From generalized factorials to greedoids, or meditations on the Vandermonde determinant

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joint work with Fedor Petrov

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This talk is being recorded!

slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/greedtalk-em2020.pdf
extended abstract with further references: http://www.cip.ifi.lmu.de/~grinberg/algebra/fps20gfv.pdf

1.

Bhargava's generalized factorials: an introduction

References:

- Manjul Bhargava, *P-orderings and polynomial functions on arbitrary subsets of Dedekind rings*, J. reine. angew. Math. **490** (1997), 101–127.
- Manjul Bhargava, *The Factorial Function and Generalizations*, Amer. Math. Month. **107** (2000), 783–799. (Recommended!)
- Manjul Bhargava, *On P-orderings, rings of integer-valued polynomials, and ultrametric analysis*, Journal of the AMS **22** (2009), 963–993.

It begins with a Vandermonde

- **Theorem** (classical exercise)**:**
  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,
  $$0! \cdot 1! \cdot 2! \cdot \cdots \cdot n! \mid \prod_{i>j} (a_i - a_j).$$

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  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,
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  $$12 = (1-0)(2-1)(2-0)(3-2)(3-1)(3-0) \\ \mid (a_1 - a_0)(a_2 - a_1)(a_2 - a_0)(a_3 - a_2)(a_3 - a_1)(a_3 - a_0);$$
  and so on.

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- **Hint to proof 1:** Show that
  $$\frac{\text{RHS}}{\text{LHS}} = \det \left(\binom{a_i}{j} \right)_{i,j \in \{0,1,\ldots,n\}} = \det \begin{pmatrix} \binom{a_0}{0} & \binom{a_0}{1} & \cdots & \binom{a_0}{n} \\ \binom{a_1}{0} & \binom{a_1}{1} & \cdots & \binom{a_1}{n} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{a_n}{0} & \binom{a_n}{1} & \cdots & \binom{a_n}{n} \end{pmatrix} .$$

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This might remind you of the Vandermonde determinant, which says that

$$\prod_{i>j} (a_i - a_j) = \det\left(a_i^j\right)_{i,j\in\{0,1,\ldots,n\}} = \det\begin{pmatrix} a_0^0 & a_0^1 & \cdots & a_0^n \\ a_1^0 & a_1^1 & \cdots & a_1^n \\ \vdots & \vdots & \ddots & \vdots \\ a_n^0 & a_n^1 & \cdots & a_n^n \end{pmatrix}.$$

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  Both are particular cases of the general fact that if $a_0, a_1, \ldots, a_n$ are numbers, and $P_0, P_1, \ldots, P_n$ are polynomials with $\deg P_j \leq j$ for each j, then
  $$\det \left(\left(P_j \left(a_i \right) \right)_{i,j\in\{0,1,\ldots,n\}} \right) = \ell_0 \ell_1 \cdots \ell_n \prod_{i>j} (a_i - a_j) ,$$
  where ℓ_j is the x^j-coefficient of P_j. [**Exercise:** Prove this!]

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- **Hint to proof 2:** WLOG assume $0 \leq a_0 < a_1 < \cdots < a_n$.
  (Otherwise, move each a_i preserving $a_i \mod \text{LHS}$.)

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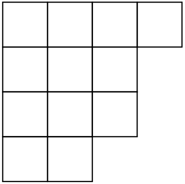
- **Hint to proof 2:** WLOG assume $0 \leq a_0 < a_1 < \cdots < a_n$.
  Consider an array of $n+1$ left-justified rows with lengths $a_0 - 0, a_1 - 1, \ldots, a_n - n$ from bottom to top:
  e.g., if $n = 3$ and $(a_0, a_1, \ldots, a_n) = (2, 4, 5, 7)$, then it is



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  Now fill this array with numbers $\in \{1, 2, \ldots, n+1\}$ that **increase weakly along rows** and **increase strictly down columns**, e.g.:

1	1	2	4
2	2	3	
3	4	4	
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(a "semistandard tableau") .

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  The number of such fillings is $\dfrac{\text{RHS}}{\text{LHS}}$.
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("Weyl's character formula" in type A; see MathOverflow question #106606.) **Question:** Bijective proof?

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  $$\prod_{i>j} (i-j) \mid \prod_{i>j} (a_i - a_j).$$
- **Hint to proof 3:** To show that $u \mid v$, it suffices to prove that every prime p divides v at least as often as it does u.
  Now get your hands dirty.

- **Theorem** (Bhargava?)**:**
  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,
  $$\frac{0! \cdot 2! \cdot \cdots \cdot (2n)!}{2^n} \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$
  (Typo in Bhargava corrected.)

- **Theorem** (slightly restated)**:**
  Let $a_0, a_1, \ldots, a_n \in \mathbb{Z}$. Then,

$$\prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

What about squares?

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- Analogues of the 3 above proofs work (I believe).

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- **Question:** Do we also have

$$\prod_{i>j} \left(i^3 - j^3\right) \mid \prod_{i>j} \left(a_i^3 - a_j^3\right) \ ?$$

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- Already for $n=6$, there is no choice of $a_0,a_1,\ldots,a_n$ that attains the gcd (such as $0,1,\ldots,n$ was for first powers and for squares).

- **General question** (Bhargava, 1997)**:** Let S be a set of integers. Fix $n \geq 0$. What is

$$\gcd \left\{ \prod_{i>j} (a_i - a_j) \;\middle|\; a_0, a_1, \ldots, a_n \in S \right\} ?$$

More generally...

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$$\gcd\left\{\prod_{i>j}(a_i - a_j) \;\middle|\; a_0, a_1, \ldots, a_n \in S\right\}\text{ ?}$$

- Enough to work out each prime p separately, because:

p-valuation

- Let p be a prime. Set $\mathbb{N} := \{0, 1, 2, \ldots\}$.
- For each nonzero $n \in \mathbb{Z}$, let $v_p(n)$ (the *p-valuation* of n) be the highest $k \in \mathbb{N}$ such that $p^k \mid n$.
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- **Examples:**

$v_3(18) = 2;$	$v_3(17) = 0;$
$v_2(14) = 1;$	$v_2(16) = 4.$

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- **Rules for p-valuations:**

$v_p(1) = 0;$	$v_p(ab) = v_p(a) + v_p(b);$
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- Define the *p-distance $d_p(a, b)$* between two integers a and b by
$$d_p(a, b) = -v_p(a - b).$$
Then, the last rule rewrites as
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The p-distance is not very geometric: For instance, 2 is closer to $p + 2$ than to 1, and even closer to $p^2 + 2$.
Cf. the p-adic solenoid. Also, artistic rendition by Fomenko.

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- Two integers u and v satisfy $u \mid v$ if and only if

$$v_p(u) \leq v_p(v) \qquad \text{for each prime } p.$$

Thus, checking divisibility is reduced to a "local" problem.

- **Equivalent problem:** Let S be a set of integers. Let p be a prime. Fix $n \geq 0$. What is

$$\min \left\{ v_p \left(\prod_{i>j} (a_i - a_j) \right) \;\middle|\; a_0, a_1, \ldots, a_n \in S \right\} ?$$

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- **Equivalent problem:** Let S be a set of integers. Let p be a prime. Fix $n \geq 0$. What is

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- We can WLOG assume that $a_0, a_1, \ldots, a_n$ are distinct.

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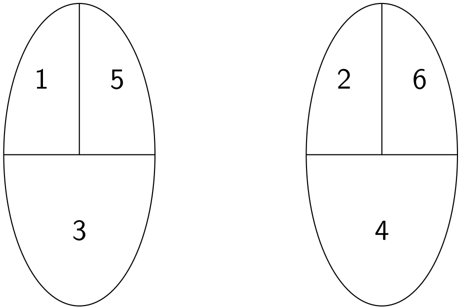

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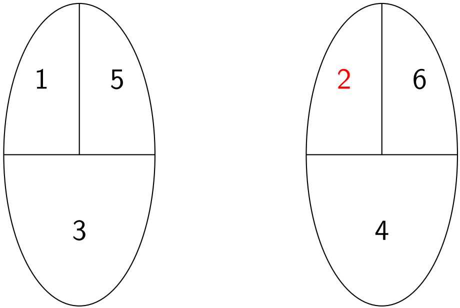

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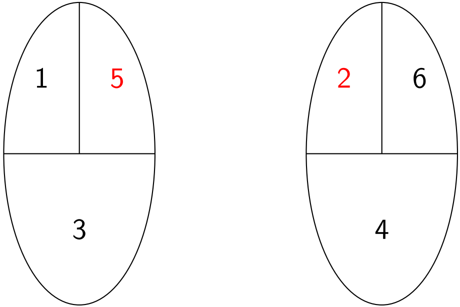

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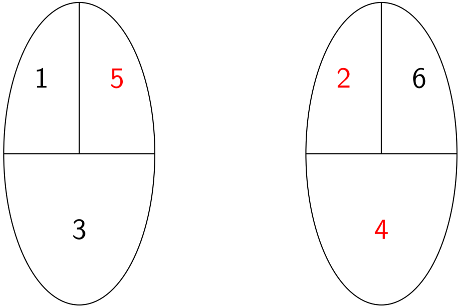

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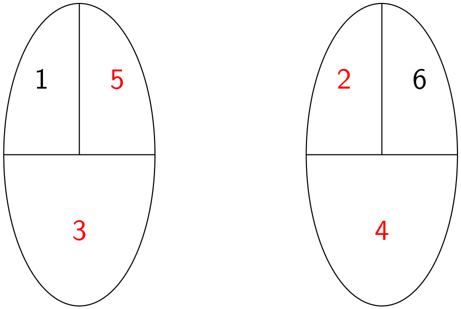

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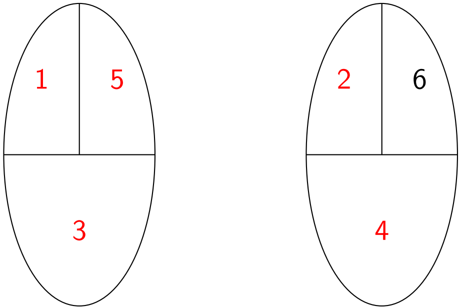

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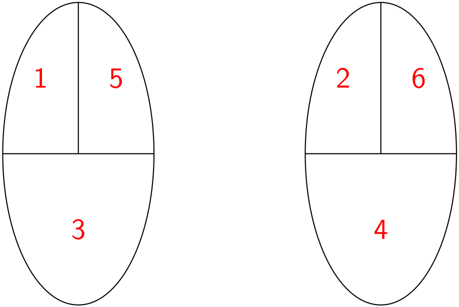

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- Thus, the choice of a_n tactically (near-sightedly) maximizes $\sum_{n \geq i > j} d_p(a_i, a_j)$ for fixed $a_0, a_1, \ldots, a_{n-1}$. (Thus "greedy".) But is it strategically optimal?

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- **Note:** There is such a sequence for each prime p, but there isn't always such a sequence that works for all p simultaneously.

Prime numbers be gone

- In his first (1997) paper on the subject, Bhargava already noticed that p is a red herring: The properties of d_p are all that is needed.

> "*We note that the above results (i.e. Theorem 1, Lemmas 1 and 2) do not rely on any special properties of P or R; they depend only on the fact that R becomes an ultrametric space when given the P-adic metric. Hence these results could be viewed more generally as statements about certain special sequences in ultrametric spaces. For convenience, however, we have chosen to present these statements only in the relevant context. In particular, we note that our proof of Theorem 1 shall be a purely algebraic one, involving no inequalities.*"

(Theorem 1 is a slight generalization of the above Theorem.)

2.

Ultra triples

References:

- Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.
- Darij Grinberg, *The Bhargava greedoid as a Gaussian elimination greedoid*, arXiv:2001.05535.
- Alex J. Lemin, *The category of ultrametric spaces is isomorphic to the category of complete, atomic, tree-like, and real graduated lattices LAT**, Algebra univers. **50** (2003), pp. 35–49.

Ultra triples

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- More generally, we can replace $\mathbb{R}$ by any totally ordered abelian group $\mathbb{V}$.

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- We will only consider ultra triples with **finite** ground set E. (Bhargava's E is infinite, but results adapt easily.)

Ultra triples, examples: 1 (congruence)

- **Example:** Let $E \subseteq \mathbb{Z}$ and $n \in \mathbb{Z}$. Define a map $w : E \to \mathbb{R}$ arbitrarily. Define a map $d : E \underline{\times} E \to \mathbb{R}$ by

$$d(a, b) = \begin{cases} 0, & \text{if } a \equiv b \mod n; \\ 1, & \text{if } a \not\equiv b \mod n \end{cases} \qquad \text{for all } (a, b) \in E \underline{\times} E.$$

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where ε and α are fixed reals with $\varepsilon \leq \alpha$. Then, (E, w, d) is an ultra triple.

Ultra triples, examples: 2 (p-adic distance)

- Let p be a prime. Let $E \subseteq \mathbb{Z}$. Define the weights $w(e) \in \mathbb{R}$ arbitrarily. Then, (E, w, d_p) is an ultra triple.
  Here, d_p is as before:
  $$d_p(a, b) = -v_p(a - b).$$
- This is the case of relevance to Bhargava's problem!
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- More generally, we can replace $p^0, p^1, p^2, \ldots$ with any unbounded sequence $r_0 \mid r_1 \mid r_2 \mid \cdots$ of integers.

- Let E be the set of all living organisms. Let

$$d(e,f) = \begin{cases} 0, & \text{if } e = f; \\ 1, & \text{if } e \text{ and } f \text{ belong to the same species;} \\ 2, & \text{if } e \text{ and } f \text{ belong to the same genus;} \\ 3, & \text{if } e \text{ and } f \text{ belong to the same family;} \\ \ldots \end{cases}$$

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Ultra triples, examples: 3 (Linnaeus)

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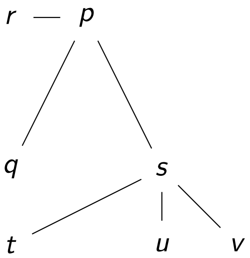
- More generally, any "nested" family of equivalence relations on E gives a distance function for an ultra triple.

Ultra triples, examples: 4 (Darwin)

- Let T be a (finite, undirected) tree. For each edge e of T, let $\lambda(e) \geq 0$ be a real. We shall call this real the *weight* of e.



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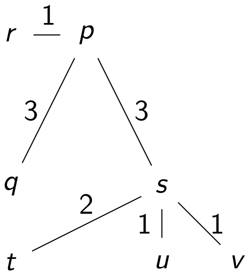

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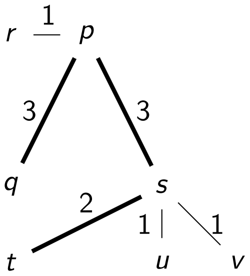

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- Fix any vertex r of T. Let E be any subset of the vertex set of T. Set

  $$d(x, y) = \lambda(x, y) - \lambda(x, r) - \lambda(y, r) \qquad \text{for each } (x, y) \in E \underline{\times} E.$$

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- **Hint to proof:** Use the well-known fact ("four-point condition") saying that if x, y, z, w are four vertices of T, then the two largest of the three numbers

  $$\lambda(x, y) + \lambda(z, w), \quad \lambda(x, z) + \lambda(y, w), \quad \lambda(x, w) + \lambda(y, z)$$

  are equal. [**Exercise:** Prove this!]

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  Then, (E, w, d) is an ultra triple for any $w : E \to \mathbb{R}$.
- This is particularly useful when T is a *phylogenetic tree* and E is a set of its leaves.
  Actually, this is the general case: Any (finite) ultra triple can be translated back into a phylogenetic tree. It is "essentially" an inverse operation.
  (The idea is not new; see, e.g., Lemin 2003.)

- Let (E, w, d) be an ultra triple, and $S \subseteq E$ be any subset. Then, the *perimeter* of S is defined to be

$$\operatorname{PER}(S) := \underbrace{\sum_{x \in S} w(x)}_{|S| \text{ addends}} + \underbrace{\sum_{\substack{\{x,y\} \subseteq S; \\ x \neq y}} d(x, y)}_{\binom{|S|}{2} \text{ addends}}.$$

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- Thus,

$$\begin{aligned}
\operatorname{PER} \varnothing &= 0; \\
\operatorname{PER} \{x\} &= w(x); \\
\operatorname{PER} \{x, y\} &= w(x) + w(y) + d(x, y); \\
\operatorname{PER} \{x, y, z\} &= w(x) + w(y) + w(z) \\
&\qquad + d(x, y) + d(x, z) + d(y, z).
\end{aligned}$$

Perimeters in ultra triples

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- **Bhargava's problem (generalized):** Given an ultra triple (E, w, d) and an $n \in \mathbb{N}$, find the maximum perimeter of an n-element subset of E, and find the subsets that attain it. (The n here corresponds to the $n + 1$ before.)

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- For $E \subseteq \mathbb{Z}$ and $w(e) = 0$ and $d_p(a, b) = -v_p(a - b)$, this is Bhargava's problem.

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- For Linnaeus or Darwin ultra triples, this is a "Noah's Ark" problem: What choices of n organisms maximize biodiversity? A similar problem has been studied in: Vincent Moulton, Charles Semple, Mike Steel, *Optimizing phylogenetic diversity under constraints*, J. Theor. Biol. **246** (2007), pp. 186–194.

3.

Solving the problem

References:

- Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.
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Greedy permutations: definition

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- Let $m \in \mathbb{N}$. A *greedy m-permutation* of E is a list $(c_1, c_2, \ldots, c_m)$ of m distinct elements of E such that for each $i \in \{1, 2, \ldots, m\}$ and each $x \in E \setminus \{c_1, c_2, \ldots, c_{i-1}\}$, we have

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- In other words, a greedy m-permutation of E is what you obtain if you try to greedily construct a maximum-perimeter m-element subset of E, by starting with $\varnothing$ and adding new points one at a time.

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- **Theorem (Petrov, G.):** Let $m, k \in \mathbb{N}$ with $|E| \geq m \geq k$. Let A be a k-element subset of E that has maximum perimeter among all such.
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- **Exercise:** Use this to prove

$$\prod_{i>j} (i-j) \mid \prod_{i>j} (a_i - a_j) \qquad \text{and} \qquad \prod_{i>j} \left(i^2 - j^2\right) \mid \prod_{i>j} \left(a_i^2 - a_j^2\right).$$

4.

Greedoids

References:

- Bernhard Korte, László Lovász, Rainer Schrader, *Greedoids*, Algorithms and Combinatorics #4, Springer 1991.
- Anders Björner, Günter M. Ziegler, *Introd. to Greedoids*, in: Neil White (ed.), *Matroid applications*, CUP 1992.
- Darij Grinberg, Fedor Petrov, *A greedoid and a matroid inspired by Bhargava's p-orderings*, arXiv:1909.01965.
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- Victor Bryant, Ian Sharpe, *Gaussian, Strong and Transversal Greedoids*, Europ. J. Comb. **20** (1999), pp. 259–262.

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Greedoids: introduction

- So the maximum-perimeter k-element subsets in an ultra triple are not just a random bunch of sets: They are accessible by a greedy algorithm.
- This is characteristic of a *greedoid* – a "noncommutative analogue" of a matroid.
- I will now define greedoids.
  Warning: some abstraction to follow.

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- **Example:** $E = \{1, 2, 3, 4, 5\}$ and

$$\mathcal{F} = \Big\{ \varnothing, \{1\}, \{2\}, \{5\}, \{1,2\}, \{1,5\}, \{2,5\}, \{4,5\}, \{1,2,3\}, \{1,2,5\}, \{2,4,5\}, \{3,4,5\} \Big\}.$$

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Axiom 3. holds for $A = \{4, 5\}$ and $B = \{2, 4, 5\}$, since we can take $b = 2$.
(More generally, Axiom 3. always holds if $A \subseteq B$.)

- If you have seen matroids:
  Let M be a matroid on a ground set E. Then,

  $$\{\text{independent sets of } M\}$$

  is a greedoid on E.
  We shall call this a *matroid greedoid*.

Greedoids, examples: 2 (Gaussian elimination)

- Let A be an $m \times n$-matrix over a field $\mathbb{K}$. Let $E = \{1, 2, \ldots, n\}$. Then,

  $$\left\{ F \subseteq E \ \mid \ \text{we have } |F| \leq n \text{ and } \det\left(\operatorname{sub}^{F}_{\{1,2,\ldots,|F|\}} A\right) \neq 0 \right\}$$

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- For example, if $\mathbb{K} = \mathbb{Q}$ and $m = 5$ and $n = 5$ and

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ then}$$

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$$\{2, 5\} \in \operatorname{GEG}(A), \qquad \text{since } \det\left(\operatorname{sub}_{\{1,2\}}^{\{2,5\}} A\right) \neq 0.$$

- Let A be an $m \times n$-matrix over a field $\mathbb{K}$. Let $E = \{1, 2, \ldots, n\}$. Then,

$$\left\{ F \subseteq E \mid \text{ we have } |F| \leq n \text{ and } \det\left(\operatorname{sub}^F_{\{1,2,\ldots,|F|\}} A\right) \neq 0 \right\}$$

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$$\{1, 2, 3, 5\} \in \mathrm{GEG}(A), \qquad \text{since } \det\left(\operatorname{sub}^{\{1,2,3,5\}}_{\{1,2,3,4\}} A\right) \neq 0.$$

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$$\operatorname{GEG}(A) = \left\{ \varnothing, \{2\}, \{3\}, \{5\}, \{1,2\}, \{1,3\}, \{1,5\}, \{2,3\}, \{2,5\}, \{1,2,3\}, \{1,2,5\}, \{1,2,3,5\} \right\}.$$

Greedoids, examples: 3 (order ideals)

- Let P be a finite poset. Let J be the set of all *order ideals* of P (that is, of all subsets I of P such that $(b \in I) \wedge (a \leq b) \Longrightarrow (a \in I)$).
- Then, J is a greedoid on P. [**Exercise:** Prove this!] We shall call this an *order ideal greedoid*.

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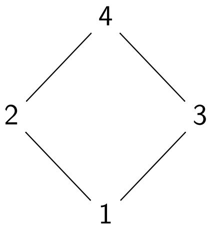
- **Example:** If P is the poset with Hasse diagram



then

$$G = \left\{ \varnothing, \{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}, \{1,2,3,4\} \right\}.$$

Greedoids, examples: 4 (complements of subtrees)

- Let T be a tree with vertex set V. Let G be the set of all subsets $U \subseteq V$ such that the induced subgraph on $V \setminus U$ is connected (i.e., no vertex in U lies on the path between two vertices in $V \setminus U$).
- Then, G is a greedoid on V. [**Exercise:** Prove this!]

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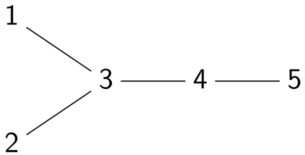

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- Back to our setting: For any ultra triple (E, w, d), define

$$\begin{aligned}\mathcal{B}(E, w, d) &= \{A \subseteq E \mid A \text{ has maximum perimeter among} \\ &\qquad \text{all } |A|\text{-element subsets of } E\} \\ &= \{A \subseteq E \mid \mathrm{PER}(A) \geq \mathrm{PER}(B) \text{ for} \\ &\qquad \text{all } B \subseteq E \text{ satisfying } |B| = |A|\}.\end{aligned}$$

We call this the *Bhargava greedoid* of (E, w, d).

- Back to our setting: For any ultra triple (E, w, d), define

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  We call this the *Bhargava greedoid* of (E, w, d).
- **Theorem (G., Petrov):** This Bhargava greedoid $\mathcal{B}(E, w, d)$ is a greedoid indeed.

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- Recall: A *greedoid* on a set E means a set system $\mathcal{F}$ on E such that
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- **Remark:** In axiom 3., we can replace the condition "$|B| = |A| + 1$" by the weaker "$|B| > |A|$"; the axiom stays equivalent.
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- Strong greedoids are also known as "*Gauss greedoids*" (not to be confused with Gaussian elimination greedoids).

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- **Not** all greedoids from trees (Example 4 above) are strong greedoids. [**Exercise:** Show that the greedoid of a tree T is a strong greedoid if and only if T is a star.]
- **Theorem (Bryant, Sharpe):** Let $\mathcal{F}$ be a strong greedoid, and $k \in \mathbb{N}$. Then, the k-element sets that belong to $\mathcal{F}$ are the bases of a matroid (unless there are none of them).
  If $\mathcal{F}$ is a Gaussian elimination greedoid, then the latter matroid is representable.

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- Note that this Theorem yields the previous one, which is thus proved twice.
- **Stronger theorem (G.):** Let (E, w, d) be an ultra triple. Let $\mathbb{K}$ be any field of size $|\mathbb{K}| \geq \operatorname{mcs}(E, w, d)$, where $\operatorname{mcs}(E, w, d)$ is the *maximum clique size* of E (that is, the maximum size of a subset $C \subseteq E$ such that $d\mid_{C \underline{\times} C}$ is constant).
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- Note that this Theorem yields the previous one, which is thus proved twice.
- **Converse theorem (G.):** Assume that the map w is constant. Let $\mathbb{K}$ be a field. Then, the Bhargava greedoid $\mathcal{B}(E, w, d)$ is (up to renaming the elements of E) a Gaussian elimination greedoid over $\mathbb{K}$ **if and only if** $|\mathbb{K}| \geq \operatorname{mcs}(E, w, d)$.

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- **1st step:** If (E, w, d) is a Bhargava-type ultra triple (E, w, d_p) for some prime p and some $E \subseteq \mathbb{Z}$, then we can explicitly find a matrix A over $\mathbb{F}_p$ that gives $\mathcal{B}(E, w, d)$ as its Gaussian elimination greedoid.

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  Even better, this matrix A is the projection of a matrix $\widetilde{A}$ over $\mathbb{Z}$ that satisfies
  $$v_p\left(\det\left(\operatorname{sub}_{\{1,2,\ldots,|F|\}}^{F} \widetilde{A}\right)\right) = (\text{max. possible perimeter}) - \operatorname{PER}(F)$$
  for each subset F of E.
  (The matrix $\widetilde{A}$ is a Vandermonde-like matrix, with entries $\dfrac{1}{p^{\text{something}}}(a_i - e_1)(a_i - e_2)\cdots(a_i - e_j)$.)

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- **2nd step:** So we know how to deal with Bhargava-type ultra triples. It would be nice if all ultra triples were isomorphic to some of them!

  I'm not sure this is true, but I can prove something close that suffices:

A few words about the proofs, 2

- **2nd step, continued:** Replace $\mathbb{Z}$ by the "polynomial ring" $\mathbb{K}[t]$, except that all powers t^a with $a \in \mathbb{R}_+$ are allowed (not just for integer a).
  For example,
  $$3 + 2t^{0.5} - 7t^{0.8} + 4t^{3.2} \qquad \text{lies in this ring.}$$

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  $$v_t\left(3t^{0.2} + 2t^{0.5} - 7t^{0.8} + 4t^{3.2}\right) = 0.2.$$

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  (The proof proceeds by strong induction, decomposing the ultra triple into smaller ones. Iterating this decomposition again reveals the connection to phylogenetic trees.)
- **Note:** In proving the general case, we had to come back to our original example, the (generalized) Vandermonde determinant!

Questions

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- It is not too hard to define a multiset analogue of greedoids (e.g., by lifting the "simple" requirement on greedoid languages). How much of the theory adapts?

Thank you!

- **Fedor Petrov** for getting this started by answering my MathOverflow question #314130.
- **Alexander Postnikov** for interesting conversations.
- **Doron Zeilberger** for the invitation.
- **you** for your patience and typo hunting.