Noncommutative Abel-like identities

Darij Grinberg

detailed version
draft, January 15, 2018

Contents

1. Introduction

In this (self-contained) note, we are going to prove three identities that hold in arbitrary noncommutative rings, and generalize some well-known combinatorial identities (known as the *Abel-Hurwitz identities*).

In their simplest and least general versions, the identities we are generalizing are equalities between polynomials in $\mathbb{Z}[X,Y,Z]$; namely, they state that

$$\sum_{k=0}^{n} \binom{n}{k} (X+kZ)^k (Y-kZ)^{n-k} = \sum_{k=0}^{n} \frac{n!}{k!} (X+Y)^k Z^{n-k}; \tag{1}$$

$$\sum_{k=0}^{n} \binom{n}{k} X (X+kZ)^{k-1} (Y-kZ)^{n-k} = (X+Y)^n; \tag{2}$$

$$\sum_{k=0}^{n} \binom{n}{k} X (X+kZ)^{k-1} Y (Y+(n-k)Z)^{n-k-1} = (X+Y)(X+Y+nZ)^{n-1} \tag{3}$$

for every nonnegative integer n. [1] These identities have a long history; for example, (2) goes back to Abel [Abel26], who observed that it is a generalization of the binomial formula (obtained by specializing Z to 0). The equality (1) is ascribed to Cauchy in Riordan's text [Riorda68, §1.5, Cauchy's identity] (at least in the specialization $Z=1$; but the general version can be recovered from this specialization by dehomogenization). The equality (3) is also well-known in combinatorics, and tends to appear in the context of tree enumeration (see, e.g., [Grinbe17, Theorem 2]) and of umbral calculus (see, e.g., [Roman84, Section 2.6, Example 3]).

The identities (1), (2) and (3) have been generalized by various authors in different directions. The most famous generalization is due to Hurwitz [Hurwit02], who replaced Z by n commuting indeterminates $Z_1, Z_2, \ldots, Z_n$. More precisely, the equalities (IV), (II) and (III) in [Hurwit02] say (in a more modern language) that if n is a nonnegative integer and V denotes the set $\{1,2,\ldots,n\}$, then

$$\sum_{S\subseteq V} \left(X+\sum_{s\in S} Z_s\right)^{|S|} \left(Y-\sum_{s\in S} Z_s\right)^{n-|S|} = \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{n-k} Z_{i_1} Z_{i_2} \cdots Z_{i_k}; \tag{4}$$

$$\sum_{S\subseteq V} X \left(X+\sum_{s\in S} Z_s\right)^{|S|-1} \left(Y-\sum_{s\in S} Z_s\right)^{n-|S|} = (X+Y)^n; \tag{5}$$

$$\sum_{S\subseteq V} X \left(X+\sum_{s\in S} Z_s\right)^{|S|-1} Y \left(Y+\sum_{s\in V\setminus S} Z_s\right)^{n-|S|-1} = (X+Y)\left(X+Y+\sum_{s\in V} Z_s\right)^{n-1} \tag{6}$$

[1] The pedantic reader will have observed that two of these identities contain "fractional" terms like X^{-1} and Y^{-1} and thus should be regarded as identities in the function field $\mathbb{Q}(X,Y,Z)$ rather than in the polynomial ring $\mathbb{Z}[X,Y,Z]$. However, this is a false alarm, because all these "fractional" terms are cancelled. For example, the addend for $k=0$ in the sum on the left hand side of (2) contains the "fractional" term $(X+0Z)^{0-1} = X^{-1}$, but this term is cancelled by the factor X directly to its left. Similarly, all the other "fractional" terms disappear. Thus, all three identities are actually identities in $\mathbb{Z}[X,Y,Z]$.

in the polynomial ring $\mathbb{Z}[X, Y, Z_1, Z_2, \ldots, Z_n]$. [2] It is easy to see that setting all indeterminates $Z_1, Z_2, \ldots, Z_n$ equal to a single indeterminate Z transforms these three identities (4), (5) and (6) into the original three identities (1), (2) and (3).

In this note, we shall show that the three identities (4), (5) and (6) can be further generalized to a noncommutative setting: Namely, the commuting indeterminates $X, Y, Z_1, Z_2, \ldots, Z_n$ can be replaced by arbitrary elements $X, Y, x_1, x_2, \ldots, x_n$ of any noncommutative ring $\mathbb{L}$, provided that a centrality assumption holds (for the identities (4) and (5), the sum $X + Y$ needs to lie in the center of $\mathbb{L}$, whereas for (6), the sum $X + Y + \sum_{s \in V} x_s$ needs to lie in the center of $\mathbb{L}$), and provided that the product $Y \left(Y + \sum_{s \in V \setminus S} Z_s \right)^{n-|S|-1}$ in (6) is replaced by $\left(Y + \sum_{s \in V \setminus S} Z_s \right)^{n-|S|-1} Y$. These generalized versions of (4), (5) and (6) are Theorem 2.2, Theorem 2.4 and Theorem 2.7 below, and will be proven by a not-too-complicated induction on n.

Acknowledgments

This note was prompted by an enumerative result of Gjergji Zaimi [Zaimi17]. The computer algebra SageMath [SageMath] (specifically, its `FreeAlgebra` class) was used to make conjectures. Thanks to Dennis Stanton for making me aware of [Johns96].

2. The identities

Let us now state our results.

Convention 2.1. Let $\mathbb{L}$ be a noncommutative ring with unity.

We claim that the following four theorems hold:[3]

Theorem 2.2. Let V be a finite set. Let $n = |V|$. For each $s \in V$, let x_s be an element of $\mathbb{L}$. Let X and Y be two elements of $\mathbb{L}$ such that $X + Y$ lies in the center

[2] The sum on the right hand side of (4) ranges over all nonnegative integers k and all k-tuples $(i_1, i_2, \ldots, i_k)$ of distinct elements of V. This includes the case of $k = 0$ and the empty 0-tuple (which contributes the addend $(X + Y)^{n-0}$ (empty product) $= (X + Y)^n$). Notice that many of the addends in this sum will be equal (indeed, if two k-tuples $(i_1, i_2, \ldots, i_k)$ and $(j_1, j_2, \ldots, j_k)$ are permutations of each other, then they produce equal addends).

Once again, "fractional" terms appear in two of these identities, but are all cancelled.

[3] We promised three identities, but we are stating four theorems. This is not a mistake, since Theorem 2.7 is just an equivalent version of Theorem 2.6 (more precisely, it is obtained from Theorem 2.6 by replacing Y with $Y + \sum_{s \in V} x_s$) and so should not be considered a separate identity. We are stating these two theorems on an equal footing since we have no opinion on which of them is the "better" one.

of $\mathbb{L}$. Then,

$$\sum_{S\subseteq V}\left(X+\sum_{s\in S}x_s\right)^{|S|}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|} = \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{n-k}\, x_{i_1}x_{i_2}\cdots x_{i_k}.$$

(Here, the sum on the right hand side ranges over all nonnegative integers k and all k-tuples $(i_1, i_2, \ldots, i_k)$ of distinct elements of V. In particular, it has an addend corresponding to $k = 0$ and $(i_1, i_2, \ldots, i_k) = ()$ (the empty 0-tuple); this addend is $\underbrace{(X+Y)^{n-0}}_{=(X+Y)^n} \cdot \underbrace{(\text{empty product})}_{=1} = (X+Y)^n$.)

Example 2.3. In the case when $V = \{1, 2\}$, the claim of Theorem 2.2 takes the following form (for any two elements x_1 and x_2 of $\mathbb{L}$, and any two elements X and Y of L such that $X + Y$ lies in the center of $\mathbb{L}$):

$$\begin{aligned} &X^0Y^2 + (X+x_1)^1(Y-x_1)^1 + (X+x_2)^1(Y-x_2)^1 \\ &\qquad + (X+x_1+x_2)^2(Y-(x_1+x_2))^0 \\ &= (X+Y)^2 + (X+Y)^1 x_1 + (X+Y)^1 x_2 + (X+Y)^0 x_1x_2 + (X+Y)^0 x_2x_1. \end{aligned}$$

If we try to verify this identity by subtracting the right hand side from the left hand side and expanding, we can quickly realize that it boils down to

$$[x_1 + x_2 + X, X + Y] = 0,$$

where $[a, b]$ denotes the commutator of two elements a and b of $\mathbb{L}$ (that is, $[a, b] = ab - ba$). Since $X + Y$ is assumed to lie in the center of $\mathbb{L}$, this equality is correct. This example shows that the requirement that $X + Y$ should lie in the center of $\mathbb{L}$ cannot be lifted from Theorem 2.2.

This example might suggest that we can replace this requirement by the weaker condition that $\left[\sum_{s\in V} x_s + X, X + Y\right] = 0$; but this would not suffice for $n = 3$.

Theorem 2.4. Let V be a finite set. Let $n = |V|$. For each $s \in V$, let x_s be an element of $\mathbb{L}$. Let X and Y be two elements of $\mathbb{L}$ such that $X + Y$ lies in the center of $\mathbb{L}$. Then,

$$\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|} = (X+Y)^n.$$

(Here, the product $X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}$ has to be interpreted as 1 when $S = \varnothing$.)

Example 2.5. In the case when $V = \{1, 2\}$, the claim of Theorem 2.4 takes the following form (for any two elements x_1 and x_2 of $\mathbb{L}$, and any two elements X and Y of L such that $X + Y$ lies in the center of $\mathbb{L}$):

$$\begin{aligned} & XX^{-1}Y^2 + X(X + x_1)^0 (Y - x_1)^1 + X(X + x_2)^0 (Y - x_2)^1 \\ & \qquad + X(X + x_1 + x_2)^1 (Y - (x_1 + x_2))^0 \\ & = (X + Y)^2 . \end{aligned}$$

(As explained in Theorem 2.4, we should interpret the product XX^{-1} as 1, so we don't need X to be invertible.) This identity boils down to $XY = YX$, which is a consequence of $X + Y$ lying in the center of $\mathbb{L}$. Computations with $n \geq 3$ show that merely assuming $XY = YX$ (without requiring that $X + Y$ lie in the center of $\mathbb{L}$) is not sufficient.

Theorem 2.6. Let V be a finite set. Let $n = |V|$. For each $s \in V$, let x_s be an element of $\mathbb{L}$. Let X and Y be two elements of $\mathbb{L}$ such that $X + Y$ lies in the center of $\mathbb{L}$. Then,

$$\begin{aligned} & \sum_{S \subseteq V} X \left(X + \sum_{s \in S} x_s \right)^{|S|-1} \left(Y - \sum_{s \in S} x_s \right)^{n-|S|-1} \left(Y - \sum_{s \in V} x_s \right) \\ & = \left(X + Y - \sum_{s \in V} x_s \right) (X + Y)^{n-1} . \end{aligned}$$

(Here,

- the product $X \left(X + \sum_{s \in S} x_s \right)^{|S|-1}$ has to be interpreted as 1 when $S = \varnothing$;
- the product $\left(Y - \sum_{s \in S} x_s \right)^{n-|S|-1} \left(Y - \sum_{s \in V} x_s \right)$ has to be interpreted as 1 when $|S| = n$;
- the product $\left(X + Y - \sum_{s \in V} x_s \right) (X + Y)^{n-1}$ has to be interpreted as 1 when $n = 0$.)

Theorem 2.7. Let V be a finite set. Let $n = |V|$. For each $s \in V$, let x_s be an element of $\mathbb{L}$. Let X and Y be two elements of $\mathbb{L}$ such that $X + Y + \sum_{s \in V} x_s$ lies in the center of $\mathbb{L}$. Then,

$$\sum_{S\subseteq V} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1} \left(Y+\sum_{s\in V\setminus S} x_s\right)^{n-|S|-1} Y$$
$$= (X+Y)\left(X+Y+\sum_{s\in V} x_s\right)^{n-1}.$$

(Here,

- the product $X\left(X+\sum\limits_{s\in S} x_s\right)^{|S|-1}$ has to be interpreted as 1 when $S=\varnothing$;
- the product $\left(Y+\sum\limits_{s\in V\setminus S} x_s\right)^{n-|S|-1} Y$ has to be interpreted as 1 when $|S| = n$;
- the product $(X+Y)\left(X+Y+\sum\limits_{s\in V} x_s\right)^{n-1}$ has to be interpreted as 1 when $n=0$.)

Before we prove these theorems, let us cite some appearances of their particular cases in the literature:

- Theorem 2.2 generalizes [Grinbe09, Problem 4] (which is obtained by setting $\mathbb{L} = \mathbb{Z}[X,Y]$ and $x_s = 1$) and [Riorda68, §1.5, Cauchy's identity] (which is obtained by setting $\mathbb{L} = \mathbb{Z}[X,Y]$ and $X = x$ and $Y = y+n$ and $x_s = 1$).
- Theorem 2.4 generalizes [Comtet74, Chapter III, Theorem B] (which is obtained by setting $\mathbb{L} = \mathbb{Z}[X,Y]$ and $x_s = z$) and [Grinbe09, Theorem 4] (which is obtained by setting $\mathbb{L} = \mathbb{Z}[X,Y]$ and $x_s = 1$) and [Kalai79, (11)] (which is obtained by setting $\mathbb{L} = \mathbb{Z}[x,y]$ and $X = x$ and $Y = n+y$) and [KelPos08, 1.3] (which is obtained by setting $\mathbb{L} = \mathbb{Z}[z,y,x(a) \mid a\in V]$ and $X = y$ and $Y = z + x(V)$ and $x_s = x(s)$) and "Hurwitz's formula" in [Knuth97, solution to Section 1.2.6, Exercise 51] (which is obtained by setting $V = \{1,2,\ldots,n\}$ and $X = x$ and $Y = y$ and $x_s = z_s$) and [Riorda68, §1.5, (13)] (which is obtained by setting $\mathbb{L} = \mathbb{Z}[X,Y,a]$ and $X = x$ and $Y = y+na$ and $x_s = a$) and [Stanle99, Exercise 5.31 **b**] (which is obtained by setting $\mathbb{L} = \mathbb{Z}[x_1,x_2,\ldots,x_{n+2}]$ and $X = x_{n+1}$ and $Y = \sum\limits_{i=1}^{n} x_i + x_{n+2}$).
- Theorem 2.7 generalizes [Comtet74, Chapter III, Exercise 20] (which is obtained when $\mathbb{L}$ is commutative) and [KelPos08, 1.2] (which is obtained by

setting $\mathbb{L} = \mathbb{Z}[z, y, x(a) \mid a \in V]$ and $X = y$ and $Y = z$ and $x_s = x(s)$) and [Knuth97, Section 2.3.4.4, Exercise 30] (which is obtained by setting $V = \{1, 2, \ldots, n\}$ and $X = x$ and $Y = y$ and $x_s = z_s$).

3. The proofs

We now come to the proofs of the identities stated above.

Convention 3.1. We shall use the notation $\mathbb{N}$ for the set $\{0, 1, 2, \ldots\}$.

3.1. Proofs of Theorems 2.2 and 2.4

We shall prove Theorem 2.2 and Theorem 2.4 together, by a simultaneous induction. In order to make the computations more palatable, let us first introduce a convenient notation:

Definition 3.2. Assume that V is a finite set. Assume that x_s is an element of $\mathbb{L}$ for each $s \in V$. Assume that S is a subset of V. Then, $x(S)$ shall denote the element $\sum\limits_{s \in S} x_s$ of $\mathbb{L}$.

We make a trivial observation:

Lemma 3.3. Let V be a finite set. For each $s \in V$, let x_s be an element of $\mathbb{L}$.
(a) We have $x(\varnothing) = 0$.
(b) Let $t \in V$. Let S be a subset of $V \setminus \{t\}$. Then, $x(S \cup \{t\}) = x_t + x(S)$.

Proof of Lemma 3.3. **(a)** The definition of $x(\varnothing)$ yields $x(\varnothing) = \sum\limits_{s \in \varnothing} x_s = (\text{empty sum}) = 0$. This proves Lemma 3.3 **(a)**.

(b) We have $t \notin S$ [4]. Hence, $(S \cup \{t\}) \setminus \{t\} = S$. Also, $t \in \{t\} \subseteq S \cup \{t\}$. The definition of $x(S)$ yields $x(S) = \sum\limits_{s \in S} x_s$.

[4] *Proof.* Assume the contrary. Thus, $t \in S$. Hence, $t \in S \subseteq V \setminus \{t\}$. In other words, $t \in V$ and $t \notin \{t\}$. But $t \notin \{t\}$ contradicts $t \in \{t\}$. This contradiction shows that our assumption was false. Qed.

Now, the definition of $x(S \cup \{t\})$ yields

$$x(S \cup \{t\}) = \sum_{s \in S \cup \{t\}} x_s = x_t + \underbrace{\sum_{\substack{s \in S \cup \{t\}; \\ s \neq t}}}_{\substack{= \sum\limits_{s \in (S \cup \{t\}) \setminus \{t\}} = \sum\limits_{s \in S} \\ \text{(since } (S \cup \{t\}) \setminus \{t\} = S\text{)}}} x_s$$

$$\left(\begin{array}{c}\text{here, we have split off the addend for } s = t \\ \text{from the sum (since } t \in S \cup \{t\}\text{)}\end{array}\right)$$

$$= x_t + \underbrace{\sum_{s \in S} x_s}_{= x(S)} = x_t + x(S).$$

This proves Lemma 3.3 **(b)**. □

Now, let us restate Theorem 2.2 and Theorem 2.4 together in a form that will be convenient for us to prove:

Lemma 3.4. Let V be a finite set. Let $n = |V|$. For each $s \in V$, let x_s be an element of $\mathbb{L}$. Let X and Y be two elements of $\mathbb{L}$ such that $X + Y$ lies in the center of $\mathbb{L}$. Then,

$$\sum_{S \subseteq V} (X + x(S))^{|S|} (Y - x(S))^{n - |S|} = \sum_{\substack{i_1, i_2, \ldots, i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X + Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k} \tag{7}$$

and

$$Y^n + \sum_{\substack{S \subseteq V; \\ S \neq \varnothing}} X (X + x(S))^{|S|-1} (Y - x(S))^{n-|S|} = (X + Y)^n. \tag{8}$$

(Here, the sum on the right hand side of (7) has to be interpreted as in Theorem 2.2.)

Note that the equalities (7) and (8) are restatements of the claims of Theorem 2.2 and of Theorem 2.4, respectively. Unlike the claim of Theorem 2.4, however, the equality (8) does not require any convention about how to interpret the term $X \left(X + \sum\limits_{s \in S} x_s \right)^{|S|-1}$ when $S = \varnothing$, because the sum on the left hand side (8) has no addend for $S = \varnothing$.

Proof of Lemma 3.4. Let us first notice that all terms appearing in Lemma 3.4 are well-defined[5].

[5] *Proof.* This follows from the following four observations:

We shall prove Lemma 3.4 by induction over n.

Lemma 3.4 holds when $n = 0$ [6]. This completes the induction base.

Induction step: Let N be a positive integer. Assume (as the induction hypothesis) that Lemma 3.4 holds when $n = N - 1$. We must then prove that Lemma 3.4 holds

---

- For every subset S of V, the term $|S|$ is well-defined. (*Proof:* Let S be a subset of V. Then, S is a subset of the finite set V, and therefore itself is finite. Hence, the term $|S|$ is well-defined. Qed.)
- For every subset S of V, the term $(Y - x(S))^{n-|S|}$ is well-defined. (*Proof:* Let S be a subset of V. Then, $|S| \leq |V| = n$, so that $n - |S| \geq 0$. In other words, $n - |S| \in \mathbb{N}$. Hence, the term $(Y - x(S))^{n-|S|}$ is well-defined. Qed.)
- Whenever $i_1, i_2, \ldots, i_k$ are distinct elements of V, the term $(X+Y)^{n-k}$ is well-defined. (*Proof:* Let $i_1, i_2, \ldots, i_k$ be distinct elements of V. Then, $|\{i_1, i_2, \ldots, i_k\}| = k$ (since $i_1, i_2, \ldots, i_k$ are k distinct elements). But $\{i_1, i_2, \ldots, i_k\} \subseteq V$ (since $i_1, i_2, \ldots, i_k$ are elements of V) and therefore $|\{i_1, i_2, \ldots, i_k\}| \leq |V| = n$. Hence, $n \geq |\{i_1, i_2, \ldots, i_k\}| = k$, so that $n - k \geq 0$. In other words, $n - k \in \mathbb{N}$. Thus, the term $(X+Y)^{n-k}$ is well-defined. Qed.)
- For every subset S of V satisfying $S \neq \varnothing$, the term $(X + x(S))^{|S|-1}$ is well-defined. (*Proof:* Let S be a subset of V satisfying $S \neq \varnothing$. Then, $|S| > 0$ (since $S \neq \varnothing$) and thus $|S| \geq 1$ (since $|S|$ is an integer). In other words, $|S| - 1 \in \mathbb{N}$. Hence, the term $(X + x(S))^{|S|-1}$ is well-defined. Qed.)

[6] *Proof.* Assume that $n = 0$. We must prove that Lemma 3.4 holds.

We have $|V| = n = 0$. Hence, $V = \varnothing$. Thus, the only subset of V is the empty set $\varnothing$. Hence, the sum $\sum_{S \subseteq V} (X + x(S))^{|S|} (Y - x(S))^{n-|S|}$ has only one addend, namely the addend for $S = \varnothing$. Therefore, this sum simplifies as follows:

$$\begin{aligned}
&\sum_{S \subseteq V} (X + x(S))^{|S|} (Y - x(S))^{n-|S|} \\
&= (X + x(\varnothing))^{|\varnothing|} (Y - x(\varnothing))^{n-|\varnothing|} = \underbrace{(X + x(\varnothing))^{0}}_{=1} \underbrace{(Y - x(\varnothing))^{0-0}}_{=(Y-x(\varnothing))^0=1} \\
&\qquad (\text{since } |\varnothing| = 0 \text{ and } n = 0) \\
&= 1.
\end{aligned} \tag{9}$$

On the other hand, the set V has no elements (since $V = \varnothing$). Thus, the only k-tuple $(i_1, i_2, \ldots, i_k)$ of distinct elements of V is the empty 0-tuple $()$. Therefore, the sum $\sum_{\substack{i_1, i_2, \ldots, i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k}$ has only one addend, namely the addend for $k = 0$ and $(i_1, i_2, \ldots, i_k) = ()$. Thus, this sum simplifies as follows:

$$\begin{aligned}
&\sum_{\substack{i_1, i_2, \ldots, i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&= (X+Y)^{n-0} \underbrace{(\text{empty product})}_{=1} = (X+Y)^{n-0} = (X+Y)^{0-0} \qquad (\text{since } n = 0) \\
&= (X+Y)^{0} = 1.
\end{aligned}$$

when $n = N$.

Let V, n, x_s, X and Y be as in Lemma 3.4. Assume that $n = N$. We are going to prove the equalities (7) and (8).

Let $t \in V$ be arbitrary. The set $V \setminus \{t\}$ is a subset of the set V, and thus is finite (since the set V is finite). From $t \in V$, we obtain $|V \setminus \{t\}| = \underbrace{|V|}_{=n=N} -1 = N - 1$. In other words, $N - 1 = |V \setminus \{t\}|$. Also, recall that x_s is an element of $\mathbb{L}$ for each $s \in V$. Hence, x_s is an element of $\mathbb{L}$ for each $s \in V \setminus \{t\}$ (since each $s \in V \setminus \{t\}$ is an element of V). Finally, of course, we have $N - 1 = N - 1$.

Thus, we can apply Lemma 3.4 to $V \setminus \{t\}$ and $N - 1$ instead of V and n (since we have assumed that Lemma 3.4 holds when $n = N - 1$). As the result, we conclude

---

Comparing this with (9), we obtain

$$\sum_{S \subseteq V} (X + x(S))^{|S|} (Y - x(S))^{n-|S|} = \sum_{\substack{i_1, i_2, \ldots, i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X + Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

In other words, (7) holds.

Recall that the only subset of V is the empty set $\varnothing$. Hence, there exist no subset of V distinct from $\varnothing$. In other words, there exists no subset S of V satisfying $S \neq \varnothing$. Thus, the sum $\sum_{\substack{S \subseteq V; \\ S \neq \varnothing}} X (X + x(S))^{|S|-1} (Y - x(S))^{n-|S|}$ is empty. Hence,

$$\sum_{\substack{S \subseteq V; \\ S \neq \varnothing}} X (X + x(S))^{|S|-1} (Y - x(S))^{n-|S|} = (\text{empty sum}) = 0.$$

Thus,

$$Y^n + \underbrace{\sum_{\substack{S \subseteq V; \\ S \neq \varnothing}} X (X + x(S))^{|S|-1} (Y - x(S))^{n-|S|}}_{=0} = Y^n = Y^0 \qquad (\text{since } n = 0)$$
$$= 1.$$

Comparing this with

$$(X + Y)^n = (X + Y)^0 \qquad (\text{since } n = 0)$$
$$= 1,$$

we obtain

$$Y^n + \sum_{\substack{S \subseteq V; \\ S \neq \varnothing}} X (X + x(S))^{|S|-1} (Y - x(S))^{n-|S|} = (X + Y)^n.$$

In other words, (8) holds.

Thus, we have shown that both (7) and (8) hold. In other words, Lemma 3.4 holds. Qed.

that

$$\begin{aligned} & \sum_{S\subseteq V\setminus\{t\}} (X + x(S))^{|S|} (Y - x(S))^{N-1-|S|} \\ & = \sum_{\substack{i_1, i_2, \ldots, i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}} (X+Y)^{N-1-k} x_{i_1} x_{i_2} \cdots x_{i_k} \end{aligned} \tag{10}$$

and

$$Y^{N-1} + \sum_{\substack{S\subseteq V\setminus\{t\}; \\ S\neq\varnothing}} X (X + x(S))^{|S|-1} (Y - x(S))^{N-1-|S|} = (X+Y)^{N-1}. \tag{11}$$

Subtracting Y^{N-1} from both sides of (11), we obtain

$$\sum_{\substack{S\subseteq V\setminus\{t\}; \\ S\neq\varnothing}} X (X + x(S))^{|S|-1} (Y - x(S))^{N-1-|S|} = (X+Y)^{N-1} - Y^{N-1}. \tag{12}$$

But x_t is an element of $\mathbb{L}$ (since x_s is an element of $\mathbb{L}$ for each $s \in V$). Thus, $X + x_t$ and $Y - x_t$ are two elements of $\mathbb{L}$. Their sum $(X + x_t) + (Y - x_t)$ lies in the center of $\mathbb{L}$ (because this sum equals $(X + x_t) + (Y - x_t) = X + Y$, but we know that $X + Y$ lies in the center of $\mathbb{L}$). Hence, we can apply Lemma 3.4 to $V \setminus \{t\}$, $N - 1$, $X + x_t$ and $Y - x_t$ instead of V, n, X and Y (since we have assumed that Lemma 3.4 holds when $n = N - 1$). As a result, we conclude that

$$\begin{aligned} & \sum_{S\subseteq V\setminus\{t\}} (X + x_t + x(S))^{|S|} (Y - x_t - x(S))^{N-1-|S|} \\ & = \sum_{\substack{i_1, i_2, \ldots, i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}} (X + x_t + Y - x_t)^{N-1-k} x_{i_1} x_{i_2} \cdots x_{i_k} \end{aligned} \tag{13}$$

and

$$\begin{aligned} & Y^{N-1} + \sum_{\substack{S\subseteq V\setminus\{t\}; \\ S\neq\varnothing}} (X + x_t) (X + x_t + x(S))^{|S|-1} (Y - x_t - x(S))^{N-1-|S|} \\ & = (X + x_t + Y - x_t)^{N-1}. \end{aligned}$$

The equality (13) becomes

$$\begin{aligned}
&\sum_{S\subseteq V\setminus\{t\}} \left(X+x_t+x\left(S\right)\right)^{|S|}\left(Y-x_t-x\left(S\right)\right)^{N-1-|S|} \\
&= \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}} \Bigg(\underbrace{X+x_t+Y-x_t}_{=X+Y}\Bigg)^{N-1-k} x_{i_1}x_{i_2}\cdots x_{i_k} \\
&= \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V\setminus\{t\}}} \left(X+Y\right)^{N-1-k} x_{i_1}x_{i_2}\cdots x_{i_k} \qquad (14)\\
&= \sum_{S\subseteq V\setminus\{t\}} \left(X+x\left(S\right)\right)^{|S|}\left(Y-x\left(S\right)\right)^{N-1-|S|} \qquad \text{(by (10))} \qquad (15)\\
&= \underbrace{\left(X+x\left(\varnothing\right)\right)^{|\varnothing|}}_{\substack{=(X+x(\varnothing))^0 \\ (\text{since } |\varnothing|=0)}} \Bigg(Y-\underbrace{x\left(\varnothing\right)}_{\substack{=0 \\ (\text{by Lemma 3.3 } \textbf{(a)})}}\Bigg)^{N-1-|\varnothing|} \\
&\qquad + \sum_{\substack{S\subseteq V\setminus\{t\}; \\ S\neq\varnothing}} \left(X+x\left(S\right)\right)^{|S|} \underbrace{\left(Y-x\left(S\right)\right)^{N-1-|S|}}_{\substack{=(Y-x(S))^{N-|S|-1} \\ (\text{since } N-1-|S|=N-|S|-1)}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addend for } S=\varnothing \text{ from the sum} \\ (\text{since } \varnothing \text{ is a subset of } V\setminus\{t\}\,)\end{array}\right) \\
&= \underbrace{\left(X+x\left(\varnothing\right)\right)^{0}}_{=1} \underbrace{\left(Y-0\right)^{N-1-|\varnothing|}}_{\substack{=Y^{N-1-0} \\ (\text{since } Y-0=Y \text{ and } |\varnothing|=0)}} + \sum_{\substack{S\subseteq V\setminus\{t\}; \\ S\neq\varnothing}} \left(X+x\left(S\right)\right)^{|S|}\left(Y-x\left(S\right)\right)^{N-|S|-1} \\
&= \underbrace{Y^{N-1-0}}_{=Y^{N-1}} + \sum_{\substack{S\subseteq V\setminus\{t\}; \\ S\neq\varnothing}} \left(X+x\left(S\right)\right)^{|S|}\left(Y-x\left(S\right)\right)^{N-|S|-1} \\
&= Y^{N-1} + \sum_{\substack{S\subseteq V\setminus\{t\}; \\ S\neq\varnothing}} \left(X+x\left(S\right)\right)^{|S|}\left(Y-x\left(S\right)\right)^{N-|S|-1}. \qquad (16)
\end{aligned}$$

Let us record the following (well-known and simple) fact: The map

$$\begin{aligned} \{S\subseteq V \mid t\notin S\} &\to \{S\subseteq V \mid t\in S\}, \\ S &\mapsto S\cup\{t\} \end{aligned} \qquad (17)$$

is a bijection[7].

[7] Its inverse is the map

$$\begin{aligned} \{S\subseteq V \mid t\in S\} &\to \{S\subseteq V \mid t\notin S\}, \\ T &\mapsto T\setminus\{t\}. \end{aligned}$$

Any subset S of V satisfying $t \notin S$ must satisfy

$$N - |S| \geq 1 \tag{18}$$

[8].

Any subset S of V satisfying $t \in S$ must automatically satisfy $S \neq \varnothing$ (because S has at least one element (namely, t)). Thus, for a subset S of V, the condition ($t \in S$ and $S \neq \varnothing$) is equivalent to the condition ($t \in S$). Hence, we have the following equality of summation signs:

$$\sum_{\substack{S \subseteq V; \\ t \in S; \\ S \neq \varnothing}} = \sum_{\substack{S \subseteq V; \\ t \in S}} . \tag{19}$$

Now, every subset S of V satisfying $S \neq \varnothing$ must satisfy $|S| - 1 \geq 0$ [9]. Hence,

---

[8] *Proof of (18):* Let S be a subset of V satisfying $t \notin S$. We have $S \subseteq V \setminus \{t\}$ (since $S \subseteq V$ and $t \notin S$), and thus $|S| \leq |V \setminus \{t\}| = N - 1$. In other words, $(N-1) - |S| \geq 0$. Thus, $N - |S| - 1 = (N-1) - |S| \geq 0$, so that $N - |S| \geq 1$. This proves (18).

[9] *Proof.* Let S be a subset of V satisfying $S \neq \varnothing$. Then, $|S| > 0$ (since $S \neq \varnothing$). Hence, $|S| \geq 1$ (since $|S|$ is an integer), so that $|S| - 1 \geq 0$, qed.

$(X + x(S))^{|S|-1}$ is a well-defined element of $\mathbb{L}$ for every such S. We have

$$\begin{aligned}
&\underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\\ t\in S}}}_{\substack{=\sum_{\substack{S\subseteq V;\\ t\in S;\\ S\neq\varnothing}} = \sum_{\substack{S\subseteq V;\\ t\in S}} \\ \text{(by (19))}}} (X + x(S))^{|S|-1} (Y - x(S))^{N-|S|} \\
&= \sum_{\substack{S\subseteq V;\\ t\in S}} (X + x(S))^{|S|-1} (Y - x(S))^{N-|S|} \\
&= \underbrace{\sum_{\substack{S\subseteq V;\\ t\notin S}}}_{=\sum_{S\subseteq V\setminus\{t\}}} \underbrace{(X + x(S\cup\{t\}))^{|S\cup\{t\}|-1} (Y - x(S\cup\{t\}))^{N-(|S\cup\{t\}|)}}_{\substack{=(X+x(S\cup\{t\}))^{(|S|+1)-1}(Y-x(S\cup\{t\}))^{N-(|S|+1)} \\ \text{(since } |S\cup\{t\}|=|S|+1 \text{ (since } t\notin S))}} \\
&\qquad \left(\begin{array}{c}\text{here, we have substituted } S\cup\{t\} \text{ for } S \text{ in the sum,} \\ \text{since the map (17) is a bijection}\end{array}\right) \\
&= \sum_{S\subseteq V\setminus\{t\}} \left(X + \underbrace{x(S\cup\{t\})}_{\substack{=x_t+x(S) \\ \text{(by Lemma 3.3 } \mathbf{(b)})}}\right)^{(|S|+1)-1} \left(Y - \underbrace{x(S\cup\{t\})}_{\substack{=x_t+x(S) \\ \text{(by Lemma 3.3 } \mathbf{(b)})}}\right)^{N-(|S|+1)} \\
&= \sum_{S\subseteq V\setminus\{t\}} \underbrace{(X + x_t + x(S))^{(|S|+1)-1}}_{\substack{=(X+x_t+x(S))^{|S|} \\ \text{(since } (|S|+1)-1=|S|)}} \underbrace{(Y - (x_t + x(S)))^{N-(|S|+1)}}_{\substack{=(Y-x_t-x(S))^{N-1-|S|} \\ \text{(since } Y-(x_t+x(S))=Y-x_t-x(S) \\ \text{and } N-(|S|+1)=N-1-|S|)}} \\
&= \sum_{S\subseteq V\setminus\{t\}} \underbrace{(X + x_t + x(S))^{(|S|+1)-1}}_{\substack{=(X+x_t+x(S))^{|S|} \\ \text{(since } (|S|+1)-1=|S|)}} \underbrace{(Y - (x_t + x(S)))^{N-(|S|+1)}}_{\substack{=(Y-x_t-x(S))^{N-1-|S|} \\ \text{(since } Y-(x_t+x(S))=Y-x_t-x(S) \\ \text{and } N-(|S|+1)=N-1-|S|)}} \\
&= \sum_{S\subseteq V\setminus\{t\}} (X + x_t + x(S))^{|S|} (Y - x_t - x(S))^{N-1-|S|} \qquad (20) \\
&= Y^{N-1} + \sum_{\substack{S\subseteq V\setminus\{t\};\\ S\neq\varnothing}} (X + x(S))^{|S|} (Y - x(S))^{N-|S|-1} \qquad \text{(by (16))}.
\end{aligned}$$

Multiplying both sides of this equality by X, we obtain

$$\begin{aligned}
& X \sum_{\substack{S \subseteq V; \\ S \neq \varnothing; \\ t \in S}} \left(X + x\left(S\right)\right)^{\left|S\right|-1} \left(Y - x\left(S\right)\right)^{N-\left|S\right|} \\
&= X \left(Y^{N-1} + \sum_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} \left(X + x\left(S\right)\right)^{\left|S\right|} \left(Y - x\left(S\right)\right)^{N-\left|S\right|-1} \right) \\
&= XY^{N-1} + \underbrace{X \sum_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} \left(X + x\left(S\right)\right)^{\left|S\right|} \left(Y - x\left(S\right)\right)^{N-\left|S\right|-1}}_{= \sum_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|}\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1}} \\
&= XY^{N-1} + \sum_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} X \underbrace{\left(X + x\left(S\right)\right)^{\left|S\right|}}_{=\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(X+x\left(S\right)\right)} \left(Y - x\left(S\right)\right)^{N-\left|S\right|-1} \\
&= XY^{N-1} + \sum_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} X \left(X + x\left(S\right)\right)^{\left|S\right|-1} \left(X + x\left(S\right)\right) \left(Y - x\left(S\right)\right)^{N-\left|S\right|-1}.
\end{aligned} \tag{21}$$

Each subset S of V satisfies either $t \in S$ or $t \notin S$ (but not both). Hence,

$$\begin{aligned}
&\sum_{\substack{S \subseteq V;\\ S \neq \varnothing}} X\left(X + x\left(S\right)\right)^{\left|S\right|-1}\left(Y - x\left(S\right)\right)^{N-\left|S\right|} \\
&= \underbrace{\sum_{\substack{S \subseteq V;\\ S \neq \varnothing;\\ t \in S}} X\left(X + x\left(S\right)\right)^{\left|S\right|-1}\left(Y - x\left(S\right)\right)^{N-\left|S\right|}}_{\substack{= X \sum\limits_{\substack{S \subseteq V;\\ S \neq \varnothing;\\ t \in S}} \left(X + x(S)\right)^{|S|-1}\left(Y - x(S)\right)^{N-|S|} \\ = XY^{N-1} + \sum\limits_{\substack{S \subseteq V \setminus \{t\};\\ S \neq \varnothing}} X\left(X + x(S)\right)^{|S|-1}\left(X + x(S)\right)\left(Y - x(S)\right)^{N-|S|-1} \\ \text{(by (21))}}} \\
&\qquad + \underbrace{\sum_{\substack{S \subseteq V;\\ S \neq \varnothing\\ t \notin S}}}_{\substack{= \sum\limits_{\substack{S \subseteq V;\\ t \notin S;\\ S \neq \varnothing}} = \sum\limits_{\substack{S \subseteq V \setminus \{t\};\\ S \neq \varnothing}} \\ \text{(since the subsets } S \text{ of } V \\ \text{satisfying } t \notin S \text{ are precisely} \\ \text{the subsets of } V \setminus \{t\})}} X\left(X + x\left(S\right)\right)^{\left|S\right|-1} \underbrace{\left(Y - x\left(S\right)\right)^{N-\left|S\right|}}_{\substack{= \left(Y - x(S)\right)\left(Y - x(S)\right)^{N-|S|-1} \\ \text{(since } N - |S| \geq 1 \\ \text{(by (18)))}}} \\
&= XY^{N-1} + \sum_{\substack{S \subseteq V \setminus \{t\};\\ S \neq \varnothing}} X\left(X + x\left(S\right)\right)^{\left|S\right|-1}\left(X + x\left(S\right)\right)\left(Y - x\left(S\right)\right)^{N-\left|S\right|-1} \\
&\qquad + \sum_{\substack{S \subseteq V \setminus \{t\};\\ S \neq \varnothing}} X\left(X + x\left(S\right)\right)^{\left|S\right|-1}\left(Y - x\left(S\right)\right)\left(Y - x\left(S\right)\right)^{N-\left|S\right|-1}.
\end{aligned}$$

Subtracting XY^{N-1} from both sides of this equality, we obtain

$$
\begin{aligned}
&\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{N-\left|S\right|} - XY^{N-1} \\
&= \sum_{\substack{S\subseteq V\setminus\{t\};\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(X+x\left(S\right)\right)\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1} \\
&\qquad + \sum_{\substack{S\subseteq V\setminus\{t\};\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1} \\
&= \sum_{\substack{S\subseteq V\setminus\{t\};\\ S\neq\varnothing}} \Big(X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(X+x\left(S\right)\right)\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1} \\
&\qquad\qquad + X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1}\Big) \\
&= \sum_{\substack{S\subseteq V\setminus\{t\};\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\underbrace{\left(\left(X+x\left(S\right)\right)+\left(Y-x\left(S\right)\right)\right)}_{=X+Y}\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1} \\
&\qquad \left(\begin{array}{c}\text{since each subset } S \text{ of } V\setminus\{t\} \text{ satisfying } S\neq\varnothing \text{ satisfies} \\ X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(X+x\left(S\right)\right)\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1} \\ +X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1} \\ = X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(\left(X+x\left(S\right)\right)+\left(Y-x\left(S\right)\right)\right)\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1}\end{array}\right) \\
&= \sum_{\substack{S\subseteq V\setminus\{t\};\\ S\neq\varnothing}} \underbrace{X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(X+Y\right)}_{\substack{=(X+Y)X(X+x(S))^{|S|-1} \\ \text{(since } X+Y \text{ commutes with } X(X+x(S))^{|S|-1} \\ \text{(since } X+Y \text{ lies in the center of } \mathbb{L}))}} \underbrace{\left(Y-x\left(S\right)\right)^{N-\left|S\right|-1}}_{\substack{=(Y-x(S))^{N-1-|S|} \\ \text{(since } N-|S|-1=N-1-|S|)}} \\
&= \sum_{\substack{S\subseteq V\setminus\{t\};\\ S\neq\varnothing}} \left(X+Y\right)X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{N-1-\left|S\right|} \\
&= \left(X+Y\right)\underbrace{\sum_{\substack{S\subseteq V\setminus\{t\};\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{N-1-\left|S\right|}}_{\substack{=(X+Y)^{N-1}-Y^{N-1} \\ \text{(by (12))}}} \\
&= \left(X+Y\right)\left(\left(X+Y\right)^{N-1}-Y^{N-1}\right) \\
&= \underbrace{\left(X+Y\right)\left(X+Y\right)^{N-1}}_{=(X+Y)^{N}} - \underbrace{\left(X+Y\right)Y^{N-1}}_{=XY^{N-1}+YY^{N-1}} \\
&= \left(X+Y\right)^{N} - \left(XY^{N-1}+YY^{N-1}\right) = \left(X+Y\right)^{N} - XY^{N-1} - YY^{N-1}.
\end{aligned}
$$

Adding XY^{N-1} to both sides of this equality, we obtain

$$\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{N-\left|S\right|}$$
$$=\left(X+Y\right)^{N}-XY^{N-1}-YY^{N-1}+XY^{N-1}=\left(X+Y\right)^{N}-\underbrace{YY^{N-1}}_{=Y^{N}}$$
$$=\left(X+Y\right)^{N}-Y^{N}. \qquad (22)$$

Now, forget that we fixed t. We thus have proven the equalities (20), (22) and (14) for each $t\in V$.

The set V is nonempty (since $\left|V\right|=N>0$ (since N is a positive integer)). Hence, there exists some $q\in V$. Consider this q. Applying (22) to $t=q$, we obtain

$$\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{N-\left|S\right|}=\left(X+Y\right)^{N}-Y^{N}.$$

Adding Y^{N} to both sides of this equality, we obtain

$$Y^{N}+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{N-\left|S\right|}=\left(X+Y\right)^{N}. \qquad (23)$$

Since $n=N$, we can rewrite this equality as follows:

$$Y^{n}+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|}=\left(X+Y\right)^{n}.$$

In other words, the equality (8) holds.

On the other hand, every $t\in V$ satisfies

$$\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\\ t\in S}} \left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{N-\left|S\right|}$$
$$=\sum_{S\subseteq V\setminus\left\{t\right\}}\left(X+x_{t}+x\left(S\right)\right)^{\left|S\right|}\left(Y-x_{t}-x\left(S\right)\right)^{N-1-\left|S\right|} \qquad \text{(by (20))}$$
$$=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\left\{t\right\}}}\left(X+Y\right)^{N-1-k}x_{i_1}x_{i_2}\cdots x_{i_k} \qquad \text{(by (14))}. \qquad (24)$$

Now, every subset S of V satisfies

$$x(S) = \underbrace{\sum_{s \in S}}_{\substack{= \sum\limits_{\substack{s \in V; \\ s \in S}} \\ (\text{since } S \subseteq V)}} x_s \qquad \text{(by the definition of } x(S)\text{)}$$
$$= \sum_{\substack{s \in V; \\ s \in S}} x_s = \sum_{\substack{t \in V; \\ t \in S}} x_t \tag{25}$$

(here, we have substituted t for s in the sum).

But every subset S of V satisfying $S \neq \varnothing$ must satisfy

$$|S| - 1 \geq 0 \tag{26}$$

[10]. Hence, $(X+x(S))^{|S|-1}$ is a well-defined element of $\mathbb{L}$ for every such S. Now,

$$\begin{aligned}
&\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} \underbrace{x(S)}_{\substack{=\sum_{\substack{t\in V;\\ t\in S}} x_t \\ \text{(by (25))}}} (X+x(S))^{|S|-1}(Y-x(S))^{N-|S|} \\
&= \sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} \left(\sum_{\substack{t\in V;\\ t\in S}} x_t\right)(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|} \\
&= \underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} \sum_{\substack{t\in V;\\ t\in S}}}_{=\sum_{t\in V}\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\\ t\in S}}} x_t (X+x(S))^{|S|-1}(Y-x(S))^{N-|S|} \\
&= \sum_{t\in V} \underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\\ t\in S}} x_t (X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}}_{=x_t \sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\\ t\in S}} (X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}} \\
&= \sum_{t\in V} x_t \underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\\ t\in S}} (X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}}_{\substack{=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}} (X+Y)^{N-1-k} x_{i_1}x_{i_2}\cdots x_{i_k} \\ \text{(by (24))}}}
\end{aligned}$$

[10]*Proof of (26):* Let S be a subset of V satisfying $S\neq\varnothing$. Then, $|S|>0$ (since $S\neq\varnothing$). Hence, $|S|\geq 1$ (since $|S|$ is an integer), so that $|S|-1\geq 0$, qed.

$$= \sum_{t\in V} \underbrace{x_t \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}} (X+Y)^{N-1-k} x_{i_1}x_{i_2}\cdots x_{i_k}}_{= \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}} x_t(X+Y)^{N-1-k} x_{i_1}x_{i_2}\cdots x_{i_k}}$$

$$= \sum_{t\in V} \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}} \underbrace{x_t\left(X+Y\right)^{N-1-k}}_{\substack{=(X+Y)^{N-1-k}x_t \\ \text{(since } (X+Y)^{N-1-k} \text{ commutes with } x_t \\ \text{(since } (X+Y)^{N-1-k} \text{ lies in the center of } \mathbb{L} \\ \text{(since } X+Y \text{ lies in the center of } \mathbb{L}, \\ \text{but the center of } \mathbb{L} \text{ is a subring of } \mathbb{L})))}} x_{i_1}x_{i_2}\cdots x_{i_k}$$

$$= \sum_{t\in V} \underbrace{\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}}}_{= \sum_{k\in\mathbb{N}} \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}}} \left(X+Y\right)^{N-1-k} x_t\left(x_{i_1}x_{i_2}\cdots x_{i_k}\right)$$

$$= \sum_{t\in V} \sum_{k\in\mathbb{N}} \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}} \left(X+Y\right)^{N-1-k} x_t\left(x_{i_1}x_{i_2}\cdots x_{i_k}\right). \qquad (27)$$

Let us next observe that if $i_1, i_2, \ldots, i_k$ are distinct elements of V (for some $k \in \mathbb{N}$), then $N - k \in \mathbb{N}$ [11], and therefore the term $(X+Y)^{N-k}$ is a well-defined element of $\mathbb{L}$. Hence, the sum $\sum\limits_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}} (X+Y)^{N-k} x_{i_1}x_{i_2}\cdots x_{i_k}$ is well-defined.

On the other hand, consider the sum $\sum\limits_{\substack{i_1,i_2,\ldots,i_0 \text{ are}\\ \text{distinct}\\ \text{elements of } V}} (X+Y)^N$. This sum has only one addend (since there exists only one 0-tuple $(i_1, i_2, \ldots, i_0)$ of distinct elements of V, namely the empty 0-tuple $()$), namely the addend for $(i_1, i_2, \ldots, i_0) = ()$. Thus, this sum simplifies as follows:

$$\sum_{\substack{i_1,i_2,\ldots,i_0 \text{ are}\\ \text{distinct}\\ \text{elements of } V}} (X+Y)^N = (X+Y)^N.$$

[11] *Proof.* Let $i_1, i_2, \ldots, i_k$ be distinct elements of V. Then, $\{i_1, i_2, \ldots, i_k\}$ is a subset of V. Hence, $|\{i_1, i_2, \ldots, i_k\}| \leq |V| = n = N$. But the elements $i_1, i_2, \ldots, i_k$ are distinct; hence, $|\{i_1, i_2, \ldots, i_k\}| = k$. Thus, $k = |\{i_1, i_2, \ldots, i_k\}| \leq N$. Hence, $N - k \geq 0$. Therefore, $N - k \in \mathbb{N}$. Qed.

Hence,

$$\sum_{\substack{i_1,i_2,\ldots,i_0 \text{ are} \\ \text{distinct} \\ \text{elements of } V}} \underbrace{(X+Y)^{N-0}}_{=(X+Y)^N} \underbrace{x_{i_1}x_{i_2}\cdots x_{i_0}}_{=(\text{empty product})=1} = \sum_{\substack{i_1,i_2,\ldots,i_0 \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^N = (X+Y)^N .$$

Now,

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{N-k} x_{i_1}x_{i_2}\cdots x_{i_k} \\
&= \underbrace{\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V; \\ k=0}} (X+Y)^{N-k} x_{i_1}x_{i_2}\cdots x_{i_k}}_{\substack{=\sum_{\substack{i_1,i_2,\ldots,i_0 \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{N-0} x_{i_1}x_{i_2}\cdots x_{i_0} \\ =(X+Y)^N}} + \underbrace{\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V; \\ k\neq 0}}}_{\substack{=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V; \\ k>0}} \\ \text{(because for any } k\text{-tuple } (i_1,i_2,\ldots,i_k), \\ \text{the condition } (k\neq 0) \text{ is equivalent to} \\ \text{the condition } (k>0) \text{ (since } k\in\mathbb{N}))}} (X+Y)^{N-k} x_{i_1}x_{i_2}\cdots x_{i_k} \\
&\qquad \left(\text{since every } k\text{-tuple } (i_1,i_2,\ldots,i_k) \text{ satisfies either } k=0 \text{ or } k\neq 0 \text{ (but not both)}\right) \\
&= (X+Y)^N + \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V; \\ k>0}} (X+Y)^{N-k} x_{i_1}x_{i_2}\cdots x_{i_k}.
\end{aligned}$$

Subtracting $(X+Y)^N$ from both sides of this equality, we obtain

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{N-k} x_{i_1} x_{i_2} \cdots x_{i_k} - (X+Y)^N \\
&= \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V; \\ k>0}} (X+Y)^{N-k} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&= \sum_{k>0} \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{N-k} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&= \sum_{k\in\mathbb{N}} \sum_{\substack{i_1,i_2,\ldots,i_{k+1} \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{N-(k+1)} \underbrace{x_{i_1} x_{i_2} \cdots x_{i_{k+1}}}_{\substack{=x_{i_1}\left(x_{i_2}x_{i_3}\cdots x_{i_{k+1}}\right) \\ (\text{since } k\in\mathbb{N})}} \\
&\qquad \left(\text{here, we substituted } k+1 \text{ for } k \text{ in the outer sum}\right) \\
&= \sum_{k\in\mathbb{N}} \underbrace{\sum_{\substack{i_1,i_2,\ldots,i_{k+1} \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{N-(k+1)} x_{i_1} \left(x_{i_2}x_{i_3}\cdots x_{i_{k+1}}\right)}_{\substack{= \sum_{\substack{t,i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{N-(k+1)} x_t \left(x_{i_1}x_{i_2}\cdots x_{i_k}\right) \\ (\text{here, we renamed the summation index } (i_1,i_2,\ldots,i_{k+1}) \text{ as } (t,i_1,i_2,\ldots,i_k))}} \\
&= \sum_{k\in\mathbb{N}} \underbrace{\sum_{\substack{t,i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}}}_{\substack{= \sum_{\substack{t,i_1,i_2,\ldots,i_k \text{ are elements of } V; \\ \text{the elements } t,i_1,i_2,\ldots,i_k \text{ are distinct}}}}} (X+Y)^{N-(k+1)} x_t \left(x_{i_1}x_{i_2}\cdots x_{i_k}\right) \\
&= \sum_{k\in\mathbb{N}} \sum_{\substack{t,i_1,i_2,\ldots,i_k \text{ are elements of } V; \\ \text{the elements } t,i_1,i_2,\ldots,i_k \text{ are distinct}}} (X+Y)^{N-(k+1)} x_t \left(x_{i_1}x_{i_2}\cdots x_{i_k}\right).
\end{aligned} \tag{28}$$

But let $k \in \mathbb{N}$ be arbitrary. If $(t, i_1, i_2, \ldots, i_k)$ is any $(k+1)$-tuple of elements of V, then we have the following chain of equivalences:

$$\begin{aligned}
&\left(\text{the elements } t, i_1, i_2, \ldots, i_k \text{ are distinct}\right) \\
\Longleftrightarrow\ &\left(\text{the elements } i_1, i_2, \ldots, i_k \text{ are distinct and differ from } t\right) \\
\Longleftrightarrow\ &\left(\text{the elements } i_1, i_2, \ldots, i_k \text{ are distinct}\right) \wedge \underbrace{\left(\text{the elements } i_1, i_2, \ldots, i_k \text{ differ from } t\right)}_{\Longleftrightarrow \left(\text{the elements } i_1,i_2,\ldots,i_k \text{ belong to } V\setminus\{t\}\right)} \\
\Longleftrightarrow\ &\left(\text{the elements } i_1, i_2, \ldots, i_k \text{ are distinct}\right) \wedge \left(\text{the elements } i_1, i_2, \ldots, i_k \text{ belong to } V \setminus \{t\}\right) \\
\Longleftrightarrow\ &\left(\text{the elements } i_1, i_2, \ldots, i_k \text{ are distinct elements of } V \setminus \{t\}\right).
\end{aligned}$$

Hence, we have the following equality of summation signs:

$$\begin{aligned}
\sum_{\substack{t,i_1,i_2,\ldots,i_k \text{ are elements of } V;\\ \text{the elements } t,i_1,i_2,\ldots,i_k \text{ are distinct}}} &= \sum_{\substack{t,i_1,i_2,\ldots,i_k \text{ are elements of } V;\\ \text{the elements } i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct elements of } V\setminus\{t\}}} \\
&= \sum_{t\in V} \underbrace{\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are elements of } V;\\ \text{the elements } i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct elements of } V\setminus\{t\}}}}_{\substack{= \sum\limits_{\substack{i_1,i_2,\ldots,i_k \text{ are distinct elements of } V\setminus\{t\};\\ \text{the elements } i_1,i_2,\ldots,i_k \text{ are elements of } V}} \\ = \sum\limits_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct elements of } V\setminus\{t\}}} \\ \text{(because if } i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct elements of } V\setminus\{t\}\text{, then}\\ i_1,i_2,\ldots,i_k \text{ must automatically}\\ \text{be elements of } V\\ \text{(since } i_1,i_2,\ldots,i_k \text{ are elements of } V\setminus\{t\}\text{,}\\ \text{but } V\setminus\{t\} \text{ is a subset of } V\text{))}}} \\
&= \sum_{t\in V} \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct elements of } V\setminus\{t\}}} . \qquad (29)
\end{aligned}$$

Now, forget that we fixed k. We thus have proven the equality (29) for each $k \in \mathbb{N}$. Now, (28) becomes

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}} (X+Y)^{N-k} x_{i_1}x_{i_2}\cdots x_{i_k} - (X+Y)^N \\
&= \sum_{k\in\mathbb{N}} \underbrace{\sum_{\substack{t,i_1,i_2,\ldots,i_k \text{ are elements of } V;\\ \text{the elements } t,i_1,i_2,\ldots,i_k \text{ are distinct}}}}_{\substack{= \sum\limits_{t\in V} \sum\limits_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct elements of } V\setminus\{t\}}} \\ \text{(by (29))}}} \underbrace{(X+Y)^{N-(k+1)}}_{\substack{=(X+Y)^{N-1-k}\\ \text{(since } N-(k+1)=N-1-k\text{)}}} x_t \left(x_{i_1}x_{i_2}\cdots x_{i_k}\right) \\
&= \underbrace{\sum_{k\in\mathbb{N}} \sum_{t\in V}}_{= \sum\limits_{t\in V} \sum\limits_{k\in\mathbb{N}}} \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct elements of } V\setminus\{t\}}} (X+Y)^{N-1-k} x_t \left(x_{i_1}x_{i_2}\cdots x_{i_k}\right) \\
&= \sum_{t\in V} \sum_{k\in\mathbb{N}} \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct elements of } V\setminus\{t\}}} (X+Y)^{N-1-k} x_t \left(x_{i_1}x_{i_2}\cdots x_{i_k}\right) \\
&= \sum_{t\in V} \sum_{k\in\mathbb{N}} \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V\setminus\{t\}}} (X+Y)^{N-1-k} x_t \left(x_{i_1}x_{i_2}\cdots x_{i_k}\right).
\end{aligned}$$

Comparing this with (27), we obtain

$$\begin{aligned}
&\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} x\left(S\right)\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{N-|S|} \\
&= \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}} \left(X+Y\right)^{N-k} x_{i_1}x_{i_2}\cdots x_{i_k} - \left(X+Y\right)^{N}. \qquad (30)
\end{aligned}$$

Adding this equality to (23), we obtain

$$\begin{aligned}
&Y^N + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{N-|S|} \\
&\qquad + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} x\left(S\right)\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{N-|S|} \\
&= \left(X+Y\right)^{N} + \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}} \left(X+Y\right)^{N-k} x_{i_1}x_{i_2}\cdots x_{i_k} - \left(X+Y\right)^{N} \\
&= \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}} \left(X+Y\right)^{N-k} x_{i_1}x_{i_2}\cdots x_{i_k}. \qquad (31)
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{S\subseteq V}(X+x(S))^{|S|}(Y-x(S))^{N-|S|}\\
&=\underbrace{(X+x(\varnothing))^{|\varnothing|}}_{\substack{=(X+x(\varnothing))^0\\ (\text{since } |\varnothing|=0)}}\left(Y-\underbrace{x(\varnothing)}_{\substack{=0\\ (\text{by Lemma 3.3 } \textbf{(a)})}}\right)^{N-|\varnothing|}+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}(X+x(S))^{|S|}(Y-x(S))^{N-|S|}\\
&\qquad\left(\begin{array}{c}\text{here, we have split off the addend for } S=\varnothing \text{ from the sum}\\ (\text{since } \varnothing \text{ is a subset of } V)\end{array}\right)\\
&=\underbrace{(X+x(\varnothing))^0}_{=1}\underbrace{(Y-0)^{N-|\varnothing|}}_{\substack{=Y^{N-0}\\ (\text{since } Y-0=Y \text{ and } |\varnothing|=0)}}+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}(X+x(S))^{|S|}(Y-x(S))^{N-|S|}\\
&=\underbrace{Y^{N-0}}_{=Y^N}+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}\underbrace{(X+x(S))^{|S|}}_{\substack{=(X+x(S))(X+x(S))^{|S|-1}\\ (\text{since } |S|-1\geq 0\\ (\text{by } (26)))}}(Y-x(S))^{N-|S|}\\
&=Y^N+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}\underbrace{(X+x(S))(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}}_{=X(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}+x(S)(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}}\\
&=Y^N+\underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}\left(X(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}+x(S)(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}\right)}_{=\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}X(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}x(S)(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}}\\
&=Y^N+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}X(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}\\
&\qquad+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}}x(S)(X+x(S))^{|S|-1}(Y-x(S))^{N-|S|}\\
&=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}}(X+Y)^{N-k}x_{i_1}x_{i_2}\cdots x_{i_k}\qquad(\text{by } (31)).
\end{aligned}$$

Since $n=N$, we can rewrite this equality as follows:

$$\sum_{S\subseteq V}(X+x(S))^{|S|}(Y-x(S))^{n-|S|}=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}}(X+Y)^{n-k}x_{i_1}x_{i_2}\cdots x_{i_k}.$$

In other words, the equality (7) holds.

We have thus shown that the equalities (7) and (8) hold.

Now, forget that we fixed V, n, x_s, X and Y and assumed that $n = N$. We thus have proven that if V, n, x_s, X and Y are as in Lemma 3.4, and if we have $n = N$, then the equalities (7) and (8) hold. In other words, Lemma 3.4 holds when $n = N$. This completes the induction step. Thus, Lemma 3.4 is proven by induction. □

We can now prove Theorem 2.2 and Theorem 2.4 in their original forms:

Proof of Theorem 2.2. From (7), we obtain

$$\begin{aligned}
&\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&= \sum_{S \subseteq V} \Bigg(X + \underbrace{x(S)}_{\substack{= \sum_{s \in S} x_s \\ \text{(by the definition of } x(S))}} \Bigg)^{|S|} \Bigg(Y - \underbrace{x(S)}_{\substack{= \sum_{s \in S} x_s \\ \text{(by the definition of } x(S))}} \Bigg)^{n-|S|} \\
&= \sum_{S \subseteq V} \left(X + \sum_{s \in S} x_s \right)^{|S|} \left(Y - \sum_{s \in S} x_s \right)^{n-|S|}.
\end{aligned}$$

In other words,

$$\sum_{S \subseteq V} \left(X + \sum_{s \in S} x_s \right)^{|S|} \left(Y - \sum_{s \in S} x_s \right)^{n-|S|} = \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V}} (X+Y)^{n-k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

This proves Theorem 2.2. □

Proof of Theorem 2.4. From (8), we obtain

$$\begin{aligned}
&(X+Y)^n \\
&= Y^n + \sum_{\substack{S \subseteq V; \\ S \neq \varnothing}} X \Bigg(X + \underbrace{x(S)}_{\substack{= \sum_{s \in S} x_s \\ \text{(by the definition of } x(S))}} \Bigg)^{|S|-1} \Bigg(Y - \underbrace{x(S)}_{\substack{= \sum_{s \in S} x_s \\ \text{(by the definition of } x(S))}} \Bigg)^{n-|S|} \\
&= Y^n + \sum_{\substack{S \subseteq V; \\ S \neq \varnothing}} X \left(X + \sum_{s \in S} x_s \right)^{|S|-1} \left(Y - \sum_{s \in S} x_s \right)^{n-|S|}.
\end{aligned}$$

Comparing this with

$$\begin{aligned}
&\sum_{S\subseteq V} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y-\sum_{s\in S} x_s\right)^{n-|S|} \\
&= \underbrace{X\left(X+\sum_{s\in\varnothing} x_s\right)^{|\varnothing|-1}}_{\substack{=1 \\ \text{(according to our convention for} \\ \text{interpreting } X\left(X+\sum\limits_{s\in S} x_s\right)^{|S|-1} \\ \text{when } S=\varnothing)}} \left(Y-\underbrace{\sum_{s\in\varnothing} x_s}_{=(\text{empty sum})=0}\right)^{n-|\varnothing|} \\
&\qquad + \sum_{\substack{S\subseteq V; \\ S\neq\varnothing}} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y-\sum_{s\in S} x_s\right)^{n-|S|} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addend for } S=\varnothing \text{ from the sum} \\ \text{(since } \varnothing \text{ is a subset of } V\text{)}\end{array}\right) \\
&= \underbrace{(Y-0)^{n-|\varnothing|}}_{\substack{=Y^{n-0} \\ \text{(since } Y-0=Y \\ \text{and } |\varnothing|=0)}} + \sum_{\substack{S\subseteq V; \\ S\neq\varnothing}} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y-\sum_{s\in S} x_s\right)^{n-|S|} \\
&= \underbrace{Y^{n-0}}_{=Y^n} + \sum_{\substack{S\subseteq V; \\ S\neq\varnothing}} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y-\sum_{s\in S} x_s\right)^{n-|S|} \\
&= Y^n + \sum_{\substack{S\subseteq V; \\ S\neq\varnothing}} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y-\sum_{s\in S} x_s\right)^{n-|S|},
\end{aligned}$$

we obtain

$$\sum_{S\subseteq V} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y-\sum_{s\in S} x_s\right)^{n-|S|} = (X+Y)^n .$$

This proves Theorem 2.4. □

3.2. Proofs of Theorems 2.6 and 2.7

Now, we are going to prepare for the proof of Theorem 2.6. Let us first restate this theorem (or, more precisely, its case when V is nonempty) in a more convenient form:

Lemma 3.5. Let V be a finite nonempty set. Let $n = |V|$. For each $s \in V$, let x_s be an element of $\mathbb{L}$. Let X and Y be two elements of $\mathbb{L}$ such that $X + Y$ lies in the center of $\mathbb{L}$. Then,

$$\begin{aligned} & Y^{n-1} (Y - x(V)) + X (X + x(V))^{n-1} \\ & \qquad + \sum_{\substack{S \subseteq V; \\ S \neq \varnothing;\ S \neq V}} X (X + x(S))^{|S|-1} (Y - x(S))^{n-|S|-1} (Y - x(V)) \\ & = (X + Y - x(V)) (X + Y)^{n-1} . \end{aligned}$$

Here, we are again using the notation from Definition 3.2.

Notice that the claim of Lemma 3.5 (unlike the claim of Theorem 2.6) does not require any convention about how to interpret the term $X\left(X + \sum_{s \in S} x_s\right)^{|S|-1}$ when $S = \varnothing$ (and any other such conventions), because all expressions appearing in Lemma 3.5 are well-defined a-priori.

Proof of Lemma 3.5. Let us first notice that all terms appearing in Lemma 3.5 are well-defined[12].

For every subset S of V, we have

$$Y - x(V) = (Y - x(S)) - \sum_{\substack{t \in V; \\ t \notin S}} x_t \tag{32}$$

[12] *Proof.* This follows from the following four observations:

- The terms Y^{n-1}, $X (X + x(V))^{n-1}$ and $(X + Y)^{n-1}$ are well-defined. (*Proof:* We know that the set V is nonempty. Hence, $|V| > 0$. Thus, $|V| \geq 1$ (since $|V|$ is an integer). Therefore, $n = |V| \geq 1$, so that $n - 1 \geq 0$ and thus $n - 1 \in \mathbb{N}$. Hence, the terms Y^{n-1}, $X (X + x(V))^{n-1}$ and $(X + Y)^{n-1}$ are well-defined. Qed.)
- For every subset S of V, the term $|S|$ is well-defined. (*Proof:* Let S be a subset of V. Then, S is a subset of the finite set V, and therefore itself is finite. Hence, the term $|S|$ is well-defined. Qed.)
- For every subset S of V satisfying $S \neq \varnothing$ and $S \neq V$, the term $(X + x(S))^{|S|-1}$ is well-defined. (*Proof:* Let S be a subset of V satisfying $S \neq \varnothing$ and $S \neq V$. Then, $|S| > 0$ (since $S \neq \varnothing$) and thus $|S| \geq 1$ (since $|S|$ is an integer). In other words, $|S| - 1 \in \mathbb{N}$. Hence, the term $(X + x(S))^{|S|-1}$ is well-defined. Qed.)
- For every subset S of V satisfying $S \neq \varnothing$ and $S \neq V$, the term $(Y - x(S))^{n-|S|-1}$ is well-defined. (*Proof:* Let S be a subset of V satisfying $S \neq \varnothing$ and $S \neq V$. Then, S is a proper subset of V (since S is a subset of V satisfying $S \neq V$). It is well-known that if B is a finite set, and if A is a proper subset of B, then $|A| < |B|$. Applying this to $A = S$ and $B = V$, we obtain $|S| < |V| = n$, so that $n - |S| > 0$. Since $n - |S|$ is an integer, this entails that $n - |S| \geq 1$. Hence, $n - |S| - 1 \geq 0$. In other words, $n - |S| - 1 \in \mathbb{N}$. Hence, the term $(Y - x(S))^{n-|S|-1}$ is well-defined. Qed.)

[13].

The definition of $x(V)$ yields

$$x(V) = \sum_{s\in V} x_s = \sum_{t\in V} x_t \tag{33}$$

(here, we have renamed the summation index s as t).

From $n = |V|$, we obtain $n - |V| = 0$.

Fix any $t \in V$. Every subset S of V satisfying $t \notin S$ must automatically satisfy $S \neq V$ [14]. Hence, for any subset S of V, we have the following logical equivalence:

$$(t \notin S \text{ and } S \neq \varnothing \text{ and } S \neq V) \iff (t \notin S \text{ and } S \neq \varnothing).$$

Thus, we have the following equality of summation signs:

$$\sum_{\substack{S\subseteq V;\\ t\notin S;\\ S\neq\varnothing;\ S\neq V}} = \sum_{\substack{S\subseteq V;\\ t\notin S;\\ S\neq\varnothing}} = \sum_{\substack{S\subseteq V\setminus\{t\};\\ S\neq\varnothing}} \tag{34}$$

(since the subsets S of V satisfying $t \notin S$ are precisely the subsets of $V \setminus \{t\}$).

The set $V \setminus \{t\}$ is a subset of the finite set V, and thus is itself finite. Moreover, from $t \in V$, we obtain $|V \setminus \{t\}| = \underbrace{|V|}_{=n} - 1 = n - 1$. Hence, we can apply the

[13] *Proof of (32):* Let S be a subset of V. The definition of $x(S)$ yields $x(S) = \sum_{s\in S} x_s$. But the definition of $x(V)$ yields

$$\begin{aligned} x(V) &= \sum_{s\in V} x_s = \underbrace{\sum_{\substack{s\in V;\\ s\in S}}}_{\substack{=\sum_{s\in S}\\ \text{(since } S \text{ is a subset of } V\text{)}}} x_s + \underbrace{\sum_{\substack{s\in V;\\ s\notin S}} x_s}_{\substack{=\sum_{t\in V;\ t\notin S} x_t\\ \text{(here, we have substituted } t \text{ for } s\\ \text{in the sum)}}} \\ &\qquad \left(\begin{array}{c}\text{since each } s\in V \text{ satisfies either } s\in S\\ \text{or } s\notin S \text{ (but not both)}\end{array}\right) \\ &= \underbrace{\sum_{s\in S} x_s}_{=x(S)} + \sum_{\substack{t\in V;\\ t\notin S}} x_t = x(S) + \sum_{\substack{t\in V;\\ t\notin S}} x_t. \end{aligned}$$

Thus,

$$Y - \underbrace{x(V)}_{=x(S)+\sum_{t\in V;\ t\notin S} x_t} = Y - \left(x(S) + \sum_{\substack{t\in V;\\ t\notin S}} x_t\right) = (Y - x(S)) - \sum_{\substack{t\in V;\\ t\notin S}} x_t.$$

This proves (32).

[14] *Proof.* Let S be a subset of V satisfying $t \notin S$. If we had $S = V$, then we would have $t \in V = S$, which would contradict $t \notin S$. Hence, we cannot have $S = V$. Thus, we have $S \neq V$. Qed.

equality (8) to $V \setminus \{t\}$ and $n-1$ instead of V and n. We thus obtain

$$Y^{n-1} + \sum_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} X \left(X + x\left(S\right)\right)^{|S|-1} \left(Y - x\left(S\right)\right)^{n-1-|S|} = \left(X+Y\right)^{n-1}.$$

Subtracting Y^{n-1} from both sides of this equality, we obtain

$$\sum_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} X \left(X + x\left(S\right)\right)^{|S|-1} \left(Y - x\left(S\right)\right)^{n-1-|S|} = \left(X+Y\right)^{n-1} - Y^{n-1}. \tag{35}$$

Now,

$$\begin{aligned}
&\underbrace{\sum_{\substack{S \subseteq V; \\ S \neq \varnothing;\ S \neq V; \\ t \notin S}}}_{\substack{= \sum\limits_{\substack{S \subseteq V; \\ t \notin S; \\ S \neq \varnothing;\ S \neq V}} = \sum\limits_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} \\ \text{(by (34))}}} X \left(X + x\left(S\right)\right)^{|S|-1} \underbrace{\left(Y - x\left(S\right)\right)^{n-|S|-1}}_{\substack{= \left(Y - x\left(S\right)\right)^{n-1-|S|} \\ \text{(since } n-|S|-1 = n-1-|S|\text{)}}} x_t \\
&= \sum_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} X \left(X + x\left(S\right)\right)^{|S|-1} \left(Y - x\left(S\right)\right)^{n-1-|S|} x_t \\
&= \underbrace{\left(\sum_{\substack{S \subseteq V \setminus \{t\}; \\ S \neq \varnothing}} X \left(X + x\left(S\right)\right)^{|S|-1} \left(Y - x\left(S\right)\right)^{n-1-|S|} \right)}_{\substack{= \left(X+Y\right)^{n-1} - Y^{n-1} \\ \text{(by (35))}}} x_t \\
&= \left(\left(X+Y\right)^{n-1} - Y^{n-1} \right) x_t.
\end{aligned} \tag{36}$$

Now, let us forget that we fixed t. We thus have proven the equality (36) for each $t \in V$.

Subtracting Y^n from both sides of the equality (8), we obtain

$$\sum_{\substack{S \subseteq V; \\ S \neq \varnothing}} X \left(X + x\left(S\right)\right)^{|S|-1} \left(Y - x\left(S\right)\right)^{n-|S|} = \left(X+Y\right)^{n} - Y^{n}. \tag{37}$$

But the set V is nonempty. In other words, $V \neq \varnothing$. Hence, V is a subset of V satisfying $V \neq \varnothing$. In other words, V is a subset S of V satisfying $S \neq \varnothing$. Hence, the sum $\sum\limits_{\substack{S \subseteq V; \\ S \neq \varnothing}} X \left(X + x\left(S\right)\right)^{|S|-1} \left(Y - x\left(S\right)\right)^{n-|S|}$ has an addend for $S = V$. If we split off this addend from this sum, then we obtain

$$\begin{aligned}
&\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|}\\
&= X\underbrace{\left(X+x\left(V\right)\right)^{\left|V\right|-1}}_{\substack{=\left(X+x\left(V\right)\right)^{n-1}\\ \text{(since } \left|V\right|=n\text{)}}}\underbrace{\left(Y-x\left(V\right)\right)^{n-\left|V\right|}}_{\substack{=\left(Y-x\left(V\right)\right)^{0}\\ \text{(since } n-\left|V\right|=0\text{)}}}\\
&\qquad+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|}\\
&= X\left(X+x\left(V\right)\right)^{n-1}\underbrace{\left(Y-x\left(V\right)\right)^{0}}_{=1}+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|}\\
&= X\left(X+x\left(V\right)\right)^{n-1}+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|}.
\end{aligned}$$

Subtracting $X\left(X+x\left(V\right)\right)^{n-1}$ from this equality, we obtain

$$\begin{aligned}
&\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|}-X\left(X+x\left(V\right)\right)^{n-1}\\
&=\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|}\\
&=\underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|}}_{\substack{=\left(X+Y\right)^{n}-Y^{n}\\ \text{(by (37))}}}-X\left(X+x\left(V\right)\right)^{n-1}\\
&=\left(X+Y\right)^{n}-Y^{n}-X\left(X+x\left(V\right)\right)^{n-1}. \qquad (38)
\end{aligned}$$

Now,

$$\begin{aligned}
&\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|-1}\underbrace{\left(Y-x\left(V\right)\right)}_{\substack{=\left(Y-x\left(S\right)\right)-\sum\limits_{\substack{t\in V;\\ t\notin S}} x_t \\ \text{(by (32))}}}\\
&=\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} \underbrace{X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|-1}\left(\left(Y-x\left(S\right)\right)-\sum_{\substack{t\in V;\\ t\notin S}} x_t\right)}_{\substack{=X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|-1}\left(Y-x\left(S\right)\right)\\ -X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|-1}\sum\limits_{\substack{t\in V;\\ t\notin S}} x_t}}\\
&=\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} \Bigg(X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|-1}\left(Y-x\left(S\right)\right)\\
&\qquad\qquad -X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|-1}\sum_{\substack{t\in V;\\ t\notin S}} x_t\Bigg)\\
&=\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\underbrace{\left(Y-x\left(S\right)\right)^{n-\left|S\right|-1}\left(Y-x\left(S\right)\right)}_{=\left(Y-x\left(S\right)\right)^{n-\left|S\right|}}\\
&\qquad -\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} \underbrace{X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|-1}\sum_{\substack{t\in V;\\ t\notin S}} x_t}_{=\sum\limits_{\substack{t\in V;\\ t\notin S}} X\left(X+x\left(S\right)\right)^{\left|S\right|-1}\left(Y-x\left(S\right)\right)^{n-\left|S\right|-1}x_t}
\end{aligned}$$

$$= \underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{n-|S|}}_{\substack{=(X+Y)^n-Y^n-X(X+x(V))^{n-1}\\ \text{(by (38))}}}$$

$$- \underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} \sum_{\substack{t\in V;\\ t\notin S}}}_{=\sum_{t\in V} \sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V;\\ t\notin S}}} X\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{n-|S|-1} x_t$$

$$= \left(X+Y\right)^n - Y^n - X\left(X+x\left(V\right)\right)^{n-1}$$

$$- \sum_{t\in V} \underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V;\\ t\notin S}} X\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{n-|S|-1} x_t}_{\substack{=\left((X+Y)^{n-1}-Y^{n-1}\right)x_t\\ \text{(by (36))}}}$$

$$= \left(X+Y\right)^n - Y^n - X\left(X+x\left(V\right)\right)^{n-1} - \underbrace{\sum_{t\in V}\left(\left(X+Y\right)^{n-1}-Y^{n-1}\right)x_t}_{=\left((X+Y)^{n-1}-Y^{n-1}\right)\sum_{t\in V} x_t}$$

$$= \left(X+Y\right)^n - Y^n - X\left(X+x\left(V\right)\right)^{n-1} - \left(\left(X+Y\right)^{n-1}-Y^{n-1}\right)\underbrace{\sum_{t\in V} x_t}_{\substack{=x(V)\\ \text{(by (33))}}}$$

$$= \underbrace{\left(X+Y\right)^n}_{=(X+Y)^{n-1}(X+Y)} - \underbrace{Y^n}_{=Y^{n-1}Y} - X\left(X+x\left(V\right)\right)^{n-1} - \underbrace{\left(\left(X+Y\right)^{n-1}-Y^{n-1}\right)x\left(V\right)}_{=(X+Y)^{n-1}x(V)-Y^{n-1}x(V)}$$

$$= \left(X+Y\right)^{n-1}\left(X+Y\right) - Y^{n-1}Y - X\left(X+x\left(V\right)\right)^{n-1} - \left(\left(X+Y\right)^{n-1}x\left(V\right) - Y^{n-1}x\left(V\right)\right)$$

$$= \left(X+Y\right)^{n-1}\left(X+Y\right) - Y^{n-1}Y - X\left(X+x\left(V\right)\right)^{n-1} - \left(X+Y\right)^{n-1}x\left(V\right) + Y^{n-1}x\left(V\right)$$

$$= \underbrace{\left(\left(X+Y\right)^{n-1}\left(X+Y\right) - \left(X+Y\right)^{n-1}x\left(V\right)\right)}_{=(X+Y)^{n-1}((X+Y)-x(V))} - \underbrace{\left(Y^{n-1}Y - Y^{n-1}x\left(V\right)\right)}_{=Y^{n-1}(Y-x(V))} - X\left(X+x\left(V\right)\right)^{n-1}$$

$$= \underbrace{\left(X+Y\right)^{n-1}\left(\left(X+Y\right)-x\left(V\right)\right)}_{\substack{=((X+Y)-x(V))(X+Y)^{n-1}\\ \text{(since } (X+Y)^{n-1} \text{ commutes with } (X+Y)-x(V)\\ \text{(since } (X+Y)^{n-1} \text{ lies in the center of } \mathbb{L}\\ \text{(since } X+Y \text{ lies in the center of } \mathbb{L},\\ \text{but the center of } \mathbb{L} \text{ is a subring of } \mathbb{L})))}} - Y^{n-1}\left(Y-x\left(V\right)\right) - X\left(X+x\left(V\right)\right)^{n-1}$$

$$= \left(\left(X+Y\right)-x\left(V\right)\right)\left(X+Y\right)^{n-1} - Y^{n-1}\left(Y-x\left(V\right)\right) - X\left(X+x\left(V\right)\right)^{n-1}.$$

Solving this equality for $((X+Y)-x(V))(X+Y)^{n-1}$, we obtain

$$\begin{aligned}
&((X+Y)-x(V))(X+Y)^{n-1}\\
&=\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X(X+x(S))^{|S|-1}(Y-x(S))^{n-|S|-1}(Y-x(V))\\
&\qquad + Y^{n-1}(Y-x(V)) + X(X+x(V))^{n-1}\\
&= Y^{n-1}(Y-x(V)) + X(X+x(V))^{n-1}\\
&\qquad + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X(X+x(S))^{|S|-1}(Y-x(S))^{n-|S|-1}(Y-x(V)).
\end{aligned}$$

Hence,

$$\begin{aligned}
&Y^{n-1}(Y-x(V)) + X(X+x(V))^{n-1}\\
&\qquad + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X(X+x(S))^{|S|-1}(Y-x(S))^{n-|S|-1}(Y-x(V))\\
&= \underbrace{((X+Y)-x(V))}_{=X+Y-x(V)}(X+Y)^{n-1}\\
&= (X+Y-x(V))(X+Y)^{n-1}.
\end{aligned}$$

This proves Lemma 3.5. □

Proof of Theorem 2.6. Theorem 2.6 holds in the case when $V=\varnothing$ [15]. Hence, for the rest of this proof, we can WLOG assume that we don't have $V=\varnothing$. Assume this.

---

[15] *Proof.* Assume that $V=\varnothing$. We must prove that Theorem 2.6 holds.

We have $V=\varnothing$. Hence, $|V|=|\varnothing|=0$. Thus, $n=|V|=0$.

But V is the empty set (since $V=\varnothing$). Hence, the only subset S of V is the empty set $\varnothing$. Thus, the sum $\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}\left(Y-\sum_{s\in V}x_s\right)$ has only one addend: namely, the addend for $S=\varnothing$. Therefore, this sum simplifies as follows:

$$\begin{aligned}
&\sum_{S\subseteq V} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}\left(Y-\sum_{s\in V}x_s\right)\\
&= \underbrace{X\left(X+\sum_{s\in\varnothing}x_s\right)^{|\varnothing|-1}}_{\substack{=1\\ \text{(according to our convention for}\\ \text{interpreting } X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\\ \text{when } S=\varnothing)}}\ \underbrace{\left(Y-\sum_{s\in\varnothing}x_s\right)^{n-|\varnothing|-1}\left(Y-\sum_{s\in V}x_s\right)}_{\substack{=1\\ \text{(according to our convention for}\\ \text{interpreting } \left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}\left(Y-\sum_{s\in V}x_s\right)\\ \text{when } |S|=n)}}\\
&= 1.
\end{aligned}$$

We have $V \neq \varnothing$ (since we don't have $V = \varnothing$). Hence, the set V is nonempty. Thus, Lemma 3.5 yields

$$\begin{aligned}
&Y^{n-1}\left(Y - x\left(V\right)\right) + X\left(X + x\left(V\right)\right)^{n-1} \\
&\qquad + \sum_{\substack{S \subseteq V; \\ S \neq \varnothing;\ S \neq V}} X\left(X + x\left(S\right)\right)^{|S|-1}\left(Y - x\left(S\right)\right)^{n-|S|-1}\left(Y - x\left(V\right)\right) \\
&= \left(X + Y - x\left(V\right)\right)\left(X + Y\right)^{n-1}. \qquad (39)
\end{aligned}$$

We have $n - \underbrace{|\varnothing|}_{=0} - 1 = n - 0 - 1 = n - 1$.

Every subset S of V satisfies

$$x\left(S\right) = \sum_{s \in S} x_s \qquad (40)$$

(by the definition of $x\left(S\right)$). Applying this to $S = V$, we obtain

$$x\left(V\right) = \sum_{s \in V} x_s. \qquad (41)$$

But the set V is a subset S of V satisfying $S \neq \varnothing$ (since V is a subset of V and since $V \neq \varnothing$). Hence, the sum $\sum\limits_{\substack{S \subseteq V; \\ S \neq \varnothing}} X\left(X + \sum\limits_{s \in S} x_s\right)^{|S|-1}\left(Y - \sum\limits_{s \in S} x_s\right)^{n-|S|-1}\left(Y - \sum\limits_{s \in V} x_s\right)$

Comparing this with

$$\left(X + Y - \sum_{s \in V} x_s\right)\left(X + Y\right)^{n-1} = 1 \qquad \left(\begin{array}{c}\text{according to our convention for} \\ \text{interpreting } \left(X + Y - \sum\limits_{s \in V} x_s\right)\left(X + Y\right)^{n-1} \\ \text{when } n = 0\end{array}\right),$$

we obtain

$$\begin{aligned}
&\sum_{S \subseteq V} X\left(X + \sum_{s \in S} x_s\right)^{|S|-1}\left(Y - \sum_{s \in S} x_s\right)^{n-|S|-1}\left(Y - \sum_{s \in V} x_s\right) \\
&= \left(X + Y - \sum_{s \in V} x_s\right)\left(X + Y\right)^{n-1}.
\end{aligned}$$

In other words, Theorem 2.6 holds. Qed.

has an addend for $S = V$. If we split off this addend from the sum, then we obtain

$$\begin{aligned}
&\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+\sum_{s\in S}x_s\right)^{|S|-1}\left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}\left(Y-\sum_{s\in V}x_s\right)\\
&= X\underbrace{\left(X+\sum_{s\in V}x_s\right)^{|V|-1}}_{\substack{=\left(X+\sum_{s\in V}x_s\right)^{n-1}\\ \text{(since } |V|=n\text{)}}}\underbrace{\left(Y-\sum_{s\in V}x_s\right)^{n-|V|-1}\left(Y-\sum_{s\in V}x_s\right)}_{\substack{=1\\ \text{(according to our convention for}\\ \text{interpreting } \left(Y-\sum_{s\in S}x_s\right)^{n-|S|-1}\left(Y-\sum_{s\in V}x_s\right)\\ \text{when } |S|=n\text{)}}}\\
&\qquad + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+\underbrace{\sum_{s\in S}x_s}_{\substack{=x(S)\\ \text{(by (40))}}}\right)^{|S|-1}\left(Y-\underbrace{\sum_{s\in S}x_s}_{\substack{=x(S)\\ \text{(by (40))}}}\right)^{n-|S|-1}\left(Y-\underbrace{\sum_{s\in V}x_s}_{\substack{=x(V)\\ \text{(by (41))}}}\right)\\
&= X\left(X+\underbrace{\sum_{s\in V}x_s}_{\substack{=x(V)\\ \text{(by (41))}}}\right)^{n-1} + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{n-|S|-1}\left(Y-x\left(V\right)\right)\\
&= X\left(X+x\left(V\right)\right)^{n-1} + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{n-|S|-1}\left(Y-x\left(V\right)\right).
\end{aligned} \tag{42}$$

Now,

$$\sum_{S\subseteq V} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1} \left(Y-\sum_{s\in S} x_s\right)^{n-|S|-1} \left(Y-\sum_{s\in V} x_s\right)$$

$$= \underbrace{X\left(X+\sum_{s\in\varnothing} x_s\right)^{|\varnothing|-1}}_{\substack{=1 \\ \text{(according to our convention for} \\ \text{interpreting } X\left(X+\sum_{s\in S} x_s\right)^{|S|-1} \\ \text{when } S=\varnothing)}} \left(Y-\underbrace{\sum_{s\in\varnothing} x_s}_{=(\text{empty sum})=0}\right)^{n-|\varnothing|-1} \left(Y-\sum_{s\in V} x_s\right)$$

$$+ \underbrace{\sum_{\substack{S\subseteq V;\\ S\neq\varnothing}} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1} \left(Y-\sum_{s\in S} x_s\right)^{n-|S|-1} \left(Y-\sum_{s\in V} x_s\right)}_{\substack{=X(X+x(V))^{n-1}+\sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X(X+x(S))^{|S|-1}(Y-x(S))^{n-|S|-1}(Y-x(V)) \\ \text{(by (42))}}}$$

$$\left(\begin{array}{c}\text{here, we have split off the addend for } S=\varnothing \text{ from the sum} \\ \text{(since } \varnothing \text{ is a subset of } V\text{)}\end{array}\right)$$

$$= \underbrace{(Y-0)^{n-|\varnothing|-1}}_{\substack{=Y^{n-1} \\ \text{(since } Y-0=0 \text{ and } n-|\varnothing|-1=n-1)}} \left(Y-\underbrace{\sum_{s\in V} x_s}_{\substack{=x(V)\\ \text{(by (41))}}}\right)$$

$$+ X\left(X+x\left(V\right)\right)^{n-1} + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{n-|S|-1}\left(Y-x\left(V\right)\right)$$

$$= Y^{n-1}\left(Y-x\left(V\right)\right) + X\left(X+x\left(V\right)\right)^{n-1} + \sum_{\substack{S\subseteq V;\\ S\neq\varnothing;\ S\neq V}} X\left(X+x\left(S\right)\right)^{|S|-1}\left(Y-x\left(S\right)\right)^{n-|S|-1}\left(Y-x\left(V\right)\right)$$

$$= \left(X+Y-\underbrace{x\left(V\right)}_{\substack{=\sum_{s\in V} x_s \\ \text{(by (41))}}}\right)\left(X+Y\right)^{n-1} \qquad \text{(by (39))}$$

$$= \left(X+Y-\sum_{s\in V} x_s\right)\left(X+Y\right)^{n-1}.$$

This proves Theorem 2.6. □

Proof of Theorem 2.7. We know that $X + Y + \sum_{s \in V} x_s$ lies in the center of $\mathbb{L}$. In other words, $X + \left(Y + \sum_{s \in V} x_s \right)$ lies in the center of $\mathbb{L}$ (since $X + \left(Y + \sum_{s \in V} x_s \right) = X + Y + \sum_{s \in V} x_s$). Thus, Theorem 2.6 (applied to $Y + \sum_{s \in V} x_s$ instead of Y) yields

$$\begin{aligned}
&\sum_{S \subseteq V} X \left(X + \sum_{s \in S} x_s \right)^{|S|-1} \left(Y + \sum_{s \in V} x_s - \sum_{s \in S} x_s \right)^{n-|S|-1} \left(Y + \sum_{s \in V} x_s - \sum_{s \in V} x_s \right) \\
&= \left(X + \underbrace{Y + \sum_{s \in V} x_s - \sum_{s \in V} x_s}_{=Y} \right) \left(X + Y + \sum_{s \in V} x_s \right)^{n-1} \\
&= (X + Y) \left(X + Y + \sum_{s \in V} x_s \right)^{n-1} .
\end{aligned} \tag{43}$$

But every subset S of V satisfies

$$\sum_{s \in V} x_s - \sum_{s \in S} x_s = \sum_{s \in V \setminus S} x_s \tag{44}$$

[16]. Now,

$$\sum_{S\subseteq V} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y+\underbrace{\sum_{s\in V} x_s-\sum_{s\in S} x_s}_{\substack{=\sum\limits_{s\in V\setminus S} x_s \\ \text{(by (44))}}}\right)^{n-|S|-1}\underbrace{\left(Y+\sum_{s\in V} x_s-\sum_{s\in V} x_s\right)}_{=Y}$$

$$=\sum_{S\subseteq V} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y+\sum_{s\in V\setminus S} x_s\right)^{n-|S|-1} Y.$$

Comparing this with (43), we obtain

$$\sum_{S\subseteq V} X\left(X+\sum_{s\in S} x_s\right)^{|S|-1}\left(Y+\sum_{s\in V\setminus S} x_s\right)^{n-|S|-1} Y$$
$$=(X+Y)\left(X+Y+\sum_{s\in V} x_s\right)^{n-1}.$$

This proves Theorem 2.7. □

4. Applications

4.1. Polarization identities

Let us show how a rather classical identity in noncommutative rings follows as a particular case from Theorem 2.2. Namely, we shall prove the following *polarization identity*:

[16]*Proof of (44):* Let S be a subset of V. Then, each element $s\in V$ satisfies either $s\in S$ or $s\notin S$ (but not both). Hence,

$$\sum_{s\in V} x_s=\underbrace{\sum_{\substack{s\in V;\\ s\in S}}}_{\substack{=\sum\limits_{s\in S} \\ \text{(since } S \text{ is a subset of } V\text{)}}} x_s+\underbrace{\sum_{\substack{s\in V;\\ s\notin S}}}_{=\sum\limits_{s\in V\setminus S}} x_s=\sum_{s\in S} x_s+\sum_{s\in V\setminus S} x_s.$$

Hence,

$$\underbrace{\sum_{s\in V} x_s}_{=\sum\limits_{s\in S} x_s+\sum\limits_{s\in V\setminus S} x_s}-\sum_{s\in S} x_s=\sum_{s\in S} x_s+\sum_{s\in V\setminus S} x_s-\sum_{s\in S} x_s=\sum_{s\in V\setminus S} x_s.$$

This proves (44).

Corollary 4.1. Let V be a finite set. Let $n = |V|$. For each $s \in V$, let x_s be an element of $\mathbb{L}$. Let $X \in \mathbb{L}$. Then,

$$\sum_{S \subseteq V} (-1)^{n-|S|} \left(X + \sum_{s \in S} x_s \right)^n = \sum_{\substack{(i_1,i_2,\ldots,i_n) \text{ is a list} \\ \text{of all elements of } V \\ \text{(with no repetitions)}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

Proof of Corollary 4.1. If $(i_1, i_2, \ldots, i_k)$ is a k-tuple of distinct elements of V for some $k \in \mathbb{N}$, then the statement $(n \leq k)$ is equivalent to the statement $(k = n)$ [17]. Hence, we have the following equality of summation signs:

$$\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V; \\ n \leq k}} = \sum_{\substack{i_1,i_2,\ldots,i_k \text{ are} \\ \text{distinct} \\ \text{elements of } V; \\ k = n}} . \qquad (45)$$

Furthermore, let us define a set $\mathfrak{A}$ by

$$\mathfrak{A} = \{(i_1, i_2, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \text{ is a list of all elements of } V \text{ (with no repetitions)}\}.$$

Let us also define a set $\mathfrak{B}$ by

$$\mathfrak{B} = \{(i_1, i_2, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \text{ is a list of distinct elements of } V\}.$$

[17] *Proof.* Let $(i_1, i_2, \ldots, i_k)$ be a k-tuple of distinct elements of V for some $k \in \mathbb{N}$. We must prove that the statement $(n \leq k)$ is equivalent to the statement $(k = n)$.

The elements $i_1, i_2, \ldots, i_k$ are elements of V (since $(i_1, i_2, \ldots, i_k)$ is a k-tuple of elements of V). In other words, $\{i_1, i_2, \ldots, i_k\} \subseteq V$. Hence, $|\{i_1, i_2, \ldots, i_k\}| \leq |V| = n$. Thus, $n \geq |\{i_1, i_2, \ldots, i_k\}| \geq k$. Hence, if $n \leq k$, then $n = k$ (because combining $n \leq k$ with $n \geq k$ yields $n = k$). In other words, the implication $(n \leq k) \Longrightarrow (n = k)$ holds. Conversely, the implication $(n = k) \Longrightarrow (n \leq k)$ holds (obviously). Combining this implication with the implication $(n \leq k) \Longrightarrow (n = k)$, we obtain the equivalence $(n \leq k) \Longleftrightarrow (n = k)$. Hence, we obtain the following chain of equivalences: $(n \leq k) \Longleftrightarrow (n = k) \Longleftrightarrow (k = n)$. Thus, the statement $(n \leq k)$ is equivalent to the statement $(k = n)$. Qed.

Then, $\mathfrak{A} \subseteq \mathfrak{B}$ [18] and $\mathfrak{B} \subseteq \mathfrak{A}$ [19]. Combining these two inclusions, we obtain

[18] *Proof.* Let $j \in \mathfrak{A}$. Thus,

$$\begin{aligned} j \in \mathfrak{A} &= \{(i_1, i_2, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \text{ is a list of all elements of } V \text{ (with no repetitions)}\} \\ &= \{(j_1, j_2, \ldots, j_n) \mid (j_1, j_2, \ldots, j_n) \text{ is a list of all elements of } V \text{ (with no repetitions)}\} \end{aligned}$$

(here, we have renamed the index $(i_1, i_2, \ldots, i_n)$ as $(j_1, j_2, \ldots, j_n)$). In other words, j can be written in the form $j = (j_1, j_2, \ldots, j_n)$, where $(j_1, j_2, \ldots, j_n)$ is a list of all elements of V (with no repetitions). Consider this $(j_1, j_2, \ldots, j_n)$.

The list $(j_1, j_2, \ldots, j_n)$ is a list of elements of V, and its entries are distinct (since $(j_1, j_2, \ldots, j_n)$ is a list with no repetitions). In other words, $(j_1, j_2, \ldots, j_n)$ is a list of distinct elements of V. Thus, $(j_1, j_2, \ldots, j_n)$ has the form $(j_1, j_2, \ldots, j_n) = (i_1, i_2, \ldots, i_n)$, where $(i_1, i_2, \ldots, i_n)$ is a list of distinct elements of V (namely, $(i_1, i_2, \ldots, i_n) = (j_1, j_2, \ldots, j_n)$). In other words,

$$(j_1, j_2, \ldots, j_n) \in \{(i_1, i_2, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \text{ is a list of distinct elements of } V\}.$$

In light of $j = (j_1, j_2, \ldots, j_n)$ and $\mathfrak{B} = \{(i_1, i_2, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \text{ is a list of distinct elements of } V\}$, this rewrites as $j \in \mathfrak{B}$.

Now, forget that we fixed j. We thus have shown that $j \in \mathfrak{B}$ for each $j \in \mathfrak{A}$. In other words, $\mathfrak{A} \subseteq \mathfrak{B}$.

[19] *Proof.* Let $j \in \mathfrak{B}$. Thus,

$$\begin{aligned} j \in \mathfrak{B} &= \{(i_1, i_2, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \text{ is a list of distinct elements of } V\} \\ &= \{(j_1, j_2, \ldots, j_n) \mid (j_1, j_2, \ldots, j_n) \text{ is a list of distinct elements of } V\} \end{aligned}$$

(here, we have renamed the index $(i_1, i_2, \ldots, i_n)$ as $(j_1, j_2, \ldots, j_n)$). In other words, j can be written in the form $j = (j_1, j_2, \ldots, j_n)$, where $(j_1, j_2, \ldots, j_n)$ is a list of distinct elements of V. Consider this $(j_1, j_2, \ldots, j_n)$.

The n elements $j_1, j_2, \ldots, j_n$ are distinct (since $(j_1, j_2, \ldots, j_n)$ is a list of distinct elements of V). Thus, $|\{j_1, j_2, \ldots, j_n\}| = n$. But $j_1, j_2, \ldots, j_n$ are elements of V (since $(j_1, j_2, \ldots, j_n)$ is a list of elements of V). Hence, $\{j_1, j_2, \ldots, j_n\}$ is a subset of V.

It is well-known that if B is a finite set, and if A is a subset of B satisfying $|A| = |B|$, then $A = B$. Applying this to $B = V$ and $A = \{j_1, j_2, \ldots, j_n\}$, we obtain $\{j_1, j_2, \ldots, j_n\} = V$ (since $|\{j_1, j_2, \ldots, j_n\}| = n = |V|$). Thus, the list $(j_1, j_2, \ldots, j_n)$ is a list of all elements of V. Furthermore, this list $(j_1, j_2, \ldots, j_n)$ has no repetitions (since the n elements $j_1, j_2, \ldots, j_n$ are distinct). Thus, the list $(j_1, j_2, \ldots, j_n)$ is a list of all elements of V (with no repetitions). Thus, $(j_1, j_2, \ldots, j_n)$ has the form $(j_1, j_2, \ldots, j_n) = (i_1, i_2, \ldots, i_n)$, where $(i_1, i_2, \ldots, i_n)$ is a list of all elements of V (with no repetitions) (namely, $(i_1, i_2, \ldots, i_n) = (j_1, j_2, \ldots, j_n)$). In other words,

$$(j_1, j_2, \ldots, j_n) \in \{(i_1, i_2, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \text{ is a list of all elements of } V \text{ (with no repetitions)}\}.$$

In light of $j = (j_1, j_2, \ldots, j_n)$ and $\mathfrak{A} = \{(i_1, i_2, \ldots, i_n) \mid (i_1, i_2, \ldots, i_n) \text{ is a list of all elements of } V \text{ (with no repetitions)}\}$, this rewrites as $j \in \mathfrak{A}$.

Now, forget that we fixed j. We thus have shown that $j \in \mathfrak{A}$ for each $j \in \mathfrak{B}$. In other words, $\mathfrak{B} \subseteq \mathfrak{A}$.

$\mathfrak{A} = \mathfrak{B}$. Now, we have the following equality of summation signs:

$$\begin{aligned}
&\sum_{\substack{(i_1,i_2,\ldots,i_n) \text{ is a list} \\ \text{of all elements of } V \\ \text{(with no repetitions)}}} \\
&= \sum_{(i_1,i_2,\ldots,i_n)\in\mathfrak{A}} \\
&\qquad \left(\begin{array}{c}\text{since} \\ \mathfrak{A} = \{(i_1,i_2,\ldots,i_n) \mid (i_1,i_2,\ldots,i_n) \text{ is a list of all elements of } V \text{ (with no repetitions)}\}\end{array}\right) \\
&= \sum_{(i_1,i_2,\ldots,i_n)\in\mathfrak{B}} \qquad (\text{since } \mathfrak{A} = \mathfrak{B}) \\
&= \sum_{\substack{(i_1,i_2,\ldots,i_n) \text{ is a list} \\ \text{of distinct elements of } V}} \\
&\qquad (\text{since } \mathfrak{B} = \{(i_1,i_2,\ldots,i_n) \mid (i_1,i_2,\ldots,i_n) \text{ is a list of distinct elements of } V\}) \\
&= \sum_{\substack{i_1,i_2,\ldots,i_n \text{ are} \\ \text{distinct} \\ \text{elements of } V}} . \qquad (46)
\end{aligned}$$

We have $X + (-X) = 0$. Thus, the element $X + (-X)$ belongs to the center of $\mathbb{L}$ (since the element 0 belongs to the center of $\mathbb{L}$). Hence, Theorem 2.2 (applied to

$Y = -X$) yields

$$\begin{aligned}
&\sum_{S\subseteq V}\left(X+\sum_{s\in S}x_s\right)^{|S|}\left(-X-\sum_{s\in S}x_s\right)^{n-|S|}\\
&=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}}\left(\underbrace{X+(-X)}_{=0}\right)^{n-k}x_{i_1}x_{i_2}\cdots x_{i_k}=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V}}0^{n-k}x_{i_1}x_{i_2}\cdots x_{i_k}\\
&=\underbrace{\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V;\\ n\leq k}}}_{\substack{=\sum\limits_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V;\\ k=n}}\\ \text{(by (45))}}}0^{n-k}x_{i_1}x_{i_2}\cdots x_{i_k}+\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V;\\ n>k}}\underbrace{0^{n-k}}_{\substack{=0\\ \text{(since } n-k>0\\ \text{(since } n>k))}}x_{i_1}x_{i_2}\cdots x_{i_k}\\
&\qquad\left(\begin{array}{c}\text{since each } k\text{-tuple } (i_1,i_2,\ldots,i_k) \text{ of distinct elements of } V\\ \text{satisfies either } n\leq k \text{ or } n>k \text{ (but not both)}\end{array}\right)\\
&=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V;\\ k=n}}0^{n-k}x_{i_1}x_{i_2}\cdots x_{i_k}+\underbrace{\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V;\\ n>k}}0x_{i_1}x_{i_2}\cdots x_{i_k}}_{=0}\\
&=\sum_{\substack{i_1,i_2,\ldots,i_k \text{ are}\\ \text{distinct}\\ \text{elements of } V;\\ k=n}}0^{n-k}x_{i_1}x_{i_2}\cdots x_{i_k}=\underbrace{\sum_{\substack{i_1,i_2,\ldots,i_n \text{ are}\\ \text{distinct}\\ \text{elements of } V}}}_{\substack{=\sum\limits_{\substack{(i_1,i_2,\ldots,i_n) \text{ is a list}\\ \text{of all elements of } V\\ \text{(with no repetitions)}}}\\ \text{(by (46))}}}\underbrace{0^{n-n}}_{=0^0=1}x_{i_1}x_{i_2}\cdots x_{i_n}\\
&=\sum_{\substack{(i_1,i_2,\ldots,i_n) \text{ is a list}\\ \text{of all elements of } V\\ \text{(with no repetitions)}}}x_{i_1}x_{i_2}\cdots x_{i_n}.
\end{aligned}$$

Comparing this with

$$\begin{aligned}
&\sum_{S\subseteq V}\left(X+\sum_{s\in S}x_s\right)^{|S|}\left(\underbrace{-X-\sum_{s\in S}x_s}_{=-\left(X+\sum_{s\in S}x_s\right)}\right)^{n-|S|}\\
&=\sum_{S\subseteq V}\left(X+\sum_{s\in S}x_s\right)^{|S|}\underbrace{\left(-\left(X+\sum_{s\in S}x_s\right)\right)^{n-|S|}}_{=(-1)^{n-|S|}\left(X+\sum_{s\in S}x_s\right)^{n-|S|}}\\
&=\sum_{S\subseteq V}\underbrace{\left(X+\sum_{s\in S}x_s\right)^{|S|}(-1)^{n-|S|}}_{=(-1)^{n-|S|}\left(X+\sum_{s\in S}x_s\right)^{|S|}}\left(X+\sum_{s\in S}x_s\right)^{n-|S|}\\
&=\sum_{S\subseteq V}(-1)^{n-|S|}\underbrace{\left(X+\sum_{s\in S}x_s\right)^{|S|}\left(X+\sum_{s\in S}x_s\right)^{n-|S|}}_{\substack{=\left(X+\sum_{s\in S}x_s\right)^{|S|+(n-|S|)}=\left(X+\sum_{s\in S}x_s\right)^{n}\\ \text{(since } |S|+(n-|S|)=n\text{)}}}\\
&=\sum_{S\subseteq V}(-1)^{n-|S|}\left(X+\sum_{s\in S}x_s\right)^{n},
\end{aligned}$$

we obtain

$$\sum_{S\subseteq V}(-1)^{n-|S|}\left(X+\sum_{s\in S}x_s\right)^{n}=\sum_{\substack{(i_1,i_2,\ldots,i_n)\text{ is a list}\\ \text{of all elements of } V\\ \text{(with no repetitions)}}}x_{i_1}x_{i_2}\cdots x_{i_n}.$$

This proves Corollary 4.1. □

Corollary 4.1 has a companion result:

Corollary 4.2. Let V be a finite set. Let $n=|V|$. For each $s\in V$, let x_s be an element of $\mathbb{L}$. Let $X\in\mathbb{L}$. Let $m\in\mathbb{N}$ be such that $m<n$. Then,

$$\sum_{S\subseteq V}(-1)^{n-|S|}\left(X+\sum_{s\in S}x_s\right)^{m}=0.$$

Proof of Corollary 4.2. For any subset W of V, we define an element $s(W) \in \mathbb{L}$ by

$$s(W) = \sum_{\substack{(i_1,i_2,\ldots,i_n) \text{ is a list} \\ \text{of all elements of } W \\ \text{(with no repetitions)}}} x_{i_1} x_{i_2} \cdots x_{i_n}. \tag{47}$$

If W is any subset of V, and if Y is any element of $\mathbb{L}$, then we define an element $r(Y, W) \in \mathbb{L}$ by

$$r(Y, W) = \sum_{S \subseteq W} (-1)^{|W|-|S|} \left(Y + \sum_{s \in S} x_s \right)^m. \tag{48}$$

Now, from Corollary 4.1, we can easily deduce the following claim:

Claim 1: Let W be any subset of V satisfying $|W| = m$. Let $Y \in \mathbb{L}$. Then, $r(Y, W) = s(W)$.

[*Proof of Claim 1:* We have $m = |W|$. Hence, Corollary 4.1 (applied to W, Y and m instead of V, X and n) yields

$$\sum_{S \subseteq W} (-1)^{m-|S|} \left(X + \sum_{s \in S} x_s \right)^m = \sum_{\substack{(i_1,i_2,\ldots,i_n) \text{ is a list} \\ \text{of all elements of } W \\ \text{(with no repetitions)}}} x_{i_1} x_{i_2} \cdots x_{i_n} = s(W)$$

(by (47)). Thus, (48) becomes

$$\begin{aligned} r(Y, W) &= \sum_{S \subseteq W} \underbrace{(-1)^{|W|-|S|}}_{\substack{=(-1)^{m-|S|} \\ \text{(since } |W|=m)}} \left(Y + \sum_{s \in S} x_s \right)^m \\ &= \sum_{S \subseteq W} (-1)^{m-|S|} \left(X + \sum_{s \in S} x_s \right)^m = s(W). \end{aligned}$$

This proves Claim 1.]

A more interesting claim is the following:

Claim 2: Let W be a subset of V. Let $t \in W$. Let $Y \in \mathbb{L}$. Then,

$$r(Y, W) = r(Y + x_t, W \setminus \{t\}) - r(Y, W \setminus \{t\}).$$

[*Proof of Claim 2:* Let us recall the following (well-known and simple) fact (which holds for any set W and any element $t \in W$): The map

$$\begin{aligned} \{S \subseteq W \mid t \notin S\} &\to \{S \subseteq W \mid t \in S\}, \\ S &\mapsto S \cup \{t\} \end{aligned} \tag{49}$$

is a bijection[20].

The definition of $r(Y, W \setminus \{t\})$ yields

$$r(Y, W \setminus \{t\}) = \underbrace{\sum_{S \subseteq W \setminus \{t\}}}_{\substack{= \sum_{\substack{S \subseteq W; \\ t \notin S}} \\ \text{(since the subsets of } W \setminus \{t\} \\ \text{are precisely the subsets } S \text{ of } W \\ \text{satisfying } t \notin S)}} \underbrace{(-1)^{|W \setminus \{t\}| - |S|}}_{\substack{= (-1)^{|W|-1-|S|} \\ \text{(since } |W \setminus \{t\}| = |W|-1 \\ \text{(since } t \in W))}} \left(Y + \sum_{s \in S} x_s\right)^m$$

$$= \sum_{\substack{S \subseteq W; \\ t \notin S}} (-1)^{|W|-1-|S|} \left(Y + \sum_{s \in S} x_s\right)^m . \tag{50}$$

The same argument (applied to $Y + x_t$ instead of Y) yields

$$r(Y + x_t, W \setminus \{t\}) = \sum_{\substack{S \subseteq W; \\ t \notin S}} (-1)^{|W|-1-|S|} \left(Y + x_t + \sum_{s \in S} x_s\right)^m . \tag{51}$$

But every subset S of W satisfying $t \notin S$ satisfies

$$|W| - 1 - |S| = |W| - |S \cup \{t\}| \tag{52}$$

[21] and

$$x_t + \sum_{s \in S} x_s = \sum_{s \in S \cup \{t\}} x_s \tag{53}$$

[20] Its inverse is the map

$$\{S \subseteq W \mid t \in S\} \to \{S \subseteq W \mid t \notin S\},$$
$$T \mapsto T \setminus \{t\}.$$

[21] *Proof of (52):* Let S be a subset of W satisfying $t \notin S$. From $t \notin S$, we obtain $|S \cup \{t\}| = |S| + 1$. Hence, $|W| - \underbrace{|S \cup \{t\}|}_{=|S|+1} = |W| - (|S| + 1) = |W| - 1 - |S|$. This proves (52).

[22]. Hence, (51) becomes

$$r\left(Y+x_t,W\setminus\{t\}\right)=\sum_{\substack{S\subseteq W;\\ t\notin S}}\underbrace{(-1)^{|W|-1-|S|}}_{\substack{=(-1)^{|W|-|S\cup\{t\}|}\\ \text{(since } |W|-1-|S|=|W|-|S\cup\{t\}| \\ \text{(by (52)))}}}\left(\underbrace{Y+x_t+\sum_{s\in S}x_s}_{\substack{=\sum_{s\in S\cup\{t\}}x_s\\ \text{(by (53))}}}\right)^m$$
$$=\sum_{\substack{S\subseteq W;\\ t\notin S}}(-1)^{|W|-|S\cup\{t\}|}\left(Y+\sum_{s\in S\cup\{t\}}x_s\right)^m. \tag{54}$$

[22]*Proof of (53):* Let S be a subset of W satisfying $t\notin S$. Then, $(S\cup\{t\})\setminus\{t\}=S$ (since $t\notin S$) and $t\in\{t\}\subseteq S\cup\{t\}$. Now,

$$\sum_{s\in S\cup\{t\}}x_s=x_t+\underbrace{\sum_{\substack{s\in S\cup\{t\};\\ s\neq t}}}_{\substack{=\sum_{s\in(S\cup\{t\})\setminus\{t\}}=\sum_{s\in S}\\ \text{(since } (S\cup\{t\})\setminus\{t\}=S)}}x_s$$
$$\left(\begin{array}{c}\text{here, we have split off the addend for } s=t\\ \text{from the sum (since } t\in S\cup\{t\}\text{)}\end{array}\right)$$
$$=x_t+\sum_{s\in S}x_s.$$

This proves (53).

But the definition of $r(Y,W)$ yields

$$r(Y,W) = \sum_{S\subseteq W} (-1)^{|W|-|S|} \left(Y + \sum_{s\in S} x_s\right)^m$$

$$= \underbrace{\sum_{\substack{S\subseteq W;\\ t\in S}} (-1)^{|W|-|S|} \left(Y + \sum_{s\in S} x_s\right)^m}_{\substack{= \sum\limits_{\substack{S\subseteq W;\\ t\notin S}} (-1)^{|W|-|S\cup\{t\}|} \left(Y + \sum\limits_{s\in S\cup\{t\}} x_s\right)^m \\ \text{(here, we have substituted } S\cup\{t\} \text{ for } S \text{ in the sum,} \\ \text{since the map (49) is a bijection)}}} + \sum_{\substack{S\subseteq W;\\ t\notin S}} \underbrace{(-1)^{|W|-|S|}}_{\substack{=-(-1)^{|W|-|S|-1} \\ =-(-1)^{|W|-1-|S|} \\ \text{(since } |W|-|S|-1=|W|-1-|S|)}} \left(Y + \sum_{s\in S} x_s\right)^m$$

$$\left(\begin{array}{c}\text{since each subset } S \text{ of } W \text{ satisfies either } t\in S \text{ or } t\notin S \\ \text{(but not both)}\end{array}\right)$$

$$= \sum_{\substack{S\subseteq W;\\ t\notin S}} (-1)^{|W|-|S\cup\{t\}|} \left(Y + \sum_{s\in S\cup\{t\}} x_s\right)^m + \underbrace{\sum_{\substack{S\subseteq W;\\ t\notin S}} \left(-(-1)^{|W|-1-|S|}\right) \left(Y + \sum_{s\in S} x_s\right)^m}_{= -\sum\limits_{\substack{S\subseteq W;\\ t\notin S}} (-1)^{|W|-1-|S|} \left(Y + \sum\limits_{s\in S} x_s\right)^m}$$

$$= \underbrace{\sum_{\substack{S\subseteq W;\\ t\notin S}} (-1)^{|W|-|S\cup\{t\}|} \left(Y + \sum_{s\in S\cup\{t\}} x_s\right)^m}_{\substack{= r(Y+x_t, W\setminus\{t\}) \\ \text{(by (54))}}} + \left(- \underbrace{\sum_{\substack{S\subseteq W;\\ t\notin S}} (-1)^{|W|-1-|S|} \left(Y + \sum_{s\in S} x_s\right)^m}_{\substack{= r(Y, W\setminus\{t\}) \\ \text{(by (50))}}}\right)$$

$$= r(Y+x_t, W\setminus\{t\}) + (-r(Y, W\setminus\{t\})) = r(Y+x_t, W\setminus\{t\}) - r(Y, W\setminus\{t\}).$$

This proves Claim 2.]

Now, the following is easy to show by induction:

> *Claim 3:* Let W be a subset of V satisfying $|W| > m$. Let $Y \in \mathbb{L}$. Then, $r(Y,W) = 0$.

[*Proof of Claim 3:* We shall prove Claim 3 by strong induction over $|W|$:

Induction step: Let $k \in \mathbb{N}$. Assume that Claim 3 is proven in the case when $|W| < k$. We must show that Claim 3 holds in the case when $|W| = k$.

We have assumed that Claim 3 is proven in the case when $|W| < k$. In other words,

$$\left(\begin{array}{c}\text{if } W \text{ is any subset of } V \text{ satisfying } |W| > m \text{ and } |W| < k, \\ \text{and if } Y \in \mathbb{L}, \text{ then } r(Y,W) = 0\end{array}\right). \tag{55}$$

Now, let W be any subset of V satisfying $|W| > m$ and $|W| = k$. Let $Y \in \mathbb{L}$. We shall show that $r(Y, W) = 0$.

We have $|W| > m \geq 0$ (since $m \in \mathbb{N}$). Hence, the set W is nonempty. In other words, there exists some $t \in W$. Consider this t. Clearly, $W \setminus \{t\}$ is a subset of V (since $W \setminus \{t\} \subseteq W \subseteq V$).

From $|W| > m$, we obtain $|W| \geq m + 1$ (since $|W|$ and m are integers).

From $t \in W$, we obtain $|W \setminus \{t\}| = \underbrace{|W|}_{\geq m+1} - 1 \geq (m+1) - 1 = m$. Thus, we are in one of the following two cases:

Case 1: We have $|W \setminus \{t\}| = m$.

Case 2: We have $|W \setminus \{t\}| > m$.

Let us first consider Case 1. In this case, we have $|W \setminus \{t\}| = m$. Hence, Claim 1 (applied to $W \setminus \{t\}$ instead of W) yields $r(Y, W \setminus \{t\}) = s(W \setminus \{t\})$. Also, Claim 1 (applied to $W \setminus \{t\}$ and $Y + x_t$ instead of W and Y) yields $r(Y + x_t, W \setminus \{t\}) = s(W \setminus \{t\})$. Now, Claim 2 yields

$$r(Y, W) = \underbrace{r(Y + x_t, W \setminus \{t\})}_{=s(W \setminus \{t\})} - \underbrace{r(Y, W \setminus \{t\})}_{=s(W \setminus \{t\})} = s(W \setminus \{t\}) - s(W \setminus \{t\}) = 0.$$

Thus, $r(Y, W) = 0$ is proven in Case 1.

Let us now consider Case 2. In this case, we have $|W \setminus \{t\}| > m$. Also, $|W \setminus \{t\}| = |W| - 1 < |W| = k$. Hence, (55) (applied to $W \setminus \{t\}$ instead of W) yields $r(Y, W \setminus \{t\}) = 0$. Also, (55) (applied to $W \setminus \{t\}$ and $Y + x_t$ instead of W and Y) yields $r(Y + x_t, W \setminus \{t\}) = 0$. Now, Claim 2 yields

$$r(Y, W) = \underbrace{r(Y + x_t, W \setminus \{t\})}_{=0} - \underbrace{r(Y, W \setminus \{t\})}_{=0} = 0 - 0 = 0.$$

Thus, $r(Y, W) = 0$ is proven in Case 2.

We have now proven $r(Y, W) = 0$ in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, we thus conclude that $r(Y, W) = 0$ always holds.

Now, let us forget that we fixed W and Y. We thus have proven that if W is any subset of V satisfying $|W| > m$ and $|W| = k$, and if $Y \in \mathbb{L}$, then $r(Y, W) = 0$. In other words, Claim 3 holds in the case when $|W| = k$. This completes the induction step. Thus, Claim 3 is proven by strong induction.]

Now, recall that V is a subset of V satisfying $|V| = n > m$ (since $m < n$). Hence, Claim 3 (applied to $W = V$ and $Y = X$) yields $r(X, V) = 0$. But the definition of $r(X, V)$ yields

$$r(X, V) = \sum_{S \subseteq V} \underbrace{(-1)^{|V| - |S|}}_{\substack{=(-1)^{n-|S|} \\ (\text{since } |V| = n)}} \left(X + \sum_{s \in S} x_s \right)^m = \sum_{S \subseteq V} (-1)^{n-|S|} \left(X + \sum_{s \in S} x_s \right)^m .$$

Hence,

$$\sum_{S \subseteq V} (-1)^{n-|S|} \left(X + \sum_{s \in S} x_s \right)^m = r(X, V) = 0.$$

This proves Corollary 4.2. □

5. Questions

The above results are not the first generalizations of the classical Abel-Hurwitz identities; there are various others. In particular, generalizations appear in [Strehl92], [Johns96], [Kalai79], [Pitman02], [Riorda68, §1.6] (see the end of [Grinbe09] for some of these) and [KelPos08]. We have not tried to lift these generalizations into our noncommutative setting, but we suspect that this is possible.

References

[Abel26] N. H. Abel, *Beweis eines Ausdrucks von welchem die Binomial-Formel ein einzelner Fall ist*, Journal für die reine und angewandte Mathematik **1** (1826), pp. 159–160.
`https://eudml.org/doc/183023`

[Comtet74] Louis Comtet, *Advanced Combinatorics*, Revised and enlarged edition, D. Reidel, 1974.

[Grinbe09] Darij Grinberg, *6th QEDMO 2009, Problem 4 (the Cauchy identity)*.
`http://www.cip.ifi.lmu.de/~grinberg/QEDMO6P4long.pdf`

[Grinbe17] Darij Grinberg, *MathOverflow post #273459 (answer to: Identity with binomial coefficients and k^k)*.
`https://mathoverflow.net/q/273459`

[Hurwit02] A. Hurwitz, *Über Abel's Verallgemeinerung der binomischen Formel*, Acta Math. **26** (1902), pp. 199–203.
`https://doi.org/10.1007/BF02415491`

[Johns96] Warren P. Johnson, *q-Extensions of identities of Abel-Rothe type*, Discrete Mathematics **159** (1996), pp. 161–177.
`https://doi.org/10.1016/0012-365X(95)00108-9`

[Kalai79] Gill Kalai, *A Note on an Evaluation of Abel Sums*, Journal of Combinatorial Theory, Series A, Volume 27, Issue 2, September 1979, pp. 213–217.
`https://doi.org/10.1016/0097-3165(79)90047-5`

[KelPos08] Alexander Kelmans, Alexander Postnikov, *Generalizations of Abel's and Hurwitz's identities*, European Journal of Combinatorics **29** (2008), pp. 1535–1543.
`https://doi.org/10.1016/j.ejc.2007.12.003`

[Knuth97] Donald Ervin Knuth, *The Art of Computer Programming, Volume 1 / Fundamental Algorithms*, 3rd edition, Addison Wesley 1997.

[Pitman02] Jim Pitman, *Forest Volume Decompositions and Abel-Cayley-Hurwitz Multinomial Expansions*, Journal of Combinatorial Theory, Series A **98**, Issue 1, April 2002, pp. 175–191.
`https://doi.org/10.1006/jcta.2001.3238`

[Riorda68] John Riordan, *Combinatorial Identities*, John Wiley & Sons, 1968.

[Roman84] Steven Roman, *The Umbral Calculus*, Academic Press 1984.

[SageMath] The Sage Developers, *SageMath, the Sage Mathematics Software System (Version 8.0)*, 2017.

[Stanle99] Richard P. Stanley, *Enumerative Combinatorics, volume 2*, Cambridge University Press 1999.

[Strehl92] Volker Strehl, *Identities of Rothe-Abel-Schläfli-Hurwitz-type*, Discrete Mathematics, Volume 99, Issues 1–3, 2 April 1992, pp. 321–340.
`https://doi.org/10.1016/0012-365X(92)90379-T`

[Zaimi17] Gjergji Zaimi, *MathOverflow post #275383 (Counting some binary trees with lots of extra stucture)*.
`https://mathoverflow.net/q/275383`