

18.745: Introduction to Lie algebras (Spring term 2012 at MIT)

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0.1. Version

ver:S

These notes are not exactly what Lusztig did in class! I have sometimes "reconstructed" proofs on my own (as Lusztig goes too fast for me to LaTeX everything on the fly), added remarks, moved things around, etc.

0.2. Introduction

The best textbook on the subject of Lie algebras is

Nicolas Bourbaki, *Groupes et algèbres de Lie*.

Another useful source is

James E. Humphreys, *Introduction to Lie Algebras and Representation Theory*, Third Printing 1980.

We will not follow any of these texts, but will distribute some notes during classes.

There will be 10 homework sheets, 1 quiz and 1 final project (write about some topic not covered in class).

The study of Lie algebras is one part of Lie theory; the other is Lie groups. These two parts are naturally related, Lie algebras being a kind of "shadow" (more precisely, linearization) of Lie groups; but Lie algebras can be studied using purely algebraic

methods, whereas Lie groups require analysis and geometry (or algebraic geometry). Hence, we are going to study Lie algebras independently of Lie groups in this course, and we will not make use of any geometry or analysis.

[Historical remarks omitted.]

0.3. Definition of Lie algebras

Let k be a field. (Eventually we will restrict ourselves to the case when k is the field of complex numbers, but as for now any field will do.)

A k -algebra will mean a k -vector space A endowed with a bilinear map $A \times A \rightarrow A$ (called the *product*), which will be denoted by $(a, b) \mapsto ab$ (or, occasionally, by $(a, b) \mapsto a \cdot b$). This is the least restrictive of all common definitions of k -algebras; most texts require certain properties to hold for this bilinear map in order for A to be a k -algebra. We will, however, not require any further properties in the definition of k -algebra. Instead we define two more restrictive notions:

- An *associative k -algebra* will mean a k -algebra A which satisfies $(ab)c = a(bc)$ for any $a \in A$, $b \in A$ and $c \in A$. (Note that, unlike many authors, we don't require associative k -algebras to have a unity.)
- A *Lie algebra* will mean a k -algebra A which satisfies

$$\begin{cases} a(bc) + b(ca) + c(ab) = 0 & \text{(the so-called Jacobi identity)} \\ \text{for all } a \in A, b \in A \text{ and } c \in A; \\ aa = 0 & \text{for all } a \in A. \end{cases} \quad (1)$$

A quick consequence of the definition of a Lie algebra:

Proposition 1. If a and b are two elements of some Lie algebra, then $ab = -ba$.

Proof of Proposition 1. Let a and b be two elements of some Lie algebra.

By the second condition in the definition of a Lie algebra, we have $aa = 0$, $bb = 0$ and $(a + b)(a + b) = 0$. But by the bilinearity condition, $(a + b)(a + b) = aa + ab + ba + bb$, so that $(a + b)(a + b) = 0$ becomes $aa + ab + ba + bb = 0$. In view of $aa = 0$ and $bb = 0$, this simplifies to $ab + ba = 0$, so that $ab = -ba$, and thus Proposition 1 is proven.

Note that, as our proof of Proposition 1 has shown, Proposition 1 follows from the second condition in the definition of a Lie algebra (along with the bilinearity of the product), without using the Jacobi identity. If $\text{char } k \neq 2$, then the second condition in the definition of a Lie algebra can actually be *replaced* by Proposition 1 (because applying Proposition 1 to $b = a$, we get $aa = -aa$, so that $2aa = 0$, which yields $aa = 0$ if $\text{char } k \neq 2$).

It is easy to obtain a Lie algebra from any associative algebra:

Proposition 2. If A is an associative algebra, then we can define a new "product" $[\cdot, \cdot]$ on A by

$$[a, b] = ab - ba \quad \text{for all } a \in A \text{ and } b \in A.$$

The vector space A , endowed with this "product" (*not* the product of the associative algebra A), is then a Lie algebra.

Proof of Proposition 2. The Jacobi identity holds, because for all $a \in A$, $b \in A$ and $c \in A$, we have

$$\begin{aligned}
& [a, [b, c]] + [b, [c, a]] + [c, [a, b]] \\
&= a(bc - cb) - (bc - cb)a \\
&\quad + b(ca - ac) - (ca - ac)b \\
&\quad + c(ab - ba) - (ab - ba)c \\
&= abc - acb - bca + cba \\
&\quad + bca - bac - cab + acb \\
&\quad + cab - cba - abc + bac \\
&= 0 \quad (\text{since all terms cancel out}).
\end{aligned}$$

The second condition is also satisfied, since every $a \in A$ satisfies $[a, a] = aa - aa = 0$. Thus, A is a Lie algebra. Proposition 2 is proven.

Convention 3. In the future, we are always going to denote the product of a Lie algebra by $(a, b) \mapsto [a, b]$ rather than $(a, b) \mapsto ab$ (or $(a, b) \mapsto a \cdot b$). This will save us a lot of confusion, since the notation $(a, b) \mapsto ab$ is often used for the product of an associative algebra, and if we have an associative algebra structure *and* a Lie algebra structure *on the same vector space*, we must use different notations for the products of these two structures, lest they get confused.

According to Convention 3, the axioms (1) of a Lie algebra rewrite as follows:

$$\left\{ \begin{array}{ll} [a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0 & (\text{the so-called Jacobi identity}) \\ \text{for all } a \in A, b \in A \text{ and } c \in A; & \\ [a, a] = 0 & \text{for all } a \in A. \end{array} \right. \quad (2)$$

And Proposition 1 rewrites as follows: If a and b are two elements of some Lie algebra, then

$$[a, b] = -[b, a]. \quad (3)$$

Convention 4. We will refer to the product of a Lie algebra as a "bracket" (or "Lie bracket") rather than as a "product". (Thus, for example, in the situation of Proposition 2, the "product of A " means the product of the associative algebra A , whereas the "bracket of A " means the bracket of the Lie algebra A .)

Here is an example of the construction of Proposition 2:

Definition 5. Let V be a vector space over k . Then, the associative algebra $\text{End } V$ (whose product is the composition of maps) becomes a Lie algebra according to Proposition 2. This Lie algebra $\text{End } V$ is also denoted by $\mathfrak{gl}(V)$ (usually to distinguish it from the associative algebra $\text{End } V$).

Definition 6. If $\dim V < \infty$, then the subspace $\{A \in \text{End } V \mid \text{Tr } A = 0\}$ of $\text{End } V$ is also a Lie algebra with the bracket $[A, B] = AB - BA$ (since $\text{Tr}(AB - BA) = 0$ for any matrices A and B). Note that this subspace is not a subalgebra of the *associative* algebra $\text{End } V$.

Here come some boilerplate definitions that the reader has probably already seen for groups, rings and commutative algebras. We formulate them for general algebras, although we will mainly use them for Lie algebras. Note that, since we will be talking about general algebras in these definitions, we will denote the product by $(a, b) \mapsto ab$ (or $(a, b) \mapsto a \cdot b$), but when applying the definitions to Lie algebras, this notation should be replaced by $(a, b) \mapsto [a, b]$.

Definition 7. Let A and A' be two k -algebras.

(a) A k -algebra homomorphism from A to A' means a k -linear map $f : A \rightarrow A'$ such that

$$f(ab) = f(a)f(b) \quad \text{for all } a \in A \text{ and } b \in A.$$

(b) A k -algebra isomorphism from A to A' means a k -algebra homomorphism from A to A' which is a k -vector space isomorphism at the same time. Equivalently, a k -algebra isomorphism from A to A' means a k -algebra homomorphism from A to A' which has an inverse, which also is a k -algebra homomorphism.

At this place, it is useful to point out that our k -algebras are not required to have a unity even when they are associative; thus, k -algebra homomorphisms aren't required to preserve unities either (even when these exist).

Definition 8. Let A be a k -algebra. A k -subalgebra of A means a k -vector subspace B of A such that

$$ab \in B \quad \text{for any } a \in B \text{ and } b \in B.$$

If A is a Lie algebra, then k -subalgebras of A are called k -Lie subalgebras of A .

Definition 9. Let A be a k -algebra. An ideal of A means a k -vector subspace I of A such that

$$ab \in I \quad \text{for any } a \in I \text{ and } b \in A$$

and

$$ab \in I \quad \text{for any } a \in A \text{ and } b \in I.$$

Note that every ideal of a k -algebra is clearly a subalgebra of it as well.

Proposition 10. If V is a finite-dimensional vector space of k , then the subspace $\{A \in \text{End } V \mid \text{Tr } A = 0\}$ of $\text{End } V$ is an ideal of the Lie algebra $\text{End } V$, since $\text{Tr}(AB - BA) = 0$ for any matrices A and B . (But it is not an ideal of the associative algebra $\text{End } V$.)

Proposition 11. Let A be an algebra, and let I be an ideal of A . Then, there exists a unique k -algebra structure on the k -vector space A/I which satisfies

$$(a + I)(b + I) = ab + I \quad \text{for any } a \in A \text{ and } b \in A. \quad (4)$$

Proof of Proposition 11. For every $x \in A/I$ and $y \in A/I$, define an element $xy \in A/I$ as follows: Write x in the form $x = a + I$ for some $a \in A$ (this is possible since $x \in A/I$). Write y in the form $y = b + I$ for some $b \in A$ (this is possible since

$y \in A/I$). Then, the value of ab (generally) depends on the particular choice of a and b , but the value of $ab + I$ only depends on the values of x and y and not on the choice of a and b ¹. Hence, we can define the element xy to be $ab + I$. This defines a product $(x, y) \mapsto xy$ on A/I . This product is immediately seen to satisfy (4), and is easily seen to be bilinear. Thus, there exists a k -algebra structure on the k -vector space A/I which satisfies (4). This structure is also easily seen to be unique, and thus Proposition 11 is proven.

Definition 12. Let A be an algebra, and let I be an ideal of A .

(a) According to Proposition 11, there exists a unique k -algebra structure on the k -vector space A/I which satisfies

$$(a + I)(b + I) = ab + I \quad \text{for any } a \in I \text{ and } b \in I.$$

This makes A/I into a k -algebra. This algebra is called the *quotient algebra* of A modulo the ideal I .

(b) It is easy to see that A/I is associative if A is associative, and that A/I is a Lie algebra if A is a Lie algebra.

Proposition 13. Let A and A' be two k -algebras. Let $f : A \rightarrow A'$ be a k -algebra homomorphism. Then, $\text{Ker } f$ is an ideal of A .

The proof of Proposition 13 is very easy and left to the reader (who has probably already seen it in the commutative case anyway). Proposition 13 has something like a converse:

Proposition 14. Let A be a k -algebra. Let I be an ideal of A . Then, the canonical projection $A \rightarrow A/I$ is a k -algebra homomorphism, and its kernel is I .

This, again, can be proven by the reader.

Let us now give some further examples of Lie algebras:

Proposition 15. Let V be a vector space, and let $(\cdot, \cdot) : V \times V \rightarrow k$ be a bilinear form. Let L be the vector space

$$\{A \in \text{End } V \mid (Ax, y) + (x, Ay) = 0 \text{ for all } x, y \in V\}.$$

¹*Proof.* Let us write x not only in the form $x = a + I$, but also in the form $x = a' + I$ for some (possibly different) $a' \in A$. Besides, let us write y not only in the form $y = b + I$, but also in the form $y = b' + I$ for some (possibly different) $b' \in A$. Then, if we can show that $ab + I = a'b' + I$, then we are done (because this will show that $ab + I$ only depends on the values of x and y and not on the choice of a and b).

Since $a + I = x = a' + I$, we have $a - a' \in I$. Similarly, $b - b' \in I$. Since I is an ideal, $(a - a')b \in I$ (since $a - a' \in I$ and $b \in A$) and $a'(b - b') \in I$ (since $a' \in A$ and $b - b' \in I$). Thus,

$$\begin{aligned} ab - a'b' &= \underbrace{(ab - a'b)}_{=(a-a')b} + \underbrace{(a'b - a'b')}_{=a'(b-b')} = \underbrace{(a - a')b}_{\in I} + \underbrace{a'(b - b')}_{\in I} \\ &\in I + I = I \quad \text{(since } I \text{ is a } k\text{-vector space),} \end{aligned}$$

so that $ab + I = a'b' + I$, qed.

This is easily seen to be a vector subspace of $\text{End } V$. But actually, L is a Lie subalgebra of the Lie algebra $\text{End } V$ (although not a subalgebra of the associative algebra $\text{End } V$ in general).

Proof of Proposition 15. Let $A \in L$ and $B \in L$. We now have to prove that $AB - BA \in L$.

Notice that

$$(Ax, y) + (x, Ay) = 0 \quad \text{for all } x, y \in V \quad (5)$$

(since $A \in L$), and

$$(Bx, y) + (x, By) = 0 \quad \text{for all } x, y \in V. \quad (6)$$

But in fact, for any $x, y \in V$, we have

$$\begin{aligned} & ((AB - BA)x, y) + (x, (AB - BA)y) \\ &= (A(B(x)) - B(A(x)), y) + (x, A(B(y)) - B(A(y))) \\ &= \underbrace{(A(B(x)), y)}_{\substack{=-(B(x), A(y)) \\ \text{(by (5), applied to } B(x) \\ \text{instead of } x)}} - \underbrace{(B(A(x)), y)}_{\substack{=-(B(x), A(y)) \\ \text{(by (6), applied to } A(x) \\ \text{instead of } x)}} + \underbrace{(x, A(B(y)))}_{\substack{=-(A(x), B(y)) \\ \text{(by (5), applied to } B(y) \\ \text{instead of } y)}} - \underbrace{(x, B(A(y)))}_{\substack{=-(B(x), A(y)) \\ \text{(by (6), applied to } A(y) \\ \text{instead of } y)}} \\ &= -(B(x), A(y)) + (B(x), A(y)) - (A(x), B(y)) + (A(x), B(y)) = 0, \end{aligned}$$

so that $AB - BA \in L$ (by the definition of L). Thus, Proposition 15 is proven.

Many other examples of Lie algebras are given by the following fact:

Proposition 16. Let A be a k -algebra. A linear map $d : A \rightarrow A$ is called a *derivation* if it satisfies

$$d(xy) = d(x) \cdot y + x \cdot d(y) \quad \text{for all } x \in A \text{ and } y \in A. \quad (7)$$

(The condition (7) is called the *Leibniz rule*.) Let $\text{Der } A$ be the set of all derivations $A \rightarrow A$.

Now, any $d \in \text{Der } A$ and any $d' \in \text{Der } A$ satisfy $dd' - d'd \in \text{Der } A$. Thus, $\text{Der } A$ becomes a Lie algebra.

Proof of Proposition 16. We only have to prove that any $d \in \text{Der } A$ and any $d' \in \text{Der } A$ satisfy $dd' - d'd \in \text{Der } A$. To prove this, let $d \in \text{Der } A$ and $d' \in \text{Der } A$. Then,

any $x \in A$ and $y \in A$ satisfy

$$\begin{aligned}
(dd' - d'd)(xy) &= d \underbrace{(d'(xy))}_{=d'(x) \cdot y + x \cdot d'(y)} - d' \underbrace{(d(xy))}_{=d(x) \cdot y + x \cdot d(y)} \\
&\quad \text{(by (7), applied to } d' \text{ instead of } d) \qquad \text{(by (7))} \\
&= d(d'(x) \cdot y + x \cdot d'(y)) - d'(d(x) \cdot y + x \cdot d(y)) \\
&= \underbrace{d(d'(x) \cdot y)}_{=d(d'(x)) \cdot y + d'(x) \cdot d(y)} + \underbrace{d(x \cdot d'(y))}_{=d(x) \cdot d'(y) + x \cdot d(d'(y))} \\
&\quad \text{(by (7), applied to } d'(x) \text{ instead of } x) \qquad \text{(by (7), applied to } d'(y) \text{ instead of } y) \\
&\quad - \underbrace{d'(d(x) \cdot y)}_{=d'(d(x)) \cdot y + d(x) \cdot d'(y)} - \underbrace{d'(x \cdot d(y))}_{=d'(x) \cdot d(y) + x \cdot d'(d(y))} \\
&\quad \text{(by (7), applied to } d' \text{ and } d(x) \text{ instead of } d \text{ and } x) \qquad \text{(by (7), applied to } d' \text{ and } d(y) \text{ instead of } d \text{ and } y) \\
&= (d(d'(x)) \cdot y + d'(x) \cdot d(y)) + (d(x) \cdot d'(y) + x \cdot d(d'(y))) \\
&\quad - (d'(d(x)) \cdot y + d(x) \cdot d'(y)) - (d'(x) \cdot d(y) + x \cdot d'(d(y))) \\
&= \underbrace{d(d'(x)) \cdot y - d'(d(x)) \cdot y}_{=(d(d'(x)) - d'(d(x))) \cdot y} + \underbrace{x \cdot d(d'(y)) - x \cdot d'(d(y))}_{=x \cdot (d(d'(y)) - d'(d(y)))} \\
&= \underbrace{(d(d'(x)) - d'(d(x))) \cdot y}_{=(dd' - d'd)(x)} + \underbrace{x \cdot (d(d'(y)) - d'(d(y)))}_{=(dd' - d'd)(y)} \\
&= (dd' - d'd)(x) \cdot y + x \cdot (dd' - d'd)(y).
\end{aligned}$$

In other words, $dd' - d'd$ is a derivation, so that $dd' - d'd \in \text{Der } A$. This proves Proposition 16.

Now, what happens to the construction of Proposition 16 if A itself is a Lie algebra?

Proposition 17. Let L be a k -Lie algebra. As we know from Proposition 16, the space $\text{Der } L$ is a Lie algebra.

Now, for every $x \in L$, define a map $\text{ad}(x) : L \rightarrow L$ by

$$\text{ad}(x)(y) = [x, y] \quad \text{for all } y \in L.$$

Then:

- (a) The map $\text{ad}(x)$ is a derivation for every $x \in L$.
- (b) The map $L \rightarrow \text{Der } L$, $x \mapsto \text{ad}(x)$ is a Lie algebra homomorphism.
- (c) The kernel of the map $L \rightarrow \text{Der } L$, $x \mapsto \text{ad}(x)$ is the ideal

$$\{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}$$

of L . This kernel is denoted by ZL and called the *center* of L .

Proof of Proposition 17. (a) Let $x \in L$. Then, for any $y \in L$ and $z \in L$, we have

$$\begin{aligned}
& \text{ad}(x)([y, z]) \\
&= [x, [y, z]] = - \left(\left[\begin{array}{c} y, \underbrace{[z, x]}_{=-[x, z]} \\ \text{(by (3))} \end{array} \right] + [z, [x, y]] \right) \\
&\quad \text{(since the Jacobi identity says that } [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0) \\
&= \left[\begin{array}{c} y, \underbrace{[x, z]}_{=\text{ad}(x)(z)} \\ \text{(by (3))} \end{array} \right] - \underbrace{[z, [x, y]]}_{=-[[x, y], z]} = [y, \text{ad}(x)(z)] + \left[\begin{array}{c} \underbrace{[x, y]}_{=\text{ad}(x)(y)}, z \\ \text{(by (3))} \end{array} \right] \\
&= [y, \text{ad}(x)(z)] + [\text{ad}(x)(y), z].
\end{aligned}$$

This means that $\text{ad}(x)$ is a derivation, so that Proposition 17 **(a)** is proven.

(b) We must prove that $\text{ad}([y, z]) = [\text{ad}(y), \text{ad}(z)]$ for any $y \in L$ and $z \in L$. But this is clear, since every $w \in L$ satisfies

$$\begin{aligned}
\text{ad}([y, z])(w) &= [[y, z], w] = -[w, [y, z]] && \text{(by (3))} \\
&= [y, [z, w]] + \left[\begin{array}{c} z, \underbrace{[w, y]}_{=-[y, w]} \\ \text{(by (3))} \end{array} \right] \\
&\quad \text{(since the Jacobi identity says that } [w, [y, z]] + [y, [z, w]] + [z, [w, y]] = 0) \\
&= \underbrace{[y, [z, w]]}_{=\text{ad}(y)([z, w])} - \underbrace{[z, [y, w]]}_{=\text{ad}(z)([y, w])} = \text{ad}(y)(\underbrace{[z, w]}_{=\text{ad}(z)(w)}) - \text{ad}(z)(\underbrace{[y, w]}_{=\text{ad}(y)(w)}) \\
&= \text{ad}(y)(\text{ad}(z)(w)) - \text{ad}(z)(\text{ad}(y)(w)) = [\text{ad}(y), \text{ad}(z)](w).
\end{aligned}$$

Thus, Proposition 17 **(b)** is proven.

(c) The kernel of the map $L \rightarrow \text{Der } L$, $x \mapsto \text{ad}(x)$ is clearly the set

$$\begin{aligned}
\{x \in L \mid \text{ad}(x) = 0\} &= \left\{ x \in L \mid \underbrace{\text{ad}(x)(y)}_{=[x, y]} = 0 \text{ for all } y \in L \right\} \\
&= \{x \in L \mid [x, y] = 0 \text{ for all } y \in L\}.
\end{aligned}$$

This is an ideal of L (since the map $L \rightarrow \text{Der } L$, $x \mapsto \text{ad}(x)$ is a Lie algebra homomorphism according to part **(b)**, and thus its kernel is an ideal by Proposition 13). Proposition 17 **(c)** is proven.

For another example of Lie algebras, we consider flags of subspaces:

Proposition 18. Let V be a vector space of dimension n .

Let $V_0 \subseteq V_1 \subseteq \dots \subseteq V_n$ be a sequence of subspaces of V of dimensions $0, 1, \dots, n$, respectively (this means that $\dim V_i = i$ for every $i \in \{0, 1, \dots, n\}$).

Let

$$\mathfrak{t} = \{A \in \text{End } V \mid AV_i \subseteq V_i \text{ for all } i\}$$

and

$$\mathfrak{n} = \{A \in \text{End } V \mid AV_i \subseteq V_{i-1} \text{ for all } i = 1, 2, \dots, n\}.$$

Then, \mathfrak{t} and \mathfrak{n} are Lie subalgebras of $\text{End } V$, and \mathfrak{n} is an ideal of \mathfrak{t} .

Proof of Proposition 18. To prove that \mathfrak{t} is a Lie subalgebra of $\text{End } V$, we need to show that $AB - BA \in \mathfrak{t}$ for any $A \in \mathfrak{t}$ and any $B \in \mathfrak{t}$.

This is easy: If $A \in \mathfrak{t}$ and $B \in \mathfrak{t}$, then $AV_i \subseteq V_i$ and $BV_i \subseteq V_i$ for every i (by the very definition of \mathfrak{t}), so that

$$(AB - BA)V_i \subseteq A \underbrace{BV_i}_{\subseteq V_i} - B \underbrace{AV_i}_{\subseteq V_i} \subseteq \underbrace{AV_i}_{\subseteq V_i} - \underbrace{BV_i}_{\subseteq V_i} \subseteq V_i - V_i \subseteq V_i \text{ for every } i,$$

and thus $AB - BA \in \mathfrak{t}$ (by the definition of \mathfrak{t}).

Thus we have shown that \mathfrak{t} is a Lie subalgebra of $\text{End } V$.

Next let us prove that \mathfrak{n} is an ideal of \mathfrak{t} . First of all,

$$\begin{aligned} \mathfrak{n} &= \{A \in \text{End } V \mid AV_i \subseteq V_{i-1} \text{ for all } i = 1, 2, \dots, n\} \\ &\subseteq \{A \in \text{End } V \mid AV_i \subseteq V_i \text{ for all } i = 1, 2, \dots, n\} \\ &\quad \left(\begin{array}{l} \text{because every } A \in \text{End } V \text{ which satisfies } AV_i \subseteq V_{i-1} \text{ for all } i = 1, 2, \dots, n \\ \text{must also satisfy } AV_i \subseteq V_i \text{ for all } i = 1, 2, \dots, n \text{ (since } AV_{i-1} \subseteq V_i) \end{array} \right) \\ &= \{A \in \text{End } V \mid AV_i \subseteq V_i \text{ for all } i = 1, 2, \dots, n, \text{ and also } AV_0 \subseteq V_0\} \\ &\quad \left(\begin{array}{l} \text{because the assertion that } AV_0 \subseteq V_0 \text{ is a tautology} \\ \text{(since } V_0 \text{ has dimension 0 and thus } V_0 = 0) \end{array} \right) \\ &= \{A \in \text{End } V \mid AV_i \subseteq V_i \text{ for all } i\} = \mathfrak{t}. \end{aligned}$$

Now, every $A \in \mathfrak{t}$ and $B \in \mathfrak{n}$ satisfy $AB - BA \in \mathfrak{n}$.² Hence, \mathfrak{n} is an ideal of \mathfrak{t} . In particular, this yields that \mathfrak{n} is a Lie subalgebra of \mathfrak{t} (since every ideal is a subalgebra), thus a Lie subalgebra of $\text{End } V$. Proposition 18 is now completely proven.

Note that the Lie subalgebras \mathfrak{t} and \mathfrak{n} of Proposition 18 can be described in more familiar terms if we choose a basis (e_1, e_2, \dots, e_n) of V which satisfies

$$V_i = \langle e_1, e_2, \dots, e_i \rangle \quad \text{for all } i$$

(such a basis can easily be chosen by recursion: first choose a basis (e_1) of V_1 , then extend it to a basis (e_1, e_2) of V_2 , then extend it to a basis (e_1, e_2, e_3) of V_3 , etc.). In fact, having chosen such a basis, we can identify $\text{End } V$ with the ring of $n \times n$ matrices over k . Now, \mathfrak{t} is the space of all upper-triangular $n \times n$ matrices, while \mathfrak{n} is the subspace of all *strictly* upper-triangular $n \times n$ matrices.

²*Proof.* Let $A \in \mathfrak{t}$ and $B \in \mathfrak{n}$. Then, $AV_i \subseteq V_i$ for all i (by the definition of \mathfrak{t} , since $A \in \mathfrak{t}$), and $BV_i \subseteq V_{i-1}$ for all $i = 1, 2, \dots, n$ (by the definition of \mathfrak{n} , since $B \in \mathfrak{n}$). Since $AV_i \subseteq V_i$ for all i , we also have $AV_{i-1} \subseteq V_{i-1}$ for all $i = 1, 2, \dots, n$. Now, for all $i = 1, 2, \dots, n$, we have

$$(AB - BA)V_i \subseteq A \underbrace{BV_i}_{\subseteq V_{i-1}} - B \underbrace{AV_i}_{\subseteq V_i} \subseteq \underbrace{AV_{i-1}}_{\subseteq V_{i-1}} - \underbrace{BV_i}_{\subseteq V_{i-1}} \subseteq V_{i-1} - V_{i-1} \subseteq V_{i-1}.$$

Hence, by the definition of \mathfrak{n} , we have $AB - BA \in \mathfrak{n}$.

0.4. Simple Lie algebras

Simple Lie algebras are not called this way because they are particularly simple to write down or to classify. They are called "simple" since they are the building blocks for more complicated Lie algebras.

Definition 19. A Lie algebra L is said to be *simple* if it satisfies the following two conditions:

- 1) We have $L \neq 0$.
- 2) If I is an ideal of L , then $I = 0$ or $I = L$.

It is pretty clear that the Lie algebra k (with Lie bracket given by $[a, b] = 0$ for all $a \in k$ and $b \in k$) is simple. But this is not a particularly interesting example of a simple Lie algebra. Here is the first interesting one:

Proposition 20. Let

$$L = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(k) \mid a, b, c, d \in k; a + d = 0 \right\}.$$

If $\text{char } k \neq 2$, then L is a simple Lie algebra.

Proof of Proposition 20. First, clearly $L \neq 0$.

Define three elements

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

An easy computation gives

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

Assume that $\text{char } k \neq 2$.

Now, suppose I is a nonzero ideal of L . In order to show that L is simple, we must prove that $I = L$.

Since $I \neq 0$, there exists some nonzero $x \in I$. Consider this x and write it in the form $x = ae + bf + ch$ with $a, b, c \in k$ not all zero. Since I is an ideal, we have $[e, x] \in I$

(since $x \in I$) and thus $[e, [e, x]] \in I$. But a simple computation gives $\left[e, \underbrace{[e, x]}_{=bh-2ce} \right] = [e, bh - 2ce] = 2be$, so that $2be \in I$ and thus $be \in I$ (here we are using $\text{char } k \neq 2$).

Similarly, we can show that $[f, [f, x]] \in I$ but also $\left[f, \underbrace{[f, x]}_{=ah+2cf} \right] = [f, ah + 2cf] = 2af$, so that $2af \in I$ and hence $af \in I$ (again because of $\text{char } k \neq 2$).

If $a \neq 0$, then $af \in I$ becomes $f \in I$ and thus $h = [e, f] \in I$ and thus $e = \frac{1}{2}[h, e] \in I$, so that $I = L$.

If $b \neq 0$, then $be \in I$ becomes $e \in I$ and thus $h = [e, f] \in I$ and thus $e = \frac{1}{2}[h, e] \in I$, so that $I = L$.

Hence, we are done in each of the cases $a \neq 0$ and $b \neq 0$. Let us finish the remaining case now: the case when both a and b are 0. In this case, $x = ch$, so that $ch = I$ and thus $h \in I$ (since $c \neq 0$), which yields $e = \frac{1}{2} [h, e] \in I$ and $f = -\frac{1}{2} [h, f] \in I$, and again we get $I = L$. Hence we have obtained $I = L$ in each possible case, and Proposition 20 is proven.

Note that the simple Lie algebra L of Proposition 20 is a very important one: it appears as a "building block" in many other Lie algebras, and this is the key to the analysis of these Lie algebras.

0.5. A lemma on nilpotent derivations

We record a lemma for future use.

Lemma 21. Assume that k has characteristic 0. Let A be a k -algebra (not necessarily associative). Let $d : A \rightarrow A$ be a derivation. Assume that $d^n = 0$ for some $n \geq 0$. Define a map $e^d : A \rightarrow A$ by

$$e^d x = x + dx + \frac{d^2 x}{2!} + \frac{d^3 x}{3!} + \dots = \sum_{n \in \mathbb{N}} \frac{d^n x}{n!} \quad \text{for every } x \in A.$$

Then, e^d is an algebra isomorphism.

Proof of Lemma 21. [...]

0.6. Normalizers

Definition 22. Let L be a Lie algebra. Let $K \subseteq L$ be a vector subspace. Then, we define the *normalizer* $N_L(K)$ of K in L to be the subspace

$$\{x \in L \mid [x, y] \in K \text{ for every } y \in K\}$$

of L .

Proposition 23. Let L be a Lie algebra. Let $K \subseteq L$ be a vector subspace. Then, $N_L(K)$ is a Lie subalgebra of L .

Proof of Proposition 23. Let $x \in N_L(K)$ and $x' \in N_L(K)$. Then, $[x, y] \in K$ for every $y \in K$, and $[x', y] \in K$ for every $y \in K$ (both by the definition of $N_L(K)$). Hence,

$$[[x, x'], y] = [[x, y], x'] + [x, [x', y]]$$

[...]

0.7. Solvable and nilpotent Lie algebras

First of all, a definition:

Definition 24. Let L be a Lie algebra. Let I and I' be two vector subspaces of L . Then, $[I, I']$ will denote the vector subspace of L spanned by all elements $[x, x']$ with $x \in I$ and $x' \in I'$.

Proposition 25. Let L be a Lie algebra. Let I and I' be two ideals of L . Then, $[I, I']$ is an ideal of L .

Proof of Proposition 25. [...]

Definition 26. Let L be a Lie algebra. We define a sequence of ideals $(L^{(0)}, L^{(1)}, L^{(2)}, \dots)$ of L by induction: Let

$$\begin{aligned} L^{(0)} &= L, & L^{(1)} &= [L^{(0)}, L^{(0)}], & L^{(2)} &= [L^{(1)}, L^{(1)}], \\ L^{(3)} &= [L^{(2)}, L^{(2)}], & & \dots, & & \\ L^{(i)} &= [L^{(i-1)}, L^{(i-1)}] & & \text{for all } i \geq 1. \end{aligned}$$

Then, by induction (using Proposition 25) we see that $L^{(i)}$ is indeed an ideal of L for every $i \in \mathbb{N}$. Hence, $L^{(i)} \supseteq [L, L^{(i)}] \supseteq [L^{(i)}, L^{(i)}] = L^{(i+1)}$ for every $i \in \mathbb{N}$. In other words, $L^{(0)} \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \dots$

The Lie algebra L is said to be *solvable* if $L^{(i)} = 0$ for some i .

The Lie algebra L is said to be *abelian* if $L^{(1)} = 0$, i. e., if $[x, y] = 0$ for all $x \in L$ and $y \in L$.

Definition 27. Let L be a Lie algebra. We define a sequence of ideals (L^0, L^1, L^2, \dots) of L by induction: Let

$$\begin{aligned} L^0 &= L, & L^1 &= [L, L^0], & L^2 &= [L, L^1], \\ L^3 &= [L, L^2], & & \dots, & & \\ L^i &= [L, L^{i-1}] & & \text{for all } i \geq 1. \end{aligned}$$

Then, by induction (using Proposition 25) we see that L^i is indeed an ideal of L for every $i \in \mathbb{N}$. Hence, $L^i \supseteq [L, L^i] = L^{i+1}$ for every $i \in \mathbb{N}$. In other words, $L^0 \supseteq L^1 \supseteq L^2 \supseteq \dots$

The Lie algebra L is said to be *nilpotent* if $L^i = 0$ for some i .

Proposition 28. Let L be a Lie algebra.

(a) Every $i \in \mathbb{N}$ satisfies $L^{(i)} \subseteq L^i$. Moreover, $L^{(0)} = L^0$ and $L^{(1)} = L^1$.

(b) If L is nilpotent, then L is also solvable.

Proof of Proposition 28. (a) This is easily proven by induction over i : If we know that $L^{(i-1)} \subseteq L^{i-1}$, then $L^{(i)} = \left[\underbrace{L^{(i-1)}}_{\subseteq L}, \underbrace{L^{(i-1)}}_{\subseteq L^{i-1}} \right] \subseteq [L, L^{i-1}] = L^i$.

(b) If L is nilpotent, then $L^i = 0$ for some i (by the definition of "nilpotent"), thus $L^{(i)} = 0$ for some i (namely, for the same i , by Proposition 28 (a)), and thus L is solvable (by the definition of "solvable"). The proof of Proposition 28 (b) is thus complete.

Here is an example of solvable and nilpotent Lie algebras:

Proposition 29. Let V be a vector space with $\dim V = n$.

Let $V_0 \subseteq V_1 \subseteq \dots \subseteq V_n$ be a sequence of subspaces of V of dimensions $0, 1, \dots, n$, respectively (this means that $\dim V_i = i$ for every $i \in \{0, 1, \dots, n\}$).

Consider the Lie algebras \mathfrak{t} and \mathfrak{n} defined in Proposition 18. Then, \mathfrak{t} is solvable, and \mathfrak{n} is nilpotent.

Proof of Proposition 29. a) The Lie algebra \mathfrak{n} is nilpotent.

Proof. [...]

b) We have $[\mathfrak{t}, \mathfrak{t}] \subseteq \mathfrak{n}$.

[...]

In general, \mathfrak{t} is not nilpotent in Proposition 29.

Let us formulate some basic facts about solvable and nilpotent Lie algebras:

Proposition 30. Let L be a Lie algebra, and L' a Lie subalgebra of L . Then, $L'^{(i)} \subseteq L^{(i)}$ for every $i \in \mathbb{N}$ (this is proven by induction). Hence, L' is solvable if L is solvable.

Proposition 31. Let L be a Lie algebra, and I an ideal of L . Then, L/I is solvable if L is solvable.

Proposition 32. Let L be a Lie algebra, and I an ideal of L . If I and L/I are both solvable, then L is solvable.

Proof of Proposition 32. [...]

Note that we cannot replace "solvable" by "nilpotent" in Proposition 32.

Proposition 33. Let L be a Lie algebra. Let I and J be ideals. Assume that both I and J are solvable (as Lie algebras). Then, $I + J$ is also a solvable Lie algebra and an ideal of L .

Proof of Proposition 33. [...]

Corollary 34. Let L be a Lie algebra. If $\dim L < \infty$, then there is a unique ideal of L which contains every solvable ideal of L . This ideal itself is solvable.

Proof of Corollary 34. Let I be a solvable ideal of L of maximum dimension. Let I' be a solvable ideal of L . By Proposition 33, it is clear that $I + I'$ is a solvable ideal of L . Since I already has maximum dimension and $I + I' \supseteq I$, we must have $I + I' = I$, thus $I' \subseteq I$. Hence, I contains every solvable ideal of L . This proves the existence. The uniqueness is clear. This proves Corollary 34.

Definition 35. Let L be a Lie algebra.

(a) The unique ideal of L which contains every solvable ideal of L is called the *radical* of L and denoted by $\text{rad } L$.

(b) We say that L is *semisimple* if $\text{rad } L = 0$.

We continue with basic properties of nilpotent Lie algebras:

Proposition 36. Let L be a Lie algebra, and L' a Lie subalgebra of L . Then, $L'^i \subseteq L^i$ for every $i \in \mathbb{N}$ (this is proven by induction). Hence, L' is nilpotent if L is nilpotent.

Proposition 37. Let L be a Lie algebra, and I an ideal of L . Then, L/I is nilpotent if L is nilpotent.

Proposition 38. Let L be a Lie algebra. Assume that $L/Z(L)$ nilpotent. Then, L is nilpotent.

Proof of Proposition 38. [...]

Theorem 39 (Engel's Theorem). **(1)** Let L be a Lie algebra such that $\dim L < \infty$. Let V be a vector space such that $0 < \dim V < \infty$. Let $\rho : L \rightarrow \text{End } V$ be a Lie algebra homomorphism such that $\rho(x) : V \rightarrow V$ is nilpotent for all $x \in L$. Then there exists some $v \in V \setminus \{0\}$ such that $\rho(x)(v) = 0$ for all $x \in L$.

(2) Let L be a Lie algebra such that $\dim L < \infty$. Assume that $\text{ad}(x) : L \rightarrow L$ is nilpotent for every $x \in L$. Then, L is nilpotent.

Before we prove this, a lemma:

Lemma 40. Let V be a vector space with $\dim V < \infty$. Let $x : V \rightarrow V$ be a nilpotent endomorphism. Consider x as an element of $\mathfrak{gl}(V)$. Then, $\text{ad}(x) : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$ is also nilpotent.

We have $\text{ad}(x)(y) = xy - yx$ for every $y \in \mathfrak{gl}(V)$.

Thus, we can write the map ad in the form $\text{ad} = T - S$, where $T : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$ is the map sending every y to xy , and $S : \mathfrak{gl}(V) \rightarrow \mathfrak{gl}(V)$ is the map sending every y to yx . Then, every $y \in \mathfrak{gl}(V)$ satisfies $T^n y = \underbrace{x(x(\dots(xy)))}_{n \text{ times}} = x^n y$ for every $n \in \mathbb{N}$.

Thus, T is nilpotent (since x is nilpotent). Similarly, S is nilpotent. Also, $TS = ST$ is very easy to see (since every $y \in \mathfrak{gl}(V)$ satisfies $TSy = xyx = STy$).

Since T and S are nilpotent and commute ($TS = ST$), their difference $T - S$ also is nilpotent.³ In other words, [...]

Proof of Theorem 39. (1) [...]

0.8. Homework sheet #1

Exercise 41 (1.1). Let V be a vector space of dimension n . Show that the derived algebra of the Lie algebra of endomorphisms of V is exactly the set of endomorphisms with trace zero.

³*Proof.* This is a particular case of a known fact that if α and β are two commuting nilpotent elements of an associative algebra, then $\alpha - \beta$ also is nilpotent. (To prove this fact, pick $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfying $\alpha^n = 0$ and $\beta^m = 0$, and notice that $(\alpha - \beta)^{n+m-1} = \sum_{k=0}^{n+m-1} \binom{n+m-1}{k} (-1)^{n+m-1-k} \alpha^k \beta^{n+m-1-k}$ by the binomial formula. Each addend on the right hand side either contains an α^n or a β^m , and thus is 0 in both cases. Hence, $(\alpha - \beta)^{n+m-1} = 0$, so that $\alpha - \beta$ is nilpotent, qed.)

Exercise 42 (1.2). Let V be a vector space of dimension n . Show that the centre of the Lie algebra of endomorphisms of V is the subset consisting of scalar multiples of the identity.

Exercise 43 (1.3). Let L be a Lie algebra. Show that L is solvable if and only if there exists a sequence of subalgebras $L = L_0, L_1, L_2, \dots, L_k = 0$ such that L_{i+1} is an ideal of L_i and L_i/L_{i+1} is abelian for $i = 0, 1, \dots, k - 1$.

Exercise 44 (1.4). Let L be a nilpotent Lie algebra such that $L \neq 0$. Show that L has some ideal of codimension 1.