetilde{\mathfrak{h}}$, we must show that $E_{i,i}^{\mathfrak{gl}_n}\psi_m = \widetilde{\omega}_m\left(E_{i,i}^{\mathfrak{gl}_n}\right)\psi_m$ for every $i \in \{1, 2, ..., n\}$. (In fact, this is enough, because the relations $K\psi_m = \widetilde{\omega}_m(K)\psi_m$ and $d\psi_m = \widetilde{\omega}_m(d)\psi_m$ follow directly from $\widehat{\rho}(K) = \mathrm{id}$ and $d\psi_m = 0$.)

Let $i \in \{1, 2, ..., n\}$. Use $\mathrm{Toep}_n\left(E_{i,i}^{\mathfrak{gl}_n}\right) = \sum\limits_{j \equiv i \bmod n} E_{j,j}^{\mathfrak{gl}_\infty}$ to conclude that

$$\widehat{\rho}\left(E_{i,i}^{\mathfrak{gl}_n}\right)\psi_m = \underbrace{\left(\begin{cases} 1, & \text{if } i \leq \overline{m}; \\ 0, & \text{if } i > \overline{m} \end{cases} + \frac{m - \overline{m}}{n}\right)}_{= \widetilde{\omega}_m\left(E_{i,i}^{\mathfrak{gl}_n}\right)}\psi_m = \widetilde{\omega}_m\left(E_{i,i}^{\mathfrak{gl}_n}\right)\psi_m,$$

where $\overline{m}$ is the element of $\{0, 1, ..., n-1\}$ satisfying $m \equiv \overline{m} \bmod n$.

Thus, we have checked that $\widetilde{\mathfrak{n}_+}\psi_m = 0$ and $x\psi_m = \widetilde{\omega}_m(x)\psi_m$ for every $x \in \widetilde{\mathfrak{h}}$. Thus, ψ_m is a singular vector of weight $\widetilde{\omega}_m$. In other words, $\psi_m \in \mathrm{Sing}_{\widetilde{\omega}_m}\left(\mathcal{F}^{(m)}\right)$. By Lemma 2.7.8, we thus have a canonical isomorphism

$$\mathrm{Hom}_{\widetilde{\mathfrak{gl}_n}}\left(M_{\widetilde{\omega}_m}^+, \mathcal{F}^{(m)}\right) \to \mathrm{Sing}_{\widetilde{\omega}_m}\left(\mathcal{F}^{(m)}\right),$$
$$\phi \mapsto \phi\left(v_{\widetilde{\omega}_m}^+\right).$$

Thus, since $\psi_m \in \mathrm{Sing}_{\widetilde{\omega}_m}\left(\mathcal{F}^{(m)}\right)$, there exists a $\widetilde{\mathfrak{gl}_n}$-module homomorphism $\phi : M_{\widetilde{\omega}_m}^+ \to \mathcal{F}^{(m)}$ such that $\phi\left(v_{\widetilde{\omega}_m}^+\right) = \psi_m$. Consider this ϕ.

Since $\mathcal{F}^{(m)}$ is generated by ψ_m as a $\widehat{\mathfrak{gl}_n}$-module (this was proven in the proof of Proposition 4.2.6), it is clear that $\mathcal{F}^{(m)}$ is generated by ψ_m as a $\widetilde{\mathfrak{gl}_n}$-module as well. Thus, ϕ must be surjective (because $\psi_m = \phi\left(v_{\widetilde{\omega}_m}^+\right) \in \phi\left(M_{\widetilde{\omega}_m}^+\right)$). Hence, $\mathcal{F}^{(m)}$ is (isomorphic to) a quotient of the $\widetilde{\mathfrak{gl}_n}$-module $M_{\widetilde{\omega}_m}^+$. In other words, $\mathcal{F}^{(m)}$ is a highest-weight module with highest weight $\widetilde{\omega}_m$. Combined with the irreducibility of $\mathcal{F}^{(m)}$, this proves Proposition 4.2.7.

4.2.4. The $\widetilde{\mathfrak{gl}_n}$-module $\mathcal{B}^{(m)}$

By applying the Boson-Fermion correspondence σ to Proposition 4.2.6, we obtain:

Proposition 4.2.8. Let $m \in \mathbb{Z}$. Let ψ'_m be the element $\sigma^{-1}(v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ...) \in \mathcal{B}^{(m)}$ (the highest-weight vector of $\mathcal{B}^{(m)}$).

There exists a unique extension of the $\widehat{\mathfrak{gl}_n}$-representation on $\mathcal{B}^{(m)}$ to $\widetilde{\mathfrak{gl}_n}$ such that $d\psi'_m = 0$. The action of d in this extension is given by

$$d\left(\sigma^{-1}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)\right) = \left(\sum_{k \geq 0}\left(\left\lceil \frac{m-k}{n} \right\rceil - \left\lceil \frac{i_k}{n} \right\rceil\right)\right) \cdot \sigma^{-1}(v_{i_0} \wedge v_{i_1} \wedge v_{i_2} \wedge ...)$$

for every m-degression $(i_0, i_1, i_2, ...)$.

By applying the Boson-Fermion correspondence σ to Proposition 4.2.7, we obtain:

Proposition 4.2.9. Let $m \in \mathbb{Z}$. Let ψ'_m be the element $\sigma^{-1}(v_m \wedge v_{m-1} \wedge v_{m-2} \wedge ...) \in \mathcal{B}^{(m)}$ (the highest-weight vector of $\mathcal{B}^{(m)}$).

(a) The $\widehat{\mathfrak{gl}_n}$-module $\mathcal{B}^{(m)}$ is irreducible.

(b) Let $\widehat{\rho} \mid_{\widetilde{\mathfrak{gl}_n}} : \widetilde{\mathfrak{gl}_n} \to \operatorname{End}\left(\mathcal{B}^{(m)}\right)$ denote the unique extension of the $\widehat{\mathfrak{gl}_n}$-representation on $\mathcal{B}^{(m)}$ to $\widetilde{\mathfrak{gl}_n}$ such that $d\psi'_m = 0$. (This is well-defined due to Proposition 4.2.8.)

The $\widetilde{\mathfrak{gl}_n}$-module $\left(\mathcal{B}^{(m)}, \widehat{\rho} \mid_{\widetilde{\mathfrak{gl}_n}}\right)$ is irreducible with highest weight $\widetilde{\omega}_m$.

4.2.5. $\widetilde{\mathfrak{sl}_n}$ and its action on $\mathcal{B}^{(m)}$

We have $\left[I_n t, \widehat{\mathfrak{sl}_n}\right] = 0$ in the Lie algebra $\widehat{\mathfrak{gl}_n}$ (this is because $[I_n t, L\mathfrak{sl}_n] = 0$ in the Lie algebra $L\mathfrak{gl}_n$, and because $\omega(I_n t, L\mathfrak{sl}_n) = 0$ where the 2-cocycle ω is the one defined in Proposition 4.1.4). Since $I_n t \in \widehat{\mathfrak{gl}_n}$ acts on $\mathcal{F}$ by the operator $\widehat{\operatorname{Toep}_n}(I_n t) = T^n$ (more precisely, by the action of T^n on $\mathcal{F}$, but let us abbreviate this by T^n here), this yields that the action of T^n on $\mathcal{F}$ is an $\widehat{\mathfrak{sl}_n}$-module homomorphism. Thus, the action of T^n on $\mathcal{B}$ also is an $\widehat{\mathfrak{sl}_n}$-module homomorphism. As a consequence, the restriction to $\widehat{\mathfrak{sl}_n}$ of the representation $\mathcal{B}^{(m)}$ is not irreducible.

But ψ'_m is still a highest-weight vector with highest weight $\widetilde{\omega}_m$. Let us look at how this representation $\mathcal{B}^{(m)}$ decomposes.

Definition 4.2.10. Let $h_i = E^{\mathfrak{gl}_n}_{i,i} - E^{\mathfrak{gl}_n}_{i+1,i+1}$ for $i \in \{1, 2, ..., n-1\}$, and let $h_0 = K - h_1 - h_2 - ... - h_{n-1}$. Then, $(h_0, h_1, ..., h_{n-1}, d)$ is a basis of $\widetilde{\mathfrak{h}} \cap \widetilde{\mathfrak{sl}_n}$ (which is the 0-th homogeneous component of $\widetilde{\mathfrak{sl}_n}$).

Definition 4.2.11. For every $m \in \mathbb{Z}$, define the weight $\omega_m \in \left(\widetilde{\mathfrak{h}} \cap \widetilde{\mathfrak{sl}_n}\right)^*$ to be the restriction $\widetilde{\omega}_m \mid_{\widetilde{\mathfrak{h}} \cap \widetilde{\mathfrak{sl}_n}}$ of $\widetilde{\omega}_m$ to the 0-th homogeneous component of $\widetilde{\mathfrak{sl}_n}$.

This weight ω_m does not depend on m but only depends on the residue class of m modulo n. In fact, it satisfies

$$\omega_m(h_i) = \widetilde{\omega}_m(h_i) = \begin{cases} 1, & \text{if } i \equiv m \bmod n; \\ 0, & \text{if } i \not\equiv m \bmod n \end{cases} \qquad \text{for all } i \in \{0, 1, ..., n-1\};$$
$$\omega_m(d) = \widetilde{\omega}_m(d) = 0.$$

Definition 4.2.12. Let $\mathcal{A}^{(n)}$ be the Lie subalgebra $\langle K \rangle + \langle a_{ni} \mid i \in \mathbb{Z} \rangle$ of $\mathcal{A}$.

Note that the map

$$\begin{aligned} &\mathcal{A} \to \mathcal{A}^{(n)}, \\ &a_i \mapsto a_{ni} \qquad \text{for every } i \in \mathbb{Z}, \\ &K \mapsto nK \end{aligned}$$

is a Lie algebra isomorphism. But we still consider $\mathcal{A}^{(n)}$ as a Lie subalgebra of $\mathcal{A}$, and we won't identify it with $\mathcal{A}$ via this isomorphism.

Since $\mathcal{A}^{(n)}$ is a Lie subalgebra of $\mathcal{A}$, both $\mathcal{A}$-modules $\mathcal{F}$ and $\mathcal{B}$ become $\mathcal{A}^{(n)}$-modules.

Let us consider the direct sum $\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}$ of Lie algebras. Let us denote by K_1 the element $(K, 0)$ of $\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}$ (where the K means the element K of $\widehat{\mathfrak{sl}_n}$), and let us denote by K_2 the element $(0, K)$ of $\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}$ (where the K means the element K of $\mathcal{A}^{(n)}$). Note that both elements $K_1 = (K, 0)$ and $K_2 = (0, K)$ lie in the center of $\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}$; hence, so does their difference $K_1 - K_2 = (K, -K)$. Thus, $\langle K_1 - K_2 \rangle$ (the $\mathbb{C}$-linear span of the set $\{K_1 - K_2\}$) is an ideal of $\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}$. Thus, $\left(\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}\right) \diagup (K_1 - K_2)$ is a Lie algebra.

Proposition 4.2.13. The Lie algebras $\widehat{\mathfrak{gl}_n}$ and $\left(\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}\right) \diagup (K_1 - K_2)$ are isomorphic. More precisely, the maps

$$\begin{aligned} \left(\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}\right) \diagup (K_1 - K_2) &\to \widehat{\mathfrak{gl}_n}, \\ \overline{(At^\ell, 0)} &\mapsto At^\ell \qquad \text{for every } A \in \mathfrak{sl}_n \text{ and } \ell \in \mathbb{Z}, \\ \overline{(0, a_{n\ell})} &\mapsto \operatorname{id}_n t^\ell \qquad \text{for every } \ell \in \mathbb{Z}, \\ \overline{K_1} = \overline{K_2} &\mapsto K \end{aligned}$$

and

$$\begin{aligned} \widehat{\mathfrak{gl}_n} &\to \left(\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}\right) \diagup (K_1 - K_2), \\ At^\ell &\mapsto \overline{\left(\left(A - \frac{1}{n}(\operatorname{Tr} A) \cdot \operatorname{id}_n\right) t^\ell, \left(\frac{1}{n} \operatorname{Tr} A\right) a_{n\ell}\right)} \qquad \text{for every } A \in \mathfrak{gl}_n \text{ and } \ell \in \mathbb{Z}, \\ K &\mapsto \overline{K_1} = \overline{K_2}. \end{aligned}$$

are mutually inverse isomorphisms of Lie algebras.

The proof of this proposition is left to the reader (it is completely straightforward). This isomorphism $\widehat{\mathfrak{gl}_n} \cong \left(\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}\right) \diagup (K_1 - K_2)$ allows us to consider any $\widehat{\mathfrak{gl}_n}$-module as an $\left(\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}\right) \diagup (K_1 - K_2)$-module, i. e., as an $\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}$-module on which K_1 and K_2 act the same way. In particular, $\mathcal{F}$ and $\mathcal{B}$ become $\widehat{\mathfrak{sl}_n} \oplus \mathcal{A}^{(n)}$-modules. Of course, the actions of the two addends $\widehat{\mathfrak{sl}_n}$ and $\mathcal{A}^{(n)}$ on $\mathcal{F}$ and $\mathcal{B}$ are exactly the actions of $\widehat{\mathfrak{sl}_n}$ and $\mathcal{A}^{(n)}$ on $\mathcal{F}$ and $\mathcal{B}$ that result from the canonical inclusions $\widehat{\mathfrak{sl}_n} \subseteq \widehat{\mathfrak{gl}_n} \subseteq \mathfrak{a}_\infty$ and $\mathcal{A}^{(n)} \subseteq \mathcal{A} \cong \widehat{\mathfrak{gl}_1} \subseteq \mathfrak{a}_\infty$. (This is clear for the action of $\widehat{\mathfrak{sl}_n}$, and is very easy to see for the action of $\mathcal{A}^{(n)}$.)

We checked above that the action of T^n on $\mathcal{B}$ is an $\widehat{\mathfrak{sl}_n}$-module homomorphism. This easily generalizes: For every integer i, the action of T^{ni} on $\mathcal{B}$ is an $\widehat{\mathfrak{sl}_n}$-module homomorphism.[197] Thus, the subspace $\mathcal{B}_0^{(m)} = \left\{v \in \mathcal{B}^{(m)} \mid T^{ni}v = 0 \text{ for all } i > 0\right\}$ of

[197] *Proof.* Let i be an integer. We have $\left[I_n t^i, \widehat{\mathfrak{sl}_n}\right] = 0$ in the Lie algebra $\widehat{\mathfrak{gl}_n}$ (this is because $\left[I_n t^i, L\mathfrak{sl}_n\right] = 0$ in the Lie algebra $L\mathfrak{gl}_n$, and because $\omega\left(I_n t^i, L\mathfrak{sl}_n\right) = 0$ where the 2-cocycle ω is the one defined in Proposition 4.1.4). Since $I_n t^i \in \widehat{\mathfrak{gl}_n}$ acts on $\mathcal{F}$ by the operator $\widehat{\operatorname{Toep}_n}\left(I_n t^i\right) = T^{ni}$ (more precisely, by the action of T^{ni} on $\mathcal{F}$, but let us abbreviate this by T^{ni} here), this yields that the action of T^{ni} on $\mathcal{F}$ is an $\widehat{\mathfrak{sl}_n}$-module homomorphism. Thus, the action of T^{ni} on $\mathcal{B}$ also is an $\widehat{\mathfrak{sl}_n}$-module homomorphism.

$\mathcal{B}^{(m)}$ is an $\widehat{\mathfrak{sl}_n}$-submodule. Recalling that $\mathcal{B}^{(m)} = \mathbb{C}[x_1, x_2, x_3, ...]$, with T^{ni} acting as $ni\dfrac{\partial}{\partial x_{ni}}$, we have $\mathcal{B}_0^{(m)} \cong \mathbb{C}[x_j \mid n \nmid j]$.

Theorem 4.2.14. This $\mathcal{B}_0^{(m)}$ is an irreducible $\widehat{\mathfrak{sl}_n}$-module (or $\widetilde{\mathfrak{sl}_n}$-module; this doesn't matter) with highest weight $\omega_{\overline{m}}$ (this means that $\mathcal{B}_0^{(m)} \cong L_{\omega_{\overline{m}}}$) and depends only on $\overline{m}$ (the remainder of m modulo n) rather than on m. Moreover, $\mathcal{B}^{(m)} \cong \mathcal{B}_0^{(m)} \otimes \widetilde{F}_m$, where $\widetilde{F}_m$ is the appropriate Fock module over $\mathcal{A}^{(n)}$.

Proof of Theorem 4.2.14. We clearly have such a decomposition as vector spaces, $\widetilde{F}_m = \mathbb{C}[x_n, x_{2n}, x_{3n}, ...]$. Each of the two Lie algebras acts in its own factor: $\mathcal{A}^{(n)}$ acts in $\widetilde{F}_m$, and $\widehat{\mathfrak{gl}_n}$ commutes with $\mathcal{A}^{(n)}$. Since the tensor product is irreducible, each factor is irreducible, so that $\mathcal{B}_0^{(m)}$ is irreducible.

4.2.6. [unfinished] Classification of unitary highest-weight $\widehat{\mathfrak{sl}_n}$-modules

We can now classify unitary highest-weight representations of $\widehat{\mathfrak{sl}_n}$:

Proposition 4.2.15. The highest-weight representation L_{ω_m} is unitary for each $m \in \{0, 1, ..., n-1\}$.

Proof. The contravariant Hermitian form on L_{ω_m} is the restriction of the form on $\mathcal{B}^{(m)}$.

Corollary 4.2.16. If $k_0, k_1, ..., k_{n-1}$ are nonnegative integers, then $L_{k_0\omega_0 + k_1\omega_1 + ... + k_{n-1}\omega_{n-1}}$ is unitary (of level $k_0 + k_1 + ... + k_{n-1}$).

Proof. The tensor product $L_{\omega_0}^{\otimes k_0} \otimes L_{\omega_1}^{\otimes k_1} \otimes ... \otimes L_{\omega_{n-1}}^{\otimes k_{n-1}}$ is unitary (being a tensor product of unitary representations), and thus is a direct sum of irreducible representations. Clearly, $L_{k_0\omega_0 + k_1\omega_1 + ... + k_{n-1}\omega_{n-1}}$ is a summand of this module, and thus also unitary, qed.

Theorem 4.2.17. These $L_{k_0\omega_0 + k_1\omega_1 + ... + k_{n-1}\omega_{n-1}}$ (with $k_0, k_1, ..., k_{n-1}$ being nonnegative integers) are the only unitary highest-weight representations of $\widehat{\mathfrak{sl}_n}$.

To prove this, first a lemma:

Lemma 4.2.18. Consider the antilinear $\mathbb{R}$-antiinvolution $\dagger : \mathfrak{sl}_2 \to \mathfrak{sl}_2$ defined by $e^\dagger = f$, $f^\dagger = e$ and $h^\dagger = h$. Let $\lambda \in \mathfrak{h}^*$. We identify the function $\lambda \in \mathfrak{h}^*$ with the value $\lambda(h) \in \mathbb{C}$. Then, L_λ is a unitary representation of $\mathfrak{sl}_2$ if and only if $\lambda \in \mathbb{Z}_+$.

Proof of Lemma 4.2.18. Assume that L_λ is a unitary representation of $\mathfrak{sl}_2$. Let $v_\lambda = v_\lambda^+$. Since L_λ is unitary, the form $(\cdot, \cdot)$ is positive definite, so that $(v_\lambda, v_\lambda) > 0$.

Every $n \in \mathbb{N}$ satisfies

$$(f^n v_\lambda, f^n v_\lambda) = n!\overline{\lambda}\left(\overline{\lambda} - 1\right) ... \left(\overline{\lambda} - n + 1\right)(v_\lambda, v_\lambda)$$

(the proof of this is analogous to the proof of (72), but uses $e^\dagger = f$). Since $(\cdot, \cdot)$ is positive definite, we must have $(f^n v_\lambda, f^n v_\lambda) \geq 0$ for every $n \in \mathbb{N}$. Thus, every $n \in \mathbb{N}$ satisfies $n!\overline{\lambda}\left(\overline{\lambda} - 1\right) ... \left(\overline{\lambda} - n + 1\right)(v_\lambda, v_\lambda) = (f^n v_\lambda, f^n v_\lambda) \geq 0$, so that

$\overline{\lambda}\left(\overline{\lambda}-1\right)...\left(\overline{\lambda}-n+1\right)\geq 0$ (since $(v_\lambda, v_\lambda) > 0$). Applied to $n = 1$, this yields $\overline{\lambda} \geq 0$, so that $\overline{\lambda} \in \mathbb{R}$ and thus $\lambda \in \mathbb{R}$. Hence, $\overline{\lambda} \geq 0$ becomes $\lambda \geq 0$.

Every $n \in \mathbb{N}$ satisfies $\lambda(\lambda-1)...(\lambda-n+1) = \overline{\lambda}\left(\overline{\lambda}-1\right)...\left(\overline{\lambda}-n+1\right) \geq 0$. Thus, $\lambda \in \mathbb{Z}_+$ (otherwise, $\lambda(\lambda-1)...(\lambda-n+1)$ would alternate in sign for each sufficiently large n).

This proves one direction of Lemma 4.2.18. The converse direction is classical and easy. Lemma 4.2.18 is proven.

Corollary 4.2.19. Let $\lambda \in \mathbb{C}$. If $\mathfrak{g}$ is a Lie algebra with antilinear $\mathbb{R}$-antiinvolution $\dagger$ and $\mathfrak{sl}_2$ is a Lie subalgebra of $\mathfrak{g}$, and if $\dagger\mid_{\mathfrak{sl}_2}$ sends e, f, h to f, e, h, and if V is a unitary representation of $\mathfrak{g}$, and if some $v \in V$ satisfies $ev = 0$ and $hv = \lambda v$, then $\lambda \in \mathbb{Z}_+$.

Proof of Theorem 4.2.17. For every $i \in \{0, 1, ..., n-1\}$, we have an $\mathfrak{sl}_2$-subalgebra:

$$h_i = \begin{cases} E_{i,i} - E_{i+1,i+1}, & \text{if } i \neq 0; \\ K + E_{n,n} - E_{1,1}, & \text{if } i = 0 \end{cases},$$

$$e_i = \begin{cases} E_{i,i+1}, & \text{if } i \neq 0; \\ E_{n,1}t, & \text{if } i = 0 \end{cases};$$

$$f_i = \begin{cases} E_{i+1,i}, & \text{if } i \neq 0; \\ E_{1,n}t^{-1}, & \text{if } i = 0 \end{cases}$$

[198] (these form an $\mathfrak{sl}_2$-triple, as can be easily checked). These satisfy $e_i^\dagger = f_i$, $f_i^\dagger = e_i$ and $h_i^\dagger = h_i$. Thus, if L_λ is a unitary representation of $\widehat{\mathfrak{sl}_n}$, then $\lambda(h_i) \in \mathbb{Z}_+$. But ω_i are a basis for the weights, and namely the dual basis to the basis of the h_i. Thus, $\lambda = \sum\limits_{i=0}^{n-1} \lambda(h_i)\,\omega_i$. Hence, $\lambda = \sum\limits_{i=0}^{n-1} k_i\omega_i$ with $k_i \in \mathbb{Z}_+$. Qed.

Remark 4.2.20. Relation between $\widehat{\mathfrak{sl}_n}$-modules and $\mathfrak{sl}_n$-modules:

Let L_λ be a unitary $\widehat{\mathfrak{sl}_n}$-module, with $\lambda = k_0\omega_0 + k_1\omega_1 + ... + k_{n-1}\omega_{n-1}$.

Then, $U(\mathfrak{sl}_n)\,v_\lambda = L_{\overline{\lambda}}$ where $\overline{\lambda} = k_1\omega_1 + k_2\omega_2 + ... + k_{n-1}\omega_{n-1}$ is a weight for $\mathfrak{sl}_n$. And if the level of L_λ was k, then we must have $k_1 + k_2 + ... + k_{n-1} \leq k$.

4.3. The Sugawara construction

We will now study the Sugawara construction. It constructs a Vir action on a $\widehat{\mathfrak{g}}$-module (under some conditions), and it generalizes the action of Vir on the μ-Fock representation F_μ (that was constructed in Proposition 3.2.13).

Definition 4.3.1. Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{C}$-Lie algebra equipped with a $\mathfrak{g}$-invariant symmetric bilinear form $(\cdot,\cdot)$. (This form needs not be nondegenerate; it is even allowed to be 0.)

Consider the 2-cocycle $\omega : \mathfrak{g}[t, t^{-1}] \times \mathfrak{g}[t, t^{-1}] \to \mathbb{C}$ defined by

$$\omega(a, b) = \operatorname{Res}_{t=0}(a', b)\,dt \qquad \text{for all } a \in \mathfrak{g}\left[t, t^{-1}\right] \text{ and } b \in \mathfrak{g}\left[t, t^{-1}\right].$$

[198] Here, $E_{i,j}$ means $E_{i,j}^{\mathfrak{gl}_n}$.

(This is the 2-cocycle ω in Definition 1.7.1. We just slightly rewrote the definition.) Also consider the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ defined through this cocycle ω.

Let Kil denote the Killing form on $\mathfrak{g}$, defined by

$$\operatorname{Kil}(a, b) = \operatorname{Tr}(\operatorname{ad}(a) \cdot \operatorname{ad}(b)) \qquad \text{for all } a, b \in \mathfrak{g}.$$

An element $k \in \mathbb{C}$ is said to be *non-critical for* $(\mathfrak{g}, (\cdot, \cdot))$ if and only if the form $k \cdot (\cdot, \cdot) + \frac{1}{2} \operatorname{Kil}$ is nondegenerate.

Definition 4.3.2. Let M be a $\widehat{\mathfrak{g}}$-module.

We say that M is *admissible* if for every $v \in M$, there exists some $N \in \mathbb{N}$ such that every integer $n \geq N$ and every $a \in \mathfrak{g}$ satisfy $at^n \cdot v = 0$.

If $k \in \mathbb{C}$, then we say that M is *of level* k if $K \mid_M = k \cdot \operatorname{id}$.

Proposition 4.3.3. Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{C}$-Lie algebra equipped with a $\mathfrak{g}$-invariant symmetric bilinear form $(\cdot, \cdot)$. Consider the affine Lie algebra $\widehat{\mathfrak{g}}$ defined as in Definition 4.3.1.

(a) Then, there is a natural homomorphism $\eta_{\widehat{\mathfrak{g}}} : W \to \operatorname{Der} \widehat{\mathfrak{g}}$ of Lie algebras given by

$$(\eta_{\widehat{\mathfrak{g}}}(f\partial))(g, \alpha) = (fg', 0) \qquad \text{for all } f \in \mathbb{C}\left[t, t^{-1}\right], \ g \in \mathfrak{g}\left[t, t^{-1}\right] \text{ and } \alpha \in \mathbb{C}.$$

(b) There also is a natural homomorphism $\widetilde{\eta_{\widehat{\mathfrak{g}}}} : \operatorname{Vir} \to \operatorname{Der} \widehat{\mathfrak{g}}$ of Lie algebras given by

$$(\widetilde{\eta_{\widehat{\mathfrak{g}}}}(f\partial + \lambda K))(g, \alpha) = (fg', 0) \qquad \text{for all } f \in \mathbb{C}\left[t, t^{-1}\right], \ g \in \mathfrak{g}\left[t, t^{-1}\right], \ \lambda \in \mathbb{C} \text{ and } \alpha \in \mathbb{C}.$$

This homomorphism $\widetilde{\eta_{\widehat{\mathfrak{g}}}}$ is simply the extension of the homomorphism $\eta_{\widehat{\mathfrak{g}}} : W \to \operatorname{Der} \widehat{\mathfrak{g}}$ to Vir by means of requiring that $\widetilde{\eta_{\widehat{\mathfrak{g}}}}(K) = 0$.

This homomorphism $\widetilde{\eta_{\widehat{\mathfrak{g}}}}$ makes $\widehat{\mathfrak{g}}$ a Vir-module on which Vir acts by derivations. Therefore, a Lie algebra $\operatorname{Vir} \ltimes \widehat{\mathfrak{g}}$ is defined (according to Definition 3.2.1).

The proof of Proposition 4.3.3 is left to the reader. (A proof of Proposition 4.3.3 **(a)** can be obtained by carefully generalizing the proof of Lemma 1.4.3. Actually, Proposition 4.3.3 **(a)** generalizes Lemma 1.4.3, since (as we will see in Remark 4.3.5) the Lie algebra $\widehat{\mathfrak{g}}$ generalizes $\mathcal{A}$.)

The following theorem is one of the most important facts about affine Lie algebras:

Theorem 4.3.4 (Sugawara construction)**.** Let us work in the situation of Definition 4.3.1.

Let $k \in \mathbb{C}$ be non-critical for $(\mathfrak{g}, (\cdot, \cdot))$. Let M be an admissible $\widehat{\mathfrak{g}}$-module of level k. Let $B \subseteq \mathfrak{g}$ be a basis orthonormal with respect to the form $k(\cdot, \cdot) + \frac{1}{2} \operatorname{Kil}$.

For every $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$, let us denote by x_n the element $xt^n \in \widehat{\mathfrak{g}}$.

For every $x \in \mathfrak{g}$, every $m \in \mathbb{Z}$ and $\ell \in \mathbb{Z}$, define the "normal ordered product" $: x_m x_\ell :$ in $U(\widehat{\mathfrak{g}})$ by

$$: x_m x_\ell : = \begin{cases} x_m x_\ell, & \text{if } m \leq \ell; \\ x_\ell x_m, & \text{if } m > \ell \end{cases}.$$

For every $n \in \mathbb{Z}$, define an endomorphism L_n of M by

$$L_n = \frac{1}{2} \sum_{a \in B} \sum_{m \in \mathbb{Z}} : a_m a_{n-m} : .$$

(a) This endomorphism L_n is indeed well-defined. In other words, for every $n \in \mathbb{Z}$, every $a \in B$ and every $v \in M$, the sum $\sum_{m \in \mathbb{Z}} : a_m a_{n-m} : v$ converges in the discrete topology (i. e., has only finitely many nonzero addends).

(b) For every $n \in \mathbb{Z}$, the endomorphism L_n does not depend on the choice of the orthonormal basis B.

(c) The endomorphisms L_n for $n \in \mathbb{Z}$ give rise to a Vir-representation on M with central charge

$$c = k \cdot \sum_{a \in B} (a, a) .$$

(d) These formulas (for L_n and c) extend the action of $\widehat{\mathfrak{g}}$ on M to an action of $\mathrm{Vir} \ltimes \widehat{\mathfrak{g}}$, so they satisfy $[L_n, a_m] = -m a_{n+m}$ and $[L_n, K] = 0$.

(e) We have $[L_n, a_m] = -m a_{n+m}$ for any $a \in \mathfrak{g}$ and any integers n and m.

Remark 4.3.5. We have already encountered an example of this construction: namely, the example where $\mathfrak{g}$ is the trivial Lie algebra $\mathbb{C}$, where $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ is the bilinear form $(x, y) \mapsto xy$, where $k = 1$, and where M is the $\widehat{\mathfrak{g}}$-module F_μ. (To make sense of this, notice that when $\mathfrak{g}$ is the trivial Lie algebra $\mathbb{C}$, the affine Lie algebra $\widehat{\mathfrak{g}}$ is canonically isomorphic to the Heisenberg algebra $\mathcal{A}$, through an isomorphism $\widehat{\mathfrak{g}} \to \mathcal{A}$ which takes t^n to a_n and K to K.) In this example, the operators L_n defined in Theorem 4.3.4 are exactly the operators L_n defined in Definition 3.2.8.

Before we prove Theorem 4.3.4, we formulate a number of lemmas. First, an elementary lemma on Killing forms of finite-dimensional Lie algebras:

Lemma 4.3.6. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Denote by Kil the Killing form of $\mathfrak{g}$. Let $n \in \mathbb{N}$ and $p_1, p_2, ..., p_n \in \mathfrak{g}$ and $q_1, q_2, ..., q_n \in \mathfrak{g}$ be such that the tensor $\sum_{i=1}^{n} p_i \otimes q_i \in \mathfrak{g} \otimes \mathfrak{g}$ is $\mathfrak{g}$-invariant. Then, $\sum_{i=1}^{n} [[b, p_i], q_i] = \sum_{i=1}^{n} \mathrm{Kil}(b, p_i) q_i$ for every $b \in \mathfrak{g}$.

Here, we are using the following notation:

Remark 4.3.7. Let $\mathfrak{g}$ be a Lie algebra. An element m of a $\mathfrak{g}$-module M is said to be $\mathfrak{g}$-*invariant* if and only if it satisfies ($x \rightharpoonup m = 0$ for every $x \in \mathfrak{g}$). We regard $\mathfrak{g}$ as a $\mathfrak{g}$-module by means of the adjoint action of $\mathfrak{g}$ (that is, we set $x \rightharpoonup m = [x, m]$ for every $x \in \mathfrak{g}$ and $m \in \mathfrak{g}$); thus, $\mathfrak{g} \otimes \mathfrak{g}$ becomes a $\mathfrak{g}$-module as well. Explicitly, the action of $\mathfrak{g}$ on $\mathfrak{g} \otimes \mathfrak{g}$ is given by

$$x \rightharpoonup \left(\sum_{i=1}^{n} p_i \otimes q_i \right) = \sum_{i=1}^{n} [x, p_i] \otimes q_i + \sum_{i=1}^{n} p_i \otimes [x, q_i]$$

for every tensor $\sum_{i=1}^{n} p_i \otimes q_i \in \mathfrak{g} \otimes \mathfrak{g}$. Hence, a tensor $\sum_{i=1}^{n} p_i \otimes q_i \in \mathfrak{g} \otimes \mathfrak{g}$ is $\mathfrak{g}$-invariant if and only if every $x \in \mathfrak{g}$ satisfies $\sum_{i=1}^{n} [x, p_i] \otimes q_i + \sum_{i=1}^{n} p_i \otimes [x, q_i] = 0$. In other words, a tensor $\sum\limits_{i=1}^{n} p_i \otimes q_i \in \mathfrak{g} \otimes \mathfrak{g}$ is $\mathfrak{g}$-invariant if and only if every $x \in \mathfrak{g}$ satisfies $\sum\limits_{i=1}^{n} [p_i, x] \otimes q_i = -\sum\limits_{i=1}^{n} p_i \otimes [q_i, x]$.

Proof of Lemma 4.3.6. Let $(c_1, c_2, ..., c_m)$ be a basis of the vector space $\mathfrak{g}$, and let $(c_1^*, c_2^*, ..., c_m^*)$ be the dual basis of $\mathfrak{g}^*$. Then, every $i \in \{1, 2, ..., n\}$ satisfies

$$\operatorname{Kil}(b, p_i) = \operatorname{Tr}((\operatorname{ad} b) \circ (\operatorname{ad} p_i)) = \sum_{j=1}^{m} c_j^* (((\operatorname{ad} b) \circ (\operatorname{ad} p_i))(c_j)) = \sum_{j=1}^{m} c_j^* ([b, [p_i, c_j]]) .$$

Hence,

$$\begin{aligned}
\sum_{i=1}^{n} \operatorname{Kil}(b, p_i) q_i &= \sum_{i=1}^{n} \sum_{j=1}^{m} c_j^* ([b, [p_i, c_j]]) q_i = \sum_{j=1}^{m} \sum_{i=1}^{n} c_j^* ([b, [p_i, c_j]]) q_i \\
&= -\sum_{j=1}^{m} \sum_{i=1}^{n} c_j^* ([b, p_i]) [q_i, c_j] \\
&\qquad \left(\begin{array}{c} \text{since } \sum\limits_{i=1}^{n} p_i \otimes q_i \text{ is } \mathfrak{g}\text{-invariant, so that} \\ \sum\limits_{i=1}^{n} [p_i, c_j] \otimes q_i = -\sum\limits_{i=1}^{n} p_i \otimes [q_i, c_j] \text{ for every } j \in \{1, 2, ..., m\} \text{, and thus} \\ \sum\limits_{i=1}^{n} c_j^* ([b, [p_i, c_j]]) q_i = -\sum\limits_{i=1}^{n} c_j^* ([b, p_i]) [q_i, c_j] \text{ for every } j \in \{1, 2, ..., m\} \end{array} \right) \\
&= -\sum_{j=1}^{m} \sum_{i=1}^{n} \left[q_i, c_j^* ([b, p_i]) c_j \right] = -\sum_{i=1}^{n} \left[q_i, \underbrace{\sum_{j=1}^{m} c_j^* ([b, p_i]) c_j}_{\substack{= [b, p_i] \\ \text{(since } (c_1^*, c_2^*, ..., c_m^*) \text{ is the dual basis} \\ \text{to the basis } (c_1, c_2, ..., c_m))}} \right] \\
&= -\sum_{i=1}^{n} [q_i, [b, p_i]] = \sum_{i=1}^{n} [[b, p_i], q_i] ,
\end{aligned}$$

which proves Lemma 4.3.6.

Here comes another lemma on $\mathfrak{g}$-invariant bilinear forms:

Lemma 4.3.8. Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{C}$-Lie algebra equipped with a $\mathfrak{g}$-invariant symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $B \subseteq \mathfrak{g}$ be a basis orthonormal with respect to the form $\langle \cdot, \cdot \rangle$.

(a) Then, the tensor $\sum\limits_{a \in B} a \otimes a$ is $\mathfrak{g}$-invariant in $\mathfrak{g} \otimes \mathfrak{g}$.

(b) Let B' also be a basis of $\mathfrak{g}$ orthonormal with respect to the form $\langle \cdot, \cdot \rangle$. Then, $\sum\limits_{a \in B} a \otimes a = \sum\limits_{a \in B'} a \otimes a$.

Proof of Lemma 4.3.8. The bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate (since it has an orthonormal basis).

(a) For every $v \in \mathfrak{g}$, let $v^* : \mathfrak{g} \to \mathbb{C}$ be the $\mathbb{C}$-linear map which sends every $w \in \mathfrak{g}$ to $\langle v, w \rangle$. Then, $\mathfrak{g}^* = \{v^* \mid v \in \mathfrak{g}\}$ (since the form $\langle \cdot, \cdot \rangle$ is nondegenerate).

Let $b \in \mathfrak{g}$. We will now prove that $h\left(\sum\limits_{a \in B} ([b, a] \otimes a + a \otimes [b, a])\right) = 0$ for every $h \in (\mathfrak{g} \otimes \mathfrak{g})^*$.

In fact, let $h \in (\mathfrak{g} \otimes \mathfrak{g})^*$. Since $\mathfrak{g}$ is finite-dimensional, we have $(\mathfrak{g} \otimes \mathfrak{g})^* = \mathfrak{g}^* \otimes \mathfrak{g}^*$, so that $h \in \mathfrak{g}^* \otimes \mathfrak{g}^*$. We can WLOG assume that $h = f_1 \otimes f_2$ for some $f_1 \in \mathfrak{g}^*$ and $f_2 \in \mathfrak{g}^*$ (because every tensor in $\mathfrak{g}^* \otimes \mathfrak{g}^*$ is a $\mathbb{C}$-linear combination of pure tensors, and the assertion which we want to prove (namely, the equality $h\left(\sum\limits_{a \in B} ([b, a] \otimes a + a \otimes [b, a])\right) = 0$) is $\mathbb{C}$-linear in h). Assume this.

Since $f_1 \in \mathfrak{g}^* = \{v^* \mid v \in \mathfrak{g}\}$, there exists some $v_1 \in \mathfrak{g}$ such that $f_1 = v_1^*$. Consider this v_1.

Since $f_2 \in \mathfrak{g}^* = \{v^* \mid v \in \mathfrak{g}\}$, there exists some $v_2 \in \mathfrak{g}$ such that $f_2 = v_2^*$. Consider this v_2.

Since B is an orthonormal basis with respect to $\langle \cdot, \cdot \rangle$, we have $\sum\limits_{a \in B} a \langle [b, v_2], a \rangle = [b, v_2]$ and $\sum\limits_{a \in B} \langle [b, v_1], a \rangle a = [b, v_1]$.

Now, $h = \underbrace{f_1}_{=v_1^*} \otimes \underbrace{f_2}_{=v_2^*} = v_1^* \otimes v_2^*$, so that

$$h\left(\sum_{a\in B}([b,a]\otimes a + a\otimes [b,a])\right)$$
$$= (v_1^* \otimes v_2^*)\left(\sum_{a\in B}([b,a]\otimes a + a\otimes [b,a])\right)$$
$$= \sum_{a\in B}\left(\underbrace{v_1^*([b,a])}_{\substack{=\langle v_1,[b,a]\rangle \\ \text{(by the definition of } v_1^*)}} \cdot \underbrace{v_2^*(a)}_{\substack{=\langle v_2,a\rangle \\ \text{(by the definition of } v_2^*)}} + \underbrace{v_1^*(a)}_{\substack{=\langle v_1,a\rangle \\ \text{(by the definition of } v_1^*)}} \cdot \underbrace{v_2^*([b,a])}_{\substack{=\langle v_2,[b,a]\rangle \\ \text{(by the definition of } v_2^*)}}\right)$$
$$= \sum_{a\in B}\left(\underbrace{\langle v_1,[b,a]\rangle}_{\substack{=-\langle [b,v_1],a\rangle \\ \text{(since } \langle\cdot,\cdot\rangle \text{ is invariant)}}} \cdot \langle v_2,a\rangle + \langle v_1,a\rangle \cdot \underbrace{\langle v_2,[b,a]\rangle}_{\substack{=-\langle [b,v_2],a\rangle \\ \text{(since } \langle\cdot,\cdot\rangle \text{ is invariant)}}}\right)$$
$$= \sum_{a\in B}\left(-\langle [b,v_1],a\rangle \cdot \langle v_2,a\rangle - \langle v_1,a\rangle \cdot \langle [b,v_2],a\rangle\right)$$
$$= -\underbrace{\sum_{a\in B}\langle [b,v_1],a\rangle \cdot \langle v_2,a\rangle}_{=\left\langle v_2,\sum_{a\in B}\langle [b,v_1],a\rangle a\right\rangle} - \underbrace{\sum_{a\in B}\langle v_1,a\rangle \cdot \langle [b,v_2],a\rangle}_{=\left\langle v_1,\sum_{a\in B} a\langle [b,v_2],a\rangle\right\rangle}$$
$$= -\left\langle v_2, \underbrace{\sum_{a\in B}\langle [b,v_1],a\rangle a}_{=[b,v_1]}\right\rangle - \left\langle v_1, \underbrace{\sum_{a\in B} a\langle [b,v_2],a\rangle}_{=[b,v_2]}\right\rangle$$
$$= -\underbrace{\langle v_2,[b,v_1]\rangle}_{\substack{=\langle [b,v_1],v_2\rangle \\ \text{(since } \langle\cdot,\cdot\rangle \text{ is symmetric)}}} - \underbrace{\langle v_1,[b,v_2]\rangle}_{\substack{=-\langle [b,v_1],v_2\rangle \\ \text{(since } \langle\cdot,\cdot\rangle \text{ is invariant)}}} = -\langle [b,v_1],v_2\rangle - (-\langle [b,v_1],v_2\rangle) = 0.$$

We thus have proven that $h\left(\sum\limits_{a\in B}([b,a]\otimes a + a\otimes [b,a])\right) = 0$ for every $h \in (\mathfrak{g}\otimes\mathfrak{g})^*$. Consequently, $\sum\limits_{a\in B}([b,a]\otimes a + a\otimes [b,a]) = 0$.

Hence, we have shown that $\sum\limits_{a\in B}([b,a]\otimes a + a\otimes [b,a]) = 0$ for every $b \in \mathfrak{g}$. In other words, the tensor $\sum\limits_{a\in B} a\otimes a$ is $\mathfrak{g}$-invariant. Lemma 4.3.8 **(a)** is proven.

(b) For every $a \in B$ and $b \in B'$, let $\xi_{a,b}$ be the b-coordinate of a with respect to the basis B'. Then, every $a \in B$ satisfies $a = \sum\limits_{b\in B'} \xi_{a,b} b$. Thus, $(\xi_{a,b})_{(a,b)\in B\times B'}$ (this is a matrix whose rows and columns are indexed by elements of B and B', respectively) is the matrix which represents the change of bases from B' to B (or from B to B', depending on how you define the matrix representing a change of basis). Since both B and B' are two orthonormal bases with respect to the same bilinear form $\langle\cdot,\cdot\rangle$, this matrix must thus be orthogonal. Hence, every $b \in B'$ and $b' \in B'$ satisfy $\sum\limits_{a\in B} \xi_{a,b}\xi_{a,b'} =$

$\delta_{b,b'}$ (where $\delta_{b,b'}$ is the Kronecker delta of b and b'). Now, since every $a \in B$ satisfies $a = \sum_{b \in B'} \xi_{a,b} b$ and $a = \sum_{b \in B'} \xi_{a,b} b = \sum_{b' \in B'} \xi_{a,b'} b'$ (here, we renamed b as b' in the sum), we have

$$\begin{aligned}
&\sum_{a \in B} \underbrace{a}_{= \sum_{b \in B'} \xi_{a,b} b} \otimes \underbrace{a}_{= \sum_{b' \in B'} \xi_{a,b'} b'} \\
&= \sum_{a \in B} \left(\sum_{b \in B'} \xi_{a,b} b \right) \otimes \left(\sum_{b' \in B'} \xi_{a,b'} b' \right) = \sum_{a \in B} \sum_{b \in B'} \sum_{b' \in B'} \xi_{a,b} \xi_{a,b'} b \otimes b' \\
&= \sum_{b \in B'} \sum_{b' \in B'} \underbrace{\sum_{a \in B} \xi_{a,b} \xi_{a,b'}}_{= \delta_{b,b'}} b \otimes b' = \sum_{b \in B'} \underbrace{\sum_{b' \in B'} \delta_{b,b'} b \otimes b'}_{= b \otimes b} = \sum_{b \in B'} b \otimes b \\
&= \sum_{a \in B'} a \otimes a \qquad \text{(here, we renamed } b \text{ as } a \text{ in the sum)}.
\end{aligned}$$

This proves Lemma 4.3.8 **(b)**.

As a consequence of this lemma, we get:

Lemma 4.3.9. Let $\mathfrak{g}$ be a finite-dimensional $\mathbb{C}$-Lie algebra equipped with a $\mathfrak{g}$-invariant symmetric bilinear form $(\cdot, \cdot)$. Denote by Kil the Killing form of $\mathfrak{g}$. Let $B \subseteq \mathfrak{g}$ be a basis orthonormal with respect to the form $k(\cdot, \cdot) + \frac{1}{2} \operatorname{Kil}$. Let $b \in \mathfrak{g}$.

(a) We have $\sum_{a \in B} ([b, a] \otimes a + a \otimes [b, a]) = 0$.

(b) We have $\frac{1}{2} \sum_{a \in B} [[b, a], a] + k \sum_{a \in B} (b, a) a = b$.

(c) We have $([b, a], a) = 0$ for every $a \in \mathfrak{g}$.

Proof of Lemma 4.3.9. The basis B is orthonormal with respect to a symmetric $\mathfrak{g}$-invariant bilinear form (namely, the form $k(\cdot, \cdot) + \frac{1}{2} \operatorname{Kil}$). As a consequence, the tensor $\sum_{a \in B} a \otimes a$ is $\mathfrak{g}$-invariant in $\mathfrak{g} \otimes \mathfrak{g}$ (by Lemma 4.3.8 **(a)**, applied to $\langle \cdot, \cdot \rangle = k(\cdot, \cdot) + \frac{1}{2} \operatorname{Kil}$). In other words, $\sum_{a \in B} ([b, a] \otimes a + a \otimes [b, a]) = 0$. This proves Lemma 4.3.9 **(a)**.

(b) If $\langle \cdot, \cdot \rangle$ is any nondegenerate inner product[199] on a finite-dimensional vector space V and B is an orthonormal basis with respect to that product, then any vector $b \in V$ is equal to $\sum_{a \in B} \langle b, a \rangle a$. Applying this fact to the inner product $\langle \cdot, \cdot \rangle = k(\cdot, \cdot) + \frac{1}{2} \operatorname{Kil}$ on the vector space $V = \mathfrak{g}$, we conclude that $b = k \sum_{a \in B} (b, a) a + \frac{1}{2} \sum_{a \in B} \operatorname{Kil}(b, a) a$.

Now, applying Lemma 4.3.6 to the $\mathfrak{g}$-invariant tensor $\sum_{a \in B} a \otimes a$ in lieu of $\sum_{i=1}^{n} p_i \otimes q_i$,

[199] By "inner product", we mean a symmetric bilinear form.

we see that $\sum_{a \in B} [[b,a],a] = \sum_{a \in B} \operatorname{Kil}(b,a) a$. Hence,

$$b = k \sum_{a \in B} (b,a) a + \frac{1}{2} \underbrace{\sum_{a \in B} \operatorname{Kil}(b,a) a}_{= \sum_{a \in B} [[b,a],a]} = \frac{1}{2} \sum_{a \in B} [[b,a],a] + k \sum_{a \in B} (b,a) a.$$

This proves Lemma 4.3.9 **(b)**.

(c) Every $c \in \mathfrak{g}$ satisfies $([a,b],c) + (b,[a,c]) = 0$ (due to the $\mathfrak{g}$-invariance of $(\cdot,\cdot)$). Applying this to $c = a$, we obtain $([a,b],a) + (b,[a,a]) = 0$. Since $[a,a] = 0$ and $[a,b] = -[b,a]$, this rewrites as $(-[b,a],a) + (b,0) = 0$. This simplifies to $-([b,a],a) = 0$. Thus, $([b,a],a) = 0$. This proves Lemma 4.3.9 **(c)**.

Next, we formulate the analogue of Remark 3.2.5:

Remark 4.3.10. Let $x \in \mathfrak{g}$. If m and n are integers such that $m \neq -n$, then $: x_m x_n : = x_m x_n$. (This is because $[x_m, x_n] = 0$ in $\widehat{\mathfrak{g}}$ when $m \neq -n$.)

In analogy to Remark 3.2.6 **(a)**, we have commutativity of normal ordered products:

Remark 4.3.11. Let $x \in \mathfrak{g}$. Any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ satisfy $: x_m x_n : = : x_n x_m :$.

Also, here is a simple way to rewrite the definition of $: x_m x_n :$:

Remark 4.3.12. Let $x \in \mathfrak{g}$. Any $m \in \mathbb{Z}$ and $n \in \mathbb{Z}$ satisfy $: x_m x_n : = x_{\min\{m,n\}} x_{\max\{m,n\}}$.

Generalizing Remark 3.2.7, we have:

Remark 4.3.13. Let $x \in \mathfrak{g}$. Let m and n be integers.

(a) Then, $: x_m x_n : = x_m x_n + n [m > 0] \delta_{m,-n} (x,x) K$. Here, when $\mathfrak{A}$ is an assertion, we denote by $[\mathfrak{A}]$ the truth value of $\mathfrak{A}$ (that is, the number $\left(\begin{cases} 1, & \text{if } \mathfrak{A} \text{ is true;} \\ 0, & \text{if } \mathfrak{A} \text{ is false} \end{cases} \right)$.

(b) For any $y \in U(\widehat{\mathfrak{g}})$, we have $[y, : x_m x_n :] = [y, x_m x_n]$ in $U(\widehat{\mathfrak{g}})$ (where $[\cdot,\cdot]$ denotes the commutator in $U(\widehat{\mathfrak{g}})$).

The proof of this is left to the reader (it follows very quickly from the definitions).

Next, here is a completely elementary lemma:

Lemma 4.3.14. Let G be an abelian group (written additively). Whenever $(u_m)_{m \in \mathbb{Z}} \in G^{\mathbb{Z}}$ is a family of elements of G, and $\mathcal{A}(m)$ is an assertion for every $m \in \mathbb{Z}$, let us abbreviate the sum $\sum_{\substack{m \in \mathbb{Z}; \\ \mathcal{A}(m)}} u_m$ (if this sum is well-defined) by $\sum_{\mathcal{A}(m)} u_m$. (For instance, we will abbreviate the sum $\sum_{\substack{m \in \mathbb{Z}; \\ 3 \leq m \leq 7}} u_m$ by $\sum_{3 \leq m \leq 7} u_m$.)

For any integers α and β such that $\alpha \leq \beta$, for any nonnegative integer N, and for any family $(u_m)_{m \in \mathbb{Z}} \in G^{\mathbb{Z}}$ of elements of G, we have

$$\sum_{|m-\beta| \leq N} u_m - \sum_{|m-\alpha| \leq N} u_m = - \sum_{\alpha - N \leq m < \beta - N} u_m + \sum_{\alpha + N < m \leq \beta + N} u_m.$$

The proof of Lemma 4.3.14 (which is merely an easy generalization of the telescope principle) is left to the reader.

Proof of Theorem 4.3.4. Let us use the notation $\sum\limits_{\mathcal{A}(m)} u_m$ defined in Lemma 4.3.14.

In the following, we will consider the topology on $\operatorname{End} M$ defined as follows: Endow M with the discrete topology, endow M^M with the product topology, and endow $\operatorname{End} M$ with a topology by viewing $\operatorname{End} M$ as a subset of the set M^M. Clearly, in this topology, a net $(a_s)_{s\in S}$ of elements of $\operatorname{End} M$ converges if and only if for every $v \in M$, the net $(a_s v)_{s\in S}$ of elements of M converges (in the discrete topology). As a consequence, whenever $(u_m)_{m\in\mathbb{Z}}$ is a family of elements of $\operatorname{End} M$ indexed by integers, the sum $\sum\limits_{m\in\mathbb{Z}} u_m$ converges with respect to the topology which we defined on $\operatorname{End} M$ if and only if for every $v \in M$, the sum $\sum\limits_{m\in\mathbb{Z}} u_m v$ converges in the discrete topology (i. e., has only finitely many nonzero addends). Consequently, the convergence of an infinite sum with respect to the topology which we defined on $\operatorname{End} M$ is equivalent to the convergence of this sum in the meaning in which we used the word "convergence" in Theorem 4.3.4.

Note that addition, composition, and scalar multiplication (in the sense of: multiplication by scalars) of maps in $\operatorname{End} M$ are continuous maps with respect to this topology.

We will use the notation $\lim\limits_{N\to\infty}$ for limits with respect to the topology on $\operatorname{End} M$. Note that, if $(u_m)_{m\in\mathbb{Z}}$ is a family of elements of $\operatorname{End} M$ indexed by integers, and if the sum $\sum\limits_{m\in\mathbb{Z}} u_m$ converges with respect to the topology which we defined on $\operatorname{End} M$, then $\sum\limits_{m\in\mathbb{Z}} u_m = \lim\limits_{N\to\infty} \sum\limits_{|m-\alpha|\leq N} u_m$ for every $\alpha \in \mathbb{R}$.

In the following, $[\cdot,\cdot]_{L\mathfrak{g}}$ will mean the Lie bracket of $L\mathfrak{g}$, whereas the notation $[\cdot,\cdot]$ without a subscript will mean either the Lie bracket of $\widehat{\mathfrak{g}}$ or the Lie bracket of $\mathfrak{g}$. Note that the use of the same notation for the Lie bracket of $\widehat{\mathfrak{g}}$ and for the Lie bracket of $\mathfrak{g}$ will not lead to conflicts, since the Lie bracket of $\mathfrak{g}$ is the restriction of the Lie bracket of $\widehat{\mathfrak{g}}$ to $\mathfrak{g}\times\mathfrak{g}$ (this follows quickly from $\omega(\mathfrak{g},\mathfrak{g}) = 0$).

Note that any $x \in \mathfrak{g}$, $y \in \mathfrak{g}$, $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfy

$$[x_n, y_m] = [x,y]_{n+m} + K\omega(x_n, y_m) \tag{291}$$

[200].

[200]This is because

$$\Bigg[\underbrace{x_n}_{=xt^n}, \underbrace{y_m}_{=yt^m}\Bigg] = [xt^n, yt^m] = \Bigg(\underbrace{\underbrace{[xt^n, yt^m]_{L\mathfrak{g}}}_{=[x,y]t^{n+m}}}_{\substack{\text{(by the definition of the Lie}\\ \text{algebra structure on } L\mathfrak{g})}}, \omega\Big(\underbrace{xt^n}_{=x_n}, \underbrace{yt^m}_{=y_m}\Big)\Bigg)$$

$$\text{(by the definition of the Lie bracket on } \widehat{\mathfrak{g}})$$

$$= \Bigg(\underbrace{[x,y]\,t^{n+m}}_{=[x,y]_{n+m}}, \omega(x_n, y_m)\Bigg) = \big([x,y]_{n+m}, \omega(x_n,y_m)\big) = [x,y]_{n+m} + K\omega(x_n, y_m).$$

(a) Let $n \in \mathbb{Z}$ and $v \in M$. We must prove that for every $a \in B$, the sum $\sum\limits_{m \in \mathbb{Z}} : a_m a_{n-m} : v$ converges in the discrete topology. We will prove a slightly more general statement: We will prove that for every $x \in \mathfrak{g}$, the sum $\sum\limits_{m \in \mathbb{Z}} : x_m x_{n-m} : v$ converges in the discrete topology.

In fact, let $x \in \mathfrak{g}$. We must prove that the sum $\sum\limits_{m \in \mathbb{Z}} : x_m x_{n-m} : v$ converges in the discrete topology.

Recall the definition of an admissible module. With slightly modified notations, it looks as follows: A $\widehat{\mathfrak{g}}$-module P is said to be *admissible* if for every $w \in P$, there exists some $\mathbf{M} \in \mathbb{N}$ such that every integer $\mathbf{m} \geq \mathbf{M}$ and every $a \in \mathfrak{g}$ satisfy $at^{\mathbf{m}} \cdot w = 0$. Hence, for every $w \in M$, there exists some $\mathbf{M} \in \mathbb{N}$ such that every integer $\mathbf{m} \geq \mathbf{M}$ and every $a \in \mathfrak{g}$ satisfy $at^{\mathbf{m}} \cdot w = 0$ (because M is admissible). Applying this to $w = v$, we see that there exists some $\mathbf{M} \in \mathbb{N}$ such that every integer $\mathbf{m} \geq \mathbf{M}$ and every $a \in \mathfrak{g}$ satisfy $at^{\mathbf{m}} \cdot v = 0$. Fix this $\mathbf{M}$. Every integer $m \geq \mathbf{M}$ satisfies

$$\underbrace{x_m}_{=xt^m} v = xt^m \cdot v = 0 \tag{292}$$

(by the equality $at^{\mathbf{m}} \cdot v = 0$, applied to $a = x$ and $\mathbf{m} = m$). Now, every integer m such that $\max\{m, n-m\} \geq \mathbf{M}$ satisfies

$$\underbrace{: x_m x_{n-m} :}_{\substack{= x_{\min\{m,n-m\}} x_{\max\{m,n-m\}} \\ \text{(by Remark 4.3.12, applied} \\ \text{to } \ell = n-m)}} v = x_{\min\{m,n-m\}} \underbrace{x_{\max\{m,n-m\}} v}_{\substack{=0 \\ \text{(by (292), applied to } \max\{m,n-m\} \\ \text{instead of } m \text{ (since } \max\{m,n-m\} \geq \mathbf{M}))}} = x_{\min\{m,n-m\}} 0 = 0.$$

Since all but finitely many integers m satisfy $\max\{m, n-m\} \geq \mathbf{M}$ (this is obvious), this shows that all but finitely many integers m satisfy $: x_m x_{n-m} : v = 0$. In other words, all but finitely many addends of the sum $\sum\limits_{m \in \mathbb{Z}} : x_m x_{n-m} : v$ are zero. Hence, the sum $\sum\limits_{m \in \mathbb{Z}} : x_m x_{n-m} : v$ converges in the discrete topology. This proves Theorem 4.3.4 **(a)**.

Note that, during the proof of Theorem 4.3.4 **(a)**, we have shown that for every $n \in \mathbb{Z}$, $x \in \mathfrak{g}$ and $v \in M$, the sum $\sum\limits_{m \in \mathbb{Z}} : x_m x_{n-m} : v$ converges in the discrete topology. In other words, for every $n \in \mathbb{Z}$ and $x \in \mathfrak{g}$, the sum $\sum\limits_{m \in \mathbb{Z}} : x_m x_{n-m} :$ converges in the topology which we defined on $\operatorname{End} M$.

(b) Let $n \in \mathbb{Z}$. Let B' be an orthonormal basis of $\mathfrak{g}$ with respect to the form $k(\cdot, \cdot) + \frac{1}{2} \operatorname{Kil}$. We are going to prove that

$$L_n = \frac{1}{2} \sum_{a \in B'} \sum_{m \in \mathbb{Z}} : a_m a_{n-m} : \tag{293}$$

(where L_n still denotes the operator $\frac{1}{2} \sum\limits_{a \in B} \sum\limits_{m \in \mathbb{Z}} : a_m a_{n-m} :$ defined in Theorem 4.3.4 using the orthonormal basis B, not the orthonormal basis B'). Once (293) is proven, it will follow that L_n does not depend on B, and thus Theorem 4.3.4 **(b)** will be proven.

Applying Lemma 4.3.8 **(b)** to $\langle \cdot, \cdot \rangle = k (\cdot, \cdot) + \frac{1}{2} \operatorname{Kil}$, we obtain $\sum\limits_{a \in B} a \otimes a = \sum\limits_{a \in B'} a \otimes a$. Thus,

$$\sum_{a \in B} a_u a_v = \sum_{a \in B'} a_u a_v \qquad \text{for any } u \in \mathbb{Z} \text{ and } v \in \mathbb{Z} \tag{294}$$

[201].

Thus, every $m \in \mathbb{Z}$ satisfies $\sum\limits_{a \in B} : a_m a_{n-m} : = \sum\limits_{a \in B'} : a_m a_{n-m} :$ [202]. Hence,

$$L_n = \frac{1}{2} \sum_{a \in B} \sum_{m \in \mathbb{Z}} : a_m a_{n-m} : = \frac{1}{2} \sum_{m \in \mathbb{Z}} \underbrace{\sum_{a \in B} : a_m a_{n-m} :}_{= \sum\limits_{a \in B'} : a_m a_{n-m} :} = \frac{1}{2} \sum_{m \in \mathbb{Z}} \sum_{a \in B'} : a_m a_{n-m} :$$
$$= \frac{1}{2} \sum_{a \in B'} \sum_{m \in \mathbb{Z}} : a_m a_{n-m} : .$$

Thus, (293) is proven. As we said, this completes the proof of Theorem 4.3.4 **(b)**.

(c) *1st step:* Let us first show that

$$[b_r, L_n] = r b_{n+r} \qquad \text{for every } b \in \mathfrak{g} \text{ and any integers } r \text{ and } n. \tag{295}$$

---

[201] This follows from applying the linear map

$$\mathfrak{g} \otimes \mathfrak{g} \to \operatorname{End} M,$$
$$x \otimes y \mapsto x_u y_v$$

to the equality $\sum\limits_{a \in B} a \otimes a = \sum\limits_{a \in B'} a \otimes a$.

[202] *Proof.* We distinguish between two cases:

Case 1: We have $m \leq n - m$.

Case 2: We have $m > n - m$.

Let us first consider Case 1. In this case, $m \leq n - m$. Hence, every $a \in \mathfrak{g}$ satisfies $: a_m a_{n-m} : = a_m a_{n-m}$. Thus,

$$\sum_{a \in B} : a_m a_{n-m} : = \sum_{a \in B} a_m a_{n-m} = \sum_{a \in B'} \underbrace{a_m a_{n-m}}_{= : a_m a_{n-m} :} \qquad \text{(by (294), applied to } u = m \text{ and } v = n - m\text{)}$$
$$= \sum_{a \in B'} : a_m a_{n-m} : .$$

This proves $\sum\limits_{a \in B} : a_m a_{n-m} : = \sum\limits_{a \in B'} : a_m a_{n-m} :$ in Case 1.

Let us now consider Case 2. In this case, $m > n - m$. Hence, every $a \in \mathfrak{g}$ satisfies $: a_m a_{n-m} : = a_{n-m} a_m$. Thus,

$$\sum_{a \in B} : a_m a_{n-m} : = \sum_{a \in B} a_{n-m} a_m = \sum_{a \in B'} \underbrace{a_{n-m} a_m}_{= : a_m a_{n-m} :} \qquad \text{(by (294), applied to } u = n - m \text{ and } v = m\text{)}$$
$$= \sum_{a \in B'} : a_m a_{n-m} : .$$

This proves $\sum\limits_{a \in B} : a_m a_{n-m} : = \sum\limits_{a \in B'} : a_m a_{n-m} :$ in Case 2.

Hence, $\sum\limits_{a \in B} : a_m a_{n-m} : = \sum\limits_{a \in B'} : a_m a_{n-m} :$ is proven in each of the cases 1 and 2. Thus, $\sum\limits_{a \in B} : a_m a_{n-m} : = \sum\limits_{a \in B'} : a_m a_{n-m} :$ always holds (since cases 1 and 2 cover all possibilities), qed.

Proof of (295): Let $b \in \mathfrak{g}$, $r \in \mathbb{Z}$ and $n \in \mathbb{Z}$.

We must be careful here with infinite sums, since not even formal algebra allows us to manipulate infinite sums like $\sum_{m \in \mathbb{Z}} [b, a]_{r+m} a_{n-m}$ (for good reasons: these are divergent in every meaning of this word). While we were working in the Heisenberg algebra $\mathcal{A}$ (which can be written as $\widehat{\mathfrak{g}}$ for $\mathfrak{g}$ being the trivial Lie algebra $\mathbb{C}$), these infinite sums made sense due to all of their addends being 0 (since $[b, a] = 0$ for all a and b lying in the trivial Lie algebra $\mathbb{C}$). But this was an exception rather than the rule, and now we need to take care.

Let us first assume that $r \geq 0$.

Since

$$L_n = \frac{1}{2} \sum_{a \in B} \underbrace{\sum_{m \in \mathbb{Z}} : a_m a_{n-m} :}_{= \lim\limits_{N \to \infty} \sum\limits_{\left| m - \frac{n}{2} \right| \leq N} : a_m a_{n-m} :} = \frac{1}{2} \sum_{a \in B} \lim_{N \to \infty} \sum_{\left| m - \frac{n}{2} \right| \leq N} : a_m a_{n-m} :$$
$$= \frac{1}{2} \lim_{N \to \infty} \sum_{a \in B} \sum_{\left| m - \frac{n}{2} \right| \leq N} : a_m a_{n-m} : ,$$

we have

$$\begin{aligned}
&[b_r, L_n] \\
&= \left[b_r, \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{\left|m-\frac{n}{2}\right|\leq N} : a_m a_{n-m} : \right] \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{\left|m-\frac{n}{2}\right|\leq N} \underbrace{[b_r, : a_m a_{n-m} :]}_{\substack{=[b_r, a_m a_{n-m}] \\ \text{(by Remark 4.3.13 }\textbf{(b)}\text{, applied to} \\ a,\ b_r \text{ and } n-m \text{ instead of } x,\ y \text{ and } n)}} \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{\left|m-\frac{n}{2}\right|\leq N} \underbrace{[b_r, a_m a_{n-m}]}_{=[b_r, a_m] a_{n-m} + a_m [b_r, a_{n-m}]} \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{\left|m-\frac{n}{2}\right|\leq N} \left(\underbrace{[b_r, a_m]}_{\substack{=[b,a]_{r+m} + K\omega(b_r, a_m) \\ \text{(by (291))}}} a_{n-m} + a_m \underbrace{[b_r, a_{n-m}]}_{\substack{=[b,a]_{n+r-m} + K\omega(b_r, a_{n-m}) \\ \text{(by (291))}}} \right) \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{\left|m-\frac{n}{2}\right|\leq N} \left([b,a]_{r+m} a_{n-m} + K\omega(b_r, a_m) a_{n-m} + a_m [b,a]_{n+r-m} + a_m K\omega(b_r, a_{n-m}) \right) \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \sum_{\left|m-\frac{n}{2}\right|\leq N} \left([b,a]_{r+m} a_{n-m} + a_m [b,a]_{n+r-m} + K\omega(b_r, a_m) a_{n-m} + a_m K\omega(b_r, a_{n-m}) \right).
\end{aligned} \tag{296}$$

Now fix $a \in B$. We now notice that for any $N \in \mathbb{N}$, the sum $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} K\omega(b_r, a_m) a_{n-m}$ (in $\operatorname{End} M$) has at most one nonzero addend (because $\omega(b_r, a_m)$ can be nonzero for at most one integer m (namely, for $m = -r$)). Hence, this sum $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} K\omega(b_r, a_m) a_{n-m}$ converges for any $N \in \mathbb{N}$. For sufficiently high N, this sum does have an addend for $m = -r$, and all other addends of this sum are 0 (since $\omega(b_r, a_m) = 0$ whenever $m \neq -r$), so that the value of this sum is $\underbrace{K}_{\substack{=k \\ \text{(since } K \text{ acts as} \\ k\cdot\mathrm{id} \text{ on } M)}} \underbrace{\omega(b_r, a_{-r})}_{\substack{=r(b,a) \\ \text{(by the definition of } \omega)}} \underbrace{a_{n-(-r)}}_{=a_{n+r}} = kr(b,a) a_{n+r}$. We thus have shown that the sum $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} K\omega(b_r, a_m) a_{n-m}$ converges for all $N \in \mathbb{N}$, and satisfies

$$\sum_{\left|m-\frac{n}{2}\right|\leq N} K\omega\left(b_r, a_m\right) a_{n-m} = kr\left(b, a\right) a_{n+r} \qquad \text{for sufficiently high } N. \tag{297}$$

Similarly, we see that the sum $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} a_m K\omega\left(b_r, a_{n-m}\right)$ converges for all $N \in \mathbb{N}$, and satisfies

$$\sum_{\left|m-\frac{n}{2}\right|\leq N} a_m K\omega\left(b_r, a_{n-m}\right) = a_{n+r} kr\left(b, a\right) \qquad \text{for sufficiently high } N. \tag{298}$$

Finally, for all $N \in \mathbb{N}$, the sum $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} \left(\left[b, a\right]_{r+m} a_{n-m} + a_m \left[b, a\right]_{n+r-m}\right)$ converges[203].

Since the sums $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} K\omega\left(b_r, a_m\right) a_{n-m}$, $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} a_m K\omega\left(b_r, a_{n-m}\right)$ and

[203] *Proof.* Let $N \in \mathbb{N}$. The sum $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} \left(\left[b, a\right]_{r+m} a_{n-m} + a_m \left[b, a\right]_{n+r-m} + K\omega\left(b_r, a_m\right) a_{n-m} + a_m K\omega\left(b_r, a_{n-m}\right)\right)$ converges (because it appears on the right hand side of (296)), and the sums $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} K\omega\left(b_r, a_m\right) a_{n-m}$ and $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} a_m K\omega\left(b_r, a_{n-m}\right)$ converge (as we have just seen). Hence, the sum

$$\sum_{\left|m-\frac{n}{2}\right|\leq N} \left(\left(\left[b, a\right]_{r+m} a_{n-m} + a_m \left[b, a\right]_{n+r-m} + K\omega\left(b_r, a_m\right) a_{n-m} + a_m K\omega\left(b_r, a_{n-m}\right)\right)\right.$$
$$\left. - K\omega\left(b_r, a_m\right) a_{n-m} - a_m K\omega\left(b_r, a_{n-m}\right)\right)$$

converges as well (since it is obtained by subtracting the latter two sums from the former sum componentwise). But this sum clearly simplifies to $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} \left(\left[b, a\right]_{r+m} a_{n-m} + a_m \left[b, a\right]_{n+r-m}\right)$.

Hence, the sum $\sum\limits_{\left|m-\frac{n}{2}\right|\leq N} \left(\left[b, a\right]_{r+m} a_{n-m} + a_m \left[b, a\right]_{n+r-m}\right)$ converges, qed.

$\sum\limits_{\left|m-\frac{n}{2}\right|\leq N}\left([b,a]_{r+m}a_{n-m}+a_m[b,a]_{n+r-m}\right)$ converge for every $N\in\mathbb{N}$, we have

$$\begin{aligned}
&\sum_{\left|m-\frac{n}{2}\right|\leq N}\left([b,a]_{r+m}a_{n-m}+a_m[b,a]_{n+r-m}+K\omega\left(b_r,a_m\right)a_{n-m}+a_mK\omega\left(b_r,a_{n-m}\right)\right)\\
&=\sum_{\left|m-\frac{n}{2}\right|\leq N}\left([b,a]_{r+m}a_{n-m}+a_m[b,a]_{n+r-m}\right)+\sum_{\left|m-\frac{n}{2}\right|\leq N}K\omega\left(b_r,a_m\right)a_{n-m}\\
&\qquad+\sum_{\left|m-\frac{n}{2}\right|\leq N}a_mK\omega\left(b_r,a_{n-m}\right)
\end{aligned}$$

for every $N\in\mathbb{N}$. Hence, for every sufficiently high $N\in\mathbb{N}$, we have

$$\begin{aligned}
&\sum_{\left|m-\frac{n}{2}\right|\leq N}\left([b,a]_{r+m}a_{n-m}+a_m[b,a]_{n+r-m}+K\omega\left(b_r,a_m\right)a_{n-m}+a_mK\omega\left(b_r,a_{n-m}\right)\right)\\
&=\sum_{\left|m-\frac{n}{2}\right|\leq N}\left([b,a]_{r+m}a_{n-m}+a_m[b,a]_{n+r-m}\right)+\underbrace{\sum_{\left|m-\frac{n}{2}\right|\leq N}K\omega\left(b_r,a_m\right)a_{n-m}}_{\substack{=kr(b,a)a_{n+r}\text{ for sufficiently high }N\\ \text{(by (297))}}}\\
&\qquad+\underbrace{\sum_{\left|m-\frac{n}{2}\right|\leq N}a_mK\omega\left(b_r,a_{n-m}\right)}_{\substack{=a_{n+r}kr(b,a)\text{ for sufficiently high }N\\ \text{(by (297))}}}\\
&=\sum_{\left|m-\frac{n}{2}\right|\leq N}\left([b,a]_{r+m}a_{n-m}+a_m[b,a]_{n+r-m}\right)+\underbrace{kr\left(b,a\right)a_{n+r}+a_{n+r}kr\left(b,a\right)}_{=2rk\cdot(b,a)a_{n+r}}\\
&=\sum_{\left|m-\frac{n}{2}\right|\leq N}\left([b,a]_{r+m}a_{n-m}+a_m[b,a]_{n+r-m}\right)+2rk\cdot\left(b,a\right)a_{n+r}. \qquad (299)
\end{aligned}$$

Now, forget that we fixed a. The equality (296) becomes

$$\begin{aligned}
&[b_r, L_n] \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \underbrace{\sum_{\left|m-\frac{n}{2}\right|\leq N} \left([b,a]_{r+m} a_{n-m} + a_m [b,a]_{n+r-m} + K\omega(b_r, a_m) a_{n-m} + a_m K\omega(b_r, a_{n-m})\right)}_{\substack{= \sum\limits_{\left|m-\frac{n}{2}\right|\leq N} \left([b,a]_{r+m} a_{n-m} + a_m [b,a]_{n+r-m}\right) + 2rk\cdot(b,a) a_{n+r} \\ \text{for sufficiently high } N \text{ (by (299))}}} \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{a\in B} \left(\sum_{\left|m-\frac{n}{2}\right|\leq N} \left([b,a]_{r+m} a_{n-m} + a_m [b,a]_{n+r-m}\right) + 2rk\cdot(b,a) a_{n+r}\right) \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{\left|m-\frac{n}{2}\right|\leq N} \left(\sum_{a\in B} [b,a]_{r+m} a_{n-m} + \sum_{a\in B} a_m [b,a]_{n+r-m}\right) + rk \sum_{a\in B} (b,a) a_{n+r}.
\end{aligned} \tag{300}$$

But since $\sum\limits_{a\in B} ([b,a] \otimes a + a \otimes [b,a]) = 0$ (by Lemma 4.3.9 **(a)**), we have $\sum\limits_{a\in B} \left([b,a]_\ell \otimes a_s + a_\ell \otimes [b,a]_s\right) = 0$ for any two integers ℓ and s. In particular, every $m \in \mathbb{Z}$ satisfies $\sum\limits_{a\in B} \left([b,a]_m \otimes a_{n+r-m} + a_m \otimes [b,a]_{n+r-m}\right) = 0$. Hence, every $m \in \mathbb{Z}$ satisfies $\sum\limits_{a\in B} \left([b,a]_m a_{n+r-m} + a_m [b,a]_{n+r-m}\right) = 0$, so that $\sum\limits_{a\in B} [b,a]_m a_{n+r-m} + \sum\limits_{a\in B} a_m [b,a]_{n+r-m} = 0$ and thus $\sum\limits_{a\in B} a_m [b,a]_{n+r-m} = -\sum\limits_{a\in B} [b,a]_m a_{n+r-m}$. Hence, (300) becomes

$$\begin{aligned}
&[b_r, L_n] \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{\left|m-\frac{n}{2}\right|\leq N} \left(\sum_{a\in B} [b,a]_{r+m} a_{n-m} + \underbrace{\sum_{a\in B} a_m [b,a]_{n+r-m}}_{= -\sum\limits_{a\in B} [b,a]_m a_{n+r-m}}\right) + rk \sum_{a\in B} (b,a) a_{n+r} \\
&= \frac{1}{2} \lim_{N\to\infty} \sum_{\left|m-\frac{n}{2}\right|\leq N} \left(\sum_{a\in B} [b,a]_{r+m} a_{n-m} - \sum_{a\in B} [b,a]_m a_{n+r-m}\right) + rk \sum_{a\in B} (b,a) a_{n+r}.
\end{aligned} \tag{301}$$

We will now transform the limit in this equation: In fact,

$$\lim_{N\to\infty} \underbrace{\sum_{\left|m-\frac{n}{2}\right|\leq N} \left(\sum_{a\in B} [b,a]_{r+m} a_{n-m} - \sum_{a\in B} [b,a]_m a_{n+r-m} \right)}_{=\sum_{a\in B}\left(\sum_{\left|m-\frac{n}{2}\right|\leq N} [b,a]_{r+m} a_{n-m} - \sum_{\left|m-\frac{n}{2}\right|\leq N} [b,a]_m a_{n+r-m}\right)}$$

$$= \lim_{N\to\infty} \sum_{a\in B} \left(\underbrace{\sum_{\left|m-\frac{n}{2}\right|\leq N} [b,a]_{r+m} a_{n-m}}_{\substack{=\sum_{\left|m-r-\frac{n}{2}\right|\leq N} [b,a]_m a_{n+r-m} \\ \text{(here, we substituted } m-r \text{ for } m \text{ in the sum)}}} - \sum_{\left|m-\frac{n}{2}\right|\leq N} [b,a]_m a_{n+r-m} \right)$$

$$= \lim_{N\to\infty} \sum_{a\in B} \underbrace{\left(\sum_{\left|m-r-\frac{n}{2}\right|\leq N} [b,a]_m a_{n+r-m} - \sum_{\left|m-\frac{n}{2}\right|\leq N} [b,a]_m a_{n+r-m} \right)}_{\substack{=-\sum_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N} [b,a]_m a_{n+r-m} + \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} [b,a]_m a_{n+r-m} \\ \text{(by Lemma 4.3.14, applied to } u_m=[b,a]_m a_{n+r-m}, \\ \alpha=\frac{n}{2} \text{ and } \beta=\frac{n}{2}+r)}}$$

$$= \lim_{N\to\infty} \sum_{a\in B} \left(- \sum_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N} [b,a]_m a_{n+r-m} + \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} [b,a]_m a_{n+r-m} \right).$$

Since every $a \in B$ satisfies $\sum_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N} [b,a]_m a_{n+r-m} \to 0$ for $N \to \infty$ [204], this

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[204] *Proof.* Let $a \in B$.

Let $w \in M$. From the proof of Theorem 4.3.4 **(a)**, recall the fact that for every $w \in M$, there exists some $\mathbf{M} \in \mathbb{N}$ such that every integer $\mathbf{m} \geq \mathbf{M}$ and every $a \in \mathfrak{g}$ satisfy $at^{\mathbf{m}} \cdot w = 0$. Applied to $w = a$, this yields that there exists some $\mathbf{M} \in \mathbb{N}$ such that

$$\text{every integer } \mathbf{m} \geq \mathbf{M} \text{ satisfies } at^{\mathbf{m}} \cdot w = 0. \tag{302}$$

becomes

$$\lim_{N\to\infty}\sum_{\left|m-\frac{n}{2}\right|\leq N}\left(\sum_{a\in B}[b,a]_{r+m}a_{n-m}-\sum_{a\in B}[b,a]_{m}a_{n+r-m}\right)$$

$$=\lim_{N\to\infty}\sum_{a\in B}\left(\underbrace{-\sum_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N}[b,a]_{m}a_{n+r-m}}_{\to 0\text{ for }N\to\infty}+\sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N}[b,a]_{m}a_{n+r-m}\right)$$

$$=\lim_{N\to\infty}\sum_{a\in B}\sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N}\underbrace{[b,a]_{m}a_{n+r-m}}_{=a_{n+r-m}[b,a]_m+\left[[b,a]_m,a_{n+r-m}\right]}$$

$$=\lim_{N\to\infty}\sum_{a\in B}\sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N}\left(a_{n+r-m}[b,a]_{m}+\left[[b,a]_{m},a_{n+r-m}\right]\right)$$

$$=\lim_{N\to\infty}\sum_{a\in B}\left(\sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N}a_{n+r-m}[b,a]_{m}+\sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N}\left[[b,a]_{m},a_{n+r-m}\right]\right).$$

Since every $a\in B$ satisfies $\sum\limits_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N}a_{n+r-m}[b,a]_{m}\to 0$ for $N\to\infty$ [205], this

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Consider this $\mathbf{M}$.

Let N be an integer such that $N\geq\mathbf{M}-\frac{n}{2}-r$. Then, $\frac{n}{2}+r+N\geq\mathbf{M}$. Now, every integer m such that $\frac{n}{2}-N\leq m<\frac{n}{2}+r-N$ must satisfy $n+r-\underbrace{m}_{\geq\frac{n}{2}-N}\leq n+r-\left(\frac{n}{2}-N\right)=\frac{n}{2}+r+N\geq\mathbf{M}$ and thus $at^{n+r-m}\cdot w=0$ (by (302), applied to $\mathbf{m}=n+r-m$), thus $[b,a]_m\underbrace{a_{n+r-m}}_{=at^{n+r-m}}w=[b,a]_m\cdot\underbrace{at^{n+r-m}\cdot w}_{=0}=0$. Hence, $\sum\limits_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N}\underbrace{[b,a]_m a_{n+r-m}w}_{=0}=\sum\limits_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N}0=0$.

Now forget that we fixed N. We thus have showed that $\sum\limits_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N}[b,a]_m a_{n+r-m}w=0$ for every integer N such that $N\geq\mathbf{M}-\frac{n}{2}-r$. Hence, $\sum\limits_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N}[b,a]_m a_{n+r-m}w=0$ for every sufficiently large N. Thus, $\sum\limits_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N}[b,a]_m a_{n+r-m}w\to 0$ for $N\to\infty$. Since this holds for every $w\in M$, we thus obtain $\sum\limits_{\frac{n}{2}-N\leq m<\frac{n}{2}+r-N}[b,a]_m a_{n+r-m}\to 0$ for $N\to\infty$, qed.

[205] *Proof.* Let $w\in M$. From the proof of Theorem 4.3.4 **(a)**, recall the fact that for every $w\in M$,

becomes

$$\lim_{N\to\infty} \sum_{\left|m-\frac{n}{2}\right|\leq N} \left(\sum_{a\in B} [b,a]_{r+m} a_{n-m} - \sum_{a\in B} [b,a]_m a_{n+r-m} \right)$$

$$= \lim_{N\to\infty} \sum_{a\in B} \left(\underbrace{\sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} a_{n+r-m} [b,a]_m}_{\to 0 \text{ for } N\to\infty} + \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \left[[b,a]_m , a_{n+r-m}\right] \right)$$

$$= \lim_{N\to\infty} \sum_{a\in B} \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \underbrace{\left[[b,a]_m , a_{n+r-m}\right]}_{\substack{=[[b,a],a]_{n+r}+K\omega\left([b,a]_m,a_{n+r-m}\right) \\ \text{(by (291), applied to } [b,a],\ a,\ m \text{ and } n+r-m \\ \text{instead of } x,\ y,\ n,\ m)}}$$

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there exists some $\mathbf{M} \in \mathbb{N}$ such that every integer $\mathbf{m} \geq \mathbf{M}$ and every $a \in \mathfrak{g}$ satisfy $at^{\mathbf{m}} \cdot w = 0$. Consider this $\mathbf{M}$. Thus,

$$\text{every integer } \mathbf{m} \geq \mathbf{M} \text{ and every } a \in \mathfrak{g} \text{ satisfy } at^{\mathbf{m}} \cdot w = 0. \tag{303}$$

Let $a \in B$.

Let N be an integer such that $N \geq \mathbf{M} - \frac{n}{2}$. Then, $\frac{n}{2} + N \geq \mathbf{M}$. Now, every integer m such that $\frac{n}{2} + N < m \leq \frac{n}{2} + r + N$ must satisfy $m > \frac{n}{2} + N \geq \mathbf{M}$ and thus $[b,a]\, t^m \cdot w = 0$ (by (303), applied to m and $[b,a]$ instead of $\mathbf{m}$ and a), thus $a_{n+r-m} \underbrace{[b,a]_m}_{=[b,a]t^m} w = a_{n+r-m} \underbrace{[b,a]\, t^m \cdot w}_{=0} = 0$. Hence, $\sum\limits_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \underbrace{a_{n+r-m} [b,a]_m w}_{=0} = \sum\limits_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} 0 = 0$.

Now forget that we fixed N. We thus have showed that $\sum\limits_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} a_{n+r-m} [b,a]_m w = 0$ for every integer N such that $N \geq \mathbf{M} - \frac{n}{2}$. Hence, $\sum\limits_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} a_{n+r-m} [b,a]_m w = 0$ for every sufficiently large N. Thus, $\sum\limits_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} a_{n+r-m} [b,a]_m w \to 0$ for $N \to \infty$. Since this holds for every $w \in M$, we thus obtain $\sum\limits_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} a_{n+r-m} [b,a]_m \to 0$ for $N \to \infty$, qed.

$$= \lim_{N\to\infty} \sum_{a\in B} \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \left([[b,a],a]_{n+r} + \underbrace{K}_{\substack{=k \\ (\text{since } K \text{ acts on } M \text{ as } k\cdot\mathrm{id})}} \omega\left([b,a]_m, a_{n+r-m}\right) \right)$$

$$= \lim_{N\to\infty} \underbrace{\sum_{a\in B} \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \left([[b,a],a]_{n+r} + k\omega\left([b,a]_m, a_{n+r-m}\right)\right)}_{=\sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \sum_{a\in B} [[b,a],a]_{n+r} + k \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \sum_{a\in B} \omega\left([b,a]_m, a_{n+r-m}\right)}$$

$$= \lim_{N\to\infty} \left(\underbrace{\sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \sum_{a\in B} [[b,a],a]_{n+r}}_{=r\sum_{a\in B}[[b,a],a]_{n+r}} + k \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \sum_{a\in B} \omega\left([b,a]_m, a_{n+r-m}\right) \right)$$

$$= \lim_{N\to\infty} \left(r\sum_{a\in B} [[b,a],a]_{n+r} + k \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \sum_{a\in B} \omega\left([b,a]_m, a_{n+r-m}\right) \right).$$

Since every integer m and every $a \in B$ satisfy $\omega\left([b,a]_m, a_{n+r-m}\right) = 0$ [206], this simplifies to

$$\lim_{N\to\infty} \sum_{\left|m-\frac{n}{2}\right|\leq N} \left(\sum_{a\in B} [b,a]_{r+m} a_{n-m} - \sum_{a\in B} [b,a]_m a_{n+r-m} \right)$$

$$= \lim_{N\to\infty} \left(r\sum_{a\in B} [[b,a],a]_{n+r} + k \sum_{\frac{n}{2}+N<m\leq\frac{n}{2}+r+N} \sum_{a\in B} \underbrace{\omega\left([b,a]_m, a_{n+r-m}\right)}_{=0} \right)$$

$$= \lim_{N\to\infty} r\sum_{a\in B} [[b,a],a]_{n+r} = r\sum_{a\in B} [[b,a],a]_{n+r}.$$

[206] *Proof.* Let m be an integer, and let $a \in B$. From Lemma 4.3.9 **(c)**, we have $([b,a],a) = 0$, so that $m([b,a],a) = 0$. But by the definition of ω, we have

$$\omega\left([b,a]_m, a_{n+r-m}\right) = \begin{cases} m([b,a],a), & \text{if } m = -(n+r-m); \\ 0, & \text{if } m \neq -(n+r-m) \end{cases} = \underbrace{\begin{cases} 0, & \text{if } m = -(n+r-m); \\ 0, & \text{if } m \neq -(n+r-m) \end{cases}}_{}$$

(since $m([b,a],a) = 0$)

$$= 0,$$

qed.

Thus, (301) becomes

$$\begin{aligned}
&[b_r, L_n] \\
&= \underbrace{\frac{1}{2} \lim_{N\to\infty} \sum_{\left|m-\frac{n}{2}\right|\leq N} \left(\sum_{a\in B} [b,a]_{r+m} a_{n-m} - \sum_{a\in B} [b,a]_m a_{n+r-m} \right)}_{=r\sum_{a\in B}[[b,a],a]_{n+r}} + rk \sum_{a\in B} (b,a) a_{n+r} \\
&= \frac{1}{2} r \sum_{a\in B} \underbrace{[[b,a],a]_{n+r}}_{=[[b,a],a]t^{n+r}} + rk \sum_{a\in B} (b,a) \underbrace{a_{n+r}}_{=at^{n+r}} \\
&= rt^{n+r} \underbrace{\left(\frac{1}{2} \sum_{a\in B} [[b,a],a] + k \sum_{a\in B} (b,a) a \right)}_{\substack{=b \\ \text{(by Lemma 4.3.9 (b))}}} = r \underbrace{t^{n+r} b}_{=b_{n+r}} = r b_{n+r}.
\end{aligned}$$

This proves (295) in the case when $r \geq 0$. The case when $r \leq 0$ is handled analogously (except that this time we have to apply Lemma 4.3.14 to $u_m = [b,a]_m a_{n+r-m}$, $\alpha = \frac{n}{2} + r$ and $\beta = \frac{n}{2}$ instead of applying it to $u_m = [b,a]_m a_{n+r-m}$, $\alpha = \frac{n}{2}$ and $\beta = \frac{n}{2} + r$). Altogether, the proof of (295) is thus complete.

2nd step: It is clear that

$$[L_n, a_m] = -m a_{n+m} \qquad \text{for any } a \in \mathfrak{g} \text{ and any integers } n \text{ and } m \tag{304}$$

(since (295) (applied to $r = m$ and $a = b$) yields $[a_m, L_n] = m a_{n+m}$, so that $[L_n, a_m] = -\underbrace{[a_m, L_n]}_{=ma_{n+m}} = -m a_{n+m}$). Also, it is clear that

$$[L_n, K] = 0 \qquad \text{for any integer } n \tag{305}$$

(since K acts as a scalar on M).

3rd step: Now, we will prove that

$$[L_n, L_m] = (n-m) L_{n+m} + \frac{n^3 - n}{12} \delta_{n,-m} k \cdot \sum_{a\in B} (a,a) \qquad \text{for any integers } n \text{ and } m \tag{306}$$

(as an identity in $\operatorname{End} M$).

Proof of (306): We know that every $n \in \mathbb{Z}$ satisfies

$$L_n = \frac{1}{2} \sum_{a\in B} \sum_{m\in\mathbb{Z}} : a_m a_{n-m} : = \frac{1}{2} \sum_{a\in B} \sum_{m\in\mathbb{Z}} : a_{-m} a_{n+m} : \tag{307}$$

(here, we substituted $-m$ for m in the second sum).

Repeat the Second Proof of Proposition 3.2.13, with the following changes:

- Reprove Lemma 3.2.10 with F_μ replaced by M and with an additional "Let $a \in \mathfrak{g}$ be arbitrary." condition. (The proof will be slightly different from the proof of the original Lemma 3.2.10 because M is no longer a polynomial ring, but this time we can use the admissibility of M instead.)

- Replace every F_μ by M.
- Instead of the equality (95), use the equality (307) (which differs from the equality (95) only in the presence of a $\sum\limits_{a\in B}$ sign). As a consequence, $\sum\limits_{a\in B}$ signs need to be dragged along through the computations (but they don't complicate the calculation).
- Instead of using Remark 3.2.5, use Remark 4.3.10.
- Instead of using Remark 3.2.6 **(a)**, use Remark 4.3.11.
- Instead of using Remark 3.2.7, use Remark 4.3.13.
- Instead of using Proposition 3.2.12, use (304).
- Instead of the equality $a_{m-\ell}a_{n+\ell} = : a_{m-\ell}a_{n+\ell} : - (n+\ell)[\ell < m]\,\delta_{m,-n}\,\mathrm{id}$, check the equality $a_{m-\ell}a_{n+\ell} = : a_{m-\ell}a_{n+\ell} : - (n+\ell)[\ell < m]\,\delta_{m,-n}\,(a,a)\,k$ for every $a \in B$.
- Instead of the equality $a_{-\ell}a_{m+n+\ell} = : a_{-\ell}a_{m+n+\ell} : - \ell\,[\ell < 0]\,\delta_{m,-n}\,\mathrm{id}$, check the equality $a_{-\ell}a_{m+n+\ell} = : a_{-\ell}a_{m+n+\ell} : - \ell\,[\ell < 0]\,\delta_{m,-n}\,(a,a)\,k$ for every $a \in B$.

Once these changes (most of which are automatic) are made, we have obtained a proof of (306).

4th step: From (306), it is clear that the endomorphisms L_n for $n \in \mathbb{Z}$ give rise to a Vir-representation on M with central charge

$$c = k \cdot \sum_{a\in B} (a,a).$$

This proves Theorem 4.3.4 **(c)**.

(d) From (304) and (305), it follows that the formulas for L_n and c we have given in Theorem 4.3.4 extend the action of $\widehat{\mathfrak{g}}$ on M to an action of $\mathrm{Vir} \ltimes \widehat{\mathfrak{g}}$. Theorem 4.3.4 **(d)** thus is proven.

(e) Theorem 4.3.4 **(e)** follows immediately from (304).

Thus, the proof of Theorem 4.3.4 is complete.

We are now going to specialize these results to the case of $\mathfrak{g}$ being simple. In this case, the so-called *dual Coxeter number* of the simple Lie algebra $\mathfrak{g}$ comes into play. Let us explain what this is:

Definition 4.3.15. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Let θ be the maximal root of $\mathfrak{g}$. (In other words, let θ be the highest weight of the adjoint representation of $\mathfrak{g}$.) Let $\rho = \dfrac{1}{2} \sum\limits_{\substack{\alpha \text{ root of } \mathfrak{g};\\ \alpha>0}} \alpha$ be the half-sum of all positive roots. The *dual Coxeter number* $h^\vee$ of $\mathfrak{g}$ is defined by $h^\vee = 1 + (\theta, \rho)$. It is easy to show that $h^\vee$ is a positive integer.

Definition 4.3.16. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. The *standard form* on $\mathfrak{g}$ will mean the scalar multiple of the Killing form under which (α, α) (under the inverse form on $\mathfrak{g}^*$) equals 2 for long roots α. (We do not care to define what a long root is, but it is enough to say that the maximal root θ is a long root, and this is clearly enough to define the standard form.)

(The *inverse form* of a nondegenerate bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$ means the bilinear form on $\mathfrak{g}^* = \mathfrak{h}^* \oplus \mathfrak{n}_+^* \oplus \mathfrak{n}_-^*$ obtained by dualizing the bilinear form $(\cdot, \cdot)$ on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ using itself.)

We are going to denote the standard form by $(\cdot, \cdot)$.

Lemma 4.3.17. Let B be an orthonormal basis of $\mathfrak{g}$ with respect to the standard form. Let $C = \sum\limits_{a \in B} a^2 \in U(\mathfrak{g})$. This element C is known to be central in $U(\mathfrak{g})$ (this is easily checked), and is called the *quadratic Casimir*.

Then:

(1) For every $\lambda \in \mathfrak{h}^*$, the element $C \in U(\mathfrak{g})$ acts on L_λ by $(\lambda, \lambda + 2\rho) \cdot \mathrm{id}$. (Here, L_λ means L_λ^+, but actually can be replaced by any highest-weight module with highest weight λ.)

(2) The element $C \in U(\mathfrak{g})$ acts on the adjoint representation $\mathfrak{g}$ by $2h^\vee \cdot \mathrm{id}$.

Proof of Lemma 4.3.17. If $(b_i)_{i \in I}$ is any basis of $\mathfrak{g}$, and $(b_i^*)_{i \in I}$ is the dual basis of $\mathfrak{g}$ with respect to the standard form $(\cdot, \cdot)$, then

$$C = \sum_{i \in I} b_i b_i^*. \tag{308}$$

[207]

(1) Let $\lambda \in \mathfrak{h}^*$.

Let us refine the triangular decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ to the weight space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus\limits_{\alpha > 0} \mathfrak{g}_\alpha \right) \oplus \left(\bigoplus\limits_{\alpha < 0} \mathfrak{g}_\alpha \right)$, where $\mathfrak{g}_\alpha = \mathbb{C} e_\alpha$ for roots $\alpha > 0$, and $\mathfrak{g}_{-\alpha} = \mathbb{C} f_\alpha$ for roots $\alpha > 0$. (This is standard theory of simple Lie algebras.) Normalize the f_α in such a way that $(e_\alpha, f_\alpha) = 1$. As usual, denote $h_\alpha = [e_\alpha, f_\alpha]$ for every root $\alpha > 0$.

Fix an orthonormal basis $(x_i)_{i \in \{1,2,\ldots,r\}}$ of $\mathfrak{h}$. Clearly, $(x_i)_{i \in \{1,2,\ldots,r\}} \cup (e_\alpha)_{\alpha > 0} \cup (f_\alpha)_{\alpha > 0}$ (where the index α runs over positive roots only) is a basis of $\mathfrak{g}$. Since

$$\begin{aligned}
&(e_\alpha, x_i) = (f_\alpha, x_i) = 0 && \text{for all } i \in \{1, 2, ..., r\} \text{ and roots } \alpha > 0; \\
&(e_\alpha, f_\beta) = 0 && \text{for any two distinct roots } \alpha > 0 \text{ and } \beta > 0; \\
&(e_\alpha, e_\gamma) = (f_\alpha, f_\gamma) = 0 && \text{for any roots } \alpha > 0 \text{ and } \gamma > 0; \\
&(x_i, x_j) = \delta_{i,j} && \text{for all } i \in \{1, 2, ..., r\} \text{ and } j \in \{1, 2, ..., r\}; \\
&(e_\alpha, f_\alpha) = (f_\alpha, e_\alpha) = 1 && \text{for any root } \alpha > 0,
\end{aligned}$$

we see that $(x_i)_{i \in \{1,2,\ldots,r\}} \cup (f_\alpha)_{\alpha > 0} \cup (e_\alpha)_{\alpha > 0}$ is the dual basis to this basis $(x_i)_{i \in \{1,2,\ldots,r\}} \cup (e_\alpha)_{\alpha > 0} \cup (f_\alpha)_{\alpha > 0}$ with respect to the standard form $(\cdot, \cdot)$. Thus, (308) yields

$$C = \sum_{i=1}^r x_i^2 + \sum_{\alpha > 0} (f_\alpha e_\alpha + e_\alpha f_\alpha),$$

[207] This is a well-known property of the quadratic Casimir.

so that (denoting v_λ^+ by v_λ) we have

$$\begin{aligned}
Cv_\lambda &= \sum_{i=1}^{r} \underbrace{x_i^2 v_\lambda}_{=\lambda(x_i)^2 v_\lambda} + \sum_{\alpha>0} \left(f_\alpha e_\alpha + e_\alpha f_\alpha\right) v_\lambda = \underbrace{\sum_{i=1}^{r} \lambda\left(x_i\right)^2}_{=(\lambda,\lambda)} v_\lambda + \sum_{\alpha>0} \left(f_\alpha \underbrace{e_\alpha v_\lambda}_{=0} + \underbrace{e_\alpha f_\alpha}_{=f_\alpha e_\alpha + [e_\alpha, f_\alpha]} v_\lambda \right) \\
&= (\lambda, \lambda) v_\lambda + \sum_{\alpha>0} \left(f_\alpha e_\alpha + [e_\alpha, f_\alpha]\right) v_\lambda \\
&= (\lambda, \lambda) v_\lambda + \sum_{\alpha>0} f_\alpha \underbrace{e_\alpha v_\lambda}_{=0} + \sum_{\alpha>0} \underbrace{[e_\alpha, f_\alpha]}_{=h_\alpha} v_\lambda \\
&= (\lambda, \lambda) v_\lambda + \sum_{\alpha>0} \underbrace{h_\alpha v_\lambda}_{=\lambda(h_\alpha) v_\lambda} = (\lambda, \lambda) v_\lambda + \sum_{\alpha>0} \underbrace{\lambda\left(h_\alpha\right)}_{=(\lambda,\alpha)} v_\lambda \\
&= (\lambda, \lambda) v_\lambda + \sum_{\alpha>0} (\lambda, \alpha) v_\lambda = \underbrace{\left((\lambda, \lambda) + \sum_{\alpha>0} (\lambda, \alpha) \right)}_{\substack{= \left(\lambda, \lambda + \sum_{\alpha>0} \alpha\right) = (\lambda, \lambda+2\rho) \\ \text{(since } \sum_{\alpha>0} \alpha = 2\rho)}} v_\lambda = (\lambda, \lambda + 2\rho) v_\lambda.
\end{aligned}$$

Thus, every $a \in U(\mathfrak{g})$ satisfies

$$\begin{aligned}
Cav_\lambda &= a \underbrace{Cv_\lambda}_{=(\lambda,\lambda+2\rho)v_\lambda} \qquad \text{(since } C \text{ is central in } U(\mathfrak{g})) \\
&= (\lambda, \lambda + 2\rho) a v_\lambda.
\end{aligned}$$

Hence, C acts as $(\lambda, \lambda + 2\rho) \cdot \mathrm{id}$ on L_λ (because every element of L_λ has the form av_λ for some $a \in U(\mathfrak{g})$). This proves Lemma 4.3.17 **(1)**.

(2) We have $\mathfrak{g} = L_\theta$, and thus Lemma 4.3.17 **(1)** yields

$$C \mid_{L_\theta} = (\theta, \theta + 2\rho) = \underbrace{(\theta, \theta)}_{=2} + 2(\theta, \rho) = 2 + 2(\theta, \rho) = 2h^\vee.$$

This proves Lemma 4.3.17 **(2)**.

Here is a little table of dual Coxeter numbers, depending on the root system type of $\mathfrak{g}$:

For A_{n-1}, we have $h^\vee = n$.
For B_n, we have $h^\vee = 2n - 1$.
For C_n, we have $h^\vee = n + 1$.
For D_n, we have $h^\vee = 2n - 2$.
For E_6, we have $h^\vee = 12$.
For E_7, we have $h^\vee = 18$.
For E_8, we have $h^\vee = 30$.
For F_4, we have $h^\vee = 9$.
For G_2, we have $h^\vee = 4$.

Every Lie theorist is supposed to remember these by heart.

Lemma 4.3.18. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Then,

$$\operatorname{Kil}(a,b) = 2h^\vee \cdot (a,b) \qquad \text{for any } a,b \in \mathfrak{g}.$$

Proof of Lemma 4.3.18. Let B be an orthonormal basis of $\mathfrak{g}$ with respect to the standard form. Define the quadratic Casimir $C = \sum\limits_{a\in B} a^2$ as in Lemma 4.3.17. Then,

$$\operatorname{Tr}_{\mathfrak{g}}(C) = \sum_{a\in B} \underbrace{\operatorname{Tr}_{\mathfrak{g}}\left(a^2\right)}_{=\operatorname{Tr}((\operatorname{ad} a)\circ(\operatorname{ad} a))=\operatorname{Kil}(a,a)} = \sum_{a\in B} \operatorname{Kil}(a,a).$$

Comparing this with

$$\begin{aligned} \operatorname{Tr}_{\mathfrak{g}}(C) &= 2h^\vee \underbrace{\operatorname{Tr}_{\mathfrak{g}}(\operatorname{id})}_{=\dim\mathfrak{g}} \qquad (\text{since } C\mid_{\mathfrak{g}} = 2h^\vee \operatorname{id} \text{ by Lemma 4.3.17 } \mathbf{(2)}) \\ &= 2h^\vee \underbrace{\dim\mathfrak{g}}_{\substack{=|B|=\sum_{a\in B}1=\sum_{a\in B}(a,a) \\ (\text{since every } a\in B \text{ satisfies } (a,a)=1)}} = 2h^\vee \sum_{a\in B}(a,a), \end{aligned}$$

we obtain $\sum\limits_{a\in B} \operatorname{Kil}(a,a) = 2h^\vee \sum\limits_{a\in B}(a,a)$. Since Kil is a scalar multiple of $(\cdot,\cdot)$ (because there is only one $\mathfrak{g}$-invariant symmetric bilinear form on $\mathfrak{g}$ up to scaling), this yields $\operatorname{Kil} = 2h^\vee \cdot (\cdot,\cdot)$ (because $\sum\limits_{a\in B} \underbrace{(a,a)}_{=1} = \sum\limits_{a\in B} 1 = |B| \neq 0$). Lemma 4.3.18 is proven.

So let us now look at the Sugawara construction when $\mathfrak{g}$ is simple finite-dimensional. First of all, k is non-critical if and only if $k \neq -h^\vee$. (The value $k = -h^\vee$ is called the *critical level.*)

If B' is an orthonormal basis under $(\cdot,\cdot)$ (rather than under $k(\cdot,\cdot)+\frac{1}{2}\operatorname{Kil} = (k+h^\vee)(\cdot,\cdot)$), then we have

$$\begin{aligned} L_n &= \frac{1}{2(k+h^\vee)} \sum_{a\in B'} \sum_{m\in\mathbb{Z}} : a_m a_{n-m} : \qquad \text{and} \\ c &= \frac{k}{k+h^\vee} \underbrace{\sum_{a\in B'} (a,a)}_{\substack{=|B'| \\ (\text{since } (a,a)=1 \text{ for every } a\in B')}} = \frac{k}{k+h^\vee} \underbrace{|B'|}_{=\dim\mathfrak{g}} = \frac{k\dim\mathfrak{g}}{k+h^\vee}. \end{aligned} \tag{309}$$

In particular, this induces an internal grading on any $\widehat{\mathfrak{g}}$-module which is a quotient of M_λ^+ by eigenvalues of L_0, whenever λ is a weight of $\widehat{\mathfrak{g}}$. This is a grading by complex numbers, since eigenvalues of L_0 are not necessarily integers. (Note that this does not work for general admissible modules in lieu of quotients of M_λ^+.)

What happens at the critical level $k = -h^\vee$? The above formulas with $k + h^\vee$ in the denominators clearly don't work at this level anymore. We can, however, remove the denominators, i. e., consider the operators

$$T_n = \frac{1}{2} \sum_{a\in B'} \sum_{m\in\mathbb{Z}} : a_m a_{n-m} : .$$

Then, the same calculations as we did in the proof of Theorem 4.3.4 tell us that these T_n satisfy $[T_n, a_m] = 0$ and $[T_n, T_m] = 0$; they are thus central "elements" of $U(\widehat{\mathfrak{g}})$ (except that they are not actually elements of $U(\widehat{\mathfrak{g}})$, but of some completion of $U(\widehat{\mathfrak{g}})$ acting on admissible modules).

For any complex numbers $\gamma_1, \gamma_2, \gamma_3, ...$, we can construct a $\widehat{\mathfrak{g}}$-module $M_\lambda \diagup \left(\sum\limits_{m \geq 1} ((T_m - \gamma_m) M_\lambda) \right)$, which does not have a grading. So, at the critical level, we do not automatically get gradings on quotients of M_λ anymore. This is one reason why representations at the critical level are considered more difficult than those at non-critical levels.

4.4. The Sugawara construction and unitarity

We now will show that the Sugawara construction preserves unitarity:

Proposition 4.4.1. Consider the situation of Theorem 4.3.4. If M is a unitary admissible module for $\widehat{\mathfrak{g}}$, then M is a unitary $\mathrm{Vir} \ltimes \widehat{\mathfrak{g}}$-module. (We recall that the Virasoro algebra had its unitary structure given by $L_n^\dagger = L_{-n}$.)

But for M to be unitary for $\widehat{\mathfrak{g}}$, we need $k \in \mathbb{Z}_+$ (this is easy to prove; we proved it for $\mathfrak{sl}_n$, and the general case is similar). Since for $k = 0$, there is only the trivial representation, we really must require $k \geq 1$ to get something interesting. And since $c = \dfrac{k \dim \mathfrak{g}}{k + h^\vee}$, the c is then ≥ 1, since $\dim \mathfrak{g} \geq 1 + h^\vee$. These modules are already known to us to be unitary, so this construction does not help us in constructing new unitary modules.

But there is a way to amend this by a variation of the Sugawara construction: the Goddard-Kent-Olive construction.

4.5. The Goddard-Kent-Olive construction (a.k.a. the coset construction)

Definition 4.5.1. Let $\mathfrak{g}$ and $\mathfrak{p}$ be two finite-dimensional Lie algebras such that $\mathfrak{g} \supseteq \mathfrak{p}$. Let $(\cdot, \cdot)$ be a $\mathfrak{g}$-invariant form (possibly degenerate) on $\mathfrak{g}$. We can restrict this form to $\mathfrak{p}$, and obtain a $\mathfrak{p}$-invariant form on $\mathfrak{p}$. Construct an affine Lie algebra $\widehat{\mathfrak{g}}$ as in Definition 4.3.1 using the $\mathfrak{g}$-invariant form $(\cdot, \cdot)$ on $\mathfrak{g}$, and similarly construct an affine Lie algebra $\widehat{\mathfrak{p}}$ using the restriction of this form to $\mathfrak{p}$. Then, $\widehat{\mathfrak{g}} \supseteq \widehat{\mathfrak{p}}$. Choose a level k which is non-critical for both $\mathfrak{g}$ and $\mathfrak{p}$.

Let M be an admissible $\widehat{\mathfrak{g}}$-module at level k. Then, M automatically becomes an admissible $\widehat{\mathfrak{p}}$-module at level k. Hence, on M, we have two Virasoro actions: one which is obtained from the $\widehat{\mathfrak{g}}$-action, and one which is obtained from the $\widehat{\mathfrak{p}}$-action. We will denote these actions by $(L_i^{\mathfrak{g}})_{i \in \mathbb{Z}}$ and $\left(L_i^{\mathfrak{p}}\right)_{i \in \mathbb{Z}}$, respectively (that is, for every $i \in \mathbb{Z}$, we denote by $L_i^{\mathfrak{g}}$ the action of $L_i \in \mathrm{Vir}$ obtained from the $\widehat{\mathfrak{g}}$-module structure on M, and we denote by $L_i^{\mathfrak{p}}$ the action of $L_i \in \mathrm{Vir}$ obtained from the $\widehat{\mathfrak{p}}$-module structure on M), and we will denote their central charges by $c_{\mathfrak{g}}$ and $c_{\mathfrak{p}}$, respectively.

Theorem 4.5.2. Consider the situation of Definition 4.5.1. Let $L_i = L_i^{\mathfrak{g}} - L_i^{\mathfrak{p}}$ for all $i \in \mathbb{Z}$.

(a) Then, $(L_i)_{i\in\mathbb{Z}}$ is a Vir-action on M with central charge $c = c_{\mathfrak{g}} - c_{\mathfrak{p}}$.

(b) Also, $[L_n, \widehat{p}] = 0$ for all $\widehat{p} \in \widehat{\mathfrak{p}}$ and $n \in \mathbb{Z}$.

(c) Moreover, $[L_n, L_m^{\mathfrak{p}}] = 0$ for all $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$.

Proof of Theorem 4.5.2. **(b)** Let $n \in \mathbb{Z}$. Every $p \in \mathfrak{p}$ and $m \in \mathbb{Z}$ satisfy

$$\left[\underbrace{L_n}_{=L_n^{\mathfrak{g}}-L_n^{\mathfrak{p}}}, p_m\right] = \underbrace{[L_n^{\mathfrak{g}}, p_m]}_{\substack{=-mp_{n+m} \\ \text{(by Theorem 4.3.4 (e),} \\ \text{applied to } p \text{ instead of } a)}} - \underbrace{[L_n^{\mathfrak{p}}, p_m]}_{\substack{=-mp_{n+m} \\ \text{(by Theorem 4.3.4 (e),} \\ \text{applied to } p \text{ and } \mathfrak{p} \text{ instead of } a \text{ and } \mathfrak{g})}} = (-mp_{m+n}) - (-mp_{m+n}) = 0.$$

Combined with the fact that $[L_n, K] = 0$ (this is trivial, since K acts as $k \cdot \mathrm{id}$ on M), this yields that $[L_n, \widehat{p}] = 0$ for all $\widehat{p} \in \widehat{\mathfrak{p}}$ and $n \in \mathbb{Z}$ (because every $\widehat{p} \in \widehat{\mathfrak{p}}$ is a $\mathbb{C}$-linear combination of terms of the form p_m (with $p \in \mathfrak{p}$ and $m \in \mathbb{Z}$) and K). Thus, Theorem 4.5.2 **(b)** is proven.

(c) Let $n \in \mathbb{Z}$. We recall that $L_n^{\mathfrak{p}}$ was defined by $L_n^{\mathfrak{p}} = \dfrac{1}{2} \sum\limits_{a\in B} \sum\limits_{m\in\mathbb{Z}} : a_m a_{n-m} :$, where B is an orthonormal basis of $\mathfrak{p}$ with respect to a certain bilinear form on $\mathfrak{p}$. Thus, $L_n^{\mathfrak{p}}$ is a sum of products of elements of $\widehat{\mathfrak{p}}$ (or, more precisely, their actions on M).

Now, let $m \in \mathbb{Z}$. We have just seen that $L_n^{\mathfrak{p}}$ is a sum of products of elements of $\widehat{\mathfrak{p}}$ (or, more precisely, their actions on M). Similarly, $L_m^{\mathfrak{p}}$ is a sum of products of elements of $\widehat{\mathfrak{p}}$ (or, more precisely, their actions on M). Since we know that L_n commutes with every element of $\widehat{\mathfrak{p}}$ (due to Theorem 4.5.2 **(b)**), this yields that L_n commutes with $L_m^{\mathfrak{p}}$. In other words, $[L_n, L_m^{\mathfrak{p}}] = 0$. Theorem 4.5.2 **(c)** is thus established.

(a) Any $n \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfy

$$\left[L_n, \underbrace{L_m}_{=L^{\mathfrak{g}}_m - L^{\mathfrak{p}}_m}\right]$$

$$= \left[L_n, L^{\mathfrak{g}}_m - L^{\mathfrak{p}}_m\right] = \left[L_n, L^{\mathfrak{g}}_m\right] - \underbrace{\left[L_n, L^{\mathfrak{p}}_m\right]}_{\substack{=0 \\ \text{(by Theorem 4.5.2 } \mathbf{(c)}\text{)}}} = \left[\underbrace{L_n}_{=L^{\mathfrak{g}}_n - L^{\mathfrak{p}}_n}, L^{\mathfrak{g}}_m\right]$$

$$= \left[L^{\mathfrak{g}}_n - L^{\mathfrak{p}}_n, L^{\mathfrak{g}}_m\right] = \left[L^{\mathfrak{g}}_n, L^{\mathfrak{g}}_m\right] - \underbrace{\left[L^{\mathfrak{p}}_n, L^{\mathfrak{g}}_m\right]}_{\substack{=\left[L^{\mathfrak{p}}_n, L^{\mathfrak{g}}_m - L^{\mathfrak{p}}_m\right] + \left[L^{\mathfrak{p}}_n, L^{\mathfrak{p}}_m\right] \\ \text{(since } L^{\mathfrak{g}}_m = \left(L^{\mathfrak{g}}_m - L^{\mathfrak{p}}_m\right) + L^{\mathfrak{p}}_m\text{)}}}$$

$$= \left[L^{\mathfrak{g}}_n, L^{\mathfrak{g}}_m\right] - \left[L^{\mathfrak{p}}_n, \underbrace{L^{\mathfrak{g}}_m - L^{\mathfrak{p}}_m}_{=L_m}\right] - \left[L^{\mathfrak{p}}_n, L^{\mathfrak{p}}_m\right]$$

$$= \underbrace{\left[L^{\mathfrak{g}}_n, L^{\mathfrak{g}}_m\right]}_{\substack{=(n-m)L^{\mathfrak{g}}_{n+m} - \frac{n^3-n}{12} c_{\mathfrak{g}} \delta_{n,-m} \\ \text{(by Theorem 4.3.4 } \mathbf{(c)}\text{)}}} - \underbrace{\left[L^{\mathfrak{p}}_n, L_m\right]}_{=-\left[L_m, L^{\mathfrak{p}}_n\right]} - \underbrace{\left[L^{\mathfrak{p}}_n, L^{\mathfrak{p}}_m\right]}_{\substack{=(n-m)L^{\mathfrak{p}}_{n+m} - \frac{n^3-n}{12} c_{\mathfrak{p}} \delta_{n,-m} \\ \text{(by Theorem 4.3.4 } \mathbf{(c)}\text{,} \\ \text{applied to } \mathfrak{p} \text{ instead of } \mathfrak{g}\text{)}}}$$

$$= \left((n-m)\, L^{\mathfrak{g}}_{n+m} - \frac{n^3-n}{12} c_{\mathfrak{g}} \delta_{n,-m}\right) + \left[L_m, L^{\mathfrak{p}}_n\right] - \left((n-m)\, L^{\mathfrak{p}}_{n+m} - \frac{n^3-n}{12} c_{\mathfrak{p}} \delta_{n,-m}\right)$$

$$= (n-m) \underbrace{\left(L^{\mathfrak{g}}_{n+m} - L^{\mathfrak{p}}_{n+m}\right)}_{=L_{n+m}} - \frac{n^3-n}{12} \left(c_{\mathfrak{g}} - c_{\mathfrak{p}}\right) \delta_{n,-m} + \underbrace{\left[L_m, L^{\mathfrak{p}}_n\right]}_{\substack{=0 \\ \text{(by Theorem 4.5.2 } \mathbf{(c)}\text{,} \\ \text{applied to } m \text{ and } n \text{ instead of } n \text{ and } m\text{)}}}$$

$$= (n-m)\, L_{n+m} - \frac{n^3-n}{12} \left(c_{\mathfrak{g}} - c_{\mathfrak{p}}\right) \delta_{n,-m}.$$

Hence, $(L_i)_{i \in \mathbb{Z}}$ is a Vir-action on M with central charge $c = c_{\mathfrak{g}} - c_{\mathfrak{p}}$. Theorem 4.5.2 **(a)** is thus proven. This completes the proof of Theorem 4.5.2.

Example 4.5.3. Let $\mathfrak{a}$ be a simple finite-dimensional Lie algebra. Let $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}$, and let $\mathfrak{p} = \mathfrak{a}_{\text{diag}} \subseteq \mathfrak{a} \oplus \mathfrak{a}$ (where $\mathfrak{a}_{\text{diag}}$ denotes the Lie subalgebra $\{(x,x) \mid x \in \mathfrak{a}\}$ of $\mathfrak{a} \oplus \mathfrak{a}$). Consider the standard form $(\cdot, \cdot)$ on $\mathfrak{a}$. Define a symmetric bilinear form on $\mathfrak{a} \oplus \mathfrak{a}$ as the direct sum of the standard forms on $\mathfrak{a}$ and $\mathfrak{a}$.

Let V' and V'' be admissible $\widehat{\mathfrak{a}}$-modules at levels k' and k''. Theorem 4.3.4 endows these vector spaces V' and V'' with Vir-module structures. These Vir-module structures have central charges $c'_{\mathfrak{a}} = \dfrac{k' \dim \mathfrak{a}}{k' + h^\vee}$ and $c''_{\mathfrak{a}} = \dfrac{k'' \dim \mathfrak{a}}{k'' + h^\vee}$, respectively (by (309)). Let $(L'_i)_{i \in \mathbb{Z}}$ and $(L''_i)_{i \in \mathbb{Z}}$ denote the actions of Vir on these modules.

Then, $V' \otimes V''$ is an admissible $\widehat{\mathfrak{g}}$-module at level $k' + k''$. Thus, by Theorem 4.3.4, this vector space $V' \otimes V''$ becomes a Vir-module. The action $(L^{\mathfrak{g}}_i)_{i \in \mathbb{Z}}$ of Vir on this Vir-module $V' \otimes V''$ is given by $L^{\mathfrak{g}}_i = L'_i + L''_i$ (or, more precisely, $L^{\mathfrak{g}}_i = L'_i \otimes \mathrm{id} + \mathrm{id} \otimes L''_i$). The central charge $c_{\mathfrak{g}}$ of this Vir-module $V' \otimes V''$ is

$$c_{\mathfrak{g}} = c'_{\mathfrak{a}} + c''_{\mathfrak{a}} = \frac{k' \dim \mathfrak{a}}{k' + h^\vee} + \frac{k'' \dim \mathfrak{a}}{k'' + h^\vee}.$$

Since $\widehat{\mathfrak{p}} = \widehat{\mathfrak{a}}$ acts on $V' \otimes V''$ by diagonal action, we also get a Vir-module structure $\left(L_i^{\mathfrak{p}}\right)_{i\in\mathbb{Z}}$ on $V' \otimes V''$ by applying Theorem 4.3.4 to $\mathfrak{p}$ instead of $\mathfrak{g}$. The central charge of this Vir-module is

$$c_{\mathfrak{p}} = \frac{k' + k''}{k' + k'' + h^\vee} \dim \mathfrak{a}$$

(since the level of the $\widehat{\mathfrak{p}}$-module $V' \otimes V''$ is $k' + k''$).

Thus, the central charge c of the Vir-action on $V' \otimes V''$ given by Theorem 4.5.2 is

$$\begin{aligned} c = c'_{\mathfrak{a}} + c''_{\mathfrak{a}} - c_{\mathfrak{p}} &= \frac{k' \dim \mathfrak{a}}{k' + h^\vee} + \frac{k'' \dim \mathfrak{a}}{k'' + h^\vee} - \frac{k' + k''}{k' + k'' + h^\vee} \dim \mathfrak{a} \\ &= \left(\frac{k'}{k' + h^\vee} + \frac{k''}{k'' + h^\vee} - \frac{k' + k''}{k' + k'' + h^\vee} \right) \dim \mathfrak{a}. \end{aligned}$$

We can use this construction to obtain, for every positive integer m, a unitary representation of Vir with central charge $1 - \dfrac{6}{(m+2)(m+3)}$: In fact, let $\mathfrak{a} = \mathfrak{sl}_2$, so that $h^\vee = 2$, and let $k' = 1$ and $k'' = m$. Then,

$$c = 3 \left(\frac{1}{3} + \frac{m}{m+2} - \frac{m+1}{m+3} \right) = 1 - \frac{6}{(m+2)(m+3)}.$$

So we get unitary representations of Vir with central charge c for these values of c.

4.6. Preliminaries to simple and Kac-Moody Lie algebras

Our next goal is defining and studying the Kac-Moody Lie algebras. Before we do this, however, we will recollect some properties of simple finite-dimensional Lie algebras (which are, in some sense, the prototypical Kac-Moody Lie algebras); and yet before that, we show some general results from the theory of Lie algebras which will be used in our later proofs.

[This whole Section 4.6 is written by Darij and aims at covering the gap between introductory courses in Lie algebras and Etingof's class. It states some folklore facts about Lie algebras which will be used later.]

4.6.1. A basic property of $\mathfrak{sl}_2$-modules

We begin with a lemma from the representation theory of $\mathfrak{sl}_2$:

Lemma 4.6.1. Let e, f and h mean the classical basis elements of $\mathfrak{sl}_2$. Let $\lambda \in \mathbb{C}$. We consider any $\mathfrak{sl}_2$-module as a $U(\mathfrak{sl}_2)$-module.

(a) Let V be an $\mathfrak{sl}_2$-module. Let $x \in V$ be such that $ex = 0$ and $hx = \lambda x$. Then, every $n \in \mathbb{N}$ satisfies $e^n f^n x = n! \lambda (\lambda - 1) \ldots (\lambda - n + 1) x$.

(b) Let V be an $\mathfrak{sl}_2$-module. Let $x \in V$ be such that $fx = 0$ and $hx = \lambda x$. Then, every $n \in \mathbb{N}$ satisfies $f^n e^n x = n! \lambda (\lambda + 1) \ldots (\lambda + n - 1) x$.

(c) Let V be a finite-dimensional $\mathfrak{sl}_2$-module. Let x be a nonzero element of V satisfying $ex = 0$ and $hx = \lambda x$. Then, $\lambda \in \mathbb{N}$ and $f^{\lambda+1} x = 0$.

Proof of Lemma 4.6.1. **(a)** *1st step:* We will see that

$$h f^m x = (\lambda - 2m) f^m x \qquad \text{for every } m \in \mathbb{N}. \tag{310}$$

Proof of (310): We will prove (310) by induction over m:

Induction base: For $m = 0$, we have $hf^m x = hf^0 x = hx = \lambda x$ and $(\lambda - 2m) f^m x = (\lambda - 2 \cdot 0) f^0 x = \lambda x$, so that $hf^m x = (\lambda - 2m) f^m x$ holds for $m = 0$. In other words, (310) holds for $m = 0$. This completes the induction base.

Induction step: Let $M \in \mathbb{N}$. Assume that (310) holds for $m = M$. We must then prove that (310) holds for $m = M + 1$ as well.

Since (310) holds for $m = M$, we have $hf^M x = (\lambda - 2M) f^M x$. Now,

$$\begin{aligned} h \underbrace{f^{M+1}}_{=ff^M} x &= \underbrace{hf}_{=fh+[h,f]} f^M x = (fh + [h,f]) f^M x = f \underbrace{hf^M x}_{=(\lambda-2M)f^M x} + \underbrace{[h,f]}_{=-2f} f^M x \\ &= (\lambda - 2M) \underbrace{ff^M}_{=f^{M+1}} x - 2 \underbrace{ff^M}_{=f^{M+1}} x = (\lambda - 2M) f^{M+1} x - 2 f^{M+1} x \\ &= \underbrace{(\lambda - 2M - 2)}_{=\lambda - 2(M+1)} f^{M+1} x = (\lambda - 2(M+1)) f^{M+1} x. \end{aligned}$$

Thus, (310) holds for $m = M + 1$ as well. This completes the induction step. The induction proof of (310) is thus complete.

2nd step: We will see that

$$ef^m x = m (\lambda - m + 1) f^{m-1} x \qquad \text{for every positive } m \in \mathbb{N}. \tag{311}$$

Proof of (311): We will prove (311) by induction over m:

Induction base: For $m = 1$, we have

$$ef^m x = \underbrace{ef^1}_{=ef=[e,f]+fe} x = ([e,f] + fe) x = \underbrace{[e,f]}_{=h} x + f \underbrace{ex}_{=0} = hx + f0 = hx = \lambda x$$

and $m (\lambda - m + 1) f^{m-1} x = 1 \underbrace{(\lambda - 1 + 1)}_{=\lambda} \underbrace{f^{1-1}}_{=1} x = \lambda x$, so that $ef^m x = m (\lambda - m + 1) f^{m-1} x$ holds for $m = 1$. In other words, (311) holds for $m = 1$. This completes the induction base.

Induction step: Let $M \in \mathbb{N}$ be positive. Assume that (311) holds for $m = M$. We must then prove that (311) holds for $m = M + 1$ as well.

Since (311) holds for $m = M$, we have $ef^M x = M (\lambda - M + 1) f^{M-1} x$. Now,

$$\begin{aligned} e \underbrace{f^{M+1}}_{=ff^M} x &= \underbrace{ef}_{=fe+[e,f]} f^M x = (fe + [e,f]) f^M x = f \underbrace{ef^M}_{=M(\lambda-M+1)f^{M-1}x} x + \underbrace{[e,f]}_{=h} f^M x \\ &= M (\lambda - M + 1) \underbrace{ff^{M-1}}_{=f^M} x + \underbrace{hf^M x}_{\substack{=(\lambda-2M)f^M x \\ \text{(by (310), applied to } m=M)}} \\ &= M (\lambda - M + 1) f^M x + (\lambda - 2M) f^M x = \underbrace{(M (\lambda - M + 1) + (\lambda - 2M))}_{=(M+1)(\lambda-(M+1)+1)} f^M x \\ &= (M + 1) (\lambda - (M + 1) + 1) f^M x. \end{aligned}$$

Thus, (311) holds for $m = M + 1$ as well. This completes the induction step. The induction proof of (311) is thus complete.

3rd step: We will see that

$$e^n f^n x = n!\lambda(\lambda-1)...(\lambda-n+1)x \qquad \text{for every } n\in\mathbb{N}. \tag{312}$$

Proof of (312): We will prove (312) by induction over n:

Induction base: For $n=0$, we have $e^n f^n x = e^0 f^0 x = x$ and $n!\lambda(\lambda-1)...(\lambda-n+1)x = \underbrace{0!}_{=1}\underbrace{\lambda(\lambda-1)...(\lambda-0+1)}_{=(\text{empty product})=1}x = x$, so that $e^n f^n x = n!\lambda(\lambda-1)...(\lambda-n+1)x$ holds for $n=0$. In other words, (312) holds for $n=0$. This completes the induction base.

Induction step: Let $N\in\mathbb{N}$. Assume that (312) holds for $n=N$. We must then prove that (312) holds for $n=N+1$ as well.

Since (312) holds for $n=N$, we have $e^N f^N x = N!\lambda(\lambda-1)...(\lambda-N+1)x$. Now,

$$\begin{aligned}
\underbrace{e^{N+1}}_{=e^N e} f^{N+1}x &= e^N \underbrace{ef^{N+1}x}_{\substack{=(N+1)(\lambda-(N+1)+1)f^{(N+1)-1}x \\ (\text{by (311), applied to } m=N+1)}} = (N+1)(\lambda-(N+1)+1)e^N \underbrace{f^{(N+1)-1}}_{=f^N}x \\
&= (N+1)(\lambda-(N+1)+1)\underbrace{e^N f^N x}_{=N!\lambda(\lambda-1)...(\lambda-N+1)x} \\
&= (N+1)(\lambda-(N+1)+1)\cdot N!\lambda(\lambda-1)...(\lambda-N+1)x \\
&= \underbrace{((N+1)\cdot N!)}_{=(N+1)!}\cdot\underbrace{(\lambda(\lambda-1)...(\lambda-N+1))\cdot(\lambda-(N+1)+1)}_{=\lambda(\lambda-1)...(\lambda-(N+1)+1)}x \\
&= (N+1)!\lambda(\lambda-1)...(\lambda-(N+1)+1)x.
\end{aligned}$$

Thus, (312) holds for $n=N+1$ as well. This completes the induction step. The induction proof of (312) is thus complete.

Lemma 4.6.1 **(a)** immediately follows from (312).

(b) The proof of Lemma 4.6.1 **(b)** is analogous to the proof of Lemma 4.6.1 **(a)**.

(c) By assumption, $\dim V<\infty$. Now, the endomorphism $h\mid_V$ of V has at most $\dim V$ distinct eigenvalues (since an endomorphism of any finite-dimensional vector space W has at most $\dim W$ distinct eigenvalues). From this, it is easy to conclude that $f^{\dim V}x=0$ [208]. Thus, there exists a smallest $m\in\mathbb{N}$ satisfying $f^m x=0$. Denote this m by u. Then, $f^u x=0$. Since $f^0 x = x\neq 0$, this u is $\neq 0$, so that $f^{u-1}x$ is well-defined. Moreover, $f^{u-1}x\neq 0$ (since u is the smallest $m\in\mathbb{N}$ satisfying $f^m x=0$).

Lemma 4.6.1 **(a)** (applied to $n=u$) yields $e^u f^u x = u!\lambda(\lambda-1)...(\lambda-u+1)x$. Since $e^u\underbrace{f^u x}_{=0}=0$, this rewrites as $u!\lambda(\lambda-1)...(\lambda-u+1)x=0$. Since $\operatorname{char}\mathbb{C}=0$, we can divide this equation by $u!$, and obtain $\lambda(\lambda-1)...(\lambda-u+1)x=0$. Since $x\neq 0$, this yields $\lambda(\lambda-1)...(\lambda-u+1)=0$. Thus, one of the numbers λ, $\lambda-1$, ..., $\lambda-u+1$ must be 0. In other words, $\lambda\in\{0,1,...,u-1\}$. Hence, $\lambda\in\mathbb{N}$ and $\lambda\leq u-1$.

[208] *Proof.* Assume the opposite. Then, $f^{\dim V}x\neq 0$.

Now, let $m\in\{0,1,...,\dim V\}$ be arbitrary. We will prove that $\lambda-2m$ is an eigenvalue of $h\mid_V$.

In fact, $m\leq\dim V$, so that $f^{\dim V-m}(f^m x) = f^{\dim V-m+m}x = f^{\dim V}x\neq 0$ and thus $f^m x\neq 0$. Since $hf^m x = (\lambda-2m)f^m x$ (by (310)), this yields that $f^m x$ is a nonzero eigenvector of $h\mid_V$ with eigenvalue $\lambda-2m$. Thus, $\lambda-2m$ is an eigenvalue of $h\mid_V$.

Now forget that we fixed m. Thus, we have proven that $\lambda-2m$ is an eigenvalue of $h\mid_V$ for every $m\in\{0,1,...,\dim V\}$. Thus we have found $\dim V+1$ pairwise distinct eigenvalues of $h\mid_V$. This contradicts the fact that $h\mid_V$ has at most $\dim V$ distinct eigenvalues. This contradiction shows that our assumption was wrong, qed.

Applying (310) to $m = u - 1$, we obtain $hf^{u-1}x = (\lambda - 2(u-1)) f^{u-1}x$. Denote $\lambda - 2(u-1)$ by μ. Then, $hf^{u-1}x = \underbrace{(\lambda - 2(u-1))}_{=\mu} f^{u-1}x = \mu f^{u-1}x$. Also, $ff^{u-1}x = f^u x = 0$. Thus, we can apply Lemma 4.6.1 **(b)** to μ, $f^{u-1}x$ and $u-1$ instead of λ, x and n. Thus, we obtain

$$f^{u-1}e^{u-1}f^{u-1}x = (u-1)!\mu(\mu+1)...(\mu+(u-1)-1) f^{u-1}x.$$

But $\mu = \underbrace{\lambda}_{\leq u-1} - 2(u-1) \leq (u-1) - 2(u-1) = -(u-1)$, so that each of the integers μ, $\mu+1$, ..., $\mu+(u-1)-1$ is nonzero. Thus, their product $\mu(\mu+1)...(\mu+(u-1)-1)$ also is $\neq 0$. Combined with $(u-1)! \neq 0$, this yields $(u-1)!\mu(\mu+1)...(\mu+(u-1)-1) \neq 0$. Combined with $f^{u-1}x \neq 0$, this yields $(u-1)!\mu(\mu+1)...(\mu+(u-1)-1) f^{u-1}x \neq 0$. Thus,

$$f^{u-1}e^{u-1}f^{u-1}x = (u-1)!\mu(\mu+1)...(\mu+(u-1)-1) f^{u-1}x \neq 0,$$

so that $e^{u-1}f^{u-1}x \neq 0$.

But Lemma 4.6.1 **(a)** (applied to $n = u-1$) yields $e^{u-1}f^{u-1}x = (u-1)!\lambda(\lambda-1)...(\lambda-(u-1)+1)x$. Thus,

$$(u-1)!\lambda(\lambda-1)...(\lambda-(u-1)+1)x = e^{u-1}f^{u-1}x \neq 0.$$

Hence, $\lambda(\lambda-1)...(\lambda-(u-1)+1) \neq 0$. Hence, $\binom{\lambda}{u-1} = \dfrac{1}{(u-1)!}\underbrace{\lambda(\lambda-1)...(\lambda-(u-1)+1)}_{\neq 0} \neq 0$, so that $u-1 \leq \lambda$ (because otherwise, we would have $\binom{\lambda}{u-1} = 0$, contradicting $\binom{\lambda}{u-1} \neq 0$). Combined with $u-1 \geq \lambda$, this yields $u-1 = \lambda$. Thus, $u = \lambda+1$. Hence, $f^u x = 0$ rewrites as $f^{\lambda+1}x = 0$. This proves Lemma 4.6.1 **(c)**.

4.6.2. Q-graded Lie algebras

The following generalization of the standard definition of a $\mathbb{Z}$-graded Lie algebra suggests itself:

Definition 4.6.2. Let Q be an abelian group, written additively.

(a) A *Q-graded vector space* will mean a vector space V equipped with a family $(V[\alpha])_{\alpha\in Q}$ of vector subspaces $V[\alpha]$ of V (indexed by elements of Q) satisfying $V = \bigoplus_{\alpha\in Q} V[\alpha]$. For every $\alpha \in Q$, the subspace $V[\alpha]$ is called the *α-th homogeneous component* of the Q-graded vector space V. The family $(V[\alpha])_{\alpha\in Q}$ is called a *Q-grading* on the vector space V.

(b) A *Q-graded Lie algebra* will mean a Lie algebra $\mathfrak{g}$ equipped with a family $(\mathfrak{g}[\alpha])_{\alpha\in Q}$ of vector subspaces $\mathfrak{g}[\alpha]$ of $\mathfrak{g}$ (indexed by elements of Q) satisfying $\mathfrak{g} = \bigoplus_{\alpha\in Q} \mathfrak{g}[\alpha]$ and satisfying

$$[\mathfrak{g}[\alpha], \mathfrak{g}[\beta]] \subseteq \mathfrak{g}[\alpha+\beta] \qquad \text{for all } \alpha, \beta \in Q.$$

In this case, Q is called the *root lattice* of this Q-graded Lie algebra $\mathfrak{g}$. (This does not mean that Q actually has to be a lattice of roots of $\mathfrak{g}$, or that Q must be related in any way to the roots of $\mathfrak{g}$.) Clearly, any Q-graded Lie algebra is a Q-graded vector space. Thus, the notion of the α-th homogeneous component of a Q-graded Lie algebra makes sense for every $\alpha \in Q$.

Convention 4.6.3. Whenever Q is an abelian group, α is an element of Q, and V is a Q-graded vector space or a Q-graded Lie algebra, we will denote the α-th homogeneous component of V by $V[\alpha]$.

In the context of a Q-graded vector space (or Lie algebra) V, one often writes V_α instead of $V[\alpha]$ for the α-th homogeneous component of V. This notation, however, can sometimes be misunderstood.

When a group homomorphism from Q to $\mathbb{Z}$ is given, a Q-graded Lie algebra canonically becomes a $\mathbb{Z}$-graded Lie algebra:

Proposition 4.6.4. Let Q be an abelian group. Let $\ell : Q \to \mathbb{Z}$ be a group homomorphism. Let $\mathfrak{g}$ be a Q-graded Lie algebra.

(a) For every $m \in \mathbb{Z}$, the internal direct sum $\bigoplus\limits_{\substack{\alpha\in Q;\\ \ell(\alpha)=m}} \mathfrak{g}[\alpha]$ is well-defined.

(b) Denote this internal direct sum $\bigoplus\limits_{\substack{\alpha\in Q;\\ \ell(\alpha)=m}} \mathfrak{g}[\alpha]$ by $\mathfrak{g}_{[m]}$. Then, the Lie algebra $\mathfrak{g}$ equipped with the grading $\left(\mathfrak{g}_{[m]}\right)_{m\in\mathbb{Z}}$ is a $\mathbb{Z}$-graded Lie algebra.

(This grading $\left(\mathfrak{g}_{[m]}\right)_{m\in\mathbb{Z}}$ is called the *principal grading* on $\mathfrak{g}$ induced by the given Q-grading on $\mathfrak{g}$ and the map ℓ.)

The proof of this proposition is straightforward and left to the reader.

4.6.3. A few lemmas on generating subspaces of Lie algebras

We proceed with some facts about generating sets of Lie algebras (free or not):

Lemma 4.6.5. Let $\mathfrak{g}$ be a Lie algebra, and let T be a vector subspace of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is generated by T as a Lie algebra.

Let U be a vector subspace of $\mathfrak{g}$ such that $T \subseteq U$ and $[T, U] \subseteq U$. Then, $U = \mathfrak{g}$.

Notice that Lemma 4.6.5 is not peculiar to Lie algebras. A similar result holds (for instance) if "Lie algebra" is replaced by "commutative nonunital algebra" and "$[T, U]$" is replaced by "TU".

The following proof is written merely for the sake of completeness; intuitively, Lemma 4.6.5 should be obvious from the observation that all iterated Lie brackets of elements of T can be written as linear combinations of Lie brackets of the form $[t_1, [t_2, [..., [t_{k-1}, t_k]]]]$ (with $t_1, t_2, ..., t_k \in T$) by applying the Jacobi identity iteratively.

Proof of Lemma 4.6.5. Define a sequence $(T_n)_{n\geq 1}$ of vector subspaces of $\mathfrak{g}$ recursively as follows: Let $T_1 = T$, and for every positive integer n, set $T_{n+1} = [T, T_n]$.

We have

$$[T_i, T_j] \subseteq T_{i+j} \qquad \text{for any positive integers } i \text{ and } j. \tag{313}$$

[209] Now, let S be the vector subspace $\sum\limits_{i\geq 1} T_i$ of $\mathfrak{g}$. Then, every positive integer k satisfies $T_k \subseteq S$. In particular, $T_1 \subseteq S$. Since $S = \sum\limits_{i\geq 1} T_i$ and $S = \sum\limits_{i\geq 1} T_i = \sum\limits_{j\geq 1} T_j$, we have

$$[S, S] = \left[\sum_{i\geq 1} T_i, \sum_{j\geq 1} T_j\right] = \sum_{i\geq 1}\sum_{j\geq 1} \underbrace{[T_i, T_j]}_{\substack{\subseteq T_{i+j} \subseteq S \\ \text{(since every positive} \\ \text{integer } k \text{ satisfies } T_k \subseteq S)}} \subseteq \sum_{i\geq 1}\sum_{j\geq 1} S \subseteq S$$

(since S is a vector space). Thus, S is a Lie subalgebra of $\mathfrak{g}$. Since $T = T_1 \subseteq S$, this yields that S is a Lie subalgebra of $\mathfrak{g}$ containing T as a subset. Since the smallest Lie subalgebra of $\mathfrak{g}$ containing T as a subset is $\mathfrak{g}$ itself (because $\mathfrak{g}$ is generated by T as a Lie algebra), this yields that $S \supseteq \mathfrak{g}$. In other words, $S = \mathfrak{g}$.

Now, it is easy to see that

$$T_i \subseteq U \text{ for every positive integer } i. \tag{315}$$

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[209] *Proof of (313):* We will prove (313) by induction over i.

Induction base: For any positive integer j, we have $T_{j+1} = [T, T_j]$ (by the definition of T_{j+1}) and thus $\left[\underbrace{T_1}_{=T}, T_j\right] = [T, T_j] = T_{j+1} = T_{1+j}$. In other words, (313) holds for $i = 1$. This completes the induction base.

Induction step: Let k be a positive integer. Assume that (313) is proven for $i = k$. We now will prove (313) for $i = k + 1$.

Since (313) is proven for $i = k$, we have

$$[T_k, T_j] \subseteq T_{k+j} \qquad \text{for any positive integer } j. \tag{314}$$

Now, let j be a positive integer. Then, $T_{k+j+1} = [T, T_{k+j}]$ (by the definition of T_{k+j+1}) and $T_{j+1} = [T, T_j]$ (by the definition of T_{j+1}). Now, any $x \in T$, $y \in T_k$ and $z \in T_j$ satisfy

$$\begin{aligned}
[[x, y], z] &= -\underbrace{[[y, z], x]}_{=-[x,[y,z]]} - \left[\underbrace{[z, x]}_{=-[x,z]}, y\right] \qquad \text{(by the Jacobi identity)} \\
&= \underbrace{-(-[x, [y, z]])}_{=[x,[y,z]]} - \underbrace{[-[x, z], y]}_{=-[[x,z],y]=[y,[x,z]]} = \left[\underbrace{x}_{\in T}, \left[\underbrace{y}_{\in T_k}, \underbrace{z}_{\in T_j}\right]\right] - \left[\underbrace{y}_{\in T_k}, \left[\underbrace{x}_{\in T}, \underbrace{z}_{\in T_j}\right]\right] \\
&\in \left[T, \underbrace{[T_k, T_j]}_{\substack{\subseteq T_{k+j} \\ \text{(by (314))}}}\right] + \left[T_k, \underbrace{[T, T_j]}_{=T_{j+1}}\right] \subseteq \underbrace{[T, T_{k+j}]}_{=T_{k+j+1}} + \underbrace{[T_k, T_{j+1}]}_{\substack{\subseteq T_{k+j+1} \\ \text{(by (314), applied to} \\ j+1 \text{ instead of } j)}} \\
&\subseteq T_{k+j+1} + T_{k+j+1} \subseteq T_{k+j+1} \qquad \text{(since } T_{k+j+1} \text{ is a vector space)} \\
&= T_{(k+1)+j}.
\end{aligned}$$

Hence, $[[T, T_k], T_j] \subseteq T_{(k+1)+j}$ (since $T_{(k+1)+j}$ is a vector space). Since $[T, T_k] = T_{k+1}$ (by the definition of T_{k+1}), this rewrites as $[T_{k+1}, T_j] \subseteq T_{(k+1)+j}$. Since we have proven this for every positive integer j, we have thus proven (313) for $i = k + 1$. The induction step is thus complete. This finishes the proof of (313).

[210] Hence,

$$\mathfrak{g} = S = \sum_{i \geq 1} \underbrace{T_i}_{\subseteq U} \subseteq \sum_{i \geq 1} U \subseteq U$$

(since U is a vector space). Thus, $U = \mathfrak{g}$, and this proves Lemma 4.6.5.

The next result is related:

Theorem 4.6.6. Let $\mathfrak{g}$ be a $\mathbb{Z}$-graded Lie algebra. Let T be a vector subspace of $\mathfrak{g}[1]$ such that $\mathfrak{g}$ is generated by T as a Lie algebra. Then, $T = \mathfrak{g}[1]$.

The proof of this theorem proceeds by defining the sequence $(T_n)_{n \geq 1}$ as in the proof of Lemma 4.6.5, and showing that $T_i \subseteq \mathfrak{g}[i]$ for every positive integer i. The details are left to the reader.

Generating subspaces can help in proving that Lie algebra homomorphisms are Q-graded:

Proposition 4.6.7. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two Q-graded Lie algebras. Let T be a Q-graded vector subspace of $\mathfrak{g}$. Assume that $\mathfrak{g}$ is generated by T as a Lie algebra.

Let $f : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism. Assume that $f\mid_T : T \to \mathfrak{h}$ is a Q-graded map.

Then, the map f is Q-graded.

The proof of this is left to the reader.

Next, a result on free Lie algebras:

Proposition 4.6.8. Let V be a vector space. We let $\operatorname{FreeLie} V$ denote the free Lie algebra on the vector space V (not on the set V), and let $T(V)$ denote the tensor algebra of V. Then, there exists a canonical algebra isomorphism $U(\operatorname{FreeLie} V) \to T(V)$, which commutes with the canonical injections of V into $U(\operatorname{FreeLie} V)$ and into $T(V)$.

We are going to prove Proposition 4.6.8 by combining the universal properties of the universal enveloping algebra, the free Lie algebra, and the tensor algebra. Let us first formulate these properties. First, the universal property of the universal enveloping algebra:

Theorem 4.6.9. Let $\mathfrak{g}$ be a Lie algebra. We denote by $\iota_{\mathfrak{g}}^U : \mathfrak{g} \to U(\mathfrak{g})$ the canonical map from $\mathfrak{g}$ into $U(\mathfrak{g})$. (This map $\iota_{\mathfrak{g}}^U$ is injective by the Poincaré-Birkhoff-Witt theorem, but this is not relevant to the current theorem.) For any algebra B and

[210] *Proof of (315):* We will prove (315) by induction over i.

Induction base: We have $T_1 = T \subseteq U$. Thus, (315) holds for $i = 1$. This completes the induction base.

Induction step: Let k be a positive integer. Assume that (315) holds for $i = k$. We now will prove (315) for $i = k + 1$.

Since (315) holds for $i = k$, we have $T_k \subseteq U$. Since $T_{k+1} = [T, T_k]$ (by the definition of T_{k+1}), we have $T_{k+1} = \left[T, \underbrace{T_k}_{\subseteq U}\right] \subseteq [T, U] \subseteq U$. In other words, (315) holds for $i = k+1$. This completes the induction step. Thus, (315) is proven.

any Lie algebra homomorphism $f : \mathfrak{g} \to B$ (where the Lie algebra structure on B is defined by the commutator of the multiplication of B), there exists a unique algebra homomorphism $F : U(\mathfrak{g}) \to B$ satisfying $f = F \circ \iota^U_{\mathfrak{g}}$.

Next, the universal property of the free Lie algebra:

Theorem 4.6.10. Let V be a vector space. We denote by $\iota^{\mathrm{FreeLie}}_V : V \to \operatorname{FreeLie} V$ the canonical map from V into $\operatorname{FreeLie} V$. (The construction of $\operatorname{FreeLie} V$ readily shows that this map $\iota^{\mathrm{FreeLie}}_V$ is injective.) For any Lie algebra $\mathfrak{h}$ and any linear map $f : V \to \mathfrak{h}$, there exists a unique Lie algebra homomorphism $F : \operatorname{FreeLie} V \to \mathfrak{h}$ satisfying $f = F \circ \iota^{\mathrm{FreeLie}}_V$.

Finally, the universal property of the tensor algebra:

Theorem 4.6.11. Let V be a vector space. We denote by $\iota^T_V : V \to T(V)$ the canonical map from V into $T(V)$. (This map ι^T_V is known to be injective.) For any algebra B and any linear map $f : V \to B$, there exists a unique algebra homomorphism $F : T(V) \to B$ satisfying $f = F \circ \iota^T_V$.

Proof of Proposition 4.6.8. The algebra $T(V)$ canonically becomes a Lie algebra (by defining the Lie bracket on $T(V)$ as the commutator of the multiplication). Similarly, the algebra $U(\operatorname{FreeLie} V)$ becomes a Lie algebra.

Applying Theorem 4.6.10 to $\mathfrak{h} = T(V)$ and $f = \iota^T_V$, we obtain that there exists a unique Lie algebra homomorphism $F : \operatorname{FreeLie} V \to T(V)$ satisfying $\iota^T_V = F \circ \iota^{\mathrm{FreeLie}}_V$. Denote this Lie algebra homomorphism F by h. Then, $h : \operatorname{FreeLie} V \to T(V)$ is a Lie algebra homomorphism satisfying $\iota^T_V = h \circ \iota^{\mathrm{FreeLie}}_V$.

Applying Theorem 4.6.9 to $\mathfrak{g} = \operatorname{FreeLie} V$, $B = T(V)$ and $f = h$, we obtain that there exists a unique algebra homomorphism $F : U(\operatorname{FreeLie} V) \to T(V)$ satisfying $h = F \circ \iota^U_{\operatorname{FreeLie} V}$. Denote this algebra homomorphism F by α. Then, $\alpha : U(\operatorname{FreeLie} V) \to T(V)$ is an algebra homomorphism satisfying $h = \alpha \circ \iota^U_{\operatorname{FreeLie} V}$.

Applying Theorem 4.6.11 to $B = U(\operatorname{FreeLie} V)$ and $f = \iota^U_{\operatorname{FreeLie} V} \circ \iota^{\mathrm{FreeLie}}_V$, we obtain that there exists a unique algebra homomorphism $F : T(V) \to U(\operatorname{FreeLie} V)$ satisfying $\iota^U_{\operatorname{FreeLie} V} \circ \iota^{\mathrm{FreeLie}}_V = F \circ \iota^T_V$. Denote this algebra homomorphism F by β. Then, $\beta : T(V) \to U(\operatorname{FreeLie} V)$ is an algebra homomorphism satisfying $\iota^U_{\operatorname{FreeLie} V} \circ \iota^{\mathrm{FreeLie}}_V = \beta \circ \iota^T_V$.

Both α and β are algebra homomorphisms, and therefore Lie algebra homomorphisms. Also, $\iota^U_{\operatorname{FreeLie} V}$ is a Lie algebra homomorphism.

We have

$$\beta \circ \underbrace{\alpha \circ \iota^U_{\operatorname{FreeLie} V}}_{=h} \circ \iota^{\mathrm{FreeLie}}_V = \beta \circ \underbrace{h \circ \iota^{\mathrm{FreeLie}}_V}_{=\iota^T_V} = \beta \circ \iota^T_V = \iota^U_{\operatorname{FreeLie} V} \circ \iota^{\mathrm{FreeLie}}_V$$

and

$$\alpha \circ \underbrace{\beta \circ \iota^T_V}_{=\iota^U_{\operatorname{FreeLie} V} \circ \iota^{\mathrm{FreeLie}}_V} = \underbrace{\alpha \circ \iota^U_{\operatorname{FreeLie} V}}_{=h} \circ \iota^{\mathrm{FreeLie}}_V = h \circ \iota^{\mathrm{FreeLie}}_V = \iota^T_V.$$

Now, applying Theorem 4.6.10 to $\mathfrak{h} = U(\operatorname{FreeLie} V)$ and $f = \iota^U_{\operatorname{FreeLie} V} \circ \iota^{\mathrm{FreeLie}}_V$, we obtain that there exists a unique Lie algebra homomorphism $F : \operatorname{FreeLie} V \to U(\operatorname{FreeLie} V)$ satisfying $\iota^U_{\operatorname{FreeLie} V} \circ \iota^{\mathrm{FreeLie}}_V = F \circ \iota^{\mathrm{FreeLie}}_V$. Thus, any two Lie algebra homomorphisms $F : \operatorname{FreeLie} V \to U(\operatorname{FreeLie} V)$ satisfying $\iota^U_{\operatorname{FreeLie} V} \circ \iota^{\operatorname{FreeLie}}_V = F \circ \iota^{\operatorname{FreeLie}}_V$ must be equal. Since $\beta \circ \alpha \circ \iota^U_{\operatorname{FreeLie} V}$ and $\iota^U_{\operatorname{FreeLie} V}$ are two such Lie algebra homomorphisms (because we know that $\beta \circ \alpha \circ \iota^U_{\operatorname{FreeLie} V} \circ \iota^{\operatorname{FreeLie}}_V = \iota^U_{\operatorname{FreeLie} V} \circ \iota^{\operatorname{FreeLie}}_V$ and clearly $\iota^U_{\operatorname{FreeLie} V} \circ \iota^{\operatorname{FreeLie}}_V = \iota^U_{\operatorname{FreeLie} V} \circ \iota^{\operatorname{FreeLie}}_V$), this yields that $\beta \circ \alpha \circ \iota^U_{\operatorname{FreeLie} V}$ and $\iota^U_{\operatorname{FreeLie} V}$ must be equal. In other words,

$$\beta \circ \alpha \circ \iota^U_{\operatorname{FreeLie} V} = \iota^U_{\operatorname{FreeLie} V}.$$

Next, applying Theorem 4.6.9 to $\mathfrak{g} = \operatorname{FreeLie} V$, $B = U(\operatorname{FreeLie} V)$ and $f = \iota^U_{\operatorname{FreeLie} V}$, we obtain that there exists a unique algebra homomorphism $F : U(\operatorname{FreeLie} V) \to U(\operatorname{FreeLie} V)$ satisfying $\iota^U_{\operatorname{FreeLie} V} = F \circ \iota^U_{\operatorname{FreeLie} V}$. Thus, any two algebra homomorphisms $F : U(\operatorname{FreeLie} V) \to U(\operatorname{FreeLie} V)$ satisfying $\iota^U_{\operatorname{FreeLie} V} = F \circ \iota^U_{\operatorname{FreeLie} V}$ must be equal. Since $\beta \circ \alpha$ and $\operatorname{id}_{U(\operatorname{FreeLie} V)}$ are two such algebra homomorphisms (because $\beta \circ \alpha \circ \iota^U_{\operatorname{FreeLie} V} = \iota^U_{\operatorname{FreeLie} V}$ and $\operatorname{id}_{U(\operatorname{FreeLie} V)} \circ \iota^U_{\operatorname{FreeLie} V} = \iota^U_{\operatorname{FreeLie} V}$), this yields that $\beta \circ \alpha$ and $\operatorname{id}_{U(\operatorname{FreeLie} V)}$ must be equal. Thus,

$$\beta \circ \alpha = \operatorname{id}_{U(\operatorname{FreeLie} V)}.$$

On the other hand, applying Theorem 4.6.11 to $B = T(V)$ and $f = \iota^T_V$, we obtain that there exists a unique algebra homomorphism $F : T(V) \to T(V)$ satisfying $\iota^T_V = F \circ \iota^T_V$. Therefore, any two algebra homomorphisms $F : T(V) \to T(V)$ satisfying $\iota^T_V = F \circ \iota^T_V$ must be equal. Since $\alpha \circ \beta$ and $\operatorname{id}_{T(V)}$ are two such algebra homomorphisms (because we know that $\alpha \circ \beta \circ \iota^T_V = \iota^T_V$ and $\operatorname{id}_{T(V)} \circ \iota^T_V = \iota^T_V$), this yields that $\alpha \circ \beta$ and $\operatorname{id}_{T(V)}$ must be equal. In other words, $\alpha \circ \beta = \operatorname{id}_{T(V)}$. Combined with $\beta \circ \alpha = \operatorname{id}_{U(\operatorname{FreeLie} V)}$, this yields that α and β are mutually inverse, and thus α and β are algebra isomorphisms. Hence, $\alpha : U(\operatorname{FreeLie} V) \to T(V)$ is a canonical algebra isomorphism. Also, α commutes with the canonical injections of V into $U(\operatorname{FreeLie} V)$ and into $T(V)$, because

$$\underbrace{\alpha \circ \iota^U_{\operatorname{FreeLie} V}}_{=h} \circ \iota^{\operatorname{FreeLie}}_V = h \circ \iota^{\operatorname{FreeLie}}_V = \iota^T_V.$$

Hence, there exists a canonical algebra isomorphism $U(\operatorname{FreeLie} V) \to T(V)$, which commutes with the canonical injections of V into $U(\operatorname{FreeLie} V)$ and into $T(V)$ (namely, α). Proposition 4.6.8 is proven.

4.6.4. Universality of the tensor algebra with respect to derivations

Next, let us notice that the universal property of the tensor algebra (Theorem 4.6.11) has an analogue for derivations in lieu of algebra homomorphisms:

Theorem 4.6.12. Let V be a vector space. We denote by $\iota^T_V : V \to T(V)$ the canonical map from V into $T(V)$. (This map ι^T_V is known to be injective.) For any $T(V)$-bimodule M and any linear map $f : V \to M$, there exists a unique derivation $F : T(V) \to M$ satisfying $f = F \circ \iota^T_V$.

It should be noticed that "derivation" means "$\mathbb{C}$-linear derivation" here.

Before we prove this theorem, let us extend its uniqueness part a bit:

Proposition 4.6.13. Let A be an algebra. Let M be an A-bimodule, and $d : A \to M$ and $e : A \to M$ two derivations. Let S be a subset of A which generates A as an algebra. Assume that $d\mid_S = e\mid_S$. Then, $d = e$.

Proof of Proposition 4.6.13. Let U be the subset $\operatorname{Ker}(d-e)$ of A. Clearly, U is a vector space (since $d-e$ is a linear map (since d and e are linear)).

It is known that any derivation $f : A \to M$ satisfies $f(1) = 0$. Applying this to $f = d$, we get $d(1) = 0$. Similarly, $e(1) = 0$. Thus, $(d-e)(1) = \underbrace{d(1)}_{=0} - \underbrace{e(1)}_{=0} = 0$, so that $1 \in \operatorname{Ker}(d-e) = U$.

Now let $b \in U$ and $c \in U$. Since $b \in U = \operatorname{Ker}(d-e)$, we have $(d-e)(b) = 0$. Thus, $d(b) - e(b) = (d-e)(b) = 0$, so that $d(b) = e(b)$. Similarly, $d(c) = e(c)$.

Now, since d is a derivation, the Leibniz formula yields $d(bc) = d(b)\cdot c + b \cdot d(c)$. Similarly, $e(bc) = e(b)\cdot c + b\cdot e(c)$. Hence,

$$\begin{aligned}(d-e)(bc) &= \underbrace{d(bc)}_{=d(b)\cdot c+b\cdot d(c)} - \underbrace{e(bc)}_{=e(b)\cdot c+b\cdot e(c)} = \left(\underbrace{d(b)}_{=e(b)}\cdot c + b\cdot \underbrace{d(c)}_{=e(c)}\right) - (e(b)\cdot c + b\cdot e(c)) \\ &= (e(b)\cdot c + b\cdot e(c)) - (e(b)\cdot c + b\cdot e(c)) = 0.\end{aligned}$$

In other words, $bc \in \operatorname{Ker}(d-e) = U$.

Now forget that we fixed b and c. We have thus showed that any $b \in U$ and $c \in U$ satisfy $bc \in U$. Combined with the fact that U is a vector space and that $1 \in U$, this yields that U is a subalgebra of A. Since $S \subseteq U$ (because every $s \in S$ satisfies

$$(d-e)(s) = \underbrace{d(s)}_{=(d\mid_S)(s)} - \underbrace{e(s)}_{=(e\mid_S)(s)} = \underbrace{(d\mid_S)}_{=e\mid_S}(s) - (e\mid_S)(s) = (e\mid_S)(s) - (e\mid_S)(s) = 0$$

and thus $s \in \operatorname{Ker}(d-e) = U$), this yields that U is a subalgebra of A containing S as a subset. But since the smallest subalgebra of A containing S as a subset is A itself (because S generates A as an algebra), this yields that $U \supseteq A$. Hence, $A \subseteq U = \operatorname{Ker}(d-e)$, so that $d - e = 0$ and thus $d = e$. Proposition 4.6.13 is proven.

Proof of Theorem 4.6.12. For any $n \in \mathbb{N}$, we can define a linear map $\Phi_n : V^{\otimes n} \to M$ by the equation

$$\left(\begin{array}{r}\Phi_n(v_1 \otimes v_2 \otimes \ldots \otimes v_n) = \sum\limits_{k=1}^{n} v_1 \cdot v_2 \cdot \ldots \cdot v_{k-1} \cdot f(v_k) \cdot v_{k+1} \cdot v_{k+2} \cdot \ldots \cdot v_n \\ \text{for all } v_1, v_2, \ldots, v_n \in V\end{array}\right) \tag{316}$$

(by the universal property of the tensor product, since the term $\sum\limits_{k=1}^{n} v_1 \cdot v_2 \cdot \ldots \cdot v_{k-1} \cdot f(v_k) \cdot v_{k+1} \cdot v_{k+2} \cdot \ldots \cdot v_n$ is clearly multilinear in v_1, v_2, ..., v_n). Define this map Φ_n. Let Φ be the map $\bigoplus\limits_{n\in\mathbb{N}} \Phi_n : \bigoplus\limits_{n\in\mathbb{N}} V^{\otimes n} \to M$. Then, every $n \in \mathbb{N}$ and every $v_1, v_2, \ldots, v_n$ satisfy

$$\begin{aligned}\Phi(v_1 \otimes v_2 \otimes \ldots \otimes v_n) &= \Phi_n(v_1 \otimes v_2 \otimes \ldots \otimes v_n) \\ &= \sum_{k=1}^{n} v_1 \cdot v_2 \cdot \ldots \cdot v_{k-1} \cdot f(v_k) \cdot v_{k+1} \cdot v_{k+2} \cdot \ldots \cdot v_n.\end{aligned} \tag{317}$$

Since $\bigoplus_{n\in\mathbb{N}} V^{\otimes n} = T(V)$, the map Φ is a map from $T(V)$ to M. We will now prove that Φ is a derivation. In fact, in order to prove this, we must show that

$$\Phi(ab) = \Phi(a) \cdot b + a \cdot \Phi(b) \qquad \text{for any } a \in T(V) \text{ and } b \in T(V). \tag{318}$$

Proof of (318): Every element of $T(V)$ is a linear combination of elements of $V^{\otimes n}$ for various $n \in \mathbb{N}$ (because $T(V) = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$). Meanwhile, every element of $V^{\otimes n}$ for any $n \in \mathbb{N}$ is a linear combination of pure tensors. Combining these two observations, we see that every element of $T(V)$ is a linear combination of pure tensors.

We need to prove the equation (318) for all $a \in T(V)$ and $b \in T(V)$. Since this equation is linear in each of a and b, we can WLOG assume that a and b are pure tensors (since every element of $T(V)$ is a linear combination of pure tensors). Assume this. Then, a is a pure tensor, so that there exists an $n \in \mathbb{N}$ and some $v_1, v_2, ..., v_n \in V$ satisfying $a = v_1 \otimes v_2 \otimes ... \otimes v_n$. Consider this n and these $v_1, v_2, ..., v_n$. Also, b is a pure tensor, so that there exists an $m \in \mathbb{N}$ and some $w_1, w_2, ..., w_m \in V$ satisfying $b = w_1 \otimes w_2 \otimes ... \otimes w_m$. Consider this m and these $w_1, w_2, ..., w_m$.

By (317) (applied to m and $w_1, w_2, ..., w_m$ instead of n and $v_1, v_2, ..., v_n$), we have

$$\begin{aligned}
\Phi(w_1 \otimes w_2 \otimes ... \otimes w_m) &= \sum_{k=1}^{m} w_1 \cdot w_2 \cdot ... \cdot w_{k-1} \cdot f(w_k) \cdot w_{k+1} \cdot w_{k+2} \cdot ... \cdot w_m \\
&= \sum_{k=n+1}^{n+m} w_1 \cdot w_2 \cdot ... \cdot w_{k-n-1} \cdot f(w_{k-n}) \cdot w_{k-n+1} \cdot w_{k-n+2} \cdot ... \cdot w_m
\end{aligned}$$

(here, we substituted $k-n$ for k in the sum).

Let $(u_1, u_2, ..., u_{n+m})$ be the $(n+m)$-tuple $(v_1, v_2, ..., v_n, w_1, w_2, ..., w_m)$. Then,

$$u_1 \otimes u_2 \otimes ... \otimes u_{n+m} = \underbrace{v_1 \otimes v_2 \otimes ... \otimes v_n}_{=a} \otimes \underbrace{w_1 \otimes w_2 \otimes ... \otimes w_m}_{=b} = a \otimes b = ab.$$

By (317) (applied to $n+m$ and $u_1, u_2, ..., u_{n+m}$ instead of n and $v_1, v_2, ..., v_n$), we have

$$\begin{aligned}
&\Phi\left(u_1\otimes u_2\otimes\ldots\otimes u_{n+m}\right)\\
&=\sum_{k=1}^{n+m} u_1\cdot u_2\cdot\ldots\cdot u_{k-1}\cdot f\left(u_k\right)\cdot u_{k+1}\cdot u_{k+2}\cdot\ldots\cdot u_{n+m}\\
&=\sum_{k=1}^{n}\underbrace{u_1\cdot u_2\cdot\ldots\cdot u_{k-1}\cdot f\left(u_k\right)\cdot u_{k+1}\cdot u_{k+2}\cdot\ldots\cdot u_{n+m}}_{\substack{=v_1\cdot v_2\cdot\ldots\cdot v_{k-1}\cdot f(v_k)\cdot v_{k+1}\cdot v_{k+2}\cdot\ldots\cdot v_n\cdot w_1\cdot w_2\cdot\ldots\cdot w_m\\ \text{(since } (u_1,u_2,\ldots,u_{n+m})=(v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_m) \text{ and } k\leq n)}}\\
&\qquad+\sum_{k=n+1}^{n+m}\underbrace{u_1\cdot u_2\cdot\ldots\cdot u_{k-1}\cdot f\left(u_k\right)\cdot u_{k+1}\cdot u_{k+2}\cdot\ldots\cdot u_{n+m}}_{\substack{=v_1\cdot v_2\cdot\ldots\cdot v_n\cdot w_1\cdot w_2\cdot\ldots\cdot w_{k-n-1}\cdot f(w_{k-n})\cdot w_{k-n+1}\cdot w_{k-n+2}\cdot\ldots\cdot w_m\\ \text{(since } (u_1,u_2,\ldots,u_{n+m})=(v_1,v_2,\ldots,v_n,w_1,w_2,\ldots,w_m) \text{ and } k>n)}}\\
&=\sum_{k=1}^{n} v_1\cdot v_2\cdot\ldots\cdot v_{k-1}\cdot f\left(v_k\right)\cdot v_{k+1}\cdot v_{k+2}\cdot\ldots\cdot v_n\cdot\underbrace{w_1\cdot w_2\cdot\ldots\cdot w_m}_{=w_1\otimes w_2\otimes\ldots\otimes w_m=b}\\
&\qquad+\sum_{k=n+1}^{n+m}\underbrace{v_1\cdot v_2\cdot\ldots\cdot v_n}_{=v_1\otimes v_2\otimes\ldots\otimes v_n=a}\cdot w_1\cdot w_2\cdot\ldots\cdot w_{k-n-1}\cdot f\left(w_{k-n}\right)\cdot w_{k-n+1}\cdot w_{k-n+2}\cdot\ldots\cdot w_m\\
&=\sum_{k=1}^{n} v_1\cdot v_2\cdot\ldots\cdot v_{k-1}\cdot f\left(v_k\right)\cdot v_{k+1}\cdot v_{k+2}\cdot\ldots\cdot v_n\cdot b\\
&\qquad+\sum_{k=n+1}^{n+m} a\cdot w_1\cdot w_2\cdot\ldots\cdot w_{k-n-1}\cdot f\left(w_{k-n}\right)\cdot w_{k-n+1}\cdot w_{k-n+2}\cdot\ldots\cdot w_m\\
&=\underbrace{\left(\sum_{k=1}^{n} v_1\cdot v_2\cdot\ldots\cdot v_{k-1}\cdot f\left(v_k\right)\cdot v_{k+1}\cdot v_{k+2}\cdot\ldots\cdot v_n\right)}_{\substack{=\Phi(v_1\otimes v_2\otimes\ldots\otimes v_n)\\ \text{(by (317))}}}\cdot b\\
&\qquad+a\cdot\underbrace{\left(\sum_{k=n+1}^{n+m} w_1\cdot w_2\cdot\ldots\cdot w_{k-n-1}\cdot f\left(w_{k-n}\right)\cdot w_{k-n+1}\cdot w_{k-n+2}\cdot\ldots\cdot w_m\right)}_{=\Phi(w_1\otimes w_2\otimes\ldots\otimes w_m)}\\
&=\Phi\left(\underbrace{v_1\otimes v_2\otimes\ldots\otimes v_n}_{=a}\right)\cdot b+a\cdot\Phi\left(\underbrace{w_1\otimes w_2\otimes\ldots\otimes w_m}_{=b}\right)=\Phi\left(a\right)\cdot b+a\cdot\Phi\left(b\right).
\end{aligned}$$

Thus, (318) is proven.

Now that we know that Φ satisfies (318), we conclude that Φ is a derivation.

Next, notice that every $v\in V$ satisfies $\iota_V^T\left(v\right)=v$ (since ι_V^T is just the inclusion map).

Hence, every $v \in V$ satisfies

$$\begin{aligned}
\left(\Phi \circ \iota_V^T\right)(v) &= \Phi\left(\underbrace{\iota_V^T(v)}_{=v}\right) = \Phi(v) \\
&= \sum_{k=1}^{1} f(v) \qquad \text{(by (317), applied to } n = 1 \text{ and } v_1 = v\text{)} \\
&= f(v).
\end{aligned}$$

Thus, $\Phi \circ \iota_V^T = f$.

So we know that Φ is a derivation satisfying $f = \Phi \circ \iota_V^T$. Thus, we have shown that there exists a derivation $F : T(V) \to M$ satisfying $f = F \circ \iota_V^T$ (namely, $F = \Phi$). In order to complete the proof of Theorem 4.6.12, we only need to check that this derivation is unique. In other words, we need to check that whenever a derivation $F : T(V) \to M$ satisfies $f = F \circ \iota_V^T$, we must have $F = \Phi$. Let us prove this now. Let $F : T(V) \to M$ be any derivation satisfying $f = F \circ \iota_V^T$. Then, every $v \in V$ satisfies

$$\begin{aligned}
(F \mid_V)(v) &= F\left(\underbrace{v}_{=\iota_V^T(v)}\right) = F\left(\iota_V^T(v)\right) = \underbrace{\left(F \circ \iota_V^T\right)}_{=f=\Phi\circ\iota_V^T}(v) = \left(\Phi \circ \iota_V^T\right)(v) \\
&= \Phi\left(\underbrace{\iota_V^T(v)}_{=v}\right) = \Phi(v) = (\Phi \mid_V)(v).
\end{aligned}$$

Thus, $F \mid_V = \Phi \mid_V$. Proposition 4.6.13 (applied to $A = T(V)$, $d = F$, $e = \Phi$ and $S = V$) thus yields $F = \Phi$ (since V generates $T(V)$ as an algebra). This completes the proof of Theorem 4.6.12 (as we have seen above).

We will later use a corollary of Proposition 4.6.13:

Corollary 4.6.14. Let A be an algebra. Let B be a subalgebra of A. Let C be a subalgebra of B. Let $d : A \to A$ be a derivation of the algebra A. Let S be a subset of C which generates C as an algebra. Assume that $d(S) \subseteq B$. Then, $d(C) \subseteq B$.

Proof of Corollary 4.6.14. Since $C \subseteq B \subseteq A$, the vector spaces A and B become C-modules.

Let $\pi : A \to A/B$ be the canonical projection. Clearly, π is a C-module homomorphism, and satisfies $\operatorname{Ker} \pi = B$. Let $d' : C \to A/B$ be the restriction of the map

$\pi \circ d : A \to A/B$ to C. It is easy to see that $d' : C \to A/B$ is a derivation[211]. On the other hand, $0 : C \to A/B$ is a derivation as well. Every $s \in S$ satisfies

$$\begin{aligned}
(d' \mid_S)(s) = d'(s) = (\pi \circ d)(s) \qquad & (\text{since } d' \text{ is the restriction of } \pi \circ d \text{ to } C) \\
= \pi(d(s)) = 0 \qquad & \left(\text{since } d\left(\underbrace{s}_{\in S}\right) \in d(S) \subseteq B = \operatorname{Ker} \pi\right) \\
= 0(s) = (0 \mid_S)(s). &
\end{aligned}$$

Thus, $d' \mid_S = 0 \mid_S$. Proposition 4.6.13 (applied to C, A/M, d' and 0 instead of A, M, d and e) therefore yields that $d' = 0$ on C. But since d' is the restriction of $\pi \circ d$ to C, we have $d' = (\pi \circ d) \mid_C$. Thus, $(\pi \circ d) \mid_C = d' = 0$, so that $(\pi \circ d)(C) = 0$. Thus, $\pi(d(C)) = (\pi \circ d)(C) = 0$, so that $d(C) \subseteq \operatorname{Ker} \pi = B$. Corollary 4.6.14 is therefore proven.

Corollary 4.6.15. Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{h}$ be a vector space equipped with both a Lie algebra structure and a $\mathfrak{g}$-module structure. Assume that $\mathfrak{g}$ acts on $\mathfrak{h}$ by derivations. Consider the semidirect product $\mathfrak{g} \ltimes \mathfrak{h}$ defined as in Definition 3.2.1 **(b)**. Consider $\mathfrak{g}$ as a Lie subalgebra of $\mathfrak{g} \ltimes \mathfrak{h}$. Consider $\mathfrak{g} \ltimes \mathfrak{h}$ as a Lie subalgebra of $U(\mathfrak{g} \ltimes \mathfrak{h})$ (where the Lie bracket on $U(\mathfrak{g} \ltimes \mathfrak{h})$ is defined as the commutator of the multiplication). Consider $\mathfrak{h}$ as a Lie subalgebra of $\mathfrak{g} \ltimes \mathfrak{h}$, whence $U(\mathfrak{h})$ becomes a subalgebra of $U(\mathfrak{g} \ltimes \mathfrak{h})$.

Then, $[\mathfrak{g}, U(\mathfrak{h})] \subseteq U(\mathfrak{h})$ (as subsets of $U(\mathfrak{g} \ltimes \mathfrak{h})$).

Proof of Corollary 4.6.15. Let $x \in \mathfrak{g}$. Define a map $\xi : U(\mathfrak{g} \ltimes \mathfrak{h}) \to U(\mathfrak{g} \ltimes \mathfrak{h})$ by

$$(\xi(y) = [x, y] \qquad \text{for every } y \in U(\mathfrak{g} \ltimes \mathfrak{h})).$$

Then, ξ is clearly a derivation of the algebra $U(\mathfrak{g} \ltimes \mathfrak{h})$.

We are identifying $\mathfrak{g}$ with a Lie subalgebra of $\mathfrak{g} \ltimes \mathfrak{h}$. Clearly, $x \in \mathfrak{g}$ corresponds to $(x, 0) \in \mathfrak{g} \ltimes \mathfrak{h}$ under this identification.

We are also identifying $\mathfrak{h}$ with a Lie subalgebra of $\mathfrak{g} \ltimes \mathfrak{h}$. Every $y \in \mathfrak{h}$ corresponds to $(0, y) \in \mathfrak{g} \ltimes \mathfrak{h}$ under this identification.

[211] *Proof.* Every $x \in C$ and $y \in C$ satisfy

$$\begin{aligned}
d'(xy) &= (\pi \circ d)(xy) \qquad (\text{since } d' \text{ is the restriction of } \pi \circ d \text{ to } C) \\
&= \pi\left(\underbrace{d(xy)}_{\substack{= d(x) \cdot y + x \cdot d(y) \\ (\text{since } d \text{ is a derivation})}}\right) = \pi(d(x) \cdot y + x \cdot d(y)) \\
&= \underbrace{\pi(d(x) \cdot y)}_{\substack{= \pi(d(x)) \cdot y \\ (\text{since } \pi \text{ is a } C\text{-module} \\ \text{homomorphism})}} + \underbrace{\pi(x \cdot d(y))}_{\substack{= x \cdot \pi(d(y)) \\ (\text{since } \pi \text{ is a } C\text{-module} \\ \text{homomorphism})}} \\
&= \underbrace{\pi(d(x))}_{\substack{= (\pi \circ d)(x) = d'(x) \\ (\text{since } d' \text{ is the restriction of} \\ \pi \circ d \text{ to } C, \text{ and since } x \in C)}} \cdot y + x \cdot \underbrace{\pi(d(y))}_{\substack{= (\pi \circ d)(y) = d'(y) \\ (\text{since } d' \text{ is the restriction of} \\ \pi \circ d \text{ to } C, \text{ and since } y \in C)}} = d'(x) \cdot y + x \cdot d'(y).
\end{aligned}$$

Thus, d' is a derivation, qed.

Thus, every $y \in \mathfrak{h}$ satisfies

$$\left[\underbrace{x}_{=(x,0)}, \underbrace{y}_{=(0,y)}\right] = [(x,0), (0,y)] = \left(\underbrace{[x,0]}_{=0}, \underbrace{[0,y]}_{=0} + x \rightharpoonup y - \underbrace{0 \rightharpoonup 0}_{=0}\right)$$
$$\text{(by the definition of the Lie bracket on } \mathfrak{g} \ltimes \mathfrak{h}\text{)}$$
$$= (0, x \rightharpoonup y) = x \rightharpoonup y \in \mathfrak{h}.$$

Hence, $\xi(y) = [x,y] \in \mathfrak{h}$ for every $y \in \mathfrak{h}$. Thus, $\xi(\mathfrak{h}) \subseteq \mathfrak{h} \subseteq U(\mathfrak{h})$.

Now, we notice that the subset $\mathfrak{h}$ of $U(\mathfrak{h})$ generates $U(\mathfrak{h})$ as an algebra. Thus, Corollary 4.6.14 (applied to $A = U(\mathfrak{g} \ltimes \mathfrak{h})$, $B = U(\mathfrak{h})$, $C = U(\mathfrak{h})$, $d = \xi$ and $S = \mathfrak{h}$) yields $\xi(U(\mathfrak{h})) \subseteq U(\mathfrak{h})$. Hence, every $u \in U(\mathfrak{h})$ satisfies $\xi(u) \in U(\mathfrak{h})$. But since $\xi(u) = [x,u]$ (by the definition of ξ), this yields that every $u \in U(\mathfrak{h})$ satisfies $[x,u] \in U(\mathfrak{h})$.

Now forget that we fixed x. We thus have shown that every $x \in \mathfrak{g}$ and every $u \in U(\mathfrak{h})$ satisfy $[x,u] \in U(\mathfrak{h})$. Thus, $[\mathfrak{g}, U(\mathfrak{h})] \subseteq U(\mathfrak{h})$ (since $U(\mathfrak{h})$ is a vector space). This proves Corollary 4.6.15.

4.6.5. Universality of the free Lie algebra with respect to derivations

Both Theorem 4.6.12 and Proposition 4.6.13 have analogues pertaining to Lie algebras in lieu of (associative) algebras.[212] We are going to formulate both of these analogues, but we start with that of Proposition 4.6.13, since it is the one we will find utile in our study of Kac-Moody Lie algebras:

Proposition 4.6.16. Let $\mathfrak{g}$ be a Lie algebra. Let M be a $\mathfrak{g}$-module, and $d : \mathfrak{g} \to M$ and $e : \mathfrak{g} \to M$ two 1-cocycles. Let S be a subset of $\mathfrak{g}$ which generates $\mathfrak{g}$ as a Lie algebra. Assume that $d\mid_S = e\mid_S$. Then, $d = e$.

The proof of Proposition 4.6.16 is analogous to that of Proposition 4.6.13.

We record a corollary of Proposition 4.6.16:

Corollary 4.6.17. Let $\mathfrak{g}$ be a Lie algebra. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Let $\mathfrak{i}$ be a Lie subalgebra of $\mathfrak{h}$. Let $d : \mathfrak{g} \to \mathfrak{g}$ be a derivation of the Lie algebra $\mathfrak{g}$. Let S be a subset of $\mathfrak{i}$ which generates $\mathfrak{i}$ as a Lie algebra. Assume that $d(S) \subseteq \mathfrak{h}$. Then, $d(\mathfrak{i}) \subseteq \mathfrak{h}$.

This corollary is analogous to Corollary 4.6.14, and proven accordingly.

Let us now state the analogue of Proposition 4.6.13 in the Lie-algebraic setting:

Theorem 4.6.18. Let V be a vector space. We denote by $\iota_V^{\mathrm{FreeLie}} : V \to \operatorname{FreeLie} V$ the canonical map from V into $\operatorname{FreeLie} V$. (This map $\iota_V^{\mathrm{FreeLie}}$ is easily seen to be injective.) For any $\operatorname{FreeLie} V$-module M and any linear map $f : V \to M$, there exists a unique 1-cocycle $F : \operatorname{FreeLie} V \to M$ satisfying $f = F \circ \iota_V^{\mathrm{FreeLie}}$.

[212] Notice that the Lie-algebraic analogue of a derivation from an algebra A into an A-bimodule is a 1-cocycle from a Lie algebra $\mathfrak{g}$ into a $\mathfrak{g}$-module.

Although we will not use this theorem anywhere in the following, let us briefly discuss how it is proven. Theorem 4.6.18 cannot be proven as directly as we proved Theorem 4.6.12. Instead, a way to prove Theorem 4.6.18 is by using the following lemma:

Lemma 4.6.19. Let $\mathfrak{g}$ be a Lie algebra. Let M be a $\mathfrak{g}$-module. Define the semidirect product $\mathfrak{g} \ltimes M$ as in Definition 1.7.7. Let $\varphi : \mathfrak{g} \to M$ be a linear map. Then, $\varphi : \mathfrak{g} \to M$ is a 1-cocycle if and only if the map

$$\mathfrak{g} \to \mathfrak{g} \ltimes M, \qquad x \mapsto (x, \varphi(x))$$

is a Lie algebra homomorphism.

This lemma helps reducing Theorem 4.6.18 to Theorem 4.6.10. We leave the details of this proof (both of the lemma and of Theorem 4.6.18) to the reader.

An alternative way to prove Theorem 4.6.18 is the following: Apply Theorem 4.6.12 to construct a derivation $F : T(V) \to M$ (of algebras) satisfying $f = F \circ \iota_V^T$, and then identify $\operatorname{FreeLie} V$ with a Lie subalgebra of $T(V)$ (because Proposition 4.6.8 $U(\operatorname{FreeLie} V) \cong T(V)$, and because the Poincaré-Birkhoff-Witt theorem entails an injection $\operatorname{FreeLie} V \to U(\operatorname{FreeLie} V)$). Then, restricting the derivation $F : T(V) \to M$ to $\operatorname{FreeLie} V$, we obtain a 1-cocycle $\operatorname{FreeLie} V \to M$ with the required properties. The uniqueness part of Theorem 4.6.18 is easy (and follows from Proposition 4.6.16 below). This proof of Theorem 4.6.18 has the disadvantage that it makes use of the Poincaré-Birkhoff-Witt theorem, which does not generalize to the case of Lie algebras over rings (whereas Theorem 4.6.18 does generalize to this case).

4.6.6. Derivations from grading

The following simple lemma will help us defining derivations on Lie algebras:

Lemma 4.6.20. Let Q be an abelian group. Let $s \in \operatorname{Hom}(Q, \mathbb{C})$ be a group homomorphism. Let $\mathfrak{n}$ be a Q-graded Lie algebra. Let $\eta : \mathfrak{n} \to \mathfrak{n}$ be a linear map satisfying

$$\eta(x) = s(w) \cdot x \qquad \text{for every } w \in Q \text{ and every } x \in \mathfrak{n}[w]. \tag{319}$$

Then, η is a derivation (of Lie algebras).

Proof of Lemma 4.6.20. In order to prove that η is a derivation, we need to check that

$$\eta([a,b]) = [\eta(a), b] + [a, \eta(b)] \qquad \text{for any } a \in \mathfrak{n} \text{ and } b \in \mathfrak{n}. \tag{320}$$

Let us prove the equation (320). Since this equation is linear in each of a and b, we can WLOG assume that a and b are homogeneous (because any element of $\mathfrak{n}$ is a sum of homogeneous elements). So, assume this. We will write the binary operation of the group Q as addition. Since a is homogeneous, we have $a \in \mathfrak{n}[u]$ for some $u \in Q$. Consider this u. Since b is homogeneous, we have $b \in \mathfrak{n}[v]$ for some $v \in Q$. Fix this v. Thus, $[a,b] \in \mathfrak{n}[u+v]$ (since $a \in \mathfrak{n}[u]$ and $b \in \mathfrak{n}[v]$ and since $\mathfrak{n}$ is Q-graded). Thus, (319) (applied to $x = a + b$ and $w = u + v$) yields $\eta([a,b]) = \underbrace{s(u+v)}_{\substack{=s(u)+s(v) \\ \text{(since } s \text{ is a group} \\ \text{homomorphism)}}} \cdot [a,b] =$

$(s(u)+s(v))\cdot[a,b]$. On the other hand, (319) (applied to $x=a$ and $w=u$) yields $\eta(a)=s(u)\cdot a$. Also, (319) (applied to $x=b$ and $y=v$) yields $\eta(b)=s(v)\cdot b$. Now,

$$\left[\underbrace{\eta(a)}_{=s(u)\cdot a},b\right]+\left[a,\underbrace{\eta(b)}_{=s(v)\cdot b}\right]=s(u)\cdot[a,b]+s(v)\cdot[a,b]=(s(u)+s(v))\cdot[a,b]=\eta([a,b]).$$

This proves (320). Now that (320) is proven, we conclude that η is a derivation. Lemma 4.6.20 is proven.

4.6.7. The commutator of derivations

The following proposition is the classical analogue of Proposition 1.4.1 for algebras in lieu of Lie algebras:

Proposition 4.6.21. Let A be an algebra. Let $f : A \to A$ and $g : A \to A$ be two derivations of A. Then, $[f,g]$ is a derivation of A. (Here, the Lie bracket is to be understood as the Lie bracket on $\operatorname{End} A$, so that we have $[f,g]=f\circ g-g\circ f$.)

The proof of this is completely analogous to that of Proposition 1.4.1. Moreover, by the same argument, the following slight generalization of Proposition 4.6.21 can be shown:

Proposition 4.6.22. Let A be a subalgebra of an algebra B. Let $f : A \to B$ and $g : B \to B$ be two derivations such that $g(A)\subseteq A$. Then, $f\circ(g\mid_A)-g\circ f : A\to B$ is a derivation.

4.7. Simple Lie algebras: a recollection

The Kac-Moody Lie algebras form a class of Lie algebras which contains all simple finite-dimensional and all affine Lie algebras, but also many more. Before we start studying them, let us recall some facts about simple Lie algebras:

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. A *Cartan subalgebra* of $\mathfrak{g}$ means a maximal commutative Lie subalgebra which consists of semisimple[213] elements. There are usually many Cartan subalgebras of $\mathfrak{g}$, but they are all conjugate under the action of the corresponding Lie group G (which satisfies $\mathfrak{g}=\operatorname{Lie}G$, and can be defined as the connected component of the identity in the group $\operatorname{Aut}\mathfrak{g}$). Thus, there is no loss of generality in picking one such subalgebra. So pick a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. We denote the dimension $\dim\mathfrak{h}$ by n and also by $\operatorname{rank}\mathfrak{g}$. This dimension $\dim\mathfrak{h}=\operatorname{rank}\mathfrak{g}$ is called the *rank* of $\mathfrak{g}$. The restriction of the Killing form on $\mathfrak{g}$ to $\mathfrak{h}\times\mathfrak{h}$ is a nondegenerate symmetric bilinear form on $\mathfrak{h}$.

For every $\alpha\in\mathfrak{h}^*$, we can define a vector subspace $\mathfrak{g}_\alpha$ of $\mathfrak{g}$ by

$$\mathfrak{g}_\alpha=\{a\in\mathfrak{g}\mid [h,a]=\alpha(h)\,a\text{ for all }h\in\mathfrak{h}\}.$$

It can be shown that $\mathfrak{g}_0=\mathfrak{h}$. Now we let Δ be the finite subset $\{\alpha\in\mathfrak{h}^*\setminus\{0\}\mid \mathfrak{g}_\alpha\neq 0\}$ of $\mathfrak{h}^*\setminus\{0\}$. Then, $\mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$ (as a direct sum of vector spaces). The subset Δ is

[213] An element of a Lie algebra is said to be *semisimple* if and only if its action on the adjoint representation is a semisimple operator.

called the *root system* of $\mathfrak{g}$. The elements of Δ are called the *roots* of $\mathfrak{g}$. It is known that for each $\alpha \in \Delta$, the vector space $\mathfrak{g}_\alpha$ is one-dimensional and can be written as $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$ for some particular $e_\alpha \in \mathfrak{g}_\alpha$.

We want to use the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus\limits_{\alpha \in \Delta} \mathfrak{g}_\alpha$ in order to construct a triangular decomposition of $\mathfrak{g}$. This can be done with the grading which we constructed in Proposition 2.5.6, but let us do it again now, with more elementary means: Fix an $\overline{h} \in \mathfrak{h}$ such that every $\alpha \in \Delta$ satisfies $\alpha\left(\overline{h}\right) \in \mathbb{R} \diagdown \{0\}$ (it can be seen that such $\overline{h}$ exists). Define $\Delta_+ = \left\{\alpha \in \Delta \mid \alpha\left(\overline{h}\right) > 0\right\}$ and $\Delta_- = \left\{\alpha \in \Delta \mid \alpha\left(\overline{h}\right) < 0\right\}$. Then, Δ is the union of two disjoint subsets Δ_+ and Δ_-, and we have $\Delta_+ = -\Delta_-$. The triangular decomposition of $\mathfrak{g}$ is now defined as $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where $\mathfrak{n}_- = \bigoplus\limits_{\alpha \in \Delta_-} \mathfrak{g}_\alpha$ and $\mathfrak{n}_+ = \bigoplus\limits_{\alpha \in \Delta_+} \mathfrak{g}_\alpha$. This decomposition depends on the choice of $\overline{h}$ (and $\mathfrak{h}$, of course). The elements of Δ_+ are called *positive roots* of $\mathfrak{g}$, and the elements of Δ_- are called *negative roots* of $\mathfrak{g}$. If α is a root of $\mathfrak{g}$, then we write $\alpha > 0$ if α is a positive root, and we write $\alpha < 0$ if α is a negative root.

Let us now construct the grading on $\mathfrak{g}$ which yields this triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. This grading was already constructed in Proposition 2.5.6, but now we are going to do this in detail:

We define the *simple roots* of $\mathfrak{g}$ as the elements of Δ_+ which cannot be written as sums of more than one element of Δ_+. It can be shown that there are exactly n of these simple roots, and they form a basis of $\mathfrak{h}^*$. Denote these simple roots as α_1, α_2, ..., α_n. Every root $\alpha \in \Delta_+$ can now be written in the form $\alpha = \sum\limits_{i=1}^{n} k_i(\alpha)\alpha_i$ for a unique n-tuple $(k_1(\alpha), k_2(\alpha), ..., k_n(\alpha))$ of nonnegative integers.

For all $\alpha, \beta \in \Delta$ with $\alpha + \beta \notin \Delta \cup \{0\}$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$. For all $\alpha, \beta \in \mathfrak{h}^*$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. In particular, for every $\alpha \in \mathfrak{h}^*$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subseteq \mathfrak{h}$. Better yet, we can show that for every $\alpha \in \Delta$, there exists some nonzero $h_\alpha \in \mathfrak{h}$ such that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C}h_\alpha$.

For every $i \in \{1, 2, ..., n\}$, pick a generator e_i of the vector space $\mathfrak{g}_{\alpha_i}$ and a generator f_i of the vector space $\mathfrak{g}_{-\alpha_i}$.

It is possible to normalize e_i and f_i in such a way that $[h_i, e_i] = 2e_i$ and $[h_i, f_i] = -2f_i$, where $h_i = [e_i, f_i]$. This h_i will, of course, lie in $\mathfrak{h}$ and be a scalar multiple of h_{α_i}. We can normalize h_{α_i} in such a way that $h_i = h_{\alpha_i}$. We suppose that all these normalizations are done. Then:

Proposition 4.7.1. With the notations introduced above, we have:

(a) The family $(h_1, h_2, ..., h_n)$ is a basis of $\mathfrak{h}$.

(b) For any i and j in $\{1, 2, ..., n\}$, denote $\alpha_j(h_i)$ by $a_{i,j}$. The Lie algebra $\mathfrak{g}$ is generated (as a Lie algebra) by the elements e_i, f_i and h_i with $i \in \{1, 2, ..., n\}$ (a total of $3n$ elements), and the following relations hold:

$$[h_i, h_j] = 0 \qquad \text{for all } i, j \in \{1, 2, ..., n\};$$
$$[h_i, e_j] = \alpha_j(h_i)e_j = a_{i,j}e_j \qquad \text{for all } i, j \in \{1, 2, ..., n\};$$
$$[h_i, f_j] = -\alpha_j(h_i)f_j = a_{i,j}f_j \qquad \text{for all } i, j \in \{1, 2, ..., n\};$$
$$[e_i, f_j] = \delta_{i,j}h_i \qquad \text{for all } i, j \in \{1, 2, ..., n\}.$$

(This does not mean that no more relations hold. In fact, additional relations, the so-called Serre relations, do hold in $\mathfrak{g}$; we will see these relations later, in Theorem 4.7.3.)

The $n \times n$ matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ is called the *Cartan matrix* of $\mathfrak{g}$.

Let $(\cdot, \cdot)$ denote the standard form on $\mathfrak{g}$ (defined in Definition 4.3.16). Then, $(\cdot, \cdot)$ is a nonzero scalar multiple of the Killing form on $\mathfrak{g}$ (since any two nonzero invariant symmetric bilinear forms on $\mathfrak{g}$ are scalar multiples of each other). Hence, the restriction of $(\cdot, \cdot)$ to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate (since the restriction of the Killing form to $\mathfrak{h} \times \mathfrak{h}$ is nondegenerate). Thus, this restriction gives rise to a vector space isomorphism $\mathfrak{h} \to \mathfrak{h}^*$. This isomorphism sends h_i to $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$ for every i (where we denote by $(\cdot, \cdot)$ not only the standard form, but also the inverse form of its restriction to $\mathfrak{h}$). Thus, $a_{i,j} = \alpha_j (h_i) = \dfrac{2 (\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}$ for all i and j. (Note that the latter equality would still hold if $(\cdot, \cdot)$ would mean the Killing form rather than the standard form.)

The elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n are called *Chevalley generators* of $\mathfrak{g}$.

Properties of the matrix A:

1) We have $a_{i,i} = 2$ for all $i \in \{1, 2, ..., n\}$.

2) Any two distinct $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., n\}$ satisfy $a_{i,j} \leq 0$ and $a_{i,j} \in \mathbb{Z}$. Also, $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.

3) The matrix A is indecomposable (i. e., if conjugation of A by a permutation matrix brings A into a block-diagonal form $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, then either A_1 or A_2 is a 0×0 matrix).

4) The matrix A is positive. Here is what we mean by this: There exists a diagonal $n \times n$ matrix D with positive diagonal entries such that DA is a symmetric and positive definite matrix.

Theorem 4.7.2. An $n \times n$ matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ satisfies the four properties **1)**, **2)**, **3)** and **4)** of Proposition 4.7.1 if and only if it is a Cartan matrix of a simple Lie algebra.

Such matrices (and thus, simple finite-dimensional Lie algebras) can be encoded by so-called *Dynkin diagrams*. The *Dynkin diagram* of a simple Lie algebra $\mathfrak{g}$ is defined as the graph with vertex set $\{1, 2, ..., n\}$, and the following rules for drawing edges[214]:

- If $a_{i,j} = 0$, then the vertices i and j are not connected by any edge (directed or undirected).
- If $a_{i,j} = a_{j,i} = -1$, then the vertices i and j are connected by exactly one edge, and this edge is undirected.
- If $a_{i,j} = -2$ and $a_{j,i} = -1$, then the vertices i and j are connected by two directed edges from j to i (and no other edges).

[214]The notion of a graph we are using here is slightly different from the familiar notions of a graph in graph theory, since this graph can have both directed and undirected edges.

- If $a_{i,j} = -3$ and $a_{j,i} = -1$, then the vertices i and j are connected by three directed edges from j to i (and no other edges).

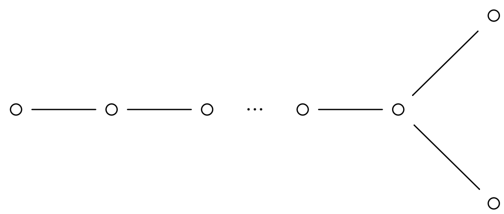
Here is a classification of simple finite-dimensional Lie algebras by their Dynkin diagrams:

$A_n = \mathfrak{sl}(n+1)$ for $n \geq 1$; the Dynkin diagram is $\circ - \circ - \circ \ \cdots \ \circ - \circ - \circ$ (with n nodes).

$B_n = \mathfrak{so}(2n+1)$ for $n \geq 2$; the Dynkin diagram is $\circ - \circ - \circ \ \cdots \ \circ - \circ \Longrightarrow \circ$ (with n nodes, only the last edge being directed and double). (Note that $\mathfrak{so}(3) \cong \mathfrak{sl}(2)$.)

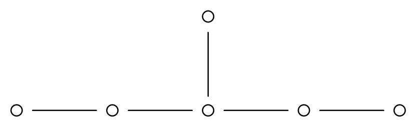
$C_n = \mathfrak{sp}(2n)$ for $n \geq 2$; the Dynkin diagram is $\circ - \circ - \circ \ \cdots \ \circ - \circ \Longleftarrow \circ$ (with n nodes, only the last edge being directed and double). (Note that $\mathfrak{sp}(2) \cong \mathfrak{sl}(2)$ and $\mathfrak{sp}(4) \cong \mathfrak{so}(5)$.)

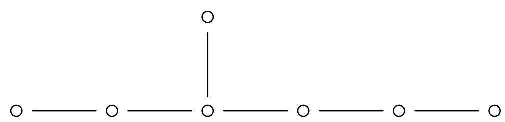
$D_n = \mathfrak{so}(2n)$ for $n \geq 4$; the Dynkin diagram is

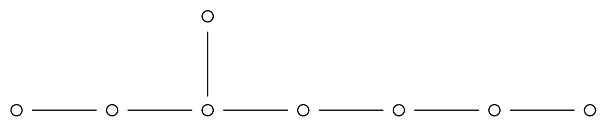


(with n nodes). (Note that $\mathfrak{so}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$ and $\mathfrak{so}(6) \cong \mathfrak{sl}(4)$.)

Exceptional Lie algebras:

E_6; the Dynkin diagram is  .

E_7; the Dynkin diagram is  .

E_8; the Dynkin diagram is  .

F_4; the Dynkin diagram is $\circ - \circ \Longrightarrow \circ - \circ$.

G_2; the Dynkin diagram is $\circ \Lleftarrow \circ$.

Now to the Serre relations, which we have not yet written down:

Theorem 4.7.3. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. Use the notations introduced in Proposition 4.7.1.

(a) Let i and j be two distinct elements of $\{1, 2, ..., n\}$. Then, in $\mathfrak{g}$, we have $(\operatorname{ad}(e_i))^{1-a_{i,j}} e_j = 0$ and $(\operatorname{ad}(f_i))^{1-a_{i,j}} f_j = 0$. These relations (totalling up to $2n(n-1)$ relations, because there are $n(n-1)$ pairs (i,j) of distinct elements of $\{1, 2, ..., n\}$) are called the *Serre relations* for $\mathfrak{g}$.

(b) Combined with the relations

$$\begin{cases} [h_i, h_j] = 0 & \text{for all } i, j \in \{1, 2, ..., n\}; \\ [h_i, e_j] = a_{i,j} e_j & \text{for all } i, j \in \{1, 2, ..., n\}; \\ [h_i, f_j] = -a_{i,j} f_j & \text{for all } i, j \in \{1, 2, ..., n\}; \\ [e_i, f_j] = \delta_{i,j} h_i & \text{for all } i, j \in \{1, 2, ..., n\} \end{cases} \tag{321}$$

of Proposition 4.7.1, the Serre relations form a set of defining relations for $\mathfrak{g}$. This means that, if $\widetilde{\mathfrak{g}}$ denotes the quotient Lie algebra

$$\operatorname{FreeLie}\left(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\}\right) \diagup \left(\text{the relations (321)}\right),$$

then $\widetilde{\mathfrak{g}} \diagup (\text{Serre relations}) \cong \mathfrak{g}$. (Here, $\operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\})$ denotes the free Lie algebra with $3n$ generators e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n.)

Remark 4.7.4. If $\mathfrak{g} \cong \mathfrak{sl}_2$, then $\mathfrak{g}$ has no Serre relations (because $n = 1$), and thus the claim of Theorem 4.7.3 **(b)** rewrites as $\widetilde{\mathfrak{g}} \cong \mathfrak{g}$ (where $\widetilde{\mathfrak{g}}$ is defined as in Theorem 4.7.3). But in all other cases, the Lie algebra $\widetilde{\mathfrak{g}}$ is infinite-dimensional, and while it clearly projects onto $\mathfrak{g}$, it is much bigger than $\mathfrak{g}$.

We will give a partial proof of Theorem 4.7.3: We will only prove part **(a)**.

*Proof of Theorem 4.7.3 **(a)**.* Define a $\mathbb{C}$-linear map

$$\begin{aligned} \Phi_i : \mathfrak{sl}_2 &\to \mathfrak{g}, \\ e &\mapsto e_i, \\ f &\mapsto f_i, \\ h &\mapsto h_i. \end{aligned}$$

Since $[e_i, f_i] = h_i$, $[h_i, e_i] = 2e_i$ and $[h_i, f_i] = -2f_i$, this map Φ_i is a Lie algebra homomorphism.

But $\mathfrak{g}$ is a $\mathfrak{g}$-module (by the adjoint representation of $\mathfrak{g}$), and thus becomes an $\mathfrak{sl}_2$-module by means of $\Phi_i : \mathfrak{sl}_2 \to \mathfrak{g}$. This $\mathfrak{sl}_2$-module satisfies

$$ef_j = \underbrace{(\Phi_i(e))}_{=e_i} f_j = (\operatorname{ad}(e_i)) f_j = [e_i, f_j] = 0 \qquad \text{(since } i \neq j\text{)}$$

and

$$hf_j = \underbrace{(\Phi_i(h))}_{=h_i} f_j = (\operatorname{ad}(h_i)) f_j = [h_i, f_j] = -a_{i,j} f_j.$$

Hence, Lemma 4.6.1 **(c)** (applied to $V = \mathfrak{g}$, $\lambda = -a_{i,j}$ and $x = f_j$) yields that $-a_{i,j} \in \mathbb{N}$ and $f^{-a_{i,j}+1} f_j = 0$. Since

$$f^{-a_{i,j}+1} f_j = f^{1-a_{i,j}} f_j = \left(\underbrace{\Phi_i(f)}_{=f_i} \right)^{1-a_{i,j}} f_j = (\operatorname{ad}(f_i))^{1-a_{i,j}} f_j,$$

this rewrites as $(\operatorname{ad}(f_i))^{1-a_{i,j}} f_j = 0$. Similarly, $(\operatorname{ad}(e_i))^{1-a_{i,j}} e_j = 0$. Theorem 4.7.3 **(a)** is thus proven.

As we said, we are not going to prove Theorem 4.7.3 **(b)** here.

4.8. [unfinished] Kac-Moody Lie algebras: definition and construction

Now forget about our simple Lie algebra $\mathfrak{g}$. Let us first define the notion of contragredient Lie algebras by axioms; we will construct these algebras later.

Definition 4.8.1. Suppose that $A = (a_{i,j})_{1 \leq i,j \leq n}$ is any $n \times n$ matrix of complex numbers.

Let Q be the free abelian group generated by n symbols α_1, α_2, ..., α_n (that is, $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus ... \oplus \mathbb{Z}\alpha_n$). These symbols are just symbols, not weights of any Lie algebra (at the moment). We write the group Q additively.

A *contragredient Lie algebra* corresponding to A is a Q-graded $\mathbb{C}$-Lie algebra $\mathfrak{g}$ which is (as a Lie algebra) generated by some elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n which satisfy the following three conditions:

(1) These elements satisfy the relations (321).

(2) The vector space $\mathfrak{g}[0]$ has $(h_1, h_2, ..., h_n)$ as a $\mathbb{C}$-vector space basis, and we have $\mathfrak{g}[\alpha_i] = \mathbb{C}e_i$ and $\mathfrak{g}[-\alpha_i] = \mathbb{C}f_i$ for all $i \in \{1, 2, ..., n\}$.

(3) Every nonzero Q-graded ideal in $\mathfrak{g}$ has a nonzero intersection with $\mathfrak{g}[0]$.

(Here, we are using the notation $\mathfrak{g}[\alpha]$ for the α-th homogeneous component of the Q-graded Lie algebra $\mathfrak{g}$, just as in Definition 4.6.2.)

Just as in the case of $\mathbb{Z}$-graded Lie algebras, we will denote $\mathfrak{g}[0]$ by $\mathfrak{h}$.

Note that the condition **(3)** is satisfied for simple finite-dimensional Lie algebras $\mathfrak{g}$ (graded by their weight spaces, where Q is the root lattice[215] of $\mathfrak{g}$, and A is the Cartan matrix); hence, simple finite-dimensional Lie algebras (graded by their weight spaces) are contragredient.

Theorem 4.8.2. Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be a (fixed) $n \times n$ matrix of complex numbers.

(a) Then, there exists a unique (up to Q-graded isomorphism respecting the generators e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n) contragredient Lie algebra $\mathfrak{g}$ corresponding to A.

(b) If A is a Cartan matrix, then the contragredient Lie algebra $\mathfrak{g}$ corresponding to A is finite-dimensional and simple.

Definition 4.8.3. Let A be an $n \times n$ matrix of complex numbers. Then, the unique (up to isomorphism) contragredient Lie algebra $\mathfrak{g}$ corresponding to A is denoted by $\mathfrak{g}(A)$.

The proof of Theorem 4.8.2 rests upon the following fact:

Theorem 4.8.4. Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be an $n \times n$ matrix of complex numbers. Let e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n be $3n$ distinct symbols (which are, a priori, new and unrelated to the vectors e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n in Definition 4.8.1). Let $\widetilde{\mathfrak{g}}$ be the quotient Lie algebra

$$\operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\}) \diagup (\text{the relations (321)}).$$

(Here, $\operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\})$ denotes the free Lie algebra with $3n$ generators e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n.)

By abuse of notation, we will denote the projections of the elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n onto the quotient Lie algebra $\widetilde{\mathfrak{g}}$ by the same letters e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n.

[215] in the meaning which this word has in the theory of simple Lie algebras

Let Q be the free abelian group generated by n symbols α_1, α_2, ..., α_n (that is, $Q = \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus ... \oplus \mathbb{Z}\alpha_n$). These symbols are just symbols, not weights of any Lie algebra (at the moment).

(a) We can make $\widetilde{\mathfrak{g}}$ uniquely into a Q-graded Lie algebra by setting

$$\deg(e_i) = \alpha_i, \qquad \deg(f_i) = -\alpha_i \qquad \text{and } \deg(h_i) = 0 \qquad \text{for all } i \in \{1, 2, ..., n\}.$$

(b) Let $\widetilde{\mathfrak{n}}_+ = \operatorname{FreeLie}(e_i \mid i \in \{1, 2, ..., n\})$ (this means the free Lie algebra with n generators e_1, e_2, ..., e_n).

Let $\widetilde{\mathfrak{n}}_- = \operatorname{FreeLie}(f_i \mid i \in \{1, 2, ..., n\})$ (this means the free Lie algebra with n generators f_1, f_2, ..., f_n).

Let $\widetilde{\mathfrak{h}}$ be the free vector space with basis $h_1, h_2, ..., h_n$. Consider $\widetilde{\mathfrak{h}}$ as an abelian Lie algebra.

Then, we have well-defined canonical Lie algebra homomorphisms $\iota_+ : \widetilde{\mathfrak{n}}_+ \to \widetilde{\mathfrak{g}}$ and $\iota_- : \widetilde{\mathfrak{n}}_- \to \widetilde{\mathfrak{g}}$ given by sending the generators e_1, e_2, ..., e_n (in the case of ι_+), respectively, f_1, f_2, ..., f_n (in the case of ι_-) to the corresponding generators e_1, e_2, ..., e_n (in the case of ι_+), respectively, f_1, f_2, ..., f_n (in the case of ι_-). Moreover, we have a well-defined linear map $\iota_0 : \widetilde{\mathfrak{h}} \to \widetilde{\mathfrak{g}}$ given by sending the generators h_1, h_2, ..., h_n to h_1, h_2, ..., h_n, respectively.

These maps ι_+, ι_- and ι_0 are injective Lie algebra homomorphisms.

(c) We have $\widetilde{\mathfrak{g}} = \iota_+(\widetilde{\mathfrak{n}}_+) \oplus \iota_-(\widetilde{\mathfrak{n}}_-) \oplus \iota_0\left(\widetilde{\mathfrak{h}}\right)$.

(d) Both $\iota_+(\widetilde{\mathfrak{n}}_+) \oplus \iota_0\left(\widetilde{\mathfrak{h}}\right)$ and $\iota_-(\widetilde{\mathfrak{n}}_-) \oplus \iota_0\left(\widetilde{\mathfrak{h}}\right)$ are Lie subalgebras of $\widetilde{\mathfrak{g}}$.

(e) The 0-th homogeneous component of $\widetilde{\mathfrak{g}}$ (in the Q-grading) is $\iota_0\left(\widetilde{\mathfrak{h}}\right)$. That is, $\widetilde{\mathfrak{g}}[0] = \iota_0\left(\widetilde{\mathfrak{h}}\right)$. Moreover,

$$\bigoplus_{\substack{\alpha \text{ is a } \mathbb{Z}\text{-linear combination} \\ \text{of } \alpha_1, \alpha_2, ..., \alpha_n \text{ with nonnegative} \\ \text{coefficients; } \alpha \neq 0}} \widetilde{\mathfrak{g}}[\alpha] = \iota_+(\widetilde{\mathfrak{n}}_+)$$

and

$$\bigoplus_{\substack{\alpha \text{ is a } \mathbb{Z}\text{-linear combination} \\ \text{of } \alpha_1, \alpha_2, ..., \alpha_n \text{ with nonpositive} \\ \text{coefficients; } \alpha \neq 0}} \widetilde{\mathfrak{g}}[\alpha] = \iota_-(\widetilde{\mathfrak{n}}_-).$$

(f) There exists an involutive Lie algebra automorphism of $\widetilde{\mathfrak{g}}$ which sends e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n to f_1, f_2, ..., f_n, e_1, e_2, ..., e_n, $-h_1$, $-h_2$, ..., $-h_n$, respectively.

(g) Every $i \in \{1, 2, ..., n\}$ satisfies $\widetilde{\mathfrak{g}}[\alpha_i] = \mathbb{C}e_i$ and $\widetilde{\mathfrak{g}}[-\alpha_i] = \mathbb{C}f_i$.

(h) Let I be the sum of all Q-graded ideals in $\widetilde{\mathfrak{g}}$ which have zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$. Then, I itself is a Q-graded ideal in $\widetilde{\mathfrak{g}}$ which has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$.

(i) Let $\mathfrak{g} = \widetilde{\mathfrak{g}} / I$. Clearly, $\mathfrak{g}$ is a Q-graded Lie algebra. The projections of the elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n of $\widetilde{\mathfrak{g}}$ on the quotient Lie algebra $\widetilde{\mathfrak{g}} / I = \mathfrak{g}$ will still be denoted by e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n. Then, $\mathfrak{g}$ is a contragredient Lie algebra corresponding to A.

Definition 4.8.5. Let A be an $n \times n$ matrix of complex numbers. Then, the Lie algebra $\widetilde{\mathfrak{g}}$ defined in Theorem 4.8.4 is denoted by $\widetilde{\mathfrak{g}}(A)$.

Proof of Theorem 4.8.4. First of all, for the sake of clarity, let us make a convention: In the following proof, the word "Lie derivation" will always mean "derivation of Lie algebras", whereas the word "derivation" without the word "Lie" directly in front of it will always mean "derivation of algebras". The only exception to this will be the formulation "$\mathfrak{a}$ acts on $\mathfrak{b}$ by derivations" where $\mathfrak{a}$ and $\mathfrak{b}$ are two Lie algebras; this formulation has been defined in Definition 3.2.1 **(a)**.

(f) The relations

$$\begin{cases} [-h_i, -h_j] = 0 & \text{for all } i, j \in \{1, 2, ..., n\} ; \\ [-h_i, f_j] = a_{i,j} f_j & \text{for all } i, j \in \{1, 2, ..., n\} ; \\ [-h_i, e_j] = -a_{i,j} e_j & \text{for all } i, j \in \{1, 2, ..., n\} ; \\ [f_i, e_j] = \delta_{i,j} (-h_i) & \text{for all } i, j \in \{1, 2, ..., n\} \end{cases}$$

are satisfied in $\widetilde{\mathfrak{g}}$ (since they are easily seen to be equivalent to the relations (321), and the relations (321) are satisfied in $\widetilde{\mathfrak{g}}$ by the definition of $\widetilde{\mathfrak{g}}$). Hence, we can define a Lie algebra homomorphism

$$\omega : \operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\}) \diagup (\text{the relations } (321)) \to \widetilde{\mathfrak{g}}$$

by requiring

$$\begin{cases} \omega(e_i) = f_i & \text{for every } i \in \{1, 2, ..., n\} ; \\ \omega(f_i) = e_i & \text{for every } i \in \{1, 2, ..., n\} ; \\ \omega(h_i) = -h_i & \text{for every } i \in \{1, 2, ..., n\} \end{cases} .$$

Consider this ω. Since $\operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\}) \diagup (\text{the relations } (321)) = \widetilde{\mathfrak{g}}$, this homomorphism ω is a Lie algebra endomorphism of $\widetilde{\mathfrak{g}}$. It is easy to see that the Lie algebra homomorphisms ω^2 and id are equal on the generators e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n of $\widetilde{\mathfrak{g}}$. Hence, these must be identical, i. e., we have $\omega^2 = \mathrm{id}$. Thus, ω is an involutive Lie algebra automorphism of $\widetilde{\mathfrak{g}}$, and as we know from its definition, it sends e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n to f_1, f_2, ..., f_n, e_1, e_2, ..., e_n, $-h_1$, $-h_2$, ..., $-h_n$, respectively. This proves Theorem 4.8.4 **(f)**.

(a) In order to define a Q-grading on a free Lie algebra, it is enough to choose the degrees of its free generators. Thus, we can define a Q-grading on the Lie algebra $\operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\})$ by setting

$$\deg(e_i) = \alpha_i, \qquad \deg(f_i) = -\alpha_i \qquad \text{and } \deg(h_i) = 0 \qquad \text{for all } i \in \{1, 2, ..., n\} .$$

The relations (321) are homogeneous with respect to this Q-grading; hence, the quotient Lie algebra $\operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\}) \diagup (\text{the relations } (321))$ inherits the Q-grading from $\operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\})$. Since this quotient Lie algebra is $\widetilde{\mathfrak{g}}$, we thus have constructed a Q-grading on $\widetilde{\mathfrak{g}}$ which satisfies

$$\deg(e_i) = \alpha_i, \qquad \deg(f_i) = -\alpha_i \qquad \text{and } \deg(h_i) = 0 \qquad \text{for all } i \in \{1, 2, ..., n\} . \tag{322}$$

Since this grading is clearly the only one to satisfy (322) (because $\widetilde{\mathfrak{g}}$ is generated as a Lie algebra by e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n), this proves Theorem 4.8.4 **(a)**.

(b) *1st step: Definitions and identifications.*

Let N_+ be the free vector space with basis $e_1, e_2, ..., e_n$. Since $\widetilde{\mathfrak{n}}_+ = \operatorname{FreeLie}(e_i \mid i \in \{1, 2, ..., n\})$, we then have a canonical isomorphism $\widetilde{\mathfrak{n}}_+ \cong \operatorname{FreeLie}(N_+)$ (where $\operatorname{FreeLie}(N_+)$ means the free Lie algebra over the vector space (not the set) N_+). We identify $\widetilde{\mathfrak{n}}_+$ with $\operatorname{FreeLie}(N_+)$ along this isomorphism. Due to the construction of the free Lie algebra, we have a canonical injection $N_+ \to \operatorname{FreeLie}(N_+) = \widetilde{\mathfrak{n}}_+$. We will regard this injection as an inclusion (so that $N_+ \subseteq \widetilde{\mathfrak{n}}_+$).

By Proposition 4.6.8 (applied to $V = N_+$), there exists a canonical algebra isomorphism $U(\operatorname{FreeLie}(N_+)) \to T(N_+)$. We identify $U(\widetilde{\mathfrak{n}}_+) = U(\operatorname{FreeLie}(N_+))$ with $T(N_+)$ along this isomorphism.

Let N_- be the free vector space with basis $f_1, f_2, ..., f_n$. Since $\widetilde{\mathfrak{n}}_- = \operatorname{FreeLie}(f_i \mid i \in \{1, 2, ..., n\})$, we then have a canonical isomorphism $\widetilde{\mathfrak{n}}_- \cong \operatorname{FreeLie}(N_-)$ (where $\operatorname{FreeLie}(N_-)$ means the free Lie algebra over the vector space (not the set) N_-). We identify $\widetilde{\mathfrak{n}}_-$ with $\operatorname{FreeLie}(N_-)$ along this isomorphism. Due to the construction of the free Lie algebra, we have a canonical injection $N_- \to \operatorname{FreeLie}(N_-) = \widetilde{\mathfrak{n}}_-$. We will regard this injection as an inclusion (so that $N_- \subseteq \widetilde{\mathfrak{n}}_-$).

By Proposition 4.6.8 (applied to $V = N_-$), there exists a canonical algebra isomorphism $U(\operatorname{FreeLie}(N_-)) \to T(N_-)$. We identify $U(\widetilde{\mathfrak{n}}_-) = U(\operatorname{FreeLie}(N_-))$ with $T(N_-)$ along this isomorphism.

A consequence of the Poincaré-Birkhoff-Witt theorem says that for any Lie algebra $\mathfrak{i}$, the canonical map $\mathfrak{i} \to U(\mathfrak{i})$ is injective. Thus, the canonical map $\widetilde{\mathfrak{n}}_+ \to U(\widetilde{\mathfrak{n}}_+)$ and the canonical map $\widetilde{\mathfrak{n}}_- \to U(\widetilde{\mathfrak{n}}_-)$ are injective. We will therefore regard these maps as inclusions.

Let us identify the group Q with $\mathbb{Z}^n$ by means of identifying α_i with the column vector $e_i = \left(\underbrace{0, 0, ..., 0}_{i-1 \text{ zeroes}}, 1, \underbrace{0, 0, ..., 0}_{n-i \text{ zeroes}}\right)^T$ for every $i \in \{1, 2, ..., n\}$. As a consequence, for every $i \in \{1, 2, ..., n\}$, the row vector $e_i^T A$ is an element of the group $\operatorname{Hom}(Q, \mathbb{C})$ of group homomorphisms from Q to $\mathbb{C}$. Thus, for every $w \in Q$ and every $i \in \{1, 2, ..., n\}$, the product $e_i^T A w$ is a complex number.

2nd step: Defining a Q-grading on $\widetilde{\mathfrak{n}}_-$.

Let us define a Q-grading on the vector space N_- by setting

$$\deg(f_i) = -\alpha_i \qquad \text{for all } i \in \{1, 2, ..., n\}.$$

(This is well-defined since $(f_1, f_2, ..., f_n)$ is a basis of N_-.) Then, the free Lie algebra $\operatorname{FreeLie}(N_-) = \widetilde{\mathfrak{n}}_-$ canonically becomes a Q-graded Lie algebra, and the grading on this Lie algebra also satisfies

$$\deg(f_i) = -\alpha_i \qquad \text{for all } i \in \{1, 2, ..., n\}.$$

(This grading clearly makes the map ι_- graded. We will not use this fact, however.) We will later use this grading to define certain Lie derivations η_1, η_2, ..., η_n of the Lie algebra $\widetilde{\mathfrak{n}}_-$.

3rd step: Defining an $\widetilde{\mathfrak{h}}$-module $\widetilde{\mathfrak{n}}_-$.

For every $i \in \{1, 2, ..., n\}$, let us define a linear map $\eta_i : \widetilde{\mathfrak{n}}_- \to \widetilde{\mathfrak{n}}_-$ by setting

$$\left(\eta_i(x) = \left(e_i^T A w\right) \cdot x \qquad \text{for every } w \in Q \text{ and every } x \in \widetilde{\mathfrak{n}}_-[w]\right). \tag{323}$$

This map η_i is well-defined (because in order to define a linear map from a Q-graded vector space, it is enough to define it linearly on every homogeneous component) and graded (because it multiplies any homogeneous element of $\widetilde{\mathfrak{n}}_-$ by a scalar). Actually, η_i acts as a scalar on each homogeneous component of $\widetilde{\mathfrak{n}}_-$. Moreover, for every $i \in \{1, 2, ..., n\}$, Lemma 4.6.20 (applied to $s = e_i^T A$, $\mathfrak{n} = \widetilde{\mathfrak{n}}_-$ and $\eta = \eta_i$) yields that η_i is a Lie derivation. That is, $\eta_i \in \operatorname{Der}(\widetilde{\mathfrak{n}}_-)$. One can directly see that

$$\eta_i(f_j) = -a_{i,j} f_j \qquad \text{for any } i \in \{1, 2, ..., n\} \text{ and } j \in \{1, 2, ..., n\} \tag{324}$$

[216].

[Note that, while we defined the η_i using the grading, there is also an alternative way to define them, by applying Theorem 4.6.18.]

It is easy to see that

$$[\eta_i, \eta_j] = 0 \qquad \text{for all } i \in \{1, 2, ..., n\} \text{ and } j \in \{1, 2, ..., n\} \tag{325}$$

(since each of the maps η_i and η_j acts as a scalar on each homogeneous component of $\widetilde{\mathfrak{n}}_-$).

Define a linear map $\Xi : \widetilde{\mathfrak{h}} \to \operatorname{Der}(\widetilde{\mathfrak{n}}_-)$ by

$$\left(\Xi(h_i) = \eta_i \qquad \text{for every } i \in \{1, 2, ..., n\}\right)$$

(this map is well-defined, since $(h_1, h_2, ..., h_n)$ is a basis of $\widetilde{\mathfrak{h}}$). Then, Ξ is a Lie algebra homomorphism (this follows from (325)), and thus makes $\widetilde{\mathfrak{n}}_-$ into an $\widetilde{\mathfrak{h}}$-module on which $\widetilde{\mathfrak{h}}$ acts by derivations. Thus, a Lie algebra $\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-$ is well-defined (according to Definition 3.2.1). Both Lie algebras $\widetilde{\mathfrak{h}}$ and $\widetilde{\mathfrak{n}}_-$ canonically inject (by Lie algebra homomorphisms) into this Lie algebra $\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-$. Therefore, both $\widetilde{\mathfrak{h}}$ and $\widetilde{\mathfrak{n}}_-$ will be considered as Lie subalgebras of $\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-$.

In the Lie algebra $\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-$, every $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., n\}$ satisfy

$$\begin{aligned} [h_i, f_j] &= h_i \rightharpoonup f_j \qquad \left(\text{where } \rightharpoonup \text{ denotes the action of } \widetilde{\mathfrak{h}} \text{ on } \widetilde{\mathfrak{n}}_-\right) \\ &= \underbrace{(\Xi(h_i))}_{=\eta_i}(f_j) = \eta_i(f_j) = -a_{i,j} f_j \qquad (\text{by (324)}). \end{aligned} \tag{326}$$

From (321), we see that the same relation is satisfied in the Lie algebra $\widetilde{\mathfrak{g}}$.

Since $\widetilde{\mathfrak{n}}_-$ is a Lie subalgebra of $\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-$, the universal enveloping algebra $U(\widetilde{\mathfrak{n}}_-)$ is a subalgebra of $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$. This makes $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ into a $U(\widetilde{\mathfrak{n}}_-)$-bimodule. Since $U(\widetilde{\mathfrak{n}}_-) = T(N_-)$, this means that $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ is a $T(N_-)$-bimodule.

[216] *Proof of (324):* Let $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., n\}$. By the definition of our grading on $\widetilde{\mathfrak{n}}_-$, we have $\deg(f_j) = -\underbrace{\alpha_j}_{=e_j} = -e_j$, so that $f_j \in \widetilde{\mathfrak{n}}_-[-e_j]$. Hence, (323) (applied to $x = f_j$ and $w = -\alpha_j$) yields $\eta_i(f_j) = \left(e_i^T A(-e_j)\right) \cdot f_j = -\underbrace{\left(e_i^T A e_j\right)}_{=a_{i,j}} \cdot f_j = -a_{i,j} f_j$. This proves (324).

4th step: Defining an action of $\widetilde{\mathfrak{g}}$ on $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$.

We are going to construct an action of the Lie algebra $\widetilde{\mathfrak{g}}$ on $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ (but not by derivations). First, let us define some further maps.

Let $\iota_{N_-}^T : N_- \to T\left(N_-\right)$ be the canonical inclusion map. Notice that we are regarding $\iota_{N_-}^T$ as an inclusion.

For every $i \in \{1, 2, ..., n\}$, let us define a derivation[217] $\varepsilon_i : U\left(\widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ by requiring that

$$\left(\varepsilon_i\left(f_j\right) = \delta_{i,j} h_i \qquad \text{for every } j \in \{1, 2, ..., n\}\right).$$

[218]

Let $\rho : U\left(\widetilde{\mathfrak{n}}_-\right) \otimes U\left(\widetilde{\mathfrak{h}}\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ be the vector space homomorphism defined by

$$\rho\left(\alpha \otimes \beta\right) = \alpha\beta \qquad \text{for all } \alpha \in U\left(\widetilde{\mathfrak{n}}_-\right) \text{ and } \beta \in U\left(\widetilde{\mathfrak{h}}\right)$$

(this is clearly well-defined). Since $\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_- = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ as vector spaces, Corollary 2.4.2 (applied to $k = \mathbb{C}$, $\mathfrak{c} = \widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-$, $\mathfrak{a} = \widetilde{\mathfrak{n}}_-$ and $\mathfrak{b} = \widetilde{\mathfrak{h}}$) yields that ρ is an isomorphism of filtered vector spaces, of left $U\left(\widetilde{\mathfrak{n}}_-\right)$-modules and of right $U\left(\widetilde{\mathfrak{h}}\right)$-modules. Thus, ρ^{-1} also is an isomorphism of filtered vector spaces, of left $U\left(\widetilde{\mathfrak{n}}_-\right)$-modules and of right $U\left(\widetilde{\mathfrak{h}}\right)$-modules.

For every $i \in \{1, 2, ..., n\}$, define a linear map $E_i : U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ by

$$\left(E_i\left(u_- u_0\right) = \varepsilon_i\left(u_-\right) u_0 \qquad \text{for every } u_- \in U\left(\widetilde{\mathfrak{n}}_-\right) \text{ and } u_0 \in U\left(\widetilde{\mathfrak{h}}\right)\right). \tag{327}$$

[219]

[217] Here, by "derivation", we mean a derivation of algebras, not of Lie algebras.

[218] Why is this well-defined? We know that $U\left(\widetilde{\mathfrak{n}}_-\right) = T\left(N_-\right)$. Hence, (by Theorem 4.6.12, applied to $V = N_-$ and $M = U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$) we can lift any linear map $f : N_- \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ to a derivation $U\left(\widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$. Taking f equal to the linear map $N_- \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ which sends every f_j to $\delta_{i,j} h_i$, we obtain a derivation $U\left(\widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ which sends every f_j to $\delta_{i,j} h_i$. This is why ε_i is well-defined.

[219] Why is this well-defined? Since ρ is an isomorphism, we have $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \cong U\left(\widetilde{\mathfrak{n}}_-\right) \otimes U\left(\widetilde{\mathfrak{h}}\right)$. In order to define a linear map $U\left(\widetilde{\mathfrak{n}}_-\right) \otimes U\left(\widetilde{\mathfrak{h}}\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$, we just need to take a bilinear map $U\left(\widetilde{\mathfrak{n}}_-\right) \times U\left(\widetilde{\mathfrak{h}}\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ and apply the universal property of the tensor product. Taking

$$U\left(\widetilde{\mathfrak{n}}_-\right) \times U\left(\widetilde{\mathfrak{h}}\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right), \qquad \left(u_-, u_0\right) \mapsto \varepsilon_i\left(u_-\right) u_0$$

as this bilinear map, we obtain (by the universal property) a linear map $U\left(\widetilde{\mathfrak{n}}_-\right) \otimes U\left(\widetilde{\mathfrak{h}}\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ which sends $u_- \otimes u_0$ to $\varepsilon_i\left(u_-\right) u_0$ for every $u_- \in U\left(\widetilde{\mathfrak{n}}_-\right)$ and $u_0 \in U\left(\widetilde{\mathfrak{h}}\right)$. Composing this map with the isomorphism $\rho^{-1} : U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{n}}_-\right) \otimes U\left(\widetilde{\mathfrak{h}}\right)$, we obtain a linear map $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ which sends $u_- u_0$ to $\varepsilon_i\left(u_-\right) u_0$ for every $u_- \in U\left(\widetilde{\mathfrak{n}}_-\right)$ and $u_0 \in U\left(\widetilde{\mathfrak{h}}\right)$. Therefore, E_i is well-defined.

For every $i \in \{1, 2, ..., n\}$, define a linear map $F_i : U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ by

$$\left(F_i(u) = f_i u \qquad \text{for every } u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right).$$

Clearly, F_i is a right $U\left(\widetilde{\mathfrak{h}}\right)$-module homomorphism.

For every $i \in \{1, 2, ..., n\}$, define a linear map $H_i : U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ by

$$\left(H_i(u) = h_i u \qquad \text{for every } u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right).$$

Clearly, H_i is a right $U\left(\widetilde{\mathfrak{h}}\right)$-module homomorphism.

Our next goal is to prove the relations

$$\begin{cases} [H_i, H_j] = 0 & \text{for all } i, j \in \{1, 2, ..., n\}; \\ [H_i, E_j] = a_{i,j} E_j & \text{for all } i, j \in \{1, 2, ..., n\}; \\ [H_i, F_j] = -a_{i,j} F_j & \text{for all } i, j \in \{1, 2, ..., n\}; \\ [E_i, F_j] = \delta_{i,j} H_i & \text{for all } i, j \in \{1, 2, ..., n\} \end{cases} \tag{328}$$

in $\operatorname{End}\left(U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right)$. Once these relations are proven, it will follow that a Lie algebra homomorphism $\widetilde{\mathfrak{g}} \to \operatorname{End}\left(U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right)$ mapping h_i, e_i, f_i to H_i, E_i, F_i for all i exists (and is unique), and this map will make $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ into a $\widetilde{\mathfrak{g}}$-module. This $\widetilde{\mathfrak{g}}$-module structure will then yield Theorem 4.8.4 **(b)** by a rather simple argument. But we must first verify (328).

5th step: Verifying the relations (328).

We will verify the four relations (328) one after the other:

Proof of the relation $[H_i, H_j] = 0$ for all $i, j \in \{1, 2, ..., n\}$:

Let i and j be two elements of $\{1, 2, ..., n\}$. Every $u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ satisfies

$$\begin{aligned} \underbrace{[H_i, H_j]}_{= H_i \circ H_j - H_j \circ H_i} u &= (H_i \circ H_j - H_j \circ H_i)(u) = H_i \underbrace{(H_j u)}_{\substack{= h_j u \\ \text{(by the definition} \\ \text{of } H_j)}} - H_j \underbrace{(H_i u)}_{\substack{= h_i u \\ \text{(by the definition} \\ \text{of } H_i)}} \\ &= \underbrace{H_i(h_j u)}_{\substack{= h_i(h_j u) \\ \text{(by the definition} \\ \text{of } H_i)}} - \underbrace{H_j(h_i u)}_{\substack{= h_j(h_i u) \\ \text{(by the definition} \\ \text{of } H_j)}} \\ &= h_i(h_j u) - h_j(h_i u) = \underbrace{(h_i h_j - h_j h_i)}_{\substack{= [h_i, h_j] = 0 \\ \text{in } U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \\ \text{(since } [h_i, h_j] = 0 \text{ in } \widetilde{\mathfrak{h}})}} u = 0. \end{aligned}$$

Thus, $[H_i, H_j] = 0$.

Now forget that we fixed i and j. We have thus proven the relation $[H_i, H_j] = 0$ for all $i, j \in \{1, 2, ..., n\}$.

Proof of the relation $[H_i, E_j] = a_{i,j} E_j$ *for all* $i, j \in \{1, 2, ..., n\}$*:*

This will be the most difficult among the four relations that we must prove.

Applying Corollary 4.6.15 to $\widetilde{\mathfrak{h}}$ and $\widetilde{\mathfrak{n}}_-$ instead of $\mathfrak{g}$ and $\mathfrak{h}$, we obtain $\left[\widetilde{\mathfrak{h}}, U\left(\widetilde{\mathfrak{n}}_-\right)\right] \subseteq U\left(\widetilde{\mathfrak{n}}_-\right)$ in $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$.

Let us consider $U\left(\widetilde{\mathfrak{n}}_-\right)$ as $\widetilde{\mathfrak{n}}_-$-module via the adjoint action. Then, $\widetilde{\mathfrak{n}}_- \subseteq U\left(\widetilde{\mathfrak{n}}_-\right)$ as $\widetilde{\mathfrak{n}}_-$-modules.

Let i be any element of $\{1, 2, ..., n\}$. Define a map $\zeta_i : U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ by

$$\left(\zeta_i(u) = [h_i, u] \qquad \text{for every } u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right).$$

Clearly, ζ_i is a derivation of algebras.

Now, using the relation $\left[\widetilde{\mathfrak{h}}, U\left(\widetilde{\mathfrak{n}}_-\right)\right] \subseteq U\left(\widetilde{\mathfrak{n}}_-\right)$, it is easy to check that $\zeta_i\left(U\left(\widetilde{\mathfrak{n}}_-\right)\right) \subseteq U\left(\widetilde{\mathfrak{n}}_-\right)$ [220].

Now, let $j \in \{1, 2, ..., n\}$ be arbitrary. Recall that $\zeta_i : U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ and $\varepsilon_j : U\left(\widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ are derivations satisfying $\zeta_i\left(U\left(\widetilde{\mathfrak{n}}_-\right)\right) \subseteq U\left(\widetilde{\mathfrak{n}}_-\right)$. Thus, Proposition 4.6.22 (applied to $U\left(\widetilde{\mathfrak{n}}_-\right)$, $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$, ε_j and ζ_i instead of A, B, f and g) yields that $\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j : U\left(\widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ is a derivation (of algebras).

On the other hand, $-a_{i,j}\varepsilon_j : U\left(\widetilde{\mathfrak{n}}_-\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ is also a derivation (of algebras), since ε_j is a derivation.

We will now prove that

$$\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j = -a_{i,j}\varepsilon_j. \tag{329}$$

[221] This will bring us very close to the proof of the relation $[H_i, E_j] = a_{i,j} E_j$.

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[220] *Proof.* Every $u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ satisfies

$$\zeta_i(u) = \Bigg[\underbrace{h_i}_{\in \widetilde{\mathfrak{h}}}, \underbrace{u}_{\in U(\widetilde{\mathfrak{n}}_-)}\Bigg] \in \left[\widetilde{\mathfrak{h}}, U\left(\widetilde{\mathfrak{n}}_-\right)\right] \subseteq U\left(\widetilde{\mathfrak{n}}_-\right).$$

In other words, $\zeta_i\left(U\left(\widetilde{\mathfrak{n}}_-\right)\right) \subseteq U\left(\widetilde{\mathfrak{n}}_-\right)$, qed.

[221] Note that the term $\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right)$ in this equality is well-defined because $\left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right)\left(U\left(\widetilde{\mathfrak{n}}_-\right)\right) = \zeta_i\left(U\left(\widetilde{\mathfrak{n}}_-\right)\right) \subseteq U\left(\widetilde{\mathfrak{n}}_-\right)$.

Proof of (329): Every $k \in \{1, 2, ..., n\}$ satisfies

$$\left(\left(\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j\right) \mid_{N_-}\right) (f_k)$$
$$= \varepsilon_j \left(\underbrace{\zeta_i (f_k)}_{\substack{=[h_i, f_k] \\ \text{(by the definition of } \zeta_i)}} \right) - \underbrace{\zeta_i (\varepsilon_j (f_k))}_{\substack{=[h_i, \varepsilon_j (f_k)] \\ \text{(by the definition of } \zeta_i)}}$$
$$= \varepsilon_j \left(\underbrace{[h_i, f_k]}_{\substack{=-a_{i,k} f_k \\ \text{(by (326), applied to} \\ k \text{ instead of } j)}} \right) - \left[h_i, \underbrace{\varepsilon_j (f_k)}_{\substack{=\delta_{j,k} h_j \\ \text{(by the definition of } \varepsilon_j)}} \right]$$
$$= \underbrace{\varepsilon_j (-a_{i,k} f_k)}_{=-a_{i,k} \varepsilon_j (f_k)} - \underbrace{[h_i, \delta_{j,k} h_j]}_{\substack{=0 \\ \text{(since } \widetilde{\mathfrak{h}} \text{ is an abelian Lie algebra)}}} = -a_{i,k} \underbrace{\varepsilon_j (f_k)}_{\substack{=\delta_{j,k} h_j \\ \text{(by the definition of } \varepsilon_j)}}$$
$$= -\underbrace{a_{i,k} \delta_{j,k}}_{=a_{i,j} \delta_{j,k}} h_j = -a_{i,j} \underbrace{\delta_{j,k} h_j}_{\substack{=\varepsilon_j (f_k) \\ \text{(by the definition of } \varepsilon_j)}} = -a_{i,j} \varepsilon_j (f_k) = \left((-a_{i,j} \varepsilon_j) \mid_{N_-}\right) (f_k) .$$

In other words, the maps $\left(\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j\right) \mid_{N_-}$ and $(-a_{i,j} \varepsilon_j) \mid_{N_-}$ are equal to each other on each of the elements f_1, f_2, ..., f_n of N_-. Since $(f_1, f_2, ..., f_n)$ is a basis of N_-, this yields that

$$\left(\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j\right) \mid_{N_-} = (-a_{i,j} \varepsilon_j) \mid_{N_-}$$

(because the maps $\left(\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j\right) \mid_{N_-}$ and $(-a_{i,j} \varepsilon_j) \mid_{N_-}$ are linear). Hence, since the set N_- generates $U(\widetilde{\mathfrak{n}}_-)$ as an algebra (because $U(\widetilde{\mathfrak{n}}_-) = T(N_-)$), Proposition 4.6.13 (applied to $U(\widetilde{\mathfrak{n}}_-)$, N_-, $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$, $\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j$ and $-a_{i,j} \varepsilon_j$ instead of A, S, M, d and e) yields that $\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j = -a_{i,j} \varepsilon_j$ (since $\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j$ and $-a_{i,j} \varepsilon_j$ are derivations). This proves (329).

Now, we will show that

$$[h_i, \varepsilon_j (u_-)] - \varepsilon_j ([h_i, u_-]) = a_{i,j} \varepsilon_j (u_-) \qquad \text{for every } u_- \in U(\widetilde{\mathfrak{n}}_-) . \tag{330}$$

Proof of (330): Let $u_- \in U(\widetilde{\mathfrak{n}}_-)$. Then,

$$\left(\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j\right) (u_-) = \varepsilon_j \left(\underbrace{\left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) (u_-)}_{\substack{=\zeta_i (u_-) = [h_i, u_-] \\ \text{(by the definition of } \zeta_i)}} \right) - \underbrace{\zeta_i (\varepsilon_j (u_-))}_{\substack{=[h_i, \varepsilon_j (u_-)] \\ \text{(by the definition of } \zeta_i)}}$$
$$= \varepsilon_j ([h_i, u_-]) - [h_i, \varepsilon_j (u_-)] .$$

Comparing this with

$$\underbrace{\left(\varepsilon_j \circ \left(\zeta_i \mid_{U(\widetilde{\mathfrak{n}}_-)}\right) - \zeta_i \circ \varepsilon_j\right)}_{\substack{=-a_{i,j} \varepsilon_j \\ \text{(by (329))}}} (u_-) = -a_{i,j} \varepsilon_j (u_-) ,$$

we obtain $-a_{i,j}\varepsilon_j(u_-) = \varepsilon_j([h_i,u_-]) - [h_i,\varepsilon_j(u_-)]$. In other words, $[h_i,\varepsilon_j(u_-)] - \varepsilon_j([h_i,u_-]) = a_{i,j}\varepsilon_j(u_-)$. This proves (330).

Now, let us finally prove that $[H_i,E_j] = a_{i,j}E_j$.

Indeed, let $u_- \in U(\widetilde{\mathfrak{n}}_-)$ and $u_0 \in U\left(\widetilde{\mathfrak{h}}\right)$. Then, $\left[\underbrace{h_i}_{\in\mathfrak{h}}, \underbrace{u_-}_{\in U(\widetilde{\mathfrak{n}}_-)}\right] \in \left[\widetilde{\mathfrak{h}}, U(\widetilde{\mathfrak{n}}_-)\right] \subseteq U(\widetilde{\mathfrak{n}}_-)$. Thus, (327) (applied to $[h_i,u_-]$ and j instead of u_- and i) yields

$$E_j([h_i,u_-]u_0) = \varepsilon_j([h_i,u_-])u_0.$$

On the other hand, $\underbrace{h_i}_{\in\widetilde{\mathfrak{h}}}\underbrace{u_0}_{\in U\left(\widetilde{\mathfrak{h}}\right)} \in \widetilde{\mathfrak{h}}U\left(\widetilde{\mathfrak{h}}\right) \subseteq U\left(\widetilde{\mathfrak{h}}\right)$. Hence, (327) (applied to h_iu_0 and j instead of u_0 and i) yields

$$E_j(u_-h_iu_0) = \varepsilon_j(u_-)h_iu_0.$$

But $\underbrace{h_iu_-}_{=[h_i,u_-]+u_-h_i} u_0 = [h_i,u_-]u_0 + u_-h_iu_0$, so that

$$\begin{aligned} E_j(h_iu_-u_0) &= E_j([h_i,u_-]u_0 + u_-h_iu_0) = \underbrace{E_j([h_i,u_-]u_0)}_{=\varepsilon_j([h_i,u_-])u_0} + \underbrace{E_j(u_-h_iu_0)}_{=\varepsilon_j(u_-)h_iu_0} \\ &= \varepsilon_j([h_i,u_-])u_0 + \varepsilon_j(u_-)h_iu_0. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\underbrace{[H_i,E_j]}_{=H_i\circ E_j - E_j\circ H_i}(u_-u_0) = (H_i\circ E_j - E_j\circ H_i)(u_-u_0) \\ &= H_i\left(\underbrace{E_j(u_-u_0)}_{\substack{=\varepsilon_j(u_-)u_0 \\ \text{(by (327), applied} \\ \text{to } j \text{ instead of } i)}}\right) - E_j\left(\underbrace{H_i(u_-u_0)}_{\substack{=h_iu_-u_0 \\ \text{(by the definition of } H_i)}}\right) \\ &= \underbrace{H_i(\varepsilon_j(u_-)u_0)}_{\substack{=h_i\varepsilon_j(u_-)u_0 \\ \text{(by the definition of } H_i)}} - \underbrace{E_j(h_iu_-u_0)}_{\substack{=\varepsilon_j([h_i,u_-])u_0+\varepsilon_j(u_-)h_iu_0 \\ \text{(as we saw above)}}} \\ &= h_i\varepsilon_j(u_-)u_0 - (\varepsilon_j([h_i,u_-])u_0 + \varepsilon_j(u_-)h_iu_0) = h_i\varepsilon_j(u_-)u_0 - \varepsilon_j(u_-)h_iu_0 - \varepsilon_j([h_i,u_-])u_0 \\ &= \left(\underbrace{h_i\varepsilon_j(u_-) - \varepsilon_j(u_-)h_i}_{=[h_i,\varepsilon_j(u_-)]} - \varepsilon_j([h_i,u_-])\right)u_0 = \underbrace{([h_i,\varepsilon_j(u_-)] - \varepsilon_j([h_i,u_-]))}_{\substack{=a_{i,j}\varepsilon_j(u_-) \\ \text{(by (330))}}}u_0 \\ &= a_{i,j}\underbrace{\varepsilon_j(u_-)u_0}_{\substack{=E_j(u_-u_0) \\ \text{(since (327) (applied to } j \text{ instead of } i) \\ \text{yields } E_j(u_-u_0)=\varepsilon_j(u_-)u_0)}} = a_{i,j}E_j(u_-u_0). \end{aligned}$$

Now, forget that we fixed u_- and u_0. We thus have proven that

$$[H_i, E_j](u_-u_0) = (a_{i,j}E_j)(u_-u_0) \qquad \text{for all } u_- \in U(\widetilde{\mathfrak{n}}_-) \text{ and } u_0 \in U\left(\widetilde{\mathfrak{h}}\right).$$

Since the vector space $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ is generated by products u_-u_0 with $u_- \in U(\widetilde{\mathfrak{n}}_-)$ and $u_0 \in U\left(\widetilde{\mathfrak{h}}\right)$ (this is because $\rho : U(\widetilde{\mathfrak{n}}_-) \otimes U\left(\widetilde{\mathfrak{h}}\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ is an isomorphism), this yields that $[H_i, E_j] = a_{i,j}E_j$.

Now forget that we fixed i and j. We have thus proven the relation $[H_i, E_j] = a_{i,j}E_j$ for all $i, j \in \{1, 2, ..., n\}$.

Proof of the relation $[H_i, F_j] = -a_{i,j}F_j$ for all $i, j \in \{1, 2, ..., n\}$:

The proof of the relation $[H_i, F_j] = -a_{i,j}F_j$ for all $i, j \in \{1, 2, ..., n\}$ is analogous to the above-given proof of the relation $[H_i, H_j] = 0$ for all $i, j \in \{1, 2, ..., n\}$ (except that this time, instead of using the equality $[h_i, h_j] = 0$ in $\widetilde{\mathfrak{h}}$, need to apply the equality $[h_i, f_j] = -a_{i,j}f_j$ in $\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-$; the latter equality is a consequence of (326)).

Proof of the relation $[E_i, F_j] = \delta_{i,j}H_i$ for all $i, j \in \{1, 2, ..., n\}$:

Let i and j be two elements of $\{1, 2, ..., n\}$. Let $u_- \in U(\widetilde{\mathfrak{n}}_-)$ and $u_0 \in U\left(\widetilde{\mathfrak{h}}\right)$. Since $f_j \in \widetilde{\mathfrak{n}}_-$ and $u_- \in U(\widetilde{\mathfrak{n}}_-)$, we have $f_ju_- \in \widetilde{\mathfrak{n}}_- \cdot U(\widetilde{\mathfrak{n}}_-) \subseteq U(\widetilde{\mathfrak{n}}_-)$. Thus, we can apply (327) to f_ju_- instead of u_-, and obtain

$$\begin{aligned} E_i(f_ju_-u_0) &= \underbrace{\varepsilon_i(f_ju_-)}_{\substack{=\varepsilon_i(f_j)u_- + f_j\varepsilon_i(u_-) \\ \text{(since } \varepsilon_i \text{ is a derivation)}}} u_0 = (\varepsilon_i(f_j)u_- + f_j\varepsilon_i(u_-))u_0 \\ &= \underbrace{\varepsilon_i(f_j)}_{\substack{=\delta_{i,j}h_i \\ \text{(by the definition of } \varepsilon_i)}} u_-u_0 + f_j\varepsilon_i(u_-)u_0 = \delta_{i,j}h_iu_-u_0 + f_j\varepsilon_i(u_-)u_0. \end{aligned}$$

But

$$\begin{aligned} \underbrace{[E_i, F_j]}_{=E_i\circ F_j - F_j\circ E_i} (u_-u_0) &= (E_i \circ F_j - F_j \circ E_i)(u_-u_0) \\ &= E_i\left(\underbrace{F_j(u_-u_0)}_{\substack{=f_ju_-u_0 \\ \text{(by the definition of } F_j)}}\right) - F_j\left(\underbrace{E_i(u_-u_0)}_{\substack{=\varepsilon_i(u_-)u_0 \\ \text{(by (327))}}}\right) \\ &= \underbrace{E_i(f_ju_-u_0)}_{\substack{=\delta_{i,j}h_iu_-u_0 + f_j\varepsilon_i(u_-)u_0 \\ \text{(as we have proven above)}}} - \underbrace{F_j(\varepsilon_i(u_-)u_0)}_{\substack{=f_j\varepsilon_i(u_-)u_0 \\ \text{(by the definition of } F_j)}} \\ &= \delta_{i,j}h_iu_-u_0 + f_j\varepsilon_i(u_-)u_0 - f_j\varepsilon_i(u_-)u_0 = \delta_{i,j}h_iu_-u_0 \\ &= \delta_{i,j} \underbrace{h_iu_-u_0}_{\substack{=H_i(u_-u_0) \\ \text{(since the definition of } H_i \text{ yields} \\ H_i(u_-u_0) = h_iu_-u_0)}} = \delta_{i,j}H_i(u_-u_0). \end{aligned}$$

Now, forget that we fixed u_- and u_0. We thus have proven that

$$[E_i, F_j](u_- u_0) = \delta_{i,j} H_i (u_- u_0) \qquad \text{for all } u_- \in U(\widetilde{\mathfrak{n}}_-) \text{ and } u_0 \in U\left(\widetilde{\mathfrak{h}}\right).$$

Since the vector space $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ is generated by products $u_- u_0$ with $u_- \in U(\widetilde{\mathfrak{n}}_-)$ and $u_0 \in U\left(\widetilde{\mathfrak{h}}\right)$ (this is because $\rho : U(\widetilde{\mathfrak{n}}_-) \otimes U\left(\widetilde{\mathfrak{h}}\right) \to U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ is an isomorphism), this yields that $[E_i, F_j] = \delta_{i,j} H_i$.

Now forget that we fixed i and j. We have thus proven the relation $[E_i, F_j] = \delta_{i,j} H_i$ for all $i, j \in \{1, 2, ..., n\}$.

Altogether, we have thus verified all four relations (328). Now, let us define a Lie algebra homomorphism $\xi' : \operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\}) \to \operatorname{End}\left(U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right)$ by the relations

$$\begin{cases} \xi'(e_i) = E_i & \text{for every } i \in \{1, 2, ..., n\}; \\ \xi'(f_i) = F_i & \text{for every } i \in \{1, 2, ..., n\}; \\ \xi'(h_i) = H_i & \text{for every } i \in \{1, 2, ..., n\} \end{cases}.$$

This ξ' is clearly well-defined (because a Lie algebra homomorphism from a free Lie algebra over a set can be defined by arbitrarily choosing its values at the elements of this set). This homomorphism ξ' clearly maps the four relations (321) to the four relations (328). Since we know that the four relations (328) are satisfied in $\operatorname{End}\left(U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right)$, we conclude that the homomorphism ξ' factors through the Lie algebra $\operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\}) \diagup (\text{the relations (321)}) = \widetilde{\mathfrak{g}}$. In other words, there exists a Lie algebra homomorphism $\xi : \widetilde{\mathfrak{g}} \to \operatorname{End}\left(U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right)$ such that

$$\begin{cases} \xi(e_i) = E_i & \text{for every } i \in \{1, 2, ..., n\}; \\ \xi(f_i) = F_i & \text{for every } i \in \{1, 2, ..., n\}; \\ \xi(h_i) = H_i & \text{for every } i \in \{1, 2, ..., n\} \end{cases}.$$

Consider this ξ. Clearly, the Lie algebra homomorphism $\xi : \widetilde{\mathfrak{g}} \to \operatorname{End}\left(U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right)$ makes the vector space $U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$ into a $\widetilde{\mathfrak{g}}$-module.

6th step: Proving the injectivity of ι_-.

We are very close to proving Theorem 4.8.4 **(b)** now.

Let ξ_- be the map $\xi \circ \iota_- : \widetilde{\mathfrak{n}}_- \to \operatorname{End}\left(U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right)$. Then, ξ_- is a Lie algebra homomorphism (since ξ and ι_- are Lie algebra homomorphisms).

Every $i \in \{1, 2, ..., n\}$ satisfies $\underbrace{\xi_-}_{=\xi \circ \iota_-}(f_i) = (\xi \circ \iota_-)(f_i) = \xi\left(\underbrace{\iota_-(f_i)}_{\substack{=f_i \\ \text{(by the definition of } \iota_-)}}\right) = \xi(f_i) = F_i$ (by the definition of ξ).

Let $\mathfrak{s}$ be the subset

$$\left\{ s \in \widetilde{\mathfrak{n}}_- \mid (\xi_-(s))(u) = su \text{ for all } u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right) \right\}$$

of $\widetilde{\mathfrak{n}}_-$. Every $i \in \{1, 2, ..., n\}$ satisfies

$$\underbrace{(\xi_- (f_i))}_{=F_i} (u) = F_i (u) = f_i u \qquad \text{(by the definition of } F_i\text{)}$$

for all $u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$, and therefore

$$f_i \in \left\{ s \in \widetilde{\mathfrak{n}}_- \mid (\xi_- (s)) (u) = su \text{ for all } u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right\} = \mathfrak{s}.$$

In other words, $\mathfrak{s}$ contains the elements f_1, f_2, ..., f_n.

On the other hand, it is very easy to see that $\mathfrak{s}$ is a Lie subalgebra of $\widetilde{\mathfrak{n}}_-$. (In fact, all that is needed to prove this is knowing that ξ_- is a Lie algebra homomorphism. The details are left to the reader.)

But now recall that $\widetilde{\mathfrak{n}}_- = \operatorname{FreeLie}(f_i \mid i \in \{1, 2, ..., n\})$. Hence, the elements f_1, f_2, ..., f_n generate $\widetilde{\mathfrak{n}}_-$ as a Lie algebra. Thus, every Lie subalgebra of $\widetilde{\mathfrak{n}}_-$ which contains the elements f_1, f_2, ..., f_n must be $\widetilde{\mathfrak{n}}_-$ itself. Since we know that $\mathfrak{s}$ is a Lie subalgebra of $\widetilde{\mathfrak{n}}_-$ and contains the elements f_1, f_2, ..., f_n, this yields that $\mathfrak{s}$ must be $\widetilde{\mathfrak{n}}_-$ itself. In other words, $\mathfrak{s} = \widetilde{\mathfrak{n}}_-$.

Now, let $s' \in \widetilde{\mathfrak{n}}_-$ be such that $\iota_- (s') = 0$. Then, $\underbrace{\xi_-}_{=\xi \circ \iota_-} (s') = (\xi \circ \iota_-) (s') = \xi \left(\underbrace{\iota_- (s')}_{=0} \right) = \xi (0) = 0$. But since

$$s' \in \widetilde{\mathfrak{n}}_- = \mathfrak{s} = \left\{ s \in \widetilde{\mathfrak{n}}_- \mid (\xi_- (s)) (u) = su \text{ for all } u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)\right\},$$

we have $(\xi_- (s')) (u) = s'u$ for all $u \in U\left(\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-\right)$. Applied to $u = 1$, this yields $(\xi_- (s')) (1) = s' \cdot 1 = s'$. Compared with $\underbrace{(\xi_- (s'))}_{=0} (1) = 0$, this yields $s' = 0$.

Now forget that we fixed s'. We have thus shown that every $s' \in \widetilde{\mathfrak{n}}_-$ such that $\iota_- (s') = 0$ must satisfy $s' = 0$. In other words, ι_- is injective.

7th step: Proving the injectivity of ι_0.

A similar, but even simpler, argument shows that ι_0 is injective. Again, the reader can fill in the details.

8th step: Proving the injectivity of ι_+.

We have proven the injectivity of the maps ι_- and ι_0 above. The proof of the injectivity of the map ι_+ is analogous to the above proof of the injectivity of the map ι_-. (Alternately, the injectivity of ι_+ can be obtained from that of ι_- using the involutive Lie algebra automorphism constructed in Theorem 4.8.4 **(f)**.)

(c) *1st step: The existence of the direct sum in question.*

Define a relation $\leq$ on Q by positing that two n-tuples $(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{Z}^n$ and $(\mu_1, \mu_2, ..., \mu_n) \in \mathbb{Z}^n$ satisfy $\lambda_1 \alpha_1 + \lambda_2 \alpha_2 + ... + \lambda_n \alpha_n \leq \mu_1 \alpha_1 + \mu_2 \alpha_2 + ... + \mu_n \alpha_n$ if and only if every $i \in \{1, 2, ..., n\}$ satisfies $\lambda_i \leq \mu_i$. It is clear that this relation $\leq$ is a non-strict partial order. Define $\geq$ to be the opposite of $\leq$. Define $>$ and $<$ to be the strict versions of the relations $\geq$ and $\leq$, respectively; thus, any $\alpha \in Q$ and $\beta \in Q$ satisfy $\alpha > \beta$ if and only if ($\alpha \neq \beta$ and $\alpha \geq \beta$).

The elements α of Q satisfying $\alpha > 0$ are exactly the nonzero sums $\lambda_1\alpha_1 + \lambda_2\alpha_2 + ... + \lambda_n\alpha_n$ with λ_1, λ_2, ..., λ_n being nonnegative integers. The elements α of Q satisfying $\alpha < 0$ are exactly the nonzero sums $\lambda_1\alpha_1 + \lambda_2\alpha_2 + ... + \lambda_n\alpha_n$ with λ_1, λ_2, ..., λ_n being nonpositive integers.

Let $\widetilde{\mathfrak{g}}[<0] = \bigoplus\limits_{\substack{\alpha\in Q;\\ \alpha<0}} \widetilde{\mathfrak{g}}[\alpha]$ and $\widetilde{\mathfrak{g}}[>0] = \bigoplus\limits_{\substack{\alpha\in Q;\\ \alpha>0}} \widetilde{\mathfrak{g}}[\alpha]$. Then, $\widetilde{\mathfrak{g}}[0]$, $\widetilde{\mathfrak{g}}[<0]$ and $\widetilde{\mathfrak{g}}[>0]$ are Q-graded Lie subalgebras of $\widetilde{\mathfrak{g}}$ (this is easy to see from the fact that $\widetilde{\mathfrak{g}}$ is a Q-graded Lie algebra).

It is easy to see that the (internal) direct sum $\widetilde{\mathfrak{g}}[>0] \oplus \widetilde{\mathfrak{g}}[<0] \oplus \widetilde{\mathfrak{g}}[0]$ is well-defined.[222]

Every $i \in \{1, 2, ..., n\}$ satisfies

$$\begin{aligned} f_i &\in \widetilde{\mathfrak{g}}[-\alpha_i] && (\text{since } \deg(f_i) = -\alpha_i) \\ &\subseteq \bigoplus_{\substack{\alpha\in Q;\\ \alpha<0}} \widetilde{\mathfrak{g}}[\alpha] && (\text{since } -\alpha_i < 0) \\ &= \widetilde{\mathfrak{g}}[<0]. \end{aligned}$$

Hence, the Lie algebra $\widetilde{\mathfrak{g}}[<0]$ contains the elements f_1, f_2, ..., f_n. But now, recall that $\widetilde{\mathfrak{n}}_- = \operatorname{FreeLie}(f_i \mid i \in \{1, 2, ..., n\})$. Hence, the elements f_1, f_2, ..., f_n of $\widetilde{\mathfrak{n}}_-$ generate $\widetilde{\mathfrak{n}}_-$ as a Lie algebra. Thus, the elements f_1, f_2, ..., f_n of $\widetilde{\mathfrak{g}}$ generate $\iota_-(\widetilde{\mathfrak{n}}_-)$ as a Lie algebra (because the elements f_1, f_2, ..., f_n of $\widetilde{\mathfrak{g}}$ are the images of the elements f_1, f_2, ..., f_n of $\widetilde{\mathfrak{n}}_-$ under the map ι_-). Thus, every Lie subalgebra of $\widetilde{\mathfrak{g}}$ which contains the elements f_1, f_2, ..., f_n must contain $\iota_-(\widetilde{\mathfrak{n}}_-)$ as a subset. Since we know that $\widetilde{\mathfrak{g}}[<0]$ is a Lie subalgebra of $\widetilde{\mathfrak{g}}$ and contains the elements f_1, f_2, ..., f_n, this yields that $\widetilde{\mathfrak{g}}[<0]$ must contain $\iota_-(\widetilde{\mathfrak{n}}_-)$ as a subset. In other words, $\iota_-(\widetilde{\mathfrak{n}}_-) \subseteq \widetilde{\mathfrak{g}}[<0]$. Similarly (by considering the elements e_1, e_2, ..., e_n instead of f_1, f_2, ..., f_n), we can show $\iota_+(\widetilde{\mathfrak{n}}_+) \subseteq \widetilde{\mathfrak{g}}[>0]$.

A similar argument proves $\iota_0\left(\widetilde{\mathfrak{h}}\right) \subseteq \widetilde{\mathfrak{g}}[0]$.

[222] *Proof.* We have $\widetilde{\mathfrak{g}} = \bigoplus\limits_{\alpha\in Q} \widetilde{\mathfrak{g}}[\alpha]$ (since $\widetilde{\mathfrak{g}}$ is Q-graded). But every $\alpha \in Q$ satisfies **exactly one** of the four assertions $\alpha > 0$, $\alpha < 0$, $\alpha = 0$ and (neither $\alpha < 0$ nor $\alpha > 0$ nor $\alpha = 0$). Thus,

$$\begin{aligned} \bigoplus_{\alpha\in Q} \widetilde{\mathfrak{g}}[\alpha] &= \underbrace{\left(\bigoplus_{\substack{\alpha\in Q;\\ \alpha>0}} \widetilde{\mathfrak{g}}[\alpha]\right)}_{=\widetilde{\mathfrak{g}}[>0]} \oplus \underbrace{\left(\bigoplus_{\substack{\alpha\in Q;\\ \alpha<0}} \widetilde{\mathfrak{g}}[\alpha]\right)}_{=\widetilde{\mathfrak{g}}[<0]} \oplus \underbrace{\left(\bigoplus_{\substack{\alpha\in Q;\\ \alpha=0}} \widetilde{\mathfrak{g}}[\alpha]\right)}_{=\widetilde{\mathfrak{g}}[0]} \oplus \left(\bigoplus_{\substack{\alpha\in Q;\\ \text{neither } \alpha<0\\ \text{nor } \alpha>0 \text{ nor } \alpha=0}} \widetilde{\mathfrak{g}}[\alpha]\right) \\ &= \widetilde{\mathfrak{g}}[>0] \oplus \widetilde{\mathfrak{g}}[<0] \oplus \widetilde{\mathfrak{g}}[0] \oplus \left(\bigoplus_{\substack{\alpha\in Q;\\ \text{neither } \alpha<0\\ \text{nor } \alpha>0 \text{ nor } \alpha=0}} \widetilde{\mathfrak{g}}[\alpha]\right). \end{aligned}$$

Thus, the (internal) direct sum $\widetilde{\mathfrak{g}}[>0] \oplus \widetilde{\mathfrak{g}}[<0] \oplus \widetilde{\mathfrak{g}}[0]$ is well-defined (because it is a partial sum of the direct sum $\widetilde{\mathfrak{g}}[>0] \oplus \widetilde{\mathfrak{g}}[<0] \oplus \widetilde{\mathfrak{g}}[0] \oplus \left(\bigoplus\limits_{\substack{\alpha\in Q;\\ \text{neither } \alpha<0\\ \text{nor } \alpha>0 \text{ nor } \alpha=0}} \widetilde{\mathfrak{g}}[\alpha]\right)$).

Since the internal direct sum $\widetilde{\mathfrak{g}}[<0] \oplus \widetilde{\mathfrak{g}}[>0] \oplus \widetilde{\mathfrak{g}}[0]$ is well-defined, the internal direct sum $\iota_+(\widetilde{\mathfrak{n}}_+) \oplus \iota_-(\widetilde{\mathfrak{n}}_-) \oplus \iota_0\left(\widetilde{\mathfrak{h}}\right)$ must also be well-defined (because $\iota_+(\widetilde{\mathfrak{n}}_+) \subseteq \widetilde{\mathfrak{g}}[>0]$, $\iota_-(\widetilde{\mathfrak{n}}_-) \subseteq \widetilde{\mathfrak{g}}[<0]$ and $\iota_0\left(\widetilde{\mathfrak{h}}\right) \subseteq \widetilde{\mathfrak{g}}[0]$). We now must prove that this direct sum is $\widetilde{\mathfrak{g}}$.

2nd step: Identifications.

Since the maps ι_+, ι_- and ι_0 are injective Lie algebra homomorphisms, and since their images are linearly disjoint (because the direct sum $\iota_+(\widetilde{\mathfrak{n}}_+) \oplus \iota_-(\widetilde{\mathfrak{n}}_-) \oplus \iota_0\left(\widetilde{\mathfrak{h}}\right)$ is well-defined), we can regard these maps ι_+, ι_- and ι_0 as inclusions of Lie algebras. Let us do this from now on. Thus, $\widetilde{\mathfrak{n}}_+$, $\widetilde{\mathfrak{n}}_-$ and $\widetilde{\mathfrak{h}}$ are Lie subalgebras of $\widetilde{\mathfrak{g}}$. The identification of $\widetilde{\mathfrak{n}}_-$ with the Lie subalgebra $\iota_-(\widetilde{\mathfrak{n}}_-)$ of $\widetilde{\mathfrak{g}}$ eliminates the need of distinguishing between the elements f_i of $\widetilde{\mathfrak{n}}_-$ and the elements f_i of $\widetilde{\mathfrak{g}}$ (because for every $i \in \{1, 2, ..., n\}$, the element f_i of $\widetilde{\mathfrak{g}}$ is the image of the element f_i of $\widetilde{\mathfrak{n}}_-$ under the map ι_-, and since we regard this map ι_- as inclusion, these two elements f_i are therefore equal). Similarly, we don't have to distinguish between the elements e_i of $\widetilde{\mathfrak{n}}_+$ and the elements e_i of $\widetilde{\mathfrak{g}}$, nor is it necessary to distinguish between the elements h_i of $\widetilde{\mathfrak{h}}$ and the elements h_i of $\widetilde{\mathfrak{g}}$.

Since we regard the maps ι_+, ι_- and ι_0 as inclusions, we have $\iota_+(\widetilde{\mathfrak{n}}_+) = \widetilde{\mathfrak{n}}_+$, $\iota_-(\widetilde{\mathfrak{n}}_-) = \widetilde{\mathfrak{n}}_-$ and $\iota_0\left(\widetilde{\mathfrak{h}}\right) = \widetilde{\mathfrak{h}}$. Hence, $\iota_+(\widetilde{\mathfrak{n}}_+) \oplus \iota_-(\widetilde{\mathfrak{n}}_-) \oplus \iota_0\left(\widetilde{\mathfrak{h}}\right) = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$. This shows that the internal direct sums $\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ and $\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}$ are well-defined (since they are partial sums of the direct sum $\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$).

3rd step: Proving that $\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ is a Lie subalgebra of $\widetilde{\mathfrak{g}}$.

We now will prove part **(d)** of Theorem 4.8.4 before we come back and finish the proof of part **(c)**.

Indeed, let us first prove that $\left[\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{n}}_-\right] \subseteq \widetilde{\mathfrak{n}}_-$.

In fact, in order to prove this, it is enough to show that $[h_i, \widetilde{\mathfrak{n}}_-] \subseteq \widetilde{\mathfrak{n}}_-$ for every $i \in \{1, 2, ..., n\}$ (since the elements h_1, h_2, ..., h_n of $\widetilde{\mathfrak{h}}$ span the vector space $\widetilde{\mathfrak{h}}$). So let $i \in \{1, 2, ..., n\}$. Let $\xi_i : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ be the map defined by

$$\left(\xi_i(x) = [h_i, x] \qquad \text{for any } x \in \widetilde{\mathfrak{g}}\right).$$

Then, ξ_i is a Lie derivation of the Lie algebra $\widetilde{\mathfrak{g}}$. On the other hand, the subset $\{f_1, f_2, ..., f_n\}$ of $\widetilde{\mathfrak{n}}_-$ generates $\widetilde{\mathfrak{n}}_-$ as a Lie algebra (since the elements f_1, f_2, ..., f_n of $\widetilde{\mathfrak{n}}_-$ generate $\widetilde{\mathfrak{n}}_-$ as a Lie algebra), and we can easily check that $\xi_i(\{f_1, f_2, ..., f_n\}) \subseteq \widetilde{\mathfrak{n}}_-$ [223]. Hence, Corollary 4.6.17 (applied to $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{n}}_-$, $\widetilde{\mathfrak{n}}_-$, ξ_i and $\{f_1, f_2, ..., f_n\}$ instead of $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{i}$, d and S) yields that $\xi_i(\widetilde{\mathfrak{n}}_-) \subseteq \widetilde{\mathfrak{n}}_-$. But by the definition of ξ_i, we have $\xi_i(\widetilde{\mathfrak{n}}_-) = [h_i, \widetilde{\mathfrak{n}}_-]$. Hence, $[h_i, \widetilde{\mathfrak{n}}_-] = \xi_i(\widetilde{\mathfrak{n}}_-) \subseteq \widetilde{\mathfrak{n}}_-$. Now forget that we fixed i. We thus have proven that $[h_i, \widetilde{\mathfrak{n}}_-] \subseteq \widetilde{\mathfrak{n}}_-$ for every $i \in \{1, 2, ..., n\}$. As explained above, this yields $\left[\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{n}}_-\right] \subseteq \widetilde{\mathfrak{n}}_-$.

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[223] *Proof.* For every $j \in \{1, 2, ..., n\}$, we have

$$\begin{aligned} \xi_i(f_j) &= [h_i, f_j] && \text{(by the definition of } \xi_i\text{)} \\ &= -a_{i,j} \underbrace{f_j}_{\in \widetilde{\mathfrak{n}}_-} && \text{(by the relations (321))} \\ &\in -a_{i,j} \widetilde{\mathfrak{n}}_- \subseteq \widetilde{\mathfrak{n}}_-. \end{aligned}$$

Thus, $\xi_i(\{f_1, f_2, ..., f_n\}) \subseteq \widetilde{\mathfrak{n}}_-$, qed.

Now, $\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = \widetilde{\mathfrak{n}}_- + \widetilde{\mathfrak{h}}$, so that

$$\begin{aligned}
\left[\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}, \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}\right] &= \left[\widetilde{\mathfrak{n}}_- + \widetilde{\mathfrak{h}}, \widetilde{\mathfrak{n}}_- + \widetilde{\mathfrak{h}}\right] \\
&= \underbrace{\left[\widetilde{\mathfrak{n}}_-, \widetilde{\mathfrak{n}}_-\right]}_{\substack{\subseteq \widetilde{\mathfrak{n}}_- \\ \text{(since } \widetilde{\mathfrak{n}}_- \text{ is a Lie algebra)}}} + \underbrace{\left[\widetilde{\mathfrak{n}}_-, \widetilde{\mathfrak{h}}\right]}_{= -\left[\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{n}}_-\right] \subseteq \left[\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{n}}_-\right] \subseteq \widetilde{\mathfrak{n}}_-} + \underbrace{\left[\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{n}}_-\right]}_{\subseteq \widetilde{\mathfrak{n}}_-} + \underbrace{\left[\widetilde{\mathfrak{h}}, \widetilde{\mathfrak{h}}\right]}_{\substack{\subseteq \widetilde{\mathfrak{h}} \\ \text{(since } \widetilde{\mathfrak{h}} \text{ is a Lie algebra)}}} \\
&\qquad \text{(since the Lie bracket is bilinear)} \\
&\subseteq \underbrace{\widetilde{\mathfrak{n}}_- + \widetilde{\mathfrak{n}}_- + \widetilde{\mathfrak{n}}_-}_{\subseteq \widetilde{\mathfrak{n}}_-} + \widetilde{\mathfrak{h}} \subseteq \widetilde{\mathfrak{n}}_- + \widetilde{\mathfrak{h}} = \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}.
\end{aligned}$$

Thus, $\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ is a Lie subalgebra of $\widetilde{\mathfrak{g}}$.

(Note that the map $(\iota_-, \iota_0) : \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \to \widetilde{\mathfrak{g}}$ is actually a Lie algebra isomorphism from the semidirect product $\widetilde{\mathfrak{h}} \ltimes \widetilde{\mathfrak{n}}_-$ (which was constructed during our proof of Theorem 4.8.4 **(b)**) to $\widetilde{\mathfrak{g}}$. But we will not need this fact, so we will not prove it either.)

So we have shown that $\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ is a Lie subalgebra of $\widetilde{\mathfrak{g}}$. A similar argument (but with $\widetilde{\mathfrak{n}}_-$ replaced by $\widetilde{\mathfrak{n}}_+$, and with f_j replaced by e_j, and with $-a_{i,j}$ replaced by $a_{i,j}$) shows that $\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}$ is a Lie subalgebra of $\widetilde{\mathfrak{g}}$.

We now know that $\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ and $\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}$ are Lie subalgebras of $\widetilde{\mathfrak{g}}$. Since $\widetilde{\mathfrak{n}}_- = \iota_- (\widetilde{\mathfrak{n}}_-)$, $\widetilde{\mathfrak{n}}_+ = \iota_+ (\widetilde{\mathfrak{n}}_+)$ and $\widetilde{\mathfrak{h}} = \iota_0 \left(\widetilde{\mathfrak{h}}\right)$, this rewrites as follows: $\iota_- (\widetilde{\mathfrak{n}}_-) \oplus \iota_0 \left(\widetilde{\mathfrak{h}}\right)$ and $\iota_+ (\widetilde{\mathfrak{n}}_+) \oplus \iota_0 \left(\widetilde{\mathfrak{h}}\right)$ are Lie subalgebras of $\widetilde{\mathfrak{g}}$. This proves Theorem 4.8.4 **(d)**.

4th step: Finishing the proof of Theorem 4.8.4 ***(c)****.*

We know that the internal direct sum $\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ makes sense. Denote this direct sum $\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ as V. We know that V is a vector subspace of $\widetilde{\mathfrak{g}}$. We need to prove that $V = \widetilde{\mathfrak{g}}$.

Let N be the vector subspace of $\widetilde{\mathfrak{g}}$ spanned by the $3n$ elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n. Then, $\widetilde{\mathfrak{g}}$ is generated by N as a Lie algebra (because the elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n generate $\widetilde{\mathfrak{g}}$ as a Lie algebra).

We will now prove that $[N, V] \subseteq V$.

Indeed, since $N = \sum\limits_{i=1}^{n} (e_i \mathbb{C}) + \sum\limits_{i=1}^{n} (f_i \mathbb{C}) + \sum\limits_{i=1}^{n} (h_i \mathbb{C})$ (because N is the vector subspace of $\widetilde{\mathfrak{g}}$ spanned by the $3n$ elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n) and $V = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = \widetilde{\mathfrak{n}}_+ + \widetilde{\mathfrak{n}}_- + \widetilde{\mathfrak{h}}$ (since direct sums are sums), we have

$$\begin{aligned}
[N, V] &= \left[\sum_{i=1}^{n} (e_i \mathbb{C}) + \sum_{i=1}^{n} (f_i \mathbb{C}) + \sum_{i=1}^{n} (h_i \mathbb{C}),\ \widetilde{\mathfrak{n}}_+ + \widetilde{\mathfrak{n}}_- + \widetilde{\mathfrak{h}}\right] \\
&\subseteq \sum_{i=1}^{n} \left[e_i \mathbb{C},\ \widetilde{\mathfrak{n}}_+\right] + \sum_{i=1}^{n} \left[e_i \mathbb{C},\ \widetilde{\mathfrak{n}}_-\right] + \sum_{i=1}^{n} \left[e_i \mathbb{C},\ \widetilde{\mathfrak{h}}\right] \\
&\qquad + \sum_{i=1}^{n} \left[f_i \mathbb{C},\ \widetilde{\mathfrak{n}}_+\right] + \sum_{i=1}^{n} \left[f_i \mathbb{C},\ \widetilde{\mathfrak{n}}_-\right] + \sum_{i=1}^{n} \left[f_i \mathbb{C},\ \widetilde{\mathfrak{h}}\right] \\
&\qquad + \sum_{i=1}^{n} \left[h_i \mathbb{C},\ \widetilde{\mathfrak{n}}_+\right] + \sum_{i=1}^{n} \left[h_i \mathbb{C},\ \widetilde{\mathfrak{n}}_-\right] + \sum_{i=1}^{n} \left[h_i \mathbb{C},\ \widetilde{\mathfrak{h}}\right] \qquad (331)
\end{aligned}$$

(since the Lie bracket is bilinear).

We will now prove that each summand of each of the nine sums on the right hand side of (331) is $\subseteq V$.

Proof that every $i \in \{1, 2, ..., n\}$ satisfies $[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$:

For every $i \in \{1, 2, ..., n\}$, we have $e_i \in \widetilde{\mathfrak{n}}_+$ and thus $e_i\mathbb{C} \subseteq \widetilde{\mathfrak{n}}_+$, so that

$$\begin{aligned} [e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] &\subseteq [\widetilde{\mathfrak{n}}_+,\ \widetilde{\mathfrak{n}}_+] \subseteq \widetilde{\mathfrak{n}}_+ \qquad \text{(since } \widetilde{\mathfrak{n}}_+ \text{ is a Lie algebra)} \\ &\subseteq \widetilde{\mathfrak{n}}_+ + \widetilde{\mathfrak{n}}_- + \widetilde{\mathfrak{h}} = V. \end{aligned}$$

We have thus proven that every $i \in \{1, 2, ..., n\}$ satisfies $[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$.

Proof that every $i \in \{1, 2, ..., n\}$ satisfies $[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq V$:

Let $i \in \{1, 2, ..., n\}$. Define a map $\psi_i : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}$ by

$$\left(\psi_i(x) = [e_i, x] \qquad \text{for every } x \in \widetilde{\mathfrak{g}}\right).$$

Then, ψ_i is a Lie derivation of the Lie algebra $\widetilde{\mathfrak{g}}$. On the other hand, the subset $\{f_1, f_2, ..., f_n\}$ of $\widetilde{\mathfrak{n}}_-$ generates $\widetilde{\mathfrak{n}}_-$ as a Lie algebra (since the elements f_1, f_2, ..., f_n of $\widetilde{\mathfrak{n}}_-$ generate $\widetilde{\mathfrak{n}}_-$ as a Lie algebra), and we can easily check that $\psi_i(\{f_1, f_2, ..., f_n\}) \subseteq \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ [224]. Hence, Corollary 4.6.17 (applied to $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$, $\widetilde{\mathfrak{n}}_-$, ψ_i and $\{f_1, f_2, ..., f_n\}$ instead of $\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{i}$, d and S) yields that $\psi_i(\widetilde{\mathfrak{n}}_-) \subseteq \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ (since $\widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$ is a Lie subalgebra of $\widetilde{\mathfrak{g}}$). But by the definition of ψ_i, we have

$$\begin{aligned} \psi_i(\widetilde{\mathfrak{n}}_-) &= [e_i, \widetilde{\mathfrak{n}}_-] = [e_i, \widetilde{\mathfrak{n}}_-]\,\mathbb{C} \qquad \left(\text{since } [e_i, \widetilde{\mathfrak{n}}_-] \text{ is a vector space}\right) \\ &= [e_i\mathbb{C}, \widetilde{\mathfrak{n}}_-] \qquad \text{(since the Lie bracket is bilinear)}. \end{aligned}$$

Thus, $[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} \subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = V$. Now, forget that we fixed i. We thus have shown that $[e_i\mathbb{C}, \widetilde{\mathfrak{n}}_-] \subseteq V$ for every $i \in \{1, 2, ..., n\}$.

Proof that every $i \in \{1, 2, ..., n\}$ satisfies $\left[e_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right] \subseteq V$:

Every $i \in \{1, 2, ..., n\}$ satisfies $e_i\mathbb{C} \subseteq \widetilde{\mathfrak{n}}_+$ (since $e_i \in \widetilde{\mathfrak{n}}_+$). Thus, every $i \in \{1, 2, ..., n\}$ satisfies

$$\begin{aligned} \left[\underbrace{e_i\mathbb{C}}_{\subseteq \widetilde{\mathfrak{n}}_+ \subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}},\ \underbrace{\widetilde{\mathfrak{h}}}_{\subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}}\right] &\subseteq \left[\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}},\ \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}\right] \\ &\subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \qquad \left(\text{since } \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \text{ is a Lie subalgebra of } \widetilde{\mathfrak{g}}\right) \\ &\subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = V. \end{aligned}$$

Proof that every $i \in \{1, 2, ..., n\}$ satisfies $[f_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$:

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[224] *Proof.* For every $j \in \{1, 2, ..., n\}$, we have

$$\begin{aligned} \psi_i(f_j) &= [e_i, f_j] \qquad \text{(by the definition of } \psi_i\text{)} \\ &= \delta_{i,j} \underbrace{h_i}_{\in \widetilde{\mathfrak{h}}} \qquad \text{(by the relations (321))} \\ &\in \widetilde{\mathfrak{h}} \subseteq \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}. \end{aligned}$$

Thus, $\psi_i(\{f_1, f_2, ..., f_n\}) \subseteq \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}}$, qed.

We have proven above that every $i \in \{1,2,...,n\}$ satisfies $[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq V$. An analogous argument (or an invocation of the automorphism guaranteed by Theorem 4.8.4 **(f)**) shows that every $i \in \{1,2,...,n\}$ satisfies $[f_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$.

Proof that every $i \in \{1,2,...,n\}$ *satisfies* $[f_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq V$*:*

We have proven above that every $i \in \{1,2,...,n\}$ satisfies $[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$. A similar argument (but with $\widetilde{\mathfrak{n}}_+$ replaced by $\widetilde{\mathfrak{n}}_-$, and with e_i replaced by f_i) shows that every $i \in \{1,2,...,n\}$ satisfies $[f_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq V$.

Proof that every $i \in \{1,2,...,n\}$ *satisfies* $\left[f_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right] \subseteq V$*:*

We have proven above that every $i \in \{1,2,...,n\}$ satisfies $\left[e_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right] \subseteq V$. A similar argument (but with $\widetilde{\mathfrak{n}}_+$ replaced by $\widetilde{\mathfrak{n}}_-$, and with e_i replaced by f_i) shows that every $i \in \{1,2,...,n\}$ satisfies $\left[f_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right] \subseteq V$.

Proof that every $i \in \{1,2,...,n\}$ *satisfies* $[h_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$*:*

Every $i \in \{1,2,...,n\}$ satisfies $h_i\mathbb{C} \subseteq \widetilde{\mathfrak{h}}$ (since $h_i \in \widetilde{\mathfrak{h}}$). Thus, every $i \in \{1,2,...,n\}$ satisfies

$$\begin{aligned}
\left[\underbrace{h_i\mathbb{C}}_{\subseteq \widetilde{\mathfrak{n}}_+ \subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}},\ \underbrace{\widetilde{\mathfrak{n}}_+}_{\subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}}\right] &\subseteq \left[\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}},\ \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}\right] \\
&\subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \qquad \left(\text{since } \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \text{ is a Lie subalgebra of } \widetilde{\mathfrak{g}}\right) \\
&\subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = V.
\end{aligned}$$

Proof that every $i \in \{1,2,...,n\}$ *satisfies* $[h_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq V$*:*

We have proven above that every $i \in \{1,2,...,n\}$ satisfies $[h_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$. A similar argument (but with $\widetilde{\mathfrak{n}}_+$ replaced by $\widetilde{\mathfrak{n}}_-$) shows that every $i \in \{1,2,...,n\}$ satisfies $[h_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq V$.

Proof that every $i \in \{1,2,...,n\}$ *satisfies* $\left[h_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right] \subseteq V$*:*

Every $i \in \{1,2,...,n\}$ satisfies $h_i\mathbb{C} \subseteq \widetilde{\mathfrak{h}}$ (since $h_i \in \widetilde{\mathfrak{h}}$). Thus, every $i \in \{1,2,...,n\}$ satisfies

$$\begin{aligned}
\left[\underbrace{h_i\mathbb{C}}_{\subseteq \widetilde{\mathfrak{n}}_+ \subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}},\ \underbrace{\widetilde{\mathfrak{h}}}_{\subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}}\right] &\subseteq \left[\widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}},\ \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}}\right] \\
&\subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \qquad \left(\text{since } \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{h}} \text{ is a Lie subalgebra of } \widetilde{\mathfrak{g}}\right) \\
&\subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = V.
\end{aligned}$$

Thus, we have proven that every $i \in \{1,2,...,n\}$ satisfies the nine relations $[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$, $[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq V$, $\left[e_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right] \subseteq V$, $[f_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$, $[f_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq V$, $\left[f_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right] \subseteq V$,

$[h_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+] \subseteq V$, $[h_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-] \subseteq V$, and $\left[h_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right] \subseteq V$. Thus, (331) becomes

$$\begin{aligned}
[N,V] \subseteq & \sum_{i=1}^n \underbrace{[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+]}_{\subseteq V} + \sum_{i=1}^n \underbrace{[e_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-]}_{\subseteq V} + \sum_{i=1}^n \underbrace{\left[e_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right]}_{\subseteq V} \\
& + \sum_{i=1}^n \underbrace{[f_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+]}_{\subseteq V} + \sum_{i=1}^n \underbrace{[f_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-]}_{\subseteq V} + \sum_{i=1}^n \underbrace{\left[f_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right]}_{\subseteq V} \\
& + \sum_{i=1}^n \underbrace{[h_i\mathbb{C},\ \widetilde{\mathfrak{n}}_+]}_{\subseteq V} + \sum_{i=1}^n \underbrace{[h_i\mathbb{C},\ \widetilde{\mathfrak{n}}_-]}_{\subseteq V} + \sum_{i=1}^n \underbrace{\left[h_i\mathbb{C},\ \widetilde{\mathfrak{h}}\right]}_{\subseteq V} \\
\subseteq & \sum_{i=1}^n V + \sum_{i=1}^n V + \sum_{i=1}^n V + \sum_{i=1}^n V + \sum_{i=1}^n V + \sum_{i=1}^n V + \sum_{i=1}^n V + \sum_{i=1}^n V + \sum_{i=1}^n V \\
\subseteq & V
\end{aligned}$$

(since V is a vector space). This proves $[N,V] \subseteq V$.

Moreover,

$$\begin{aligned}
N = & \sum_{i=1}^n \underbrace{(e_i\mathbb{C})}_{\substack{\subseteq V \\ (\text{since } e_i \in \widetilde{\mathfrak{n}}_+ \subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = V)}} + \sum_{i=1}^n \underbrace{(f_i\mathbb{C})}_{\substack{\subseteq V \\ (\text{since } f_i \in \widetilde{\mathfrak{n}}_- \subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = V)}} + \sum_{i=1}^n \underbrace{(h_i\mathbb{C})}_{\substack{\subseteq V \\ (\text{since } h_i \in \widetilde{\mathfrak{h}} \subseteq \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = V)}} \\
\subseteq & \sum_{i=1}^n V + \sum_{i=1}^n V + \sum_{i=1}^n V \subseteq V
\end{aligned}$$

(since V is a vector space).

So we know that N and V are vector subspaces of $\widetilde{\mathfrak{g}}$ such that $\widetilde{\mathfrak{g}}$ is generated by N as a Lie algebra and such that $N \subseteq V$ and $[N,V] \subseteq V$. Hence, Lemma 4.6.5 (applied to $\widetilde{\mathfrak{g}}$, N and V instead of $\mathfrak{g}$, T and U) yields $V = \widetilde{\mathfrak{g}}$. Thus, $\widetilde{\mathfrak{g}} = V = \widetilde{\mathfrak{n}}_+ \oplus \widetilde{\mathfrak{n}}_- \oplus \widetilde{\mathfrak{h}} = \iota_+ (\widetilde{\mathfrak{n}}_+) \oplus \iota_- (\widetilde{\mathfrak{n}}_-) \oplus \iota_0 \left(\widetilde{\mathfrak{h}}\right)$ (since $\widetilde{\mathfrak{n}}_- = \iota_- (\widetilde{\mathfrak{n}}_-)$, $\widetilde{\mathfrak{n}}_+ = \iota_+ (\widetilde{\mathfrak{n}}_+)$ and $\widetilde{\mathfrak{h}} = \iota_0 \left(\widetilde{\mathfrak{h}}\right)$). This proves Theorem 4.8.4 **(c)**.

(d) During the proof of Theorem 4.8.4 **(c)**, we have already proven Theorem 4.8.4 **(d)**.

(e) We will use the notations we introduced in our proof of Theorem 4.8.4 **(d)**. During this proof, we have shown that $\iota_+ (\widetilde{\mathfrak{n}}_+) \subseteq \widetilde{\mathfrak{g}}\,[> 0]$, $\iota_- (\widetilde{\mathfrak{n}}_-) \subseteq \widetilde{\mathfrak{g}}\,[< 0]$ and $\iota_0 \left(\widetilde{\mathfrak{h}}\right) \subseteq \widetilde{\mathfrak{g}}\,[0]$. Also, we know that $\widetilde{\mathfrak{g}} = \iota_+ (\widetilde{\mathfrak{n}}_+) \oplus \iota_- (\widetilde{\mathfrak{n}}_-) \oplus \iota_0 \left(\widetilde{\mathfrak{h}}\right)$. Finally, we know that the internal direct sum $\widetilde{\mathfrak{g}}\,[> 0] \oplus \widetilde{\mathfrak{g}}\,[< 0] \oplus \widetilde{\mathfrak{g}}\,[0]$ is well-defined.

Now, a simple fact from linear algebra says the following: If U_1, U_2, U_3, V_1, V_2, V_3 are six vector subspaces of a vector space V satisfying the four relations $U_1 \subseteq V_1$, $U_2 \subseteq V_2$, $U_3 \subseteq V_3$ and $V = U_1 \oplus U_2 \oplus U_3$, and if the internal direct sum $V_1 \oplus V_2 \oplus V_3$ is well-defined, then we must have $U_1 = V_1$, $U_2 = V_2$ and $U_3 = V_3$.

If we apply this fact to $\widetilde{\mathfrak{g}}$, $\iota_+ (\widetilde{\mathfrak{n}}_+)$, $\iota_- (\widetilde{\mathfrak{n}}_-)$, $\iota_0 \left(\widetilde{\mathfrak{h}}\right)$, $\widetilde{\mathfrak{g}}\,[> 0]$, $\widetilde{\mathfrak{g}}\,[< 0]$, $\widetilde{\mathfrak{g}}\,[0]$ instead of V, U_1, U_2, U_3, V_1, V_2, V_3, then we obtain that $\iota_+ (\widetilde{\mathfrak{n}}_+) = \widetilde{\mathfrak{g}}\,[> 0]$, $\iota_- (\widetilde{\mathfrak{n}}_-) = \widetilde{\mathfrak{g}}\,[< 0]$

and $\iota_0\left(\widetilde{\mathfrak{h}}\right) = \widetilde{\mathfrak{g}}[0]$ (because we know that $\iota_+(\widetilde{\mathfrak{n}}_+)$, $\iota_-(\widetilde{\mathfrak{n}}_-)$, $\iota_0\left(\widetilde{\mathfrak{h}}\right)$, $\widetilde{\mathfrak{g}}[>0]$, $\widetilde{\mathfrak{g}}[<0]$, $\widetilde{\mathfrak{g}}[0]$ are six vector subspaces of $\widetilde{\mathfrak{g}}$ satisfying the four relations $\iota_+(\widetilde{\mathfrak{n}}_+) \subseteq \widetilde{\mathfrak{g}}[>0]$, $\iota_-(\widetilde{\mathfrak{n}}_-) \subseteq \widetilde{\mathfrak{g}}[<0]$, $\iota_0\left(\widetilde{\mathfrak{h}}\right) \subseteq \widetilde{\mathfrak{g}}[0]$ and $\widetilde{\mathfrak{g}} = \iota_+(\widetilde{\mathfrak{n}}_+) \oplus \iota_-(\widetilde{\mathfrak{n}}_-) \oplus \iota_0\left(\widetilde{\mathfrak{h}}\right)$, and we know that the internal direct sum $\widetilde{\mathfrak{g}}[>0] \oplus \widetilde{\mathfrak{g}}[<0] \oplus \widetilde{\mathfrak{g}}[0]$ is well-defined).

So we have proven that $\widetilde{\mathfrak{g}}[0] = \iota_0\left(\widetilde{\mathfrak{h}}\right)$. In other words, the 0-th homogeneous component of $\widetilde{\mathfrak{g}}$ (in the Q-grading) is $\iota_0\left(\widetilde{\mathfrak{h}}\right)$.

On the other hand, we have proven that $\iota_+(\widetilde{\mathfrak{n}}_+) = \widetilde{\mathfrak{g}}[>0]$. Thus,

$$\iota_+(\widetilde{\mathfrak{n}}_+) = \widetilde{\mathfrak{g}}[>0] = \bigoplus_{\substack{\alpha\in Q;\\ \alpha>0}} \widetilde{\mathfrak{g}}[\alpha] = \bigoplus_{\substack{\alpha \text{ is a } \mathbb{Z}\text{-linear combination}\\ \text{of } \alpha_1,\ \alpha_2,\ ...,\ \alpha_n \text{ with nonnegative}\\ \text{coefficients; } \alpha\neq 0}} \widetilde{\mathfrak{g}}[\alpha]$$

(since an element $\alpha \in Q$ satisfies $\alpha > 0$ if and only if α is a $\mathbb{Z}$-linear combination of α_1, α_2, ..., α_n with nonnegative coefficients such that $\alpha \neq 0$).

Similarly, $\iota_-(\widetilde{\mathfrak{n}}_-) = \bigoplus_{\substack{\alpha \text{ is a } \mathbb{Z}\text{-linear combination}\\ \text{of } \alpha_1,\ \alpha_2,\ ...,\ \alpha_n \text{ with nonpositive}\\ \text{coefficients; } \alpha\neq 0}} \widetilde{\mathfrak{g}}[\alpha]$.

This completes the proof of Theorem 4.8.4 **(e)**.

(g) Define a $\mathbb{Z}$-linear map $\ell : Q \to \mathbb{Z}$ by

$$(\ell(\alpha_i) = 1 \text{ for every } i \in \{1, 2, ..., n\}).$$

(This is well-defined since Q is a free abelian group with generators α_1, α_2, ..., α_n.) Then, ℓ is a group homomorphism.

We will use the notations we introduced in our proof of Theorem 4.8.4 **(c)**. As shown in the proof of Theorem 4.8.4 **(e)**, we have $\iota_+(\widetilde{\mathfrak{n}}_+) = \widetilde{\mathfrak{g}}[>0]$, $\iota_-(\widetilde{\mathfrak{n}}_-) = \widetilde{\mathfrak{g}}[<0]$ and $\iota_0\left(\widetilde{\mathfrak{h}}\right) = \widetilde{\mathfrak{g}}[0]$.

Just as in the proof of Theorem 4.8.4 **(c)**, we will regard the maps ι_+, ι_- and ι_0 as inclusions. Thus, $\iota_+(\widetilde{\mathfrak{n}}_+) = \widetilde{\mathfrak{n}}_+$, $\iota_-(\widetilde{\mathfrak{n}}_-) = \widetilde{\mathfrak{n}}_-$ and $\iota_0\left(\widetilde{\mathfrak{h}}\right) = \widetilde{\mathfrak{h}}$.

From the proof of Theorem 4.8.4 **(c)**, we know that $\widetilde{\mathfrak{g}}[0]$, $\widetilde{\mathfrak{g}}[<0]$ and $\widetilde{\mathfrak{g}}[>0]$ are Q-graded Lie subalgebras of $\widetilde{\mathfrak{g}}$. Since $\widetilde{\mathfrak{g}}[0] = \iota_0\left(\widetilde{\mathfrak{h}}\right) = \widetilde{\mathfrak{h}}$, $\widetilde{\mathfrak{g}}[<0] = \iota_-(\widetilde{\mathfrak{n}}_-) = \widetilde{\mathfrak{n}}_-$ and $\widetilde{\mathfrak{g}}[>0] = \iota_+(\widetilde{\mathfrak{n}}_+) = \widetilde{\mathfrak{n}}_+$, this rewrites as follows: $\widetilde{\mathfrak{h}}$, $\widetilde{\mathfrak{n}}_-$ and $\widetilde{\mathfrak{n}}_+$ are Q-graded Lie subalgebras of $\widetilde{\mathfrak{g}}$.

Fix $i \in \{1, 2, ..., n\}$. Since $\alpha_i > 0$, the space $\widetilde{\mathfrak{g}}[\alpha_i]$ is an addend in the direct sum $\bigoplus_{\substack{\alpha\in Q;\\ \alpha>0}} \widetilde{\mathfrak{g}}[\alpha]$ (namely, the addend for $\alpha = \alpha_i$). Hence, $\widetilde{\mathfrak{g}}[\alpha_i] \subseteq \bigoplus_{\substack{\alpha\in Q;\\ \alpha>0}} \widetilde{\mathfrak{g}}[\alpha] = \widetilde{\mathfrak{g}}[>0] = \widetilde{\mathfrak{n}}_+$. But since $\widetilde{\mathfrak{n}}_+$ is a Q-graded vector subspace of $\widetilde{\mathfrak{g}}$, we have $\widetilde{\mathfrak{n}}_+[\alpha_i] = (\widetilde{\mathfrak{g}}[\alpha_i]) \cap \widetilde{\mathfrak{n}}_+ = \widetilde{\mathfrak{g}}[\alpha_i]$ (since $\widetilde{\mathfrak{g}}[\alpha_i] \subseteq \widetilde{\mathfrak{n}}_+$).

Now, $\widetilde{\mathfrak{n}}_+$ is a Q-graded Lie algebra, and ℓ is a group homomorphism. Hence, we can apply Proposition 4.6.4 to $\widetilde{\mathfrak{n}}_+$ instead of $\widetilde{\mathfrak{g}}$. Applying Proposition 4.6.4 **(a)** to $\widetilde{\mathfrak{n}}_+$ instead of $\mathfrak{g}$, we see that for every $m \in \mathbb{Z}$, the internal direct sum $\bigoplus_{\substack{\alpha\in Q;\\ \ell(\alpha)=m}} \widetilde{\mathfrak{n}}_+[\alpha]$ is well-defined. Denote this internal direct sum $\bigoplus_{\substack{\alpha\in Q;\\ \ell(\alpha)=m}} \widetilde{\mathfrak{n}}_+[\alpha]$ by $\widetilde{\mathfrak{n}}_{+[m]}$. Applying Proposition

4.6.4 **(b)** to $\widetilde{\mathfrak{n}}_+$ instead of $\mathfrak{g}$, we see that the Lie algebra $\widetilde{\mathfrak{n}}_+$ equipped with the grading $\left(\widetilde{\mathfrak{n}}_{+[m]}\right)_{m\in\mathbb{Z}}$ is a $\mathbb{Z}$-graded Lie algebra.

Let N_+ be the free vector space with basis $e_1, e_2, ..., e_n$. Since $\widetilde{\mathfrak{n}}_+ = \operatorname{FreeLie}(e_i \mid i \in \{1, 2, ..., n\})$, we then have a canonical isomorphism $\widetilde{\mathfrak{n}}_+ \cong \operatorname{FreeLie}(N_+)$ (where $\operatorname{FreeLie}(N_+)$ means the free Lie algebra over the vector space (not the set) N_+). We identify $\widetilde{\mathfrak{n}}_+$ with $\operatorname{FreeLie}(N_+)$ along this isomorphism. Due to the construction of the free Lie algebra, we have a canonical injection $N_+ \to \operatorname{FreeLie}(N_+) = \widetilde{\mathfrak{n}}_+$. We will regard this injection as an inclusion (so that $N_+ \subseteq \widetilde{\mathfrak{n}}_+$).

Since $\widetilde{\mathfrak{n}}_+ = \operatorname{FreeLie}(N_+)$, it is clear that $\widetilde{\mathfrak{n}}_+$ is generated by N_+ as a Lie algebra.

Clearly, $e_j \in \widetilde{\mathfrak{n}}_+[\alpha_j] \subseteq \widetilde{\mathfrak{n}}_{+[1]}$ for every $j \in \{1, 2, ..., n\}$. Thus, $N_+ \subseteq \widetilde{\mathfrak{n}}_{+[1]}$. Combining this with the fact that $\widetilde{\mathfrak{n}}_+$ is generated by N_+ as a Lie algebra, we see that we can apply Theorem 4.6.6 to the Lie algebra $\widetilde{\mathfrak{n}}_+$ (with the $\mathbb{Z}$-grading $\left(\widetilde{\mathfrak{n}}_{+[m]}\right)_{m\in\mathbb{Z}}$, not with the original Q-grading) and N_+ instead of the Lie algebra $\mathfrak{g}$ and T. As a result, we obtain $N_+ = \widetilde{\mathfrak{n}}_{+[1]}$. Since $\widetilde{\mathfrak{g}}[\alpha_i] = \widetilde{\mathfrak{n}}_+[\alpha_i] \subseteq \widetilde{\mathfrak{n}}_{+[1]} = N_+$, we have $\widetilde{\mathfrak{g}}[\alpha_i] = N_+[\alpha_i]$ (since N_+ is a Q-graded subspace of $\widetilde{\mathfrak{g}}$). But $N_+[\alpha_i] = \mathbb{C}e_i$ (this is clear from the fact that N_+ has basis $e_1, e_2, ..., e_n$, and each of the vectors in this basis has a different degree in the Q-grading). Hence, $\widetilde{\mathfrak{g}}[\alpha_i] = N_+[\alpha_i] = \mathbb{C}e_i$. A similar argument (with $-\ell$ taking the role of ℓ) shows that $\widetilde{\mathfrak{g}}[-\alpha_i] = \mathbb{C}f_i$. This proves Theorem 4.8.4 **(g)**.

(h) It is clear that I (being a sum of Q-graded ideals) is a Q-graded ideal. We only need to prove that I has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$.

Let $\pi_0 : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}[0]$ be the canonical projection from the Q-graded vector space $\widetilde{\mathfrak{g}}$ on its 0-th homogeneous component $\widetilde{\mathfrak{g}}[0]$.

For every Q-graded vector subspace M of $\widetilde{\mathfrak{g}}$, we have $\pi_0(M) = M \cap (\widetilde{\mathfrak{g}}[0])$ (this is just an elementary property of Q-graded vector spaces). Since $\widetilde{\mathfrak{g}}[0] = \iota_0\left(\widetilde{\mathfrak{h}}\right)$ (by Theorem 4.8.4 **(e)**), this rewrites as follows: For every Q-graded vector subspace M of $\widetilde{\mathfrak{g}}$, we have $\pi_0(M) = M \cap \iota_0\left(\widetilde{\mathfrak{h}}\right)$. Thus, every Q-graded ideal $\mathfrak{i}$ of $\widetilde{\mathfrak{g}}$ which has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$ satisfies $\pi_0(\mathfrak{i}) = \mathfrak{i} \cap \iota_0\left(\widetilde{\mathfrak{h}}\right) = 0$. Therefore, the sum I of all such ideals also satisfies $\pi_0(I) = 0$ (since π_0 is linear). But since $\pi_0(I) = I \cap \iota_0\left(\widetilde{\mathfrak{h}}\right)$ (because for every Q-graded vector subspace M of $\widetilde{\mathfrak{g}}$, we have $\pi_0(M) = M \cap \iota_0\left(\widetilde{\mathfrak{h}}\right)$), this rewrites as $I \cap \iota_0\left(\widetilde{\mathfrak{h}}\right) = 0$. In other words, I has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$. Theorem 4.8.4 **(h)** is proven.

(i) First, we notice that the Lie algebra $\widetilde{\mathfrak{g}}$ is generated by its elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n (since

$$\widetilde{\mathfrak{g}} = \operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\}) \diagup (\text{the relations (321)})$$

). Hence, the Lie algebra $\mathfrak{g}$ is generated by its elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n as well (since $\mathfrak{g} = \widetilde{\mathfrak{g}} \diagup I$).

In order to prove that $\mathfrak{g}$ is a contragredient Lie algebra corresponding to A, we must prove that it satisfies the conditions **(1)**, **(2)** and **(3)** of Definition 4.8.1.

*Proof of condition **(1)**:* The relations (321) are satisfied in $\widetilde{\mathfrak{g}}$ (by the definition of $\widetilde{\mathfrak{g}}$ as the quotient Lie algebra $\operatorname{FreeLie}(h_i, f_i, e_i) \diagup (\text{the relations (321)})$) and thus also in

$\mathfrak{g}$ (since $\mathfrak{g}$ is a quotient Lie algebra of $\widetilde{\mathfrak{g}}$). This proves condition **(1)** for our Q-graded Lie algebra $\mathfrak{g}$.

Proof of condition (2): By Theorem 4.8.4 **(e)**, we have $\widetilde{\mathfrak{g}}[0] = \iota_0\left(\widetilde{\mathfrak{h}}\right)$. We know that $h_1, h_2, ..., h_n$ is a basis of the vector space $\widetilde{\mathfrak{h}}$ (since $\widetilde{\mathfrak{h}}$ was defined as the free vector space with basis $h_1, h_2, ..., h_n$). Since ι_0 is injective, this yields that $h_1, h_2, ..., h_n$ is a basis of $\iota_0\left(\widetilde{\mathfrak{h}}\right)$ (because we identify the images of the vectors h_1, h_2, ..., h_n under ι_0 with h_1, h_2, ..., h_n). Thus, in particular, the vectors $h_1, h_2, ..., h_n$ in $\widetilde{\mathfrak{g}}$ span the vector space $\iota_0\left(\widetilde{\mathfrak{h}}\right) = \widetilde{\mathfrak{g}}[0]$. As a consequence, the vectors h_1, h_2, ..., h_n in $\mathfrak{g}$ span the vector space $\mathfrak{g}[0]$ (because $\mathfrak{g} = \widetilde{\mathfrak{g}}/I$).

The vectors h_1, h_2, ..., h_n in $\mathfrak{g}$ are linearly independent[225]. Hence, $h_1, h_2, ..., h_n$ is a basis of the vector space $\mathfrak{g}[0]$ (since the vectors h_1, h_2, ..., h_n in $\mathfrak{g}$ span the vector space $\mathfrak{g}[0]$ and are linearly independent). In other words, the vector space $\mathfrak{g}[0]$ has $(h_1, h_2, ..., h_n)$ as a $\mathbb{C}$-vector space basis.

Let $i \in \{1, 2, ..., n\}$. Theorem 4.8.4 **(g)** yields $\widetilde{\mathfrak{g}}[\alpha_i] = \mathbb{C}e_i$. Projecting this onto $\widetilde{\mathfrak{g}}/I = \mathfrak{g}$, we obtain $\mathfrak{g}[\alpha_i] = \mathbb{C}e_i$ (since the projection of e_i onto $\mathfrak{g}$ is also called e_i). Similarly, $\mathfrak{g}[-\alpha_i] = \mathbb{C}f_i$.

Condition **(2)** is thus verified for our Q-graded Lie algebra $\mathfrak{g}$.

Proof of condition (3): Let J be a nonzero Q-graded ideal in $\mathfrak{g}$. Assume that $J \cap (\mathfrak{g}[0]) = 0$.

Recall that I has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$. That is, $I \cap \iota_0\left(\widetilde{\mathfrak{h}}\right) = 0$.

Let $\operatorname{proj} : \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}/I = \mathfrak{g}$ be the canonical projection. Then, proj is a Q-graded Lie algebra homomorphism, so that $\operatorname{proj}^{-1}(J)$ is a Q-graded ideal of $\widetilde{\mathfrak{g}}$ (since J is a Q-graded ideal of $\mathfrak{g}$). Also, $\operatorname{Ker}\operatorname{proj} = I$ (since proj is the canonical projection $\widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}/I$).

Let $x \in \operatorname{proj}^{-1}(J) \cap \iota_0\left(\widetilde{\mathfrak{h}}\right)$. Then, $x \in \operatorname{proj}^{-1}(J)$ and $x \in \iota_0\left(\widetilde{\mathfrak{h}}\right)$. Since $x \in \operatorname{proj}^{-1}(J)$, we have $\operatorname{proj}(x) \in J$. Since $x \in \iota_0\left(\widetilde{\mathfrak{h}}\right) = \widetilde{\mathfrak{g}}[0]$ (by Theorem 4.8.4 **(e)**), we have $\operatorname{proj}(x) \in \mathfrak{g}[0]$ (since proj is Q-graded). Combined with $\operatorname{proj}(x) \in J$, this yields $\operatorname{proj}(x) \in J \cap (\mathfrak{g}[0]) = 0$, so that $\operatorname{proj}(x) = 0$, thus $x \in \operatorname{Ker}\operatorname{proj} = I$. Combined with $x \in \iota_0\left(\widetilde{\mathfrak{h}}\right)$, this yields $x \in I \cap \left(\iota_0\left(\widetilde{\mathfrak{h}}\right)\right) = 0$, so that $x = 0$.

Forget that we fixed x. We thus have proven that every $x \in \operatorname{proj}^{-1}(J) \cap \iota_0\left(\widetilde{\mathfrak{h}}\right)$ satisfies $x = 0$. Hence, $\operatorname{proj}^{-1}(J) \cap \iota_0\left(\widetilde{\mathfrak{h}}\right) = 0$. Thus, $\operatorname{proj}^{-1}(J)$ is a Q-graded ideal in

[225] *Proof.* Let $(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n$ be such that $\lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n = 0$ in $\mathfrak{g}$. Then, $\lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n \in I$ in $\widetilde{\mathfrak{g}}$ (since $\mathfrak{g} = \widetilde{\mathfrak{g}}/I$). Combined with $\lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n \in \widetilde{\mathfrak{g}}[0] = \iota_0\left(\widetilde{\mathfrak{h}}\right)$, this yields $\lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n \in I \cap \iota_0\left(\widetilde{\mathfrak{h}}\right) = 0$ (since Theorem 4.8.4 **(h)** yields that I has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$). Thus, $\lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n = 0$ in $\iota_0\left(\widetilde{\mathfrak{h}}\right)$. Since $h_1, h_2, ..., h_n$ is a basis of $\iota_0\left(\widetilde{\mathfrak{h}}\right)$, this yields $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$.

Now forget that we fixed $(\lambda_1, \lambda_2, ..., \lambda_n)$. We have thus shown that every $(\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{C}^n$ such that $\lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n = 0$ in $\mathfrak{g}$ satisfies $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$. In other words, the vectors h_1, h_2, ..., h_n in $\mathfrak{g}$ are linearly independent, qed.

$\widetilde{\mathfrak{g}}$ which has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$. Hence,

$$\operatorname{proj}^{-1}(J) \subseteq \left(\text{sum of all } Q\text{-graded ideals in } \widetilde{\mathfrak{g}} \text{ which have zero intersection with } \iota_0\left(\widetilde{\mathfrak{h}}\right)\right) = I.$$

Now let $y \in J$ be arbitrary. Since $y \in J \subseteq \mathfrak{g} = \widetilde{\mathfrak{g}} \diagup I$, there exists a $y' \in \widetilde{\mathfrak{g}}$ such that $y = \operatorname{proj}(y')$. Consider this y. Since $\operatorname{proj}(y') = y \in J$, we have $y' \in \operatorname{proj}^{-1}(J) \subseteq I = \operatorname{Ker}\operatorname{proj}$, so that $\operatorname{proj}(y') = 0$. Thus, $y = \operatorname{proj}(y') = 0$. Now, forget that we fixed y. We thus have proven that every $y \in J$ satisfies $y = 0$. Thus, $J = 0$, contradicting to the fact that J is nonzero.

This contradiction shows that our assumption (that $J \cap (\mathfrak{g}[0]) = 0$) was wrong. In other words, $J \cap (\mathfrak{g}[0]) \neq 0$.

Now forget that we fixed J. We thus have proven that every nonzero Q-graded ideal J in $\mathfrak{g}$ satisfies $J \cap (\mathfrak{g}[0]) \neq 0$. In other words, every nonzero Q-graded ideal in $\mathfrak{g}$ has a nonzero intersection with $\mathfrak{g}[0]$. This proves that Condition **(3)** holds for our Q-graded Lie algebra $\mathfrak{g}$.

Now that we have checked all three conditions **(1)**, **(2)** and **(3)** for our Q-graded Lie algebra $\mathfrak{g}$, we conclude that $\mathfrak{g}$ indeed is a contragredient Lie algebra corresponding to A. Theorem 4.8.4 **(i)** is proven.

Proof of Theorem 4.8.2. **(a)** Let the Q-graded Lie algebra $\mathfrak{g}$ be defined as in Theorem 4.8.4. According to Theorem 4.8.4 **(i)**, this $\mathfrak{g}$ is a contragredient Lie algebra corresponding to A. Thus, there exists at least one contragredient Lie algebra corresponding to A, namely this $\mathfrak{g}$. Now, it only remains to prove that it is the only such Lie algebra (up to isomorphism). In other words, it remains to prove that whenever $\mathfrak{g}'$ is a contragredient Lie algebra corresponding to A, then there exists a Q-graded Lie algebra isomorphism $\mathfrak{g} \to \mathfrak{g}'$ which sends the generators e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n of $\mathfrak{g}$ to the respective generators e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n of $\mathfrak{g}'$.

So let $\mathfrak{g}'$ be a contragredient Lie algebra. Then, condition **(1)** of Definition 4.8.1 is satisfied for $\mathfrak{g}'$. Thus, the relations (321) are satisfied in $\mathfrak{g}'$.

Define a Lie algebra homomorphism $\psi : \widetilde{\mathfrak{g}} \to \mathfrak{g}'$ by

$$\begin{cases} \psi(e_i) = e_i & \text{for every } i \in \{1, 2, ..., n\}; \\ \psi(f_i) = f_i & \text{for every } i \in \{1, 2, ..., n\}; \\ \psi(h_i) = h_i & \text{for every } i \in \{1, 2, ..., n\} \end{cases}.$$

This ψ is well-defined because the relations (321) are satisfied in $\mathfrak{g}'$ (and because $\widetilde{\mathfrak{g}} = \operatorname{FreeLie}(h_i, f_i, e_i \mid i \in \{1, 2, ..., n\}) \diagup (\text{the relations (321)})$).

Since the Lie algebra $\mathfrak{g}'$ is generated by its elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n (by the definition of a contragredient Lie algebra), the homomorphism ψ is surjective (since all of the elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n clearly lie in the image of ψ).

Since $\mathfrak{g}'$ is a contragredient Lie algebra, the condition **(2)** of Definition 4.8.1 is satisfied for $\mathfrak{g}'$. In other words, the vector space $\mathfrak{g}'[0]$ has $(h_1, h_2, ..., h_n)$ as a $\mathbb{C}$-vector space basis, and we have $\mathfrak{g}'[\alpha_i] = \mathbb{C}e_i$ and $\mathfrak{g}'[-\alpha_i] = \mathbb{C}f_i$ for all $i \in \{1, 2, ..., n\}$. This yields that the elements e_i, f_i and h_i of $\mathfrak{g}'$ satisfy

$$\deg(e_i) = \alpha_i, \qquad \deg(f_i) = -\alpha_i \qquad \text{and } \deg(h_i) = 0 \qquad \text{for all } i \in \{1, 2, ..., n\}.$$

Of course, the elements e_i, f_i and h_i of $\widetilde{\mathfrak{g}}$ satisfy the same relations (because of the definition of the Q-grading on $\widetilde{\mathfrak{g}}$). As a consequence, it is easy to see that Lie algebra homomorphism ψ is Q-graded[226]. As a consequence, $\operatorname{Ker}\psi$ is a Q-graded Lie ideal of $\widetilde{\mathfrak{g}}$.

Define $\widetilde{\mathfrak{h}}$, I and ι_0 as in Theorem 4.8.4. Then, $\widetilde{\mathfrak{h}}$ is the free vector space with basis $h_1, h_2, ..., h_n$. Thus, the vector space $\widetilde{\mathfrak{h}}$ is spanned by $h_1, h_2, ..., h_n$. As a consequence, the vector space $\iota_0\left(\widetilde{\mathfrak{h}}\right)$ is spanned by $h_1, h_2, ..., h_n$ (since ι_0 maps the elements $h_1, h_2, ..., h_n$ of $\widetilde{\mathfrak{h}}$ to the elements $h_1, h_2, ..., h_n$ of $\widetilde{\mathfrak{g}}$). Now, it is easy to see that $(\operatorname{Ker}\psi) \cap \iota_0\left(\widetilde{\mathfrak{h}}\right) = 0$ [227]. Hence, $\operatorname{Ker}\psi$ is a Q-graded Lie ideal of $\widetilde{\mathfrak{g}}$ which has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$.

But I is the sum of all Q-graded ideals in $\widetilde{\mathfrak{g}}$ which have zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$. Thus, every Q-graded ideal of $\widetilde{\mathfrak{g}}$ which has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$ must be a subset of I. Since $\operatorname{Ker}\psi$ is a Q-graded Lie ideal of $\widetilde{\mathfrak{g}}$ which has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$, this yields that $\operatorname{Ker}\psi \subseteq I$.

We will now prove the reverse inclusion, i. e., we will show that $I \subseteq \operatorname{Ker}\psi$.

We know that I is Q-graded (by Theorem 4.8.4 **(h)**). Since ψ is Q-graded, this yields that $\psi(I)$ is a Q-graded vector subspace of $\mathfrak{g}'$. On the other hand, since I is Q-graded, we have $I[0] = I \cap \underbrace{\left(\widetilde{\mathfrak{g}}[0]\right)}_{\substack{=\iota_0\left(\widetilde{\mathfrak{h}}\right) \\ \text{(by Theorem 4.8.4 (e))}}} = I \cap \iota_0\left(\widetilde{\mathfrak{h}}\right) = 0$ (since Theorem 4.8.4 **(h)** yields that I has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$).

Since $\mathfrak{g}'$ is a contragredient Lie algebra, the condition **(3)** of Definition 4.8.1 is satisfied for $\mathfrak{g}'$. In other words, every nonzero Q-graded ideal in $\mathfrak{g}'$ has a nonzero

[226] *Proof.* Let T be the vector subspace of $\widetilde{\mathfrak{g}}$ spanned by the elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n. Then, $\widetilde{\mathfrak{g}}$ is generated by T as a Lie algebra (because $\widetilde{\mathfrak{g}}$ is generated by the elements e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n as a Lie algebra). Due to the relations

$$\deg(e_i) = \alpha_i, \qquad \deg(f_i) = -\alpha_i \qquad \text{and } \deg(h_i) = 0 \qquad \text{for all } i \in \{1, 2, ..., n\}$$

holding both in $\widetilde{\mathfrak{g}}$ and in $\mathfrak{g}'$, it is clear that the map $\psi\mid_T$ is Q-graded. Proposition 4.6.7 (applied to $\widetilde{\mathfrak{g}}$, $\mathfrak{g}'$ and ψ instead of $\mathfrak{g}$, $\mathfrak{h}$ and f) now yields that ψ is Q-graded, qed.

[227] *Proof.* Let $x \in (\operatorname{Ker}\psi) \cap \iota_0\left(\widetilde{\mathfrak{h}}\right)$. Then, $x \in \operatorname{Ker}\psi$ and $x \in \iota_0\left(\widetilde{\mathfrak{h}}\right)$. Since $x \in \iota_0\left(\widetilde{\mathfrak{h}}\right)$, there exist some elements λ_1, λ_2, ..., λ_n of $\mathbb{C}$ such that $x = \lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n$ (since the vector space $\iota_0\left(\widetilde{\mathfrak{h}}\right)$ is spanned by $h_1, h_2, ..., h_n$). Consider these λ_1, λ_2, ..., λ_n. Since $x \in \operatorname{Ker}\psi$, we have $\psi(x) = 0$, so that

$$\begin{aligned} 0 = \psi(x) &= \psi(\lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n) = \lambda_1 \psi(h_1) + \lambda_2 \psi(h_2) + ... + \lambda_n \psi(h_n) \\ &= \lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n \qquad (\text{since } \psi(h_i) = h_i \text{ for every } i \in \{1, 2, ..., n\}) \end{aligned}$$

in $\mathfrak{g}'$. But since the elements h_1, h_2, ..., h_n of $\mathfrak{g}'$ are linearly independent (because the vector space $\mathfrak{g}'[0]$ has $(h_1, h_2, ..., h_n)$ as a $\mathbb{C}$-vector space basis), this yields that $\lambda_1 = \lambda_2 = ... = \lambda_n = 0$. Thus, $x = \lambda_1 h_1 + \lambda_2 h_2 + ... + \lambda_n h_n$ becomes $x = 0h_1 + 0h_2 + ... + 0h_n = 0$.

Now forget that we fixed x. We thus have seen that every $x \in (\operatorname{Ker}\psi) \cap \iota_0\left(\widetilde{\mathfrak{h}}\right)$ satisfies $x = 0$. In other words, $(\operatorname{Ker}\psi) \cap \iota_0\left(\widetilde{\mathfrak{h}}\right) = 0$, qed.

intersection with $\mathfrak{g}'[0]$. Since I is an ideal of $\widetilde{\mathfrak{g}}$, the image $\psi(I)$ is an ideal of $\mathfrak{g}'$ (because ψ is a surjective homomorphism of Lie algebras, and because the image of an ideal under a **surjective** homomorphism of Lie algebras must always be an ideal of the target Lie algebra). Assume that $\psi(I) \neq 0$. Clearly, $\psi(I)$ is Q-graded (since I is Q-graded (by Theorem 4.8.4 **(h)**) and since ψ is Q-graded). Thus, $\psi(I)$ is a nonzero Q-graded ideal in $\mathfrak{g}'$. Thus, $\psi(I)$ has a nonzero intersection with $\mathfrak{g}'[0]$ (because every nonzero Q-graded ideal in $\mathfrak{g}'$ has a nonzero intersection with $\mathfrak{g}'[0]$). In other words, $\psi(I) \cap (\mathfrak{g}'[0]) \neq 0$.

The following is a known and easy fact from linear algebra: If A and B are two Q-graded vector spaces, and $\Phi : A \to B$ is a Q-graded linear map, then $\Phi(A[\beta]) = (\Phi(A))[\beta]$ for every $\beta \in Q$. Applying this fact to $A = I$, $B = \mathfrak{g}'$, $\Phi = \psi$ and $\beta = 0$, we obtain $\psi(I[0]) = (\psi(I))[0]$. But since $I[0] = 0$, this rewrites as $\psi(0) = (\psi(I))[0]$. Hence, $(\psi(I))[0] = \psi(0) = 0$.

But since $\psi(I)$ is a Q-graded vector subspace of $\mathfrak{g}'$, we have $\psi(I) \cap (\mathfrak{g}'[0]) = (\psi(I))[0] = 0$. This contradicts the fact that $\psi(I) \cap (\mathfrak{g}'[0]) \neq 0$. Hence, our assumption (that $\psi(I) \neq 0$) must have been wrong. In other words, $\psi(I) = 0$, so that $I \subseteq \operatorname{Ker}\psi$. Combined with $\operatorname{Ker}\psi \subseteq I$, this yields $I = \operatorname{Ker}\psi$.

Since the Q-graded Lie algebra homomorphism $\psi : \widetilde{\mathfrak{g}} \to \mathfrak{g}'$ is surjective, it factors (according to the homomorphism theorem) through a Q-graded Lie algebra isomorphism $\widetilde{\mathfrak{g}} \diagup (\operatorname{Ker}\psi) \to \mathfrak{g}'$. Since $\widetilde{\mathfrak{g}} \diagup \underbrace{(\operatorname{Ker}\psi)}_{=I} = \widetilde{\mathfrak{g}} \diagup I = \mathfrak{g}$, this means that ψ factors through a Q-graded Lie algebra isomorphism $\mathfrak{g} \to \mathfrak{g}'$. This Q-graded Lie algebra isomorphism $\mathfrak{g} \to \mathfrak{g}'$ clearly sends the generators $e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n$ of $\mathfrak{g}$ to the respective generators $e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n$ of $\mathfrak{g}'$.

We have thus proven that there exists a Q-graded Lie algebra isomorphism $\mathfrak{g} \to \mathfrak{g}'$ which sends the generators $e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n$ of $\mathfrak{g}$ to the respective generators $e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n$ of $\mathfrak{g}'$. This completes the proof of Theorem 4.8.2 **(a)**.

(b) Let A be the Cartan matrix of a simple finite-dimensional Lie algebra. Clearly it is enough to prove that this Lie algebra is a contragredient Lie algebra corresponding to A, that is, is generated by $e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n$ as a Lie algebra and satisfies the conditions **(1)**, **(2)** and **(3)** of Definition 4.8.1. But this follows from the standard theory of roots of simple finite-dimensional Lie algebras[228]. Theorem 4.8.2 **(b)** is thus proven.

Remark 4.8.6. Let $A = (a_{i,j})_{1 \leq i,j \leq n}$ be a complex $n \times n$ matrix such that every $i \in \{1, 2, ..., n\}$ satisfies $a_{i,i} = 2$. One can show that the Lie algebra $\mathfrak{g}(A)$ is finite-dimensional if and only if A is the Cartan matrix of a semisimple finite-dimensional Lie algebra. (In this case, $\mathfrak{g}(A)$ is exactly this semisimple Lie algebra, and the ideal I of Theorem 4.8.4 is generated by the left hand sides $(\operatorname{ad}(e_i))^{1-a_{i,j}} e_j$ and $(\operatorname{ad}(f_i))^{1-a_{i,j}} f_j$ of the Serre relations.)

[...]

[Add something about the total degree on $\widetilde{\mathfrak{g}}$, since this will later be used for the bilinear form. $\widetilde{\mathfrak{g}}[\operatorname{tot} 0] = \widetilde{\mathfrak{g}}[0] = \widetilde{\mathfrak{h}}$, $\widetilde{\mathfrak{g}}[\operatorname{tot} < 0] ...$, $\widetilde{\mathfrak{g}}[1] = ...$]

[228] For instance, condition **(3)** follows from the fact that the Lie algebra in question is simple and thus contains no ideals other than 0 and itself.

Remark 4.8.7. Let A_1 and A_2 be two square complex matrices. As usual, we denote by $A_1 \oplus A_2$ the block-diagonal matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. Then, $\mathfrak{g}(A_1 \oplus A_2) \cong \mathfrak{g}(A_1) \oplus \mathfrak{g}(A_2)$ as Lie algebras naturally.

Proof of Remark 4.8.7 (sketched). Say A_1 is an $\ell \times \ell$ matrix, and A_2 is an $m \times m$ matrix. Let $n = \ell + m$ and $A = A_1 \oplus A_2$. Introduce the notations $\widetilde{\mathfrak{g}}$, $\widetilde{\mathfrak{h}}$, $\widetilde{\mathfrak{n}}_+$, $\widetilde{\mathfrak{n}}_-$, ι_0, ι_+, ι_- and I as in Theorem 4.8.4. Let $\mathfrak{j}_+$ be the ideal of the Lie algebra $\widetilde{\mathfrak{n}}_+$ generated by all elements of the form $[e_i, e_j]$ with $i \in \{1, 2, ..., \ell\}$ and $j \in \{\ell+1, \ell+2, ..., n\}$. Let $\mathfrak{j}_-$ be the ideal of the Lie algebra $\widetilde{\mathfrak{n}}_-$ generated by all elements of the form $[f_i, f_j]$ with $i \in \{1, 2, ..., \ell\}$ and $j \in \{\ell+1, \ell+2, ..., n\}$. Prove that $\iota_+(\mathfrak{j}_+)$ and $\iota_-(\mathfrak{j}_-)$ are actually Q-graded ideals of $\widetilde{\mathfrak{g}}$ (and not only of $\iota_+(\widetilde{\mathfrak{n}}_+)$ and $\iota_-(\widetilde{\mathfrak{n}}_-)$), so that both $\iota_+(\mathfrak{j}_+)$ and $\iota_-(\mathfrak{j}_-)$ are subsets of I. For every $i \in \{1, 2\}$, let $\widetilde{\mathfrak{g}}_i$ be the Lie algebra constructed analogously to $\widetilde{\mathfrak{g}}$ but for the matrix A_i instead of A. Notice that $\widetilde{\mathfrak{g}} \diagup (\iota_+(\mathfrak{j}_+) + \iota_-(\mathfrak{j}_-)) \cong \widetilde{\mathfrak{g}}_1 \oplus \widetilde{\mathfrak{g}}_2$. Conclude the proof by noticing that if J is a Q-graded ideal in $\widetilde{\mathfrak{g}}$ which has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$, and K is the sum of all Q-graded ideals in $\widetilde{\mathfrak{g}} \diagup J$ which have zero intersection with the projection of $\iota_0\left(\widetilde{\mathfrak{h}}\right)$ on $\widetilde{\mathfrak{g}} \diagup J$, then $(\widetilde{\mathfrak{g}} \diagup J) \diagup K \cong \widetilde{\mathfrak{g}} \diagup I = \mathfrak{g}$. The details are left to the reader.

4.9. [unfinished] Kac-Moody algebras for generalized Cartan matrices

For general A, we do not know much about $\mathfrak{g}(A)$; its definition was not even constructive (find that I !). It is not known in general how to obtain generators for I. But for some particular cases – not only Cartan matrices of semisimple Lie algebras –, things behave well. Here is the most important such case:

Definition 4.9.1. An $n \times n$ matrix $A = (a_{i,j})_{1 \leq i,j \leq n}$ of complex numbers is said to be a *generalized Cartan matrix* if it satisfies:

(1) We have $a_{i,i} = 2$ for all $i \in \{1, 2, ..., n\}$.

(2) For every i and j, the number $a_{i,j}$ is a nonpositive integer. Also, $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.

(3) The matrix A is symmetrizable, i. e., there exists a diagonal matrix $D > 0$ such that $(DA)^T = DA$.

Note that a Cartan matrix is the same as a generalized Cartan matrix A with $DA > 0$.

Example 4.9.2. Let $A = \begin{pmatrix} 2 & -m \\ -1 & 2 \end{pmatrix}$ for $m \geq 1$. This matrix A is a generalized Cartan matrix, since $\begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} 2 & -m \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & -m \\ -m & 2m \end{pmatrix}$. Note that $\det \begin{pmatrix} 2 & -m \\ -1 & 2 \end{pmatrix} = 4 - m$.

For $m = 1$, we have $\mathfrak{g}(A) \cong A_2 = \mathfrak{sl}_3$.

For $m = 2$, we have $\mathfrak{g}(A) \cong B_2 \cong C_2 \cong \mathfrak{sp}_4 \cong \mathfrak{so}_5$.

For $m = 3$, we have $\mathfrak{g}(A) \cong G_2$.

For $m \geq 4$, the Lie algebra $\mathfrak{g}(A)$ is infinite-dimensional.

For $m = 4$, it is a twisted version of $\widehat{\mathfrak{sl}_2}$, called A_2^2.

For $m \geq 5$, the Lie algebra $\mathfrak{g}(A)$ is big (in the sense of having exponential growth).

This strange behaviour is related to the behaviour of the m-subspaces problem (finite for $m \leq 3$, tame for $m = 4$, wild for $m \geq 5$). More generally, Kac-Moody algebras are related to representation theory of quivers.

Definition 4.9.3. A *symmetrizable Kac-Moody algebra* is a Lie algebra of the form $\mathfrak{g}(A)$ for a generalized Cartan matrix A.

Theorem 4.9.4 (Gabber-Kac)**.** If A is a generalized Cartan matrix, then the ideal $I \subseteq \widetilde{\mathfrak{g}}(A)$ is generated by the Serre relations (where the notation I comes from Theorem 4.8.4).

Partial proof of Theorem 4.9.4. Proving this theorem requires showing two assertions: first, that the Serre relations are contained in I; second, that they actually generate I. We will only prove the first of these two assertions.

Set $I_+ = I \cap \widetilde{\mathfrak{n}}_+$ and $I_- = I \cap \widetilde{\mathfrak{n}}_-$. Denote $\widetilde{\mathfrak{g}}(A)$ by $\widetilde{\mathfrak{g}}$ as in Theorem 4.8.4.

We know (from Theorem 4.8.4 **(h)**) that I is a Q-graded ideal in $\widetilde{\mathfrak{g}}$ which has zero intersection with $\iota_0\left(\widetilde{\mathfrak{h}}\right)$ (where the notations are those of Theorem 4.8.4). Since $\widetilde{\mathfrak{g}}[0] = \iota_0\left(\widetilde{\mathfrak{h}}\right)$ (by Theorem 4.8.4 **(e)**), this rewrites as follows: I is a Q-graded ideal in $\widetilde{\mathfrak{g}}$ which has zero intersection with $\widetilde{\mathfrak{g}}[0]$. Thus, $I = I_+ \oplus I_-$.

Let us show that $(\operatorname{ad}(f_i))^{1-a_{i,j}} f_j \in I_-$.

To do that, it is sufficient to show that $\left[e_k, (\operatorname{ad}(f_i))^{1-a_{i,j}} f_j\right] = 0$ for all k. (If we grade $\widetilde{\mathfrak{g}}$ by setting $\deg(f_i) = -1$, $\deg(e_i) = 1$ and $\deg(h_i) = 0$ (this is called the *principal grading*), then f_k can only lower degree, so that the Lie ideal generated by $(\operatorname{ad}(f_i))^{1-a_{i,j}} f_j$ will lie entirely in negative degrees, and thus $(\operatorname{ad}(f_i))^{1-a_{i,j}} f_j$ will lie in I_-.)

Case 1: We have $k \neq i, j$. This case is clear since e_k commutes with f_i and f_j (by our relations).

Case 2: We have $k = j$. In this case,

$$\begin{aligned}\left[e_k, (\operatorname{ad}(f_i))^{1-a_{i,j}} f_j\right] &= \left[e_j, (\operatorname{ad}(f_i))^{1-a_{i,j}} f_j\right] = (\operatorname{ad}(f_i))^{1-a_{i,j}} ([e_j, f_j]) \\ &\qquad \left(\text{since } \operatorname{ad}(f_i) \text{ and } \operatorname{ad}(e_j) \text{ commute, due to } i \neq j\right) \\ &= (\operatorname{ad}(f_i))^{1-a_{i,j}} h_j.\end{aligned}$$

We now distinguish between two cases according to whether $a_{i,j}$ is $= 0$ or < 0:

Case 2a: We have $a_{i,j} = 0$. Then, $a_{j,i} = 0$ by the definition of generalized Cartan matrices. Thus, $[f_i, h_j] = -[h_j, f_i] = -a_{j,i} f_i = 0$, and we are done.

Case 2b: We have $a_{i,j} < 0$. Then, $1 - a_{i,j} \geq 2$. Now, $(\operatorname{ad}(f_i))^2 h_j = (\operatorname{ad}(f_i))(c f_i) = 0$ for some constant c.

Case 3: We have $k = i$. Let $(\mathfrak{sl}_2)_i = \langle e_i, f_i, h_i \rangle$. Let M be the $(\mathfrak{sl}_2)_i$-submodule in $\widetilde{\mathfrak{g}}(A)$ generated by f_j.

We have $[h_i, f_j] = -a_{i,j} f_j = m f_j$, where $m = -a_{i,j} \geq 0$. Together with $[e_i, f_j] = 0$, this shows that $f_j =: v$ is a highest-weight vector of M with weight m. Thus, $f_i^{m+1} v =$

$(\mathrm{ad}(f_i))^{1-a_{i,j}} f_j$ is a singular vector for $(\mathfrak{sl}_2)_i$ (by representation theory of $\mathfrak{sl}_2$ [229]).

So much for our part of the proof of Theorem 4.9.4.

Of course, simple Lie algebras are Kac-Moody algebras. The next class of Kac-Moody algebras we are interested in is the *affine Lie algebras*:

Remark 4.9.5. Let $\sigma \in S_n$ be a permutation, and A be an $n \times n$ complex matrix. Then, $\mathfrak{g}(A) \cong \mathfrak{g}(\sigma A \sigma^{-1})$.

Definition 4.9.6. A generalized Cartan matrix A is said to be *indecomposable* if it cannot be written in the form $\sigma(A_1 \oplus A_2)\sigma^{-1}$ for some permutation σ and nontrivial square matrices A_1 and A_2. Due to the above remark and to Remark 4.8.7, we need to only consider indecomposable generalized Cartan matrices.

Definition 4.9.7. A generalized Cartan matrix A is said to be *affine* if $DA \geq 0$ but $DA \not> 0$ (thus, $\det(DA) = 0$).

Definition 4.9.8. If A is an affine generalized Cartan matrix, then $\mathfrak{g}(A)$ is called an *affine Kac-Moody algebra.*

Now let A be the (usual) Cartan matrix of a simple Lie algebra, and let $\mathfrak{g} = \mathfrak{g}(A)$ be this simple Lie algebra. Let $L\mathfrak{g} = \mathfrak{g}[t, t^{-1}]$, and let $\widehat{\mathfrak{g}} = L\mathfrak{g} \oplus \mathbb{C}K$ as defined long ago.

Theorem 4.9.9. This $\widehat{\mathfrak{g}}$ is an affine Kac-Moody algebra with generalized Cartan matrix $\widetilde{A}$ whose $(1,1)$-entry is 2 and whose submatrix obtained by omitting the first row and the first column is A. (We do not yet say what the remaining entries are.)

Proof of Theorem. Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$. Let $r = \dim \mathfrak{h}$; thus, r is the rank of $\mathfrak{g}$. Let $(h_1, h_2, ..., h_r)$ be a corresponding basis of $\mathfrak{h}$, and let e_i, f_i be standard generators for every $i \in \{1, 2, ..., r\}$.

Let θ be the maximal root.

Let us now define elements $e_0 = f_\theta \cdot t$, $f_0 = e_\theta \cdot t^{-1}$ and $h_0 = [e_0, f_0] = -h_\theta + \underbrace{(f_\theta, e_\theta)}_{=1 \text{ (due to our normalization)}} K = K - h_\theta$ of $\widehat{\mathfrak{g}}$ (the commutator is computed in $\widehat{\mathfrak{g}}$, not in $L\mathfrak{g}$).

Add these elements to our system of generators.

Why do we then get a system of generators of $\widehat{\mathfrak{g}}$?

First, h_i for $i \in \{0, 1, ..., r\}$ are a basis of $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K$.

Also, $\mathfrak{g}t^0$ is generated by e_i, f_i, h_i for $i \in \{1, 2, ..., r\}$. Now, $\mathfrak{g}t^1$ is an irreducible $\mathfrak{g}$-module with lowest-weight vector $f_\theta \cdot t$.

$\Longrightarrow U(\mathfrak{g}) \cdot f_\theta t = \mathfrak{g}t$. Now, $\mathfrak{g}t$ generates $\mathfrak{g}t\mathbb{C}[t]$ (since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$). Similarly, $U(\mathfrak{g}) \cdot e_\theta t^{-1} = \mathfrak{g}t^{-1}$, and $\mathfrak{g}t^{-1}$ generates $\mathfrak{g}t^{-1}\mathbb{C}[t^{-1}]$. $\Longrightarrow$ our e_i, f_i, h_i (including $i = 0$) generate all of $\widehat{\mathfrak{g}}$.

Now to the relations.

$[h_i, h_j] = 0$ is clear for all $(i, j) \in \{0, 1, ..., r\}^2$.

We have $[h_0, e_0] = [K - h_\theta, f_\theta t] = -[h_\theta, f_\theta] t = 2f_\theta t = 2e_0$.

[229] What we are using is the following: Consider the module $M_\lambda = \mathbb{C}[f] v$ over $\mathfrak{sl}_2$. Then, $ef^n v = n(\lambda - n + 1) f^{n-1} v$. Thus, when $n = m + 1$ and $\lambda = m$, we get $ef^n v = 0$.

We have $[h_0, f_0] = -2f_0$ similarly.

We have $[e_0, f_0] = h_0$.

We have $[h_0, e_i] = [K - h_\theta, e_i] = -\alpha_i (h_\theta) e_i = -(\alpha_i, \theta) e_i \Longrightarrow a_{0,i} = -(\alpha_i, \theta) =$ (some nonpositive integer).

We have $[h_0, f_i] = (\alpha_i, \theta) f_i$, same argument.

We have $[h_i, e_0] = [h_i, f_\theta t] = -\theta (h_i) f_\theta t = -\theta (h_i) e_0 = -(\alpha_i^\vee, \theta) e_0$ (where $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)}$) $\Longrightarrow a_{i,0} = -(\alpha_i^\vee, \theta)$.

We have $[h_i, f_0] = (\alpha_i^\vee, \theta) f_0$, same argument.

We have $[e_0, f_i] = [f_\theta t, f_i] = 0$.

We have $[e_i, f_0] = [e_i, e_\theta t^{-1}] = 0$.

Thus, all basic relations are satisfied.

Now let us define a grading: $\widehat{Q} = Q \oplus \mathbb{Z}\delta$, where Q is the root lattice of $\mathfrak{g}$. Define $\alpha_0 = \delta - \theta$. $\delta \mid_{\widehat{\mathfrak{h}}} = 0$. So if we think of α_0 as an element of $\widehat{\mathfrak{h}}^*$, then $\alpha_0, \alpha_1, ..., \alpha_r$ is neither linearly independent nor spanning. So the direct sum $Q \oplus \mathbb{Z}\delta$ is an external direct sum, not an internal one!!

$\widehat{Q}$-grading: $\deg (e_i) = \alpha_i$, $\deg (f_i) = -\alpha_i$ and $\deg (h_i) = 0$ for $i = 0, 1, ..., r$. Also $\deg \left(at^k\right) = \deg a + k\delta$ (so, so to speak, "$\deg t = \delta$").

So we have $\widehat{\mathfrak{g}} [0] = \widehat{\mathfrak{h}}$ and $\widehat{\mathfrak{g}} [\alpha_i] = \langle e_i \rangle$ and $\widehat{\mathfrak{g}} [-\alpha_i] = \langle f_i \rangle$.

Note (which we won't use): $[h, a] = \alpha (h) a$, $a \in \widehat{\mathfrak{g}} [\alpha]$ "if you define things this way".

The only thing we now have to do is to show that $I = 0$ in $\widehat{\mathfrak{g}}$.

Let $\overline{I}$ be the projection of I to $L\mathfrak{g} = \widehat{\mathfrak{g}} \diagup (K)$. Clearly, $\overline{I} \cap \mathfrak{h} = 0$.

We must prove that $\overline{I} = 0$.

But there is a **claim** that any $\widehat{Q}$-graded ideal in $L\mathfrak{g}$ is 0 or $L\mathfrak{g}$. (*Proof:* If J is a $\widehat{Q}$-graded ideal of $L\mathfrak{g}$ different from 0, then there exists a nonzero $a \in \mathfrak{g}$ and an $m \in \mathbb{Z}$ such that $at^m \in J$. But at^m generates $L\mathfrak{g}$ under the action of $L\mathfrak{g}$, since $[bt^{n-m}, at^m] = [b, a] t^n$ and $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.)

Proof of Theorem complete.

Let us show how Dynkin diagrams look like for these affine Kac-Moody algebras.

Consider the case of $A_{n-1} = \mathfrak{sl}_n$. Then, $\theta = (1, 0, 0, ..., 0, -1)$. Also, $\alpha_1 = (1, -1, 0, 0, ..., 0)$, $\alpha_2 = (0, 1, -1, 0, 0, ..., 0)$, ..., $\alpha_{n-1} = (0, 0, ..., 0, 1, -1)$. Also, $\alpha = \alpha^\vee$ for all simple roots α. We thus have $(\theta, \alpha_i) = 1$ if $\alpha \in \{1, n-1\}$ and $= 0$ otherwise. The Dynkin diagram of $\widehat{A_{n-1}} = A_{n-1}^1 = \widehat{\mathfrak{sl}_n}$ (these are just three notations for one and the same thing) is thus $\circ$ —— $\circ$ —— $\circ$ $\cdots$ $\circ$ —— $\circ$ —— $\circ$ with a cyclically connected dot underneath.

The case $n = 2$ is special: double link. $\circ = \circ$ double link.

Now let us consider other types. Suppose that θ is a fundamental weight, i. e., satisfies $(\theta, \alpha_i^\vee) = 1$ for some i and satisfies $(\theta, \alpha_i^\vee) = 0$ for all other i. (This happens for a lot of simple Lie algebras.)

To get $\widehat{D_n} = \widehat{\mathfrak{so}_{2n}}$, need to attach a new vertex to the second vertex from the left.

To get $\widehat{C_n} = \widehat{\mathfrak{sp}_{2n}}$, need to attach a new vertex **doubly-linked** to the first vertex from the left. (The arrow points to the right, i. e., to the C_n diagram.)

For $\widehat{G_2}$, attach a vertex on the left (where the arrow points to the right).

For $\widehat{F_4}$, attach a vertex on the left (where the arrow points to the right).

For $\widehat{E_6}$, attach a vertex to the "bottom" (the vertex off the line).

For $\widehat{E_7}$, attach a vertex to the short leg (to make the graph symmetric).

For $\widehat{E_8}$, attach a vertex to the long leg.

These are untwisted affine Lie algebras ($\widehat{\mathfrak{g}}$).

There are also twisted ones: A_2^2 with Cartan matrix $\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$ and Dynkin diagram $\circ$ (4 arrows pointing rightward) $\circ$. We will not discuss this kind of Lie algebras here.

4.10. [unfinished] Representation theory of $\mathfrak{g}(A)$

We will now work out the representation theory of $\mathfrak{g}(A)$.

Let us start with the case of $\mathfrak{g}(A)$ being finite-dimensional. In contrast with usual courses on Lie algebras, we will not restrict ourselves to finite-dimensional representations. We define a Category $\mathcal{O}$ which is analogous but (in its details) somewhat different from the one we defined above. In future, we will use only the new definition.

Definition 4.10.1. The objects of *category* $\mathcal{O}$ will be $\mathfrak{g}$-modules M such that:

1) The module M is $\mathfrak{h}$-diagonalizable. By this we mean that $M = \bigoplus\limits_{\mu\in\mathfrak{h}^*} M[\mu]$ (where $M[\mu]$ means the μ-weight space of M), and every $\mu \in \mathfrak{h}^*$ satisfies $\dim(M[\mu]) < \infty$.

2) Let $\operatorname{Supp} M$ denote the set of all $\mu \in \mathfrak{h}^*$ such that $M[\mu] \neq 0$. Then, there exist finitely many $\lambda_1, \lambda_2, ..., \lambda_n \in \mathfrak{h}^*$ such that $\operatorname{Supp} M \subseteq D(\lambda_1) \cup D(\lambda_2) \cup ... \cup D(\lambda_n)$, where for every $\lambda \in \mathfrak{h}^*$, we denote by $D(\lambda)$ the subset

$$\{\lambda - k_1\alpha_1 - k_2\alpha_2 - ... - k_r\alpha_r \mid (k_1, k_2, ..., k_r) \in \mathbb{N}^r\} \qquad \text{of } \mathfrak{h}^*.$$

The *morphisms of category* $\mathcal{O}$ will be $\mathfrak{g}$-module homomorphisms.

Examples of modules in Category $\mathcal{O}$ are Verma modules $M_\lambda = M_\lambda^+$ and their irreducible quotients L_λ (and all of their quotients). Category $\mathcal{O}$ is an abelian category (in our case, this simply means it is closed under taking subquotients and direct sums).

Definition 4.10.2. Let $M \in \mathcal{O}$ be a $\mathfrak{g}$-module. Then, the *formal character* of M denotes the sum $\operatorname{ch} M = \sum\limits_{\mu\in\mathfrak{h}^*} \dim(M[\mu])\, e^\mu$. Here $\mathbb{C}[\mathfrak{h}^*]$ denotes the group algebra of the additive group $\mathfrak{h}^*$, where this additive group $\mathfrak{h}^*$ is written multiplicatively and every $\mu \in \mathfrak{h}^*$ is renamed as e^μ.

Where does this sum $\sum\limits_{\mu\in\mathfrak{h}^*} \dim(M[\mu])\, e^\mu$ lie?

Let Γ be a coset of Q (the root lattice) in $\mathfrak{h}^*$. Then, let R_Γ denote the space $\lim\limits_{\mu\in\Gamma} e^\mu \mathbb{C}[[e^{-\alpha_1}, e^{-\alpha_2}, ..., e^{-\alpha_r}]]$ (this is a union, but not a disjoint union, since $R_\mu \subseteq R_{\mu+\alpha_i}$ for all i and μ). Let $R = \bigoplus\limits_{\Gamma\in\mathfrak{h}^*/Q} R_\Gamma$. This R is a ring. We view $\operatorname{ch} M$ as an element of R.

Now, for an example, let us compute the formal character $\operatorname{ch}(M_\lambda)$ of the Verma module $M_\lambda = U(\mathfrak{n}_-)\, v_\lambda$.

Recall that $U(\mathfrak{n}_-)$ has a Poincaré-Birkhoff-Witt basis consisting of all elements of the form $f_{\alpha^{(1)}}^{m_1} f_{\alpha^{(2)}}^{m_2} ... f_{\alpha^{(\ell)}}^{m_\ell}$ where $\alpha^{(1)}, \alpha^{(2)}, ..., \alpha^{(\ell)}$ are all positive roots of $\mathfrak{g}$, and $\ell = \dim(\mathfrak{n}_-)$. The weight of this element $f_{\alpha^{(1)}}^{m_1} f_{\alpha^{(2)}}^{m_2} ... f_{\alpha^{(\ell)}}^{m_\ell}$ is $-\left(m_1\alpha^{(1)} + m_2\alpha^{(2)} + ... + m_\ell\alpha^{(\ell)}\right)$. Thus, the weight of $f_{\alpha^{(1)}}^{m_1} f_{\alpha^{(2)}}^{m_2} ... f_{\alpha^{(\ell)}}^{m_\ell} v_\lambda$ is $\lambda - \left(m_1\alpha^{(1)} + m_2\alpha^{(2)} + ... + m_\ell\alpha^{(\ell)}\right)$.

Thus, $\dim(M_\lambda[\lambda-\beta])$ is the number of partitions of β into positive roots. We denote this by $p(\beta)$, and call p the *Kostant partition function.*

Now, it is very easy (using geometric series) to see that

$$\sum_{\beta\in Q_+} p(\beta)\,e^{-\beta} = \prod_{\substack{\alpha \text{ root;}\\ a>0}} \frac{1}{1-e^{-\alpha}}.$$

Thus,

$$\operatorname{ch}(M_\lambda) = \sum_{\beta\in Q_+} p(\beta)\,e^{\lambda-\beta} = e^\lambda \underbrace{\sum_{\beta\in Q_+} p(\beta)\,e^{-\beta}}_{=\prod_{\substack{\alpha \text{ root;}\\ a>0}} \frac{1}{1-e^{-\alpha}}} = e^\lambda \prod_{\substack{\alpha \text{ root;}\\ a>0}} \frac{1}{1-e^{-\alpha}}.$$

Example: Let $\mathfrak{g}=\mathfrak{sl}_2$. Then,

$$\operatorname{ch}(M_\lambda) = \frac{e^\lambda}{1-e^{-\alpha}} = e^\lambda + e^{\lambda-\alpha} + e^{\lambda-2\alpha} + \dots.$$

Classically, one identifies weights of $\mathfrak{sl}_2$ with elements of $\mathbb{C}$ (by $\omega_1 \mapsto 1$ and thus $\alpha\mapsto 2$). Write x for e^{ω_1}. Then,

$$\operatorname{ch}(M_\lambda) = \frac{x^\lambda}{1-x^{-2}} = x^\lambda + x^{\lambda-2} + x^{\lambda-4} + \dots.$$

The quotient L_λ has weights λ, $\lambda-2$, ..., $-\lambda$ and thus satisfies

$$\operatorname{ch}(L_\lambda) = x^\lambda + x^{\lambda-2} + \dots + x^{-\lambda} = \frac{x^{\lambda+1}-x^{-\lambda-1}}{x-x^{-1}}.$$

Back to the general case of finite-dimensional $\mathfrak{g}(A)$. First of all, category $\mathcal{O}$ has tensor products, and they make it into a tensor category.

Proposition 4.10.3. 1) We have $\operatorname{ch}(M_1\otimes M_2) = \operatorname{ch}(M_1)\cdot\operatorname{ch}(M_2)$.
2) If $N\subseteq M$ are both in $\mathcal{O}$, then $\operatorname{ch} M = \operatorname{ch} N + \operatorname{ch}(M\diagup N)$.

Proof of Proposition. **1)**

$$(M_1\otimes M_2)[\mu] = \bigoplus_{\mu_1+\mu_2=\mu} M_1[\mu_1]\otimes M_2[\mu_2].$$

2)

$$(M\diagup N)[\mu] = M[\mu]\diagup N[\mu].$$

Now, let us generalize to the case of Kac-Moody Lie algebras (or $\mathfrak{g}(A)$ for general A). Here we run into troubles: For example, for $\widehat{\mathfrak{sl}_2}$, we have $M_\lambda = U(\widetilde{\mathfrak{n}}_-)\,v_\lambda$, and the vectors $ht^{-1}v_\lambda, ht^{-2}v_\lambda, \dots$ all have weight λ with respect to $\widehat{\mathfrak{h}} = \langle h_0,h_1\rangle$ with $h_1=h$, $h_0=K-h$. This yields that weight spaces are infinite-dimensional, and we cannot define characters.

Let us work around this by adding derivations.

Assume that A is an $r \times r$ complex matrix. Let $\mathfrak{g}_{\mathrm{ext}}(A) = \mathfrak{g}(A) \oplus \bigoplus_{i=1}^{r} \mathbb{C}D_i$ with new relations

$$\begin{aligned} [D_i, D_j] &= 0 && \text{for all } i, j; \\ [D_i, e_j] &= 0 && \text{for all } i \neq j; \\ [D_i, f_j] &= 0 && \text{for all } i \neq j; \\ [D_i, h_j] &= 0 && \text{for all } i \neq j; \\ [D_i, e_i] &= e_i; \\ [D_i, f_i] &= -f_i; \\ [D_i, h_i] &= 0. \end{aligned}$$

Note that this definition is equivalent to making $\mathfrak{g}_{\mathrm{ext}}(A)$ a semidirect product, so there is no cancellation here.

We have $\mathfrak{g}_{\mathrm{ext}}(A) = \mathfrak{n}_+ \oplus \mathfrak{h}_{\mathrm{ext}} \oplus \mathfrak{n}_-$ where $\mathfrak{h}_{\mathrm{ext}} = \mathbb{C}^r \oplus \mathfrak{h}$ (here the $\mathbb{C}^r$ is spanned by the $\mathbb{C}D_i$).

Consider α_i as maps $\mathfrak{h}_{\mathrm{ext}} \to \mathbb{C}$ given by $\alpha_i(h_j) = a_{j,i}$ and $\alpha_i(D_j) = \delta_{i,j}$.

Then, for every $h \in \mathfrak{h}_{\mathrm{ext}}$, we have $[h, e_i] = \alpha_i(h)\, e_i$ and $[h, f_i] = -\alpha_i(h)\, f_i$.

Let $F = Q \otimes_{\mathbb{Z}} \mathbb{C}$ and $P = \mathfrak{h}^* \oplus F$.

Let $\varphi : P \to \mathfrak{h}_{\mathrm{ext}}^*$ be given by $\varphi(h_i^*)(D_j) = 0$, $\varphi(h_i^*)(h_j) = \delta_{i,j}$, $\varphi(\alpha_i)(D_j) = \delta_{i,j}$, $\varphi(\alpha_i)(h_j) = a_{j,i}$.

Easy to see φ is an iso.

Now the trouble disappears. Do the same as for simple Lie algebras. Now weights lie in $\mathfrak{h}_{\mathrm{ext}}^*$.

Annoying fact: Now, even when A is a Cartan matrix and $\mathfrak{g}$ is simple finite-dimensional, this is not the same as the usual theory [what?]. But it is equivalent. Namely: Suppose $\chi \in \mathfrak{h}_{\mathrm{ext}}^*$. Let $\mathcal{O}_\chi$ be the category of modules whose weights lie in $\chi + F$. Therefore, $\mathcal{O} = \bigoplus_{\chi \in \mathfrak{h}^*} \mathcal{O}_\chi$.

Proposition 4.10.4. If $\chi_1 - \chi_2 \in \operatorname{Im}(F \to \mathfrak{h}^*)$, then $\mathcal{O}_{\chi_1} \cong \mathcal{O}_{\chi_2}$.

(See Feigin-Zelevinsky paper for proof.)

If A is invertible (in particular, for simple $\mathfrak{g}$), all $\mathcal{O}_\chi$ are the same and we just have a single category $\mathcal{O}$ (which is the category $\mathcal{O}$ we defined).

Affine case: $\operatorname{Coker}(F \to \mathfrak{h}^*)$ is 1-dimensional, so χ has one essential parameter (namely, the image k of χ in this Coker). So we get a 1-parameter category of categories, $\mathcal{O}(k)$, parametrized by a complex number k. In our old approach to $\widehat{\mathfrak{g}}$, this k is the level of representations (i. e., the eigenvalue of the action of K). So we did not get anything new, but we have got a uniform way to treat all cases of this kind.

4.11. [unfinished] Invariant bilinear forms

Now let us start developing the theory of invariant bilinear forms on $\mathfrak{g}(A)$ and $\widetilde{\mathfrak{g}}(A)$.

[We denote $\mathfrak{g}[\alpha]$ as $\mathfrak{g}_\alpha$.]

Let A be an indecomposable complex matrix. We want to see when we can have nontrivial nonzero invariant symmetric bilinear forms on $\widetilde{\mathfrak{g}}(A)$ and $\mathfrak{g}(A)$. Let us only care about forms of degree 0, which means that they send $\mathfrak{g}_\alpha \times \mathfrak{g}_\beta$ to 0 unless $\alpha + \beta = 0$. It also sounds like a good goal to have the forms nondegenerate, but this cannot always be reached. Let us impose the weaker condition that, if e_i and f_i denote generators of $\mathfrak{g}_{\alpha_i}$ and $\mathfrak{g}_{-\alpha_i}$, respectively, then $(e_i, f_i) = d_i$ for some $d_i \neq 0$.

These conditions already force some properties upon $\mathfrak{g}(A)$: First,

$$(h_i, h_j) = (h_i, [e_j, f_j]) = -([h_i, f_j], e_j) = a_{i,j}(f_j, e_j) = a_{i,j} d_j,$$

so that the symmetry of our form (and the condition $d_i \neq 0$) enforces $a_{i,j} d_j = a_{j,i} d_i$. Thus, if D denotes the matrix $\operatorname{diag}(d_1, d_2, ..., d_r)$, then $(AD)^T = AD$. This means that A is symmetrizable. (Our definition of "symmetrizable" spoke of DA instead of AD, but this is simply a matter of replacing D by D^{-1}.)

Lemma 4.11.1. Let A be an indecomposable symmetrizable matrix. Then, there is a unique diagonal matrix D satisfying $(AD)^T = AD$ up to scaling.

This lemma is purely combinatorial and more or less trivial.

Proposition 4.11.2. Let A be an indecomposable symmetrizable matrix. Then, there is at most one invariant symmetric bilinear form of degree 0 on $\widetilde{\mathfrak{g}}(A)$ up to scaling.

Note that the degree in "degree 0" is the degree with respect to Q-grading; this is a tuple.

Proof of Proposition. Let B be such a form. Then, we can view B as a $\mathfrak{g}$-module homomorphism $B^\vee : \mathfrak{g} \to \mathfrak{g}^*$. If we fix d_i (uniquely up to scaling, as we know from Lemma), then we know $B^\vee(h_i)$, $B^\vee(f_i)$ and $B^\vee(e_i)$ (because the form is of degree 0, and thus the linear maps $B^\vee(h_i)$, $B^\vee(f_i)$ and $B^\vee(e_i)$ are determined by what they do to the corresponding elements of the corresponding degree). But $\mathfrak{g}$ is generated as a $\mathfrak{g}$-module by e_i, f_i, h_i, so B is uniquely determined if it exists. Proposition is proven.

Theorem 4.11.3. Let A be a symmetrizable matrix. Then, there is a nonzero invariant bilinear symmetric form of degree 0 on $\widetilde{\mathfrak{g}}(A)$. (We know from the previous proposition that this form is unique up to scaling if A is indecomposable.)

Proof of Theorem (incomplete, as we will skip some steps). First, fix the d_i. Then, we can calculate the form by

$$\begin{aligned}
&\Bigg(\underbrace{[e_{i_1}, [e_{i_2}, ... [e_{i_{n-1}}, e_{i_n}] ...]]}_{\in \mathfrak{g}_\alpha}, \underbrace{[f_{j_1}, [f_{j_2}, ... [f_{j_{n-1}}, f_{j_n}] ...]]}_{\in \mathfrak{g}_{-\alpha}}\Bigg) \\
&= -\Bigg([e_{i_1}, ...], \underbrace{[[e_{i_2}, [e_{i_3}, ... [e_{i_{n-1}}, e_{i_n}] ...]], [f_{j_1}, [f_{j_2}, ... [f_{j_{n-1}}, f_{j_n}] ...]]]}_{\in \mathfrak{g}_{-\alpha}}\Bigg) \\
&+ ...
\end{aligned}$$

induction on α. For details and well-definedness, see page 51 of the Feigin-Zelevinsky paper.

Also, $\widetilde{\mathfrak{g}}(A)$ has such a form by pullback.

As usual, denote these forms by $(\cdot, \cdot)$.

Proposition 4.11.4. The kernel I of the canonical projection $\widetilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$ is a subset of $\operatorname{Ker}((\cdot, \cdot))$.

Proof of Proposition. We defined the form $(\cdot, \cdot)$ on $\widetilde{\mathfrak{g}}(A) \times \widetilde{\mathfrak{g}}(A)$ as the pullback of the form $(\cdot, \cdot) : \mathfrak{g}(A) \times \mathfrak{g}(A) \to \mathbb{C}$ through the canonical projection $\widetilde{\mathfrak{g}}(A) \times \widetilde{\mathfrak{g}}(A) \to \mathfrak{g}(A) \times \mathfrak{g}(A)$. Thus, it is clear that the kernel of the former form contains the kernel of the canonical projection $\widetilde{\mathfrak{g}}(A) \to \mathfrak{g}(A)$. Proposition proven.

Lemma 4.11.5. 1) The center Z of $\mathfrak{g}(A)$ is contained in $\mathfrak{h}$, and is

$$Z = \left\{ \sum_i \beta_i h_i \ \mid \ \beta_i \in \mathbb{C} \text{ for all } i, \text{ and } \sum_i \beta_i a_{i,j} = 0 \text{ for all } j \right\}.$$

2) If A is an indecomposable symmetrizable matrix, and $A \neq 0$, then any graded proper ideal in $\mathfrak{g}(A)$ is contained in Z.

3) If $a_{i,i} \neq 0$ for all i, then $[\mathfrak{g}(A), \mathfrak{g}(A)] = \mathfrak{g}(A)$.

Proof of Lemma. **1)** Let z be a nonzero central element of $\mathfrak{g}(A)$. We can WLOG assume that z is homogeneous. Then, $\mathbb{C}z$ is a graded nonzero ideal of $\mathfrak{g}(A)$, so that $\deg z$ must be 0, and thus $z \in \mathfrak{h}$. If $z = \sum\limits_i \beta_i h_i$, then every j satisfies $0 = [z, e_j] = \left[\sum\limits_i \beta_i h_i, e_j\right] = \left(\sum\limits_i \beta_i a_{i,j}\right) e_j$, so that $\sum\limits_i \beta_i a_{i,j} = 0$.

This proves that $Z \subseteq \left\{ \sum\limits_i \beta_i h_i \ \mid \ \beta_i \in \mathbb{C} \text{ for all } i, \text{ and } \sum\limits_i \beta_i a_{i,j} = 0 \text{ for all } j \right\}$. The reverse inclusion is easy to see (using $[h_i, f_j] = -a_{i,j} f_j$).

2) Let $I \neq 0$ be a graded ideal. Then, $I \cap \mathfrak{h} \neq 0$. So $I = I_+ \oplus I_0 \oplus I_-$ with I_0 being a nonzero subspace of $\mathfrak{h}$. Assume $I \not\subseteq Z$. Then we claim that $I_+ \neq 0$ or $I_- \neq 0$.

(In fact, otherwise, we would have $I_+ = 0$ and $I_- = 0$, so that $I \subseteq \mathfrak{h}$, so that there exists some $h \in I \subseteq \mathfrak{h}$ with $h \notin Z$, so that $[h, e_j] = \lambda e_j$ for some j and some $\lambda \neq 0$, so that $e_j \in I_+$, contradicting $I_+ = 0$ and $I_- = 0$.)

Let $\mathfrak{G}$ be the subset $\{e_1, e_2, ..., e_n, f_1, f_2, ..., f_n, h_1, h_2, ..., h_n\}$ of $\mathfrak{g}(A)$. As we know, this subset $\mathfrak{G}$ generates the Lie algebra $\mathfrak{g}(A)$.

So let us WLOG assume $I_+ \neq 0$. Then there exists a nonzero $a \in I_+[\alpha]$ for some $\alpha \neq 0$. Set J be the ideal generated by a. In other words, $J = U(\mathfrak{g}(A)) \cdot a$. This J is a graded ideal. Thus, $J \cap \mathfrak{h} \neq 0$. Hence, there exists $x \in U(\mathfrak{g}(A))$ such that $x \rightharpoonup a \in \mathfrak{h}$ and $x \rightharpoonup a \neq 0$. We can WLOG assume that x has degree $-\alpha$ and is a product of some elements of the set $\mathfrak{G}$ (with repetitions allowed). Of course, this product is nonempty (otherwise, a itself would be in I_0, not in I_+), and hence (by splitting off its first factor) can be written as $\xi \cdot \eta$ with ξ being an element of the set $\mathfrak{G}$ and η being a product of elements of $\mathfrak{G}$. Consider these ξ and η. We assume WLOG that η is a product of elements of $\mathfrak{G}$ with a minimum possible number of factors. Then, $\xi \notin \{h_1, h_2, ..., h_n\}$ (because otherwise, we could replace x by η, and would then, by splitting off the first factor, obtain a new η with an even smaller number of factors). So we have either $\xi = e_i$ for some i, or $\xi = f_i$ for some i. Let us WLOG assume that we are in the first case, i. e., we have $\xi = e_i$ for some i.

Let $y = \eta \rightharpoonup a$. Then, $y \in I$ (since $a \in I$ and since I is an ideal) and

$$[\xi, y] = \xi \rightharpoonup \underbrace{y}_{=\eta \rightharpoonup a} \qquad \text{(since } \xi \in \mathfrak{G} \subseteq \mathfrak{g}(A)\text{)}$$
$$= \xi \rightharpoonup (\eta \rightharpoonup a) = \underbrace{(\xi \cdot \eta)}_{=x} \rightharpoonup a = x \rightharpoonup a \in \mathfrak{h}$$

and $[\xi, y] = x \rightharpoonup a \neq 0$. Since $\xi = e_i \in \mathfrak{g}_{\alpha_i}$ and y is homogeneous, this yields that $y \in \mathfrak{g}_{-\alpha_i}$. Thus, $y = \chi \cdot f_i$ for some $\chi \in \mathbb{C}$. This χ is nonzero, since y is nonzero (since $[\xi, y] \neq 0$).

Since $y = \chi \cdot f_i$, we have $[e_i, y] = \chi \cdot \underbrace{[e_i, f_i]}_{=h_i} = \chi h_i$. Since $[e_i, y] \in I$ (because I is an ideal and $y \in I$), this becomes $\chi h_i \in I$, so that $h_i \in I$ (since χ is nonzero). Moreover, since $\chi \cdot f_i = y \in I$, we have $f_i \in I$ (since χ is nonzero). Altogether, we now know that $h_i \in I$ and $f_i \in I$.

If A is an 1×1 matrix, then $a_{i,i} \neq 0$ (since $A \neq 0$), so that $e_i = \dfrac{[h_i, e_i]}{a_{i,i}} \in I$ (because $h_i \in I$). Hence, if A is an 1×1 matrix, then all of e_i, f_i and h_i lie in I, so that $I = \mathfrak{g}(A)$ (because there exists only one i).

If the size of A is > 1, there exists some $j \neq i$ such that $a_{i,j} \neq 0$ and $a_{j,i} \neq 0$ (since A is indecomposable and symmetrizable), so that $e_j = \dfrac{[h_i, e_j]}{a_{i,j}} \in I$ (since $h_i \in I$), furthermore $f_j = -\dfrac{[h_i, f_j]}{a_{i,j}} \in I$, therefore $h_j = [e_j, f_j] \in I$, and finally $e_i = \dfrac{[h_j, e_i]}{a_{j,i}} \in I$. And for every $k \neq i$ with $a_{i,k} \neq 0$ and $a_{k,i} \neq 0$, we similarly get h_k, f_k, $e_k \in I$ etc.. By repeating this argument, we conclude that $e_\ell, f_\ell, h_\ell \in I$ for all ℓ (since A is indecomposable). That is, $\mathfrak{G} \subseteq I$. Since $\mathfrak{G}$ is a generating set of the Lie algebra $\mathfrak{g}(A)$, this entails $I = \mathfrak{g}(A)$.

3) If $a_{i,i} \neq 0$, then the relations (321) imply that all generators are in $[\mathfrak{g}(A), \mathfrak{g}(A)]$. Qed.

Proposition 4.11.6. Assume that A is symmetrizable. We have $\operatorname{Ker}\left((\cdot,\cdot)\mid_{\mathfrak{g}(A)}\right) = Z(\mathfrak{g}(A))$.

Proof of Proposition. Assume WLOG that A is indecomposable.

1) 1×1 case, $A = 0$ trivial: $[e, f] = h$, $[h, e] = [h, f] = 0$, $(e, f) = 1$. Then the kernel of this form is a graded ideal and is not $\mathfrak{g}(A)$. Hence, it must be contained in Z by the lemma. But $Z \subseteq \operatorname{Ker}\left((\cdot,\cdot)\mid_{\mathfrak{g}(A)}\right)$ is easy (because $\left(\sum\limits_i \beta_i h_i, h_j\right) = \sum\limits_i \beta_i a_{i,j} d_j = 0$).

Let $F = Q \otimes_{\mathbb{Z}} \mathbb{C} = \bigoplus_{i=1}^r \mathbb{C}\alpha_i$.

Define $\gamma : F \to \mathfrak{h}$ isomorphism by $\gamma(\alpha_i) = d_i^{-1} h_i =: h_{\alpha_i}$. Extend by linearity: $\gamma(\alpha)$ will be called h_α, $\alpha \in F$.

Claim: $(h_\alpha, h) = \overline{\alpha}(h)$, where $\overline{\alpha}$ is the image of α in $\mathfrak{h}^*$.

Proof: $(h_{\alpha_i}, h_j) = d_i^{-1}(h_i, h_j) = d_i^{-1} a_{i,j} d_j = d_i^{-1} a_{j,i} d_i = a_{j,i} = \overline{\alpha_i}(h_j) \qquad ([h_j, e_i] = a_{j,i} e_i)$.

Proposition 4.11.7. If $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, then $[x,y] = (x,y)\, h_\alpha$.

Proof of Proposition. By induction over $|\alpha|$, where $|\alpha|$ means the sum of the coordinates of α.

Base: $|\alpha| = 1$, $\alpha = \alpha_i$. Want to prove $[e_i, f_i] \stackrel{?}{=} (e_i, f_i)\, h_{\alpha_i}$. But $[e_i, f_i] = h_i$ and $(e_i, f_i)\, h_{\alpha_i} = d_i d_i^{-1} h_i$, so we are done with the base.

Step: For $x \in \mathfrak{g}_{\alpha-\alpha_i}$ and $y \in \mathfrak{g}_{\alpha-\alpha_j}$, we have

$$\begin{aligned}
&[[e_i, x], [f_j, y]] \\
&= [[e_i, [f_j, y]], x] + [e_i, [x, [f_j, y]]] \\
&= -([e_i, [f_j, y]], x)\, h_{\alpha-\alpha_i} + (e_i, [x, [f_j, y]])\, h_{\alpha_i} \\
&\qquad \text{(by the induction assumption)} \\
&= ([f_j, y], [e_i, x])\, (h_{\alpha-\alpha_i} + h_{\alpha_i}) = ([e_i, x], [f_j, y])\, h_\alpha .
\end{aligned}$$

Induction step complete. Proposition proven.

Corollary 4.11.8. If we give $\mathfrak{g}(A)$ the principal $\mathbb{Z}$-grading (so that $\mathfrak{g}(A)[n] = \bigoplus_{\substack{\alpha \in Q; \\ |\alpha| = n}} \mathfrak{g}(A)[\alpha]$), then $\mathfrak{g}(A)$ is a nondegenerate Lie algebra.

Proof. If $\lambda \in \mathfrak{h}^*$ is such that $\lambda(h_\alpha) \neq 0$, then $\lambda([x,y])$ is a nondegenerate form $\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha} \to \mathbb{C}$. Qed.

Recall $P = \mathfrak{h}^* \oplus F \cong \mathfrak{h}^*_{\text{ext}}$.

$(\cdot,\cdot)$ on P: $\left(\underbrace{\varphi}_{\in \mathfrak{h}^*} \oplus \underbrace{\alpha}_{\in F}, \underbrace{\psi}_{\in \mathfrak{h}^*} \oplus \underbrace{\beta}_{\in F} \right) = \psi(h_\alpha) + \varphi(h_\beta) + (h_\alpha, h_\beta)$

$\left(h_{\alpha_i}, h_{\alpha_j}\right) = d_i^{-1} d_j^{-1} (h_i, h_j) = d_i^{-1} d_j^{-1} a_{i,j} d_j = d_i^{-1} a_{i,j}$.

Basis $h^*_{\alpha_i} \in \mathfrak{h}^*$, $\alpha_i \in F \Longrightarrow$ matrix of the form $\begin{pmatrix} 0 & 1 \\ 1 & D^{-1}A \end{pmatrix}$.

Inverse form on $\mathfrak{h}_{\text{ext}}$: dual basis: h_{α_i}, D_i.

$(D_i, D_j) = 0$, $\left(D_i, h_{\alpha_j}\right) = \delta_{i,j}$, $\left(h_{\alpha_i}, h_{\alpha_j}\right) = d_i^{-1} a_{i,j}$.

Proposition 4.11.9. The form on $\mathfrak{g}_{\text{ext}}(A) = \mathfrak{g}(A) \oplus \mathbb{C}D_1 \oplus \mathbb{C}D_2 \oplus ... \oplus \mathbb{C}D_r$ defined by this is a nondegenerate symmetric invariant form.

4.12. [unfinished] Casimir element

We now define the Casimir element. The problem with the classical "sum of squares of orthonormal basis" construction which works well in the finite-dimensional case is that now we are infinite-dimensional and such a sum needs to be defined.

Note that it will be a generalization of the L_0 of the Sugawara construction.

Define $\rho \in \mathfrak{h}^*$ by $\rho(h_i) = \dfrac{a_{i,i}}{2}$ (in the Kac-Moody case, this becomes $\rho(h_i) = 1$).

$(\rho, \rho) = 0$.

Case of a finite-dimensional simple Lie algebra: $\Delta = \sum\limits_{a \in B} a^2 = \sum\limits_{i=1}^{r} x_i^2 + 2h_\rho + 2 \sum\limits_{\alpha > 0} f_\alpha e_\alpha$ where $(x_i)_{i=1,\ldots,r}$ is an orthonormal basis of $\mathfrak{h}$.

In the infinite-dimensional case, we fix a basis $(e_\alpha^i)_i$ of $\mathfrak{g}_\alpha$ for every α, and a dual basis $(f_\alpha^i)_i$ of $\mathfrak{g}_{-\alpha}$ under the inner product. Then define $\Delta_+ = 2\sum\limits_{\alpha>0}\sum\limits_i f_\alpha^i e_\alpha^i$ and $\Delta_0 = \sum\limits_j x_j^2 + 2h_\rho$ (where (x_j) is an orthonormal basis of $\mathfrak{h}_{\text{ext}}$). We set $\Delta = \Delta_+ + \Delta_0$.

Note that Δ_+ is an infinite sum and not in $U(\mathfrak{g}(A))$. But it becomes finite after applying to any vector in a module in category $\mathcal{O}$.

Theorem 4.12.1. 1) The operator Δ commutes with $\mathfrak{g}(A)$.
2) We have $\Delta\mid_{M_\lambda} = (\lambda, \lambda+2\rho)\operatorname{id}$.

Proof of Theorem. Let us first prove **2)** using **1)**:

2) We have $\Delta v_\lambda = \Delta_0 v_\lambda = \left(\sum\limits_j \lambda(x_j)^2 + 2\lambda(h_\rho)\right) v_\lambda = ((\lambda,\lambda) + 2(\lambda,\rho)) v_\lambda = (\lambda, \lambda+2\rho) v_\lambda$.

From **1)**, we see that every $a \in U(\mathfrak{g}(A))$ satisfies $\Delta a v_\lambda = a\Delta v_\lambda = (\lambda, \lambda+2\rho) a v_\lambda$. This proves **2)** since $M_\lambda = U(\mathfrak{g}(A)) v_\lambda$.

1) We need to show that $[\Delta, e_i] = [\Delta, f_i] = 0$.

Let us prove $[\Delta, e_i] = 0$ (the proof of $[\Delta, f_i] = 0$ is similar).

We have $[\Delta_0, e_i] = \left[\sum x_j^2 + 2h\rho, e_i\right] = \sum x_j [x_j, e_i] + \sum [x_j, e_i] x_j + 2(\alpha_i, \rho) e_i$

$= \sum x_j \underbrace{\alpha_i(x_j)}_{=(h_{\alpha_i}, x_j)} e_i + \sum \alpha_i(x_j) e_i x_j + 2(\alpha_i, \rho) e_i$

$= 2h_{\alpha_i} e_i - \sum \underbrace{\alpha_i(x_j)}_{=(\alpha_i,\alpha_i)e_i} \alpha_i(x_j) e_i + 2(\alpha_i, \rho) e_i = 2h_\alpha e_i$

$\Longrightarrow$ Our job is to show $[\Delta_+, e_i] = -2h_{\alpha_i} e_i$. But

$[\Delta_+, e_i] = 2\sum\limits_{\alpha>0} f_\alpha^j [e_\alpha^j, e_i] + 2 \underbrace{\sum\limits_{\alpha>0} \left[f_\alpha^j, e_i\right] e_\alpha^j}_{\substack{\text{for } \alpha=\alpha_i \text{ the addend is} \\ -2h_{\alpha_i}e_i \\ \text{because } f_{\alpha_i}=d_i^{-1}f_i,\ e_{\alpha_i}=e_i, \\ \left[d_i^{-1}f_i, e_i\right]e_i = -d_i^{-1}h_i e_i = -h_{\alpha_i}e_i}}$.

So we need to show that

$$\sum_{\alpha>0} f_\alpha^j \left[e_\alpha^j, e_i\right] + 2\sum_{\substack{\alpha>0; \\ \alpha\neq\alpha_i}} \left[f_\alpha^j, e_i\right] e_\alpha^j = 0.$$

For this it is enough to check

$$\sum_{\alpha>0} f_\alpha^j \otimes \left[e_\alpha^j, e_i\right] + 2\sum_{\substack{\alpha>0; \\ \alpha\neq\alpha_i}} \left[f_\alpha^j, e_i\right] \otimes e_\alpha^j = 0.$$

For this it is enough to check that $\left[e_i, e_\alpha^k\right] = \sum \left(e_\beta^k, [f_\alpha^j, e_i]\right) e_\alpha^j$. This is somehow obvious. Proof complete.

Exercise: for $\widehat{\mathfrak{g}}$ (affine), $\Delta = (k + h^\vee)(L_0 - d)$ (Sugawara).

4.13. [unfinished] Preparations for the Weyl-Kac character formula

Let A be a symmetrizable generalized Cartan matrix, WLOG indecomposable.

We consider the Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A) \subseteq \mathfrak{g}_{\text{ext}}(A)$.

Proposition 4.13.1. The Serre relations $(\mathrm{ad}\,(e_i))^{1-a_{i,j}}\,e_j = (\mathrm{ad}\,(f_i))^{1-a_{i,j}}\,f_j = 0$ hold in $\mathfrak{g}(A)$.

This is a part of Theorem 4.9.4 (actually, the part that we proved above).

Definition 4.13.2. Let A be an associative algebra (with 1, as always). Let V be an A-module.

(a) Let $v \in V$. Then, the vector v is said to be *of finite type* if $\dim(Av) < \infty$.

(b) The A-module V is said to be *locally finite* if every $v \in V$ is of finite type.

It is very easy to check that:

Proposition 4.13.3. Let A be an associative algebra (with 1, as always). Let V be an A-module. Then, V is locally finite if and only if V is a sum of finite-dimensional A-modules.

Proof of Proposition 4.13.3 (sketched). $\Longrightarrow$: Assume that V is locally finite. Then, for every $v \in V$, we have $\dim(Av) < \infty$ (since v is of finite type), so that Av is a finite-dimensional A-module. Thus, $V = \sum\limits_{v\in V} Av$ is a sum of finite-dimensional A-modules.

$\Longleftarrow$: Assume that V is a sum of finite-dimensional A-modules. Then, for every $v \in V$, the vector v belongs to a sum of **finitely many** finite-dimensional A-modules. But such a sum is finite-dimensional as well. As a consequence, for every $v \in V$, the vector v belongs to a finite-dimensional A-module, and thus $\dim(Av) < \infty$, so that v is of finite type. Thus, V is locally finite.

Proposition 4.13.3 is proven.

Convention 4.13.4. If $\mathfrak{g}$ is a Lie algebra, then "locally finite" and "of finite type" with respect to $\mathfrak{g}$ mean locally finite resp. of finite type with respect to $U(\mathfrak{g})$.

In the following, let $A = U(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{g}(A)$.

Definition 4.13.5. Let V be a $\mathfrak{g}(A)$-module. We say that V is *integrable* if V is locally finite under the $\mathfrak{sl}_2$-subalgebra $(\mathfrak{sl}_2)_i = \langle e_i, f_i, h_i\rangle$ for every $i \in \{1, 2, ..., r\}$.

To motivate the terminology "integrable", let us notice:

Proposition 4.13.6. If V is a $\mathfrak{sl}_2$-module, then V is locally finite if and only if V is isomorphic to a direct sum $\bigoplus\limits_{n=0}^{\infty} W_n \otimes V_n$, where W_n are vector spaces and V_n is the irreducible representation of $\mathfrak{sl}_2$ of highest weight n (so that $\dim(V_n) = n+1$) for every $n \in \mathbb{N}$. (In such a direct sum, we have $W_n \cong \mathrm{Hom}_{\mathfrak{sl}_2}(V_n, V)$.)

Locally-finite $\mathfrak{sl}_2$-modules can be lifted to modules over the **algebraic group** $\mathrm{SL}_2(\mathbb{C})$.

Since lifting is called "integrating" (in analogy to geometry, where an action of a Lie group gives rise to an action of the corresponding of the Lie algebra by "differentiation", and thus the converse operation, when it makes sense, is called "integration"), the last sentence of this proposition explains the name "integrable".

Proposition 4.13.7. The $\mathfrak{g}$-module $\mathfrak{g} = \mathfrak{g}(A)$ itself is integrable.

The proof of this proposition is based on the following lemma:

Lemma 4.13.8. Let $\mathfrak{a}$ be a Lie algebra, and $\mathfrak{b}$ be another Lie algebra. Assume that we are given a Lie algebra homomorphism $\mathfrak{b} \to \operatorname{Der} \mathfrak{a}$; this makes $\mathfrak{a}$ into a $\mathfrak{b}$-module. Then, if $x, y \in \mathfrak{a}$ are of finite type for $\mathfrak{b}$, then so is $[x, y]$.

Proof of Lemma 4.13.8. In $\mathfrak{a}$ (not in $U(\mathfrak{a})$), we have

$$U(\mathfrak{b}) \cdot [x, y] \subseteq \left[\underbrace{U(\mathfrak{b}) \cdot x}_{\text{finite dimensional}}, \underbrace{U(\mathfrak{b}) \cdot y}_{\text{finite dimensional}} \right].$$

Hence, $U(\mathfrak{b}) \cdot [x, y]$ is finite-dimensional. Hence, $[x, y]$ is of finite type for $\mathfrak{b}$. Lemma 4.13.8 is proven.

Proof of Proposition 4.13.7. We know that e_i is of finite type under $(\mathfrak{sl}_2)_i$ (in fact, e_i generates a 3-dimensional representation of $(\mathfrak{sl}_2)_i$), and that e_j is of finite type under $(\mathfrak{sl}_2)_i$ for every $j \neq i$ (in fact, e_j generates a representation of dimension $1 - a_{i,j}$). The same applies to f_j, and hence also to h_j (by Lemma 4.13.8). Hence (again using Lemma 4.13.8), the whole $\mathfrak{g}(A)$ is locally finite under $(\mathfrak{sl}_2)_i$. [Fix some stuff here.] Proposition 4.13.7 is proven.

Proposition 4.13.9. If V is a $\mathfrak{g}(A)$-module, then V is integrable if and only if there exists a generating family $(v_\alpha)_{\alpha \in \mathfrak{A}}$ of the $\mathfrak{g}(A)$-module V such that each v_α is of finite type under $(\mathfrak{sl}_2)_i$ for each i.

Note that this proposition could just as well be formulated for every Lie algebra $\mathfrak{g}$ instead of $\mathfrak{g}(A)$.

Proof of Proposition. $\Longleftarrow$: Let $v \in V$. We need to show that v is of finite type under $(\mathfrak{sl}_2)_i$ for all i.

Pick some $i \in \{1, 2, ..., r\}$. Let $\mathfrak{g} = \mathfrak{g}(A)$.

Fix some i. Then, there exist $i_1, i_2, ..., i_m \in \mathfrak{A}$ such that $v \in U(\mathfrak{g}) \cdot v_{i_1} + U(\mathfrak{g}) \cdot v_{i_2} + ... + U(\mathfrak{g}) \cdot v_{i_m}$. WLOG assume that $i_1 = 1$, $i_2 = 2$, ..., $i_m = m$, and denote the $\mathfrak{g}$-submodule $U(\mathfrak{g}) \cdot v_1 + U(\mathfrak{g}) \cdot v_2 + ... + U(\mathfrak{g}) \cdot v_m$ of V by V'. Then, $v \in U(\mathfrak{g}) \cdot v_{i_1} + U(\mathfrak{g}) \cdot v_{i_2} + ... + U(\mathfrak{g}) \cdot v_{i_m} = U(\mathfrak{g}) \cdot v_1 + U(\mathfrak{g}) \cdot v_2 + ... + U(\mathfrak{g}) \cdot v_m = V' \subseteq V$.

Pick a finite-dimensional $(\mathfrak{sl}_2)_i$-subrepresentation W of V' such that $v_1, v_2, ..., v_m \in W$. (This is possible because $v_1, v_2, ..., v_m$ are of finite type under $(\mathfrak{sl}_2)_i$.) Then we have a surjective homomorphism of $(\mathfrak{sl}_2)_i$-modules $U(\mathfrak{g}) \otimes W \to V'$ (namely, the homomorphism sending $x \otimes w$ to xw), where $\mathfrak{g}$ acts on $U(\mathfrak{g})$ by adjoint action, and where $(\mathfrak{sl}_2)_i$ acts on $U(\mathfrak{g})$ by restricting the $\mathfrak{g}$-action on $U(\mathfrak{g})$ to $(\mathfrak{sl}_2)_i$. So it suffices to show that $U(\mathfrak{g})$ is integrable for the adjoint action of $\mathfrak{g}$. But by the symmetrization map (which is an isomorphism by PBW), we have $U(\mathfrak{g}) \cong S(\mathfrak{g}) = \bigoplus\limits_{m \in \mathbb{N}} S^m(\mathfrak{g})$ (as $\mathfrak{g}$-modules) (this is true for every Lie algebra over a field of characteristic 0). Since $S^m(\mathfrak{g})$ injects into $\mathfrak{g}^{\otimes m}$, and since $\mathfrak{g}^{\otimes m}$ is integrable (because $\mathfrak{g}$ is (in fact, it is easy to see that if X and Y are locally finite $\mathfrak{a}$-modules, then so is $X \otimes Y$)), this yields that $U(\mathfrak{g})$ is integrable. Hence, $U(\mathfrak{g}) \otimes W$ is a locally finite $(\mathfrak{sl}_2)_i$-module, and thus V' (being a quotient module of $U(\mathfrak{g}) \otimes W$) is a locally finite $(\mathfrak{sl}_2)_i$-module also as well. Hence, v (being an element of V') is of finite type under $(\mathfrak{sl}_2)_i$.

$\Longrightarrow$: Trivial (take all vectors of V as generators).

Proposition proven.

Corollary 4.13.10. Let L_λ be the irreducible highest-weight module for $\mathfrak{g}(A)$. Then, L_λ is integrable if and only if for every $i \in \{1, 2, ..., r\}$, the value $\lambda(h_i)$ is a nonnegative integer.

Proof of Corollary. $\Longrightarrow$: Assume that L_λ is integrable. Consider the element v_λ of L_λ. Since L_λ is integrable, we know that v_λ is of finite type under $(\mathfrak{sl}_2)_i$. In other words, $U((\mathfrak{sl}_2)_i) v_\lambda$ is a finite-dimensional $(\mathfrak{sl}_2)_i$-module. Also, we know that $v_\lambda \neq 0$, $e_i v_\lambda = 0$ and $h_i v_\lambda = \lambda(h_i) v_\lambda$. Hence, Lemma 4.6.1 **(c)** (applied to $(\mathfrak{sl}_2)_i$, e_i, h_i, f_i, $U((\mathfrak{sl}_2)_i) v_\lambda$, v_λ and $\lambda(h_i)$ instead of $\mathfrak{sl}_2$, e, h, f, V, x and λ) yields that $\lambda(h_i) \in \mathbb{N}$ and $f_i^{\lambda(h_i)+1} v_\lambda = 0$. In particular, $\lambda(h_i)$ is a nonnegative integer.

$\Longleftarrow$: We have

$$\begin{aligned} e_i f_i^{\lambda(h_i)+1} v_\lambda &= (\lambda(h_i) + 1) \underbrace{(\lambda(h_i) - (\lambda(h_i) + 1) + 1)}_{=0} f_i^{\lambda(h_i)} v_\lambda \\ &\qquad \left(\text{by the formula } e_i f_i^m v_\lambda = m(\lambda(h_i) - m + 1) f_i^{m-1} v_\lambda\right) \\ &= 0. \end{aligned}$$

Hence, $f_i^{\lambda(h_i)+1} v_\lambda$ must also be zero (since otherwise, this vector would generate a proper graded submodule). This implies that v_λ generates a finite-dimensional $(\mathfrak{sl}_2)_i$-module of dimension $\lambda(h_i) + 1$ with basis $\left(v_\lambda, f_i v_\lambda, ..., f_i^{\lambda(h_i)} v_\lambda\right)$. Hence, v_λ is of finite type with respect to $(\mathfrak{sl}_2)_i$.

By the previous proposition, this yields that L_λ is integrable. Proof of Corollary complete.

Remark 4.13.11. Assume that for every $i \in \{1, 2, ..., r\}$, the value $\lambda(h_i)$ is a nonnegative integer. Then, the relations $f_i^{\lambda(h_i)+1} v_\lambda = 0$ are defining for L_λ.

We will not prove this now, but this will follow from things we do later (from the main theorem for the character formula).

Definition 4.13.12. A weight λ for which all $\lambda(h_i)$ are nonnegative integers is called *integral* (for $\mathfrak{g}(A)$ or for $\mathfrak{g}_{\text{ext}}(A)$).

Now, our next goal is to compute the character of L_λ for any dominant integral weight λ.

For finite-dimensional simple Lie algebras, these L_λ are exactly the finite-dimensional irreducible representations, and their characters can be computed by the well-known Weyl character formula. So our goal is to generalize this formula.

The Weyl character formula involves a summation over the Weyl group. So, first of all, we need to define a "Weyl group" for Kac-Moody Lie algebras.

4.14. [unfinished] Weyl group

Definition 4.14.1. Consider $P = \mathfrak{h}^* \oplus F$. We know that there is a nondegenerate form $(\cdot, \cdot)$ on P, and we have $\dim P = 2r$. Let $i \in \{1, 2, ..., r\}$. Let $r_i : P \to P$ be the map given by $r_i(\chi) = \chi - \chi(h_i)\alpha_i$.

Note that r_i is an involution, since

$$r_i^2(\chi) = \chi - \chi(h_i)\alpha_i - \chi(h_i)\alpha_i + \chi(h_i)\underbrace{\alpha_i(h_i)}_{=2}\alpha_i = \chi$$

for every $\chi \in P$. Since $r_i(\alpha_i) = -\alpha_i$, this yields $\det(r_i) = -1$.

Easy to check that $(r_i x, r_i y) = (x, y)$ for all $x, y \in P$.

Proposition 4.14.2. Let V be an integrable $\mathfrak{g}(A)$-module. Then, for each $i \in \{1, 2, ..., r\}$ and any $\mu \in P$, we have an isomorphism $V[\mu] \to V[r_i\mu]$. In particular, $\dim(V[\mu]) = \dim(V[r_i\mu])$.

Proof of Proposition. We have $r_i\mu = \mu - \mu(h_i)\alpha_i$. Since V is integrable for $(\mathfrak{sl}_2)_i$, we know that $\mu(h_i)$ is an integer. We have $(r_i\mu)(h_i) = -\mu(h_i)$. Hence, we can assume WLOG that $\mu(h_i)$ is nonnegative (because otherwise, we can switch μ with $r_i\mu$, and it will change sign). Then we have $f_i^{\mu(h_i)} : V[\mu] \to V[r_i\mu]$.

I claim that $f_i^{\mu(h_i)}$ is an isomorphism.

This follows from:

Lemma 4.14.3. If V is a locally finite $\mathfrak{sl}_2$-module, then $f^m : V[m] \to V[-m]$ is an isomorphism.

Definition 4.14.4. The *Weyl group of* $\mathfrak{g}(A)$ is defined as the subgroup of $\mathrm{GL}(P)$ generated by the r_i. This Weyl group is denoted by W. The elements r_i are called *simple reflections.*

We will not prove:

Remark 4.14.5. The Weyl group W is finite if and only if A is a Cartan matrix (of a finite-dimensional Lie algebra).

Proposition 4.14.6. 1) The form $(\cdot, \cdot)$ on P is W-invariant.

2) There exists an isomorphism $V[\mu] \to V[w\mu]$ for every $\mu \in P$, $w \in W$ and any integrable V.

3) The set of roots R is W-invariant. (We recall that a *root* means a nonzero element $\alpha \in F = Q \otimes_{\mathbb{Z}} \mathbb{C}$ such that $\mathfrak{g}_\alpha \neq 0$. We consider F as a subspace of P.)

4) We have $r_i(\alpha_i) = -\alpha_i$. Moreover, r_i induces a permutation of all positive roots except for α_i.

Proof of Proposition. **1)** and **2)** follow easily from the corresponding statement for generators proven above.

3) By part **2)**, the set of weights $P(V)$ of an integrable $\mathfrak{g}$-module V is W-invariant. (Here, "weight" means a weight whose weight subspace is nonzero.) Applied to $V = \mathfrak{g}$, this implies **3)** (since $P(\mathfrak{g}) = 0 \cup R$).

4) Proving $r_i(\alpha_i) = -\alpha_i$ is straightforward. Now for the other part:

Any positive root can be written as $\alpha = \sum_i k_i\alpha_i$ where all k_i are ≥ 0 and $\sum_i k_i > 0$. Thus, for such a root, $r_i(\alpha) = \alpha - \alpha(h_i)\alpha_i = \sum_{j\neq i} k_j\alpha_j + (k_i - \alpha(h_i))\alpha_i$.

If there exists a $j \neq i$ such that $k_j > 0$, then $r_i(\alpha)$ must be a positive root (since there is no such thing as a partly-negative-partly-positive root).

Alternative: $k_j = 0$ for all $j \neq i$. But then $\alpha = k_i\alpha_i$, so that $k_i = 1$ (because a positive multiple of a simple root is not a root, unless we are multiplying with 1), but this is the case we excluded ("except for α_i"). Proposition proven.

4.15. [unfinished] The Weyl-Kac character formula

Theorem 4.15.1 (Kac)**.** Denote by P_+ the set $\{\chi \in P \mid \chi(h_i) \in \mathbb{N} \text{ for all } i \in \{1, 2, ..., r\}\}$.

Let χ be a dominant integral weight of $\mathfrak{g}(A)$. (This means that $\chi(h_i)$ is a nonnegative integer for every $i \in \{1, 2, ..., r\}$.) Let V be an integrable highest-weight $\mathfrak{g}_{\text{ext}}(A)$-module with highest weight χ. Then:

(1) The $\mathfrak{g}$-module V is isomorphic to L_χ. (In other words, the $\mathfrak{g}$-module V is irreducible.)

(2) The character of V is

$$\operatorname{ch}(V) = \frac{\sum\limits_{w\in W} \det(w) \cdot e^{w(\chi+\rho)-\rho}}{\prod\limits_{\alpha>0} (1 - e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)}} \qquad \text{in } R.$$

Here, we recall that R is the ring $\lim\limits_{\lambda\in P_+} e^{\lambda}\mathbb{C}[[e^{-\alpha_1}, e^{-\alpha_2}, ..., e^{-\alpha_r}]]$ (note that this term increases when λ is changed to $\lambda + \alpha_i$) in which the characters are defined.

Here, ρ is the element of $\mathfrak{h}^*$ satisfying $\rho(h_i) = 1$ (as defined above). Since $\mathfrak{h}^* \subseteq P$, this ρ becomes an element of P.

Note that $\det(w)$ is always 1 or -1 (and, in fact, equals $(-1)^k$, where w is written in the form $w = r_{i_1}r_{i_2}...r_{i_k}$).

Part **(2)** of this theorem is called the *Weyl-Kac character formula*.

We want to prove this theorem.

Since χ is a dominant integral weight, we have $\chi \in P_+$.

Some comments on the theorem:

First of all, part **(2)** implies part **(1)**, since both V and L_χ satisfy the conditions of the Theorem and thus (according to part **(2)**) share the same character, but we also have a surjective homomorphism $\varphi : V \to L_\chi$, so (because of the characters being the same) it is an isomorphism. Thus, we only need to bother about proving part **(2)**.

Secondly, let us remark that the theorem yields $L_\lambda = M_\lambda \diagup \left\langle f_i^{\lambda(h_i)+1} v_\lambda \mid i \in \{1, 2, ..., r\}\right\rangle$ for all dominant integral weights λ. Indeed, denote $M_\lambda \diagup \left\langle f_i^{\lambda(h_i)+1} v_\lambda \mid i \in \{1, 2, ..., r\}\right\rangle$ by L'_λ. Then, L'_λ is integrable (as we showed above more or less; more precisely, we showed that L_λ was integrable, but this proof went exactly through proving that L'_λ is integrable), so that the theorem is still applicable to L'_λ and we obtain $L'_\lambda \cong L_\lambda$.

Our third remark: In the case of a simple finite-dimensional Lie algebra $\mathfrak{g}$, we have

$$\operatorname{ch}(M_\lambda) = \frac{e^\lambda}{\prod\limits_{\alpha>0} (1-e^{-\alpha})}.$$

The denominator can be rewritten $\prod\limits_{\alpha>0} (1-e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)}$, since $\dim(\mathfrak{g}_\alpha) = 1$ for all roots α.

In the case of Kac-Moody Lie algebras $\mathfrak{g} = \mathfrak{g}(A)$, we can use similar arguments to show that

$$\operatorname{ch}(M_\lambda) = \frac{e^\lambda}{\prod\limits_{\alpha>0} (1-e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)}}.$$

So the Weyl-Kac character formula can be written as

$$\operatorname{ch}(V) = \sum_{w\in W} \det(w) \cdot \operatorname{ch}\left(M_{w(\chi+\rho)-\rho}\right).$$

This formula can be proven using the BGG[230] resolution (in fact, it is obtained as the Euler character of that resolution), but we will take a different route here.

Another remark before we prove the formula. The Weyl-Kac character formula has the following corollary:

Corollary 4.15.2 (Weyl-Kac denominator formula). We have $\prod\limits_{\alpha>0} (1-e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)} = \sum\limits_{w\in W} \det(w) \cdot e^{w\rho-\rho}$.

Proof of Corollary (using Weyl-Kac character formula). Set $\chi = 0$. Then $L_\chi = \mathbb{C}$, so that $\operatorname{ch}(L_\chi) = 1$ but on the other hand $\operatorname{ch}(L_\chi) = \dfrac{\sum\limits_{w\in W} \det(w) \cdot e^{w\rho-\rho}}{\prod\limits_{\alpha>0} (1-e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)}}$. Thus, $\prod\limits_{\alpha>0} (1-e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)} = \sum\limits_{w\in W} \det(w) \cdot e^{w\rho-\rho}$.

To prove the Weyl-Kac character formula, we will have to show several lemmas.

Lemma 4.15.3. Let $\chi \in P_+$.

(1) Then, $W\chi \subseteq D(\chi)$ (where, as we recall, $D(\chi)$ denotes the set $\{\chi - \sum_i k_i\alpha_i \mid k_i \in \mathbb{N} \text{ for all } i\}$.

(2) If $D \subseteq D(\chi)$ is a W-invariant subset, then $D \cap P_+ \neq \varnothing$.

Proof of Lemma 4.15.3. **(1)** Consider L_χ. Since L_χ is integrable, the set $P(L_\chi)$ is W-invariant, so that $W\chi \subseteq P(L_\chi)$. But $P(L_\chi) \subseteq D(\chi)$, since any weight of L_χ is χ minus a sum of positive roots. Part **(1)** is proven.

(2) Let $\psi \in D$. Pick $w \in W$ such that $x - w\psi = \sum_i k_i\alpha_i$ with nonnegative integers k_i and minimal $\sum_i k_i$. We claim that this w satisfies $w\psi \in P_+$. This, of course, will prove part **(2)**.

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To prove $w\psi \in P_+$, assume that $w\psi \notin P_+$. Then, there exists an i such that $(w\psi, \alpha_i) = d_i^{-1} (w\psi)(h_i) < 0$. (Note that all the d_i are > 0.) Then, $r_i w\psi = w\psi - (w\psi)(h_i)\alpha_i$, so that $\chi - r_i w\psi = \chi - w\psi + (w\psi)(h_i)\alpha_i = \sum_j k_j \alpha_j + (w\psi)(h_i)\alpha_i = \sum_j k'_j \alpha_j$ and $\sum_j k'_j = \sum_j k_j + (w\psi)(h_i) < \sum_j k_j$. This contradicts the minimality in our choice of w. Part **(2)** is thus proven.

Corollary 4.15.4. Let $w \in W$ satisfy $w \neq 1$. Then, there exists i such that $w\alpha_i < 0$. (By $w\alpha_i < 0$ we mean that $w\alpha_i$ is a negative root.)

Proof of Corollary 4.15.4. Choose $\chi \in P_+$ such that $w\chi \neq \chi$. (Such a χ always exists, due to the definition of P_+). Then, $w^{-1}\chi = \chi - \sum k_i \alpha_i$ for some $k_i \in \mathbb{N}$ (by Lemma 4.15.3 **(1)**). Hence,

$$\chi = ww^{-1}\chi = w\chi - \sum k_i w\alpha_i = \left(\chi - \sum k'_i \alpha_i\right) - \sum k_i w\alpha_i.$$

Thus, $\sum k'_i \alpha_i + \sum k_i w\alpha_i = 0$. But $\sum k'_i > 0$, so there must exist an i such that $w\alpha_i < 0$. Corollary 4.15.4 is proven.

Proposition 4.15.5. Let $\varphi, \psi \in P$ be such that $\varphi(h_i) > 0$ and $\psi(h_i) \geq 0$ for each i. Let $w \in W$.

Then, $w\varphi = \psi$ if and only if $\varphi = \psi$ and $w = 1$.

Proof of Proposition 4.15.5. For every i, we have $\varphi(h_i) > 0$ if and only if $(\varphi, \alpha_i) > 0$. Now suppose that there exists a $w \neq 1$ such that $w\varphi = \psi$. Then, by Corollary 4.15.4, there exists an i such that $w\alpha_i < 0$. Then, $(\varphi, \alpha_i) > 0$ but $(\varphi, \alpha_i) = (w^{-1}\psi, \alpha_i) = (\psi, w\alpha_i) \leq 0$. This is a contradiction. Proposition 4.15.5 is proven.

Next, notice that W acts on R.

Proposition 4.15.6. Let K denote the Weyl-Kac denominator $\prod\limits_{\alpha>0} (1 - e^{-\alpha})^{\dim(\mathfrak{g}_\alpha)}$.

Then, $w \cdot K = \det(w) \cdot K$ for every $w \in W$.

Proof of Proposition 4.15.6. We can WLOG take $w = r_i$ (since det is multiplicative). Then,

$$\begin{aligned} r_i K &= e^{r_i\rho} \prod_{\alpha>0} \left(1 - e^{-r_i\alpha}\right)^{\dim(\mathfrak{g}_\alpha)} = e^{r_i\rho} \left(1 - e^{+\alpha_i}\right)^{\dim\left(\mathfrak{g}_{\alpha_i}\right)} \prod_{\substack{\alpha>0;\\ \alpha\neq\alpha_i}} \left(1 - e^{-\alpha}\right)^{\dim(\mathfrak{g}_\alpha)} \\ &\qquad \text{(by Proposition 4.14.6)} \\ &= e^{r_i\rho} \left(1 - e^{+\alpha_i}\right) \prod_{\substack{\alpha>0;\\ \alpha\neq\alpha_i}} \left(1 - e^{-\alpha}\right)^{\dim(\mathfrak{g}_\alpha)} \qquad \text{(since } \dim(\mathfrak{g}_{\alpha_i}) = 1) \\ &= \frac{e^{r_i\rho}(1 - e^{+\alpha_i})}{e^{\rho}(1 - e^{-\alpha_i})} \cdot K. \end{aligned}$$

Thus, we must only prove that $\dfrac{e^{r_i\rho}(1 - e^{+\alpha_i})}{e^{\rho}(1 - e^{-\alpha_i})} = -1$.

But this is very easy: We have $r_i\rho = \rho - \underbrace{\rho(h_i)}_{=1}\alpha_i = \rho - \alpha_i$, so that

$$\frac{e^{r_i\rho}\left(1-e^{+\alpha_i}\right)}{e^{\rho}\left(1-e^{-\alpha_i}\right)} = \frac{e^{\rho-\alpha_i}\left(1-e^{+\alpha_i}\right)}{e^{\rho}\left(1-e^{-\alpha_i}\right)} = \frac{e^{-\alpha_i}\left(1-e^{+\alpha_i}\right)}{1-e^{-\alpha_i}} = \frac{e^{-\alpha_i}-1}{1-e^{-\alpha_i}} = -1.$$

Proposition 4.15.6 is proven.

Proposition 4.15.7. Let $\mu,\nu \in P_+$ be such that $\mu \in D(\nu)$ and $\mu \neq \nu$. Then, $(\nu+\rho)^2 - (\mu+\rho)^2 > 0$. Here, λ^2 is defined to mean the inner product (λ,λ).

Proof of Proposition 4.15.7. We have $\nu - \mu = \sum\limits_i k_i\alpha_i$ for some $k_i \geq 0$ (since $\mu \in D(\nu)$). There exists an i such that $k_i > 0$ (because $\mu \neq \nu$). Now,

$$(\nu+\rho)^2 - (\mu+\rho)^2 = (\nu-\mu, \mu+\nu+2\rho) = \sum_i k_i(\alpha_i, \mu+\nu+2\rho).$$

But now use $(\alpha_i,\mu) \geq 0$ (since $\mu \in P_+$), also $(\alpha_i,\nu) \geq 0$ (since $\nu \in P_+$) and $(\alpha_i,\rho) = d_i^{-1} > 0$ to conclude that this is > 0 (since there exists an i such that $k_i > 0$). Proposition 4.15.7 is proven.

Proposition 4.15.8. Suppose that V is a $\mathfrak{g}_{\text{ext}}(A)$-module from Category $\mathcal{O}$ such that the Casimir C satisfies $\Delta\mid_V = \gamma\cdot\mathrm{id}$. Then, $\operatorname{ch}(V) = \sum c_\lambda \operatorname{ch}(M_\lambda)$, where the sum is over all λ satisfying $(\lambda,\lambda+2\rho) = \gamma$, and $c_\lambda \in \mathbb{Z}$ are some integers.

Proof of Proposition 4.15.8. The expansion is built inductively as follows:

Suppose $P(V) \subseteq D(\lambda_1)\cup D(\lambda_2)\cup\ldots\cup D(\lambda_m)$ for some weights $\lambda_1,\lambda_2,\ldots,\lambda_m$. Assume that this is a minimal such union. Then, $\lambda_i + \alpha_j \notin P(V)$ for any i,j.

Let $d_i = \dim(V[\lambda_i])$. Then, we have a homomorphism $\varphi : \bigoplus_i d_iM_{\lambda_i} \to V$ which is an isomorphism in weight λ_i. Let $K = \operatorname{Ker}\varphi$. Let $C = \operatorname{Coker}\varphi$. Clearly, both K and C lie in Category $\mathcal{O}$. We have an exact sequence $0 \to K \to \bigoplus_i d_iM_{\lambda_i} \to V \to C \to 0$. Since the alternating sum of characters in an exact sequence is 0, this yields $\operatorname{ch}V = \sum_i d_i \operatorname{ch}(M_{\lambda_i}) - \operatorname{ch}K + \operatorname{ch}C$.

Now we claim that $\Delta\mid_{M_{\lambda_i}} = (\lambda_i,\lambda_i+2\rho) = \gamma$ if $d_i \neq 0$. (Otherwise, a homomorphism φ could not exist.)

Also, $\Delta\mid_K = \Delta\mid_C = \gamma$.

But if $\mu \in P(K)\cup P(C)$, then for some i, we have $\lambda_i - \mu = \sum k_j\alpha_j$ with $\sum k_j \geq 1$.

Next step: $\sum k_i \geq 2$.

Etc.

If we run this procedure indefinitely, eventually every weight in this cone will be exhausted. Then we apply the procedure to K and C, and then to their K and C etc..

Proof of Weyl-Kac character formula. According to Proposition 4.15.8, we have

$$\operatorname{ch}(V) = \sum_{\psi\in D(\chi)} c_\psi \operatorname{ch}(M_\psi) \qquad \text{with } c_\chi = 1.$$

We will now need:

Corollary 4.15.9. If $c_\psi \neq 0$, then $(\psi+\rho)^2 = (\chi+\rho)^2$.

Proof of Corollary 4.15.9. This follows from Proposition 4.15.8.

Lemma 4.15.10. If $\psi + \rho = w(\chi+\rho)$, then $c_\psi = \det(w) \cdot c_\chi$.

Proof of Lemma 4.15.10. We have $wK = (\det w) \cdot K$ and $w \cdot \operatorname{ch} V = \operatorname{ch} V$. Hence, $w(K \cdot \operatorname{ch} V) = (\det w) \cdot (K \operatorname{ch} V)$. But since $\operatorname{ch}(M_\psi) = \dfrac{\sum c_\psi e^{\psi+\rho}}{K}$, we have $K \operatorname{ch} V = \sum\limits_{\psi \in D(\chi)} c_\psi e^{\psi+\rho} = (\det w) \cdot \sum\limits_{\psi \in D(\chi)} c_\psi e^{\psi+\rho}$. (If $\psi + \rho = w(\chi+\rho)$.) Thus, $c_\psi = (\det w) \cdot c_\chi$.

Lemma 4.15.11. Let $D = \{\psi \mid c_{\psi-\rho} \neq 0\}$. Then, $D = W(\chi+\rho)$.

Proof of Lemma 4.15.11. We have $W(\chi+\rho) \subseteq D$ by Lemma 4.15.10. Also, D is W-invariant since V is integrable.

Suppose $D \neq W(\chi+\rho)$. Then, $(D \diagdown W(\chi+\rho)) \cap P_+ \neq \varnothing$ by Lemma 4.15.3 **(2)**. Take some $\beta \in (D \diagdown W(\chi+\rho)) \cap P_+$. Then, $\beta - \rho \in D(\chi)$, so that $(\chi+\rho, \chi+\rho) - (\beta, \beta) > 0$ (by Proposition 4.15.7). Thus, β cannot occur in the sum (by Corollary 4.15.9).

Punchline: $\operatorname{ch} V = \sum_{w \in W} \dfrac{(\det w) \cdot e^{w(\chi+\rho)}}{K}$. This is exactly the Weyl-Kac character formula.

4.16. [unfinished] ...

[...]

5. [unfinished] ...

[...] [747l22.pdf]

KZ equations, consistent (define a flat connection)

$\mathfrak{g}$ simple Lie algebra

$V_1, V_2, ..., V_N$ representations of $\mathfrak{g}$ from Category $\mathcal{O}$.

$\mathbb{C}_0^N = \mathbb{C}^N \diagdown \{z_i = z_j\}$

$U \subseteq \mathbb{C}_0^N$ simply connected open set

$F(z_1, ..., z_N) \in (V_1 \otimes V_2 \otimes ... \otimes V_N)[\nu]$ holomorphic function in $z_1, ..., z_N$ for a fixed weight ν.

$x \in \mathbb{C}$ [or was it $\kappa \in \mathbb{C}$?]

$\dfrac{\partial F}{\partial z_i} - \overline{h} \sum\limits_{i \neq j} \dfrac{\Omega_{i,j}}{z_i - z_j} F$ where $\Omega_{i,j} : V_1 \otimes V_2 \otimes ... \otimes V_N \to V_1 \otimes V_2 \otimes ... \otimes V_N$

$\Omega \in (S^2 \mathfrak{g})^{\mathfrak{g}}$

Consistent means: setting $\nabla_i = \dfrac{\partial}{\partial z_i} - \overline{h} \sum\limits_{i \neq j} \dfrac{\Omega_{i,j}}{z_i - z_j}$, we have $[\nabla_i, \nabla_j] = 0$. Consistent systems are known to have locally unique-and-existent solutions.

Why is this in our course?

The reason is that these equations arise in the representation theory of affine Lie algebras.

Interpretation of KZ equations in terms of $\widehat{\mathfrak{g}}$:
Consider $L\mathfrak{g}$, $\widehat{\mathfrak{g}}$, $\widetilde{\mathfrak{g}} = \widehat{\mathfrak{g}} \rtimes \mathbb{C}d$.
Define Weyl modules:

Definition 5.0.1. Let $\lambda \in P_+$ be a dominant integral weight for a simple finite-dimensional Lie algebra $\mathfrak{g}$. Let L_λ be an irreducible finite-dimensional representation of $\mathfrak{g}$ with highest weight λ. Let us extend L_λ to a $\mathfrak{g}[t] \oplus \mathbb{C}K$-module by making $t\mathfrak{g}[t]$ act by 0 and K act by some scalar k (that is, $K\mid_{L_\lambda} = k \cdot \mathrm{id}$ for some $k \in \mathbb{C}$).

Denote this $\mathfrak{g}[t] \oplus \mathbb{C}K$-module by $L_\lambda^{(k)}$. Then, we define a $\widehat{\mathfrak{g}}$-module $V_{\lambda,k} = U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t]\oplus\mathbb{C}K)} L_\lambda^{(k)}$. This module is called a *Weyl module* for $\widehat{\mathfrak{g}}$ at level k.

By the PBW theorem, we immediately see that $U(\widehat{\mathfrak{g}}) \cong U(t^{-1}\mathfrak{g}[t^{-1}]) \otimes U(\mathfrak{g}[t] \oplus \mathbb{C}K)$ and thus $V_{\lambda,k} \cong U(t^{-1}\mathfrak{g}[t^{-1}]) \otimes L_\lambda$ (canonically, but only as vector spaces).

Assuming that $k \neq -h^\vee$, we can extend $V_{\lambda,k}$ to $\widetilde{\mathfrak{g}}$ by letting d act as $-L_0$ (from Sugawara construction).

Definition 5.0.2. If V is a $\mathfrak{g}$-module, then $V[z, z^{-1}]$ is an $L\mathfrak{g}$-module, and in fact a $\widehat{\mathfrak{g}}$-module where K acts by 0. It extends to $\widetilde{\mathfrak{g}}$ by setting $d = z\dfrac{\partial}{\partial z}$.

More generally: Can set $d(vz^n) = (n - \Delta)vz^n$ for any fixed $\Delta \in \mathbb{C}$.

Call this module $z^{-\Delta}V[z, z^{-1}]$.

Lemma 5.0.3. If $k \notin \mathbb{Q}$, then $V_{\lambda,k}$ is irreducible.

Proof of Lemma. Assume $V_{\lambda,k}$ is reducible. This $V_{\lambda,k}$ is a highest-weight module. So, it must have a singular vector in degree $\ell > 0$. Let C be the Casimir for $\widetilde{\mathfrak{g}}$. We know $C = L_0 - \deg$ (where deg returns the positive degree).

Assume that w (our singular vector) lives in an irr. repr. of $\mathfrak{g}$. Singular vector means $a(m)w = 0$ for all $m > 0$. Here $a(m)$ means at^m.

$C\mid_{V_{\lambda,k}} = \dfrac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)}$

$Cw = \left(\dfrac{(\mu, \mu + 2\rho)}{2(k + h^\vee)} - \ell\right) w$

$L_0 = \dfrac{1}{2(k + h^\vee)} \sum_{i\in\mathbb{Z}} \sum_{a\in B} : a(i)\,a(-i) : = \dfrac{1}{2(k + h^\vee)} \left(\sum_{a\in B} a(0)^2 + 2\sum_{a\in B} \sum_{m\geq 1} a(-m)\,a(m)\right)$

where $a(m) = at^m$.

$\Longrightarrow \underbrace{(\lambda, \lambda + 2\rho) = (\mu, \mu + 2\rho)}_{\in\mathbb{Z}} - 2\ell(k + h^\vee) \Longrightarrow k = -h^\vee + \dfrac{(\lambda, \lambda + 2\rho) - (\mu, \mu + 2\rho)}{2\ell} \in \mathbb{Q}$. $\Longrightarrow$ contradiction.

Corollary 5.0.4. If $k \notin \mathbb{Q}$, then $V_{\lambda,k}^*$ (restricted dual) is $U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t^{-1}]\oplus\mathbb{C}K)} L_\lambda^{*(-k)}$. (Here, $L_\lambda^{*(-k)}$ means L_λ^* with K acting as $-k$.)

Proof of Corollary. From Frobenius reciprocity, we have a homomorphism $\varphi : U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t^{-1}]\oplus\mathbb{C}K)} L_\lambda^{*(-k)} \to V_{\lambda,k}^*$ which is id in degree 0. In fact, Frobenius reciprocity tells us that

$$\mathrm{Hom}_{\widehat{\mathfrak{g}}}\left(U(\widehat{\mathfrak{g}}) \otimes_{U(\mathfrak{g}[t^{-1}]\oplus\mathbb{C}K)} L_\lambda^{*(-k)}, M\right) \cong \mathrm{Hom}_{\mathfrak{g}[t^{-1}]\oplus\mathbb{C}K}\left(L_\lambda^{*(-k)}, M\right),$$

which, in the case $M = V_{\lambda,k}^*$, becomes [...].

Because $V_{\lambda,k}$ is irreducible (here we are using $k \notin \mathbb{Q}$), $V_{\lambda,k}^*$ is irreducible as well, this homomorphism φ is surjective. This φ also preserves grading, and the characters are equal. $\Longrightarrow$ φ is an isomorphism.

Corollary 5.0.5. $\operatorname{Hom}_{\widetilde{\mathfrak{g}}}\left(V_{\lambda,k} \otimes V_{\nu,k}^*, z^{-\Delta}V\left[z, z^{-1}\right]\right) \cong \operatorname{Hom}_{\mathfrak{g}}\left(L_\lambda \otimes L_\nu^*, V\right)$ if $\Delta = \Delta(\lambda) - \Delta(\nu)$.

Proof of Corollary. Frobenius reciprocity as for the previous corollary. (Skip.)
[...]

We now cite a classical theorem on ODEs.

Theorem 5.0.6. Let $N \in \mathbb{N}$. Let $A(z) = A_0 + A_1 z + A_2 z^2 + ...$ be a holomorphic function on $\{z \in \mathbb{C} \mid |z| < 1\}$ with values in $\mathrm{M}_N(\mathbb{C})$. Assume that for any eigenvalues λ and μ of A_0 such that $\lambda \neq \mu$, one has $\lambda - \mu \notin \mathbb{Z}$. Then, the ODE $z\dfrac{dF}{dz} = A(z)F$ (which, of course, is equivalent to $\dfrac{dF}{dz} = \dfrac{A(z)}{z}F$) has a matrix solution of the form $F(z) = \left(1 + B_1 z + B_2 z^2 + ...\right) z^{A_0}$ such that the power series $1 + B_1 z + B_2 z^2 + ...$ converges for $|z| < 1$. Here, z^{A_0} means $\exp\left(A_0 \log z\right)$ (on $\mathbb{C} \diagdown \mathbb{R}_{\leq 0}$).

Remark 5.0.7. This is a development of the following basic theorem: If we are given an ODE $\dfrac{dF}{dz} = C(z)F$ with $C(z)$ holomorphic, then there exists a holomorphic F satisfying this equation and having the form $F = 1 + O(z)$ (the so-called fundamental equation).

Proof of Theorem. Plug in the solution $F(z)$ in the above formula:

$$\left(\sum_{n\geq 1} nB_n z^n\right) z^{A_0} + \left(1 + \sum_{n\geq 1} B_n z^n\right) A_0 z^{A_0} = \left(A_0 + A_1 z + A_2 z^2 + ...\right)\left(1 + B_1 z + B_2 z^2 + ...\right) z^{A_0}.$$

Cancel z^{A_0} from this to obtain

$$\sum_{n\geq 1} nB_n z^n + \left(1 + \sum_{n\geq 1} B_n z^n\right) A_0 = \left(A_0 + A_1 z + A_2 z^2 + ...\right)\left(1 + B_1 z + B_2 z^2 + ...\right).$$

This is the system of recursive equations

$$nB_n - A_0 B_n + B_n A_0 = A_1 B_{n-1} + A_2 B_{n-2} + ... + A_{n-1} B_1 + A_n.$$

This rewrites as

$$\left(n - \operatorname{ad} A_0\right)\left(B_n\right) = A_1 B_{n-1} + A_2 B_{n-2} + ... + A_{n-1} B_1 + A_n.$$

The operator $n - \operatorname{ad} A_0 : \mathrm{M}_N(\mathbb{C}) \to \mathrm{M}_N(\mathbb{C})$ is invertible (because eigenvalues of this operator are $n - (\lambda - \mu)$ for λ and μ being eigenvalues of A_0, and because of the condition that for any eigenvalues λ and μ of A_0 such that $\lambda \neq \mu$, one has $\lambda - \mu \notin \mathbb{Z}$). Hence, we can use the above equation to recursively compute B_n for all n.

This implies that a solution in the formal sense exists.

We also need to estimate radius of convergence. [...]

The following generalizes our theorem to several variables:

Theorem 5.0.8. Let $m \in \mathbb{N}$ and $N \in \mathbb{N}$. For every $i \in \{1, 2, ..., m\}$, let $A_i(\xi_1, \xi_2, ..., \xi_m)$ be a holomorphic on $\{(\xi_1, \xi_2, ..., \xi_m) \mid |\xi_j| < 1 \text{ for all } j\}$ with values in $\mathrm{M}_N(\mathbb{C})$. Consider the system of differential equations $\xi_i \dfrac{dF}{d\xi_i} = A_i(\xi) F$ for all $i \in \{1, 2, ..., m\}$ on a single function $F : \mathbb{C}^m \to \mathrm{M}_N(\mathbb{C})$. Assume

$$\left[\xi_i \frac{d}{d\xi_i} - A_i, \xi_j \frac{d}{d\xi_j} - A_j\right] = 0 \qquad \text{for all } i, j \in \{1, 2, ..., m\}$$

(this is called a *consistency condition*, aka a zero curvature equation). Then, $[A_i(0), A_j(0)] = 0$ for all $i, j \in \{1, 2, ..., m\}$, and thus the matrices $A_i(0)$ for all i can be simultaneously trigonalized. Under this trigonalization, let $\lambda_{i,1}$, $\lambda_{i,2}$, ..., $\lambda_{i,N}$ be the diagonal entries of $A_i(0)$.

Assume that the condition

$$(\lambda_{1,k} - \lambda_{1,\ell}, \lambda_{2,k} - \lambda_{2,\ell}, ..., \lambda_{m,k} - \lambda_{m,\ell}) \notin \mathbb{Z}^m \diagdown 0$$

holds for all k and ℓ. [...]