Sign functions for reduced expressions in Coxeter groups: proof of a conjecture of Bergeron, Ceballos and Labbé

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slides:
http://mit.edu/~darij/www/algebra/october06.pdf
paper: arXiv:1603.03138 or
http://mit.edu/~darij/www/algebra/bcl.pdf

A motivating example, part 1

- Fix a positive integer n.
- The symmetric group S_n can be presented by generators $s_1, s_2, \ldots, s_{n-1}$ and relations
  - $s_i^2 = 1$ (the *quadratic relations*);
  - $s_i s_j = s_j s_i$ whenever $|i - j| > 1$ (the *2-braid relations*);
  - $s_i s_j s_i = s_j s_i s_j$ whenever $|i - j| = 1$ (the *3-braid relations*).

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  (*Coxeter presentation*, aka *Moore presentation*).
- An *expression* for $w \in S_n$ is a way to write w as $s_{i_1} s_{i_2} \cdots s_{i_k}$.
- A *reduced expression* for $w \in S_n$ is an expression for w having minimum length (i.e., minimum k).
- **Example:** In S_5, the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ has reduced expressions

  $s_2 s_4 s_1 s_2$, $s_2 s_1 s_4 s_2$, $s_1 s_4 s_2 s_1$, and 5 others.

A motivating example, part 2

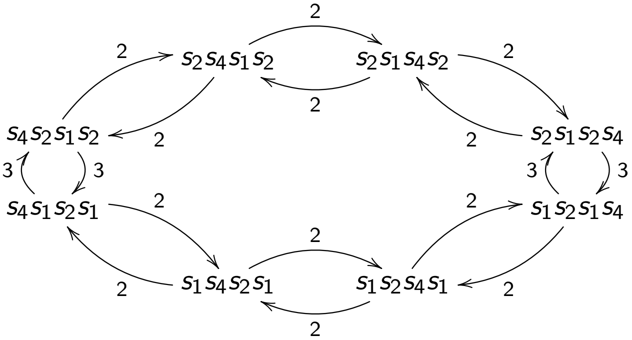
- Fix a positive integer n and a permutation $w \in S_n$.
- The braid relations give ways to transform reduced expressions into other reduced expressions:
  - $\cdots s_i s_j \cdots \mapsto \cdots s_j s_i \cdots$ for $|i-j| > 1$ (a *2-braid move*);
  - $\cdots s_i s_j s_i \cdots \mapsto \cdots s_j s_i s_j \cdots$ for $|i-j| = 1$ (a *3-braid move*).

  These are called *braid moves*.
- **Example:**

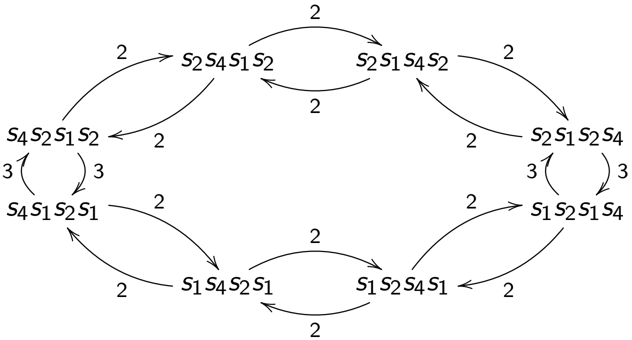
$$s_2 s_4 s_1 s_2 \xmapsto[\text{at positions 2 and 3}]{\text{2-braid move with } i=4 \text{ and } j=1} s_2 s_1 s_4 s_2.$$

- The natural thing to do: Define an edge-colored directed graph $\mathcal{R}_0(w)$ with
  - vertices = reduced expressions for w;
  - an arc going from one expression $\overrightarrow{a}$ to another expression $\overrightarrow{b}$ whenever a braid move takes $\overrightarrow{a}$ to $\overrightarrow{b}$;
  - color each arc with a 2 if we used a 2-braid move, and a 3 if we used a 3-braid move.

- **Example:** In S_5, the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ has the following $\mathcal{R}_0(w)$:
  (The number over any edge is its color.)



- What do we see on the example?



A motivating example, part 5

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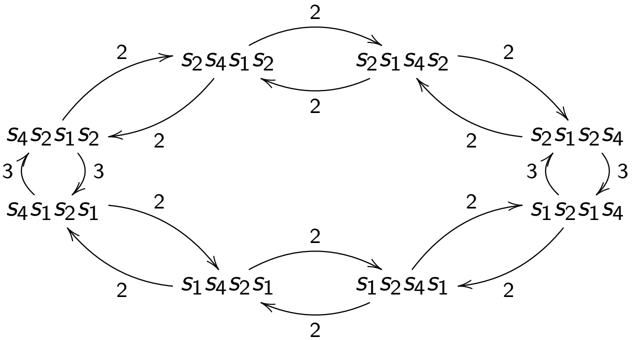

  - A single bidirected cycle.

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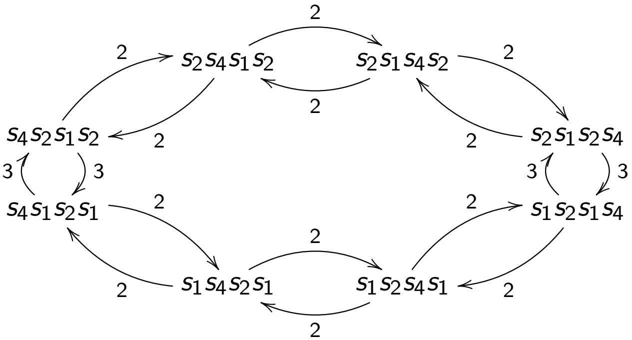
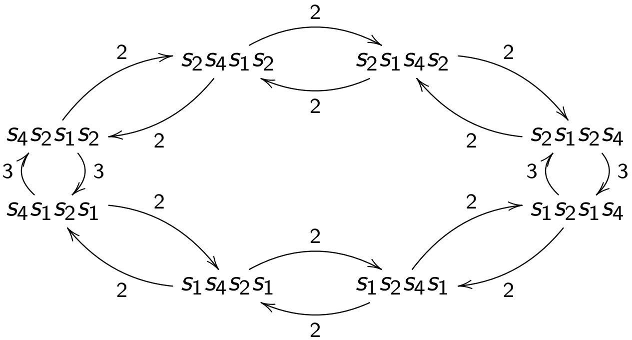


- A single bidirected cycle. (Does not generalize.)

- What do we see on the example?



- A single bidirected cycle. (Does not generalize.)
- Strongly connected.

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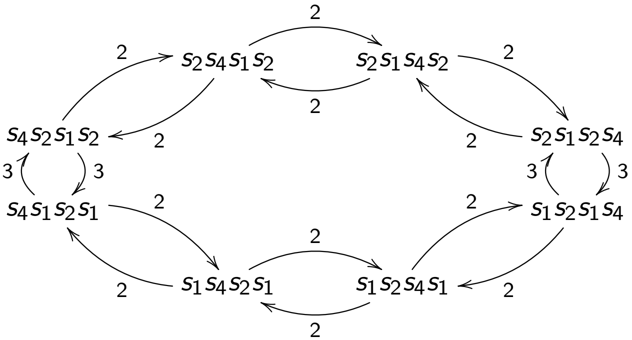
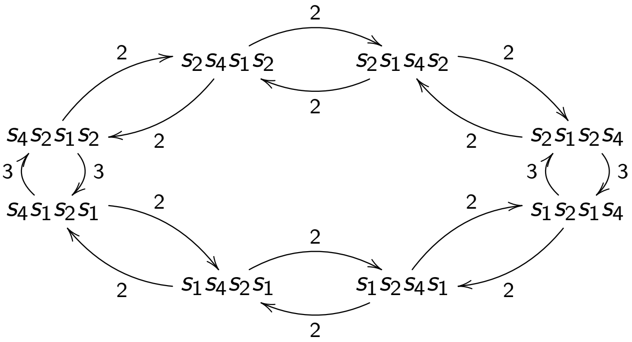


- A single bidirected cycle. (Does not generalize.)
- Strongly connected. (Generalizes to arbitrary Coxeter groups: Matsumoto-Tits theorem.)

- What do we see on the example?



- Walk down the long cycle counterclockwise.

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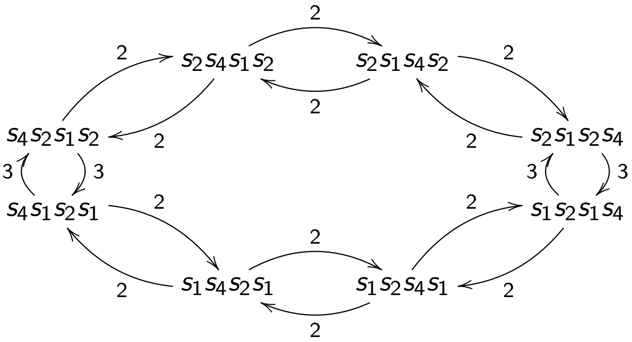

  - Walk down the long cycle counterclockwise.
    - The total number of 2-braid moves used is even.
    - The total number of 3-braid moves used is even.

A motivating example, part 7

- These latter observations do generalize:
  For any $n \geq 1$ and any $w \in S_n$, any directed cycle in $\mathcal{R}_0(w)$ uses an even number of 2-braid relations and an even number of 3-braid relations.
- This was found by Bergeron, Ceballos and Labbé (arXiv:1404.7380v2). Their proof used hyperplane arrangement geometry.

- Let (W, S) be a Coxeter group with Coxeter matrix $(m_{s,t})_{(s,t) \in S \times S}$.
- Set

$$\mathfrak{M} = \{(s, t) \in S \times S \mid s \neq t \text{ and } m_{s,t} < \infty\} .$$

- Recall that W has generators s (for $s \in S$) and relations
  - $s^2 = 1$ for all $s \in S$ (the *quadratic relations*);
  - $sts \cdots = tst \cdots$ (where both sides have $m_{s,t}$ factors) for all $(s, t) \in \mathfrak{M}$ (the *braid relations*).

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- An *expression* for $w \in W$ is a way to write w as a product $a_1 a_2 \cdots a_k$ where $a_1, a_2, \ldots, a_k \in S$.
- A *reduced expression* for $w \in W$ is an expression for w having minimum length (i.e., minimum k).

Coxeter groups: braid moves

- The braid relations give ways to transform reduced expressions into other reduced expressions:

  $$\cdots(sts\cdots)\cdots \mapsto \cdots(tst\cdots)\cdots$$

  (where both parenthesized products have $m_{s,t}$ factors) for $(s,t) \in \mathfrak{M}$.
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- We can again assemble the reduced expressions of a given $w \in W$ into an edge-colored directed graph.
  Examples: (courtesy Rob Edman, Victor Reiner)
  - H_3 (longest element);
  - B_3 (longest element);
  - A_3 (longest element).

  But we can do better than take $m_{s,t}$'s as colors.

- Define an equivalence relation $\sim$ ("simultaneous conjugation") on $\mathfrak{M}$ as follows:

$$(s, t) \sim (s', t')$$

$\iff$ there exists a $q \in W$ such that $qsq^{-1} = s'$ and $qtq^{-1} = t'$.

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- Define an edge-colored directed graph $\mathcal{R}(w)$ as follows:
  - vertices = reduced expressions for w;
  - an arc going from one expression $\overrightarrow{a}$ to another expression $\overrightarrow{b}$ whenever a braid move takes $\overrightarrow{a}$ to $\overrightarrow{b}$;
  - color each arc with the equivalence class $[(s,t)]$ if the braid move used was

$$\cdots(sts\cdots)\cdots \mapsto \cdots(tst\cdots)\cdots .$$

Main Theorem

- **Theorem (Postnikov, G.).** Let C be a directed cycle in the graph $\mathcal{R}(w)$ for some $w \in W$.
  Let $c \in \mathfrak{M}/\sim$ be an equivalence class (under simultaneous conjugation).
  Let c^{op} denote the equivalence class of the opposite pair (i.e., if $c = [(s,t)]$, then $c^{\mathrm{op}} = [(t,s)]$).

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$$\begin{aligned} & (\text{the number of arcs colored } c \text{ in } C) \\ = & (\text{the number of arcs colored } c^{\mathrm{op}} \text{ in } C). \end{aligned}$$

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- **Note:** Neither of **(a)** and **(b)** implies the other!
- Bergeron, Ceballos, Labbé proved a special case of **(b)**.

Strategy of the proof, part 1

- Let $T = \bigcup_{w \in W} wSw^{-1}$ (the set of *reflections* in W).
- Extend the relation $\sim$ to T (same definition).
- Every reduced expression $\overrightarrow{a} = a_1 a_2 \cdots a_k$ for w gives rise to a list ("inversion word", aka "reflection order")

$$\operatorname{Invs} \overrightarrow{a} = (t_1, t_2, \ldots, t_k) \in T^k, \qquad \text{where}$$

$$t_i = a_1 a_2 \cdots a_i \cdots a_2 a_1 \qquad (\text{"up to } a_i \text{ and then down"}).$$

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- If $\overrightarrow{a} \xrightarrow{\text{braid move involving } s \text{ and } t} \overrightarrow{b}$,

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  then $\operatorname{Invs} \overrightarrow{a} \xrightarrow{\text{revert a certain factor}} \operatorname{Invs} \overrightarrow{b}$.
- Which factor? Let's say the braid move replaces some $a_{i+1}a_{i+2}\cdots a_{i+k} = sts\cdots$ in $\overrightarrow{a}$ by $b_{i+1}b_{i+2}\cdots b_{i+k} = tst\cdots$.
  Then, the factor that gets reverted is in positions $i+1, i+2, \ldots, i+k$ again.

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  Then, the factor that gets reverted is in positions $i+1, i+2, \ldots, i+k$ again.
- The dihedral subgroup $\langle s, t\rangle$ has $m_{s,t}$ reflections, and two canonical ways to list them:

$$\rho_{s,t} = \left(s, sts, ststs, \cdots, \underbrace{sts\cdots s}_{2m_{s,t}-1 \text{ factors}}\right),$$

$$\rho_{t,s} = \left(t, tst, tstst, \cdots, \underbrace{tst\cdots t}_{2m_{s,t}-1 \text{ factors}}\right).$$

(These are mutually reverse.)

- If $\overrightarrow{a} \xrightarrow[\text{in positions } i+1, i+2, \ldots, i+k]{\text{braid move involving } s \text{ and } t} \overrightarrow{b}$,

  then $\operatorname{Invs} \overrightarrow{a} \xrightarrow[\text{in positions } i+1, i+2, \ldots, i+k]{\text{revert the word } q\rho_{s,t}q^{-1}} \operatorname{Invs} \overrightarrow{b}$, where $q = a_1 a_2 \cdots a_i$. (Words are conjugated letter-wise. Reverting $q\rho_{s,t}q^{-1}$ gives $q\rho_{t,s}q^{-1}$.)

- If $$\overrightarrow{a} \xrightarrow[\text{in positions } i+1, i+2, \ldots, i+k]{\text{braid move involving } s \text{ and } t} \overrightarrow{b},$$ then $$\operatorname{Invs} \overrightarrow{a} \xrightarrow[\text{in positions } i+1, i+2, \ldots, i+k]{\text{revert the word } q\rho_{s,t}q^{-1}} \operatorname{Invs} \overrightarrow{b},$$ where $q = a_1 a_2 \cdots a_i$. (Words are conjugated letter-wise. Reverting $q\rho_{s,t}q^{-1}$ gives $q\rho_{t,s}q^{-1}$.)
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- If $\overrightarrow{a} \xrightarrow[\text{in positions } i+1, i+2, \ldots, i+k]{\text{braid move involving } s \text{ and } t} \overrightarrow{b}$,

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- This is why we had to take $\sim$-conjugacy classes (and not plain pairs) as colors!
- Thus we can try a parity argument: Count how often a $q\rho_{s,t}q^{-1}$ appears as a subword in $\operatorname{Invs} \overrightarrow{a}$ (either never or once), and notice that our braid move changes this count by 1 mod 2.

Strategy of the proof, part 4

- Complications:
  - Be careful with redundant counts (counting everything twice makes mod 2 useless).
  - Subwords start out as factors, but can get broken apart by other braid moves.
  - Need to show that other braid moves never mutate our subword (even though they can spread its letters apart / move them together). Need some subtle descent/length/parabolic-coset arguments.
  - The $c = c^{\mathrm{op}}$ and $c \neq c^{\mathrm{op}}$ cases need separate proofs at the end.

See paper for details.

- What happens if we replace "reduced expression" by "expression" everywhere?
- **Conjecture 1.** Let C be a directed cycle in the graph $\mathcal{E}(w)$ (defined as $\mathcal{R}(w)$, but using all expressions) for some $w \in W$.
  Let $c \in \mathfrak{M}/\sim$ be an equivalence class (under simultaneous conjugation).
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Conjecture 2

- An attempt to explain at least part **(b)**...

- **Conjecture 2.** For every $(s,t) \in \mathfrak{M}$, let $c_{s,t} \in \{1,-1\}$.

$$\text{Assume that } c_{s,t} = c_{s',t'} \quad \text{whenever } (s,t) \sim (s',t')\,;$$
$$c_{s,t} = c_{t,s} \quad \text{for all } (s,t) \in \mathfrak{M}.$$

Let W' be the group given by:

  - *Generators:* the elements $s \in S$ and an extra generator q.
  - *Relations:*

$$\begin{aligned}
s^2 &= 1 && \text{for every } s \in S; \\
q^2 &= 1; && \\
qs &= sq && \text{for every } s \in S; \\
(st)^{m_{s,t}} &= 1 && \text{for every } (s,t) \in \mathfrak{M} \text{ satisfying } c_{s,t} = 1; \\
(st)^{m_{s,t}} &= q && \text{for every } (s,t) \in \mathfrak{M} \text{ satisfying } c_{s,t} = -1.
\end{aligned}$$

Then, $q \neq 1$ in W'. Equivalently, this sequence is exact:

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \xrightarrow{\overline{1} \mapsto q} W' \xrightarrow[q \mapsto 1]{s_i \mapsto s_i} W \longrightarrow 1$$

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Let W' be the group given by:

- *Generators:* the elements $s \in S$ and an extra generator q.
- *Relations:* (think spin symmetric groups!)

$$s^2 = 1 \qquad \text{for every } s \in S;$$
$$q^2 = 1;$$
$$qs = sq \qquad \text{for every } s \in S;$$
$$(st)^{m_{s,t}} = 1 \qquad \text{for every } (s,t) \in \mathfrak{M} \text{ satisfying } c_{s,t} = 1;$$
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- Darij Grinberg, Alexander Postnikov, *Proof of a conjecture of Bergeron, Ceballos and Labbé*, arXiv:1603.03138v2.
- Nantel Bergeron, Cesar Ceballos, Jean-Philippe Labbé, *Fan realizations of subword complexes and multi-associahedra via Gale duality*, arXiv:1404.7380v2.
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  Preprint: arXiv:0906.4768v3.

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