From Chio Pivotal Condensation to the Matrix-Tree theorem

Darij Grinberg, Karthik Karnik, Anya Zhang

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Abstract

We show a determinant identity which generalizes both the Chio pivotal condensation theorem and the Matrix-Tree theorem.

1. Introduction

The Chio pivotal condensation theorem (Theorem 2.1 below, or [Eves68, Theorem 3.6.1]) is a simple particular case of the Dodgson-Muir determinantal identity ([BerBru08, (4)]), which can be used to reduce the computation of an $n \times n$-determinant to that of an $(n-1) \times (n-1)$-determinant (provided that an entry of the matrix can be divided by[1]). On the other hand, the Matrix-Tree theorem (Theorem 2.12, or [Zeilbe85, Section 4], or [Verstr12, Theorem 1]) expresses the number of spanning trees of a graph as a determinant[2]. In this note, we show that these two results have a common generalization (Theorem 2.13). As we have tried to keep the note self-contained, using only the well-known fundamental properties of determinants, it also provides new proofs for both results.

1.1. Acknowledgments

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[1] We work with matrices over arbitrary commutative rings, so this is not a moot point. Of course, if the ring is a field, then this just means that the matrix has a nonzero entry.

[2] And not just the number; rather, a "weighted number" from which the spanning trees can be read off if the weights are chosen generically enough.

2. The theorems

We shall use the (rather standard) notations defined in [Grinbe15]. In particular, $\mathbb{N}$ means the set $\{0, 1, 2, \ldots\}$. For any $n \in \mathbb{N}$, we let S_n denote the group of permutations of the set $\{1, 2, \ldots, n\}$. The $n \times m$-matrix whose (i, j)-th entry is $a_{i,j}$ for each $(i, j) \in \{1, 2, \ldots, n\} \times \{1, 2, \ldots, m\}$ will be denoted by $\left(a_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq m}$.

Let $\mathbb{K}$ be a commutative ring. We shall regard $\mathbb{K}$ as fixed throughout this note (so we won't always write "Let $\mathbb{K}$ be a commutative ring" in our propositions); the notion "matrix" will always mean "matrix with entries in $\mathbb{K}$".

2.1. Chio Pivotal Condensation

We begin with a statement of the Chio Pivotal Condensation theorem (see, e.g., [KarZha16, Theorem 0.1] and the reference therein):

Theorem 2.1. Let $n \geq 2$ be an integer. Let $A = \left(a_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n} \in \mathbb{K}^{n \times n}$ be a matrix. Then,

$$\det\left(\left(a_{i,j}a_{n,n} - a_{i,n}a_{n,j}\right)_{1 \leq i \leq n-1,\ 1 \leq j \leq n-1}\right) = a_{n,n}^{n-2} \cdot \det\left(\left(a_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n}\right).$$

Example 2.2. If $n = 3$ and $A = \begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}$, then Theorem 2.1 says that

$$\det\begin{pmatrix} ac'' - a''c & a'c'' - a''c' \\ bc'' - b''c & b'c'' - b''c' \end{pmatrix} = \left(c''\right)^{3-2} \cdot \det\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \\ c & c' & c'' \end{pmatrix}.$$

Theorem 2.1 (originally due to Félix Chio in 1853[3]) is nowadays usually regarded either as a particular case of the Dodgson-Muir determinantal identity ([BerBru08, (4)]), or as a relatively easy exercise on row operations and the method of universal identities[4]. We, however, shall generalize it in a different direction.

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[3]See [Heinig11, footnote 2] and [Abeles14, §2] for some historical background.

[4]In more detail:

- In order to derive Theorem 2.1 from [BerBru08, (4)], it suffices to set $k = n - 1$ and recognize the right hand side of [BerBru08, (4)] as $\det\left(\left(a_{i,j}a_{n,n} - a_{i,n}a_{n,j}\right)_{1 \leq i \leq n-1,\ 1 \leq j \leq n-1}\right)$.
- A proof of Theorem 2.1 using row operations can be found in [Eves68, Theorem 3.6.1], up to a few minor issues: First of all, [Eves68, Theorem 3.6.1] proves not exactly Theorem 2.1 but the analogous identity

$$\det\left(\left(a_{i+1,j+1}a_{1,1} - a_{i+1,1}a_{1,j+1}\right)_{1 \leq i \leq n-1,\ 1 \leq j \leq n-1}\right) = a_{1,1}^{n-2} \cdot \det\left(\left(a_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n}\right).$$

2.2. Generalization, step 1

Our generalization will proceed in two steps. In the first step, we shall replace some of the n's on the left hand side by $f(i)$'s (see Theorem 2.9 below). We first define some notations:

Definition 2.3. Let n be a positive integer. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be any map such that $f(n) = n$.

We say that the map f is *n-potent* if for every $i \in \{1, 2, \ldots, n\}$, there exists some $k \in \mathbb{N}$ such that $f^k(i) = n$. (In less formal terms, f is n-potent if and only if every element of $\{1, 2, \ldots, n\}$ eventually arrives at n when being subjected to repeated application of f.)

(Note that, by definition, any n-potent map $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ must satisfy $f(n) = n$.)

Example 2.4. For this example, let $n = 3$. The map $\{1, 2, 3\} \to \{1, 2, 3\}$ sending $1, 2, 3$ to $2, 1, 3$, respectively, is not n-potent (because applying it repeatedly to 1 can only give 1 or 2, but never 3). The map $\{1, 2, 3\} \to \{1, 2, 3\}$ sending $1, 2, 3$ to $3, 3, 2$, respectively, is not n-potent (since it does not send n to n). The map $\{1, 2, 3\} \to \{1, 2, 3\}$ sending $1, 2, 3$ to $3, 1, 3$, respectively, is n-potent (indeed, every element of $\{1, 2, 3\}$ goes to 3 after at most two applications of this map).

Remark 2.5. Given a positive integer n, the n-potent maps $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ are in 1-to-1 correspondence with the trees with vertex set $\{1, 2, \ldots, n\}$. Namely, an n-potent map f corresponds to the tree whose edges are $\{i, f(i)\}$ for all $i \in \{1, 2, \ldots, n-1\}$. If we regard the tree as a rooted tree with root n, and if we direct every edge towards the root, then the edges are $(i, f(i))$ for all $i \in \{1, 2, \ldots, n-1\}$.

Remark 2.6. Let $n \geq 2$ be an integer. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be any n-potent map. Then:

(a) There exists some $g \in \{1, 2, \ldots, n-1\}$ such that $f(g) = n$.

(b) We have $\left|f^{-1}(n)\right| \geq 2$.

The (very simple) proof of Remark 2.6 can be found in the Appendix (Section 4).

Second, [Eves68, Theorem 3.6.1] assumes $a_{1,1}$ to be invertible (and all $a_{i,j}$ to belong to a field); however, assumptions like this can easily be disposed of using the method of universal identities (see [Conrad09]).

A more explicit and self-contained proof of Theorem 2.1 can be found in [KarZha16]. References to other proofs appear in [Abeles14, §2].

Definition 2.7. Let $n \geq 2$ be an integer. Let $A = \left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n} \in \mathbb{K}^{n\times n}$ be an $n \times n$-matrix. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be any n-potent map.

(a) We define an element $\operatorname{weight}_f A$ of $\mathbb{K}$ by

$$\operatorname{weight}_f A = \prod_{i=1}^{n-1} a_{i,f(i)}.$$

(b) We define an element $\operatorname{abut}_f A$ of $\mathbb{K}$ by

$$\operatorname{abut}_f A = a_{n,n}^{\left|f^{-1}(n)\right|-2} \prod_{\substack{i\in\{1,2,\ldots,n-1\};\\ f(i)\neq n}} a_{f(i),n}.$$

(This is well-defined, since Remark 2.6 **(b)** shows that $\left|f^{-1}(n)\right| - 2 \in \mathbb{N}$.)

Remark 2.8. Let n, A and f be as in Definition 2.7. Here are two slightly more intuitive ways to think of $\operatorname{abut}_f A$:

(a) If $a_{n,n} \in \mathbb{K}$ is invertible, then $\operatorname{abut}_f A$ is simply $\dfrac{1}{a_{n,n}} \prod_{i\in\{1,2,\ldots,n-1\}} a_{f(i),n}$.

(b) Remark 2.6 **(a)** shows that there exists some $g \in \{1, 2, \ldots, n-1\}$ such that $f(g) = n$. Fix such a g. Then,

$$\operatorname{abut}_f A = \prod_{\substack{i\in\{1,2,\ldots,n-1\};\\ i\neq g}} a_{f(i),n}.$$

The (nearly trivial) proof of Remark 2.8 is again found in the Appendix.

Now, we can state our first generalization of Theorem 2.1:

Theorem 2.9. Let n be a positive integer. Let $A = \left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n} \in \mathbb{K}^{n\times n}$ be an $n \times n$-matrix. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be any map such that $f(n) = n$.

Let B be the $(n-1) \times (n-1)$-matrix

$$\left(a_{i,j}a_{f(i),n} - a_{i,n}a_{f(i),j}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1} \in \mathbb{K}^{(n-1)\times(n-1)}.$$

(a) If the map f is not n-potent, then $\det B = 0$.

(b) Assume that $n \geq 2$. Assume that the map f is n-potent. Then,

$$\det B = \left(\operatorname{abut}_f A\right) \cdot \det A.$$

Example 2.10. For this example, let $n = 3$ and $A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix}$.

If $f : \{1,2,3\} \to \{1,2,3\}$ is the map sending $1,2,3$ to $3,1,3$, respectively, then the matrix B defined in Theorem 2.9 is $\begin{pmatrix} a_{1,1}a_{3,3} - a_{1,3}a_{3,1} & a_{1,2}a_{3,3} - a_{1,3}a_{3,2} \\ a_{2,1}a_{1,3} - a_{2,3}a_{1,1} & a_{2,2}a_{1,3} - a_{2,3}a_{1,2} \end{pmatrix}$. Since this map f is n-potent, Theorem 2.9 **(b)** predicts that this matrix B satisfies $\det B = \left(\operatorname{abut}_f A\right) \cdot \det A$. This is indeed easily checked (indeed, we have $\operatorname{abut}_f A = a_{1,3}$ in this case).

On the other hand, if $f : \{1,2,3\} \to \{1,2,3\}$ is the map sending $1,2,3$ to $1,1,3$, respectively, then the matrix B defined in Theorem 2.9 is $\begin{pmatrix} a_{1,1}a_{1,3} - a_{1,3}a_{1,1} & a_{1,2}a_{1,3} - a_{1,3}a_{1,2} \\ a_{2,1}a_{1,3} - a_{2,3}a_{1,1} & a_{2,2}a_{1,3} - a_{2,3}a_{1,2} \end{pmatrix}$. Since this map f is not n-potent, Theorem 2.9 **(a)** predicts that this matrix B satisfies $\det B = 0$. This, too, is easily checked (and arguably obvious in this case).

Applying Theorem 2.9 **(b)** to $f(i) = n$ yields Theorem 2.1. (The map $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ defined by $f(i) = n$ is clearly n-potent, and satisfies $\operatorname{abut}_f A = a_{n,n}^{n-2}$.)

We defer the proof of Theorem 2.9 until later; first, let us see how it can be generalized a bit further (not substantially, anymore) and how this generalization also encompasses the matrix-tree theorem.

2.3. The matrix-tree theorem

Definition 2.11. For any two objects i and j, we define an element $\delta_{i,j} \in \mathbb{K}$ by $\delta_{i,j} = \begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases}$.

Let us first state the matrix-tree theorem.

To be honest, there is no "the matrix-tree theorem", but rather a network of "matrix-tree theorems" (some less, some more general), each of which has a reasonable claim to this name. Here we shall prove the following one:

Theorem 2.12. Let $n \geq 1$ be an integer. Let $W : \{1,2,\ldots,n\} \times \{1,2,\ldots,n\} \to \mathbb{K}$ be any function. For every $i \in \{1,2,\ldots,n\}$, set

$$d^+(i) = \sum_{j=1}^{n} W(i,j).$$

Let L be the matrix $\left(\delta_{i,j} d^+(i) - W(i,j)\right)_{1 \leq i \leq n-1,\ 1 \leq j \leq n-1} \in \mathbb{K}^{(n-1)\times(n-1)}$. Then,

$$\det L = \sum_{\substack{f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}; \\ f(n)=n; \\ f \text{ is } n\text{-potent}}} \prod_{i=1}^{n-1} W(i, f(i)). \tag{1}$$

Since our notation differs from that in most other sources on the matrix-tree theorem, let us explain the equivalence between our Theorem 2.12 and one of its better-known avatars: The version of the matrix-tree theorem stated in [Zeilbe85, Section 4] involves some "weights" $a_{k,m}$, a determinant of an $(n-1)\times(n-1)$-matrix, and a sum over a set $\mathcal{T}=\mathcal{T}(n)$. These correspond (respectively) to the values $W(k,m)$, the determinant $\det L$, and the sum over all n-potent maps f in our Theorem 2.12. In fact, the only nontrivial part of this correspondence is the bijection between the trees in $\mathcal{T}$ and the n-potent maps f over which the sum in (1) ranges. This bijection is precisely the one introduced in Remark 2.5.[5]

It might seem weird to call Theorem 2.12 the "matrix-tree theorem" if the word "tree" never occurs inside it. However, as we have already noticed in Remark 2.5, the trees on the set $\{1,2,\ldots,n\}$ are in bijection with the n-potent maps $\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$, and therefore the sum on the right hand side of (1) can be viewed as a sum over all these trees. Moreover, the function W can be viewed as an $n\times n$-matrix; when this matrix is specialized to the adjacency matrix of a directed graph, the sum on the right hand side of (1) becomes the number of directed spanning trees of this directed graph directed towards the root n.

2.4. Generalization, step 2

Now, as promised, we will generalize Theorem 2.9 a step further. While the result will not be significantly stronger (we will actually derive it from Theorem 2.9 quite easily), it will lead to a short proof of Theorem 2.12:

[5] A slightly different version of the matrix-tree theorem appears in [Verstr12, Theorem 1] (and various other places); it involves a function W, a number $v\in\{1,2,\ldots,n\}$, a matrix L_v, a set $\mathcal{T}_v$ and a sum $\tau(W,v)$. Our Theorem 2.12 is equivalent to the case of [Verstr12, Theorem 1] for $v=n$; but this case is easily seen to be equivalent to the general case of [Verstr12, Theorem 1] (since the elements of $\{1,2,\ldots,n\}$ can be permuted at will). Our matrix L is the L_n of [Verstr12, Theorem 1]. Furthermore, our sum over all n-potent maps f corresponds to the sum $\tau(W,n)$ in [Verstr12], which is a sum over all n-arborescences on $\{1,2,\ldots,n\}$; the correspondence is again due to Remark 2.5.

Theorem 2.13. Let $n \geq 2$ be an integer. Let $A = \left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n} \in \mathbb{K}^{n\times n}$ and $B = \left(b_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n} \in \mathbb{K}^{n\times n}$ be $n\times n$-matrices. Write the $n\times n$-matrix BA in the form $BA = \left(c_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$.

Let G be the $(n-1)\times(n-1)$-matrix

$$\left(a_{i,j}c_{i,n} - a_{i,n}c_{i,j}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1} \in \mathbb{K}^{(n-1)\times(n-1)}.$$

Then,

$$\det G = \left(\sum_{\substack{f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\};\\ f(n)=n;\\ f \text{ is } n\text{-potent}}} \left(\operatorname{weight}_f B\right)\left(\operatorname{abut}_f A\right) \right) \cdot \det A.$$

To obtain Theorem 2.9 from Theorem 2.13, we have to define B by $B = \left(\delta_{j,f(i)}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$. Below we shall show how to obtain the matrix-tree theorem from Theorem 2.13.

Example 2.14. Let us see what Theorem 2.13 says for $n = 3$. There are three n-potent maps $f : \{1,2,3\} \to \{1,2,3\}$:

- one map f_{33} which sends both 1 and 2 to 3;
- one map f_{23} which sends 1 to 2 and 2 to 3;
- one map f_{31} which sends 2 to 1 and 1 to 3.

The definition of the $c_{i,j}$ as the entries of BA shows that $c_{i,j} = b_{i,1}a_{1,j} + b_{i,2}a_{2,j} + b_{i,3}a_{3,j}$ for all i and j. We have

$$G = \begin{pmatrix} a_{1,1}c_{1,3} - c_{1,1}a_{1,3} & a_{1,2}c_{1,3} - c_{1,2}a_{1,3} \\ a_{2,1}c_{2,3} - c_{2,1}a_{2,3} & a_{2,2}c_{2,3} - c_{2,2}a_{2,3} \end{pmatrix}.$$

Theorem 2.13 says that

$$\begin{aligned} \det G &= \left(\left(\operatorname{weight}_{f_{33}} B\right)\left(\operatorname{abut}_{f_{33}} A\right) + \left(\operatorname{weight}_{f_{23}} B\right)\left(\operatorname{abut}_{f_{23}} A\right) \right.\\ &\qquad \left. + \left(\operatorname{weight}_{f_{31}} B\right)\left(\operatorname{abut}_{f_{31}} A\right)\right) \cdot \det A \\ &= \left(b_{1,3}b_{2,3}a_{3,3} + b_{1,2}b_{2,3}a_{2,3} + b_{1,3}b_{2,1}a_{1,3}\right) \cdot \det A. \end{aligned}$$

3. The proofs

3.1. Deriving Theorem 2.13 from Theorem 2.9

Let us see how Theorem 2.13 can be proven using Theorem 2.9 (which we have not proven yet). We shall need two lemmas:

Lemma 3.1. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Let $b_{i,k}$ be an element of $\mathbb{K}$ for every $i \in \{1,2,\ldots,m\}$ and every $k \in \{1,2,\ldots,n\}$. Let $d_{i,j,k}$ be an element of $\mathbb{K}$ for every $i \in \{1,2,\ldots,m\}$, $j \in \{1,2,\ldots,m\}$ and $k \in \{1,2,\ldots,n\}$. Let G be the $m \times m$-matrix $\left(\sum\limits_{k=1}^{n} b_{i,k} d_{i,j,k}\right)_{1\leq i\leq m,\ 1\leq j\leq m}$. Then,

$$\det G = \sum_{f:\{1,2,\ldots,m\}\to\{1,2,\ldots,n\}} \left(\prod_{i=1}^{m} b_{i,f(i)}\right) \det\left(\left(d_{i,j,f(i)}\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right).$$

Lemma 3.1 is merely a scary way to state the multilinearity of the determinant as a function of its rows. See the Appendix for a proof.

Let us specialize Lemma 3.1 in a way that is closer to our goal:

Lemma 3.2. Let n be a positive integer. Let $b_{i,k}$ be an element of $\mathbb{K}$ for every $i \in \{1,2,\ldots,n-1\}$ and every $k \in \{1,2,\ldots,n\}$. Let $d_{i,j,k}$ be an element of $\mathbb{K}$ for every $i \in \{1,2,\ldots,n-1\}$, $j \in \{1,2,\ldots,n-1\}$ and $k \in \{1,2,\ldots,n\}$. Let G be the $(n-1)\times(n-1)$-matrix $\left(\sum\limits_{k=1}^{n} b_{i,k} d_{i,j,k}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}$. Then,

$$\det G = \sum_{\substack{f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\};\\ f(n)=n}} \left(\prod_{i=1}^{n-1} b_{i,f(i)}\right) \det\left(\left(d_{i,j,f(i)}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}\right).$$

Proof of Lemma 3.2. Lemma 3.1 (applied to $m = n-1$) shows that

$$\det G = \sum_{f:\{1,2,\ldots,n-1\}\to\{1,2,\ldots,n\}} \left(\prod_{i=1}^{n-1} b_{i,f(i)}\right) \det\left(\left(d_{i,j,f(i)}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}\right).$$

The only difference between this formula and the claim of Lemma 3.2 is that the sum here is over all $f : \{1,2,\ldots,n-1\} \to \{1,2,\ldots,n\}$, whereas the sum in the claim of Lemma 3.2 is over all $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ satisfying $f(n) = n$. But this is not much of a difference: Each map $\{1,2,\ldots,n-1\} \to \{1,2,\ldots,n\}$ is a restriction (to $\{1,2,\ldots,n-1\}$) of a unique map $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ satisfying $f(n) = n$, and therefore the two sums are equal. $\square$

Proof of Theorem 2.13. For every $i \in \{1, 2, \ldots, n-1\}$, $j \in \{1, 2, \ldots, n-1\}$ and $k \in \{1, 2, \ldots, n\}$, define an element $d_{i,j,k}$ of $\mathbb{K}$ by

$$d_{i,j,k} = a_{i,j} a_{k,n} - a_{i,n} a_{k,j}. \tag{2}$$

For every $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ satisfying $f(n) = n$, we have

$$\begin{aligned}
&\det \left(\left(\underbrace{d_{i,j,f(i)}}_{\substack{= a_{i,j} a_{f(i),n} - a_{i,n} a_{f(i),j} \\ \text{(by (2))}}} \right)_{1 \le i \le n-1,\ 1 \le j \le n-1} \right) \\
&= \det \left(\left(a_{i,j} a_{f(i),n} - a_{i,n} a_{f(i),j} \right)_{1 \le i \le n-1,\ 1 \le j \le n-1} \right) \\
&= \begin{cases} 0, & \text{if } f \text{ is not } n\text{-potent;} \\ \left(\operatorname{abut}_f A \right) \cdot \det A, & \text{if } f \text{ is } n\text{-potent} \end{cases}
\end{aligned} \tag{3}$$

(by Theorem 2.9, applied to the matrix $\left(a_{i,j} a_{f(i),n} - a_{i,n} a_{f(i),j} \right)_{1 \le i \le n-1,\ 1 \le j \le n-1}$ instead of B).

We have

$$\left(c_{i,j} \right)_{1 \le i \le n,\ 1 \le j \le n} = BA = \left(\sum_{k=1}^{n} b_{i,k} a_{k,j} \right)_{1 \le i \le n,\ 1 \le j \le n}$$

(by the definition of the product of two matrices). Thus,

$$c_{i,j} = \sum_{k=1}^{n} b_{i,k} a_{k,j} \qquad \text{for every } (i,j) \in \{1, 2, \ldots, n\}^2 . \tag{4}$$

Now, for every $(i,j) \in \{1, 2, \ldots, n-1\}^2$, we have

$$\begin{aligned}
&a_{i,j} \underbrace{c_{i,n}}_{\substack{= \sum_{k=1}^{n} b_{i,k} a_{k,n} \\ \text{(by (4), applied to } n \\ \text{instead of } j)}} - a_{i,n} \underbrace{c_{i,j}}_{\substack{= \sum_{k=1}^{n} b_{i,k} a_{k,j} \\ \text{(by (4))}}} \\
&= a_{i,j} \sum_{k=1}^{n} b_{i,k} a_{k,n} - a_{i,n} \sum_{k=1}^{n} b_{i,k} a_{k,j} = \sum_{k=1}^{n} b_{i,k} \underbrace{\left(a_{i,j} a_{k,n} - a_{i,n} a_{k,j} \right)}_{\substack{= d_{i,j,k} \\ \text{(by (2))}}} = \sum_{k=1}^{n} b_{i,k} d_{i,j,k}.
\end{aligned}$$

Hence,

$$G = \left(\underbrace{a_{i,j} c_{i,n} - a_{i,n} c_{i,j}}_{= \sum_{k=1}^{n} b_{i,k} d_{i,j,k}} \right)_{1 \le i \le n-1,\ 1 \le j \le n-1} = \left(\sum_{k=1}^{n} b_{i,k} d_{i,j,k} \right)_{1 \le i \le n-1,\ 1 \le j \le n-1} .$$

Hence, Lemma 3.2 yields

$$\det G = \sum_{\substack{f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\};\\ f(n)=n}} \underbrace{\left(\prod_{i=1}^{n-1} b_{i,f(i)}\right)}_{\substack{=\operatorname{weight}_f B\\ \text{(by the definition}\\ \text{of weight}_f B)}} \underbrace{\det\left(\left(d_{i,j,f(i)}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}\right)}_{=\begin{cases} 0, & \text{if } f \text{ is not } n\text{-potent;}\\ \left(\operatorname{abut}_f A\right)\cdot\det A, & \text{if } f \text{ is } n\text{-potent}\end{cases}}$$

$$= \sum_{\substack{f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\};\\ f(n)=n}} \left(\operatorname{weight}_f B\right) \begin{cases} 0, & \text{if } f \text{ is not } n\text{-potent;}\\ \left(\operatorname{abut}_f A\right)\cdot\det A, & \text{if } f \text{ is } n\text{-potent}\end{cases}$$

$$= \sum_{\substack{f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\};\\ f(n)=n;\\ f \text{ is } n\text{-potent}}} \left(\operatorname{weight}_f B\right)\left(\operatorname{abut}_f A\right)\cdot\det A$$

$$= \left(\sum_{\substack{f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\};\\ f(n)=n;\\ f \text{ is } n\text{-potent}}} \left(\operatorname{weight}_f B\right)\left(\operatorname{abut}_f A\right)\right)\cdot\det A.$$

$\square$

3.2. Deriving Theorem 2.12 from Theorem 2.13

Now let us see why Theorem 2.13 generalizes the matrix-tree theorem.

Proof of Theorem 2.12. WLOG assume that $n \geq 2$ (since the case $n = 1$ is easy to check by hand). Define an $n \times n$-matrix A by $A = \left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$, where

$$a_{i,j} = \delta_{i,j} + \delta_{j,n}\left(1 - \delta_{i,n}\right).$$

(This scary formula hides a simple idea: this is the matrix whose entries on the diagonal and in its last column are 1, and all other entries are 0. Thus,

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 1\\ 0 & 1 & 0 & 0 & \cdots & 0 & 1\\ 0 & 0 & 1 & 0 & \cdots & 0 & 1\\ 0 & 0 & 0 & 1 & \cdots & 0 & 1\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & 1 & 1\\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

) Note that every $(i,j) \in \{1,2,\ldots,n-1\}^2$ satisfies

$$a_{i,j} = \delta_{i,j} + \underbrace{\delta_{j,n}}_{\substack{=0 \\ \text{(since } j \neq n \\ \text{(since } j \in \{1,2,\ldots,n-1\}))}} (1 - \delta_{i,n}) = \delta_{i,j}. \tag{5}$$

Also, every $i \in \{1,2,\ldots,n-1\}$ satisfies

$$\begin{aligned} a_{i,n} &= \underbrace{\delta_{i,n}}_{\substack{=0 \\ \text{(since } i \neq n)}} + \underbrace{\delta_{n,n}}_{\substack{=1 \\ \text{(since } n = n)}} \left(1 - \underbrace{\delta_{i,n}}_{\substack{=0 \\ \text{(since } i \neq n)}}\right) \qquad \text{(by the definition of } a_{i,n}\text{)} \\ &= 0 + 1(1-0) = 1. \end{aligned} \tag{6}$$

Also, let B be the $n \times n$-matrix $(W(i,j))_{1 \leq i \leq n,\ 1 \leq j \leq n}$. Write the $n \times n$-matrix BA in the form $BA = (c_{i,j})_{1 \leq i \leq n,\ 1 \leq j \leq n}$. Then, it is easy to see that every $(i,j) \in \{1,2,\ldots,n\}^2$ satisfies

$$c_{i,j} = W(i,j) + \delta_{j,n}\left(d^+(i) - W(i,n)\right) \tag{7}$$

[6].

[6]*Proof of (7):* For every $i \in \{1,2,\ldots,n\}$, we have

$$\begin{aligned} d^+(i) &= \sum_{j=1}^{n} W(i,j) \qquad \left(\text{by the definition of } d^+(i)\right) \\ &= \sum_{j=1}^{n-1} W(i,j) + W(i,n) = \sum_{k=1}^{n-1} W(i,k) + W(i,n) \end{aligned}$$

(here, we renamed the summation index j as k) and thus

$$\sum_{k=1}^{n-1} W(i,k) = d^+(i) - W(i,n). \tag{8}$$

But

$$(c_{i,j})_{1 \leq i \leq n,\ 1 \leq j \leq n} = BA = \left(\sum_{k=1}^{n} W(i,k)\, a_{k,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n}$$

(by the definition of the product of two matrices, since $B = (W(i,j))_{1 \leq i \leq n,\ 1 \leq j \leq n}$ and $A =$

Thus, for every $(i,j) \in \{1,2,\ldots,n-1\}^2$, we have

$$\underbrace{a_{i,j}}_{\substack{=\delta_{i,j} \\ \text{(by (5))}}} \underbrace{c_{i,n}}_{\substack{=W(i,n)+\delta_{n,n}\left(d^+(i)-W(i,n)\right) \\ \text{(by (7), applied} \\ \text{to } j \text{ instead of } n)}} - \underbrace{a_{i,n}}_{\substack{=1 \\ \text{(by (6))}}} \underbrace{c_{i,j}}_{\substack{=W(i,j)+\delta_{j,n}\left(d^+(i)-W(i,n)\right) \\ \text{(by (7))}}}$$

$$= \delta_{i,j}\left(W(i,n) + \underbrace{\delta_{n,n}}_{=1}\left(d^+(i) - W(i,n)\right)\right) - \left(W(i,j) + \underbrace{\delta_{j,n}}_{\substack{=0 \\ \text{(since } j<n)}}\left(d^+(i) - W(i,n)\right)\right)$$

$$= \delta_{i,j}\underbrace{\left(W(i,n) + \left(d^+(i) - W(i,n)\right)\right)}_{=d^+(i)} - W(i,j) = \delta_{i,j}d^+(i) - W(i,j).$$

Hence,

$$\left(a_{i,j}c_{i,n} - a_{i,n}c_{i,j}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1} = \left(\delta_{i,j}d^+(i) - W(i,j)\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1} = L.$$

In other words, L is the matrix $\left(a_{i,j}c_{i,n} - a_{i,n}c_{i,j}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1} \in \mathbb{K}^{(n-1)\times(n-1)}$.

$\left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$). Hence, every $(i,j) \in \{1,2,\ldots,n\}^2$ satisfies

$$\begin{aligned}
c_{i,j} &= \sum_{k=1}^{n} W(i,k) \underbrace{a_{k,j}}_{\substack{=\delta_{k,j}+\delta_{j,n}\left(1-\delta_{k,n}\right) \\ \text{(by the definition of } a_{k,j})}} \\
&= \sum_{k=1}^{n} W(i,k)\left(\delta_{k,j} + \delta_{j,n}\left(1-\delta_{k,n}\right)\right) \\
&= \underbrace{\sum_{k=1}^{n} W(i,k)\,\delta_{k,j}}_{\substack{=W(i,j) \\ \text{(because the factor } \delta_{k,j} \text{ in the sum} \\ \text{kills every addend except the one for } k=j)}} + \delta_{j,n} \underbrace{\sum_{k=1}^{n} W(i,k)\left(1-\delta_{k,n}\right)}_{=\sum_{k=1}^{n-1} W(i,k)\left(1-\delta_{k,n}\right)+W(i,n)\left(1-\delta_{n,n}\right)} \\
&= W(i,j) + \delta_{j,n}\left(\sum_{k=1}^{n-1} W(i,k)\left(1 - \underbrace{\delta_{k,n}}_{\substack{=0 \\ \text{(since } k<n)}}\right) + W(i,n)\underbrace{\left(1-\delta_{n,n}\right)}_{\substack{=0 \\ \text{(since } \delta_{n,n}=1)}}\right) \\
&= W(i,j) + \delta_{j,n}\left(\sum_{k=1}^{n-1} W(i,k)\underbrace{(1-0)}_{=1} + \underbrace{W(i,n)\,0}_{=0}\right) \\
&= W(i,j) + \delta_{j,n}\underbrace{\sum_{k=1}^{n-1} W(i,k)}_{\substack{=d^+(i)-W(i,n) \\ \text{(by (8))}}} = W(i,j) + \delta_{j,n}\left(d^+(i) - W(i,n)\right),
\end{aligned}$$

and thus (7) is proven.

Thus, Theorem 2.13 (applied to $G = L$) yields

$$\det L = \left(\sum_{\substack{f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\};\\ f(n)=n;\\ f \text{ is } n\text{-potent}}} \underbrace{\left(\operatorname{weight}_f B\right)}_{=\prod_{i=1}^{n-1} W(i,f(i))} \underbrace{\left(\operatorname{abut}_f A\right)}_{=1} \right) \cdot \underbrace{\det A}_{=1}$$
$$= \sum_{\substack{f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\};\\ f(n)=n;\\ f \text{ is } n\text{-potent}}} \prod_{i=1}^{n-1} W\left(i, f\left(i\right)\right).$$

This proves Theorem 2.12. □

3.3. Some combinatorial lemmas

We still owe the reader a proof of Theorem 2.9. We prepare by proving some properties of maps $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Proposition 3.3. Let $n \in \mathbb{N}$. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a map. Let $i \in \{1, 2, \ldots, n\}$. Then,

$$f^k\left(i\right) \in \left\{f^s\left(i\right) \mid s \in \{0, 1, \ldots, n-1\}\right\} \qquad \text{for every } k \in \mathbb{N}.$$

Proposition 3.3 is a classical fact; we give the proof in the Appendix below.

The following three results can be easily derived from Proposition 3.3; we shall give more detailed proofs in the Appendix:

Proposition 3.4. Let n be a positive integer. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a map such that $f\left(n\right) = n$. Let $i \in \{1, 2, \ldots, n\}$. Then, $f^{n-1}\left(i\right) = n$ if and only if there exists some $k \in \mathbb{N}$ such that $f^k\left(i\right) = n$.

Proposition 3.5. Let n be a positive integer. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a map such that $f\left(n\right) = n$. Then, the map f is n-potent if and only if $f^{n-1}\left(\{1, 2, \ldots, n\}\right) = \{n\}$.

Corollary 3.6. Let n be a positive integer. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a map such that $f\left(n\right) = n$. Let $i \in \{1, 2, \ldots, n\}$. Then, $\delta_{f^{n-1}(i),n} = \delta_{f^n(i),n}$.

One consequence of Proposition 3.5 is the following: If n is a positive integer, and if $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a map such that $f\left(n\right) = n$, then we can check in finite time whether the map f is n-potent (because we can check in finite time whether $f^{n-1}\left(\{1, 2, \ldots, n\}\right) = \{n\}$). Thus, for any given positive integer n, it is possible to enumerate all n-potent maps $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Next, we shall show a property of n-potent maps:

Lemma 3.7. Let n be a positive integer. Let $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ be a map such that $f(n) = n$. Assume that f is n-potent.

Let $\sigma \in S_n$ be a permutation such that $\sigma \neq \mathrm{id}$. Then, there exists some $i \in \{1,2,\ldots,n\}$ such that $\sigma(i) \notin \{i, f(i)\}$.

Proof of Lemma 3.7. Assume the contrary. Thus, $\sigma(i) \in \{i, f(i)\}$ for every $i \in \{1,2,\ldots,n\}$.

We have $\sigma \neq \mathrm{id}$. Hence, there exists some $h \in \{1,2,\ldots,n\}$ such that $\sigma(h) \neq h$. Fix such a h. We shall prove that

$$\sigma^j(h) = f^j(h) \qquad \text{for every } j \in \mathbb{N}. \tag{9}$$

Indeed, we shall prove this by induction over j. The induction base (the case $j = 0$) is obvious. For the induction step, fix $J \in \mathbb{N}$, and assume that $\sigma^J(h) = f^J(h)$. We need to prove that $\sigma^{J+1}(h) = f^{J+1}(h)$.

We have assumed that $\sigma(i) \in \{i, f(i)\}$ for every $i \in \{1,2,\ldots,n\}$. Applying this to $i = \sigma^J(h)$, we obtain $\sigma\left(\sigma^J(h)\right) \in \left\{\sigma^J(h), f\left(\sigma^J(h)\right)\right\}$. In other words, $\sigma^{J+1}(h) \in \left\{\sigma^J(h), f\left(\sigma^J(h)\right)\right\}$. Thus, either $\sigma^{J+1}(h) = \sigma^J(h)$ or $\sigma^{J+1}(h) = f\left(\sigma^J(h)\right)$. Since $\sigma^{J+1}(h) = \sigma^J(h)$ is impossible (because in light of the invertibility of σ, this would yield $\sigma(h) = h$, which contradicts $\sigma(h) \neq h$), we thus must have $\sigma^{J+1}(h) = f\left(\sigma^J(h)\right)$. Hence, $\sigma^{J+1}(h) = f\left(\underbrace{\sigma^J(h)}_{=f^J(h)}\right) = f\left(f^J(h)\right) = f^{J+1}(h)$. This completes the induction step.

Thus, (9) is proven.

But f is n-potent. Hence, there exists some $k \in \mathbb{N}$ such that $f^k(h) = n$. Consider this k. Applying (9) to $j = k$, we obtain $\sigma^k(h) = f^k(h) = n$.

But applying (9) to $j = k+1$, we obtain $\sigma^{k+1}(h) = f^{k+1}(h) = f\left(\underbrace{f^k(h)}_{=n}\right) = f(n) = n$. Hence, $n = \sigma^{k+1}(h) = \sigma^k(\sigma(h))$, so that $\sigma^k(\sigma(h)) = n = \sigma^k(h)$. Since σ^k is invertible, this entails $\sigma(h) = h$, which contradicts $\sigma(h) \neq h$. This contradiction proves that our assumption was wrong. Thus, Lemma 3.7 is proven. □

3.4. The matrix Z_f and its determinant

Next, we assign a matrix Z_f to every such $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$:

Definition 3.8. Let n be a positive integer. Let $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ be a map. Then, we define an $n \times n$-matrix $Z_f \in \mathbb{K}^{n\times n}$ by

$$Z_f = \left(\delta_{i,j} - (1 - \delta_{i,n})\,\delta_{f(i),j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}.$$

Example 3.9. For this example, set $n = 4$, and define a map $f : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$ by $(f(1), f(2), f(3), f(4)) = (2, 4, 1, 4)$. Then,

$$Z_f = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, we claim the following:

Proposition 3.10. Let n be a positive integer. Let $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ be a map such that $f(n) = n$. Let v_f be the column vector $\left(1 - \delta_{f^{n-1}(i), n}\right)_{1 \leq i \leq n,\ 1 \leq j \leq 1} \in \mathbb{K}^{n \times 1}$. Then, $Z_f v_f = 0_{n \times 1}$.

(Recall that $0_{n \times 1}$ denotes the $n \times 1$ zero matrix, i.e., the column vector with n entries whose all entries are 0.)

Proof of Proposition 3.10. We shall prove that

$$\sum_{k=1}^{n} \left(\delta_{i,k} - (1 - \delta_{i,n}) \delta_{f(i),k}\right) \left(1 - \delta_{f^{n-1}(k),n}\right) = 0 \qquad (10)$$

for every $i \in \{1, 2, \ldots, n\}$.

Proof of (10): Let $i \in \{1, 2, \ldots, n\}$. Corollary 3.6 yields $\delta_{f^{n-1}(i),n} = \delta_{f^n(i),n}$.

On the other hand, $f(n) = n$. Thus, it is straightforward to see (by induction over h) that $f^h(n) = n$ for every $h \in \mathbb{N}$. Applying this to $h = n$, we obtain $f^n(n) = n$.

Now,

$$\begin{aligned}
&\sum_{k=1}^{n}\left(\delta_{i,k}-(1-\delta_{i,n})\,\delta_{f(i),k}\right)\left(1-\delta_{f^{n-1}(k),n}\right)\\
&= \underbrace{\sum_{k=1}^{n}\delta_{i,k}\left(1-\delta_{f^{n-1}(k),n}\right)}_{\substack{=1-\delta_{f^{n-1}(i),n}\\ \text{(because the factor } \delta_{i,k} \text{ in the sum}\\ \text{kills every addend except the one for } k=i)}} - \underbrace{\sum_{k=1}^{n}(1-\delta_{i,n})\,\delta_{f(i),k}\left(1-\delta_{f^{n-1}(k),n}\right)}_{\substack{=(1-\delta_{i,n})\left(1-\delta_{f^{n-1}(f(i)),n}\right)\\ \text{(because the factor } \delta_{f(i),k} \text{ in the sum}\\ \text{kills every addend except the one for } k=f(i))}}\\
&= \left(1-\underbrace{\delta_{f^{n-1}(i),n}}_{=\delta_{f^n(i),n}}\right)-(1-\delta_{i,n})\left(1-\underbrace{\delta_{f^{n-1}(f(i)),n}}_{=\delta_{f^n(i),n}}\right)\\
&= \left(1-\delta_{f^n(i),n}\right)-(1-\delta_{i,n})\left(1-\delta_{f^n(i),n}\right)\\
&= \underbrace{(1-(1-\delta_{i,n}))}_{=\delta_{i,n}}\left(1-\delta_{f^n(i),n}\right)=\delta_{i,n}\left(1-\delta_{f^n(i),n}\right)\\
&= \begin{cases}0, & \text{if } i\neq n;\\ 1-\delta_{f^n(n),n}, & \text{if } i=n\end{cases} = \begin{cases}0, & \text{if } i\neq n;\\ 0, & \text{if } i=n\end{cases}\\
&\qquad \left(\text{since } f^n(n)=n \text{ and thus } \delta_{f^n(n),n}=\delta_{n,n}=1 \text{ and hence } 1-\delta_{f^n(n),n}=0\right)\\
&= 0.
\end{aligned}$$

This proves (10).

Recall now that

$$Z_f=\left(\delta_{i,j}-(1-\delta_{i,n})\,\delta_{f(i),j}\right)_{1\le i\le n,\ 1\le j\le n}$$

and $v_f=\left(1-\delta_{f^{n-1}(i),n}\right)_{1\le i\le n,\ 1\le j\le 1}$. Hence, the definition of the product of two matrices yields

$$\begin{aligned}
Z_f v_f &= \left(\underbrace{\sum_{k=1}^{n}\left(\delta_{i,k}-(1-\delta_{i,n})\,\delta_{f(i),k}\right)\left(1-\delta_{f^{n-1}(k),n}\right)}_{\substack{=0\\ \text{(by (10))}}}\right)_{1\le i\le n,\ 1\le j\le 1}\\
&= (0)_{1\le i\le n,\ 1\le j\le 1}=0_{n\times 1}.
\end{aligned}$$

This proves Proposition 3.10. □

Now, we recall the following well-known properties of determinants[7]:

Lemma 3.11. Let $n \in \mathbb{N}$. Let A be an $n \times n$-matrix. Let v be a column vector with n entries. If $Av = 0_{n\times 1}$, then $\det A \cdot v = 0_{n\times 1}$.

Lemma 3.12. Let n be a positive integer. Let $A = \left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$ be an $n \times n$-matrix. Assume that

$$a_{i,n} = 0 \qquad \text{for every } i \in \{1,2,\ldots,n-1\}. \tag{11}$$

Then, $\det A = a_{n,n} \cdot \det\left(\left(a_{i,j}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}\right)$.

Now, we can prove the crucial property of the matrix Z_f:

Proposition 3.13. Let n be a positive integer. Let $f : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$ be a map satisfying $f(n) = n$.

(a) If f is n-potent, then $\det\left(Z_f\right) = 1$.

(b) If f is not n-potent, then $\det\left(Z_f\right) = 0$.

Proof of Proposition 3.13. Write the matrix Z_f in the form $\left(z_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$. Thus,

$$\left(z_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n} = Z_f = \left(\delta_{i,j} - \left(1-\delta_{i,n}\right)\delta_{f(i),j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}.$$

Hence, every $(i,j) \in \{1,2,\ldots,n\}^2$ satisfies

$$z_{i,j} = \delta_{i,j} - \underbrace{\left(1-\delta_{i,n}\right)}_{=\begin{cases}1, & \text{if } i<n;\\ 0, & \text{if } i=n\end{cases}} \delta_{f(i),j} = \delta_{i,j} - \begin{cases}1, & \text{if } i<n;\\ 0, & \text{if } i=n\end{cases} \delta_{f(i),j} \tag{12}$$

$$= \delta_{i,j} - \begin{cases}\delta_{f(i),j}, & \text{if } i<n;\\ 0, & \text{if } i=n\end{cases} = \begin{cases}\delta_{i,j} - \delta_{f(i),j}, & \text{if } i<n;\\ \delta_{i,j}, & \text{if } i=n\end{cases}. \tag{13}$$

(a) Assume that f is n-potent.

Let $\sigma \in S_n$ be a permutation such that $\sigma \neq \operatorname{id}$. Then, there exists some $i \in \{1,2,\ldots,n\}$ such that $\sigma(i) \notin \{i, f(i)\}$ (by Lemma 3.7). Hence, there exists some $i \in \{1,2,\ldots,n\}$ such that $z_{i,\sigma(i)} = 0$ [8]. Hence, the product $\prod_{i=1}^{n} z_{i,\sigma(i)}$ has at least one zero factor, and thus equals 0.

[7]For the sake of completeness: Lemma 3.11 is [Grinbe15, Corollary 5.102]; Lemma 3.12 is [Grinbe15, Corollary 5.45].

[8]*Proof.* We have just shown that there exists some $i \in \{1,2,\ldots,n\}$ such that $\sigma(i) \notin \{i, f(i)\}$. Consider this i. We have $\sigma(i) \notin \{i, f(i)\}$, thus $\sigma(i) \neq i$, and thus $\delta_{i,\sigma(i)} = 0$. Also, $\sigma(i) \notin$

Now, forget that we fixed σ. We thus have shown that

$$\prod_{i=1}^{n} z_{i,\sigma(i)} = 0 \qquad \text{for every } \sigma \in S_n \text{ such that } \sigma \neq \mathrm{id}. \tag{14}$$

On the other hand, it is easy to see that

$$\prod_{i=1}^{n} z_{i,i} = 1. \tag{15}$$

[9]

Now, the definition of $\det(Z_f)$ yields

$$\begin{aligned}
\det(Z_f) &= \sum_{\sigma\in S_n} (-1)^{\sigma} \prod_{i=1}^{n} z_{i,\sigma(i)} \qquad \left(\text{since } Z_f = (z_{i,j})_{1\leq i\leq n,\ 1\leq j\leq n}\right) \\
&= \underbrace{(-1)^{\mathrm{id}}}_{=1} \prod_{i=1}^{n} \underbrace{z_{i,\mathrm{id}(i)}}_{=z_{i,i}} + \sum_{\substack{\sigma\in S_n;\\ \sigma\neq\mathrm{id}}} (-1)^{\sigma} \underbrace{\prod_{i=1}^{n} z_{i,\sigma(i)}}_{\substack{=0\\ \text{(by (14))}}} \\
&= \prod_{i=1}^{n} z_{i,i} + \underbrace{\sum_{\substack{\sigma\in S_n;\\ \sigma\neq\mathrm{id}}} (-1)^{\sigma} 0}_{=0} = \prod_{i=1}^{n} z_{i,i} = 1 \qquad (\text{by (15)}).
\end{aligned}$$

This proves Proposition 3.13 **(a)**.

(b) Assume that f is not n-potent. Then, there exists some $i \in \{1,2,\ldots,n\}$ such that $f^{n-1}(i) \neq n$ [10]. Fix such an i, and denote it by u. Thus, $u \in \{1,2,\ldots,n\}$ is such that $f^{n-1}(u) \neq n$.

$\{i, f(i)\}$, thus $\sigma(i) \neq f(i)$, and thus $\delta_{f(i),\sigma(i)} = 0$. Now, (12) (applied to $(i,\sigma(i))$ instead of (i,j)) yields

$$z_{i,\sigma(i)} = \underbrace{\delta_{i,\sigma(i)}}_{=0} - \begin{cases} 1, & \text{if } i < n; \\ 0, & \text{if } i = n \end{cases} \underbrace{\delta_{f(i),\sigma(i)}}_{=0} = 0 - 0 = 0,$$

qed.

[9] *Proof of (15):* To prove this, it is sufficient to show that $z_{i,i} = 1$ for every $i \in \{1,2,\ldots,n\}$. This is obvious when $i = n$ (using the formula (13)), so we only need to consider the case when $i < n$. In this case, (13) (applied to (i,i) instead of (i,j)) shows that $z_{i,i} = \underbrace{\delta_{i,i}}_{=1} - \delta_{f(i),i} = 1 - \delta_{f(i),i}$.

Hence, in order to prove that $z_{i,i} = 1$, we need to show that $\delta_{f(i),i} = 0$. In other words, we need to prove that $f(i) \neq i$.

Indeed, assume the contrary. Thus, $f(i) = i$. Hence, by induction over k, we can easily see that $f^k(i) = i$ for every $k \in \mathbb{N}$. Hence, for every $k \in \mathbb{N}$, we have $f^k(i) = i \neq n$. This contradicts the fact that there exists some $k \in \mathbb{N}$ such that $f^k(i) = n$ (since f is n-potent). This contradiction proves that our assumption was wrong. Hence, (15) is proven.

[10] *Proof.* Assume the contrary. Thus, for every $i \in \{1,2,\ldots,n\}$, we have $f^{n-1}(i) = n$. Hence, for every $i \in \{1,2,\ldots,n\}$, there exists some $k \in \mathbb{N}$ such that $f^k(i) = n$ (according to the $\Longrightarrow$ direction of Proposition 3.4). In other words, the map f is n-potent. This contradicts the fact that f is not n-potent. This contradiction shows that our assumption was wrong, qed.

Define the vector v_f as in Proposition 3.10. Proposition 3.10 yields $Z_f v_f = 0_{n\times 1}$. Lemma 3.11 (applied to Z_f and v_f instead of A and v) thus yields $\det\left(Z_f\right)\cdot v_f = 0_{n\times 1}$. Thus,

$$\begin{aligned}
(0)_{1\le i\le n,\ 1\le j\le 1} = 0_{n\times 1} &= \det\left(Z_f\right)\cdot \underbrace{v_f}_{=\left(1-\delta_{f^{n-1}(i),n}\right)_{1\le i\le n,\ 1\le j\le 1}} \\
&= \det\left(Z_f\right)\cdot\left(1-\delta_{f^{n-1}(i),n}\right)_{1\le i\le n,\ 1\le j\le 1} \\
&= \left(\det\left(Z_f\right)\cdot\left(1-\delta_{f^{n-1}(i),n}\right)\right)_{1\le i\le n,\ 1\le j\le 1}.
\end{aligned}$$

In other words, $0 = \det\left(Z_f\right)\cdot\left(1-\delta_{f^{n-1}(i),n}\right)$ for each $i\in\{1,2,\ldots,n\}$. Applying this to $i = u$, we obtain

$$0 = \det\left(Z_f\right)\cdot\left(1-\underbrace{\delta_{f^{n-1}(u),n}}_{\substack{=0\\(\text{since } f^{n-1}(u)\neq n)}}\right) = \det\left(Z_f\right)\cdot 1 = \det\left(Z_f\right).$$

This proves Proposition 3.13 **(b)**. □

3.5. Proof of Theorem 2.9

Let us finally recall a particularly basic property of determinants:

> **Lemma 3.14.** Let $m\in\mathbb{N}$. Let $A = \left(a_{i,j}\right)_{1\le i\le m,\ 1\le j\le m}\in\mathbb{K}^{m\times m}$ be an $m\times m$-matrix. Let $b_1, b_2, \ldots, b_m$ be m elements of $\mathbb{K}$. Then,
>
> $$\det\left(\left(b_i a_{i,j}\right)_{1\le i\le m,\ 1\le j\le m}\right) = \left(\prod_{i=1}^{m} b_i\right)\det A.$$

(Again, see the Appendix for the proof of this lemma.)

We can now finally prove Theorem 2.9:

Proof of Theorem 2.9. The identities we want to prove (both for part **(a)** and for part **(b)**) are polynomial identities in the entries of A. Thus, we can WLOG assume that all these entries are invertible.[11] In other words, we can assume that $a_{i,j}$ is invertible for each $(i,j)\in\{1,2,\ldots,n\}^2$. Assume this.

[11]Here is a more detailed justification for this "WLOG":

Let us restrict ourselves to Theorem 2.9 **(b)**. (The argument for Theorem 2.9 **(a)** is analogous.)

Assume that Theorem 2.9 **(b)** is proven in the case when all entries of A are invertible. We now must show that Theorem 2.9 **(b)** always holds.

Let C be the $(n-1)\times(n-1)$-matrix

$$\left(\frac{a_{i,j}}{a_{i,n}}-\frac{a_{f(i),j}}{a_{f(i),n}}\right)_{1\le i\le n-1,\ 1\le j\le n-1}\in\mathbb{K}^{(n-1)\times(n-1)}.$$

Let n be a positive integer such that $n\geq 2$. Let $f:\{1,2,\ldots,n\}\to\{1,2,\ldots,n\}$ be an n-potent map. Then, Theorem 2.9 **(b)** claims that

$$\det B=\left(\operatorname{abut}_f A\right)\cdot\det A \tag{16}$$

for every $n\times n$-matrix $A=\left(a_{i,j}\right)_{1\le i\le n,\ 1\le j\le n}\in\mathbb{K}^{n\times n}$, where B is as defined in Theorem 2.9. The equality (16) rewrites as

$$\begin{aligned}&\sum_{\sigma\in S_{n-1}}\prod_{i=1}^{n-1}\left(a_{i,\sigma(i)}a_{f(i),n}-a_{i,n}a_{f(i),\sigma(i)}\right)\\&=\left(a_{n,n}^{\left|f^{-1}(n)\right|-2}\prod_{\substack{i\in\{1,2,\ldots,n-1\};\\ f(i)\neq n}}a_{f(i),n}\right)\cdot\sum_{\sigma\in S_n}\prod_{i=1}^{n}a_{i,\sigma(i)}\end{aligned}\tag{17}$$

(because we have

$$\begin{aligned}\det\underbrace{B}_{=\left(a_{i,j}a_{f(i),n}-a_{i,n}a_{f(i),j}\right)_{1\le i\le n-1,\ 1\le j\le n-1}}&=\det\left(\left(a_{i,j}a_{f(i),n}-a_{i,n}a_{f(i),j}\right)_{1\le i\le n-1,\ 1\le j\le n-1}\right)\\&=\sum_{\sigma\in S_{n-1}}\prod_{i=1}^{n-1}\left(a_{i,\sigma(i)}a_{f(i),n}-a_{i,n}a_{f(i),\sigma(i)}\right)\end{aligned}$$

and $\operatorname{abut}_f A=a_{n,n}^{\left|f^{-1}(n)\right|-2}\prod_{\substack{i\in\{1,2,\ldots,n-1\};\\ f(i)\neq n}}a_{f(i),n}$ and $\det A=\sum_{\sigma\in S_n}\prod_{i=1}^{n}a_{i,\sigma(i)}$). Thus, Theorem 2.9 **(b)** (for our given n and f) is equivalent to the claim that (17) holds for every $n\times n$-matrix $\left(a_{i,j}\right)_{1\le i\le n,\ 1\le j\le n}\in\mathbb{K}^{n\times n}$.

Now, let $\mathbb{P}$ be the polynomial ring $\mathbb{Z}\left[X_{i,j}\ \mid\ (i,j)\in\{1,2,\ldots,n\}^2\right]$ in the n^2 indeterminates $X_{i,j}$ for $(i,j)\in\{1,2,\ldots,n\}^2$. Let $\mathbb{F}$ be the quotient field of $\mathbb{P}$; this is the field $\mathbb{Q}\left(X_{i,j}\ \mid\ (i,j)\in\{1,2,\ldots,n\}^2\right)$ of rational functions in the same indeterminates (but over $\mathbb{Q}$).

Let A_X be the $n\times n$-matrix $\left(X_{i,j}\right)_{1\le i\le n,\ 1\le j\le n}\in\mathbb{P}^{n\times n}$. If we regard A_X as a matrix in $\mathbb{F}^{n\times n}$, then all entries of A_X are invertible (because they are nonzero elements of the field $\mathbb{F}$). Hence, Theorem 2.9 **(b)** can be applied to $\mathbb{F}$, A_X, $X_{i,j}$ and B_X instead of $\mathbb{K}$, A, $a_{i,j}$ and B (because we have assumed that Theorem 2.9 **(b)** is proven in the case when all entries of A are invertible). As we know, this means that (17) holds for $a_{i,j}=X_{i,j}$. In other words, we have

$$\begin{aligned}&\sum_{\sigma\in S_{n-1}}\prod_{i=1}^{n-1}\left(X_{i,\sigma(i)}X_{f(i),n}-X_{i,n}X_{f(i),\sigma(i)}\right)\\&=\left(X_{n,n}^{\left|f^{-1}(n)\right|-2}\prod_{\substack{i\in\{1,2,\ldots,n-1\};\\ f(i)\neq n}}X_{f(i),n}\right)\cdot\sum_{\sigma\in S_n}\prod_{i=1}^{n}X_{i,\sigma(i)}.\end{aligned}\tag{18}$$

Thus, Lemma 3.14 (applied to $n-1$, C, $\dfrac{a_{i,j}}{a_{i,n}} - \dfrac{a_{f(i),j}}{a_{f(i),n}}$ and $a_{i,n}a_{f(i),n}$ instead of m, A, $a_{i,j}$ and b_i) yields

$$\det\left(\left(a_{i,n}a_{f(i),n}\left(\frac{a_{i,j}}{a_{i,n}} - \frac{a_{f(i),j}}{a_{f(i),n}}\right)\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}\right) = \left(\prod_{i=1}^{n-1}\left(a_{i,n}a_{f(i),n}\right)\right)\det C.$$

Comparing this with

$$\det\left(\left(\underbrace{a_{i,n}a_{f(i),n}\left(\frac{a_{i,j}}{a_{i,n}} - \frac{a_{f(i),j}}{a_{f(i),n}}\right)}_{=a_{i,j}a_{f(i),n}-a_{i,n}a_{f(i),j}}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}\right) = \det\left(\underbrace{\left(a_{i,j}a_{f(i),n} - a_{i,n}a_{f(i),j}\right)_{1\leq i\leq n-1,\ 1\leq j\leq n-1}}_{=B}\right) = \det B,$$

we find

$$\det B = \left(\prod_{i=1}^{n-1}\left(a_{i,n}a_{f(i),n}\right)\right)\det C. \qquad (19)$$

It remains to compute $\det C$.

For every $(i,j) \in \{1,2,\ldots,n\}^2$, define an element $d_{i,j} \in \mathbb{K}$ by

$$d_{i,j} = \begin{cases} \dfrac{a_{i,j}}{a_{i,n}} - \dfrac{a_{f(i),j}}{a_{f(i),n}}, & \text{if } i < n; \\ \dfrac{a_{i,j}}{a_{i,n}}, & \text{if } i = n \end{cases}.$$

Now, let $\left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n} \in \mathbb{K}^{n\times n}$ be an $n\times n$-matrix. The equality (18) is an identity between polynomials in the polynomial ring $\mathbb{P}$. Thus, we can substitute $a_{i,j}$ for each $X_{i,j}$ in this equality. As a result, we obtain the equality (17).

Thus we have shown that (17) holds for every $n\times n$-matrix $\left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n} \in \mathbb{K}^{n\times n}$. As we have already explained, this is just a restatement of Theorem 2.9 **(b)**; hence, Theorem 2.9 **(b)** is proven in full generality.

(The justification above is a typical use of the "method of universal identities". See [Conrad09] for examples of similar justifications, albeit used in different settings.)

For every $i \in \{1, 2, \ldots, n-1\}$, the definition of $d_{i,n}$ yields

$$d_{i,n} = \begin{cases} \dfrac{a_{i,n}}{a_{i,n}} - \dfrac{a_{f(i),n}}{a_{f(i),n}}, & \text{if } i < n; \\ \dfrac{a_{i,n}}{a_{i,n}}, & \text{if } i = n \end{cases} = \underbrace{\frac{a_{i,n}}{a_{i,n}}}_{=1} - \underbrace{\frac{a_{f(i),n}}{a_{f(i),n}}}_{=1} \qquad (\text{since } i < n)$$
$$= 1 - 1 = 0.$$

Moreover, the definition of $d_{n,n}$ yields

$$d_{n,n} = \begin{cases} \dfrac{a_{n,n}}{a_{n,n}} - \dfrac{a_{f(n),n}}{a_{f(n),n}}, & \text{if } n < n; \\ \dfrac{a_{n,n}}{a_{n,n}}, & \text{if } n = n \end{cases} = \frac{a_{n,n}}{a_{n,n}} \qquad (\text{since } n = n)$$
$$= 1.$$

Finally, every $i \in \{1, 2, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, n\}$ satisfy

$$d_{i,j} = \begin{cases} \dfrac{a_{i,j}}{a_{i,n}} - \dfrac{a_{f(i),j}}{a_{f(i),n}}, & \text{if } i < n; \\ \dfrac{a_{i,j}}{a_{i,n}}, & \text{if } i = n \end{cases} = \frac{a_{i,j}}{a_{i,n}} - \frac{a_{f(i),j}}{a_{f(i),n}} \tag{20}$$

(since $i < n$).

Now, let D be the $n \times n$-matrix

$$\left(d_{i,j}\right)_{1 \leq i \leq n,\ 1 \leq j \leq n} \in \mathbb{K}^{n \times n}.$$

Recall that $d_{i,n} = 0$ for every $i \in \{1, 2, \ldots, n-1\}$. Hence, Lemma 3.12 (applied to D and $d_{i,j}$ instead of A and $a_{i,j}$) shows that

$$\det D = \underbrace{d_{n,n}}_{=1} \det \left(\left(\underbrace{d_{i,j}}_{\substack{= \frac{a_{i,j}}{a_{i,n}} - \frac{a_{f(i),j}}{a_{f(i),n}} \\ \text{(by (20))}}} \right)_{1 \leq i \leq n-1,\ 1 \leq j \leq n-1} \right)$$
$$= \det \underbrace{\left(\left(\frac{a_{i,j}}{a_{i,n}} - \frac{a_{f(i),j}}{a_{f(i),n}} \right)_{1 \leq i \leq n-1,\ 1 \leq j \leq n-1} \right)}_{=C} = \det C.$$

Hence, (19) becomes

$$\begin{aligned}\det B &= \left(\prod_{i=1}^{n-1}\left(a_{i,n}a_{f(i),n}\right)\right)\underbrace{\det C}_{=\det D} \\ &= \left(\prod_{i=1}^{n-1}\left(a_{i,n}a_{f(i),n}\right)\right)\det D. \qquad (21)\end{aligned}$$

Hence, we only need to compute $\det D$. How do we do this?

Let E be the $n\times n$-matrix $\left(\dfrac{a_{i,j}}{a_{i,n}}\right)_{1\leq i\leq n,\ 1\leq j\leq n}\in\mathbb{K}^{n\times n}$.

Recall that $A=\left(a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$. Lemma 3.14 (applied to $m=n$ and $b_i=\dfrac{1}{a_{i,n}}$) thus yields

$$\det\left(\left(\frac{1}{a_{i,n}}a_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}\right)=\left(\prod_{i=1}^{n}\frac{1}{a_{i,n}}\right)\det A.$$

Compared with

$$\det\left(\left(\underbrace{\frac{1}{a_{i,n}}a_{i,j}}_{=\frac{a_{i,j}}{a_{i,n}}}\right)_{1\leq i\leq n,\ 1\leq j\leq n}\right)=\det\left(\underbrace{\left(\frac{a_{i,j}}{a_{i,n}}\right)_{1\leq i\leq n,\ 1\leq j\leq n}}_{=E}\right)=\det E,$$

this yields

$$\det E=\left(\prod_{i=1}^{n}\frac{1}{a_{i,n}}\right)\det A. \qquad (22)$$

On the other hand, recall that we have defined an $n\times n$-matrix Z_f in Definition 3.8. We now claim that

$$D=Z_fE. \qquad (23)$$

Proof of (23): We have $Z_f=\left(\delta_{i,j}-\left(1-\delta_{i,n}\right)\delta_{f(i),j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$ and $E=\left(\dfrac{a_{i,j}}{a_{i,n}}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$. Thus, the definition of the product of two matrices yields

$$Z_fE=\left(\sum_{k=1}^{n}\left(\delta_{i,k}-\left(1-\delta_{i,n}\right)\delta_{f(i),k}\right)\frac{a_{k,j}}{a_{k,n}}\right)_{1\leq i\leq n,\ 1\leq j\leq n}.$$

Since every $(i,j) \in \{1,2,\ldots,n\}^2$ satisfies

$$\begin{aligned}
&\sum_{k=1}^{n} \left(\delta_{i,k} - (1-\delta_{i,n})\,\delta_{f(i),k}\right) \frac{a_{k,j}}{a_{k,n}} \\
&= \underbrace{\sum_{k=1}^{n} \delta_{i,k} \frac{a_{k,j}}{a_{k,n}}}_{\substack{= \frac{a_{i,j}}{a_{i,n}} \\ \text{(because the factor } \delta_{i,k} \text{ in the sum} \\ \text{kills every addend except the one for } k=i)}} - \sum_{k=1}^{n} \underbrace{(1-\delta_{i,n})\,\delta_{f(i),k} \frac{a_{k,j}}{a_{k,n}}}_{\substack{=(1-\delta_{i,n}) \frac{a_{f(i),j}}{a_{f(i),n}} \\ \text{(because the factor } \delta_{f(i),k} \text{ in the sum} \\ \text{kills every addend except the one for } k=f(i))}} \\
&= \frac{a_{i,j}}{a_{i,n}} - \underbrace{(1-\delta_{i,n})}_{=\begin{cases}1, & \text{if } i<n;\\ 0, & \text{if } i=n\end{cases}} \frac{a_{f(i),j}}{a_{f(i),n}} = \frac{a_{i,j}}{a_{i,n}} - \begin{cases}1, & \text{if } i<n;\\ 0, & \text{if } i=n\end{cases} \frac{a_{f(i),j}}{a_{f(i),n}} \\
&= \begin{cases} \frac{a_{i,j}}{a_{i,n}} - \frac{a_{f(i),j}}{a_{f(i),n}}, & \text{if } i<n; \\ \frac{a_{i,j}}{a_{i,n}}, & \text{if } i=n \end{cases} = d_{i,j} \qquad \left(\text{by the definition of } d_{i,j}\right),
\end{aligned}$$

this rewrites as

$$Z_f E = \left(d_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}.$$

Comparing this with $D = \left(d_{i,j}\right)_{1\leq i\leq n,\ 1\leq j\leq n}$, we obtain $D = Z_f E$. This proves (23).

Now, we can prove parts **(a)** and **(b)** of Theorem 2.9:

(a) Assume that the map f is not n-potent. Taking determinants on both sides of (23), we obtain

$$\det D = \det\left(Z_f E\right) = \underbrace{\det\left(Z_f\right)}_{\substack{=0 \\ \text{(by Proposition 3.13 (b))}}} \cdot \det E = 0.$$

Thus, (21) becomes

$$\det B = \left(\prod_{i=1}^{n-1} \left(a_{i,n} a_{f(i),n}\right)\right) \underbrace{\det D}_{=0} = 0.$$

This proves Theorem 2.9 **(a)**.

(b) Assume that the map f is n-potent. Taking determinants on both sides of

(23), we obtain

$$\begin{aligned}
\det D &= \det\left(Z_f E\right) = \underbrace{\det\left(Z_f\right)}_{\substack{=1 \\ \text{(by Proposition 3.13 (a))}}} \cdot \det E = \det E \\
&= \underbrace{\left(\prod_{i=1}^{n} \frac{1}{a_{i,n}}\right)}_{=\left(\prod_{i=1}^{n-1} \frac{1}{a_{i,n}}\right) \cdot \frac{1}{a_{n,n}}} \det A \qquad \text{(by (22))} \\
&= \left(\prod_{i=1}^{n-1} \frac{1}{a_{i,n}}\right) \cdot \frac{1}{a_{n,n}} \det A.
\end{aligned}$$

Thus, (21) becomes

$$\begin{aligned}
\det B &= \left(\prod_{i=1}^{n-1} \left(a_{i,n} a_{f(i),n}\right)\right) \underbrace{\det D}_{=\left(\prod_{i=1}^{n-1} \frac{1}{a_{i,n}}\right) \cdot \frac{1}{a_{n,n}} \det A} \\
&= \underbrace{\left(\prod_{i=1}^{n-1} \left(a_{i,n} a_{f(i),n}\right)\right) \left(\prod_{i=1}^{n-1} \frac{1}{a_{i,n}}\right)}_{=\prod_{i=1}^{n-1} a_{f(i),n} = \prod_{i \in \{1,2,\ldots,n-1\}} a_{f(i),n}} \cdot \frac{1}{a_{n,n}} \det A \\
&= \underbrace{\left(\prod_{i \in \{1,2,\ldots,n-1\}} a_{f(i),n}\right) \cdot \frac{1}{a_{n,n}}}_{\substack{=\frac{1}{a_{n,n}} \prod_{i \in \{1,2,\ldots,n-1\}} a_{f(i),n} = \operatorname{abut}_f A \\ \text{(by Remark 2.8 (a))}}} \det A = \left(\operatorname{abut}_f A\right) \det A.
\end{aligned}$$

This proves Theorem 2.9 **(b)**. □

3.6. Further questions

The above – rather indirect – road to the matrix-tree theorem suggests the following two questions:

- Is there a combinatorial proof of Theorem 2.9? Or, at least, is there a "division-free" proof (i.e., a proof that does not use a WLOG assumption that some of the $a_{i,j}$ are invertible or a similar trick)?
- Can we similarly obtain some of the various generalizations and variants of the matrix-tree theorem, such as the all-minors matrix-tree theorem ([Chaiken82, (2)] and [Sahi13, Theorem 6])?

4. Appendix: some standard proofs

For the sake of completeness, let us give some proofs of standard results that have been used without proof above.

Proof of Remark 2.6. **(a)** We have $1 \neq n$ (since $n \geq 2$). But the map f is n-potent. Thus, there exists some $k \in \mathbb{N}$ such that $f^k(1) = n$. Let h be the smallest such k. Then, $f^h(1) = n$. Hence, $h \neq 0$ (since $f^h(1) = n \neq 1 = f^0(1)$). Therefore, $h - 1 \in \mathbb{N}$, so that $f^{h-1}(1) \neq n$ (because h is the **smallest** $k \in \mathbb{N}$ such that $f^k(1) = n$). Hence, $f^{h-1}(1) \in \{1, 2, \ldots, n-1\}$. Thus, $f^{h-1}(1)$ is a $g \in \{1, 2, \ldots, n-1\}$ such that $f(g) = n$ (since $f\left(f^{h-1}(1)\right) = f^h(1) = n$). Therefore, such a g exists. This proves Remark 2.6 **(a)**.

(b) The map f is n-potent; thus, $f(n) = n$. Hence, $n \in f^{-1}(n)$. Remark 2.6 **(a)** shows that there exists some $g \in \{1, 2, \ldots, n-1\}$ such that $f(g) = n$. Consider this g. From $f(g) = n$, we obtain $g \in f^{-1}(n)$. From $g \in \{1, 2, \ldots, n-1\}$, we obtain $g \neq n$. Hence, g and n are two distinct elements of the set $f^{-1}(n)$. Consequently, $\left|f^{-1}(n)\right| \geq 2$. This proves Remark 2.6 **(b)**. □

Proof of Remark 2.8. **(b)** We have $n \in f^{-1}(n)$ (since $f(n) = n$) and $g \in f^{-1}(n)$ (since $f(g) = n$). Moreover, $g \neq n$ (since $g \in \{1, 2, \ldots, n-1\}$). Hence, g and n are two distinct elements of $f^{-1}(n)$. Hence, $\left|f^{-1}(n) \setminus \{n, g\}\right| = \left|f^{-1}(n)\right| - 2$. But

$$\begin{aligned}
&\{i \in \{1, 2, \ldots, n-1\} \setminus \{g\} \mid f(i) = n\} \\
&= f^{-1}(n) \cap (\{1, 2, \ldots, n-1\} \setminus \{g\}) \\
&= \underbrace{f^{-1}(n) \cap \{1, 2, \ldots, n-1\}}_{= f^{-1}(n) \setminus \{n\}} \setminus \{g\} = \left(f^{-1}(n) \setminus \{n\}\right) \setminus \{g\} \\
&= f^{-1}(n) \setminus \{n, g\}
\end{aligned}$$

so that

$$|\{i \in \{1, 2, \ldots, n-1\} \setminus \{g\} \mid f(i) = n\}| = \left|f^{-1}(n) \setminus \{n, g\}\right| = \left|f^{-1}(n)\right| - 2. \tag{24}$$

Now,

$$\prod_{\substack{i\in\{1,2,\ldots,n-1\};\\ i\neq g}} a_{f(i),n}$$

$$= \prod_{i\in\{1,2,\ldots,n-1\}\setminus\{g\}} a_{f(i),n} = \left(\prod_{\substack{i\in\{1,2,\ldots,n-1\}\setminus\{g\};\\ f(i)=n}} \underbrace{a_{f(i),n}}_{\substack{=a_{n,n}\\ \text{(since } f(i)=n)}}\right)\left(\prod_{\substack{i\in\{1,2,\ldots,n-1\}\setminus\{g\};\\ f(i)\neq n}} a_{f(i),n}\right)$$

$$= \underbrace{\left(\prod_{\substack{i\in\{1,2,\ldots,n-1\}\setminus\{g\};\\ f(i)=n}} a_{n,n}\right)}_{\substack{=a_{n,n}^{|\{i\in\{1,2,\ldots,n-1\}\setminus\{g\}\ \mid\ f(i)=n\}|}=a_{n,n}^{\left|f^{-1}(n)\right|-2}\\ \text{(by (24))}}}\left(\prod_{\substack{i\in\{1,2,\ldots,n-1\}\setminus\{g\};\\ f(i)\neq n}} a_{f(i),n}\right)$$

$$= a_{n,n}^{\left|f^{-1}(n)\right|-2}\left(\prod_{\substack{i\in\{1,2,\ldots,n-1\};\\ f(i)\neq n}} a_{f(i),n}\right) = \operatorname{abut}_f A$$

(by the definition of $\operatorname{abut}_f A$). This proves Remark 2.8 **(b)**.

(a) Assume that $a_{n,n}\in\mathbb{K}$ is invertible. Fix $g\in\{1,2,\ldots,n-1\}$ as in Remark 2.8 **(b)**. Then,

$$\prod_{i\in\{1,2,\ldots,n-1\}} a_{f(i),n} = \underbrace{a_{f(g),n}}_{\substack{=a_{n,n}\\ \text{(since } f(g)=n)}} \underbrace{\prod_{\substack{i\in\{1,2,\ldots,n-1\};\\ i\neq g}} a_{f(i),n}}_{\substack{=\operatorname{abut}_f A\\ \text{(by Remark 2.8 (b))}}} = a_{n,n}\operatorname{abut}_f A,$$

so that $\operatorname{abut}_f A = \dfrac{1}{a_{n,n}}\prod_{i\in\{1,2,\ldots,n-1\}} a_{f(i),n}$. This proves Remark 2.8 **(a)**. □

Proof of Lemma 3.1. We have $G = \left(\sum_{k=1}^{n} b_{i,k}d_{i,j,k}\right)_{1\leq i\leq m,\ 1\leq j\leq m}$. Thus, the definition of a determinant yields

$$\begin{aligned}
\det G &= \sum_{\sigma\in S_m} (-1)^\sigma \underbrace{\prod_{i=1}^{m}\left(\sum_{k=1}^{n} b_{i,k} d_{i,\sigma(i),k}\right)}_{\substack{=\sum_{f:\{1,2,\ldots,m\}\to\{1,2,\ldots,n\}} \prod_{i=1}^{m}\left(b_{i,f(i)} d_{i,\sigma(i),f(i)}\right) \\ \text{(by the product rule)}}} \\
&= \sum_{\sigma\in S_m} (-1)^\sigma \sum_{f:\{1,2,\ldots,m\}\to\{1,2,\ldots,n\}} \prod_{i=1}^{m}\left(b_{i,f(i)} d_{i,\sigma(i),f(i)}\right) \\
&= \sum_{f:\{1,2,\ldots,m\}\to\{1,2,\ldots,n\}} \sum_{\sigma\in S_m} (-1)^\sigma \underbrace{\prod_{i=1}^{m}\left(b_{i,f(i)} d_{i,\sigma(i),f(i)}\right)}_{=\left(\prod_{i=1}^{m} b_{i,f(i)}\right)\left(\prod_{i=1}^{m} d_{i,\sigma(i),f(i)}\right)} \\
&= \sum_{f:\{1,2,\ldots,m\}\to\{1,2,\ldots,n\}} \left(\prod_{i=1}^{m} b_{i,f(i)}\right) \underbrace{\sum_{\sigma\in S_m} (-1)^\sigma \left(\prod_{i=1}^{m} d_{i,\sigma(i),f(i)}\right)}_{\substack{=\det\left(\left(d_{i,j,f(i)}\right)_{1\le i\le m,\ 1\le j\le m}\right) \\ \text{(by the definition of a determinant)}}} \\
&= \sum_{f:\{1,2,\ldots,m\}\to\{1,2,\ldots,n\}} \left(\prod_{i=1}^{m} b_{i,f(i)}\right) \det\left(\left(d_{i,j,f(i)}\right)_{1\le i\le m,\ 1\le j\le m}\right).
\end{aligned}$$

□

Proof of Proposition 3.3. The elements $f^0(i), f^1(i), \ldots, f^n(i)$ are $n+1$ elements of the n-element set $\{1,2,\ldots,n\}$. Thus, by the pigeonhole principle, we see that two of these elements must be equal. In other words, there exist two elements u and v of $\{0,1,\ldots,n\}$ such that $u<v$ and $f^u(i) = f^v(i)$. Consider these u and v. We have $v\in\{0,1,\ldots,n\}$, so that $v\le n$ and thus $v-1\le n-1$. Hence, $\{0,1,\ldots,v-1\}\subseteq\{0,1,\ldots,n-1\}$.

We have $u<v$, so that $u\le v-1$ (since u and v are integers). Thus, $u\in\{0,1,\ldots,v-1\}$ (since u is a nonnegative integer). Hence, $0\le u\le v-1$, so that $0\in\{0,1,\ldots,v-1\}$.

Let S be the set $\left\{f^0(i), f^1(i), \ldots, f^{v-1}(i)\right\}$. From $u\in\{0,1,\ldots,v-1\}$, we obtain $f^u(i)\in\left\{f^0(i), f^1(i), \ldots, f^{v-1}(i)\right\} = S$. From $0\in\{0,1,\ldots,v-1\}$, we obtain $f^0(i)\in\left\{f^0(i), f^1(i), \ldots, f^{v-1}(i)\right\} = S$.

Now,

$$f(s)\in S \qquad \text{for every } s\in S \tag{25}$$

[12].

[12] *Proof of (25):* Let $s\in S$.

We have $s\in S = \left\{f^0(i), f^1(i), \ldots, f^{v-1}(i)\right\}$. In other words, $s = f^h(i)$ for some $h\in$

Now, we can easily see that

$$f^k(i) \in S \qquad \text{for every } k \in \mathbb{N} \tag{26}$$

[13].

On the other hand,

$$\begin{aligned} S &= \left\{f^0(i), f^1(i), \ldots, f^{v-1}(i)\right\} = \{f^s(i) \mid s \in \{0,1,\ldots,v-1\}\} \\ &\subseteq \{f^s(i) \mid s \in \{0,1,\ldots,n-1\}\} \qquad (\text{since } \{0,1,\ldots,v-1\} \subseteq \{0,1,\ldots,n-1\}). \end{aligned}$$

Hence, for every $k \in \mathbb{N}$, we have

$$\begin{aligned} f^k(i) &\in S \qquad (\text{by (26)}) \\ &\subseteq \{f^s(i) \mid s \in \{0,1,\ldots,n-1\}\}. \end{aligned}$$

This proves Proposition 3.3. □

Proof of Proposition 3.4. $\Longrightarrow$: Assume that $f^{n-1}(i) = n$. Thus, there exists some $k \in \mathbb{N}$ such that $f^k(i) = n$ (namely, $k = n-1$). This proves the $\Longrightarrow$ direction of Proposition 3.4.

$\Longleftarrow$: Assume that there exists some $k \in \mathbb{N}$ such that $f^k(i) = n$. Consider this k. We must show that $f^{n-1}(i) = n$.

We have $n = f^k(i) \in \{f^s(i) \mid s \in \{0,1,\ldots,n-1\}\}$ (by Proposition 3.3). In other words, $n = f^s(i)$ for some $s \in \{0,1,\ldots,n-1\}$. Consider this s.

$\{0,1,\ldots,v-1\}$. Consider this h. Thus, $f\left(\underbrace{s}_{=f^h(i)}\right) = f\left(f^h(i)\right) = f^{h+1}(i)$.

We want to prove that $f(s) \in S$. We are in one of the following two cases:

Case 1: We have $h = v-1$.

Case 2: We have $h \neq v-1$.

Let us first consider Case 1. In this case, we have $h = v-1$. Hence, $h+1 = v$. Now, $f(s) = f^{h+1}(i) = f^v(i)$ (since $h+1 = v$). Compared with $f^u(i) = f^v(i)$, this yields $f(s) = f^u(i) \in S$. Hence, $f(s) \in S$ is proven in Case 1.

Let us now consider Case 2. In this case, we have $h \neq v-1$. Combined with $h \in \{0,1,\ldots,v-1\}$, this yields $h \in \{0,1,\ldots,v-1\} \setminus \{v-1\} = \{0,1,\ldots,(v-1)-1\}$, so that $h+1 \in \{0,1,\ldots,v-1\}$. Thus, $f^{h+1}(i) \in \{f^0(i), f^1(i), \ldots, f^{v-1}(i)\} = S$. Hence, $f(s) = f^{h+1}(i) \in S$. Thus, $f(s) \in S$ is proven in Case 2.

We have now proven $f(s) \in S$ in each of the two Cases 1 and 2. Thus, $f(s) \in S$ always holds. This proves (25).

[13] *Proof of (26):* We shall prove (26) by induction over k:

Induction base: We have $f^0(i) \in S$. In other words, (26) holds for $k = 0$. This completes the induction base.

Induction step: Let $K \in \mathbb{N}$. Assume that (26) holds for $k = K$. We must prove that (26) holds for $k = K+1$.

We have assumed that (26) holds for $k = K$. In other words, $f^K(i) \in S$. Thus, (25) (applied to $s = f^K(i)$) yields $f\left(f^K(i)\right) \in S$. Thus, $f^{K+1}(i) = f\left(f^K(i)\right) \in S$. In other words, (26) holds for $k = K+1$. This completes the induction step. Hence, (26) is proven by induction.

We have $f^s(i) = n$. Using this fact (and the fact that $f(n) = n$), we can prove (by induction over h) that

$$f^h(i) = n \qquad \text{for every integer } h \geq s. \tag{27}$$

But $s \in \{0, 1, \ldots, n-1\}$, so that $s \leq n-1$ and therefore $n-1 \geq s$. Hence, (27) (applied to $h = n-1$) yields $f^{n-1}(i) = n$. This proves the $\Longleftarrow$ direction of Proposition 3.4. □

Proof of Proposition 3.5. $\Longleftarrow$: Assume that $f^{n-1}(\{1, 2, \ldots, n\}) = \{n\}$. For every $i \in \{1, 2, \ldots, n\}$, we have

$$f^{n-1}\left(\underbrace{i}_{\in\{1,2,\ldots,n\}}\right) \in f^{n-1}(\{1, 2, \ldots, n\}) = \{n\}$$

and thus $f^{n-1}(i) = n$. Hence, for every $i \in \{1, 2, \ldots, n-1\}$, there exists some $k \in \mathbb{N}$ such that $f^k(i) = n$ (namely, $k = n-1$). In other words, the map f is n-potent. This proves the $\Longleftarrow$ direction of Proposition 3.5.

$\Longrightarrow$: Assume that the map f is n-potent. Let $i \in \{1, 2, \ldots, n-1\}$. Then, there exists some $k \in \mathbb{N}$ such that $f^k(i) = n$ (since f is n-potent). Thus, $f^{n-1}(i) = n$ (by the $\Longleftarrow$ direction of Proposition 3.4).

Now, forget that we fixed i. We thus have shown that $f^{n-1}(i) = n$ for each $i \in \{1, 2, \ldots, n\}$. Hence,

$$\left\{f^{n-1}(1), f^{n-1}(2), \ldots, f^{n-1}(n)\right\} = \left\{\underbrace{n, n, \ldots, n}_{n \text{ times } n}\right\} = \{n\}.$$

Thus, $f^{n-1}(\{1, 2, \ldots, n\}) = \left\{f^{n-1}(1), f^{n-1}(2), \ldots, f^{n-1}(n)\right\} = \{n\}$. This proves the $\Longrightarrow$ direction of Proposition 3.5. □

Proof of Corollary 3.6. We are in one of the following two cases:

Case 1: We have $f^{n-1}(i) = n$.

Case 2: We have $f^{n-1}(i) \neq n$.

Let us consider Case 1 first. In this case, we have $f^{n-1}(i) = n$. Thus, $\delta_{f^{n-1}(i),n} = 1$. But $f^n(i) = f\left(\underbrace{f^{n-1}(i)}_{=n}\right) = f(n) = n$, so that $\delta_{f^n(i),n} = 1$. Hence, $\delta_{f^{n-1}(i),n} = 1 = \delta_{f^n(i),n}$. Thus, Corollary 3.6 is proven in Case 1.

Let us now consider Case 2. In this case, we have $f^{n-1}(i) \neq n$. Thus, $\delta_{f^{n-1}(i),n} = 0$. On the other hand, we have $f^n(i) \neq n$ [14]. Hence, $\delta_{f^n(i),n} = 0$. Hence, $\delta_{f^{n-1}(i),n} = 0 = \delta_{f^n(i),n}$. Thus, Corollary 3.6 is proven in Case 2.

[14] *Proof.* Assume the contrary. Thus, $f^n(i) = n$. Hence, there exists some $k \in \mathbb{N}$ such that $f^k(i) = n$ (namely, $k = n$). Thus, $f^{n-1}(i) = n$ (according to the $\Longleftarrow$ direction of Proposition 3.4). This contradicts $f^{n-1}(i) \neq n$. This contradiction proves that our assumption was wrong, qed.

Now, we have proven Corollary 3.6 in each of the two Cases 1 and 2. Hence, Corollary 3.6 always holds. □

Proof of Lemma 3.14. The definition of $\det A$ yields $\det A = \sum\limits_{\sigma \in S_m} (-1)^\sigma \prod\limits_{i=1}^{m} a_{i,\sigma(i)}$ (since $A = (a_{i,j})_{1 \leq i \leq m,\ 1 \leq j \leq m}$). On the other hand, the definition of $\det \left((b_i a_{i,j})_{1 \leq i \leq m,\ 1 \leq j \leq m} \right)$ yields

$$\begin{aligned}
\det \left((b_i a_{i,j})_{1 \leq i \leq m,\ 1 \leq j \leq m} \right) &= \sum_{\sigma \in S_m} (-1)^\sigma \underbrace{\prod_{i=1}^{m} \left(b_i a_{i,\sigma(i)} \right)}_{= \left(\prod_{i=1}^{m} b_i \right) \left(\prod_{i=1}^{m} a_{i,\sigma(i)} \right)} \\
&= \sum_{\sigma \in S_m} (-1)^\sigma \left(\prod_{i=1}^{m} b_i \right) \left(\prod_{i=1}^{m} a_{i,\sigma(i)} \right) \\
&= \left(\prod_{i=1}^{m} b_i \right) \underbrace{\sum_{\sigma \in S_m} (-1)^\sigma \prod_{i=1}^{m} a_{i,\sigma(i)}}_{= \det A} = \left(\prod_{i=1}^{m} b_i \right) \det A.
\end{aligned}$$

This proves Lemma 3.14. □

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The numbering of theorems and formulas in this link might shift

when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see `https://github.com/darijgr/detnotes/releases/tag/2017-02-15`.

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