Petrie symmetric functions

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detailed version
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Abstract. For any positive integer k and nonnegative integer m, we consider the symmetric function $G(k,m)$ defined as the sum of all monomials of degree m that involve only exponents smaller than k. We call $G(k,m)$ a *Petrie symmetric function* in honor of Flinders Petrie, as the coefficients in its expansion in the Schur basis are determinants of Petrie matrices (and thus belong to $\{0,1,-1\}$ by a classical result of Gordon and Wilkinson). More generally, we prove a Pieri-like rule for expanding a product of the form $G(k,m) \cdot s_\mu$ in the Schur basis whenever μ is a partition; all coefficients in this expansion belong to $\{0,1,-1\}$. We also show that $G(k,1), G(k,2), G(k,3), \ldots$ form an algebraically independent generating set for the symmetric functions when $1-k$ is invertible in the base ring, and we prove a conjecture of Liu and Polo about the expansion of $G(k,2k-1)$ in the Schur basis.

Keywords: symmetric functions, Schur functions, Schur polynomials, combinatorial Hopf algebras, Petrie matrices, Pieri rules, Murnaghan–Nakayama rule.

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Contents

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Considered as a ring, the *symmetric functions* (which is short for "formal power series in countably many indeterminates $x_1, x_2, x_3, \ldots$ that are of bounded degree and fixed under permutations of the indeterminates") are hardly a remarkable object: By a classical result essentially known to Gauss, they form a polynomial ring in countably many indeterminates. The true theory of symmetric functions is rather the study of specific *families* of symmetric functions, often defined by combinatorial formulas (e.g., as multivariate generating functions) but interacting deeply with many other fields of mathematics. Classical families are, for example, the *monomial symmetric functions* m_λ, the *complete homogeneous symmetric functions* h_n, the *power-sum symmetric functions* p_n, and the *Schur functions* s_λ. Some of these families – such as the monomial symmetric functions m_λ and the Schur functions s_λ – form bases of the ring of symmetric functions (as a module over the base ring).

In this paper, we introduce a new family $(G(k,m))_{k\geq 1;\ m\geq 0}$ of symmetric functions, which we call the *Petrie symmetric functions* in honor of Flinders Petrie. For any integers $k \geq 1$ and $m \geq 0$, we define $G(k,m)$ as the sum of all monomials of degree m (in $x_1, x_2, x_3, \ldots$) that involve only exponents smaller than k. When $G(k,m)$ is expanded in the Schur basis (i.e., as a linear combination of Schur functions s_λ), all coefficients belong to $\{0, 1, -1\}$ by a classical result of Gordon and Wilkinson, as they are determinants of so-called *Petrie matrices* (whence our name for $G(k,m)$). We give an explicit combinatorial description for the coefficients as well. More generally, we prove a Pieri-like rule for expanding a product of the form $G(k,m) \cdot s_\mu$ in the Schur basis whenever μ is a partition; all coefficients in this expansion again belong to $\{0, 1, -1\}$ (although we have no explicit combinatorial rule for them). We show some further properties of $G(k,m)$ and prove that if k is a fixed positive integer such that $1-k$ is invertible in the base ring, then $G(k,1), G(k,2), G(k,3), \ldots$ form an algebraically independent generating set for the symmetric functions. We prove a conjecture of Liu and Polo in [LiuPol19, Remark 1.4.5] about the expansion of $G(k, 2k-1)$ in the Schur basis.

This paper begins with Section 1, in which we introduce the notions and notations that the paper will rely on. (Further notations will occasionally be introduced as the need arises.) The rest of the paper consists of two essentially independent parts. The first part comprises Section 2, in which we define the Petrie symmetric functions $G(k,m)$ (and the related power series $G(k)$) and state several of their properties, and Section 3, in which we prove said properties. The second part is Section 4, which is devoted to proving the conjecture of Liu and Polo.[1] A final Section 5 adds comments, formulates two conjectures, and (in its last subsection) explores a more general family of symmetric functions that still shares some of the properties of the Petrie functions $G(k,m)$. (As a byproduct of the latter generalization, a formula for the antipode of $G(k,m)$ – Corollary 5.37 – emerges.)

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[1] This proof is independent of the first part of the paper, except that it uses the very simple Proposition 2.3 **(c)**.

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Remarks

1. A short exposition of the main results of this paper (without proofs), along with an additional question motivated by it, can be found in [Grinbe20a].

2. While finishing this work, I have become aware of three independent discoveries of the Petrie symmetric functions $G(k,m)$:

- **(a)** In [DotWal92, §3.3], Stephen Doty and Grant Walker define a *modular complete symmetric function* h'_d, which is precisely our Petrie symmetric function $G(k,m)$ up to a renaming of variables (namely, their m and d correspond to our k and m). Some of our results appear in their work: Our Theorem 2.22 is (a slight generalization of) [DotWal92, Corollary 3.9]; our Theorem 2.29 is (part of) [DotWal92, Proposition 3.15] restated in the language of Hopf algebras. The h'_d are studied further in Walker's follow-up paper [Walker94], some of whose results mirror ours again (in particular, the maps ψ^p and ψ_p from [Walker94] are our $\mathbf{f}_p$ and $\mathbf{v}_p$).

- **(b)** The preprint [FuMei20] by Houshan Fu and Zhousheng Mei introduces the Petrie symmetric functions $G(k,m)$ and refers to them as *truncated homogeneous symmetric functions* $h_m^{[k-1]}$. Some results below are also independently obtained in [FuMei20]. In particular, Theorem 2.9 is a formula in [FuMei20, §2], and Theorem 2.15 is equivalent to [FuMei20, Proposition 2.9]. The particular case of Theorem 2.22 when $\mathbf{k} = \mathbb{Q}$ is part of [FuMei20, Theorem 2.7].

- **(c)** The paper [BaAhBe18] by Bazeniar, Ahmia and Belbachir introduces the symmetric functions $G(k,m)$ as well, or rather their evaluations $(G(k,m))(x_1, x_2, \ldots, x_n)$ at finitely many variables; it denotes them by $E_m^{(k-1)}(n) = E_m^{(k-1)}(x_1, x_2, \ldots, x_n)$. Ahmia and Merca continue the study of these $E_m^{(k-1)}(x_1, x_2, \ldots, x_n)$ in [AhmMer20]. Our Theorem 2.21 is equivalent to the second formula in [AhmMer20, Theorem 3.3] (although we are using infinitely many variables).

- **(d)** The formal power series $G(k)$ also appears in [FulLan85, Chapter I, §6], under the guise of *Bott's cannibalistic class* $\theta^j(e)$ (for $j = k$ and rewritten in the language of λ-ring operations[2]); it is used there to prove an abstract Riemann–Roch theorem. An application to group representations appears in [AtiTal69].

3. The Petrie symmetric functions have been added to Per Alexandersson's collection of symmetric functions at `https://www.symmetricfunctions.com/petrie.htm` .

Remark on alternative versions

You are reading the detailed version of this paper. For the standard version (which is shorter by virtue of omitting some straightforward or well-known proofs), see [Grinbe20b].

1. Notations

We will use the following notations (most of which are also used in [GriRei20, §2.1]):

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- The words "positive", "larger", etc. will be used in their Anglophone meaning (so that 0 is neither positive nor larger than itself).
- We fix a commutative ring $\mathbf{k}$; we will use this $\mathbf{k}$ as the base ring in what follows.
- A *weak composition* means an infinite sequence of nonnegative integers that contains only finitely many nonzero entries (i.e., a sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathbb{N}^\infty$ such that all but finitely many $i \in \{1, 2, 3, \ldots\}$ satisfy $\alpha_i = 0$).
- We let WC denote the set of all weak compositions.
- For any weak composition α and any positive integer i, we let α_i denote the i-th entry of α (so that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$). More generally, we use this notation whenever α is an infinite sequence of any kind of objects.
- The *size* $|\alpha|$ of a weak composition α is defined to be $\alpha_1 + \alpha_2 + \alpha_3 + \cdots \in \mathbb{N}$.

[2]See [Hazewi08, §16.74] for the connection between symmetric functions (over $\mathbb{Z}$) and universal operations on λ-rings. To be specific: If a is an element of a λ-ring A, then the canonical λ-ring morphism $\Lambda_{\mathbb{Z}} \to A$ (where $\Lambda_{\mathbb{Z}}$ is the ring of symmetric functions over $\mathbb{Z}$) that sends $e_1 = x_1 + x_2 + x_3 + \cdots \in \Lambda_{\mathbb{Z}}$ to $a \in A$ will send the Petrie symmetric function $G(k, m)$ to the "m-th graded component" of Bott's cannibalistic class $\theta^k(a)$. (Bott's cannibalistic class $\theta^k(a)$ itself is defined only if a is a "positive element" in the sense of [FulLan85] (or can only be defined in an appropriate closure of A). When it is defined, it is the image of the series $G(k)$. Otherwise, its "graded components" are the right object to consider.)

- A *partition* means a weak composition whose entries weakly decrease (i.e., a weak composition α satisfying $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$).

- If $n \in \mathbb{Z}$, then a *partition of* n means a partition α having size n (that is, satisfying $|\alpha| = n$).

- We let Par denote the set of all partitions. For each $n \in \mathbb{Z}$, we let Par_n denote the set of all partitions of n.

- We will sometimes omit trailing zeroes from partitions: i.e., a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ will be identified with the k-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ whenever $k \in \mathbb{N}$ satisfies $\lambda_{k+1} = \lambda_{k+2} = \lambda_{k+3} = \cdots = 0$. For example, $(3, 2, 1, 0, 0, 0, \ldots) = (3, 2, 1) = (3, 2, 1, 0)$.

- The partition $(0, 0, 0, \ldots) = ()$ is called the *empty partition* and denoted by $\varnothing$.

- A *part* of a partition λ means a nonzero entry of λ. For example, the parts of the partition $(3, 1, 1) = (3, 1, 1, 0, 0, 0, \ldots)$ are $3, 1, 1$.

- We will use the notation 1^k for "$\underbrace{1, 1, \ldots, 1}_{k \text{ times}}$" in partitions. (For example, $\left(2, 1^4\right) = (2, 1, 1, 1, 1)$. This notation is a particular case of the more general notation m^k for "$\underbrace{m, m, \ldots, m}_{k \text{ times}}$" in partitions, used, e.g., in [GriRei20, Definition 2.2.1].)

- We let Λ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, x_3, \ldots$ over $\mathbf{k}$. This is a subring of the ring $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ of formal power series. To be more specific, Λ consists of all power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ that are symmetric (i.e., invariant under permutations of the variables) and of bounded degree (see [GriRei20, §2.1] for the precise meaning of this).

- A *monomial* shall mean a formal expression of the form $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$ with $\alpha \in \mathrm{WC}$. Formal power series are formal infinite $\mathbf{k}$-linear combinations of such monomials.

- For any weak composition α, we let $\mathbf{x}^\alpha$ denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots$.

- The *degree* of a monomial $\mathbf{x}^\alpha$ is defined to be $|\alpha|$.

- A formal power series is said to be *homogeneous of degree* n (for some $n \in \mathbb{N}$) if all monomials appearing in it (with nonzero coefficient) have degree n. In particular, the power series 0 is homogeneous of any degree.

- If $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is a power series, then there is a unique family $(f_i)_{i \in \mathbb{N}} = (f_0, f_1, f_2, \ldots)$ of formal power series $f_i \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ such that each f_i is homogeneous of degree i and such that $f = \sum_{i \in \mathbb{N}} f_i$. This family $(f_i)_{i \in \mathbb{N}}$ is called the *homogeneous decomposition* of f, and its entry f_i (for any given $i \in \mathbb{N}$) is called the *i-th degree homogeneous component* of f.

- The $\mathbf{k}$-algebra Λ is graded: i.e., any symmetric function f can be uniquely written as a sum $\sum_{i \in \mathbb{N}} f_i$, where each f_i is a homogeneous symmetric function of degree i, and where all but finitely many $i \in \mathbb{N}$ satisfy $f_i = 0$.

We shall use the symmetric functions $m_\lambda, h_n, e_n, p_n, s_\lambda$ in Λ as defined in [GriRei20, Sections 2.1 and 2.2]. Let us briefly recall how they are defined:

- For any partition λ, we define the *monomial symmetric function* $m_\lambda \in \Lambda$ by[3]
$$m_\lambda = \sum \mathbf{x}^\alpha,$$
where the sum ranges over all weak compositions $\alpha \in \mathrm{WC}$ that can be obtained from λ by permuting entries[4]. For example,
$$m_{(2,2,1)} = \sum_{i<j<k} x_i^2 x_j^2 x_k + \sum_{i<j<k} x_i^2 x_j x_k^2 + \sum_{i<j<k} x_i x_j^2 x_k^2.$$
The family $(m_\lambda)_{\lambda \in \mathrm{Par}}$ (that is, the family of the symmetric functions m_λ as λ ranges over all partitions) is a basis of the $\mathbf{k}$-module Λ.

- For each $n \in \mathbb{Z}$, we define the *complete homogeneous symmetric function* $h_n \in \Lambda$ by
$$h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = n}} \mathbf{x}^\alpha = \sum_{\lambda \in \mathrm{Par}_n} m_\lambda.$$
Thus, $h_0 = 1$ and $h_n = 0$ for all $n < 0$.
We know (e.g., from [GriRei20, Proposition 2.4.1]) that the family $(h_n)_{n \geq 1} = (h_1, h_2, h_3, \ldots)$ is algebraically independent and generates Λ as a $\mathbf{k}$-algebra. In other words, Λ is freely generated by $h_1, h_2, h_3, \ldots$ as a commutative $\mathbf{k}$-algebra.

- For each $n \in \mathbb{Z}$, we define the *elementary symmetric function* $e_n \in \Lambda$ by
$$e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\substack{\alpha \in \mathrm{WC} \cap \{0,1\}^\infty; \\ |\alpha| = n}} \mathbf{x}^\alpha.$$
Thus, $e_0 = 1$ and $e_n = 0$ for all $n < 0$. If $n \geq 0$, then $e_n = m_{(1^n)}$, where, as we have agreed above, the notation (1^n) stands for the n-tuple $(1, 1, \ldots, 1)$.

[3] This definition of m_λ is not the same as the one given in [GriRei20, Definition 2.1.3]; but it is easily seen to be equivalent to the latter (i.e., it defines the same m_λ). See Subsection 3.1 below (and the proof of Proposition 3.3 in particular) for the details.

[4] Here, we understand λ to be an infinite sequence, not a finite tuple, so the entries being permuted include infinitely many 0's.

- For each positive integer n, we define the *power-sum symmetric function* $p_n \in \Lambda$ by
$$p_n = x_1^n + x_2^n + x_3^n + \cdots = m_{(n)}.$$
- For each partition λ, we define the *Schur function* $s_\lambda \in \Lambda$ by
$$s_\lambda = \sum \mathbf{x}_T,$$
where the sum ranges over all semistandard tableaux T of shape λ, and where $\mathbf{x}_T$ denotes the monomial obtained by multiplying the x_i for all entries i of T. We refer the reader to [GriRei20, Definition 2.2.1] or to [Stanle01, §7.10] for the details of this definition and further descriptions of the Schur functions. One of the most important properties of Schur functions (see, e.g., [GriRei20, (2.4.16) for $\mu = \varnothing$] or [MenRem15, Theorem 2.32] or [Stanle01, Theorem 7.16.1 for $\mu = \varnothing$] or [Sagan20, Theorem 7.2.3 (a)]) is the fact that
$$s_\lambda = \det\left(\left(h_{\lambda_i - i + j}\right)_{1 \le i \le \ell,\ 1 \le j \le \ell}\right) \tag{1}$$
for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. This is known as the *(first, straight-shape) Jacobi–Trudi formula*.

  The family $(s_\lambda)_{\lambda \in \mathrm{Par}}$ is a basis of the $\mathbf{k}$-module Λ, and is known as the *Schur basis*. It is easy to see that each $n \in \mathbb{N}$ satisfies $s_{(n)} = h_n$ and $s_{(1^n)} = e_n$. Moreover, for each partition λ, the Schur function $s_\lambda \in \Lambda$ is homogeneous of degree $|\lambda|$.

Among the many relations between these symmetric functions is an expression for the power-sum symmetric function p_n in terms of the Schur basis:

Proposition 1.1. Let n be a positive integer. Then,
$$p_n = \sum_{i=0}^{n-1} (-1)^i \, s_{\left(n-i, 1^i\right)}.$$

Proof. This is a classical formula, and appears (e.g.) in [Egge19, Problem 4.21], [GriRei20, Exercise 5.4.12(g)] and [MenRem15, Exercise 2.2]. Alternatively, this is an easy consequence of the Murnaghan–Nakayama rule (see [MenRem15, Theorem 6.3] or [Sam17, Theorem 4.4.2] or [Stanle01, Theorem 7.17.3] or [Wildon15, (1)]), applied to the product $p_n s_\varnothing$ (since $s_\varnothing = 1$). □

Finally, we will sometimes use the *Hall inner product* $\langle \cdot, \cdot \rangle : \Lambda \times \Lambda \to \mathbf{k}$ as defined in [GriRei20, Definition 2.5.12].[5] This is the $\mathbf{k}$-bilinear form on Λ that is defined by

[5]However, it is denoted by $(\cdot, \cdot)$ rather than by $\langle \cdot, \cdot \rangle$ in [GriRei20]. (That is, what we call $\langle a, b \rangle$ is denoted by (a, b) in [GriRei20].)

The Hall inner product also appears (for $\mathbf{k} = \mathbb{Z}$ and $\mathbf{k} = \mathbb{Q}$) in [Egge19, Definition 7.5], in [Stanle01, §7.9] and in [Macdon95, Section I.4] (note that it is called the "scalar product" in the latter two references). The definitions of the Hall inner product in [Stanle01, §7.9] and in [Macdon95, Section I.4] are different from ours, but they are equivalent to ours (because of [Stanle01, Corollary 7.12.2] and [Macdon95, Chapter I, (4.8)]).

the requirement that

$$\left\langle s_\lambda, s_\mu \right\rangle = \delta_{\lambda,\mu} \qquad \text{for any } \lambda, \mu \in \operatorname{Par}$$

(where $\delta_{\lambda,\mu}$ denotes the Kronecker delta). Thus, the Schur basis $(s_\lambda)_{\lambda\in\operatorname{Par}}$ of Λ is an orthonormal basis with respect to the Hall inner product. It is easy to see[6] that the Hall inner product $(\cdot,\cdot)$ is graded: i.e., we have

$$\langle f, g\rangle = 0 \tag{2}$$

if f and g are two homogeneous symmetric functions of different degrees. We shall also use the following two known evaluations of the Hall inner product:

Proposition 1.2. Let n be a positive integer. Then, $\langle h_n, p_n\rangle = 1$.

Proposition 1.3. Let n be a positive integer. Then, $\langle e_n, p_n\rangle = (-1)^{n-1}$.

See Subsection 3.3 for the proofs of these two propositions.

2. Theorems

2.1. Definitions

The main role in this paper is played by two power series that we will now define:

Definition 2.1. **(a)** For any positive integer k, we let[7]

$$G(k) = \sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha. \tag{3}$$

This is a symmetric formal power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$ (but does not belong to Λ in general).

(b) For any positive integer k and any $m \in \mathbb{N}$, we let

$$G(k,m) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha \in \Lambda. \tag{4}$$

[6]See, e.g., [GriRei20, Exercise 2.5.13(a)] for a proof.

[7]Here and in all similar situations, "for all i" means "for all positive integers i".

Example 2.2. **(a)** We have

$$\begin{aligned} G(2) &= \sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha_i < 2 \text{ for all } i}} \mathbf{x}^\alpha \\ &= 1 + x_1 + x_2 + x_3 + \cdots + x_1x_2 + x_1x_3 + x_2x_3 + \cdots \\ &\qquad + x_1x_2x_3 + x_1x_2x_4 + x_2x_3x_4 + \cdots \\ &\qquad + \cdots \\ &= \sum_{m \in \mathbb{N}} \underbrace{\sum_{1 \le i_1 < i_2 < \cdots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m}}_{= e_m} = \sum_{m \in \mathbb{N}} e_m. \end{aligned}$$

(b) For each $m \in \mathbb{N}$, we have

$$G(2, m) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i < 2 \text{ for all } i}} \mathbf{x}^\alpha = \sum_{1 \le i_1 < i_2 < \cdots < i_m} x_{i_1} x_{i_2} \cdots x_{i_m} = e_m.$$

We suggest the name *k-Petrie symmetric series* for $G(k)$ and the name (k, m)*-Petrie symmetric function* for $G(k, m)$. The reason for this naming is that the coefficients of these functions in the Schur basis of Λ are determinants of Petrie matrices, as we will see in Subsection 3.9.

2.2. Basic identities

We begin our study of the $G(k)$ and $G(k, m)$ with some simple properties:

Proposition 2.3. Let k be a positive integer.

(a) The symmetric function $G(k, m)$ is the m-th degree homogeneous component of $G(k)$ for each $m \in \mathbb{N}$.

(b) We have

$$G(k) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \mathrm{Par}; \\ \lambda_i < k \text{ for all } i}} m_\lambda = \prod_{i=1}^{\infty} \left(x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right).$$

(c) We have

$$G(k, m) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = m; \\ \lambda_i < k \text{ for all } i}} m_\lambda$$

for each $m \in \mathbb{N}$.

(d) If $m \in \mathbb{N}$ satisfies $k > m$, then $G(k, m) = h_m$.

(e) If $m \in \mathbb{N}$ and $k = 2$, then $G(k, m) = e_m$.

(f) If $m = k$, then $G(k, m) = h_m - p_m$.

We shall prove Proposition 2.3 in Subsection 3.4 below. (The easy proof is good practice in understanding the definitions of m_λ, h_n, e_n, p_n, $G(k)$ and $G(k,n)$.)

Parts **(d)** and **(e)** of Proposition 2.3 show that the Petrie symmetric functions $G(k,m)$ can be seen as interpolating between the h_m and the e_m.

2.3. The Schur expansion

The solution to [Stanle01, Exercise 7.3] gives an expansion of $G(3)$ in terms of the elementary symmetric functions (due to I. M. Gessel); this expansion can be rewritten as

$$G(3) = \sum_{n\in\mathbb{N}} e_n^2 + \sum_{m<n} c_{m,n} e_m e_n, \qquad \text{where } c_{m,n} = (-1)^{m-n} \begin{cases} 2, & \text{if } 3 \mid m-n; \\ -1, & \text{if } 3 \nmid m-n \end{cases} .$$

We shall instead expand $G(k)$ in terms of Schur functions. For this, we need to define some notations.

Convention 2.4. We shall use the *Iverson bracket notation*: i.e., if $\mathcal{A}$ is a logical statement, then $[\mathcal{A}]$ shall denote the truth value of $\mathcal{A}$ (that is, the integer $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false} \end{cases}$).

We shall furthermore use the notation $(a_{i,j})_{1\le i\le \ell,\ 1\le j\le \ell}$ for the $\ell \times \ell$-matrix whose (i,j)-th entry is $a_{i,j}$ for each $i,j \in \{1,2,\ldots,\ell\}$.

Definition 2.5. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \operatorname{Par}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \operatorname{Par}$, and let k be a positive integer. Then, the *k-Petrie number* $\operatorname{pet}_k(\lambda,\mu)$ of λ and μ is the integer defined by

$$\operatorname{pet}_k(\lambda,\mu) = \det\left(\left([0 \le \lambda_i - \mu_j - i + j < k] \right)_{1\le i\le \ell,\ 1\le j\le \ell} \right).$$

Note that this integer does not depend on the choice of ℓ (in the sense that it does not change if we enlarge ℓ by adding trailing zeroes to the representations of λ and μ); this follows from Lemma 2.7 below.

Example 2.6. Let λ be the partition $(3,2,1) \in \operatorname{Par}$, let μ be the partition $(1,1) \in \operatorname{Par}$, let $\ell = 3$, and let k be a positive integer. Then, the definition of $\operatorname{pet}_k(\lambda,\mu)$

yields

$$\begin{aligned}
&\operatorname{pet}_k(\lambda,\mu) \\
&= \det\left(\left([0 \leq \lambda_i - \mu_j - i + j < k]\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right) \\
&= \det\begin{pmatrix} [0 \leq \lambda_1 - \mu_1 < k] & [0 \leq \lambda_1 - \mu_2 + 1 < k] & [0 \leq \lambda_1 - \mu_3 + 2 < k] \\ [0 \leq \lambda_2 - \mu_1 - 1 < k] & [0 \leq \lambda_2 - \mu_2 < k] & [0 \leq \lambda_2 - \mu_3 + 1 < k] \\ [0 \leq \lambda_3 - \mu_1 - 2 < k] & [0 \leq \lambda_3 - \mu_2 - 1 < k] & [0 \leq \lambda_3 - \mu_3 < k] \end{pmatrix} \\
&= \det\begin{pmatrix} [0 \leq 3 - 1 < k] & [0 \leq 3 - 1 + 1 < k] & [0 \leq 3 - 0 + 2 < k] \\ [0 \leq 2 - 1 - 1 < k] & [0 \leq 2 - 1 < k] & [0 \leq 2 - 0 + 1 < k] \\ [0 \leq 1 - 1 - 2 < k] & [0 \leq 1 - 1 - 1 < k] & [0 \leq 1 - 0 < k] \end{pmatrix} \\
&\qquad\qquad \left(\begin{array}{c} \text{since } \lambda_1 = 3 \text{ and } \lambda_2 = 2 \text{ and } \lambda_3 = 1 \\ \text{and } \mu_1 = 1 \text{ and } \mu_2 = 1 \text{ and } \mu_3 = 0 \end{array}\right) \\
&= \det\begin{pmatrix} [0 \leq 2 < k] & [0 \leq 3 < k] & [0 \leq 5 < k] \\ [0 \leq 0 < k] & [0 \leq 1 < k] & [0 \leq 3 < k] \\ [0 \leq -2 < k] & [0 \leq -1 < k] & [0 \leq 1 < k] \end{pmatrix}.
\end{aligned}$$

Thus, taking $k = 4$, we obtain

$$\begin{aligned}
\operatorname{pet}_4(\lambda,\mu) &= \det\begin{pmatrix} [0 \leq 2 < 4] & [0 \leq 3 < 4] & [0 \leq 5 < 4] \\ [0 \leq 0 < 4] & [0 \leq 1 < 4] & [0 \leq 3 < 4] \\ [0 \leq -2 < 4] & [0 \leq -1 < 4] & [0 \leq 1 < 4] \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = 0.
\end{aligned}$$

On the other hand, taking $k = 3$, we obtain

$$\begin{aligned}
\operatorname{pet}_3(\lambda,\mu) &= \det\begin{pmatrix} [0 \leq 2 < 3] & [0 \leq 3 < 3] & [0 \leq 5 < 3] \\ [0 \leq 0 < 3] & [0 \leq 1 < 3] & [0 \leq 3 < 3] \\ [0 \leq -2 < 3] & [0 \leq -1 < 3] & [0 \leq 1 < 3] \end{pmatrix} \\
&= \det\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1.
\end{aligned}$$

Lemma 2.7. Let $\lambda \in \operatorname{Par}$ and $\mu \in \operatorname{Par}$, and let k be a positive integer. Let $\ell \in \mathbb{N}$ be such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$. Then, the determinant $\det\left(\left([0 \leq \lambda_i - \mu_j - i + j < k]\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right)$ does not depend on the choice of ℓ.

See Subsection 3.8 for the simple proof of Lemma 2.7.

Surprisingly, the k-Petrie numbers $\operatorname{pet}_k(\lambda,\mu)$ can take only three possible values:

Proposition 2.8. Let $\lambda \in \operatorname{Par}$ and $\mu \in \operatorname{Par}$, and let k be a positive integer. Then, $\operatorname{pet}_k(\lambda, \mu) \in \{-1, 0, 1\}$.

Proposition 2.8 will be proved in Subsection 3.9.

We can now expand the Petrie symmetric functions $G(k, m)$ and the power series $G(k)$ in the basis $(s_\lambda)_{\lambda \in \operatorname{Par}}$ of Λ:

Theorem 2.9. Let k be a positive integer. Then,

$$G(k) = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \varnothing) s_\lambda.$$

(Recall that $\varnothing$ denotes the empty partition $() = (0, 0, 0, \ldots)$.)

We will not prove Theorem 2.9 directly; instead, we will first show a stronger result (Theorem 2.17), and then derive Theorem 2.9 from it in Subsection 3.12.

Corollary 2.10. Let k be a positive integer. Let $m \in \mathbb{N}$. Then,

$$G(k, m) = \sum_{\lambda \in \operatorname{Par}_m} \operatorname{pet}_k(\lambda, \varnothing) s_\lambda.$$

Corollary 2.10 easily follows from Theorem 2.9 using Proposition 2.3 **(a)**; but again, we shall instead derive it from a stronger result (Corollary 2.18) in Subsection 3.12.

We will see a more explicit description of the k-Petrie numbers $\operatorname{pet}_k(\lambda, \varnothing)$ in Subsection 2.4.

Remark 2.11. Corollary 2.10, in combination with Proposition 2.8, shows that each k-Petrie function $G(k, m)$ (for any $k > 0$ and $m \in \mathbb{N}$) is a linear combination of Schur functions, with all coefficients belonging to $\{-1, 0, 1\}$. It is natural to expect the more general symmetric functions

$$\widetilde{G}(k, k', m) = \sum_{\substack{\alpha \in \operatorname{WC}; \\ |\alpha| = m; \\ k' \leq \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha, \qquad \text{where } 0 < k' \leq k,$$

to have the same property. However, this is not the case. For example,

$$\widetilde{G}(4, 2, 5) = m_{(3,2)} = -2s_{(1,1,1,1,1)} + 2s_{(2,1,1,1)} - s_{(2,2,1)} - s_{(3,1,1)} + s_{(3,2)}.$$

2.4. An explicit description of the k-Petrie numbers $\operatorname{pet}_k(\lambda, \varnothing)$

Can the k-Petrie numbers $\operatorname{pet}_k(\lambda, \varnothing)$ from Definition 2.5 be described more explicitly than as determinants? To be somewhat pedantic, the answer to this question depends on one's notion of "explicit", as determinants are not hard to compute, and another algorithm for calculating $\operatorname{pet}_k(\lambda, \varnothing)$ can be extracted from our proof of Proposition 2.8 (when combined with [GorWil74, proof of Theorem 1]). Nevertheless, there is a more explicit description. This description will be stated in Theorem 2.15 further below.

First, let us get a simple case out of the way:

Proposition 2.12. Let $\lambda \in \operatorname{Par}$, and let k be a positive integer such that $\lambda_1 \geq k$. Then, $\operatorname{pet}_k(\lambda, \varnothing) = 0$.

Proof of Proposition 2.12. Write λ as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Thus, $\ell \geq 1$ (since $\lambda_1 \geq k > 0$). Moreover, the empty partition $\varnothing$ can be written as $\varnothing = (\varnothing_1, \varnothing_2, \ldots, \varnothing_\ell)$ (since $\varnothing_i = 0$ for each integer $i > \ell$).

Thus, we have $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\varnothing = (\varnothing_1, \varnothing_2, \ldots, \varnothing_\ell)$. Hence, the definition of $\operatorname{pet}_k(\lambda, \varnothing)$ yields

$$\begin{aligned}\operatorname{pet}_k(\lambda, \varnothing) &= \det\left(\left(\left[0 \leq \lambda_i - \underbrace{\varnothing_j}_{=0} - i + j < k\right]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right) \\ &= \det\left(\left([0 \leq \lambda_i - i + j < k]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right). \qquad (5)\end{aligned}$$

But each $j \in \{1, 2, \ldots, \ell\}$ satisfies $[0 \leq \lambda_1 - 1 + j < k] = 0$ (since $\lambda_1 - 1 + \underbrace{j}_{\geq 1} \geq \lambda_1 - 1 + 1 = \lambda_1 \geq k$). In other words, the $\ell \times \ell$-matrix $\left([0 \leq \lambda_i - i + j < k]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}$ has first row $(0, 0, \ldots, 0)$. Therefore, its determinant is 0. In other words, $\operatorname{pet}_k(\lambda, \varnothing) = 0$ (since $\operatorname{pet}_k(\lambda, \varnothing)$ is its determinant[8]). This proves Proposition 2.12. $\square$

Stating Theorem 2.15 will require some notation:

Definition 2.13. For any $\lambda \in \operatorname{Par}$, we define the *transpose* of λ to be the partition $\lambda^t \in \operatorname{Par}$ determined by

$$\left(\lambda^t\right)_i = \left|\left\{j \in \{1, 2, 3, \ldots\} \mid \lambda_j \geq i\right\}\right| \qquad \text{for each } i \geq 1.$$

This partition λ^t is also known as the *conjugate* of λ, and is perhaps easiest to understand in terms of Young diagrams: To wit, the Young diagram of λ^t is obtained from that of λ by a flip across the main diagonal.

One important use of transpose partitions is the following fact (see, e.g., [GriRei20, (2.4.17) for $\mu = \varnothing$] or [MenRem15, Theorem 2.32] or [Stanle01, Theorem 7.16.2 applied to λ^t and $\varnothing$ instead of λ and μ] for proofs): We have

$$s_{\lambda^t} = \det\left(\left(e_{\lambda_i - i + j}\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right) \qquad (6)$$

[8]by (5)

for any partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. This is known as the *(second, straight-shape) Jacobi–Trudi formula*.

We will use the following notation for quotients and remainders:

Convention 2.14. Let k be a positive integer. Let $n \in \mathbb{Z}$. Then, $n\%k$ shall denote the remainder of n divided by k, whereas $n//k$ shall denote the quotient of this division (an integer). Thus, $n//k$ and $n\%k$ are uniquely determined by the three requirements that $n//k \in \mathbb{Z}$ and $n\%k \in \{0, 1, \ldots, k-1\}$ and $n = (n//k) \cdot k + (n\%k)$.

The "$//$" and "$\%$" signs bind more strongly than the "$+$" and "$-$" signs. That is, for example, the expression "$a + b\%k$" shall be understood to mean "$a + (b\%k)$" rather than "$(a+b)\%k$".

Now, we can state our "formula" for k-Petrie numbers of the form $\operatorname{pet}_k(\lambda, \varnothing)$.

Theorem 2.15. Let $\lambda \in \operatorname{Par}$, and let k be a positive integer. Let $\mu = \lambda^t$.

(a) If $\mu_k \neq 0$, then $\operatorname{pet}_k(\lambda, \varnothing) = 0$.

From now on, let us assume that $\mu_k = 0$.

Define a $(k-1)$-tuple $(\beta_1, \beta_2, \ldots, \beta_{k-1}) \in \mathbb{Z}^{k-1}$ by setting

$$\beta_i = \mu_i - i \qquad \text{for each } i \in \{1, 2, \ldots, k-1\}. \tag{7}$$

Define a $(k-1)$-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}) \in \{1, 2, \ldots, k\}^{k-1}$ by setting

$$\gamma_i = 1 + (\beta_i - 1)\%k \qquad \text{for each } i \in \{1, 2, \ldots, k-1\}. \tag{8}$$

(b) If the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are not distinct, then $\operatorname{pet}_k(\lambda, \varnothing) = 0$.

(c) Assume that the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are distinct. Let

$$g = \left| \left\{ (i,j) \in \{1, 2, \ldots, k-1\}^2 \mid i < j \text{ and } \gamma_i < \gamma_j \right\} \right|.$$

Then, $\operatorname{pet}_k(\lambda, \varnothing) = (-1)^{(\beta_1 + \beta_2 + \cdots + \beta_{k-1}) + g + (\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1})}$.

The proof of this theorem is technical and will be given in Subsection 3.13.

It is possible to restate part of Theorem 2.15 without using λ^t:

Proposition 2.16. Let $\lambda \in \operatorname{Par}$, and let k be a positive integer. Assume that $\lambda_1 < k$. Define a subset B of $\mathbb{Z}$ by

$$B = \{\lambda_i - i \mid i \in \{1, 2, 3, \ldots\}\}.$$

Let $\overline{0}, \overline{1}, \ldots, \overline{k-1}$ be the residue classes of the integers $0, 1, \ldots, k-1$ modulo k (considered as subsets of $\mathbb{Z}$). Let W be the set of all integers smaller than $k-1$.

Then, $\operatorname{pet}_k(\lambda, \varnothing) \neq 0$ if and only if each $i \in \{0, 1, \ldots, k-1\}$ satisfies $\left| \left(\overline{i} \cap W\right) \setminus B \right| \leq 1$.

In Subsection 3.13, we will outline how this proposition can be derived from Theorem 2.15.

The sets B and $\left(\bar{i} \cap W\right) \setminus B$ in Proposition 2.16 are related to the *k-modular structure* of the partition λ, such as the *β-set*, the *k-abacus*, the *k-core* and the *k-quotient* (see [Olsson93, §§1–3] for some of these concepts). Essentially equivalent concepts include the *Maya diagram* of λ (see, e.g., [Crane18, §3.3])[9] and the *first column hook lengths* of λ (see [Olsson93, Proposition (1.3)]).

2.5. A "Pieri" rule

Now, the following generalization of Theorem 2.9 holds:

Theorem 2.17. Let k be a positive integer. Let $\mu \in \operatorname{Par}$. Then,

$$G\left(k\right) \cdot s_{\mu} = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k\left(\lambda, \mu\right) s_{\lambda}.$$

Theorem 2.9 is the particular case of Theorem 2.17 for $\mu = \varnothing$.

We shall give two proofs of Theorem 2.17 in Subsections 3.10 and 3.11.

We can also generalize Corollary 2.10 to obtain a Pieri-like rule for multiplication by $G\left(k, m\right)$:

Corollary 2.18. Let k be a positive integer. Let $m \in \mathbb{N}$. Let $\mu \in \operatorname{Par}$. Then,

$$G\left(k, m\right) \cdot s_{\mu} = \sum_{\lambda \in \operatorname{Par}_{m+\left|\mu\right|}} \operatorname{pet}_k\left(\lambda, \mu\right) s_{\lambda}.$$

Corollary 2.18 follows from Theorem 2.17 by projecting onto the $\left(m + \left|\mu\right|\right)$-th graded component of Λ. (We shall explain this argument in more detail in Subsection 3.12.)

2.6. Coproducts of Petrie functions

In the following, the "$\otimes$" sign will always stand for $\otimes_{\mathbf{k}}$ (that is, tensor product of $\mathbf{k}$-modules or of $\mathbf{k}$-algebras).

The $\mathbf{k}$-algebra Λ is a Hopf algebra due to the presence of a comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$. We recall (from [GriRei20, §2.1]) one way to define this comultiplication:

[9] The *Maya diagram* of λ is a coloring of the set $\left\{z + \dfrac{1}{2} \mid z \in \mathbb{Z}\right\}$ with the colors black and white, in which the elements $\lambda_i - i + \dfrac{1}{2}$ (for all $i \in \left\{1, 2, 3, \ldots\right\}$) are colored black while all remaining elements are colored white. Borcherds's proof of the Jacobi triple product identity ([Camero94, §13.3]) also essentially constructs this Maya diagram (wording it in terms of the "*Dirac sea*" model for electrons).

Consider the rings

$$\mathbf{k}[[\mathbf{x}]] := \mathbf{k}[[x_1, x_2, x_3, \ldots]] \qquad \text{and} \qquad \mathbf{k}[[\mathbf{x}, \mathbf{y}]] := \mathbf{k}[[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]]$$

of formal power series. We shall use the notations $\mathbf{x}$ and $\mathbf{y}$ for the sequences $(x_1, x_2, x_3, \ldots)$ and $(y_1, y_2, y_3, \ldots)$ of indeterminates. If $f \in \mathbf{k}[[\mathbf{x}]]$ is any formal power series, then $f(\mathbf{y})$ shall mean the result of substituting $y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ in f. (This will be a formal power series in $\mathbf{k}[[y_1, y_2, y_3, \ldots]]$.) For the sake of symmetry, we also use the analogous notation $f(\mathbf{x})$ for the result of substituting $x_1, x_2, x_3, \ldots$ for $x_1, x_2, x_3, \ldots$ in f; of course, this $f(\mathbf{x})$ is just f. Finally, if the power series $f \in \mathbf{k}[[\mathbf{x}]]$ is symmetric, then we use the notation $f(\mathbf{x}, \mathbf{y})$ for the result of substituting the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ in f (that is, choosing some bijection $\phi : \{x_1, x_2, x_3, \ldots\} \to \{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$ [10] and substituting $\phi(x_i)$ for each x_i in f). This result does not depend on the order in which the former variables are substituted for the latter (i.e., on the choice of the bijection ϕ) because f is symmetric.

Now, the comultiplication of Λ is the map $\Delta : \Lambda \to \Lambda \otimes \Lambda$ determined as follows: For a symmetric function $f \in \Lambda$, we have

$$\Delta(f) = \sum_{i \in I} f_{1,i} \otimes f_{2,i}, \tag{9}$$

where $f_{1,i}, f_{2,i} \in \Lambda$ are such that

$$f(\mathbf{x}, \mathbf{y}) = \sum_{i \in I} f_{1,i}(\mathbf{x}) f_{2,i}(\mathbf{y}). \tag{10}$$

More precisely, if $f \in \Lambda$, if I is a finite set, and if $(f_{1,i})_{i \in I} \in \Lambda^I$ and $(f_{2,i})_{i \in I} \in \Lambda^I$ are two families satisfying (10), then $\Delta(f)$ is given by (9). [11]

For example, for any $n \in \mathbb{N}$, it is easy to see that

$$e_n(\mathbf{x}, \mathbf{y}) = \sum_{i=0}^{n} e_i(\mathbf{x}) e_{n-i}(\mathbf{y}),$$

and thus the above definition of Δ yields

$$\Delta(e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}.$$

A similar formula exists for the image of a Petrie symmetric function under Δ:

[10] Such bijections clearly exist, since the sets $\{x_1, x_2, x_3, \ldots\}$ and $\{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$ have the same cardinality (namely, $\aleph_0$). This is one of several observations commonly illustrated by the metaphor of "Hilbert's hotel".

[11] In the language of [GriRei20, §2.1], this can be restated as $\Delta(f) = f(\mathbf{x}, \mathbf{y})$, because $\Lambda \otimes \Lambda$ is identified with a certain subring of $\mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ in [GriRei20, §2.1] (via the injection $\Lambda \otimes \Lambda \to \mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ that sends any $u \otimes v \in \Lambda \otimes \Lambda$ to $u(\mathbf{x}) v(\mathbf{y}) \in \mathbf{k}[[\mathbf{x}, \mathbf{y}]]$).

Theorem 2.19. Let k be a positive integer. Let $m \in \mathbb{N}$. Then,

$$\Delta\left(G\left(k,m\right)\right) = \sum_{i=0}^{m} G\left(k,i\right) \otimes G\left(k,m-i\right).$$

The proof of Theorem 2.19 is given in Subsection 3.14; it is a simple consequence (albeit somewhat painful to explain) of (9).

It is well-known that $\Delta : \Lambda \to \Lambda \otimes \Lambda$ is a $\mathbf{k}$-algebra homomorphism. Equipping the $\mathbf{k}$-algebra Λ with the comultiplication Δ (as well as a counit $\varepsilon : \Lambda \to \mathbf{k}$, which we won't need here) yields a connected graded Hopf algebra. (See, e.g., [GriRei20, §2.1] for proofs.)

2.7. The Frobenius endomorphisms and Petrie functions

We shall next derive another formula for the Petrie symmetric functions $G\left(k,m\right)$. For this formula, we need the following definition ([GriRei20, Exercise 2.9.9]):

Definition 2.20. Let $n \in \{1,2,3,\ldots\}$. We define a map $\mathbf{f}_n : \Lambda \to \Lambda$ by

$$\left(\mathbf{f}_n\left(a\right) = a\left(x_1^n, x_2^n, x_3^n, \ldots\right) \qquad \text{for each } a \in \Lambda\right).$$

This map $\mathbf{f}_n$ is called the *n-th Frobenius endomorphism* of Λ.

Clearly, this map $\mathbf{f}_n$ is a $\mathbf{k}$-algebra endomorphism of Λ (since it amounts to a substitution of indeterminates). It is known (from [GriRei20, Exercise 2.9.9(d)]) that this map $\mathbf{f}_n : \Lambda \to \Lambda$ is a Hopf algebra endomorphism of Λ.

Using the notion of *plethysm* (see, e.g., [Stanle01, Chapter 7, Definition A2.6] or [Macdon95, §I.8][12]), we can view the map $\mathbf{f}_n$ as a plethysm with the n-th power-sum symmetric function p_n, in the sense that any $a \in \Lambda$ satisfies $\mathbf{f}_n\left(a\right) = a\left[p_n\right] = p_n\left[a\right]$ as long as $\mathbf{k} = \mathbb{Z}$. (Plethysm becomes somewhat subtle when the base ring $\mathbf{k}$ is complicated; $\mathbf{f}_n\left(a\right) = a\left[p_n\right]$ holds for any $\mathbf{k}$, while $\mathbf{f}_n\left(a\right) = p_n\left[a\right]$ relies on good properties of $\mathbf{k}$.) The plethystic viewpoint makes some properties of $\mathbf{f}_n$ clear, but we shall avoid it for reasons of elementarity.

Now, we can express the Petrie symmetric functions $G\left(k,m\right)$ using Frobenius endomorphisms as follows:

Theorem 2.21. Let k be a positive integer. Let $m \in \mathbb{N}$. Then,

$$G\left(k,m\right) = \sum_{i \in \mathbb{N}} \left(-1\right)^i h_{m-ki} \cdot \mathbf{f}_k\left(e_i\right).$$

(The sum on the right hand side of this equality is well-defined, since all sufficiently high $i \in \mathbb{N}$ satisfy $m - ki < 0$ and thus $h_{m-ki} = 0$.)

[12] Note that [Stanle01] uses the notation $f\left[g\right]$ for the plethysm of f with g, whereas [Macdon95] uses the notation $f \circ g$ for this. We shall use $f\left[g\right]$.

Theorem 2.21 will be proved in Subsection 3.15 below.

2.8. The Petrie functions as polynomial generators of Λ

We now claim the following:

Theorem 2.22. Fix a positive integer k. Assume that $1-k$ is invertible in $\mathbf{k}$.

Then, the family $(G(k,m))_{m\geq 1} = (G(k,1), G(k,2), G(k,3), \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra Λ. (In other words, the canonical $\mathbf{k}$-algebra homomorphism

$$\mathbf{k}[u_1, u_2, u_3, \ldots] \to \Lambda, \qquad u_m \mapsto G(k,m)$$

is an isomorphism.)

We shall prove Theorem 2.22 in Subsection 3.16. The proof uses the following two formulas for Hall inner products:[13]

Lemma 2.23. Let k and m be positive integers. Let $j \in \mathbb{N}$. Then, $\langle p_m, \mathbf{f}_k(e_j)\rangle = (-1)^{j-1}[m = kj]\,k$.

Proposition 2.24. Let k and m be positive integers. Then, $\langle p_m, G(k,m)\rangle = 1 - [k \mid m]\,k$.

Both of these formulas will be proved in Subsection 3.16 as well.

2.9. The Verschiebung endomorphisms

Now we recall another definition ([GriRei20, Exercise 2.9.10]):

Definition 2.25. Let $n \in \{1,2,3,\ldots\}$. We define a $\mathbf{k}$-algebra homomorphism $\mathbf{v}_n : \Lambda \to \Lambda$ by

$$\left(\mathbf{v}_n(h_m) = \begin{cases} h_{m/n}, & \text{if } n \mid m; \\ 0, & \text{if } n \nmid m \end{cases} \qquad \text{for each } m > 0\right).$$

(This is well-defined, since the sequence $(h_1, h_2, h_3, \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra Λ.)

This map $\mathbf{v}_n$ is called the *n-th Verschiebung endomorphism* of Λ.

Again, it is known ([GriRei20, Exercise 2.9.10(e)]) that this map $\mathbf{v}_n : \Lambda \to \Lambda$ is a Hopf algebra endomorphism of Λ. Moreover, the following holds ([GriRei20, Exercise 2.9.10(f)]):

[13] Here, we are again using the Iverson bracket notation.

Proposition 2.26. Let $n \in \{1, 2, 3, \ldots\}$. Then, the maps $\mathbf{f}_n : \Lambda \to \Lambda$ and $\mathbf{v}_n : \Lambda \to \Lambda$ are adjoint with respect to the Hall inner product on Λ. That is, any $a \in \Lambda$ and $b \in \Lambda$ satisfy

$$\langle a, \mathbf{f}_n(b) \rangle = \langle \mathbf{v}_n(a), b \rangle .$$

Furthermore, any positive integers n and m satisfy

$$\mathbf{v}_n(p_m) = \begin{cases} np_{m/n}, & \text{if } n \mid m; \\ 0, & \text{if } n \nmid m \end{cases} . \tag{11}$$

(This is [GriRei20, Exercise 2.9.10(a)].)

2.10. The Hopf endomorphisms U_k and V_k

In this final subsection, we shall show another way to obtain the Petrie symmetric functions $G(k, m)$ using the machinery of Hopf algebras. We refer, e.g., to [GriRei20, Chapters 1 and 2] for everything we will use about Hopf algebras.

Convention 2.27. As already mentioned, Λ is a connected graded Hopf algebra. We let S denote its antipode.

Definition 2.28. If C is a $\mathbf{k}$-coalgebra and A is a $\mathbf{k}$-algebra, and if $f, g : C \to A$ are two $\mathbf{k}$-linear maps, then the *convolution* $f \star g$ of f and g is defined to be the $\mathbf{k}$-linear map $m_A \circ (f \otimes g) \circ \Delta_C : C \to A$, where $\Delta_C : C \to C \otimes C$ is the comultiplication of the $\mathbf{k}$-coalgebra C, and where $m_A : A \otimes A \to A$ is the $\mathbf{k}$-linear map sending each pure tensor $a \otimes b \in A \otimes A$ to $ab \in A$.

We also recall Definition 2.25 and Definition 2.20. We now claim the following.

Theorem 2.29. Fix a positive integer k. Let U_k be the map $\mathbf{f}_k \circ S \circ \mathbf{v}_k : \Lambda \to \Lambda$. Let V_k be the map $\mathrm{id}_\Lambda \star U_k : \Lambda \to \Lambda$. (This is well-defined by Definition 2.28, since Λ is both a $\mathbf{k}$-coalgebra and a $\mathbf{k}$-algebra.) Then:

(a) The map U_k is a $\mathbf{k}$-Hopf algebra homomorphism.

(b) The map V_k is a $\mathbf{k}$-Hopf algebra homomorphism.

(c) We have $V_k(h_m) = G(k, m)$ for each $m \in \mathbb{N}$.

(d) We have $V_k(p_n) = (1 - [k \mid n]\, k)\, p_n$ for each positive integer n.

See Subsection 3.17 for a proof of this theorem.

Using Theorem 2.29, we can give a new proof for Theorem 2.19; see Subsection 3.18 for this.

We also obtain the following corollary from Theorem 2.19:

Corollary 2.30. Let k and n be two positive integers. Then, there exists a polynomial $f \in \mathbf{k}[x_1, x_2, x_3, \ldots]$ such that

$$(1 - [k \mid n]\, k)\, p_n = f\left(G(k,1), G(k,2), G(k,3), \ldots\right). \tag{12}$$

This corollary will be proved in Subsection 3.19.

3. Proofs

3.1. The infinite and finitary symmetric groups

We now approach the proofs of the many claims made above. First, let us briefly discuss some technicalities in the definition of monomial symmetric functions m_λ.

Let $\mathfrak{S}_\infty$ be the group of all permutations of the set $\{1, 2, 3, \ldots\}$. (The group operation is given by composition.)

A permutation $\sigma \in \mathfrak{S}_\infty$ is said to be *finitary* if it leaves all but finitely many elements of $\{1, 2, 3, \ldots\}$ invariant (i.e., if all but finitely many $i \in \{1, 2, 3, \ldots\}$ satisfy $\sigma(i) = i$).

The set of all finitary permutations $\sigma \in \mathfrak{S}_\infty$ is a subgroup of $\mathfrak{S}_\infty$; it is called the *finitary symmetric group*, and will be denoted by $\mathfrak{S}_{(\infty)}$.

The group $\mathfrak{S}_\infty$ acts on the set WC of all weak compositions by permuting their entries:

$$\sigma \cdot (\alpha_1, \alpha_2, \alpha_3, \ldots) = \left(\alpha_{\sigma^{-1}(1)}, \alpha_{\sigma^{-1}(2)}, \alpha_{\sigma^{-1}(3)}, \ldots\right) \qquad \text{for any } (\alpha_1, \alpha_2, \alpha_3, \ldots) \in \mathrm{WC} \text{ and } \sigma \in \mathfrak{S}_\infty.$$

Thus, the subgroup $\mathfrak{S}_{(\infty)}$ of $\mathfrak{S}_\infty$ acts on WC as well. We now claim the following:

Lemma 3.1. Let $\beta \in \mathrm{WC}$. Then, the orbit $\mathfrak{S}_{(\infty)}\beta$ of β under the action of $\mathfrak{S}_{(\infty)}$ is identical with the orbit $\mathfrak{S}_\infty\beta$ of β under the action of $\mathfrak{S}_\infty$.

Our proof of Lemma 3.1 will rely on a lemma about how two bijections $\varphi : X \to X'$ and $\psi : Y \to Y'$ can be combined ("glued together") to a bijection $X \cup Y \to X' \cup Y'$ as long as X and Y are disjoint and X' and Y' are disjoint:

Lemma 3.2. Let X, Y, X' and Y' be four sets. Assume that X and Y are disjoint. Assume that X' and Y' are disjoint. Let $\varphi : X \to X'$ be a bijection. Let $\psi : Y \to Y'$ be a bijection. Then, the map

$$X \cup Y \to X' \cup Y',$$
$$z \mapsto \begin{cases} \varphi(z), & \text{if } z \in X; \\ \psi(z), & \text{if } z \in Y \end{cases}$$

is well-defined and is a bijection from $X \cup Y$ to $X' \cup Y'$.

Lemma 3.2 is a basic fact in set theory, and its proof is straightforward (whence we omit it).

Proof of Lemma 3.1. From $\mathfrak{S}_{(\infty)} \subseteq \mathfrak{S}_\infty$, we obtain $\mathfrak{S}_{(\infty)}\beta \subseteq \mathfrak{S}_\infty\beta$. We shall now show that $\mathfrak{S}_\infty\beta \subseteq \mathfrak{S}_{(\infty)}\beta$.

Indeed, let $\gamma \in \mathfrak{S}_\infty\beta$. Thus, there exists some permutation $\tau \in \mathfrak{S}_\infty$ such that $\gamma = \tau\beta$. Consider this τ. The map τ is a permutation of the set $\{1,2,3,\ldots\}$ (since $\tau \in \mathfrak{S}_\infty$), and thus is a bijection from $\{1,2,3,\ldots\}$ to $\{1,2,3,\ldots\}$. We have $\gamma = (\gamma_1, \gamma_2, \gamma_3, \ldots)$ and thus

$$\begin{aligned}(\gamma_1, \gamma_2, \gamma_3, \ldots) = \gamma = \tau \underbrace{\beta}_{=(\beta_1,\beta_2,\beta_3,\ldots)} &= \tau \cdot (\beta_1, \beta_2, \beta_3, \ldots) \\ &= \left(\beta_{\tau^{-1}(1)}, \beta_{\tau^{-1}(2)}, \beta_{\tau^{-1}(3)}, \ldots\right)\end{aligned}$$

(by the definition of the action of $\mathfrak{S}_\infty$ on WC). In other words,

$$\gamma_i = \beta_{\tau^{-1}(i)} \qquad \text{for every positive integer } i. \tag{13}$$

Define two subsets B and C of $\{1,2,3,\ldots\}$ by

$$B = \{i \in \{1,2,3,\ldots\} \mid \beta_i \neq 0\} \qquad \text{and}$$
$$C = \{i \in \{1,2,3,\ldots\} \mid \gamma_i \neq 0\}.$$

Then, the set B is finite[14]. Hence, the set $B \setminus C$ is finite (since $B \setminus C$ is a subset of B).

We have $\tau(j) \in C$ for each $j \in B$ [15]. Hence, the map

$$\overline{\tau} : B \to C,$$
$$j \mapsto \tau(j)$$

is well-defined. Consider this map $\overline{\tau}$.

We have $\tau^{-1}(j) \in B$ for each $j \in C$ [16]. Hence, the map

$$\widehat{\tau} : C \to B,$$
$$j \mapsto \tau^{-1}(j)$$

[14] *Proof.* Recall that β is a weak composition. Thus, β contains only finitely many nonzero entries (by the definition of a weak composition). In other words, there are only finitely many $i \in \{1,2,3,\ldots\}$ satisfying $\beta_i \neq 0$. In other words, the set $\{i \in \{1,2,3,\ldots\} \mid \beta_i \neq 0\}$ is finite. In other words, the set B is finite (since $B = \{i \in \{1,2,3,\ldots\} \mid \beta_i \neq 0\}$).

[15] *Proof.* Let $j \in B$. Thus, $j \in B = \{i \in \{1,2,3,\ldots\} \mid \beta_i \neq 0\}$. In other words, j is an $i \in \{1,2,3,\ldots\}$ satisfying $\beta_i \neq 0$. In other words, j is an element of $\{1,2,3,\ldots\}$ and satisfies $\beta_j \neq 0$. Hence, $j \in \{1,2,3,\ldots\}$ (since j is an element of $\{1,2,3,\ldots\}$), so that $\tau(j) \in \{1,2,3,\ldots\}$ (since τ is a bijection from $\{1,2,3,\ldots\}$ to $\{1,2,3,\ldots\}$). Thus, $\tau(j)$ is a positive integer. Hence, (13) (applied to $i = \tau(j)$) yields $\gamma_{\tau(j)} = \beta_{\tau^{-1}(\tau(j))} = \beta_j$ (since $\tau^{-1}(\tau(j)) = j$). Therefore, $\gamma_{\tau(j)} = \beta_j \neq 0$. Thus, $\tau(j)$ is an $i \in \{1,2,3,\ldots\}$ satisfying $\gamma_i \neq 0$ (since $\tau(j) \in \{1,2,3,\ldots\}$ and $\gamma_{\tau(j)} \neq 0$). In other words, $\tau(j) \in \{i \in \{1,2,3,\ldots\} \mid \gamma_i \neq 0\}$. In other words, $\tau(j) \in C$ (since $C = \{i \in \{1,2,3,\ldots\} \mid \gamma_i \neq 0\}$). Qed.

[16] *Proof.* Let $j \in C$. Thus, $j \in C = \{i \in \{1,2,3,\ldots\} \mid \gamma_i \neq 0\}$. In other words, j is an $i \in \{1,2,3,\ldots\}$ satisfying $\gamma_i \neq 0$. In other words, j is an element of $\{1,2,3,\ldots\}$ and satisfies $\gamma_j \neq 0$. Hence,

is well-defined. Consider this map $\widehat{\tau}$.

We have $\overline{\tau} \circ \widehat{\tau} = \operatorname{id}$ [17] and $\widehat{\tau} \circ \overline{\tau} = \operatorname{id}$ [18]. Combining these two equalities, we conclude that the maps $\overline{\tau}$ and $\widehat{\tau}$ are mutually inverse. Hence, the map $\overline{\tau}$ is invertible, i.e., is a bijection. Thus, we have found a bijection $\overline{\tau} : B \to C$. Therefore, the set C has the same cardinality as B. Thus, the set C is finite (since the set B is finite). Hence, the set $C \setminus B$ is finite (since $C \setminus B$ is a subset of C).

We have showed that the set C has the same cardinality as B. In other words, $|C| = |B|$. But the set C is the union of the two disjoint sets $C \cap B$ and $C \setminus B$; hence, we have $|C| = |C \cap B| + |C \setminus B|$. Hence, $|C \setminus B| = |C| - |C \cap B|$. The same argument (with the roles of B and C interchanged) yields $|B \setminus C| = |B| - |B \cap C|$. Comparing this with $|C \setminus B| = \underbrace{|C|}_{=|B|} - \left|\underbrace{C \cap B}_{=B \cap C}\right| = |B| - |B \cap C|$, we obtain $|C \setminus B| = |B \setminus C|$. In other words, the two sets $C \setminus B$ and $B \setminus C$ have the same cardinality. Hence, there exists a bijection $\rho : C \setminus B \to B \setminus C$. Consider this ρ.

Elementary set theory shows that $C \cup (C \setminus B) = C \cup B = B \cup C$ and $B \cup (B \setminus C) = B \cup C$.

The sets B and $C \setminus B$ are disjoint. The sets C and $B \setminus C$ are disjoint. The maps $\overline{\tau} : B \to C$ and $\rho : C \setminus B \to B \setminus C$ are bijections. Hence, Lemma 3.2 (applied to

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$j \in \{1, 2, 3, \ldots\}$ (since j is an element of $\{1, 2, 3, \ldots\}$), so that $\tau^{-1}(j) \in \{1, 2, 3, \ldots\}$ (since τ is a bijection from $\{1, 2, 3, \ldots\}$ to $\{1, 2, 3, \ldots\}$). Thus, $\tau^{-1}(j)$ is a positive integer. Hence, (13) (applied to $i = j$) yields $\gamma_j = \beta_{\tau^{-1}(j)}$. Therefore, $\beta_{\tau^{-1}(j)} = \gamma_j \neq 0$. Thus, $\tau^{-1}(j)$ is an $i \in \{1, 2, 3, \ldots\}$ satisfying $\beta_i \neq 0$ (since $\tau^{-1}(j) \in \{1, 2, 3, \ldots\}$ and $\beta_{\tau^{-1}(j)} \neq 0$). In other words, $\tau^{-1}(j) \in \{i \in \{1, 2, 3, \ldots\} \mid \beta_i \neq 0\}$. In other words, $\tau^{-1}(j) \in B$ (since $B = \{i \in \{1, 2, 3, \ldots\} \mid \beta_i \neq 0\}$). Qed.

[17] *Proof.* Every $j \in C$ satisfies

$$(\overline{\tau} \circ \widehat{\tau})(j) = \overline{\tau}\left(\underbrace{\widehat{\tau}(j)}_{\substack{=\tau^{-1}(j) \\ \text{(by the definition of } \widehat{\tau})}}\right) = \overline{\tau}\left(\tau^{-1}(j)\right) = \tau\left(\tau^{-1}(j)\right) \qquad \text{(by the definition of } \overline{\tau})$$
$$= j = \operatorname{id}(j).$$

Thus, $\overline{\tau} \circ \widehat{\tau} = \operatorname{id}$.

[18] *Proof.* Every $j \in B$ satisfies

$$(\widehat{\tau} \circ \overline{\tau})(j) = \widehat{\tau}\left(\underbrace{\overline{\tau}(j)}_{\substack{=\tau(j) \\ \text{(by the definition of } \overline{\tau})}}\right) = \widehat{\tau}(\tau(j)) = \tau^{-1}(\tau(j)) \qquad \text{(by the definition of } \widehat{\tau})$$
$$= j = \operatorname{id}(j).$$

Thus, $\widehat{\tau} \circ \overline{\tau} = \operatorname{id}$.

$X = B$, $Y = C \setminus B$, $X' = C$, $Y' = B \setminus C$, $\varphi = \overline{\tau}$ and $\psi = \rho$) yields that the map

$$\begin{aligned} C \cup (C \setminus B) &\to B \cup (B \setminus C), \\ z &\mapsto \begin{cases} \overline{\tau}(z), & \text{if } z \in B; \\ \rho(z), & \text{if } z \in C \setminus B \end{cases} \end{aligned}$$

is well-defined and is a bijection from $C \cup (C \setminus B)$ to $B \cup (B \setminus C)$. In view of $C \cup (C \setminus B) = B \cup C$ and $B \cup (B \setminus C) = B \cup C$, we can restate this result as follows: The map

$$\begin{aligned} B \cup C &\to B \cup C, \\ z &\mapsto \begin{cases} \overline{\tau}(z), & \text{if } z \in B; \\ \rho(z), & \text{if } z \in C \setminus B \end{cases} \end{aligned}$$

is well-defined and is a bijection from $B \cup C$ to $B \cup C$. Let us denote this map by η. Thus, η is a bijection from $B \cup C$ to $B \cup C$. In other words, the map $\eta : B \cup C \to B \cup C$ is a bijection.

Clearly, B and C are subsets of $\{1, 2, 3, \ldots\}$; thus, $B \cup C$ is a subset of $\{1, 2, 3, \ldots\}$. Let D denote the complement of this subset $B \cup C$ in $\{1, 2, 3, \ldots\}$. That is, we have $D = \{1, 2, 3, \ldots\} \setminus (B \cup C)$. Thus, the set D is disjoint from $B \cup C$ and satisfies $(B \cup C) \cup D = \{1, 2, 3, \ldots\}$.

The sets $B \cup C$ and D are disjoint (since D is disjoint from $B \cup C$). The maps $\eta : B \cup C \to B \cup C$ and $\operatorname{id}_D : D \to D$ are bijections. Hence, Lemma 3.2 (applied to $X = B \cup C$, $Y = D$, $X' = B \cup C$, $Y' = D$, $\varphi = \eta$ and $\psi = \operatorname{id}_D$) yields that the map

$$\begin{aligned} (B \cup C) \cup D &\to (B \cup C) \cup D, \\ z &\mapsto \begin{cases} \eta(z), & \text{if } z \in B \cup C; \\ \operatorname{id}_D(z), & \text{if } z \in D \end{cases} \end{aligned}$$

is well-defined and is a bijection from $(B \cup C) \cup D$ to $(B \cup C) \cup D$. In view of $(B \cup C) \cup D = \{1, 2, 3, \ldots\}$, we can restate this result as follows: The map

$$\begin{aligned} \{1, 2, 3, \ldots\} &\to \{1, 2, 3, \ldots\}, \\ z &\mapsto \begin{cases} \eta(z), & \text{if } z \in B \cup C; \\ \operatorname{id}_D(z), & \text{if } z \in D \end{cases} \end{aligned}$$

is well-defined and is a bijection from $\{1, 2, 3, \ldots\}$ to $\{1, 2, 3, \ldots\}$. Let us denote this map by σ. Thus, σ is a bijection from $\{1, 2, 3, \ldots\}$ to $\{1, 2, 3, \ldots\}$. In other words, σ is a permutation of $\{1, 2, 3, \ldots\}$. In other words, $\sigma \in \mathfrak{S}_\infty$ (since $\mathfrak{S}_\infty$ is the set of all permutations of $\{1, 2, 3, \ldots\}$).

The set $B \cup C$ is finite (since it is the union of the two finite sets B and C). Hence, all but finitely many $i \in \{1, 2, 3, \ldots\}$ satisfy $i \notin B \cup C$. But each $i \in \{1, 2, 3, \ldots\}$

satisfying $i \notin B \cup C$ must satisfy $\sigma(i) = i$ [19]. Hence, all but finitely many $i \in \{1,2,3,\ldots\}$ satisfy $\sigma(i) = i$ (since all but finitely many $i \in \{1,2,3,\ldots\}$ satisfy $i \notin B \cup C$). In other words, the permutation σ leaves all but finitely many elements of $\{1,2,3,\ldots\}$ invariant. In other words, the permutation σ is finitary (by the definition of "finitary"). Hence, σ is a finitary permutation in $\mathfrak{S}_\infty$. In other words, $\sigma \in \mathfrak{S}_{(\infty)}$ (since $\mathfrak{S}_{(\infty)}$ is the set of all finitary permutations in $\mathfrak{S}_\infty$).

Now, we claim that

$$\gamma_{\sigma(j)} = \beta_j \qquad \text{for every positive integer } j. \tag{14}$$

[*Proof of (14):* Let j be a positive integer. We must prove (14).

We have $j \in \{1,2,3,\ldots\}$ (since j is a positive integer). Thus, $\sigma(j) \in \{1,2,3,\ldots\}$ (since σ is a bijection from $\{1,2,3,\ldots\}$ to $\{1,2,3,\ldots\}$). In other words, $\sigma(j)$ is a positive integer. Hence, (13) (applied to $i = \sigma(j)$) yields $\gamma_{\sigma(j)} = \beta_{\tau^{-1}(\sigma(j))}$.

We are in one of the following three cases:

Case 1: We have $j \in B$.

Case 2: We have $j \in C \setminus B$.

Case 3: We have neither $j \in B$ nor $j \in C \setminus B$.

Let us first consider Case 1. In this case, we have $j \in B$. Hence, $j \in B \subseteq B \cup C$. Now, the definition of σ yields

$$\begin{aligned}
\sigma(j) &= \begin{cases} \eta(j), & \text{if } j \in B \cup C; \\ \mathrm{id}_D(j), & \text{if } j \in D \end{cases} = \eta(j) \qquad (\text{since } j \in B \cup C) \\
&= \begin{cases} \overline{\tau}(j), & \text{if } j \in B; \\ \rho(j), & \text{if } j \in C \setminus B \end{cases} \qquad (\text{by the definition of } \eta) \\
&= \overline{\tau}(j) \qquad (\text{since } j \in B) \\
&= \tau(j) \qquad (\text{by the definition of } \overline{\tau}).
\end{aligned}$$

Hence, $\tau^{-1}(\sigma(j)) = j$. But recall that $\gamma_{\sigma(j)} = \beta_{\tau^{-1}(\sigma(j))}$. In view of $\tau^{-1}(\sigma(j)) = j$, this rewrites as $\gamma_{\sigma(j)} = \beta_j$. Thus, (14) is proved in Case 1.

Let us next consider Case 2. In this case, we have $j \in C \setminus B$. Thus, $j \in C \setminus B \subseteq$

[19] *Proof.* Let $i \in \{1,2,3,\ldots\}$ be such that $i \notin B \cup C$. Thus, $i \in \{1,2,3,\ldots\} \setminus (B \cup C) = D$ (since $D = \{1,2,3,\ldots\} \setminus (B \cup C)$). The definition of σ yields

$$\begin{aligned}
\sigma(i) &= \begin{cases} \eta(i), & \text{if } i \in B \cup C; \\ \mathrm{id}_D(i), & \text{if } i \in D \end{cases} = \mathrm{id}_D(i) \qquad (\text{since } i \in D) \\
&= i.
\end{aligned}$$

Qed.

$C \subseteq B \cup C$. Now, the definition of σ yields

$$\begin{aligned}
\sigma(j) &= \begin{cases} \eta(j), & \text{if } j \in B \cup C; \\ \operatorname{id}_D(j), & \text{if } j \in D \end{cases} = \eta(j) \qquad (\text{since } j \in B \cup C) \\
&= \begin{cases} \overline{\tau}(j), & \text{if } j \in B; \\ \rho(j), & \text{if } j \in C \setminus B \end{cases} \qquad (\text{by the definition of } \eta) \\
&= \rho\left(\underbrace{j}_{\in C \setminus B}\right) \qquad (\text{since } j \in C \setminus B) \\
&\in \rho(C \setminus B) = B \setminus C \qquad (\text{since } \rho \text{ is a bijection from } C \setminus B \text{ to } B \setminus C).
\end{aligned}$$

In other words, $\sigma(j) \in B$ and $\sigma(j) \notin C$. Thus, we can easily conclude that $\gamma_{\sigma(j)} = 0$ [20]. Also, from $j \in C \setminus B$, we obtain $j \in C$ and $j \notin B$. Hence, $\beta_j = 0$ [21]. Comparing this with $\gamma_{\sigma(j)} = 0$, we obtain $\gamma_{\sigma(j)} = \beta_j$. Thus, (14) is proved in Case 2.

Let us finally consider Case 3. In this case, we have neither $j \in B$ nor $j \in C \setminus B$. In other words, we have $j \notin B$ and $j \notin C \setminus B$. Hence, we have $j \notin B \cup (C \setminus B)$. In view of $B \cup (C \setminus B) = B \cup C$ (which follows from elementary set theory), we can restate this as $j \notin B \cup C$. In other words, $j \notin B$ and $j \notin C$. Thus, we can easily conclude that $\gamma_j = 0$ [22] and $\beta_j = 0$ [23].

From $j \in \{1, 2, 3, \ldots\}$ and $j \notin B \cup C$, we obtain $j \in \{1, 2, 3, \ldots\} \setminus (B \cup C)$. In other words, $j \in D$ (since $D = \{1, 2, 3, \ldots\} \setminus (B \cup C)$). Now, the definition of σ yields

$$\begin{aligned}
\sigma(j) &= \begin{cases} \eta(j), & \text{if } j \in B \cup C; \\ \operatorname{id}_D(j), & \text{if } j \in D \end{cases} = \operatorname{id}_D(j) \qquad (\text{since } j \in D) \\
&= j.
\end{aligned}$$

Hence, $\gamma_{\sigma(j)} = \gamma_j = 0 = \beta_j$ (since $\beta_j = 0$). Thus, (14) is proved in Case 3.

[20] *Proof.* Assume the contrary. Thus, $\gamma_{\sigma(j)} \neq 0$. Hence, $\sigma(j)$ is an $i \in \{1, 2, 3, \ldots\}$ satisfying $\gamma_i \neq 0$ (since $\sigma(j) \in \{1, 2, 3, \ldots\}$ and $\gamma_{\sigma(j)} \neq 0$). In other words, $\sigma(j) \in \{i \in \{1, 2, 3, \ldots\} \mid \gamma_i \neq 0\}$. But this contradicts $\sigma(j) \notin C = \{i \in \{1, 2, 3, \ldots\} \mid \gamma_i \neq 0\}$. This contradiction shows that our assumption was false, qed.

[21] *Proof.* Assume the contrary. Thus, $\beta_j \neq 0$. Hence, j is an $i \in \{1, 2, 3, \ldots\}$ satisfying $\beta_i \neq 0$ (since $j \in \{1, 2, 3, \ldots\}$ and $\beta_j \neq 0$). In other words, $j \in \{i \in \{1, 2, 3, \ldots\} \mid \beta_i \neq 0\}$. But this contradicts $j \notin B = \{i \in \{1, 2, 3, \ldots\} \mid \beta_i \neq 0\}$. This contradiction shows that our assumption was false, qed.

[22] *Proof.* Assume the contrary. Thus, $\gamma_j \neq 0$. Hence, j is an $i \in \{1, 2, 3, \ldots\}$ satisfying $\gamma_i \neq 0$ (since $j \in \{1, 2, 3, \ldots\}$ and $\gamma_j \neq 0$). In other words, $j \in \{i \in \{1, 2, 3, \ldots\} \mid \gamma_i \neq 0\}$. But this contradicts $j \notin C = \{i \in \{1, 2, 3, \ldots\} \mid \gamma_i \neq 0\}$. This contradiction shows that our assumption was false, qed.

[23] *Proof.* Assume the contrary. Thus, $\beta_j \neq 0$. Hence, j is an $i \in \{1, 2, 3, \ldots\}$ satisfying $\beta_i \neq 0$ (since $j \in \{1, 2, 3, \ldots\}$ and $\beta_j \neq 0$). In other words, $j \in \{i \in \{1, 2, 3, \ldots\} \mid \beta_i \neq 0\}$. But this contradicts $j \notin B = \{i \in \{1, 2, 3, \ldots\} \mid \beta_i \neq 0\}$. This contradiction shows that our assumption was false, qed.

We have now proved (14) in each of the three Cases 1, 2 and 3. Since these three Cases cover all possibilities, we thus conclude that (14) always holds.]

We have now proved (14). Hence, we have

$$\gamma_i = \beta_{\sigma^{-1}(i)} \qquad \text{for every positive integer } i. \tag{15}$$

[*Proof of (15):* Let i be a positive integer. Thus, $i \in \{1,2,3,\ldots\}$, so that $\sigma^{-1}(i) \in \{1,2,3,\ldots\}$ (since σ is a bijection from $\{1,2,3,\ldots\}$ to $\{1,2,3,\ldots\}$). In other words, $\sigma^{-1}(i)$ is a positive integer. Hence, (14) (applied to $j = \sigma^{-1}(i)$) yields $\gamma_{\sigma\left(\sigma^{-1}(i)\right)} = \beta_{\sigma^{-1}(i)}$. In other words, $\gamma_i = \beta_{\sigma^{-1}(i)}$ (since $\sigma\left(\sigma^{-1}(i)\right) = i$). This proves (15).]

Now, we have proved (15). In other words, we have proved that $\gamma_i = \beta_{\sigma^{-1}(i)}$ for every positive integer i. In other words,

$$(\gamma_1, \gamma_2, \gamma_3, \ldots) = \left(\beta_{\sigma^{-1}(1)}, \beta_{\sigma^{-1}(2)}, \beta_{\sigma^{-1}(3)}, \ldots\right). \tag{16}$$

Now, the definition of the action of $\mathfrak{S}_\infty$ on WC yields

$$\sigma \cdot (\beta_1, \beta_2, \beta_3, \ldots) = \left(\beta_{\sigma^{-1}(1)}, \beta_{\sigma^{-1}(2)}, \beta_{\sigma^{-1}(3)}, \ldots\right).$$

Hence,

$$\begin{aligned} \sigma \cdot \underbrace{\beta}_{=(\beta_1,\beta_2,\beta_3,\ldots)} &= \sigma \cdot (\beta_1, \beta_2, \beta_3, \ldots) = \left(\beta_{\sigma^{-1}(1)}, \beta_{\sigma^{-1}(2)}, \beta_{\sigma^{-1}(3)}, \ldots\right) \\ &= (\gamma_1, \gamma_2, \gamma_3, \ldots) \qquad \text{(by (16))} \\ &= \gamma. \end{aligned}$$

Thus, $\gamma = \underbrace{\sigma}_{\in \mathfrak{S}_{(\infty)}} \cdot \beta \in \mathfrak{S}_{(\infty)}\beta$.

Forget that we fixed γ. We thus have shown that $\gamma \in \mathfrak{S}_{(\infty)}\beta$ for each $\gamma \in \mathfrak{S}_\infty\beta$. In other words, $\mathfrak{S}_\infty\beta \subseteq \mathfrak{S}_{(\infty)}\beta$. Combining this with $\mathfrak{S}_{(\infty)}\beta \subseteq \mathfrak{S}_\infty\beta$, we obtain $\mathfrak{S}_{(\infty)}\beta = \mathfrak{S}_\infty\beta$. In other words, the orbit $\mathfrak{S}_{(\infty)}\beta$ of β under the action of $\mathfrak{S}_{(\infty)}$ is identical with the orbit $\mathfrak{S}_\infty\beta$ of β under the action of $\mathfrak{S}_\infty$. This proves Lemma 3.1. □

Recall that each partition is a weak composition. In other words, we have Par $\subseteq$ WC.

Now, we easily obtain the following:

Proposition 3.3. Let $\lambda \in$ Par. Our above definition of m_λ is equivalent to the definition of m_λ given in [GriRei20, Definition 2.1.3].

Proof of Proposition 3.3. We have $\lambda \in$ Par $\subseteq$ WC. Hence, Lemma 3.1 (applied to $\beta = \lambda$) shows that the orbit $\mathfrak{S}_{(\infty)}\lambda$ of λ under the action of $\mathfrak{S}_{(\infty)}$ is identical with the orbit $\mathfrak{S}_\infty \lambda$ of λ under the action of $\mathfrak{S}_\infty$. In other words, $\mathfrak{S}_{(\infty)}\lambda = \mathfrak{S}_\infty \lambda$. In other words, $\mathfrak{S}_\infty \lambda = \mathfrak{S}_{(\infty)}\lambda$.

Recall that the group $\mathfrak{S}_\infty$ acts on the set WC of all weak compositions by permuting their entries. Thus, for any weak composition α, we have the following chain of logical equivalences:

$$\begin{aligned}
&(\alpha \text{ can be obtained from } \lambda \text{ by permuting entries}) \\
\iff &(\alpha \text{ can be obtained from } \lambda \text{ by the action of some } \sigma \in \mathfrak{S}_\infty) \\
\iff &(\text{there exists some } \sigma \in \mathfrak{S}_\infty \text{ such that } \alpha = \sigma\lambda) \\
\iff &(\alpha \in \mathfrak{S}_\infty \lambda) .
\end{aligned}$$

Thus, we have the following equality of summation signs:

$$\sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha \text{ can be obtained from } \lambda \text{ by permuting entries}}} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha \in \mathfrak{S}_\infty \lambda}} = \sum_{\alpha \in \mathfrak{S}_\infty \lambda} \tag{17}$$

(since $\mathfrak{S}_\infty \lambda \subseteq \mathrm{WC}$). Hence,

$$\begin{aligned}
&\sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha \text{ can be obtained from } \lambda \text{ by permuting entries}}} \mathbf{x}^\alpha \\
&= \sum_{\alpha \in \mathfrak{S}_\infty \lambda} \mathbf{x}^\alpha = \sum_{\alpha \in \mathfrak{S}_{(\infty)} \lambda} \mathbf{x}^\alpha
\end{aligned} \tag{18}$$

(since $\mathfrak{S}_\infty \lambda = \mathfrak{S}_{(\infty)}\lambda$).

Our above definition of m_λ says that

$$m_\lambda = \sum \mathbf{x}^\alpha,$$

where the sum ranges over all weak compositions $\alpha \in \mathrm{WC}$ that can be obtained from λ by permuting entries. In other words, it says that

$$m_\lambda = \sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha \text{ can be obtained from } \lambda \text{ by permuting entries}}} \mathbf{x}^\alpha .$$

On the other hand, the definition of m_λ given in [GriRei20, Definition 2.1.3] says that

$$m_\lambda = \sum_{\alpha \in \mathfrak{S}_{(\infty)} \lambda} \mathbf{x}^\alpha . \tag{19}$$

But the right hand sides of these two equalities are equal (because of (18)). Hence, the left hand sides must be equal as well. In other words, m_λ defined according to our above definition of m_λ is equal to m_λ defined according to [GriRei20, Definition 2.1.3]. In other words, these two definitions are equivalent. This proves Proposition 3.3. $\square$

3.2. The symmetric functions h_λ

Next, let us introduce a family of symmetric functions, obtained by multiplying several h_n's:

Definition 3.4. Let λ be a partition. Write λ in the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are positive integers. Then, we define a symmetric function $h_\lambda \in \Lambda$ by

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}.$$

Note that this definition also appears in [GriRei20, Definition 2.2.1].

The symmetric function h_λ is called the *complete homogeneous symmetric function* corresponding to the partition λ.

From [GriRei20, Corollary 2.5.17(a)], we know that the families $(h_\lambda)_{\lambda \in \mathrm{Par}}$ and $(m_\lambda)_{\lambda \in \mathrm{Par}}$ are dual bases with respect to the Hall inner product. Thus,

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu} \qquad \text{for any } \lambda \in \mathrm{Par} \text{ and } \mu \in \mathrm{Par}. \tag{20}$$

Let us record a slightly different way to express h_λ:

Proposition 3.5. Let λ be a partition. Then,

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots .$$

(Here, the infinite product $h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots$ is well-defined, since every sufficiently high positive integer i satisfies $\lambda_i = 0$ and thus $h_{\lambda_i} = h_0 = 1$.)

This is how h_λ is defined in [Macdon95, Section I.2].

Proof of Proposition 3.5. Write the partition λ in the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are positive integers. Then, the definition of h_λ yields $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$. But from $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, we obtain $\lambda_{\ell+1} = \lambda_{\ell+2} = \lambda_{\ell+3} = \cdots = 0$. In other words, each $i \in \{\ell+1, \ell+2, \ell+3, \ldots\}$ satisfies $\lambda_i = 0$. Hence, each $i \in \{\ell+1, \ell+2, \ell+3, \ldots\}$ satisfies

$$\begin{aligned} h_{\lambda_i} &= h_0 \qquad (\text{since } \lambda_i = 0) \\ &= 1. \end{aligned} \tag{21}$$

Now,

$$\begin{aligned} h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots &= \left(h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}\right) \underbrace{\left(h_{\lambda_{\ell+1}} h_{\lambda_{\ell+2}} h_{\lambda_{\ell+3}} \cdots\right)}_{= \prod_{i=\ell+1}^{\infty} h_{\lambda_i}} = \left(h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}\right) \prod_{i=\ell+1}^{\infty} \underbrace{h_{\lambda_i}}_{\substack{=1 \\ (\text{by } (21))}} \\ &= \left(h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}\right) \underbrace{\prod_{i=\ell+1}^{\infty} 1}_{=1} = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}. \end{aligned}$$

Comparing this with $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}$, we obtain $h_\lambda = h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots$. This proves Proposition 3.5. $\square$

3.3. Proofs of Proposition 1.2 and Proposition 1.3

Proof of Proposition 1.2. There are myriad ways to prove this. Here is perhaps the simplest one: Let us use the notation h_λ as defined in Definition 3.4. Thus, $h_{(n)} = h_n$ (since n is a positive integer). Applying (20) to $\lambda = (n)$ and $\mu = (n)$, we obtain $\left\langle h_{(n)}, m_{(n)} \right\rangle = \delta_{(n),(n)} = 1$. In view of $h_{(n)} = h_n$ and $m_{(n)} = p_n$, this rewrites as $\langle h_n, p_n \rangle = 1$. This proves Proposition 1.2. $\square$

Proof of Proposition 1.3. This is [GriRei20, Exercise 2.8.8(a)]. But here is a self-contained proof: Proposition 1.1 yields

$$\begin{aligned}
\langle e_n, p_n \rangle &= \left\langle e_n, \sum_{i=0}^{n-1} (-1)^i s_{\left(n-i,1^i\right)} \right\rangle = \sum_{i=0}^{n-1} (-1)^i \underbrace{\left\langle e_n, s_{\left(n-i,1^i\right)} \right\rangle}_{\substack{=\left\langle s_{(1^n)}, s_{\left(n-i,1^i\right)} \right\rangle \\ \text{(since } e_n = s_{(1^n)}\text{)}}} \\
&= \sum_{i=0}^{n-1} (-1)^i \underbrace{\left\langle s_{(1^n)}, s_{\left(n-i,1^i\right)} \right\rangle}_{\substack{=\delta_{(1^n),\left(n-i,1^i\right)} \\ \text{(since the basis } (s_\lambda)_{\lambda\in\operatorname{Par}} \text{ of } \Lambda \text{ is} \\ \text{orthonormal with respect to} \\ \text{the Hall inner product)}}} = \sum_{i=0}^{n-1} (-1)^i \underbrace{\delta_{(1^n),\left(n-i,1^i\right)}}_{\substack{= \begin{cases} 1, & \text{if } (1^n) = \left(n-i,1^i\right); \\ 0, & \text{if } (1^n) \neq \left(n-i,1^i\right) \end{cases} \\ = \begin{cases} 1, & \text{if } i = n-1; \\ 0, & \text{if } i \neq n-1 \end{cases} \\ \text{(since we have } (1^n) = \left(n-i,1^i\right) \\ \text{if and only if } i = n-1\text{)}}} \\
&= \sum_{i=0}^{n-1} (-1)^i \begin{cases} 1, & \text{if } i = n-1; \\ 0, & \text{if } i \neq n-1 \end{cases} = (-1)^{n-1}.
\end{aligned}$$

$\square$

3.4. Proof of Proposition 2.3

Our next goal is to prove Proposition 2.3. Speaking frankly, the proof is obvious, but making it fully rigorous will require us to prove some lemmas first. We shall use the notations from Subsection 3.1; in particular, we recall how the group $\mathfrak{S}_\infty$ and its subgroup $\mathfrak{S}_{(\infty)}$ act on WC. It is intuitively clear that any weak composition has exactly one partition among its rearrangements (namely, the partition obtained by sorting its entries into weakly decreasing order). In other words, each orbit of the action of $\mathfrak{S}_{(\infty)}$ on WC contains exactly one partition. Let us state this as a lemma and outline a rigorous proof:

Lemma 3.6. Let $\alpha \in \mathrm{WC}$. Then, there exists a unique partition $\lambda \in \operatorname{Par}$ such that $\alpha \in \mathfrak{S}_{(\infty)} \lambda$.

Proof of Lemma 3.6 (sketched). We have $\alpha \in \mathrm{WC}$. In other words, α is a weak composition. In other words, α is a sequence $(\alpha_1, \alpha_2, \alpha_3, \ldots)$ of nonnegative integers that contains only finitely many nonzero entries. Thus, there exists some $k \in \mathbb{N}$ such that $\alpha_{k+1} = \alpha_{k+2} = \alpha_{k+3} = \cdots = 0$. Consider this k. Thus, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k, 0, 0, 0, \ldots)$. Now, by sorting the first k entries $\alpha_1, \alpha_2, \ldots, \alpha_k$ of α into weakly decreasing order, we obtain a new weak composition $\beta \in \mathrm{WC}$ that has the form $\beta = \left(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(k)}, 0, 0, 0, \ldots\right)$ for some permutation σ of $\{1, 2, \ldots, k\}$ and satisfies $\alpha_{\sigma(1)} \geq \alpha_{\sigma(2)} \geq \cdots \geq \alpha_{\sigma(k)}$. Consider this β and this σ. From $\beta = \left(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(k)}, 0, 0, 0, \ldots\right)$, we conclude that the entries of β are weakly decreasing (since $\alpha_{\sigma(1)} \geq \alpha_{\sigma(2)} \geq \cdots \geq \alpha_{\sigma(k)} \geq 0 \geq 0 \geq 0 \geq \cdots$); in other words, β is a partition. In other words, $\beta \in \mathrm{Par}$. Also, β was obtained from α by sorting the first k entries into weakly decreasing order; thus, β was obtained from α by permuting the first k entries. Hence, α can be obtained from β by permuting the first k entries. This shows that $\alpha \in \mathfrak{S}_{(\infty)} \beta$ (since permuting the first k entries of a weak composition can be achieved by the action of some permutation $\sigma \in \mathfrak{S}_{(\infty)}$). Therefore, there exists a partition $\lambda \in \mathrm{Par}$ such that $\alpha \in \mathfrak{S}_{(\infty)} \lambda$ (namely, $\lambda = \beta$). It thus remains to prove that this λ is unique. In other words, it remains to prove that there exists at most one partition $\lambda \in \mathrm{Par}$ such that $\alpha \in \mathfrak{S}_{(\infty)} \lambda$.

But this is easy: Let $\lambda \in \mathrm{Par}$ be a partition such that $\alpha \in \mathfrak{S}_{(\infty)} \lambda$. From $\alpha \in \mathfrak{S}_{(\infty)} \lambda$, we conclude that the sequence α is a rearrangement of the sequence λ. Hence, the sequences α and λ differ only in the order of their entries. Hence, for each $i \in \mathbb{N}$, we have

$$\begin{aligned} & (\text{the number of times } i \text{ appears in } \alpha) \\ & = (\text{the number of times } i \text{ appears in } \lambda) . \end{aligned} \tag{22}$$

The same argument (applied to β instead of λ) yields that for each $i \in \mathbb{N}$, we have

$$\begin{aligned} & (\text{the number of times } i \text{ appears in } \alpha) \\ & = (\text{the number of times } i \text{ appears in } \beta) \end{aligned}$$

(since $\alpha \in \mathfrak{S}_{(\infty)} \beta$). Comparing this with (22), we conclude that

$$\begin{aligned} & (\text{the number of times } i \text{ appears in } \beta) \\ & = (\text{the number of times } i \text{ appears in } \lambda) \end{aligned}$$

for each $i \in \mathbb{N}$. In other words, the two partitions β and λ contain each $i \in \mathbb{N}$ the same number of times. But this entails that the partitions β and λ are equal (because a partition is uniquely determined by how often it contains each $i \in \mathbb{N}$). In other words, $\beta = \lambda$. Thus, $\lambda = \beta$. Now, forget that we fixed λ. We thus have proved that every partition $\lambda \in \mathrm{Par}$ such that $\alpha \in \mathfrak{S}_{(\infty)} \lambda$ will satisfy $\lambda = \beta$. Hence, there exists at most one partition $\lambda \in \mathrm{Par}$ such that $\alpha \in \mathfrak{S}_{(\infty)} \lambda$. This completes the proof of Lemma 3.6. $\square$

The next lemma tells us that the action of $\mathfrak{S}_{(\infty)}$ on WC leaves some properties of a partition unchanged:

Lemma 3.7. Let $\lambda \in \operatorname{Par}$. Let $\alpha \in \mathfrak{S}_{(\infty)}\lambda$. Then:
(a) We have $|\lambda| = |\alpha|$.
(b) For every positive integer k, we have the logical equivalence[24]

$$(\lambda_i < k \text{ for all } i) \iff (\alpha_i < k \text{ for all } i).$$

(Note that we could have just as well required $\lambda \in \operatorname{WC}$ instead of $\lambda \in \operatorname{Par}$ in Lemma 3.7, and we could have required $\alpha \in \mathfrak{S}_\infty \lambda$ instead of $\alpha \in \mathfrak{S}_{(\infty)}\lambda$. But we have chosen to state the lemma in the setting in which we will be applying it later on.)

Proof of Lemma 3.7. We have $\lambda \in \operatorname{Par} \subseteq \operatorname{WC}$, so that $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$.

We have $\alpha \in \mathfrak{S}_{(\infty)}\lambda$. In other words, there exists some $\sigma \in \mathfrak{S}_{(\infty)}$ such that $\alpha = \sigma \cdot \lambda$. Consider this σ.

We have $\sigma \in \mathfrak{S}_{(\infty)} \subseteq \mathfrak{S}_\infty$. Thus, σ is a permutation of the set $\{1, 2, 3, \ldots\}$; in other words, σ is a bijection from $\{1, 2, 3, \ldots\}$ to $\{1, 2, 3, \ldots\}$. Hence, its inverse σ^{-1} is a bijection from $\{1, 2, 3, \ldots\}$ to $\{1, 2, 3, \ldots\}$ as well. In other words, σ^{-1} is a permutation of the set $\{1, 2, 3, \ldots\}$.

We have $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$. Now, $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$, so that

$$\begin{aligned}(\alpha_1, \alpha_2, \alpha_3, \ldots) = \alpha = \sigma \cdot \underbrace{\lambda}_{=(\lambda_1, \lambda_2, \lambda_3, \ldots)} &= \sigma \cdot (\lambda_1, \lambda_2, \lambda_3, \ldots) \\ &= \left(\lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \lambda_{\sigma^{-1}(3)}, \ldots\right)\end{aligned}$$

(by the definition of the action of $\mathfrak{S}_\infty$ on WC).

(a) The definition of $|\lambda|$ yields

$$\begin{aligned}|\lambda| &= \lambda_1 + \lambda_2 + \lambda_3 + \cdots = \sum_{i \in \{1,2,3,\ldots\}} \lambda_i = \sum_{i \in \{1,2,3,\ldots\}} \lambda_{\sigma^{-1}(i)} \\ &\qquad \left(\begin{array}{c}\text{here, we have substituted } \sigma^{-1}(i) \text{ for } i \text{ in the sum,} \\ \text{since } \sigma^{-1} \text{ is a bijection from } \{1,2,3,\ldots\} \text{ to } \{1,2,3,\ldots\}\end{array}\right) \\ &= \lambda_{\sigma^{-1}(1)} + \lambda_{\sigma^{-1}(2)} + \lambda_{\sigma^{-1}(3)} + \cdots.\end{aligned}$$

The definition of $|\alpha|$ yields

$$\begin{aligned}|\alpha| &= \alpha_1 + \alpha_2 + \alpha_3 + \cdots = \lambda_{\sigma^{-1}(1)} + \lambda_{\sigma^{-1}(2)} + \lambda_{\sigma^{-1}(3)} + \cdots \\ &\qquad \left(\text{since } (\alpha_1, \alpha_2, \alpha_3, \ldots) = \left(\lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \lambda_{\sigma^{-1}(3)}, \ldots\right)\right).\end{aligned}$$

[24]Here and in all similar situations, "for all i" means "for all positive integers i".

Comparing these two equalities, we find $|\lambda| = |\alpha|$. This proves Lemma 3.7 **(a)**.

(b) Recall that σ^{-1} is a permutation of the set $\{1, 2, 3, \ldots\}$. Hence, the numbers $\lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \lambda_{\sigma^{-1}(3)}, \ldots$ are precisely the numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$, except possibly in a different order. Thus, the numbers $\lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \lambda_{\sigma^{-1}(3)}, \ldots$ are all $< k$ if and only if the numbers $\lambda_1, \lambda_2, \lambda_3, \ldots$ are all $< k$. In other words, we have the following logical equivalence:

$$\begin{aligned} & \left(\text{the numbers } \lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \lambda_{\sigma^{-1}(3)}, \ldots \text{ are all } < k\right) \\ \Longleftrightarrow \ & \left(\text{the numbers } \lambda_1, \lambda_2, \lambda_3, \ldots \text{ are all } < k\right). \end{aligned} \tag{23}$$

Now, we have the following chain of logical equivalences:

$$\begin{aligned} & (\alpha_i < k \text{ for all } i) \\ \Longleftrightarrow \ & (\text{the numbers } \alpha_1, \alpha_2, \alpha_3, \ldots \text{ are all } < k) \\ \Longleftrightarrow \ & \left(\text{the numbers } \lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \lambda_{\sigma^{-1}(3)}, \ldots \text{ are all } < k\right) \\ & \qquad \left(\text{since } (\alpha_1, \alpha_2, \alpha_3, \ldots) = \left(\lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \lambda_{\sigma^{-1}(3)}, \ldots\right)\right) \\ \Longleftrightarrow \ & (\text{the numbers } \lambda_1, \lambda_2, \lambda_3, \ldots \text{ are all } < k) \qquad (\text{by (23)}) \\ \Longleftrightarrow \ & (\lambda_i < k \text{ for all } i). \end{aligned}$$

In other words, we have the equivalence $(\lambda_i < k \text{ for all } i) \Longleftrightarrow (\alpha_i < k \text{ for all } i)$. This proves Lemma 3.7 **(b)**. ☐

Proof of Proposition 2.3. **(a)** It is easy to see that for any $m \in \mathbb{N}$, the formal power series $G(k, m)$ is homogeneous of degree m [25]. Moreover, (3) yields

$$G(k) = \underbrace{\sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha_i < k \text{ for all } i}}}_{\substack{= \sum\limits_{m \in \mathbb{N}} \sum\limits_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \\ (\text{since } |\alpha| \in \mathbb{N} \text{ for each } \alpha \in \mathrm{WC})}} \mathbf{x}^\alpha = \sum_{m \in \mathbb{N}} \underbrace{\sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha}_{\substack{= G(k,m) \\ (\text{by (4)})}} = \sum_{m \in \mathbb{N}} G(k, m).$$

[25] *Proof.* Let $m \in \mathbb{N}$. For any $\alpha \in \mathrm{WC}$, the monomial $\mathbf{x}^\alpha$ is a monomial of degree $|\alpha|$. Thus, if $\alpha \in \mathrm{WC}$ satisfies $|\alpha| = m$, then $\mathbf{x}^\alpha$ is a monomial of degree m (since $|\alpha| = m$). Hence, $\sum\limits_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha$ is a sum of monomials of degree m. In view of

$$G(k, m) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha \qquad (\text{by (4)}),$$

we can restate this as follows: $G(k, m)$ is a sum of monomials of degree m. Thus, the formal power series $G(k, m)$ is homogeneous of degree m. Qed.

Thus, the family $(G(k,m))_{m\in\mathbb{N}}$ is the homogeneous decomposition of $G(k)$ (since each $G(k,m)$ is homogeneous of degree m). Hence, for each $m \in \mathbb{N}$, the power series $G(k,m)$ is the m-th degree homogeneous component of $G(k)$. This proves Proposition 2.3 **(a)**.

(b) Let us define the group $\mathfrak{S}_{(\infty)}$ and its action on the set WC as in Subsection 3.1. Then,

$$\sum_{\substack{\lambda\in\mathrm{Par};\\ \lambda_i<k \text{ for all } i}} \underbrace{m_\lambda}_{\substack{=\sum\limits_{\alpha\in\mathfrak{S}_{(\infty)}\lambda} \mathbf{x}^\alpha \\ \text{(by (19))}}} = \sum_{\substack{\lambda\in\mathrm{Par};\\ \lambda_i<k \text{ for all } i}} \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda} \mathbf{x}^\alpha. \tag{24}$$

Now, we have the following equality of summation signs:

$$\sum_{\substack{\lambda\in\mathrm{Par};\\ \lambda_i<k \text{ for all } i}} \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda} = \sum_{\lambda\in\mathrm{Par}} \underbrace{\sum_{\substack{\alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ \lambda_i<k \text{ for all } i}}}_{\substack{=\sum\limits_{\substack{\alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ \alpha_i<k \text{ for all } i}} \\ \text{(because for each } \alpha\in\mathfrak{S}_{(\infty)}\lambda\text{, we have the} \\ \text{equivalence } (\lambda_i<k \text{ for all } i) \iff (\alpha_i<k \text{ for all } i) \\ \text{(by Lemma 3.7 (b)))}}} = \sum_{\lambda\in\mathrm{Par}} \underbrace{\sum_{\substack{\alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ \alpha_i<k \text{ for all } i}}}_{\substack{=\sum\limits_{\substack{\alpha\in\mathrm{WC};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ \alpha_i<k \text{ for all } i}} \\ \text{(since } \mathfrak{S}_{(\infty)}\lambda\subseteq\mathrm{WC}\text{)}}}$$

$$= \sum_{\lambda\in\mathrm{Par}} \sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ \alpha_i<k \text{ for all } i}} = \sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i}} \sum_{\substack{\lambda\in\mathrm{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}}.$$

Hence, (24) becomes

$$\begin{aligned}
\sum_{\substack{\lambda\in\mathrm{Par};\\ \lambda_i<k \text{ for all } i}} m_\lambda &= \underbrace{\sum_{\substack{\lambda\in\mathrm{Par};\\ \lambda_i<k \text{ for all } i}} \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda}}_{=\sum\limits_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i}} \sum\limits_{\substack{\lambda\in\mathrm{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}}} \mathbf{x}^\alpha \\
&= \sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i}} \sum_{\substack{\lambda\in\mathrm{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} \mathbf{x}^\alpha.
\end{aligned} \tag{25}$$

Now, fix some $\alpha \in \mathrm{WC}$. Then, Lemma 3.6 yields that there exists a unique partition $\lambda \in \mathrm{Par}$ such that $\alpha \in \mathfrak{S}_{(\infty)}\lambda$. Thus, the sum $\sum\limits_{\substack{\lambda\in\mathrm{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} \mathbf{x}^\alpha$ has exactly one addend. Hence, this sum simplifies as follows:

$$\sum_{\substack{\lambda\in\mathrm{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} \mathbf{x}^\alpha = \mathbf{x}^\alpha. \tag{26}$$

Forget that we fixed α. We thus have proved (26) for each $\alpha \in \mathrm{WC}$. Thus, (25) becomes

$$\sum_{\substack{\lambda\in\mathrm{Par};\\ \lambda_i<k \text{ for all } i}} m_\lambda = \sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i}} \underbrace{\sum_{\substack{\lambda\in\mathrm{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} \mathbf{x}^\alpha}_{\substack{=\mathbf{x}^\alpha\\ \text{(by (26))}}} = \sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha .$$

Comparing this with (3), we obtain

$$G(k) = \sum_{\substack{\lambda\in\mathrm{Par};\\ \lambda_i<k \text{ for all } i}} m_\lambda . \qquad (27)$$

Comparing (3) with

$$\prod_{i=1}^{\infty} \underbrace{\left(x_i^0 + x_i^1 + \cdots + x_i^{k-1}\right)}_{= \sum\limits_{u\in\{0,1,\ldots,k-1\}} x_i^u}$$

$$= \prod_{i=1}^{\infty} \sum_{u\in\{0,1,\ldots,k-1\}} x_i^u = \underbrace{\sum_{\substack{(u_1,u_2,u_3,\ldots)\in\{0,1,\ldots,k-1\}^\infty; \\ \text{all but finitely many } i \text{ satisfy } u_i=0}}}_{\substack{= \sum\limits_{\substack{(u_1,u_2,u_3,\ldots)\in\{0,1,\ldots,k-1\}^\infty; \\ (u_1,u_2,u_3,\ldots)\in\mathrm{WC}}} \\ \text{(since a sequence } (u_1,u_2,u_3,\ldots) \text{ of nonnegative integers} \\ \text{satisfies the statement ``all but finitely many } i \text{ satisfy } u_i=0\text{''} \\ \text{if and only if it satisfies } (u_1,u_2,u_3,\ldots)\in\mathrm{WC}\text{)}}} x_1^{u_1} x_2^{u_2} x_3^{u_3} \cdots$$

$$\text{(by the product rule)}$$

$$= \underbrace{\sum_{\substack{(u_1,u_2,u_3,\ldots)\in\{0,1,\ldots,k-1\}^\infty; \\ (u_1,u_2,u_3,\ldots)\in\mathrm{WC}}}}_{\substack{= \sum\limits_{\substack{(u_1,u_2,u_3,\ldots)\in\mathrm{WC}; \\ (u_1,u_2,u_3,\ldots)\in\{0,1,\ldots,k-1\}^\infty}} = \sum\limits_{\substack{(u_1,u_2,u_3,\ldots)\in\mathrm{WC}; \\ u_i<k \text{ for all } i}} \\ \text{(because a weak composition } (u_1,u_2,u_3,\ldots)\in\mathrm{WC} \\ \text{satisfies the statement } (u_1,u_2,u_3,\ldots)\in\{0,1,\ldots,k-1\}^\infty \\ \text{if and only if it satisfies ``} u_i<k \text{ for all } i\text{'')}}} x_1^{u_1} x_2^{u_2} x_3^{u_3} \cdots$$

$$= \sum_{\substack{(u_1,u_2,u_3,\ldots)\in\mathrm{WC}; \\ u_i<k \text{ for all } i}} x_1^{u_1} x_2^{u_2} x_3^{u_3} \cdots = \sum_{\substack{(\alpha_1,\alpha_2,\alpha_3,\ldots)\in\mathrm{WC}; \\ \alpha_i<k \text{ for all } i}} \underbrace{x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \cdots}_{\substack{= \mathbf{x}^{(\alpha_1,\alpha_2,\alpha_3,\ldots)} \\ \text{(by the definition of } \mathbf{x}^{(\alpha_1,\alpha_2,\alpha_3,\ldots)}\text{)}}}$$

$$\left(\begin{array}{c} \text{here, we have renamed the} \\ \text{summation index } (u_1,u_2,u_3,\ldots) \text{ as } (\alpha_1,\alpha_2,\alpha_3,\ldots) \end{array}\right)$$

$$= \sum_{\substack{(\alpha_1,\alpha_2,\alpha_3,\ldots)\in\mathrm{WC}; \\ \alpha_i<k \text{ for all } i}} \mathbf{x}^{(\alpha_1,\alpha_2,\alpha_3,\ldots)} = \sum_{\substack{\alpha\in\mathrm{WC}; \\ \alpha_i<k \text{ for all } i}} \mathbf{x}^{\alpha}$$

$$\left(\begin{array}{c} \text{here, we have renamed the} \\ \text{summation index } (\alpha_1,\alpha_2,\alpha_3,\ldots) \text{ as } \alpha \end{array}\right),$$

we obtain

$$G(k) = \prod_{i=1}^{\infty} \left(x_i^0 + x_i^1 + \cdots + x_i^{k-1}\right).$$

Combining this equality with (27) and (3), we obtain

$$G(k) = \sum_{\substack{\alpha\in\mathrm{WC}; \\ \alpha_i<k \text{ for all } i}} \mathbf{x}^{\alpha} = \sum_{\substack{\lambda\in\mathrm{Par}; \\ \lambda_i<k \text{ for all } i}} m_\lambda = \prod_{i=1}^{\infty} \left(x_i^0 + x_i^1 + \cdots + x_i^{k-1}\right).$$

This proves Proposition 2.3 **(b)**.

(c) Let $m \in \mathbb{N}$. Let us define the group $\mathfrak{S}_{(\infty)}$ and its action on the set WC as in Subsection 3.1. Then,

$$\sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m; \\ \lambda_i < k \text{ for all } i}} \underbrace{m_\lambda}_{\substack{= \sum_{\alpha \in \mathfrak{S}_{(\infty)} \lambda} \mathbf{x}^\alpha \\ \text{(by (19))}}} = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m; \\ \lambda_i < k \text{ for all } i}} \sum_{\alpha \in \mathfrak{S}_{(\infty)} \lambda} \mathbf{x}^\alpha. \tag{28}$$

Now, we have the following equality of summation signs:

$$\begin{aligned}
\sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m; \\ \lambda_i < k \text{ for all } i}} \sum_{\alpha \in \mathfrak{S}_{(\infty)} \lambda} &= \sum_{\lambda \in \operatorname{Par}} \underbrace{\sum_{\substack{\alpha \in \mathfrak{S}_{(\infty)} \lambda; \\ |\lambda| = m; \\ \lambda_i < k \text{ for all } i}}}_{\substack{= \sum_{\substack{\alpha \in \mathfrak{S}_{(\infty)} \lambda; \\ |\lambda| = m; \\ \alpha_i < k \text{ for all } i}} \\ \text{(because for each } \alpha \in \mathfrak{S}_{(\infty)} \lambda \text{, we have the} \\ \text{equivalence } (\lambda_i < k \text{ for all } i) \iff (\alpha_i < k \text{ for all } i) \\ \text{(by Lemma 3.7 (b)))}}} \\
&= \sum_{\lambda \in \operatorname{Par}} \underbrace{\sum_{\substack{\alpha \in \mathfrak{S}_{(\infty)} \lambda; \\ |\lambda| = m; \\ \alpha_i < k \text{ for all } i}}}_{\substack{= \sum_{\substack{\alpha \in \mathfrak{S}_{(\infty)} \lambda; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \\ \text{(because for each } \alpha \in \mathfrak{S}_{(\infty)} \lambda \text{,} \\ \text{we have } |\lambda| = |\alpha| \\ \text{(by Lemma 3.7 (a)))}}} = \sum_{\lambda \in \operatorname{Par}} \underbrace{\sum_{\substack{\alpha \in \mathfrak{S}_{(\infty)} \lambda; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}}}_{\substack{= \sum_{\substack{\alpha \in \operatorname{WC}; \\ \alpha \in \mathfrak{S}_{(\infty)} \lambda; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \\ \text{(since } \mathfrak{S}_{(\infty)} \lambda \subseteq \operatorname{WC} \text{)}}} \\
&= \sum_{\lambda \in \operatorname{Par}} \sum_{\substack{\alpha \in \operatorname{WC}; \\ \alpha \in \mathfrak{S}_{(\infty)} \lambda; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} = \sum_{\substack{\alpha \in \operatorname{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \sum_{\substack{\lambda \in \operatorname{Par}; \\ \alpha \in \mathfrak{S}_{(\infty)} \lambda}} .
\end{aligned}$$

Hence, (28) becomes

$$\begin{aligned}
\sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m; \\ \lambda_i < k \text{ for all } i}} m_\lambda &= \underbrace{\sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m; \\ \lambda_i < k \text{ for all } i}} \sum_{\alpha \in \mathfrak{S}_{(\infty)} \lambda}}_{= \sum_{\substack{\alpha \in \operatorname{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \sum_{\substack{\lambda \in \operatorname{Par}; \\ \alpha \in \mathfrak{S}_{(\infty)} \lambda}}} \mathbf{x}^\alpha = \sum_{\substack{\alpha \in \operatorname{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \underbrace{\sum_{\substack{\lambda \in \operatorname{Par}; \\ \alpha \in \mathfrak{S}_{(\infty)} \lambda}} \mathbf{x}^\alpha}_{\substack{= \mathbf{x}^\alpha \\ \text{(by (26))}}} \\
&= \sum_{\substack{\alpha \in \operatorname{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha. \tag{29}
\end{aligned}$$

Now, (4) becomes

$$G(k,m) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=m;\\ \lambda_i<k \text{ for all } i}} m_\lambda \qquad \text{(by (29))}.$$

This proves Proposition 2.3 **(c)**.

(d) Let $m\in\mathbb{N}$ satisfy $k>m$. Then, each $\alpha\in\mathrm{WC}$ satisfying $|\alpha|=m$ must automatically satisfy $(\alpha_i<k \text{ for all } i)$ [26]. Hence, the condition "$\alpha_i<k$ for all i" under the summation sign "$\sum\limits_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}}$" is redundant and can be removed. In other words, we have the following equality between summation signs:

$$\sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}} = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m}}.$$

Now, (4) yields

$$G(k,m) = \underbrace{\sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}}}_{=\sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m}}} \mathbf{x}^\alpha = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m}} \mathbf{x}^\alpha = h_m$$

(since the definition of h_m yields $h_m = \sum\limits_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m}} \mathbf{x}^\alpha$). This proves Proposition 2.3 **(d)**.

(e) Let $m\in\mathbb{N}$, and assume that $k=2$. Then, an $\alpha\in\mathrm{WC}$ satisfies $(\alpha_i<k \text{ for all } i)$ if and only if it satisfies $\alpha\in\{0,1\}^\infty$ (because we have the chain of logical equiva-

[26] *Proof.* Let $\alpha\in\mathrm{WC}$ satisfy $|\alpha|=m$. We must prove that $\alpha_i<k$ for all i.

Indeed, let i be a positive integer. We must prove that $\alpha_i<k$.

We have $i\in\{1,2,3,\ldots\}$ (since i is a positive integer). The definition of $|\alpha|$ yields

$$\begin{aligned}|\alpha| &= \alpha_1+\alpha_2+\alpha_3+\cdots = \sum_{j\geq 1}\alpha_j = \alpha_i + \sum_{\substack{j\geq 1;\\ j\neq i}} \underbrace{\alpha_j}_{\substack{\geq 0\\ (\text{since } \alpha\in\mathrm{WC}\subseteq\mathbb{N}^\infty)}} \\ &\qquad \text{(here, we have split off the addend for } j=i \text{ from the sum)} \\ &\geq \alpha_i + \underbrace{\sum_{\substack{j\geq 1;\\ j\neq i}} 0}_{=0} = \alpha_i,\end{aligned}$$

so that $\alpha_i\leq|\alpha|=m<k$ (since $k>m$). Thus, we have proved that $\alpha_i<k$. Qed.

lences

$$\begin{aligned}(\alpha_i < k \text{ for all } i) &\iff (\alpha_i < 2 \text{ for all } i) && (\text{since } k = 2)\\ &\iff (\alpha_i \in \{0,1\} \text{ for all } i) && (\text{since } \alpha_i \in \mathbb{N} \text{ for all } i)\\ &\iff \left(\alpha \in \{0,1\}^\infty\right)\end{aligned}$$

for each $\alpha \in \mathrm{WC}$). Therefore, the condition "$\alpha_i < k$ for all i" under the summation sign "$\sum\limits_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m;\\ \alpha_i < k \text{ for all } i}}$" can be replaced by "$\alpha \in \{0,1\}^\infty$". Thus, we obtain the following equality between summation signs:

$$\sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m;\\ \alpha_i < k \text{ for all } i}} = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m;\\ \alpha \in \{0,1\}^\infty}} = \sum_{\substack{\alpha \in \mathrm{WC};\\ \alpha \in \{0,1\}^\infty;\\ |\alpha| = m}} = \sum_{\substack{\alpha \in \mathrm{WC} \cap \{0,1\}^\infty;\\ |\alpha| = m}} .$$

Now, (4) yields

$$G(k,m) = \underbrace{\sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m;\\ \alpha_i < k \text{ for all } i}}}_{= \sum\limits_{\substack{\alpha \in \mathrm{WC} \cap \{0,1\}^\infty;\\ |\alpha| = m}}} \mathbf{x}^\alpha = \sum_{\substack{\alpha \in \mathrm{WC} \cap \{0,1\}^\infty;\\ |\alpha| = m}} \mathbf{x}^\alpha = e_m$$

(since the definition of e_m yields $e_m = \sum\limits_{\substack{\alpha \in \mathrm{WC} \cap \{0,1\}^\infty;\\ |\alpha| = m}} \mathbf{x}^\alpha$). This proves Proposition 2.3 **(e)**.

(f) Let $m = k$. Thus, $p_m = p_k = m_{(k)}$ (since k is a positive integer).

Now, if $\lambda \in \mathrm{Par}$ satisfies $|\lambda| = k$, then we have the logical equivalence

$$(\lambda_i < k \text{ for all } i) \iff (\lambda \neq (k)) . \tag{30}$$

[*Proof of (30):* Let $\lambda \in \mathrm{Par}$ satisfy $|\lambda| = k$. We must prove the equivalence (30). We shall prove the "$\Longrightarrow$" and "$\Longleftarrow$" directions of this equivalence separately:

$\Longrightarrow$: Assume that $(\lambda_i < k$ for all $i)$. We must show that $\lambda \neq (k)$.

We have assumed that $(\lambda_i < k$ for all $i)$. Applying this to $i = 1$, we obtain $\lambda_1 < k$. But if we had $\lambda = (k)$, then we would have $\lambda_1 = (k)_1 = k$, which would contradict $\lambda_1 < k$. Hence, we cannot have $\lambda = (k)$. Thus, $\lambda \neq (k)$. This proves the "$\Longrightarrow$" implication of the equivalence (30).

$\Longleftarrow$: Assume that $\lambda \neq (k)$. We must show that $(\lambda_i < k$ for all $i)$.

Indeed, let i be a positive integer. Assume (for the sake of contradiction) that $\lambda_i \geq k$. But λ is a partition (since $\lambda \in \mathrm{Par}$); thus, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ and therefore $\lambda_1 \geq \lambda_i \geq k$. But $|\lambda| = k$, so that

$$\begin{aligned}k = |\lambda| &= \lambda_1 + \lambda_2 + \lambda_3 + \cdots && (\text{by the definition of } |\lambda|)\\ &= \underbrace{\lambda_1}_{\geq k} + (\lambda_2 + \lambda_3 + \lambda_4 + \cdots) \geq k + (\lambda_2 + \lambda_3 + \lambda_4 + \cdots) .\end{aligned}$$

Subtracting k from both sides of this inequality, we obtain $0 \geq \lambda_2 + \lambda_3 + \lambda_4 + \cdots$, so that $\lambda_2 + \lambda_3 + \lambda_4 + \cdots \leq 0$. In other words, the sum of the numbers $\lambda_2, \lambda_3, \lambda_4, \ldots$ is ≤ 0.

But the numbers $\lambda_2, \lambda_3, \lambda_4, \ldots$ are nonnegative integers (since λ is a partition). Hence, the only way their sum can be ≤ 0 is if they are all $= 0$. Since their sum is ≤ 0, we thus conclude that they are all $= 0$. In other words, $\lambda_2 = \lambda_3 = \lambda_4 = \cdots = 0$. Hence, $\lambda = (\lambda_1)$. Thus, $|\lambda| = \lambda_1$, so that $\lambda_1 = |\lambda| = k$. Hence, $\lambda = \left(\underbrace{\lambda_1}_{=k}\right) = (k)$. This contradicts $\lambda \neq (k)$.

This contradiction shows that our assumption (that $\lambda_i \geq k$) was false. Hence, we must have $\lambda_i < k$.

Forget that we fixed i. We have now showed that ($\lambda_i < k$ for all i). This proves the "$\Longleftarrow$" implication of the equivalence (30).

We have now proven both implications of the equivalence (30). This concludes the proof of (30).]

The logical equivalence (30) yields the following equality of summation signs:

$$\sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = k; \\ \lambda_i < k \text{ for all } i}} = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = k; \\ \lambda \neq (k)}} . \tag{31}$$

Now, one of the definitions of h_k yields

$$\begin{aligned} h_k &= \sum_{\lambda \in \operatorname{Par}_k} m_\lambda = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = k}} m_\lambda \\ &\qquad \left(\text{since } \operatorname{Par}_k \text{ is the set of all } \lambda \in \operatorname{Par} \text{ satisfying } |\lambda| = k\right) \\ &= \underbrace{m_{(k)}}_{=p_k} + \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = k; \\ \lambda \neq (k)}} m_\lambda \\ &\qquad \left(\text{here, we have split off the addend for } \lambda = (k) \text{ from the sum}\right) \\ &= p_k + \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = k; \\ \lambda \neq (k)}} m_\lambda . \end{aligned}$$

Hence,

$$h_k - p_k = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = k; \\ \lambda \neq (k)}} m_\lambda .$$

On the other hand, Proposition 2.3 **(c)** (applied to k instead of m) yields

$$G(k,k) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=k;\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha = \underbrace{\sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=k;\\ \lambda_i<k \text{ for all } i}}}_{\substack{=\sum\limits_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=k;\\ \lambda\neq(k)}} \\ \text{(by (31))}}} m_\lambda = \sum_{\substack{\lambda\in\mathrm{Par};\\ |\lambda|=k;\\ \lambda\neq(k)}} m_\lambda .$$

Comparing these two equalities, we obtain $G(k,k) = h_k - p_k$. In other words, $G(k,m) = h_m - p_m$ (since $m = k$). This proves Proposition 2.3 **(f)**. $\square$

3.5. Skew Schur functions

Let us define a classical partial order on Par (see, e.g., [GriRei20, Definition 2.3.1]):

Definition 3.8. Let λ and μ be two partitions.
We say that $\mu \subseteq \lambda$ if each $i \in \{1,2,3,\ldots\}$ satisfies $\mu_i \leq \lambda_i$.
We say that $\mu \not\subseteq \lambda$ if we don't have $\mu \subseteq \lambda$.

For example, $(3,2) \subseteq (4,2,1)$, but $(3,2,1) \not\subseteq (4,2)$ (since $(3,2,1)_3 = 1$ is not $\leq$ to $(4,2)_3 = 0$).

For any two partitions λ and μ, a symmetric function $s_{\lambda/\mu}$ called a *skew Schur function* is defined in [GriRei20, Definition 2.3.1] and in [Macdon95, §I.5] (see also [Stanle01, Definition 7.10.1] for the case when $\mu \subseteq \lambda$). We shall not recall its standard definition here, but rather state a few properties.

The first property (which can in fact be used as an alternative definition of $s_{\lambda/\mu}$) is the *first Jacobi–Trudi formula* for skew shapes; it states the following:

Theorem 3.9. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ be two partitions. Then,

$$s_{\lambda/\mu} = \det\left(\left(h_{\lambda_i-\mu_j-i+j}\right)_{1\leq i\leq\ell,\ 1\leq j\leq\ell}\right). \tag{32}$$

Theorem 3.9 appears (with proof) in [GriRei20, (2.4.16)] and in [Macdon95, Chapter I, (5.4)].

The following properties of skew Schur functions are easy to see:

- If λ is any partition, then $s_{\lambda/\varnothing} = s_\lambda$. (Recall that $\varnothing$ denotes the empty partition.)

- If λ and μ are two partitions satisfying $\mu \not\subseteq \lambda$, then $s_{\lambda/\mu} = 0$.

3.6. A Cauchy-like identity

We shall use the following identity, which connects the skew Schur functions $s_{\lambda/\mu}$, the symmetric functions h_λ from Definition 3.4 and the monomial symmetric functions m_λ:

Theorem 3.10. Recall the symmetric functions h_λ defined in Definition 3.4. Let μ be any partition. Then, in the ring $\mathbf{k}\left[\left[\mathbf{x},\mathbf{y}\right]\right]$, we have

$$\sum_{\lambda\in\operatorname{Par}} s_{\lambda/\mu}\left(\mathbf{x}\right) s_\lambda\left(\mathbf{y}\right) = s_\mu\left(\mathbf{y}\right)\cdot\sum_{\lambda\in\operatorname{Par}} h_\lambda\left(\mathbf{x}\right) m_\lambda\left(\mathbf{y}\right).$$

Here, we are using the notations introduced in Subsection 2.6.

Theorem 3.10 appears in [Macdon95, fourth display on page 70], but let us give a proof for the sake of completeness:

Proof of Theorem 3.10. A well-known identity (proved, e.g., in [Macdon95, Chapter I, (4.2)] and in [GriRei20, proof of Proposition 2.5.15]) says that

$$\prod_{i,j=1}^{\infty}\left(1-x_iy_j\right)^{-1} = \sum_{\lambda\in\operatorname{Par}} h_\lambda\left(\mathbf{x}\right) m_\lambda\left(\mathbf{y}\right). \tag{33}$$

(Here, the product sign "$\prod_{i,j=1}^{\infty}$" means "$\prod_{(i,j)\in\{1,2,3,\ldots\}^2}$".)

Another well-known identity (proved, e.g., in [Macdon95, §I.5, example 26] and in [GriRei20, Exercise 2.5.11(a)]) says that

$$\sum_{\lambda\in\operatorname{Par}} s_\lambda\left(\mathbf{x}\right) s_{\lambda/\mu}\left(\mathbf{y}\right) = s_\mu\left(\mathbf{x}\right)\cdot\prod_{i,j=1}^{\infty}\left(1-x_iy_j\right)^{-1}.$$

If we swap the roles of $\mathbf{x}=\left(x_1,x_2,x_3,\ldots\right)$ and $\mathbf{y}=\left(y_1,y_2,y_3,\ldots\right)$ in this identity, then we obtain

$$\sum_{\lambda\in\operatorname{Par}} s_\lambda\left(\mathbf{y}\right) s_{\lambda/\mu}\left(\mathbf{x}\right) = s_\mu\left(\mathbf{y}\right)\cdot\prod_{i,j=1}^{\infty}\left(1-y_ix_j\right)^{-1}.$$

In view of

$$\sum_{\lambda\in\operatorname{Par}} \underbrace{s_\lambda\left(\mathbf{y}\right) s_{\lambda/\mu}\left(\mathbf{x}\right)}_{=s_{\lambda/\mu}\left(\mathbf{x}\right)s_\lambda\left(\mathbf{y}\right)} = \sum_{\lambda\in\operatorname{Par}} s_{\lambda/\mu}\left(\mathbf{x}\right) s_\lambda\left(\mathbf{y}\right)$$

and

$$\prod_{i,j=1}^{\infty}\left(1-y_ix_j\right)^{-1}=\underbrace{\prod_{j,i=1}^{\infty}}_{=\prod_{i,j=1}^{\infty}}\left(1-\underbrace{y_jx_i}_{=x_iy_j}\right)^{-1}$$

$$\left(\begin{array}{c}\text{here, we have renamed the index } (i,j) \text{ as } (j,i)\\ \text{in the product}\end{array}\right)$$

$$=\prod_{i,j=1}^{\infty}\left(1-x_iy_j\right)^{-1}=\sum_{\lambda\in\operatorname{Par}}h_\lambda\left(\mathbf{x}\right)m_\lambda\left(\mathbf{y}\right)\qquad\left(\text{by (33)}\right),$$

this rewrites as

$$\sum_{\lambda\in\operatorname{Par}}s_{\lambda/\mu}\left(\mathbf{x}\right)s_\lambda\left(\mathbf{y}\right)=s_\mu\left(\mathbf{y}\right)\cdot\sum_{\lambda\in\operatorname{Par}}h_\lambda\left(\mathbf{x}\right)m_\lambda\left(\mathbf{y}\right).$$

This proves Theorem 3.10. □

3.7. The k-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$

Recall that the family $\left(h_n\right)_{n\geq 1}=\left(h_1,h_2,h_3,\ldots\right)$ is algebraically independent and generates Λ as a $\mathbf{k}$-algebra. Thus, Λ can be viewed as a polynomial ring in the (infinitely many) indeterminates $h_1,h_2,h_3,\ldots$. The universal property of a polynomial ring thus shows that if A is any commutative $\mathbf{k}$-algebra, and if $\left(a_1,a_2,a_3,\ldots\right)$ is any sequence of elements of A, then there is a unique $\mathbf{k}$-algebra homomorphism from Λ to A that sends h_i to a_i for all positive integers i. We shall refer to this as the *h-universal property of* Λ. It lets us make the following definition:[27]

Definition 3.11. Let k be a positive integer. The h-universal property of Λ shows that there is a unique $\mathbf{k}$-algebra homomorphism $\alpha_k:\Lambda\to\mathbf{k}$ that sends h_i to $[i<k]$ for all positive integers i. Consider this α_k.

We will use this homomorphism α_k several times in what follows; let us thus begin by stating some elementary properties of α_k.

Lemma 3.12. Let k be a positive integer.

(a) We have

$$\alpha_k\left(h_i\right)=[i<k]\qquad\text{for all } i\in\mathbb{N}. \tag{34}$$

(b) We have

$$\alpha_k\left(h_i\right)=[0\leq i<k]\qquad\text{for all } i\in\mathbb{Z}. \tag{35}$$

[27]We are using the Iverson bracket notation (see Convention 2.4) here.

(c) Let λ be a partition. Define h_λ as in Definition 3.4. Then,

$$\alpha_k(h_\lambda) = [\lambda_i < k \text{ for all } i]. \tag{36}$$

(Here, "for all i" means "for all positive integers i".)

Proof of Lemma 3.12. Note that $0 < k$ (since k is positive).

(a) Let $i \in \mathbb{N}$. We must prove that $\alpha_k(h_i) = [i < k]$. If $i > 0$, then this follows from the definition of α_k. Thus, we WLOG assume that we don't have $i > 0$. Hence, $i = 0$ (since $i \in \mathbb{N}$). Therefore, $h_i = h_0 = 1$, so that $\alpha_k(h_i) = \alpha_k(1) = 1$ (since α_k is a $\mathbf{k}$-algebra homomorphism). On the other hand, $i = 0 < k$, so that $[i < k] = 1$. Comparing this with $\alpha_k(h_i) = 1$, we obtain $\alpha_k(h_i) = [i < k]$. This proves Lemma 3.12 **(a)**.

(b) Let $i \in \mathbb{Z}$. We must prove that $\alpha_k(h_i) = [0 \leq i < k]$. If $i < 0$, then this easily follows from $0 = 0$ [28]. Hence, we WLOG assume that we don't have $i < 0$. Therefore, $i \geq 0$, so that $i \in \mathbb{N}$. Hence, Lemma 3.12 **(a)** yields $\alpha_k(h_i) = [i < k]$. On the other hand, the statement "$0 \leq i < k$" is equivalent to the statement "$i < k$" (since $0 \leq i$ holds automatically[29]); thus, $[0 \leq i < k] = [i < k]$. Comparing this with $\alpha_k(h_i) = [i < k]$, we obtain $\alpha_k(h_i) = [0 \leq i < k]$. This proves Lemma 3.12 **(b)**.

(c) We recall the following simple property of truth values: If $\ell \in \mathbb{N}$, and if $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_\ell$ are ℓ logical statements, then

$$\begin{aligned}\prod_{i=1}^{\ell} [\mathcal{A}_i] &= [\mathcal{A}_1][\mathcal{A}_2]\cdots[\mathcal{A}_\ell] = [\mathcal{A}_1 \wedge \mathcal{A}_2 \wedge \cdots \wedge \mathcal{A}_\ell] \\ &= [\mathcal{A}_i \text{ for all } i \in \{1, 2, \ldots, \ell\}].\end{aligned} \tag{37}$$

Now, write the partition λ in the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are positive integers. Then, the definition of h_λ yields

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell} = \prod_{i=1}^{\ell} h_{\lambda_i}.$$

[28] *Proof.* Assume that $i < 0$. Thus, $h_i = 0$, so that $\alpha_k(h_i) = \alpha_k(0) = 0$ (since α_k is a $\mathbf{k}$-algebra homomorphism). But the statement "$0 \leq i < k$" is false (since $i < 0$); hence, $[0 \leq i < k] = 0$. Comparing this with $\alpha_k(h_i) = 0$, we obtain $\alpha_k(h_i) = [0 \leq i < k]$, qed.

[29] because $i \geq 0$

Applying the map α_k to both sides of this equality, we find

$$\alpha_k(h_\lambda) = \alpha_k\left(\prod_{i=1}^{\ell} h_{\lambda_i}\right) = \prod_{i=1}^{\ell} \underbrace{\alpha_k(h_{\lambda_i})}_{\substack{=[\lambda_i<k] \\ \text{(by (34), applied to } \lambda_i \text{ instead of } i)}}$$

$$\left(\text{since } \alpha_k \text{ is a } \mathbf{k}\text{-algebra homomorphism}\right)$$

$$= \prod_{i=1}^{\ell} [\lambda_i < k] = [\lambda_i < k \text{ for all } i \in \{1, 2, \ldots, \ell\}] \tag{38}$$

$$\left(\text{by (37), applied to } \mathcal{A}_i = (\text{“}\lambda_i < k\text{”})\right).$$

But we have $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and thus $\lambda_{\ell+1} = \lambda_{\ell+2} = \lambda_{\ell+3} = \cdots = 0$. In other words, we have $\lambda_i = 0$ for all $i \in \{\ell+1, \ell+2, \ell+3, \ldots\}$. Hence, we have $\lambda_i < k$ for all $i \in \{\ell+1, \ell+2, \ell+3, \ldots\}$ (since $\lambda_i = 0$ implies $\lambda_i < k$ (because $0 < k$)).

Now, we have the following equivalence of logical statements:

$$\begin{aligned}
&(\lambda_i < k \text{ for all } i) \\
\iff &(\lambda_i < k \text{ for all positive integers } i) \\
\iff &\left(\lambda_i < k \text{ for all } i \in \underbrace{\{1, 2, 3, \ldots\}}_{=\{1,2,\ldots,\ell\} \cup \{\ell+1,\ell+2,\ell+3,\ldots\}}\right) \\
\iff &(\lambda_i < k \text{ for all } i \in \{1, 2, \ldots, \ell\} \cup \{\ell+1, \ell+2, \ell+3, \ldots\}) \\
\iff &(\lambda_i < k \text{ for all } i \in \{1, 2, \ldots, \ell\} \text{ and all } i \in \{\ell+1, \ell+2, \ell+3, \ldots\}) \\
\iff &(\lambda_i < k \text{ for all } i \in \{1, 2, \ldots, \ell\}) \wedge \underbrace{(\lambda_i < k \text{ for all } i \in \{\ell+1, \ell+2, \ell+3, \ldots\})}_{\substack{\iff (\text{true}) \\ (\text{since we have } \lambda_i<k \text{ for all } i\in\{\ell+1,\ell+2,\ell+3,\ldots\})}} \\
\iff &(\lambda_i < k \text{ for all } i \in \{1, 2, \ldots, \ell\}) \wedge (\text{true}) \\
\iff &(\lambda_i < k \text{ for all } i \in \{1, 2, \ldots, \ell\}).
\end{aligned}$$

Hence,

$$[\lambda_i < k \text{ for all } i] = [\lambda_i < k \text{ for all } i \in \{1, 2, \ldots, \ell\}].$$

Comparing this with (38), we obtain $\alpha_k(h_\lambda) = [\lambda_i < k \text{ for all } i]$. Thus, Lemma 3.12 **(c)** is proved. $\square$

3.8. Proof of Lemma 2.7

We shall give two proofs of Lemma 2.7: one using the homomorphism α_k from Definition 3.11, and one by direct manipulation of determinants.

First proof of Lemma 2.7. Recall that $\alpha_k : \Lambda \to \mathbf{k}$ is a $\mathbf{k}$-algebra homomorphism. Thus, α_k respects determinants; i.e., if $(a_{i,j})_{1\leq i\leq m,\ 1\leq j\leq m} \in \Lambda^{m\times m}$ is an $m \times m$-matrix over Λ (for some $m \in \mathbb{N}$), then

$$\alpha_k \left(\det \left(\left(a_{i,j} \right)_{1 \leq i \leq m,\ 1 \leq j \leq m} \right) \right) = \det \left(\left(\alpha_k \left(a_{i,j} \right) \right)_{1 \leq i \leq m,\ 1 \leq j \leq m} \right). \tag{39}$$

Applying α_k to both sides of the equality (32), we obtain

$$\begin{aligned}
\alpha_k \left(s_{\lambda/\mu} \right) &= \alpha_k \left(\det \left(\left(h_{\lambda_i - \mu_j - i + j} \right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell} \right) \right) \\
&= \det \left(\left(\underbrace{\alpha_k \left(h_{\lambda_i - \mu_j - i + j} \right)}_{\substack{= [0 \leq \lambda_i - \mu_j - i + j < k] \\ \text{(by (35), applied to } \lambda_i - \mu_j - i + j \\ \text{instead of } i)}} \right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell} \right) \\
&\qquad \left(\text{by (39), applied to } m = \ell \text{ and } a_{i,j} = h_{\lambda_i - \mu_j - i + j} \right) \\
&= \det \left(\left(\left[0 \leq \lambda_i - \mu_j - i + j < k \right] \right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell} \right).
\end{aligned} \tag{40}$$

Clearly, the integer $\alpha_k \left(s_{\lambda/\mu} \right)$ does not depend on the choice of ℓ. In view of the equality (40), we can rewrite this as follows: The integer $\det \left(\left(\left[0 \leq \lambda_i - \mu_j - i + j < k \right] \right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell} \right)$ does not depend on the choice of ℓ. This proves Lemma 2.7. $\square$

Second proof of Lemma 2.7. It suffices to show that

$$\begin{aligned}
&\det \left(\left(\left[0 \leq \lambda_i - \mu_j - i + j < k \right] \right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell} \right) \\
&= \det \left(\left(\left[0 \leq \lambda_i - \mu_j - i + j < k \right] \right)_{1 \leq i \leq \ell+1,\ 1 \leq j \leq \ell+1} \right).
\end{aligned}$$

So let us prove this.

From $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$, we obtain $\mu_{\ell+1} = 0$. From $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, we obtain $\lambda_{\ell+1} = 0$. Hence,

$$\left[0 \leq \underbrace{\lambda_{\ell+1} - \mu_{\ell+1} - (\ell+1) + (\ell+1)}_{\substack{= \lambda_{\ell+1} - \mu_{\ell+1} = 0 \\ \text{(since } \lambda_{\ell+1} = 0 \text{ and } \mu_{\ell+1} = 0)}} < k \right] = [0 \leq 0 < k] = 1$$

(since $0 \leq 0 < k$).

Recall that $\lambda_{\ell+1} = 0$. Hence, each $j \in \{1, 2, \ldots, \ell\}$ satisfies

$$\left[0 \leq \lambda_{\ell+1} - \mu_j - (\ell+1) + j < k\right] = 0 \tag{41}$$

[30]. In other words, the first ℓ entries of the last row of the matrix $\left(\left[0 \leq \lambda_i - \mu_j - i + j < k\right]\right)_{1 \leq i \leq \ell+1,\ 1 \leq j \leq \ell+1}$ are 0. Hence, if we expand the determinant of this matrix along this last row, then we obtain a sum having only one (potentially) nonzero addend, namely

$$\underbrace{\left[0 \leq \lambda_{\ell+1} - \mu_{\ell+1} - (\ell+1) + (\ell+1) < k\right]}_{=1} \cdot \det\left(\left(\left[0 \leq \lambda_i - \mu_j - i + j < k\right]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right)$$
$$= \det\left(\left(\left[0 \leq \lambda_i - \mu_j - i + j < k\right]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right).$$

Thus,

$$\det\left(\left(\left[0 \leq \lambda_i - \mu_j - i + j < k\right]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right)$$
$$= \det\left(\left(\left[0 \leq \lambda_i - \mu_j - i + j < k\right]\right)_{1 \leq i \leq \ell+1,\ 1 \leq j \leq \ell+1}\right).$$

This completes our second proof of Lemma 2.7. □

Our first proof of Lemma 2.7 has an additional consequence that will be useful to us:

Lemma 3.13. Let k be a positive integer. Let λ and μ be two partitions. Then, the homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ from Definition 3.11 satisfies

$$\alpha_k\left(s_{\lambda/\mu}\right) = \operatorname{pet}_k(\lambda, \mu). \tag{42}$$

Proof of Lemma 3.13. Write the partitions λ and μ in the forms $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ for some $\ell \in \mathbb{N}$ [31]. Then, the equality (40) (which we showed in our first proof of Lemma 2.7) yields

$$\alpha_k\left(s_{\lambda/\mu}\right) = \det\left(\left(\left[0 \leq \lambda_i - \mu_j - i + j < k\right]\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right) = \operatorname{pet}_k(\lambda, \mu)$$

(by the definition of $\operatorname{pet}_k(\lambda, \mu)$). This proves Lemma 3.13. □

[30] *Proof of (41):* Let $j \in \{1, 2, \ldots, \ell\}$. Then, $\underbrace{\lambda_{\ell+1}}_{=0} - \underbrace{\mu_j}_{\geq 0} - \underbrace{(\ell+1)}_{>\ell} + \underbrace{j}_{\leq \ell} < 0 - 0 - \ell + \ell = 0$. Hence, $0 \leq \lambda_{\ell+1} - \mu_j - (\ell+1) + j < k$ cannot hold. Thus, $\left[0 \leq \lambda_{\ell+1} - \mu_j - (\ell+1) + j < k\right] = 0$. Qed.

[31] Such an ℓ can always be found, since each of λ and μ has only finitely many nonzero entries.

3.9. Proof of Proposition 2.8

Proof of Proposition 2.8. Write the partitions λ and μ in the forms $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ for some $\ell \in \mathbb{N}$ [32]. The definition of $\operatorname{pet}_k(\lambda, \mu)$ yields

$$\operatorname{pet}_k(\lambda,\mu) = \det\left(\left(\left[\underbrace{0 \leq \lambda_i - \mu_j - i + j < k}_{\text{This is equivalent to } \mu_j - j \leq \lambda_i - i < \mu_j - j + k}\right]\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right)$$
$$= \det\left(\left(\left[\mu_j - j \leq \lambda_i - i < \mu_j - j + k\right]\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right). \qquad (43)$$

Let B be the $\ell \times \ell$-matrix $\left(\left[\mu_j - j \leq \lambda_i - i < \mu_j - j + k\right]\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell} \in \mathbf{k}^{\ell\times\ell}$. Then, (43) rewrites as follows:

$$\operatorname{pet}_k(\lambda,\mu) = \det B. \qquad (44)$$

We will use the concept of Petrie matrices (see [GorWil74, Theorem 1]). Namely, a *Petrie matrix* is a matrix whose entries all belong to $\{0, 1\}$ and such that the 1's in each column occur consecutively (i.e., as a contiguous block). In other words, a Petrie matrix is a matrix whose each column has the form

$$\left(\underbrace{0,0,\ldots,0}_{a \text{ zeroes}}, \underbrace{1,1,\ldots,1}_{b \text{ ones}}, \underbrace{0,0,\ldots,0}_{c \text{ zeroes}}\right)^T \qquad (45)$$

for some nonnegative integers a, b, c (where any of a, b, c can be 0). For example, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ is a Petrie matrix, but $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ is not.

A well-known result due to Fulkerson and Gross (first stated in [FulGro65, §8][33]) says that if a square matrix A is a Petrie matrix, then

$$\det A \in \{-1, 0, 1\}. \qquad (46)$$

Now, we shall show that B is a Petrie matrix.

Indeed, fix some $j \in \{1, 2, \ldots, \ell\}$. We have $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$ (since λ is a partition) and thus $\lambda_1 - 1 > \lambda_2 - 2 > \cdots > \lambda_\ell - \ell$. In other words, the numbers $\lambda_i - i$ for $i \in \{1, 2, \ldots, \ell\}$ decrease as i increases. Hence, the set of all $i \in \{1, 2, \ldots, \ell\}$ satisfying $\mu_j - j \leq \lambda_i - i < \mu_j - j + k$ is a (possibly empty) integer interval[34]. Let us denote this integer interval by I_j. Therefore, if we let i range over $\{1, 2, \ldots, \ell\}$, then the truth value $\left[\mu_j - j \leq \lambda_i - i < \mu_j - j + k\right]$ will be 1 for all $i \in I_j$, and 0 for

[32] Such an ℓ can always be found, since each of λ and μ has only finitely many nonzero entries.

[33] See [GorWil74, Theorem 1] for an explicit proof.

[34] An *integer interval* means a subset of $\mathbb{Z}$ that has the form $\{a, a+1, \ldots, b\}$ for some $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. (If $a > b$, then this is the empty set.)

all other i. Since I_j is an integer interval, this means that this truth value will be 1 when i lies in a certain integer interval (namely, I_j) and 0 when i lies outside it. In other words, the j-th column of the matrix B has a contiguous (but possibly empty) block of 1's (in the rows corresponding to all $i \in I_j$), while all other entries of this column are 0 (because the entries of the j-th column of B are precisely these truth values $\left[\mu_j - j \leq \lambda_i - i < \mu_j - j + k\right]$ for all $i \in \{1, 2, \ldots, \ell\}$). Therefore, this column has the form (45) for some nonnegative integers a, b, c.

Now, forget that we fixed j. We thus have proved that for each $j \in \{1, 2, \ldots, \ell\}$, the j-th column of B has the form (45) for some nonnegative integers a, b, c. In other words, each column of B has this form. Hence, B is a Petrie matrix (by the definition of a Petrie matrix). Therefore, (46) (applied to $A = B$) yields $\det B \in \{-1, 0, 1\}$. Thus, (44) becomes $\operatorname{pet}_k(\lambda, \mu) = \det B \in \{-1, 0, 1\}$. This proves Proposition 2.8. $\square$

3.10. First proof of Theorem 2.17

We are now ready for our first proof of Theorem 2.17:

First proof of Theorem 2.17. We shall use the notations $\mathbf{k}[[\mathbf{x}]]$, $\mathbf{k}[[\mathbf{x}, \mathbf{y}]]$, $\mathbf{x}$, $\mathbf{y}$, $f(\mathbf{x})$ and $f(\mathbf{y})$ introduced in Subsection 2.6. If R is any commutative ring, then $R[[\mathbf{y}]]$ shall denote the ring $R[[y_1, y_2, y_3, \ldots]]$ of formal power series in the indeterminates $y_1, y_2, y_3, \ldots$ over the ring R. We will identify the ring $\mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ with the ring $(\mathbf{k}[[\mathbf{x}]])[[\mathbf{y}]] = (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[y_1, y_2, y_3, \ldots]]$. Note that $\Lambda \subseteq \mathbf{k}[[\mathbf{x}]]$ and thus $\Lambda[[\mathbf{y}]] \subseteq (\mathbf{k}[[\mathbf{x}]])[[\mathbf{y}]] = \mathbf{k}[[\mathbf{x}, \mathbf{y}]]$. We equip the rings $\mathbf{k}[[\mathbf{y}]]$, $\Lambda[[\mathbf{y}]]$ and $\mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ with the usual topologies that are defined on rings of power series, where Λ itself is equipped with the discrete topology. This has the somewhat confusing consequence that $\Lambda[[\mathbf{y}]] \subseteq \mathbf{k}[[\mathbf{x}, \mathbf{y}]]$ is an inclusion of rings but not of topological spaces; however, this will not cause us any trouble, since all infinite sums in $\Lambda[[\mathbf{y}]]$ we will consider (such as $\sum\limits_{\lambda \in \operatorname{Par}} s_{\lambda/\mu}(\mathbf{x}) s_\lambda(\mathbf{y})$ and $\sum\limits_{\lambda \in \operatorname{Par}} h_\lambda(\mathbf{x}) m_\lambda(\mathbf{y})$) will converge to the same value in either topology.

We consider both $\mathbf{k}[[\mathbf{y}]]$ and Λ as subrings of $\Lambda[[\mathbf{y}]]$ (indeed, $\mathbf{k}[[\mathbf{y}]]$ embeds into $\Lambda[[\mathbf{y}]]$ because $\mathbf{k}$ is a subring of Λ, whereas Λ embeds into $\Lambda[[\mathbf{y}]]$ because $\Lambda[[\mathbf{y}]]$ is a ring of power series over Λ).

In this proof, the word "monomial" may refer to a monomial in any set of variables (not necessarily in $x_1, x_2, x_3, \ldots$).

Recall the $\mathbf{k}$-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ from Definition 3.11. This $\mathbf{k}$-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ induces a $\mathbf{k}[[\mathbf{y}]]$-algebra homomorphism $\alpha_k[[\mathbf{y}]] : \Lambda[[\mathbf{y}]] \to \mathbf{k}[[\mathbf{y}]]$, which is given by the formula

$$(\alpha_k[[\mathbf{y}]]) \left(\sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} f_\mathfrak{n} \mathfrak{n} \right) = \sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} \alpha_k(f_\mathfrak{n}) \mathfrak{n}$$

for any family $(f_{\mathfrak{n}})_{\mathfrak{n} \text{ is a monomial in } y_1, y_2, y_3, \ldots}$ of elements of Λ. This induced $\mathbf{k}[[\mathbf{y}]]$-algebra homomorphism $\alpha_k[[\mathbf{y}]]$ is $\mathbf{k}[[\mathbf{y}]]$-linear and continuous (with respect to the usual topologies on the power series rings $\Lambda[[\mathbf{y}]]$ and $\mathbf{k}[[\mathbf{y}]]$), and thus preserves infinite $\mathbf{k}[[\mathbf{y}]]$-linear combinations. Moreover, it extends α_k (that is, for any $f \in \Lambda$, we have $(\alpha_k[[\mathbf{y}]])(f) = \alpha_k(f)$).

Recall the skew Schur functions $s_{\lambda/\mu}$ defined in Subsection 3.5. Also, recall the symmetric functions h_λ defined in Definition 3.4. Theorem 3.10 yields

$$\begin{aligned}\sum_{\lambda\in\operatorname{Par}} s_{\lambda/\mu}(\mathbf{x})\, s_\lambda(\mathbf{y}) &= s_\mu(\mathbf{y}) \cdot \sum_{\lambda\in\operatorname{Par}} h_\lambda(\mathbf{x})\, m_\lambda(\mathbf{y}) = \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y}) \underbrace{h_\lambda(\mathbf{x})\, m_\lambda(\mathbf{y})}_{=m_\lambda(\mathbf{y}) h_\lambda(\mathbf{x})} \\ &= \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y}) \underbrace{h_\lambda(\mathbf{x})}_{=h_\lambda} = \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y})\, h_\lambda.\end{aligned}$$

Comparing this with

$$\sum_{\lambda\in\operatorname{Par}} \underbrace{s_{\lambda/\mu}(\mathbf{x})\, s_\lambda(\mathbf{y})}_{=s_\lambda(\mathbf{y}) s_{\lambda/\mu}(\mathbf{x})} = \sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y}) \underbrace{s_{\lambda/\mu}(\mathbf{x})}_{=s_{\lambda/\mu}} = \sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y})\, s_{\lambda/\mu},$$

we obtain

$$\sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y})\, s_{\lambda/\mu} = \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y})\, h_\lambda.$$

Consider this as an equality in the ring $\Lambda[[\mathbf{y}]] = \Lambda[[y_1, y_2, y_3, \ldots]]$. Apply the map $\alpha_k[[\mathbf{y}]] : \Lambda[[\mathbf{y}]] \to \mathbf{k}[[\mathbf{y}]]$ to both sides of this equality. We obtain

$$(\alpha_k[[\mathbf{y}]]) \left(\sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y})\, s_{\lambda/\mu} \right) = (\alpha_k[[\mathbf{y}]]) \left(\sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y})\, h_\lambda \right).$$

Comparing this with

$$\begin{aligned}&(\alpha_k[[\mathbf{y}]]) \left(\sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y})\, s_{\lambda/\mu} \right) \\ &= \sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y}) \cdot \underbrace{(\alpha_k[[\mathbf{y}]])\left(s_{\lambda/\mu}\right)}_{\substack{=\alpha_k(s_{\lambda/\mu}) \\ \text{(since } \alpha_k[[\mathbf{y}]] \text{ extends } \alpha_k)}} \\ &\qquad \left(\text{since the map } \alpha_k[[\mathbf{y}]] \text{ preserves infinite } \mathbf{k}[[\mathbf{y}]]\text{-linear combinations}\right) \\ &= \sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y}) \cdot \underbrace{\alpha_k\left(s_{\lambda/\mu}\right)}_{\substack{=\operatorname{pet}_k(\lambda,\mu) \\ \text{(by (42))}}} = \sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y}) \cdot \operatorname{pet}_k(\lambda,\mu) = \sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_k(\lambda,\mu) \cdot s_\lambda(\mathbf{y}),\end{aligned}$$

we obtain

$$\begin{aligned}
&\sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_k(\lambda,\mu)\cdot s_\lambda(\mathbf{y}) \\
&= (\alpha_k[[\mathbf{y}]])\left(\sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y})\, h_\lambda\right) = \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y}) \underbrace{(\alpha_k[[\mathbf{y}]])(h_\lambda)}_{\substack{=\alpha_k(h_\lambda) \\ (\text{since } \alpha_k[[\mathbf{y}]] \text{ extends } \alpha_k)}} \\
&\qquad (\text{since the map } \alpha_k[[\mathbf{y}]] \text{ preserves infinite } \mathbf{k}[[\mathbf{y}]]\text{-linear combinations}) \\
&= \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y}) \underbrace{\alpha_k(h_\lambda)}_{\substack{=[\lambda_i<k \text{ for all } i] \\ (\text{by } (36))}} = \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y})\cdot[\lambda_i<k \text{ for all } i] \\
&= \sum_{\lambda\in\operatorname{Par}} [\lambda_i<k \text{ for all } i]\cdot s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y}).
\end{aligned}$$

Renaming the indeterminates $\mathbf{y}=(y_1,y_2,y_3,\ldots)$ as $\mathbf{x}=(x_1,x_2,x_3,\ldots)$ on both sides of this equality, we obtain

$$\begin{aligned}
&\sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_k(\lambda,\mu)\cdot s_\lambda(\mathbf{x}) \\
&= \sum_{\lambda\in\operatorname{Par}} [\lambda_i<k \text{ for all } i]\cdot \underbrace{s_\mu(\mathbf{x})}_{=s_\mu}\underbrace{m_\lambda(\mathbf{x})}_{=m_\lambda} = \sum_{\lambda\in\operatorname{Par}} [\lambda_i<k \text{ for all } i]\cdot s_\mu m_\lambda \\
&= \sum_{\substack{\lambda\in\operatorname{Par}; \\ \lambda_i<k \text{ for all } i}} \underbrace{[\lambda_i<k \text{ for all } i]}_{\substack{=1 \\ (\text{since } \lambda_i<k \text{ for all } i)}} \cdot s_\mu m_\lambda + \sum_{\substack{\lambda\in\operatorname{Par}; \\ \text{not } (\lambda_i<k \text{ for all } i)}} \underbrace{[\lambda_i<k \text{ for all } i]}_{\substack{=0 \\ (\text{since we don't have "}\lambda_i<k \text{ for all } i\text{")}}} \cdot s_\mu m_\lambda \\
&\qquad (\text{since any } \lambda\in\operatorname{Par} \text{ satisfies either } (\lambda_i<k \text{ for all } i) \text{ or not } (\lambda_i<k \text{ for all } i)) \\
&= \sum_{\substack{\lambda\in\operatorname{Par}; \\ \lambda_i<k \text{ for all } i}} s_\mu m_\lambda + \underbrace{\sum_{\substack{\lambda\in\operatorname{Par}; \\ \text{not } (\lambda_i<k \text{ for all } i)}} 0 s_\mu m_\lambda}_{=0} = \sum_{\substack{\lambda\in\operatorname{Par}; \\ \lambda_i<k \text{ for all } i}} s_\mu m_\lambda.
\end{aligned}$$

Comparing this with

$$G(k)\cdot s_\mu = s_\mu\cdot \underbrace{G(k)}_{\substack{=\sum\limits_{\substack{\lambda\in\operatorname{Par}; \\ \lambda_i<k \text{ for all } i}} m_\lambda \\ (\text{by Proposition 2.3 } \textbf{(b)})}} = s_\mu\cdot \sum_{\substack{\lambda\in\operatorname{Par}; \\ \lambda_i<k \text{ for all } i}} m_\lambda = \sum_{\substack{\lambda\in\operatorname{Par}; \\ \lambda_i<k \text{ for all } i}} s_\mu m_\lambda,$$

we obtain

$$G(k)\cdot s_\mu = \sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_k(\lambda,\mu)\cdot \underbrace{s_\lambda(\mathbf{x})}_{=s_\lambda} = \sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_k(\lambda,\mu)\, s_\lambda.$$

This proves Theorem 2.17. $\square$

3.11. Second proof of Theorem 2.17

Our second proof of Theorem 2.17 will rely on [GriRei20, §2.6] and specifically on the notion of alternants:

Second proof of Theorem 2.17. If $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is any formal power series, and if $\ell \in \mathbb{N}$, then $f(x_1, x_2, \ldots, x_\ell)$ shall denote the formal power series

$$f(x_1, x_2, \ldots, x_\ell, 0, 0, 0, \ldots) \in \mathbf{k}[[x_1, x_2, \ldots, x_\ell]]$$

that is obtained by substituting $0, 0, 0, \ldots$ for the variables $x_{\ell+1}, x_{\ell+2}, x_{\ell+3}, \ldots$ in f. Equivalently, $f(x_1, x_2, \ldots, x_\ell)$ can be obtained from f by removing all monomials that contain any of the variables $x_{\ell+1}, x_{\ell+2}, x_{\ell+3}, \ldots$. This makes it clear that any $f \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ satisfies

$$f = \lim_{\ell \to \infty} f(x_1, x_2, \ldots, x_\ell) \tag{47}$$

(where the limit is taken with respect to the usual topology on $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$).

Let $\ell(\mu)$ denote the *length* of the partition μ; this is defined as the unique $i \in \mathbb{N}$ such that $\mu_1, \mu_2, \ldots, \mu_i$ are positive but $\mu_{i+1}, \mu_{i+2}, \mu_{i+3}, \ldots$ are zero. Thus, $\ell(\mu)$ is the number of parts of μ.

Fix an $\ell \in \mathbb{N}$ such that $\ell \geq \ell(\mu)$.

Let P_ℓ denote the set of all partitions with at most ℓ parts. We shall show that

$$\begin{aligned} &(G(k))(x_1, x_2, \ldots, x_\ell) \cdot s_\mu(x_1, x_2, \ldots, x_\ell) \\ &= \sum_{\lambda \in P_\ell} \operatorname{pet}_k(\lambda, \mu) s_\lambda(x_1, x_2, \ldots, x_\ell). \end{aligned} \tag{48}$$

Once this is done, the usual "let ℓ tend to ∞" argument (analogous to [GriRei20, proof of Corollary 2.6.11]) will yield the validity of Theorem 2.17.

Any partition $\lambda \in P_\ell$ satisfies $\lambda_{\ell+1} = \lambda_{\ell+2} = \lambda_{\ell+3} = \cdots = 0$ and thus $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Thus, any partition $\lambda \in P_\ell$ can be regarded as an ℓ-tuple of nonnegative integers. In other words, $P_\ell \subseteq \mathbb{N}^\ell$. More precisely, the partitions $\lambda \in P_\ell$ are precisely the ℓ-tuples $(\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{N}^\ell$ satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell$.

Define the ℓ-tuple $\rho = (\ell - 1, \ell - 2, \ldots, 0) \in \mathbb{N}^\ell$.

For any ℓ-tuple $\alpha \in \mathbb{N}^\ell$ and each $i \in \{1, 2, \ldots, \ell\}$, we shall write α_i for the i-th entry of α.

For any ℓ-tuple $\alpha \in \mathbb{N}^\ell$, we let $\mathbf{x}^\alpha$ denote the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell}$.

Let $\mathfrak{S}_\ell$ denote the symmetric group of the set $\{1, 2, \ldots, \ell\}$. This group $\mathfrak{S}_\ell$ acts by $\mathbf{k}$-algebra homomorphisms on the polynomial ring $\mathbf{k}[x_1, x_2, \ldots, x_\ell]$.

The symmetric group $\mathfrak{S}_\ell$ also acts on the set $\mathbb{N}^\ell$ by permuting the entries of an ℓ-tuple; namely,

$$\sigma \cdot \beta = \left(\beta_{\sigma^{-1}(1)}, \beta_{\sigma^{-1}(2)}, \ldots, \beta_{\sigma^{-1}(\ell)}\right) \qquad \text{for any } \sigma \in \mathfrak{S}_\ell \text{ and } \beta \in \mathbb{N}^\ell.$$

This action has the property that

$$\mathbf{x}^{\sigma\cdot\beta} = \sigma\left(\mathbf{x}^{\beta}\right) \qquad \text{for any } \sigma \in \mathfrak{S}_\ell \text{ and } \beta \in \mathbb{N}^\ell.$$

For any $\sigma \in \mathfrak{S}_\ell$, we let $(-1)^\sigma$ denote the sign of the permutation σ.

If $\alpha \in \mathbb{N}^\ell$ is any ℓ-tuple, then we define the polynomial $a_\alpha \in \mathbf{k}[x_1, x_2, \ldots, x_\ell]$ by

$$a_\alpha = \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \sigma(\mathbf{x}^\alpha) = \det\left(\left(x_j^{\alpha_i}\right)_{1\le i\le \ell,\ 1\le j\le \ell}\right).$$

[35] This polynomial a_α is called the *α-alternant*.

We define addition of ℓ-tuples $\alpha \in \mathbb{N}^\ell$ entrywise (so that $(\alpha+\beta)_i = \alpha_i + \beta_i$ for every $\alpha, \beta \in \mathbb{N}^\ell$ and $i \in \{1, 2, \ldots, \ell\}$). Thus, $\lambda + \rho \in \mathbb{N}^\ell$ for each $\lambda \in P_\ell$ (since $\lambda \in P_\ell \subseteq \mathbb{N}^\ell$). Note that

$$\mathbf{x}^\alpha \cdot \mathbf{x}^\beta = \mathbf{x}^{\alpha+\beta} \tag{49}$$

for any $\alpha, \beta \in \mathbb{N}^\ell$.

It is known ([GriRei20, Corollary 2.6.7]) that

$$s_\lambda(x_1, x_2, \ldots, x_\ell) = \frac{a_{\lambda+\rho}}{a_\rho} \tag{50}$$

for every $\lambda \in P_\ell$. (The denominator a_ρ is a non-zero-divisor in the ring $\mathbf{k}[x_1, x_2, \ldots, x_\ell]$, and the quotient $\dfrac{a_{\lambda+\rho}}{a_\rho}$ exists.)

Note that $\ell \ge \ell(\mu)$, so that $\ell(\mu) \le \ell$; in other words, the partition μ has at most ℓ parts (since $\ell(\mu)$ is the number of parts of μ). In other words, $\mu \in P_\ell$.

---

[35] Here is why the second equality sign in this equality holds:

Let $\alpha \in \mathbb{N}^\ell$ be any ℓ-tuple. Then, each $\sigma \in \mathfrak{S}_\ell$ satisfies

$$\sigma\left(\underbrace{\mathbf{x}^\alpha}_{=x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_\ell^{\alpha_\ell}}\right) = \sigma\left(x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_\ell^{\alpha_\ell}\right) = x_{\sigma(1)}^{\alpha_1}x_{\sigma(2)}^{\alpha_2}\cdots x_{\sigma(\ell)}^{\alpha_\ell}$$

(by the definition of the action of $\mathfrak{S}_\ell$ on $\mathbf{k}[x_1, x_2, \ldots, x_\ell]$). Thus,

$$\sum_{\sigma\in\mathfrak{S}_\ell} (-1)^\sigma \underbrace{\sigma(\mathbf{x}^\alpha)}_{=x_{\sigma(1)}^{\alpha_1}x_{\sigma(2)}^{\alpha_2}\cdots x_{\sigma(\ell)}^{\alpha_\ell}} = \sum_{\sigma\in\mathfrak{S}_\ell} (-1)^\sigma x_{\sigma(1)}^{\alpha_1}x_{\sigma(2)}^{\alpha_2}\cdots x_{\sigma(\ell)}^{\alpha_\ell}.$$

Comparing this with

$$\det\left(\left(x_j^{\alpha_i}\right)_{1\le i\le \ell,\ 1\le j\le \ell}\right) = \sum_{\sigma\in\mathfrak{S}_\ell} (-1)^\sigma x_{\sigma(1)}^{\alpha_1}x_{\sigma(2)}^{\alpha_2}\cdots x_{\sigma(\ell)}^{\alpha_\ell} \qquad \left(\begin{array}{c}\text{by the definition}\\ \text{of a determinant}\end{array}\right),$$

we obtain $\sum\limits_{\sigma\in\mathfrak{S}_\ell} (-1)^\sigma \sigma(\mathbf{x}^\alpha) = \det\left(\left(x_j^{\alpha_i}\right)_{1\le i\le \ell,\ 1\le j\le \ell}\right)$. Qed.

Now, define $\alpha \in \mathbb{N}^\ell$ by $\alpha = \mu + \rho$. Proposition 2.3 **(b)** yields

$$G(k) = \prod_{i=1}^{\infty} \left(x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right).$$

Substituting $0, 0, 0, \ldots$ for $x_{\ell+1}, x_{\ell+2}, x_{\ell+3}, \ldots$ in this equality, we obtain

$$\begin{aligned}
&(G(k))(x_1, x_2, \ldots, x_\ell) \\
&= \left(\prod_{i=1}^{\ell} \left(x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right) \right) \cdot \left(\prod_{i=\ell+1}^{\infty} \underbrace{\left(0^0 + 0^1 + \cdots + 0^{k-1} \right)}_{\substack{=1 \\ \text{(since } 0^0=1 \text{ and } 0^j=0 \text{ for all } j>0)}} \right) \\
&= \prod_{i=1}^{\ell} \underbrace{\left(x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right)}_{=\sum_{j=0}^{k-1} x_i^j} = \prod_{i=1}^{\ell} \sum_{j=0}^{k-1} x_i^j \qquad (51) \\
&= \sum_{(j_1, j_2, \ldots, j_\ell) \in \{0,1,\ldots,k-1\}^\ell} \underbrace{x_1^{j_1} x_2^{j_2} \cdots x_\ell^{j_\ell}}_{\substack{= \mathbf{x}^{(j_1, j_2, \ldots, j_\ell)} \\ \text{(by the definition of } \mathbf{x}^{(j_1, j_2, \ldots, j_\ell)})}} \qquad \text{(by the product rule)} \\
&= \sum_{(j_1, j_2, \ldots, j_\ell) \in \{0,1,\ldots,k-1\}^\ell} \mathbf{x}^{(j_1, j_2, \ldots, j_\ell)} = \underbrace{\sum_{\beta \in \{0,1,\ldots,k-1\}^\ell}}_{= \sum_{\substack{\beta \in \mathbb{N}^\ell; \\ \beta_i < k \text{ for all } i}}} \mathbf{x}^\beta \\
&\qquad \text{(here, we have substituted } \beta \text{ for } (j_1, j_2, \ldots, j_\ell) \text{ in the sum)} \\
&= \sum_{\substack{\beta \in \mathbb{N}^\ell; \\ \beta_i < k \text{ for all } i}} \mathbf{x}^\beta.
\end{aligned}$$

(Here and in the rest of this proof, "for all i" means "for all $i \in \{1, 2, \ldots, \ell\}$".)

From (51), we see that $(G(k))(x_1, x_2, \ldots, x_\ell)$ is a polynomial in $\mathbf{k}[x_1, x_2, \ldots, x_\ell]$ (not merely a power series in $\mathbf{k}[[x_1, x_2, \ldots, x_\ell]]$). This polynomial $(G(k))(x_1, x_2, \ldots, x_\ell) \in \mathbf{k}[x_1, x_2, \ldots, x_\ell]$ is invariant under the action of $\mathfrak{S}_\ell$ (because of (51), or alternatively, because $G(k)$ is a symmetric power series). In other words,

$$\sigma((G(k))(x_1, x_2, \ldots, x_\ell)) = (G(k))(x_1, x_2, \ldots, x_\ell) \qquad (52)$$

for any $\sigma \in \mathfrak{S}_\ell$.

But from $a_\alpha = \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \sigma(\mathbf{x}^\alpha)$, we obtain

$$\begin{aligned}
&(G(k))(x_1, x_2, \ldots, x_\ell) \cdot a_\alpha \\
&= (G(k))(x_1, x_2, \ldots, x_\ell) \cdot \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \sigma(\mathbf{x}^\alpha) \\
&= \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \underbrace{(G(k))(x_1, x_2, \ldots, x_\ell)}_{\substack{=\sigma((G(k))(x_1, x_2, \ldots, x_\ell)) \\ \text{(by (52))}}} \cdot \sigma(\mathbf{x}^\alpha) \\
&= \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \underbrace{\sigma((G(k))(x_1, x_2, \ldots, x_\ell)) \cdot \sigma(\mathbf{x}^\alpha)}_{\substack{=\sigma((G(k))(x_1, x_2, \ldots, x_\ell) \cdot \mathbf{x}^\alpha) \\ \text{(since } \mathfrak{S}_\ell \text{ acts on } \mathbf{k}[x_1, x_2, \ldots, x_\ell] \\ \text{by } \mathbf{k}\text{-algebra homomorphisms)}}} \\
&= \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \sigma\left(\underbrace{(G(k))(x_1, x_2, \ldots, x_\ell)}_{= \sum_{\substack{\beta \in \mathbb{N}^\ell; \\ \beta_i < k \text{ for all } i}} \mathbf{x}^\beta} \cdot \mathbf{x}^\alpha \right) \\
&= \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \sigma\left(\sum_{\substack{\beta \in \mathbb{N}^\ell; \\ \beta_i < k \text{ for all } i}} \mathbf{x}^\beta \cdot \mathbf{x}^\alpha \right) = \sum_{\substack{\beta \in \mathbb{N}^\ell; \\ \beta_i < k \text{ for all } i}} \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \sigma\left(\underbrace{\mathbf{x}^\beta \cdot \mathbf{x}^\alpha}_{\substack{=\mathbf{x}^\alpha \cdot \mathbf{x}^\beta \\ =\mathbf{x}^{\alpha+\beta} \\ \text{(by (49))}}} \right) \\
&= \sum_{\substack{\beta \in \mathbb{N}^\ell; \\ \beta_i < k \text{ for all } i}} \underbrace{\sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \sigma\left(\mathbf{x}^{\alpha+\beta}\right)}_{\substack{=a_{\alpha+\beta} \\ \text{(since } a_{\alpha+\beta} \text{ is defined} \\ \text{by } a_{\alpha+\beta} = \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \sigma\left(\mathbf{x}^{\alpha+\beta}\right))}} = \sum_{\substack{\beta \in \mathbb{N}^\ell; \\ \beta_i < k \text{ for all } i}} a_{\alpha+\beta} \\
&= \sum_{\substack{\gamma \in \mathbb{N}^\ell; \\ 0 \leq \gamma_i - \alpha_i < k \text{ for all } i}} a_\gamma
\end{aligned} \tag{53}$$

(here, we have substituted γ for $\alpha + \beta$ in the sum).

It is well-known (and easy to check using the properties of determinants[36]) that if an ℓ-tuple $\gamma \in \mathbb{N}^\ell$ has two equal entries, then

$$a_\gamma = 0. \tag{54}$$

Moreover, any ℓ-tuple $\gamma \in \mathbb{N}^\ell$ and any $\sigma \in \mathfrak{S}_\ell$ satisfy

$$a_{\sigma \cdot \gamma} = (-1)^\sigma \cdot a_\gamma. \tag{55}$$

[36]specifically: using the fact that a square matrix with two equal rows always has determinant 0

(This, too, follows from the properties of determinants[37].)

Let SP_ℓ denote the set of all ℓ-tuples $\delta \in \mathbb{N}^\ell$ such that $\delta_1 > \delta_2 > \cdots > \delta_\ell$. Then, the map

$$\begin{aligned} P_\ell &\to SP_\ell, \\ \lambda &\mapsto \lambda + \rho \end{aligned} \tag{56}$$

is a bijection.

If an ℓ-tuple $\gamma \in \mathbb{N}^\ell$ has no two equal entries, then γ can be uniquely written in the form $\sigma \cdot \delta$ for some $\sigma \in \mathfrak{S}_\ell$ and some $\delta \in SP_\ell$ (indeed, δ is the result of sorting γ into decreasing order, while σ is the permutation that achieves this sorting). In other words, the map

$$\begin{aligned} \mathfrak{S}_\ell \times SP_\ell &\to \left\{ \gamma \in \mathbb{N}^\ell \ \mid \ \text{the } \ell\text{-tuple } \gamma \text{ has no two equal entries} \right\}, \\ (\sigma, \delta) &\mapsto \sigma \cdot \delta \end{aligned} \tag{57}$$

is a bijection.

[37] specifically: using the fact that permuting the rows of a square matrix results in its determinant getting multiplied by the sign of the permutation

Now, (53) becomes

$$
\begin{aligned}
& (G(k))(x_1, x_2, \ldots, x_\ell) \cdot a_\alpha \\
&= \sum_{\substack{\gamma \in \mathbb{N}^\ell; \\ 0 \le \gamma_i - \alpha_i < k \text{ for all } i}} a_\gamma \\
&= \sum_{\substack{\gamma \in \mathbb{N}^\ell; \\ 0 \le \gamma_i - \alpha_i < k \text{ for all } i; \\ \text{the } \ell\text{-tuple } \gamma \text{ has no two equal entries}}} a_\gamma + \sum_{\substack{\gamma \in \mathbb{N}^\ell; \\ 0 \le \gamma_i - \alpha_i < k \text{ for all } i; \\ \text{the } \ell\text{-tuple } \gamma \text{ has two equal entries}}} \underbrace{a_\gamma}_{\substack{=0 \\ \text{(by (54))}}} \\
&= \sum_{\substack{\gamma \in \mathbb{N}^\ell; \\ 0 \le \gamma_i - \alpha_i < k \text{ for all } i; \\ \text{the } \ell\text{-tuple } \gamma \text{ has no two equal entries}}} a_\gamma \\
&= \sum_{\substack{\gamma \in \mathbb{N}^\ell; \\ \text{the } \ell\text{-tuple } \gamma \text{ has no two equal entries}}} [0 \le \gamma_i - \alpha_i < k \text{ for all } i] \cdot a_\gamma \\
&= \underbrace{\sum_{(\sigma,\delta) \in \mathfrak{S}_\ell \times SP_\ell}}_{= \sum_{\delta \in SP_\ell} \sum_{\sigma \in \mathfrak{S}_\ell}} \left[0 \le \underbrace{(\sigma \cdot \delta)_i}_{=\delta_{\sigma^{-1}(i)}} - \alpha_i < k \text{ for all } i \right] \cdot \underbrace{a_{\sigma \cdot \delta}}_{\substack{=(-1)^\sigma a_\delta \\ \text{(by (55))}}} \\
&\qquad \left(\begin{array}{c} \text{here, we have substituted } \sigma \cdot \delta \text{ for } \gamma \text{ in the sum,} \\ \text{since the map (57) is a bijection} \end{array} \right) \\
&= \sum_{\delta \in SP_\ell} \sum_{\sigma \in \mathfrak{S}_\ell} \underbrace{\left[0 \le \delta_{\sigma^{-1}(i)} - \alpha_i < k \text{ for all } i \right]}_{\substack{= \prod_{i=1}^{\ell} \left[0 \le \delta_{\sigma^{-1}(i)} - \alpha_i < k \right] \\ = \prod_{i=1}^{\ell} \left[0 \le \delta_i - \alpha_{\sigma(i)} < k \right] \\ \text{(here, we have substituted } \sigma(i) \\ \text{for } i \text{ in the product, since } \sigma \text{ is a bijection)}}} \cdot (-1)^\sigma a_\delta \\
&= \sum_{\delta \in SP_\ell} \underbrace{\sum_{\sigma \in \mathfrak{S}_\ell} \left(\prod_{i=1}^{\ell} \left[0 \le \delta_i - \alpha_{\sigma(i)} < k \right] \right) \cdot (-1)^\sigma}_{\substack{= \sum_{\sigma \in \mathfrak{S}_\ell} (-1)^\sigma \prod_{i=1}^{\ell} \left[0 \le \delta_i - \alpha_{\sigma(i)} < k \right] \\ = \det \left(\left(\left[0 \le \delta_i - \alpha_j < k \right] \right)_{1 \le i \le \ell,\ 1 \le j \le \ell} \right) \\ \text{(by the definition of a determinant)}}} a_\delta \\
&= \sum_{\delta \in SP_\ell} \det \left(\left(\left[0 \le \delta_i - \alpha_j < k \right] \right)_{1 \le i \le \ell,\ 1 \le j \le \ell} \right) a_\delta \\
&= \sum_{\lambda \in P_\ell} \det \left(\left(\left[0 \le (\lambda + \rho)_i - \alpha_j < k \right] \right)_{1 \le i \le \ell,\ 1 \le j \le \ell} \right) a_{\lambda + \rho}
\end{aligned}
\tag{58}
$$

(here, we have substituted $\lambda + \rho$ for δ in the sum, since the map (56) is a bijection).

Every $\lambda \in P_\ell$ and every $i, j \in \{1, 2, \ldots, \ell\}$ satisfy

$$\begin{aligned}
&\underbrace{(\lambda+\rho)_i}_{\substack{=\lambda_i+\rho_i=\lambda_i+\ell-i \\ \text{(since the definition of } \rho \\ \text{yields } \rho_i=\ell-i)}} - \underbrace{\alpha_j}_{\substack{=(\mu+\rho)_j \\ \text{(since } \alpha=\mu+\rho)}} \\
&= \lambda_i + \ell - i - \underbrace{(\mu+\rho)_j}_{\substack{=\mu_j+\rho_j=\mu_j+\ell-j \\ \text{(since the definition of } \rho \\ \text{yields } \rho_j=\ell-j)}} = \lambda_i + \ell - i - \left(\mu_j + \ell - j\right) \\
&= \lambda_i - \mu_j - i + j. \qquad (59)
\end{aligned}$$

Now, (50) (applied to $\lambda = \mu$) yields

$$s_\mu (x_1, x_2, \ldots, x_\ell) = \frac{a_{\mu+\rho}}{a_\rho} = \frac{a_\alpha}{a_\rho}$$

(since $\mu + \rho = \alpha$). Multiplying this equality by $(G(k))(x_1, x_2, \ldots, x_\ell)$, we find

$$\begin{aligned}
&(G(k))(x_1, x_2, \ldots, x_\ell) \cdot s_\mu (x_1, x_2, \ldots, x_\ell) \\
&= (G(k))(x_1, x_2, \ldots, x_\ell) \cdot \frac{a_\alpha}{a_\rho} \\
&= \frac{1}{a_\rho} \cdot \underbrace{(G(k))(x_1, x_2, \ldots, x_\ell) \cdot a_\alpha}_{\substack{=\sum\limits_{\lambda\in P_\ell} \det\left(\left([0\leq(\lambda+\rho)_i-\alpha_j<k]\right)_{1\leq i\leq\ell,\ 1\leq j\leq\ell}\right) a_{\lambda+\rho} \\ \text{(by (58))}}} \\
&= \frac{1}{a_\rho} \cdot \sum_{\lambda\in P_\ell} \det\left(\left([0 \leq (\lambda+\rho)_i - \alpha_j < k]\right)_{1\leq i\leq\ell,\ 1\leq j\leq\ell}\right) a_{\lambda+\rho} \\
&= \sum_{\lambda\in P_\ell} \det\left(\left(\left[0 \leq \underbrace{(\lambda+\rho)_i - \alpha_j}_{\substack{=\lambda_i-\mu_j-i+j \\ \text{(by (59))}}} < k\right]\right)_{1\leq i\leq\ell,\ 1\leq j\leq\ell}\right) \underbrace{\frac{a_{\lambda+\rho}}{a_\rho}}_{\substack{=s_\lambda(x_1,x_2,\ldots,x_\ell) \\ \text{(by (50))}}} \\
&= \sum_{\lambda\in P_\ell} \underbrace{\det\left(\left([0 \leq \lambda_i - \mu_j - i + j < k]\right)_{1\leq i\leq\ell,\ 1\leq j\leq\ell}\right)}_{\substack{=\operatorname{pet}_k(\lambda,\mu) \\ \text{(by the definition of } \operatorname{pet}_k(\lambda,\mu))}} \cdot s_\lambda (x_1, x_2, \ldots, x_\ell) \\
&= \sum_{\lambda\in P_\ell} \operatorname{pet}_k (\lambda, \mu)\, s_\lambda (x_1, x_2, \ldots, x_\ell).
\end{aligned}$$

This proves (48).

On the other hand, it is known (see, e.g., [GriRei20, Exercise 2.3.8(b)]) that if λ is a partition having more than ℓ parts, then

$$s_\lambda (x_1, x_2, \ldots, x_\ell) = 0. \qquad (60)$$

Now, each partition $\lambda \in \operatorname{Par}$ either has at most ℓ parts or has more than ℓ parts (but not both at the same time). Hence,

$$\begin{aligned}
&\sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda,\mu)\, s_\lambda(x_1,x_2,\ldots,x_\ell) \\
&= \underbrace{\sum_{\substack{\lambda \in \operatorname{Par};\\ \lambda \text{ has at most } \ell \text{ parts}}}}_{\substack{=\sum_{\lambda\in P_\ell} \\ \text{(by the definition of } P_\ell)}} \operatorname{pet}_k(\lambda,\mu)\, s_\lambda(x_1,x_2,\ldots,x_\ell) \\
&\qquad + \sum_{\substack{\lambda \in \operatorname{Par};\\ \lambda \text{ has more than } \ell \text{ parts}}} \operatorname{pet}_k(\lambda,\mu)\, \underbrace{s_\lambda(x_1,x_2,\ldots,x_\ell)}_{\substack{=0 \\ \text{(by (60))}}} \\
&= \sum_{\lambda\in P_\ell} \operatorname{pet}_k(\lambda,\mu)\, s_\lambda(x_1,x_2,\ldots,x_\ell) + \underbrace{\sum_{\substack{\lambda \in \operatorname{Par};\\ \lambda \text{ has more than } \ell \text{ parts}}} \operatorname{pet}_k(\lambda,\mu)\, 0}_{=0} \\
&= \sum_{\lambda\in P_\ell} \operatorname{pet}_k(\lambda,\mu)\, s_\lambda(x_1,x_2,\ldots,x_\ell).
\end{aligned}$$

Comparing this with (48), we obtain

$$\begin{aligned}
&(G(k))(x_1,x_2,\ldots,x_\ell) \cdot s_\mu(x_1,x_2,\ldots,x_\ell) \\
&= \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda,\mu)\, s_\lambda(x_1,x_2,\ldots,x_\ell).
\end{aligned} \tag{61}$$

Forget that we fixed ℓ. Thus, we have proved (61) for each $\ell \in \mathbb{N}$ that satisfies $\ell \geq \ell(\mu)$. Thus, for each $\ell \in \mathbb{N}$ that satisfies $\ell \geq \ell(\mu)$, we have

$$\begin{aligned}
&\left(G(k) \cdot s_\mu\right)(x_1,x_2,\ldots,x_\ell) \\
&= (G(k))(x_1,x_2,\ldots,x_\ell) \cdot s_\mu(x_1,x_2,\ldots,x_\ell) \\
&= \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda,\mu)\, s_\lambda(x_1,x_2,\ldots,x_\ell) \qquad \text{(by (61))}.
\end{aligned} \tag{62}$$

Now, (47) (applied to $f = G(k) \cdot s_\mu$) yields

$$\begin{aligned}
G(k) \cdot s_\mu &= \lim_{\ell\to\infty} \underbrace{\left(G(k) \cdot s_\mu\right)(x_1,x_2,\ldots,x_\ell)}_{\substack{=\sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_k(\lambda,\mu) s_\lambda(x_1,x_2,\ldots,x_\ell) \text{ when } \ell\geq\ell(\mu) \\ \text{(by (62))}}} \\
&= \lim_{\ell\to\infty} \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda,\mu)\, s_\lambda(x_1,x_2,\ldots,x_\ell).
\end{aligned}$$

Comparing this with

$$\sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \mu) \underbrace{s_\lambda}_{\substack{= \lim\limits_{\ell \to \infty} s_\lambda(x_1, x_2, \ldots, x_\ell) \\ \text{(by (47))}}} = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \mu) \lim_{\ell \to \infty} s_\lambda(x_1, x_2, \ldots, x_\ell)$$
$$= \lim_{\ell \to \infty} \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \mu) s_\lambda(x_1, x_2, \ldots, x_\ell)$$

[38], we obtain

$$G(k) \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \mu) s_\lambda.$$

This completes the second proof of Theorem 2.17. □

3.12. Proofs of Corollary 2.18, Theorem 2.9 and Corollary 2.10

Having proved Theorem 2.17, we can now obtain Corollary 2.18, Theorem 2.9 and Corollary 2.10 as easy consequences:

Proof of Corollary 2.18. Forget that we fixed m. If $n \in \mathbb{N}$, then the power series $\begin{cases} G(k, n - |\mu|) \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is homogeneous of degree n [39].

[38] Why were we allowed to interchange the limit with the summation sign here? One way to justify this is by realizing that each Schur function s_λ and therefore each polynomial $s_\lambda(x_1, x_2, \ldots, x_\ell)$ are homogeneous of degree $|\lambda|$, and for each $n \in \mathbb{N}$ there are only finitely many partitions $\lambda \in \operatorname{Par}$ satisfying $|\lambda| = n$. This entails that each individual monomial $\mathfrak{m}$ is affected only by finitely many addends in the sums appearing on both sides of our equation.

[39] *Proof.* Let $n \in \mathbb{N}$. We must prove that the power series $\begin{cases} G(k, n - |\mu|) \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n.

We are in one of the following two cases:

Case 1: We have $n \geq |\mu|$.

Case 2: We have $n < |\mu|$.

Let us first consider Case 1. In this case, we have $n \geq |\mu|$. Hence, $n - |\mu| \in \mathbb{N}$. But Proposition 2.3 **(a)** (applied to $m = n - |\mu|$) yields that the symmetric function $G(k, n - |\mu|)$ is the $(n - |\mu|)$-th degree homogeneous component of $G(k)$. Hence, $G(k, n - |\mu|)$ is homogeneous of degree $n - |\mu|$.

On the other hand, recall that for any $\lambda \in \operatorname{Par}$, the Schur function s_λ is homogeneous of degree $|\lambda|$. Applying this to $\lambda = \mu$, we conclude that the Schur function s_μ is homogeneous of degree $|\mu|$.

So we know that $G(k, n - |\mu|)$ is homogeneous of degree $n - |\mu|$, whereas s_μ is homogeneous of degree $|\mu|$. Since Λ is a graded algebra, this entails that the power series $G(k, n - |\mu|) \cdot s_\mu$ (being the product of $G(k, n - |\mu|)$ and s_μ) is homogeneous of degree $(n - |\mu|) + |\mu|$. In other words, the power series $G(k, n - |\mu|) \cdot s_\mu$ is homogeneous of degree n (since $(n - |\mu|) + |\mu| = n$). In other words, the power series $\begin{cases} G(k, n - |\mu|) \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n

In the proof of Proposition 2.3 **(a)**, we have shown that $G(k) = \sum_{m\in\mathbb{N}} G(k,m)$. Multiplying both sides of this equality by s_μ, we find

$$G(k) \cdot s_\mu = \left(\sum_{m\in\mathbb{N}} G(k,m)\right) \cdot s_\mu = \sum_{m\in\mathbb{N}} G(k,m) \cdot s_\mu.$$

(since

$$\begin{cases} G(k, n-|\mu|) \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases} = G(k, n-|\mu|) \cdot s_\mu \qquad \text{(because } n \geq |\mu|\text{)}$$

). Thus, we have proved in Case 1 that the power series $\begin{cases} G(k, n-|\mu|) \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n.

Let us now consider Case 2. In this case, we have $n < |\mu|$. The power series 0 is homogeneous of degree n (since 0 is homogeneous of any degree). In other words, the power series $\begin{cases} G(k, n-|\mu|) \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n (since

$$\begin{cases} G(k, n-|\mu|) \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases} = 0 \qquad \text{(because } n < |\mu|\text{)}$$

). Thus, we have proved in Case 2 that the power series $\begin{cases} G(k, n-|\mu|) \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n.

Thus, our claim (namely, that the power series $\begin{cases} G(k, n-|\mu|) \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n) has been proven in both Cases 1 and 2. Since these cases cover all possibilities, we thus conclude that our claim always holds. Qed.

Comparing this with

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} \begin{cases} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu, & \text{if } n\geq\left|\mu\right|;\\ 0, & \text{if } n<\left|\mu\right| \end{cases}\\
&=\sum_{\substack{n\in\mathbb{N};\\ n\geq\left|\mu\right|}} \underbrace{\begin{cases} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu, & \text{if } n\geq\left|\mu\right|;\\ 0, & \text{if } n<\left|\mu\right| \end{cases}}_{\substack{=G(k,n-|\mu|)\cdot s_\mu\\ \text{(since } n\geq|\mu|)}} + \sum_{\substack{n\in\mathbb{N};\\ n<\left|\mu\right|}} \underbrace{\begin{cases} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu, & \text{if } n\geq\left|\mu\right|;\\ 0, & \text{if } n<\left|\mu\right| \end{cases}}_{\substack{=0\\ \text{(since } n<|\mu|)}}\\
&\qquad \left(\begin{array}{c}\text{since each } n\in\mathbb{N} \text{ satisfies either } n\geq\left|\mu\right| \text{ or } n<\left|\mu\right|\\ \text{(but not both at the same time)}\end{array}\right)\\
&=\sum_{\substack{n\in\mathbb{N};\\ n\geq\left|\mu\right|}} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu + \underbrace{\sum_{\substack{n\in\mathbb{N};\\ n<\left|\mu\right|}} 0}_{=0} = \sum_{\substack{n\in\mathbb{N};\\ n\geq\left|\mu\right|}} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu\\
&=\sum_{m\in\mathbb{N}} G\left(k,m\right)\cdot s_\mu \qquad \left(\text{here, we have substituted } m \text{ for } n-\left|\mu\right| \text{ in the sum}\right),
\end{aligned}$$

we obtain

$$G\left(k\right)\cdot s_\mu = \sum_{n\in\mathbb{N}} \begin{cases} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu, & \text{if } n\geq\left|\mu\right|;\\ 0, & \text{if } n<\left|\mu\right| \end{cases}. \tag{63}$$

But recall that each $\begin{cases} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu, & \text{if } n\geq\left|\mu\right|;\\ 0, & \text{if } n<\left|\mu\right| \end{cases}$ is homogeneous of degree n. Thus, the equality (63) reveals that the family

$$\left(\begin{cases} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu, & \text{if } n\geq\left|\mu\right|;\\ 0, & \text{if } n<\left|\mu\right| \end{cases}\right)_{n\in\mathbb{N}}$$

is the homogeneous decomposition of $G\left(k\right)\cdot s_\mu$ (by the definition of a homogeneous decomposition). Therefore, for each $n\in\mathbb{N}$, the power series $\begin{cases} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu, & \text{if } n\geq\left|\mu\right|;\\ 0, & \text{if } n<\left|\mu\right| \end{cases}$ is the n-th degree homogeneous component of $G\left(k\right)\cdot s_\mu$.

Now, let $m\in\mathbb{N}$. We have just shown that for each $n\in\mathbb{N}$, the power series $\begin{cases} G\left(k,n-\left|\mu\right|\right)\cdot s_\mu, & \text{if } n\geq\left|\mu\right|;\\ 0, & \text{if } n<\left|\mu\right| \end{cases}$ is the n-th degree homogeneous component of $G\left(k\right)\cdot s_\mu$. Applying this to $n=m+\left|\mu\right|$, we conclude that the power series $\begin{cases} G\left(k,\left(m+\left|\mu\right|\right)-\left|\mu\right|\right)\cdot s_\mu, & \text{if } m+\left|\mu\right|\geq\left|\mu\right|;\\ 0, & \text{if } m+\left|\mu\right|<\left|\mu\right| \end{cases}$ is the $\left(m+\left|\mu\right|\right)$-th degree homogeneous component of $G(k) \cdot s_\mu$. Since

$$\begin{cases} G(k,(m+|\mu|)-|\mu|) \cdot s_\mu, & \text{if } m+|\mu| \geq |\mu|; \\ 0, & \text{if } m+|\mu| < |\mu| \end{cases}$$
$$= G\left(k, \underbrace{(m+|\mu|)-|\mu|}_{=m}\right) \cdot s_\mu \qquad \left(\text{since } m+|\mu| \geq |\mu| \text{ (because } m \geq 0)\right)$$
$$= G(k,m) \cdot s_\mu,$$

we can rewrite this as follows: The power series $G(k,m) \cdot s_\mu$ is the $(m+|\mu|)$-th degree homogeneous component of $G(k) \cdot s_\mu$. In other words,

$$G(k,m) \cdot s_\mu = \left(\text{the } (m+|\mu|)\text{-th degree homogeneous component of } G(k) \cdot s_\mu\right). \tag{64}$$

On the other hand, Theorem 2.17 yields

$$G(k) \cdot s_\mu = \underbrace{\sum_{\lambda \in \mathrm{Par}}}_{\substack{= \sum\limits_{n \in \mathbb{N}} \sum\limits_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \\ \text{(since each } \lambda \in \mathrm{Par} \\ \text{satisfies } |\lambda| \in \mathbb{N})}} \mathrm{pet}_k(\lambda,\mu) s_\lambda = \sum_{n \in \mathbb{N}} \sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_k(\lambda,\mu) s_\lambda. \tag{65}$$

For each $n \in \mathbb{N}$, the formal power series $\sum\limits_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_k(\lambda,\mu) s_\lambda$ is homogeneous of degree n [40]. Thus, the equality (65) reveals that

$$\left(\sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_k(\lambda,\mu) s_\lambda\right)_{n \in \mathbb{N}}$$

is the homogeneous decomposition of $G(k) \cdot s_\mu$. Therefore, for each $n \in \mathbb{N}$, the power series $\sum\limits_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_k(\lambda,\mu) s_\lambda$ is the n-th degree homogeneous component

[40] *Proof.* Let $n \in \mathbb{N}$. Recall that for any $\lambda \in \mathrm{Par}$, the Schur function s_λ is homogeneous of degree $|\lambda|$. Hence, if $\lambda \in \mathrm{Par}$ satisfies $|\lambda| = n$, then the Schur function s_λ is homogeneous of degree n (since $|\lambda| = n$). Thus, $\sum\limits_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_k(\lambda,\mu) s_\lambda$ is a $\mathbf{k}$-linear combination of Schur functions that are homogeneous of degree n. Therefore, $\sum\limits_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_k(\lambda,\mu) s_\lambda$ is homogeneous of degree n. Qed.

of $G(k) \cdot s_\mu$. Applying this to $n = m + |\mu|$, we conclude that the power series $\sum\limits_{\substack{\lambda \in \operatorname{Par};\\ |\lambda| = m + |\mu|}} \operatorname{pet}_k(\lambda, \mu) s_\lambda$ is the $(m + |\mu|)$-th degree homogeneous component of $G(k) \cdot s_\mu$. In other words,

$$\sum_{\substack{\lambda \in \operatorname{Par};\\ |\lambda| = m + |\mu|}} \operatorname{pet}_k(\lambda, \mu) s_\lambda$$
$$= \left(\text{the } (m + |\mu|)\text{-th degree homogeneous component of } G(k) \cdot s_\mu\right).$$

Comparing this with (64), we find

$$G(k, m) \cdot s_\mu = \underbrace{\sum_{\substack{\lambda \in \operatorname{Par};\\ |\lambda| = m + |\mu|}}}_{\substack{= \sum\limits_{\lambda \in \operatorname{Par}_{m+|\mu|}} \\ \text{(since } \operatorname{Par}_{m+|\mu|} \text{ is defined as the} \\ \text{set of all } \lambda \in \operatorname{Par} \text{ satisfying } |\lambda| = m + |\mu|)}} \operatorname{pet}_k(\lambda, \mu) s_\lambda = \sum_{\lambda \in \operatorname{Par}_{m+|\mu|}} \operatorname{pet}_k(\lambda, \mu) s_\lambda.$$

This proves Corollary 2.18. ☐

Proof of Theorem 2.9. Theorem 2.17 (applied to $\mu = \varnothing$) yields

$$G(k) \cdot s_\varnothing = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \varnothing) s_\lambda.$$

Comparing this with $G(k) \cdot \underbrace{s_\varnothing}_{=1} = G(k)$, we obtain

$$G(k) = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_k(\lambda, \varnothing) s_\lambda.$$

This proves Theorem 2.9. ☐

Proof of Corollary 2.10. Corollary 2.18 (applied to $\mu = \varnothing$) yields

$$G(k, m) \cdot s_\varnothing = \sum_{\lambda \in \operatorname{Par}_{m+|\varnothing|}} \operatorname{pet}_k(\lambda, \varnothing) s_\lambda.$$

In view of $G(k, m) \cdot \underbrace{s_\varnothing}_{=1} = G(k, m)$ and $m + \underbrace{|\varnothing|}_{=0} = m$, we can rewrite this as

$$G(k, m) = \sum_{\lambda \in \operatorname{Par}_m} \operatorname{pet}_k(\lambda, \varnothing) s_\lambda.$$

This proves Corollary 2.10. ☐

3.13. Proof of Theorem 2.15

Our proof of Theorem 2.15 will depend on two lemmas about determinants:

Lemma 3.14. Let $m \in \mathbb{N}$. Let R be a commutative ring. Let $\left(a_{i,j}\right)_{1\leq i\leq m,\ 1\leq j\leq m} \in R^{m\times m}$ be an $m \times m$-matrix.

(a) If τ is any permutation of $\{1, 2, \ldots, m\}$, then

$$\det\left(\left(a_{\tau(i),j}\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right) = (-1)^{\tau} \cdot \det\left(\left(a_{i,j}\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right).$$

Here, $(-1)^{\tau}$ denotes the sign of the permutation τ.

(b) Let $u_1, u_2, \ldots, u_m$ be m elements of R. Let $v_1, v_2, \ldots, v_m$ be m elements of R. Then,

$$\det\left(\left(u_i v_j a_{i,j}\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right) = \left(\prod_{i=1}^{m} (u_i v_i)\right) \cdot \det\left(\left(a_{i,j}\right)_{1\leq i\leq m,\ 1\leq j\leq m}\right).$$

Proof of Lemma 3.14. **(a)** This is just the well-known fact that if the rows of a square matrix are permuted using a permutation τ, then the determinant of this matrix gets multiplied by $(-1)^{\tau}$.

(b) This follows easily from the definition of the determinant. $\square$

Lemma 3.15. Let k be a positive integer. Let $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ be $k-1$ elements of the set $\{1, 2, \ldots, k\}$.

Let G be the $(k-1) \times (k-1)$-matrix

$$\left((-1)^{(\gamma_i + j)\%k} \left[(\gamma_i + j)\,\%k \in \{0, 1\}\right]\right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1}.$$

(a) If the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are not distinct, then

$$\det G = 0.$$

(b) If $\gamma_1 > \gamma_2 > \cdots > \gamma_{k-1}$, then

$$\det G = (-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.$$

(c) Assume that the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are distinct. Let

$$g = \left|\left\{(i,j) \in \{1, 2, \ldots, k-1\}^2 \mid i < j \text{ and } \gamma_i < \gamma_j\right\}\right|.$$

Then,

$$\det G = (-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.$$

Proof of Lemma 3.15. **(a)** Assume that the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are not distinct. In other words, there exist two elements u and v of $\{1, 2, \ldots, k-1\}$ such that $u < v$ and $\gamma_u = \gamma_v$. Consider these u and v. Now, from $\gamma_u = \gamma_v$, we conclude that the u-th and the v-th rows of the matrix G are equal (by the construction of G). Hence, the matrix G has two equal rows (since $u < v$). Thus, $\det G = 0$. This proves Lemma 3.15 **(a)**.

(b) Assume that $\gamma_1 > \gamma_2 > \cdots > \gamma_{k-1}$. Thus, $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are distinct. Hence, $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ is a $(k-1)$-element set. But $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ is a subset of $\{1, 2, \ldots, k\}$ (since $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are elements of $\{1, 2, \ldots, k\}$). Therefore, $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ is a $(k-1)$-element subset of $\{1, 2, \ldots, k\}$.

But $\{1, 2, \ldots, k\}$ is a k-element set. Hence, each $(k-1)$-element subset of $\{1, 2, \ldots, k\}$ has the form $\{1, 2, \ldots, k\} \setminus \{u\}$ for some $u \in \{1, 2, \ldots, k\}$. Thus, in particular, $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ has this form (since $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ is a $(k-1)$-element subset of $\{1, 2, \ldots, k\}$). In other words,

$$\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\} = \{1, 2, \ldots, k\} \setminus \{u\} \tag{66}$$

for some $u \in \{1, 2, \ldots, k\}$. Consider this u. From (66), we conclude that $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are the $k-1$ elements of the set $\{1, 2, \ldots, k\} \setminus \{u\}$, listed in decreasing order (since $\gamma_1 > \gamma_2 > \cdots > \gamma_{k-1}$). In other words,

$$(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}) = (k, k-1, \ldots, \widehat{u}, \ldots, 2, 1), \tag{67}$$

where the "hat" over the u signifies that u is omitted from the list (i.e., the expression "$(k, k-1, \ldots, \widehat{u}, \ldots, 2, 1)$" is understood to mean the $(k-1)$-element list $(k, k-1, \ldots, u+1, u-1, \ldots, 2, 1)$, which contains all k integers from 1 to k in decreasing order except for u). Thus,

$$(\gamma_1, \gamma_2, \ldots, \gamma_{k-u}) = (k, k-1, \ldots, u+1) \qquad \text{and} \tag{68}$$

$$(\gamma_{k-u+1}, \gamma_{k-u+2}, \ldots, \gamma_{k-1}) = (u-1, u-2, \ldots, 1). \tag{69}$$

Now, we claim that

$$\begin{aligned} &(-1)^{(\gamma_i+j)\%k} [(\gamma_i + j) \%k \in \{0, 1\}] \\ &= (-1)^{\gamma_i+j-k} [\gamma_i + j \in \{k, k+1\}] \end{aligned} \tag{70}$$

for any $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$.

[*Proof of (70):* Let $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$. We must prove the equality (70).

From $i \in \{1, 2, \ldots, k-1\}$, we obtain $1 \leq i \leq k-1$ and thus $k-1 \geq 1$. Thus, $k > k-1 \geq 1$. Hence, $k+1 < 2k$, so that $(k+1) \%k = 1$.

If we don't have $(\gamma_i + j) \%k \in \{0, 1\}$, then we cannot have $\gamma_i + j \in \{k, k+1\}$ either (because $\gamma_i + j \in \{k, k+1\}$ would entail $(\gamma_i + j) \%k \in \left\{ \underbrace{k\%k}_{=0}, \underbrace{(k+1) \%k}_{=1} \right\} = \{0, 1\}$). Thus, if we don't have $(\gamma_i + j) \%k \in \{0, 1\}$, then both truth values $[(\gamma_i + j) \%k \in \{0, 1\}]$ and $[\gamma_i + j \in \{k, k+1\}]$ are 0, and therefore the equality (70) simplifies to $(-1)^{(\gamma_i+j)\%k} 0 = (-1)^{\gamma_i+j-k} 0$ in this case, which is obviously true.

Hence, for the rest of this proof, we WLOG assume that we do have $(\gamma_i + j) \% k \in \{0, 1\}$.

But $\gamma_i \in \{1, 2, \ldots, k\}$, so that $1 \leq \gamma_i \leq k$. Also, $j \in \{1, 2, \ldots, k-1\}$, so that $1 \leq j \leq k-1$. Hence, $\underbrace{\gamma_i}_{\geq 1} + \underbrace{j}_{\geq 1} \geq 1 + 1 = 2$ and $\underbrace{\gamma_i}_{\leq k} + \underbrace{j}_{\leq k-1} \leq k + (k-1) = 2k-1$. Altogether, we thus obtain $2 \leq \gamma_i + j \leq 2k-1$, so that $\gamma_i + j \in \{2, 3, \ldots, 2k-1\}$.

The remainders of the numbers $2, 3, \ldots, 2k-1$ upon division by k are $2, 3, \ldots, k-1, 0, 1, \ldots, k-1$ (in this order). Thus, the only numbers $p \in \{2, 3, \ldots, 2k-1\}$ that satisfy $p\%k \in \{0, 1\}$ are k and $k+1$. In other words, for any number $p \in \{2, 3, \ldots, 2k-1\}$ satisfying $p\%k \in \{0, 1\}$, we have $p \in \{k, k+1\}$. Applying this to $p = \gamma_i + j$, we obtain $\gamma_i + j \in \{k, k+1\}$ (since $\gamma_i + j \in \{2, 3, \ldots, 2k-1\}$ and $(\gamma_i + j) \% k \in \{0, 1\}$). Hence, $k \leq \gamma_i + j \leq k+1$, so that $k \leq \gamma_i + j < 2k$ (since $k + 1 < 2k$). Thus, $(\gamma_i + j) // k = 1$. But every integer n satisfies $n = (n//k)\, k + (n\%k)$. Applying this to $n = \gamma_i + j$, we obtain $\gamma_i + j = \underbrace{((\gamma_i + j) // k)}_{=1} k + ((\gamma_i + j) \% k) = k + ((\gamma_i + j) \% k)$. Hence, $(\gamma_i + j) \% k = \gamma_i + j - k$. Thus,

$$\underbrace{(-1)^{(\gamma_i + j)\%k}}_{\substack{=(-1)^{\gamma_i + j - k} \\ (\text{since } (\gamma_i + j)\%k = \gamma_i + j - k)}} \underbrace{[(\gamma_i + j) \% k \in \{0, 1\}]}_{\substack{=1 \\ (\text{since } (\gamma_i + j)\%k \in \{0,1\})}} = (-1)^{\gamma_i + j - k}.$$

Comparing this with

$$(-1)^{\gamma_i + j - k} \underbrace{[\gamma_i + j \in \{k, k+1\}]}_{\substack{=1 \\ (\text{since } \gamma_i + j \in \{k, k+1\})}} = (-1)^{\gamma_i + j - k},$$

we obtain

$$(-1)^{(\gamma_i + j)\%k} [(\gamma_i + j) \% k \in \{0, 1\}] = (-1)^{\gamma_i + j - k} [\gamma_i + j \in \{k, k+1\}].$$

This proves (70).]

Now, G is a $(k-1) \times (k-1)$-matrix. For each $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$, we have

$$\begin{aligned}
&(\text{the } (i,j)\text{-th entry of } G) \\
&= (-1)^{(\gamma_i + j)\%k} [(\gamma_i + j) \% k \in \{0, 1\}] \qquad (\text{by the definition of } G) \\
&= (-1)^{\gamma_i + j - k} [\gamma_i + j \in \{k, k+1\}] \qquad (\text{by (70)}) \\
&= \begin{cases} 1, & \text{if } \gamma_i + j = k; \\ -1, & \text{if } \gamma_i + j = k+1; \\ 0, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } j = k - \gamma_i; \\ -1, & \text{if } j = k - \gamma_i + 1; \\ 0, & \text{otherwise} \end{cases}.
\end{aligned}$$

Thus, we can explicitly describe the matrix G as follows: For each $i \in \{1, 2, \ldots, k-1\}$, the i-th row of G has an entry equal to 1 in position $k - \gamma_i$ if $k - \gamma_i > 0$, and an entry equal to -1 in position $k-\gamma_i+1$ if $k-\gamma_i+1<k$; all remaining entries of this row are 0. Recalling (68) and (69), we thus see that G has the following form:[41]

$$G=\left(\begin{array}{ccccc|ccccc}
-1 & & & & & & & & & \\
1 & -1 & & & & & & & & \\
 & 1 & -1 & & & & & & & \\
 & & \ddots & \ddots & & & & & & \\
 & & & 1 & -1 & & & & & \\
\hline
 & & & & & 1 & -1 & & & \\
 & & & & & & 1 & -1 & & \\
 & & & & & & & 1 & -1 & \\
 & & & & & & & & \ddots & \ddots \\
 & & & & & & & & & 1
\end{array}\right),$$

where the horizontal bar separates the $(k-u)$-th row from the $(k-u+1)$-st row, while the vertical bar separates the $(k-u)$-th column from the $(k-u+1)$-st column. In other words, G can be written as a block matrix

$$G=\begin{pmatrix} A & 0_{(k-u)\times(u-1)} \\ 0_{(u-1)\times(k-u)} & B \end{pmatrix}, \tag{71}$$

where A is the $(k-u)\times(k-u)$-matrix $\begin{pmatrix} -1 & & & & \\ 1 & -1 & & & \\ & 1 & -1 & & \\ & & \ddots & \ddots & \\ & & & 1 & -1 \end{pmatrix}$ (that is, the $(k-u)\times(k-u)$-matrix whose diagonal entries are -1 and whose entries immediately below the diagonal are 1, while all its other entries are 0), and where B is the $(u-1)\times(u-1)$-matrix $\begin{pmatrix} 1 & -1 & & & \\ & 1 & -1 & & \\ & & 1 & -1 & \\ & & & \ddots & \ddots \\ & & & & 1 \end{pmatrix}$ (that is, the $(u-1)\times(u-1)$-matrix whose diagonal entries are 1 and whose entries immediately above the diagonal are -1, while all its other entries are 0). Thus, G (as written in (71)) is a block-diagonal matrix (since A and B are square matrices). Since the determinant of a block-diagonal matrix equals the product of the determinants of its diagonal

[41] Empty cells are understood to have entry 0.

blocks, we thus conclude that

$$\det G = \underbrace{\det A}_{\substack{=(-1)^{k-u} \\ \text{(since } A \text{ is a} \\ \text{lower-triangular } (k-u)\times(k-u)\text{-matrix} \\ \text{whose all diagonal entries equal } -1)}} \cdot \underbrace{\det B}_{\substack{=1 \\ \text{(since } B \text{ is an} \\ \text{upper-triangular } (u-1)\times(u-1)\text{-matrix} \\ \text{whose all diagonal entries equal } 1)}}$$
$$= (-1)^{k-u}. \tag{72}$$

But (67) yields

$$\begin{aligned} \gamma_1 + \gamma_2 + \cdots + \gamma_{k-1} &= k + (k-1) + \cdots + \widehat{u} + \cdots + 2 + 1 \\ &= \underbrace{(k + (k-1) + \cdots + 2 + 1)}_{\substack{=1+2+\cdots+k \\ =(1+2+\cdots+(k-1))+k}} - u \\ &= (1 + 2 + \cdots + (k-1)) + k - u. \end{aligned}$$

Solving this for $k-u$, we find

$$k - u = (\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1}) - (1 + 2 + \cdots + (k-1)).$$

Hence, (72) rewrites as

$$\det G = (-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.$$

This proves Lemma 3.15 **(b)**.

(c) Assume that the $k-1$ numbers $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are distinct. Then, there exists a unique permutation σ of $\{1, 2, \ldots, k-1\}$ such that $\gamma_{\sigma(1)} > \gamma_{\sigma(2)} > \cdots > \gamma_{\sigma(k-1)}$ (indeed, this is simply saying that the $(k-1)$-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_{k-1})$ can be sorted into decreasing order by a unique permutation). Consider this σ.

Let τ denote the permutation σ^{-1}. Thus, τ is a permutation of $\{1, 2, \ldots, k-1\}$ and satisfies $\sigma \circ \tau = \mathrm{id}$.

Let $\delta_1, \delta_2, \ldots, \delta_{k-1}$ denote the $k-1$ elements $\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \ldots, \gamma_{\sigma(k-1)}$ of $\{1, 2, \ldots, k\}$. Thus, for each $j \in \{1, 2, \ldots, k-1\}$, we have

$$\delta_j = \gamma_{\sigma(j)}. \tag{73}$$

Hence, the chain of inequalities $\gamma_{\sigma(1)} > \gamma_{\sigma(2)} > \cdots > \gamma_{\sigma(k-1)}$ (which is true) can be rewritten as $\delta_1 > \delta_2 > \cdots > \delta_{k-1}$.

Moreover, from (73), we obtain

$$\begin{aligned} \delta_1 + \delta_2 + \cdots + \delta_{k-1} &= \gamma_{\sigma(1)} + \gamma_{\sigma(2)} + \cdots + \gamma_{\sigma(k-1)} \\ &= \gamma_1 + \gamma_2 + \cdots + \gamma_{k-1} \end{aligned} \tag{74}$$

(since σ is a permutation of $\{1, 2, \ldots, k-1\}$).

Moreover, for each $i \in \{1,2,\ldots,k-1\}$, we have

$$\begin{aligned}
\delta_{\tau(i)} &= \gamma_{\sigma(\tau(i))} && \left(\text{by (73), applied to } j=\tau(i)\right) \\
&= \gamma_i && \left(\text{since } \sigma(\tau(i)) = \underbrace{(\sigma\circ\tau)}_{=\mathrm{id}}(i) = i\right).
\end{aligned} \tag{75}$$

Recall that an *inversion* of the permutation τ is defined to be a pair (i,j) of elements of $\{1,2,\ldots,k-1\}$ satisfying $i<j$ and $\tau(i)>\tau(j)$. Hence,

$$\begin{aligned}
&\{\text{the inversions of } \tau\} \\
&= \left\{ (i,j) \in \{1,2,\ldots,k-1\}^2 \ \mid\ i<j \text{ and } \underbrace{\tau(i)>\tau(j)}_{\substack{\text{This is equivalent to } \left(\delta_{\tau(i)}<\delta_{\tau(j)}\right) \\ (\text{since } \delta_1>\delta_2>\cdots>\delta_{k-1})}} \right\} \\
&= \left\{ (i,j) \in \{1,2,\ldots,k-1\}^2 \ \mid\ i<j \text{ and } \underbrace{\delta_{\tau(i)}}_{\substack{=\gamma_i \\ (\text{by (75)})}} < \underbrace{\delta_{\tau(j)}}_{\substack{=\gamma_j \\ (\text{by (75)})}} \right\} \\
&= \left\{ (i,j) \in \{1,2,\ldots,k-1\}^2 \ \mid\ i<j \text{ and } \gamma_i<\gamma_j \right\}.
\end{aligned} \tag{76}$$

Recall that the *length* $\ell(\tau)$ of the permutation τ is defined to be the number of inversions of τ. Thus,

$$\begin{aligned}
\ell(\tau) &= (\text{the number of inversions of } \tau) \\
&= \left|\{\text{the inversions of } \tau\}\right| \\
&= \left|\left\{ (i,j) \in \{1,2,\ldots,k-1\}^2 \ \mid\ i<j \text{ and } \gamma_i<\gamma_j \right\}\right| && (\text{by (76)}) \\
&= g && (\text{by the definition of } g).
\end{aligned}$$

Recall that the *sign* $(-1)^\tau$ of the permutation τ is defined by $(-1)^\tau = (-1)^{\ell(\tau)}$. Hence, $(-1)^\tau = (-1)^{\ell(\tau)} = (-1)^g$ (since $\ell(\tau)=g$).

Let H be the $(k-1)\times(k-1)$-matrix

$$\left((-1)^{(\delta_i+j)\%k}\left[(\delta_i+j)\%k \in \{0,1\}\right]\right)_{1\le i\le k-1,\ 1\le j\le k-1}.$$

Then, we can apply Lemma 3.15 **(b)** to δ_i and H instead of γ_i and G (since $\delta_1,\delta_2,\ldots,\delta_{k-1}$ are $k-1$ elements of $\{1,2,\ldots,k\}$ and satisfy $\delta_1>\delta_2>\cdots>\delta_{k-1}$). We thus obtain

$$\det H = (-1)^{(\delta_1+\delta_2+\cdots+\delta_{k-1})-(1+2+\cdots+(k-1))} = (-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}$$

(by (74)).

But the definition of G yields

$$G = \left(\underbrace{(-1)^{(\gamma_i+j)\%k} [(\gamma_i + j) \%k \in \{0,1\}]}_{\substack{=(-1)^{\left(\delta_{\tau(i)}+j\right)\%k}\left[\left(\delta_{\tau(i)}+j\right)\%k\in\{0,1\}\right] \\ \text{(since (75) yields } \gamma_i=\delta_{\tau(i)})}} \right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1}$$
$$= \left((-1)^{\left(\delta_{\tau(i)}+j\right)\%k} \left[\left(\delta_{\tau(i)} + j\right) \%k \in \{0,1\} \right] \right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1}.$$

Hence,

$$\det G = \det \left(\left((-1)^{\left(\delta_{\tau(i)}+j\right)\%k} \left[\left(\delta_{\tau(i)} + j\right) \%k \in \{0,1\} \right] \right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1} \right)$$
$$= \underbrace{(-1)^{\tau}}_{=(-1)^g} \det \left(\underbrace{\left((-1)^{(\delta_i+j)\%k} [(\delta_i + j) \%k \in \{0,1\}] \right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1}}_{\substack{=H \\ \text{(by the definition of } H)}} \right)$$
$$\left(\begin{array}{c} \text{by Lemma 3.14 (a), applied to } m = k-1 \text{ and } R = \mathbf{k} \\ \text{and } a_{i,j} = (-1)^{(\delta_i+j)\%k} [(\delta_i + j) \%k \in \{0,1\}] \end{array} \right)$$
$$= (-1)^g \underbrace{\det H}_{=(-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}}$$
$$= (-1)^g (-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}$$
$$= (-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.$$

This proves Lemma 3.15 **(c)**. □

Next, we recall a well-known property of symmetric functions:

Lemma 3.16. Consider the ring $\Lambda[[t]]$ of formal power series in one indeterminate t over Λ. In this ring, we have

$$1 = \left(\sum_{n\geq 0} (-1)^n e_n t^n \right) \left(\sum_{n\geq 0} h_n t^n \right). \tag{77}$$

Lemma 3.16 is a well-known identity (see, e.g., [Stanle01, proof of Theorem 7.6.1] or [GriRei20, (2.4.3)]); for the sake of completeness, let us nevertheless give a proof:

Proof of Lemma 3.16. Consider the ring $(\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]$ of formal power series in one indeterminate t over $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$. In this ring, we have the equalities

$$\prod_{i=1}^{\infty} (1 - x_i t)^{-1} = \sum_{n \geq 0} h_n t^n \tag{78}$$

and

$$\prod_{i=1}^{\infty} (1 + x_i t) = \sum_{n \geq 0} e_n t^n. \tag{79}$$

(Indeed, the first of these two equalities is [GriRei20, (2.2.18)], whereas the second is [GriRei20, (2.2.19)].)

Substituting $-t$ for t in the equality (79), we obtain

$$\prod_{i=1}^{\infty} (1 - x_i t) = \sum_{n \geq 0} e_n \underbrace{(-t)^n}_{=(-1)^n t^n} = \sum_{n \geq 0} (-1)^n e_n t^n.$$

Multiplying this equality by (78), we find

$$\left(\prod_{i=1}^{\infty} (1 - x_i t) \right) \left(\prod_{i=1}^{\infty} (1 - x_i t)^{-1} \right) = \left(\sum_{n \geq 0} (-1)^n e_n t^n \right) \left(\sum_{n \geq 0} h_n t^n \right).$$

Comparing this with

$$\left(\prod_{i=1}^{\infty} (1 - x_i t) \right) \left(\prod_{i=1}^{\infty} (1 - x_i t)^{-1} \right) = 1,$$

we obtain

$$1 = \left(\sum_{n \geq 0} (-1)^n e_n t^n \right) \left(\sum_{n \geq 0} h_n t^n \right).$$

This equality is an equality in $\Lambda[[t]]$ (since both of its sides belong to $\Lambda[[t]]$). This proves Lemma 3.16. $\square$

Next, we shall prove yet another evaluation of the homomorphism α_k:

Lemma 3.17. Let k be a positive integer such that $k > 1$. Consider the $\mathbf{k}$-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ from Definition 3.11. Also, recall Convention 2.4. Let r be an integer such that $r > -k + 1$. Then,

$$\alpha_k(e_r) = (-1)^{r + r\%k} [r\%k \in \{0, 1\}]. \tag{80}$$

Proof of Lemma 3.17. Consider the ring $\Lambda[[t]]$ of formal power series in one indeterminate t over Λ. Consider also the analogous ring $\mathbf{k}[[t]]$ over $\mathbf{k}$.

The map $\alpha_k : \Lambda \to \mathbf{k}$ is a $\mathbf{k}$-algebra homomorphism. Hence, it induces a continuous[42] $\mathbf{k}[[t]]$-algebra homomorphism

$$\alpha_k[[t]] : \Lambda[[t]] \to \mathbf{k}[[t]]$$

that sends each formal power series $\sum_{n\geq 0} a_n t^n \in \Lambda[[t]]$ (with $a_n \in \Lambda$) to $\sum_{n\geq 0} \alpha_k(a_n) t^n$. Consider this $\mathbf{k}[[t]]$-algebra homomorphism $\alpha_k[[t]]$. In particular, it satisfies

$$(\alpha_k[[t]])\left(t^i\right) = t^i \qquad \text{for each } i \in \mathbb{N}.$$

Applying the map $\alpha_k[[t]]$ to both sides of the equality (77), we obtain

$$\begin{aligned}
(\alpha_k[[t]])(1) &= (\alpha_k[[t]])\left(\left(\sum_{n\geq 0}(-1)^n e_n t^n\right)\left(\sum_{n\geq 0} h_n t^n\right)\right) \\
&= \underbrace{(\alpha_k[[t]])\left(\sum_{n\geq 0}(-1)^n e_n t^n\right)}_{\substack{=\sum_{n\geq 0}\alpha_k\left((-1)^n e_n\right)t^n \\ \text{(by the definition of } \alpha_k[[t]])}} \cdot \underbrace{(\alpha_k[[t]])\left(\sum_{n\geq 0} h_n t^n\right)}_{\substack{=\sum_{n\geq 0}\alpha_k(h_n)t^n \\ \text{(by the definition of } \alpha_k[[t]])}} \\
&\qquad (\text{since } \alpha_k[[t]] \text{ is a } \mathbf{k}[[t]]\text{-algebra homomorphism}) \\
&= \left(\sum_{n\geq 0} \underbrace{\alpha_k\left((-1)^n e_n\right)}_{\substack{=(-1)^n\alpha_k(e_n) \\ \text{(since } \alpha_k \text{ is } \mathbf{k}\text{-linear)}}} t^n\right) \cdot \left(\sum_{n\geq 0} \underbrace{\alpha_k(h_n)}_{\substack{=[n<k] \\ \text{(by (34))}}} t^n\right) \\
&= \left(\sum_{n\geq 0}(-1)^n \alpha_k(e_n) t^n\right) \cdot \underbrace{\left(\sum_{n\geq 0}[n<k] t^n\right)}_{=t^0+t^1+\cdots+t^{k-1}=\dfrac{1-t^k}{1-t}} \\
&= \left(\sum_{n\geq 0}(-1)^n \alpha_k(e_n) t^n\right) \cdot \frac{1-t^k}{1-t}.
\end{aligned}$$

Comparing this with

$$(\alpha_k[[t]])(1) = 1 \qquad (\text{since } \alpha_k[[t]] \text{ is a } \mathbf{k}[[t]]\text{-algebra homomorphism}),$$

we obtain

$$\left(\sum_{n\geq 0}(-1)^n \alpha_k(e_n) t^n\right) \cdot \frac{1-t^k}{1-t} = 1.$$

[42] Continuity is defined with respect to the usual topologies on $\Lambda[[t]]$ and $\mathbf{k}[[t]]$, where we equip both Λ and $\mathbf{k}$ with the discrete topologies.

Hence,

$$\sum_{n\geq 0}(-1)^n\alpha_k(e_n)t^n = \frac{1-t}{1-t^k} = (1-t)\cdot \underbrace{\frac{1}{1-t^k}}_{=1+t^k+t^{2k}+t^{3k}+\cdots}$$
$$= (1-t)\cdot\left(1+t^k+t^{2k}+t^{3k}+\cdots\right)$$
$$= 1-t+t^k-t^{k+1}+t^{2k}-t^{2k+1}+t^{3k}-t^{3k+1}\pm\cdots$$
$$= \sum_{n\geq 0}(-1)^{n\%k}[n\%k\in\{0,1\}]\,t^n$$

(here, we have used that $k>1$, since for $k=1$ there would be cancellations in the sum $1-t+t^k-t^{k+1}+t^{2k}-t^{2k+1}+t^{3k}-t^{3k+1}\pm\cdots$). Comparing coefficients before t^m on both sides of this equality, we obtain

$$(-1)^m\alpha_k(e_m) = (-1)^{m\%k}[m\%k\in\{0,1\}] \tag{81}$$

for each $m\in\mathbb{N}$.

Now, each $m\in\mathbb{N}$ satisfies $\underbrace{(-1)^m(-1)^m}_{=(-1)^{2m}=1}\alpha_k(e_m)=\alpha_k(e_m)$ and thus

$$\alpha_k(e_m) = (-1)^m\underbrace{(-1)^m\alpha_k(e_m)}_{\substack{=(-1)^{m\%k}[m\%k\in\{0,1\}]\\ \text{(by (81))}}} = \underbrace{(-1)^m(-1)^{m\%k}}_{=(-1)^{m+m\%k}}[m\%k\in\{0,1\}]$$
$$= (-1)^{m+m\%k}[m\%k\in\{0,1\}]. \tag{82}$$

We must prove that

$$\alpha_k(e_r) = (-1)^{r+r\%k}[r\%k\in\{0,1\}].$$

If $r\in\mathbb{N}$, then this follows by applying (82) to $m=r$. Hence, for the rest of this proof, we WLOG assume that $r\notin\mathbb{N}$. Thus, r is negative (since r is an integer). In view of $r>-k+1$, this yields $r\in\{-k+2,-k+3,\ldots,-1\}$. Hence, $r\%k\in\{2,3,\ldots,k-1\}$. Thus, $r\%k\notin\{0,1\}$. Hence, $[r\%k\in\{0,1\}]=0$. Also, $e_r=0$ (since r is negative) and thus $\alpha_k(e_r)=\alpha_k(0)=0$ (since the map α_k is $\mathbf{k}$-linear). Comparing this with $(-1)^{r+r\%k}\underbrace{[r\%k\in\{0,1\}]}_{=0}=0$, we obtain $\alpha_k(e_r)=(-1)^{r+r\%k}[r\%k\in\{0,1\}]$. This concludes the proof of Lemma 3.17. $\square$

Proof of Theorem 2.15. **(a)** Assume that $\mu_k\neq 0$. But $\mu=\lambda^t$, whence

$$\mu_k = \left(\lambda^t\right)_k = \left|\left\{j\in\{1,2,3,\ldots\}\ \mid\ \lambda_j\geq k\right\}\right| \qquad \left(\text{by the definition of }\lambda^t\right).$$

Hence,

$$\left|\left\{j\in\{1,2,3,\ldots\}\ \mid\ \lambda_j\geq k\right\}\right| = \mu_k\neq 0.$$

In other words, the set $\left\{j \in \{1,2,3,\ldots\} \mid \lambda_j \geq k\right\}$ is nonempty. Hence, there exists some $j \in \{1,2,3,\ldots\}$ satisfying $\lambda_j \geq k$. Consider this j. We have $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ (since $\lambda \in \operatorname{Par}$) and thus $\lambda_1 \geq \lambda_j$ (since $1 \leq j$). Hence, $\lambda_1 \geq \lambda_j \geq k$. Thus, Proposition 2.12 yields $\operatorname{pet}_k(\lambda, \varnothing) = 0$. This proves Theorem 2.15 **(a)**.

Now, let us prepare for the proof of parts **(b)** and **(c)**.

Consider the $\mathbf{k}$-algebra homomorphism $\alpha_k : \Lambda \to \mathbf{k}$ from Definition 3.11.

For each $i \in \{1,2,\ldots,k-1\}$, we have

$$\begin{aligned}
\gamma_i &= 1 + (\beta_i - 1)\,\%k \qquad \text{(by (8))} \\
&\in \{1,2,\ldots,k\} \qquad \left(\begin{array}{c}\text{since } (\beta_i - 1)\,\%k \in \{0,1,\ldots,k-1\} \\ \text{(by the definition of a remainder)}\end{array}\right).
\end{aligned}$$

Hence, the $(k-1)$-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_{k-1})$ really belongs to $\{1,2,\ldots,k\}^{k-1}$. In other words, $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are $k-1$ elements of the set $\{1,2,\ldots,k\}$.

Assume that $\mu_k = 0$. But $\mu \in \operatorname{Par}$ and thus $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$. Hence, from $\mu_k = 0$, we obtain $\mu_k = \mu_{k+1} = \mu_{k+2} = \cdots = 0$. Thus, $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$.

It is known that taking the transpose of the transpose of a partition returns the original partition. Thus, $\left(\lambda^t\right)^t = \lambda$. In view of $\mu = \lambda^t$, this rewrites as $\mu^t = \lambda$. Hence, $\lambda = \mu^t$. Therefore,

$$s_\lambda = s_{\mu^t} = \det\left(\left(e_{\mu_i - i + j}\right)_{1 \leq i \leq k-1,\ 1 \leq j \leq k-1}\right)$$

(by (6), applied to μ and $k-1$ instead of λ and ℓ), because $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$. Applying the map α_k to both sides of this equality, we find

$$\begin{aligned}
\alpha_k(s_\lambda) &= \alpha_k\left(\det\left(\left(\underbrace{e_{\mu_i - i + j}}_{\substack{= e_{\beta_i + j} \\ \text{(since (7) yields } \mu_i - i = \beta_i)}}\right)_{1 \leq i \leq k-1,\ 1 \leq j \leq k-1}\right)\right) \\
&= \alpha_k\left(\det\left(\left(e_{\beta_i + j}\right)_{1 \leq i \leq k-1,\ 1 \leq j \leq k-1}\right)\right) \\
&= \det\left(\left(\alpha_k\left(e_{\beta_i + j}\right)\right)_{1 \leq i \leq k-1,\ 1 \leq j \leq k-1}\right)
\end{aligned}$$

(since α_k is a $\mathbf{k}$-algebra homomorphism, and thus commutes with taking determinants of matrices). On the other hand,

$$\alpha_k\left(\underbrace{s_\lambda}_{= s_{\lambda/\varnothing}}\right) = \alpha_k\left(s_{\lambda/\varnothing}\right) = \operatorname{pet}_k(\lambda, \varnothing)$$

(by (42), applied to $\varnothing$ instead of μ). Comparing these two equalities, we obtain

$$\operatorname{pet}_k(\lambda, \varnothing) = \det\left(\left(\alpha_k\left(e_{\beta_i + j}\right)\right)_{1 \leq i \leq k-1,\ 1 \leq j \leq k-1}\right). \tag{83}$$

But each $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$ satisfy $k > 1$ [43] and

$$\underbrace{\beta_i}_{\substack{=\mu_i - i \\ \text{(by (7))}}} + j = \underbrace{\mu_i}_{\geq 0} - \underbrace{i}_{\leq k-1} + \underbrace{j}_{>0} > 0 - (k-1) + 0 = -k+1$$

and thus

$$\alpha_k \left(e_{\beta_i + j}\right) = (-1)^{(\beta_i + j) + (\beta_i + j)\%k} \left[(\beta_i + j)\,\%k \in \{0, 1\}\right] \tag{84}$$

(by (80), applied to $r = \beta_i + j$).

Furthermore, each $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$ satisfy

$$\begin{aligned} &(-1)^{(\beta_i + j) + (\beta_i + j)\%k} \left[(\beta_i + j)\,\%k \in \{0, 1\}\right] \\ &= (-1)^{\beta_i} (-1)^j (-1)^{(\gamma_i + j)\%k} \left[(\gamma_i + j)\,\%k \in \{0, 1\}\right]. \end{aligned} \tag{85}$$

[*Proof of (85):* Let $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$. The definition of γ_i yields

$$\gamma_i = 1 + \underbrace{(\beta_i - 1)\,\%k}_{\substack{\equiv \beta_i - 1 \operatorname{mod} k \\ \text{(since } u\%k \equiv u \operatorname{mod} k \text{ for any } u \in \mathbb{Z})}} \equiv 1 + (\beta_i - 1) = \beta_i \operatorname{mod} k.$$

Hence, $\underbrace{\gamma_i}_{\equiv \beta_i \operatorname{mod} k} + j \equiv \beta_i + j \operatorname{mod} k$. But if two integers are congruent modulo k, then they must leave the same remainder upon division by k. In other words, if $u \in \mathbb{Z}$ and $v \in \mathbb{Z}$ satisfy $u \equiv v \operatorname{mod} k$, then $u\%k = v\%k$. Applying this to $u = \gamma_i + j$ and $v = \beta_i + j$, we obtain $(\gamma_i + j)\,\%k = (\beta_i + j)\,\%k$. Hence,

$$\begin{aligned} &(-1)^{\beta_i} (-1)^j (-1)^{(\gamma_i + j)\%k} \left[(\gamma_i + j)\,\%k \in \{0, 1\}\right] \\ &= \underbrace{(-1)^{\beta_i} (-1)^j (-1)^{(\beta_i + j)\%k}}_{=(-1)^{(\beta_i + j) + (\beta_i + j)\%k}} \left[(\beta_i + j)\,\%k \in \{0, 1\}\right] \\ &= (-1)^{(\beta_i + j) + (\beta_i + j)\%k} \left[(\beta_i + j)\,\%k \in \{0, 1\}\right]. \end{aligned}$$

This proves (85).]

[43]Indeed, if $i \in \{1, 2, \ldots, k-1\}$, then $1 \leq i \leq k-1$ and thus $k - 1 \geq 1 > 0$, so that $k > 1$.

Now, (83) becomes

$$\begin{aligned}
&\operatorname{pet}_k(\lambda,\varnothing)\\
&=\det\left(\left(\underbrace{\alpha_k\left(e_{\beta_i+j}\right)}_{\substack{=(-1)^{(\beta_i+j)+(\beta_i+j)\%k}[(\beta_i+j)\%k\in\{0,1\}]\\ \text{(by (84))}}}\right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1}\right)\\
&=\det\left(\left(\underbrace{(-1)^{(\beta_i+j)+(\beta_i+j)\%k}\left[(\beta_i+j)\%k\in\{0,1\}\right]}_{\substack{=(-1)^{\beta_i}(-1)^j(-1)^{(\gamma_i+j)\%k}[(\gamma_i+j)\%k\in\{0,1\}]\\ \text{(by (85))}}}\right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1}\right)\\
&=\det\left(\left((-1)^{\beta_i}(-1)^j(-1)^{(\gamma_i+j)\%k}\left[(\gamma_i+j)\%k\in\{0,1\}\right]\right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1}\right)\\
&=\left(\prod_{i=1}^{k-1}\left((-1)^{\beta_i}(-1)^i\right)\right)\cdot\det\left(\left((-1)^{(\gamma_i+j)\%k}\left[(\gamma_i+j)\%k\in\{0,1\}\right]\right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1}\right)
\end{aligned}$$

(by Lemma 3.14 **(b)**, applied to $m=k-1$ and $R=\mathbf{k}$ and $a_{i,j}=(-1)^{(\gamma_i+j)\%k}\left[(\gamma_i+j)\%k\in\{0,1\}\right]$ and $u_i=(-1)^{\beta_i}$ and $v_j=(-1)^j$).

Define a $(k-1)\times(k-1)$-matrix G as in Lemma 3.15. Then, this becomes

$$\begin{aligned}
&\operatorname{pet}_k(\lambda,\varnothing)\\
&=\left(\prod_{i=1}^{k-1}\left((-1)^{\beta_i}(-1)^i\right)\right)\cdot\det\left(\underbrace{\left((-1)^{(\gamma_i+j)\%k}\left[(\gamma_i+j)\%k\in\{0,1\}\right]\right)_{1\leq i\leq k-1,\ 1\leq j\leq k-1}}_{\substack{=G\\ \text{(by the definition of }G)}}\right)\\
&=\left(\prod_{i=1}^{k-1}\left((-1)^{\beta_i}(-1)^i\right)\right)\cdot\det G. \qquad (86)
\end{aligned}$$

Now, we can readily prove parts **(b)** and **(c)** of Theorem 2.15:

(b) Assume that the $k-1$ numbers $\gamma_1,\gamma_2,\ldots,\gamma_{k-1}$ are not distinct. Then, Lemma 3.15 **(a)** yields $\det G=0$. Hence, (86) yields

$$\operatorname{pet}_k(\lambda,\varnothing)=\left(\prod_{i=1}^{k-1}\left((-1)^{\beta_i}(-1)^i\right)\right)\cdot\underbrace{\det G}_{=0}=0.$$

This proves Theorem 2.15 **(b)**.

(c) The equality (86) becomes

$$\begin{aligned}
\operatorname{pet}_k(\lambda,\varnothing) &= \underbrace{\left(\prod_{i=1}^{k-1}\left((-1)^{\beta_i}(-1)^i\right)\right)}_{=\left(\prod\limits_{i=1}^{k-1}(-1)^{\beta_i}\right)\left(\prod\limits_{i=1}^{k-1}(-1)^i\right)} \cdot \underbrace{\det G}_{\substack{=(-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))} \\ \text{(by Lemma 3.15 (c))}}} \\
&= \underbrace{\left(\prod_{i=1}^{k-1}(-1)^{\beta_i}\right)}_{=(-1)^{\beta_1+\beta_2+\cdots+\beta_{k-1}}} \underbrace{\left(\prod_{i=1}^{k-1}(-1)^i\right)}_{=(-1)^{1+2+\cdots+(k-1)}} \cdot (-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))} \\
&= (-1)^{\beta_1+\beta_2+\cdots+\beta_{k-1}}(-1)^{1+2+\cdots+(k-1)} \cdot (-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))} \\
&= (-1)^{(\beta_1+\beta_2+\cdots+\beta_{k-1})+(1+2+\cdots+(k-1))+g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))} \\
&= (-1)^{(\beta_1+\beta_2+\cdots+\beta_{k-1})+g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})}.
\end{aligned}$$

This proves Theorem 2.15 **(c)**. □

The proof of Proposition 2.16 relies on the following known fact:

Proposition 3.18. Let $\lambda \in \operatorname{Par}$. Let $\mu = \lambda^t$. Then:

(a) If i and j are two positive integers satisfying $\lambda_i \geq j$, then $\mu_j \geq i$.

(b) If i and j are two positive integers satisfying $\lambda_i < j$, then $\mu_j < i$.

(c) Any two positive integers i and j satisfy $\lambda_i + \mu_j - i - j \neq -1$.

For each positive integer i, set $\alpha_i = \lambda_i - i$. For each positive integer j, set $\beta_j = \mu_j - j$ and $\eta_j = -1 - \beta_j$. Then:

(d) The two sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \eta_3, \ldots\}$ are disjoint, and their union is $\mathbb{Z}$.

(e) Let p be an integer such that $p \geq \lambda_1$. Then, the two sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint, and their union is

$$\{\ldots, p-3, p-2, p-1\} = \{k \in \mathbb{Z} \mid k < p\}.$$

(f) Let p and q be two integers such that $p \geq \lambda_1$ and $q \geq \mu_1$. Then, the two sets $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint, and their union is

$$\{-q, -q+1, \ldots, p-1\} = \{k \in \mathbb{Z} \mid -q \leq k < p\}.$$

Note that Proposition 3.18 **(f)** is a restatement of [Macdon95, Chapter I, (1.7)].

Proof of Proposition 3.18. We have $\mu = \lambda^t$. Thus, each positive integer i satisfies

$$\begin{aligned}
\mu_i = \left(\lambda^t\right)_i &= \left|\{j \in \{1,2,3,\ldots\} \mid \lambda_j \geq i\}\right| && \text{(by Definition 2.13)} \\
&= \left|\{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq i\}\right| && (87)
\end{aligned}$$

(here, we have renamed the index j as k).

(a) Let i and j be two positive integers satisfying $\lambda_i \geq j$. We must prove that $\mu_j \geq i$.

Indeed, (87) (applied to $i = j$) yields

$$\mu_j = |\{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}|. \tag{88}$$

Now, we have $\{1,2,\ldots,i\} \subseteq \{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}$ [44] and therefore

$$|\{1,2,\ldots,i\}| \leq |\{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}| = \mu_j \qquad \text{(by (88))}.$$

Hence, $\mu_j \geq |\{1,2,\ldots,i\}| = i$. This proves Proposition 3.18 **(a)**.

(b) Let i and j be two positive integers satisfying $\lambda_i < j$. We must prove that $\mu_j < i$.

We have $\{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\} \subseteq \{1,2,\ldots,i-1\}$ [45]. Hence,

$$|\{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}| \leq |\{1,2,\ldots,i-1\}| = i-1.$$

But (87) (applied to $i = j$) yields

$$\mu_j = |\{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}| \leq i-1 < i.$$

This proves Proposition 3.18 **(b)**.

(c) Let i and j be two positive integers. We must prove that $\lambda_i + \mu_j - i - j \neq -1$.

We are in one of the following two cases:

Case 1: We have $\lambda_i \geq j$.

Case 2: We have $\lambda_i < j$.

[44] *Proof.* Let $g \in \{1,2,\ldots,i\}$. We shall show that $g \in \{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}$.

Indeed, $g \in \{1,2,\ldots,i\} \subseteq \{1,2,3,\ldots\}$ and $g \leq i$ (since $g \in \{1,2,\ldots,i\}$). But λ is a partition (since $\lambda \in \operatorname{Par}$). Hence, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. Thus, if u and v are two positive integers satisfying $u \leq v$, then $\lambda_u \geq \lambda_v$. Applying this to $u = g$ and $v = i$, we obtain $\lambda_g \geq \lambda_i$ (since $g \leq i$). Hence, $\lambda_g \geq \lambda_i \geq j$. Now, we know that g is an element of $\{1,2,3,\ldots\}$ (since $g \in \{1,2,3,\ldots\}$) and satisfies $\lambda_g \geq j$. In other words, g is a $k \in \{1,2,3,\ldots\}$ satisfying $\lambda_k \geq j$. In other words, $g \in \{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}$.

Now, forget that we fixed g. We thus have shown that $g \in \{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}$ for each $g \in \{1,2,\ldots,i\}$. In other words, we have $\{1,2,\ldots,i\} \subseteq \{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}$.

[45] *Proof.* Let $g \in \{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}$. We shall show that $g \in \{1,2,\ldots,i-1\}$.

Indeed, assume the contrary. Thus, $g \notin \{1,2,\ldots,i-1\}$.

But $g \in \{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}$. In other words, g is a $k \in \{1,2,3,\ldots\}$ satisfying $\lambda_k \geq j$. In other words, g is an element of $\{1,2,3,\ldots\}$ and satisfies $\lambda_g \geq j$. Hence, $g \in \{1,2,3,\ldots\}$. Combining this with $g \notin \{1,2,\ldots,i-1\}$, we obtain $g \in \{1,2,3,\ldots\} \setminus \{1,2,\ldots,i-1\} = \{i,i+1,i+2,\ldots\}$. Thus, $g \geq i$. Hence, $i \leq g$.

But λ is a partition (since $\lambda \in \operatorname{Par}$). Hence, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. Thus, if u and v are two positive integers satisfying $u \leq v$, then $\lambda_u \geq \lambda_v$. Applying this to $u = i$ and $v = g$, we obtain $\lambda_i \geq \lambda_g$ (since $i \leq g$). Hence, $\lambda_g \leq \lambda_i < j$. This contradicts $\lambda_g \geq j$. This contradiction shows that our assumption was false. Thus, $g \in \{1,2,\ldots,i-1\}$ is proven.

Now, forget that we fixed g. We thus have shown that $g \in \{1,2,\ldots,i-1\}$ for each $g \in \{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\}$. In other words, we have $\{k \in \{1,2,3,\ldots\} \mid \lambda_k \geq j\} \subseteq \{1,2,\ldots,i-1\}$.

Let us first consider Case 1. In this case, we have $\lambda_i \geq j$. Hence, Proposition 3.18 **(a)** yields $\mu_j \geq i$. Hence, $\underbrace{\lambda_i}_{\geq j} + \underbrace{\mu_j}_{\geq i} - i - j \geq j + i - i - j = 0 > -1$. Thus, $\lambda_i + \mu_j - i - j \neq -1$. Hence, Proposition 3.18 **(c)** is proved in Case 1.

Let us next consider Case 2. In this case, we have $\lambda_i < j$. Hence, Proposition 3.18 **(b)** yields $\mu_j < i$. Hence, $\mu_j \leq i - 1$ (since μ_j and i are integers). Thus, $\underbrace{\lambda_i}_{<j} + \underbrace{\mu_j}_{\leq i-1} - i - j < j + (i-1) - i - j = -1$. Thus, $\lambda_i + \mu_j - i - j \neq -1$. Hence, Proposition 3.18 **(c)** is proved in Case 2.

We have now proved Proposition 3.18 **(c)** in each of the two Cases 1 and 2. Since these two Cases cover all possibilities, we thus conclude that Proposition 3.18 **(c)** always holds.

(f) We have $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$ [46] and $\beta_1 > \beta_2 > \beta_3 > \cdots$ [47], hence $\eta_1 < \eta_2 < \eta_3 < \cdots$ [48].

From $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$, we obtain $\alpha_1 > \alpha_2 > \cdots > \alpha_q$. Thus, the q integers $\alpha_1, \alpha_2, \ldots, \alpha_q$ are distinct. Hence, $\left|\left\{\alpha_1, \alpha_2, \ldots, \alpha_q\right\}\right| = q$.

From $\eta_1 < \eta_2 < \eta_3 < \cdots$, we obtain $\eta_1 < \eta_2 < \cdots < \eta_p$. Thus, the p integers $\eta_1, \eta_2, \ldots, \eta_p$ are distinct. Hence, $\left|\left\{\eta_1, \eta_2, \ldots, \eta_p\right\}\right| = p$.

Let L be the finite set $\{-q, -q+1, \ldots, p-1\} = \{k \in \mathbb{Z} \mid -q \leq k < p\}$. Then,

$$\begin{aligned} |L| &= (p-1) - (-q) + 1 = \underbrace{q}_{=\left|\left\{\alpha_1, \alpha_2, \ldots, \alpha_q\right\}\right|} + \underbrace{p}_{=\left|\left\{\eta_1, \eta_2, \ldots, \eta_p\right\}\right|} \\ &= \left|\left\{\alpha_1, \alpha_2, \ldots, \alpha_q\right\}\right| + \left|\left\{\eta_1, \eta_2, \ldots, \eta_p\right\}\right|. \end{aligned}$$

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[46] *Proof.* Let $i \in \{1, 2, 3, \ldots\}$. We shall show that $\alpha_i > \alpha_{i+1}$.

We know that λ is a partition (since $\lambda \in \operatorname{Par}$), so that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. Hence, $\lambda_i \geq \lambda_{i+1}$. But the definition of α_i yields $\alpha_i = \lambda_i - i$, while the definition of α_{i+1} yields $\alpha_{i+1} = \lambda_{i+1} - (i+1)$. Hence, $\alpha_i = \underbrace{\lambda_i}_{\geq \lambda_{i+1}} - \underbrace{i}_{<i+1} > \lambda_{i+1} - (i+1) = \alpha_{i+1}$.

Now, forget that we fixed i. We thus have proved that $\alpha_i > \alpha_{i+1}$ for each $i \in \{1, 2, 3, \ldots\}$. In other words, $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$.

[47] *Proof.* Let $j \in \{1, 2, 3, \ldots\}$. We shall show that $\beta_j > \beta_{j+1}$.

We know that μ is a partition (since $\mu = \lambda^t \in \operatorname{Par}$), so that $\mu_1 \geq \mu_2 \geq \mu_3 \geq \cdots$. Hence, $\mu_j \geq \mu_{j+1}$. But the definition of β_j yields $\beta_j = \mu_j - j$, while the definition of β_{j+1} yields $\beta_{j+1} = \mu_{j+1} - (j+1)$. Hence, $\beta_j = \underbrace{\mu_j}_{\geq \mu_{j+1}} - \underbrace{j}_{<j+1} > \mu_{j+1} - (j+1) = \beta_{j+1}$.

Now, forget that we fixed j. We thus have proved that $\beta_j > \beta_{j+1}$ for each $j \in \{1, 2, 3, \ldots\}$. In other words, $\beta_1 > \beta_2 > \beta_3 > \cdots$.

[48] *Proof.* Let $j \in \{1, 2, 3, \ldots\}$. We shall show that $\eta_j < \eta_{j+1}$.

We know that $\beta_1 > \beta_2 > \beta_3 > \cdots$. Hence, $\beta_j > \beta_{j+1}$. But the definition of η_j yields $\eta_j = -1 - \beta_j$, while the definition of η_{j+1} yields $\eta_{j+1} = -1 - \beta_{j+1}$. Hence, $\eta_j = -1 - \underbrace{\beta_j}_{>\beta_{j+1}} < -1 - \beta_{j+1} = \eta_{j+1}$.

Now, forget that we fixed j. We thus have proved that $\eta_j < \eta_{j+1}$ for each $j \in \{1, 2, 3, \ldots\}$. In other words, $\eta_1 < \eta_2 < \eta_3 < \cdots$.

We have $\{\alpha_1, \alpha_2, \ldots, \alpha_q\} \subseteq L$ [49] and $\{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq L$ [50]. Furthermore, Proposition 3.18 **(c)** easily shows that the sets $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint[51].

Now, recall the following basic fact from the theory of finite sets:

[49] *Proof.* Let $i \in \{1, 2, \ldots, q\}$. We shall show that $\alpha_i \in L$.

We have $i \in \{1, 2, \ldots, q\}$, so that $1 \leq i \leq q$. Thus, $q \geq 1$. Hence, α_q is well-defined.

From $\alpha_1 > \alpha_2 > \cdots > \alpha_q$, we conclude that all the q numbers $\alpha_1, \alpha_2, \ldots, \alpha_q$ lie in the interval between α_q (inclusive) and α_1 (inclusive). In other words, $\alpha_q \leq \alpha_j \leq \alpha_1$ for each $j \in \{1, 2, \ldots, q\}$. Applying this to $j = i$, we obtain $\alpha_q \leq \alpha_i \leq \alpha_1$.

Now, the definition of α_1 yields $\alpha_1 = \lambda_1 - 1 < \lambda_1 \leq p$ (since $p \geq \lambda_1$). Now, $\alpha_i \leq \alpha_1 < p$. Also, the definition of α_q yields $\alpha_q = \underbrace{\lambda_q}_{\geq 0} - q \geq -q$, so that $-q \leq \alpha_q \leq \alpha_i$. Hence, $-q \leq \alpha_i < p$.

Thus, we know that α_i is an element of $\mathbb{Z}$ and satisfies $-q \leq \alpha_i < p$. In other words, α_i is a $k \in \mathbb{Z}$ satisfying $-q \leq k < p$. In other words, $\alpha_i \in \{k \in \mathbb{Z} \mid -q \leq k < p\}$.

But the definition of L yields $L = \{k \in \mathbb{Z} \mid -q \leq k < p\}$. Hence, $\alpha_i \in \{k \in \mathbb{Z} \mid -q \leq k < p\} = L$.

Forget that we fixed i. We thus have proven that $\alpha_i \in L$ for each $i \in \{1, 2, \ldots, q\}$. In other words, $\alpha_1, \alpha_2, \ldots, \alpha_q$ are elements of L. In other words, $\{\alpha_1, \alpha_2, \ldots, \alpha_q\} \subseteq L$.

[50] *Proof.* Let $j \in \{1, 2, \ldots, p\}$. We shall show that $\eta_j \in L$.

We have $j \in \{1, 2, \ldots, p\}$, so that $1 \leq j \leq p$. Thus, $p \geq 1$. Hence, η_p is well-defined.

From $\eta_1 < \eta_2 < \cdots < \eta_p$, we conclude that all the p numbers $\eta_1, \eta_2, \ldots, \eta_p$ lie in the interval between η_1 (inclusive) and η_p (inclusive). In other words, $\eta_1 \leq \eta_i \leq \eta_p$ for each $i \in \{1, 2, \ldots, p\}$. Applying this to $i = j$, we obtain $\eta_1 \leq \eta_j \leq \eta_p$.

Now, the definition of β_1 yields $\beta_1 = \underbrace{\mu_1}_{\substack{\leq q \\ (\text{since } q \geq \mu_1)}} - 1 \leq q - 1$. But the definition of η_1 yields $\eta_1 = -1 - \underbrace{\beta_1}_{\leq q-1} \geq -1 - (q-1) = -q$. Hence, $-q \leq \eta_1 \leq \eta_j$.

Also, the definition of β_p yields $\beta_p = \underbrace{\mu_p}_{\geq 0} - p \geq -p$. But the definition of η_p yields $\eta_p = -1 - \underbrace{\beta_p}_{\geq -p} \leq -1 - (-p) = p - 1 < p$. Hence, $\eta_j \leq \eta_p < p$.

Thus, we know that η_j is an element of $\mathbb{Z}$ and satisfies $-q \leq \eta_j < p$. In other words, η_j is a $k \in \mathbb{Z}$ satisfying $-q \leq k < p$. In other words, $\eta_j \in \{k \in \mathbb{Z} \mid -q \leq k < p\}$.

But the definition of L yields $L = \{k \in \mathbb{Z} \mid -q \leq k < p\}$. Hence, $\eta_j \in \{k \in \mathbb{Z} \mid -q \leq k < p\} = L$.

Forget that we fixed j. We thus have proven that $\eta_j \in L$ for each $j \in \{1, 2, \ldots, p\}$. In other words, $\eta_1, \eta_2, \ldots, \eta_p$ are elements of L. In other words, $\{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq L$.

[51] *Proof.* Let $\zeta \in \{\alpha_1, \alpha_2, \ldots, \alpha_q\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\}$. Then, $\zeta \in \{\alpha_1, \alpha_2, \ldots, \alpha_q\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq \{\alpha_1, \alpha_2, \ldots, \alpha_q\}$; in other words, there exists some $i \in \{1, 2, \ldots, q\}$ such that $\zeta = \alpha_i$. Consider this i. We then have $\zeta = \alpha_i = \lambda_i - i$ (by the definition of α_i).

Also, $\zeta \in \{\alpha_1, \alpha_2, \ldots, \alpha_q\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq \{\eta_1, \eta_2, \ldots, \eta_p\}$; in other words, there exists some $j \in \{1, 2, \ldots, p\}$ such that $\zeta = \eta_j$. Consider this j. We then have $\zeta = \eta_j = -1 - \beta_j$ (by the definition of η_j). But the definition of β_j yields $\beta_j = \mu_j - j$. Hence, $\zeta = -1 - \underbrace{\beta_j}_{= \mu_j - j} = -1 - (\mu_j - j) = -1 - \mu_j + j$. Comparing this with $\zeta = \lambda_i - i$, we obtain $\lambda_i - i = -1 - \mu_j + j$. In other words, $\lambda_i + \mu_j - i - j = -1$. But Proposition 3.18 **(c)** yields $\lambda_i + \mu_j - i - j \neq -1$. This contradicts $\lambda_i + \mu_j - i - j = -1$.

Forget that we fixed ζ. We thus have found a contradiction for each $\zeta \in \{\alpha_1, \alpha_2, \ldots, \alpha_q\} \cap$

> *Fact A:* Let U, V and W be three finite sets such that $V \subseteq U$ and $W \subseteq U$ and $|U| = |V| + |W|$. Assume that V and W are disjoint. Then, $V \cup W = U$.

[*Proof of Fact A:* The set V is a subset of U (since $V \subseteq U$), but is disjoint from W (since V and W are disjoint). Thus, V is a subset of $U \setminus W$. But from $W \subseteq U$, we obtain $|U \setminus W| = |U| - |W| = |V|$ (since $|U| = |V| + |W|$). Hence, $|V| = |U \setminus W|$. Thus, the set V has the same size as $U \setminus W$. Note that the set $U \setminus W$ is finite (since U is finite).

Now, recall the well-known fact that if a subset Q of a finite set R has the same size as R, then $Q = R$. We can apply this to $R = U \setminus W$ and $Q = V$ (since V is a subset of $U \setminus W$ and has the same size as $U \setminus W$), and conclude that $V = U \setminus W$. Hence, $\underbrace{V}_{=U \setminus W} \cup W = (U \setminus W) \cup W = U$ (since $W \subseteq U$). This proves Fact A.]

Now, recall that L, $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are three finite sets such that $\{\alpha_1, \alpha_2, \ldots, \alpha_q\} \subseteq L$ and $\{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq L$ and $|L| = \left|\{\alpha_1, \alpha_2, \ldots, \alpha_q\}\right| + \left|\{\eta_1, \eta_2, \ldots, \eta_p\}\right|$ and such that the sets $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint. Hence, Fact A (applied to $U = L$, $V = \{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $W = \{\eta_1, \eta_2, \ldots, \eta_p\}$) yields that $\{\alpha_1, \alpha_2, \ldots, \alpha_q\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\} = L$. In other words, the union of the two sets $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ is L.

Thus, we have shown that the two sets $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint, and their union is

$$L = \{-q, -q+1, \ldots, p-1\} = \{k \in \mathbb{Z} \mid -q \leq k < p\}.$$

This proves Proposition 3.18 **(f)**.

(e) We have $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$ (as we have shown in our above proof of Proposition 3.18 **(f)**) and $\eta_1 < \eta_2 < \cdots < \eta_p$ (as we have shown in our above proof of Proposition 3.18 **(f)**).

Let M be the set $\{\ldots, p-3, p-2, p-1\} = \{k \in \mathbb{Z} \mid k < p\}$. Then, $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \subseteq M$ [52] and $\{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq M$ [53]. Furthermore, Proposition 3.18 **(c)** easily

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$\{\eta_1, \eta_2, \ldots, \eta_p\}$. Thus, there exists no $\zeta \in \{\alpha_1, \alpha_2, \ldots, \alpha_q\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\}$. In other words, the set $\{\alpha_1, \alpha_2, \ldots, \alpha_q\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\}$ is empty. In other words, the sets $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint.

[52] *Proof.* Let $i \in \{1, 2, 3, \ldots\}$. We shall show that $\alpha_i \in M$.

From $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$, we conclude that $\alpha_1 \geq \alpha_j$ for each $j \in \{1, 2, 3, \ldots\}$. Applying this to $j = i$, we obtain $\alpha_1 \geq \alpha_i$, so that $\alpha_i \leq \alpha_1$.

Now, the definition of α_1 yields $\alpha_1 = \lambda_1 - 1 < \lambda_1 \leq p$ (since $p \geq \lambda_1$). Now, $\alpha_i \leq \alpha_1 < p$.

Thus, we know that α_i is an element of $\mathbb{Z}$ and satisfies $\alpha_i < p$. In other words, α_i is a $k \in \mathbb{Z}$ satisfying $k < p$. In other words, $\alpha_i \in \{k \in \mathbb{Z} \mid k < p\}$.

But the definition of M yields $M = \{k \in \mathbb{Z} \mid k < p\}$. Hence, $\alpha_i \in \{k \in \mathbb{Z} \mid k < p\} = M$.

Forget that we fixed i. We thus have proven that $\alpha_i \in M$ for each $i \in \{1, 2, 3, \ldots\}$. In other words, $\alpha_1, \alpha_2, \alpha_3, \ldots$ are elements of M. In other words, $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \subseteq M$.

[53] *Proof.* Let $j \in \{1, 2, \ldots, p\}$. We shall show that $\eta_j \in L$.

We have $j \in \{1, 2, \ldots, p\}$, so that $1 \leq j \leq p$. Thus, $p \geq 1$. Hence, η_p is well-defined.

From $\eta_1 < \eta_2 < \cdots < \eta_p$, we conclude that all the p numbers $\eta_1, \eta_2, \ldots, \eta_p$ lie in the interval shows that the sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint[54]. Moreover, $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\} = M$ [55]. In other words, the union of the two sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ is M.

Thus, we have shown that the two sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint, and their union is

$$M = \{\ldots, p-3, p-2, p-1\} = \{k \in \mathbb{Z} \mid k < p\}.$$

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between η_1 (inclusive) and η_p (inclusive). In other words, $\eta_1 \leq \eta_i \leq \eta_p$ for each $i \in \{1, 2, \ldots, p\}$. Applying this to $i = j$, we obtain $\eta_1 \leq \eta_j \leq \eta_p$.

The definition of β_p yields $\beta_p = \underbrace{\mu_p}_{\geq 0} - p \geq -p$. But the definition of η_p yields $\eta_p = -1 - \underbrace{\beta_p}_{\geq -p} \leq -1 - (-p) = p - 1 < p$. Hence, $\eta_j \leq \eta_p < p$.

Thus, we know that η_j is an element of $\mathbb{Z}$ and satisfies $\eta_j < p$. In other words, η_j is a $k \in \mathbb{Z}$ satisfying $k < p$. In other words, $\eta_j \in \{k \in \mathbb{Z} \mid k < p\}$.

But the definition of M yields $M = \{k \in \mathbb{Z} \mid k < p\}$. Hence, $\eta_j \in \{k \in \mathbb{Z} \mid k < p\} = M$.

Forget that we fixed j. We thus have proven that $\eta_j \in M$ for each $j \in \{1, 2, \ldots, p\}$. In other words, $\eta_1, \eta_2, \ldots, \eta_p$ are elements of M. In other words, $\{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq M$.

[54] *Proof.* Let $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\}$. Then, $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$; in other words, there exists some $i \in \{1, 2, 3, \ldots\}$ such that $\zeta = \alpha_i$. Consider this i. We then have $\zeta = \alpha_i = \lambda_i - i$ (by the definition of α_i).

Also, $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq \{\eta_1, \eta_2, \ldots, \eta_p\}$; in other words, there exists some $j \in \{1, 2, \ldots, p\}$ such that $\zeta = \eta_j$. Consider this j. We then have $\zeta = \eta_j = -1 - \beta_j$ (by the definition of η_j). But the definition of β_j yields $\beta_j = \mu_j - j$. Hence, $\zeta = -1 - \underbrace{\beta_j}_{=\mu_j - j} = -1 - (\mu_j - j) = -1 - \mu_j + j$. Comparing this with $\zeta = \lambda_i - i$, we obtain $\lambda_i - i = -1 - \mu_j + j$. In other words, $\lambda_i + \mu_j - i - j = -1$. But Proposition 3.18 **(c)** yields $\lambda_i + \mu_j - i - j \neq -1$. This contradicts $\lambda_i + \mu_j - i - j = -1$.

Forget that we fixed ζ. We thus have found a contradiction for each $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\}$. Thus, there exists no $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\}$. In other words, the set $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \ldots, \eta_p\}$ is empty. In other words, the sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint.

[55] *Proof.* Combining $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \subseteq M$ with $\{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq M$, we obtain $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq M$. We shall now prove the reverse inclusion.

Fix $m \in M$. Thus, $m \in M = \{\ldots, p-3, p-2, p-1\}$ (by the definition of M), so that $m \leq p - 1 < p$.

Let $q = \max\{\mu_1, -m\}$. Thus, $q = \max\{\mu_1, m\} \geq -m$ and $q = \max\{\mu_1, m\} \geq \mu_1$.

From $q \geq -m$, we obtain $-\underbrace{q}_{\geq -m} \leq -(-m) = m$. Hence, $-q \leq m < p$. Thus, m is an element of $\mathbb{Z}$ satisfying $-q \leq m < p$. In other words, m is a $k \in \mathbb{Z}$ satisfying $-q \leq k < p$. In other words, $m \in \{k \in \mathbb{Z} \mid -q \leq k < p\}$.

But Proposition 3.18 **(f)** yields that the two sets $\{\alpha_1, \alpha_2, \ldots, \alpha_q\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint, and their union is

$$\{-q, -q+1, \ldots, p-1\} = \{k \in \mathbb{Z} \mid -q \leq k < p\}.$$

Thus, in particular, their union is $\{-q, -q+1, \ldots, p-1\} = \{k \in \mathbb{Z} \mid -q \leq k < p\}$. In other words,

$$\{\alpha_1, \alpha_2, \ldots, \alpha_q\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\} = \{-q, -q+1, \ldots, p-1\} = \{k \in \mathbb{Z} \mid -q \leq k < p\}.$$

This proves Proposition 3.18 **(e)**.

(d) We have $\alpha_1 > \alpha_2 > \alpha_3 > \cdots$ (as we have shown in our above proof of Proposition 3.18 **(f)**) and $\eta_1 < \eta_2 < \eta_3 < \cdots$ (as we have shown in our above proof of Proposition 3.18 **(f)**).

We have $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \subseteq \mathbb{Z}$ (since $\alpha_1, \alpha_2, \alpha_3, \ldots$ are integers) and $\{\eta_1, \eta_2, \eta_3, \ldots\} \subseteq \mathbb{Z}$ (since $\eta_1, \eta_2, \eta_3, \ldots$ are integers). Furthermore, Proposition 3.18 **(c)** easily shows that the sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \eta_3, \ldots\}$ are disjoint[56]. Moreover, $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \eta_3, \ldots\} = \mathbb{Z}$ [57]. In other words, the union of the two sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$

Hence,

$$m \in \{k \in \mathbb{Z} \mid -q \leq k < p\} = \underbrace{\{\alpha_1, \alpha_2, \ldots, \alpha_q\}}_{\subseteq \{\alpha_1, \alpha_2, \alpha_3, \ldots\}} \cup \{\eta_1, \eta_2, \ldots, \eta_p\}$$
$$\subseteq \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\}.$$

Forget that we fixed m. We thus have shown that $m \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\}$ for each $m \in M$. In other words, $M \subseteq \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\}$. Combining this with $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\} \subseteq M$, we obtain $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\} = M$. Qed.

[56]*Proof.* Let $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \eta_3, \ldots\}$. Then, $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \eta_3, \ldots\} \subseteq \{\alpha_1, \alpha_2, \alpha_3, \ldots\}$; in other words, there exists some $i \in \{1, 2, 3, \ldots\}$ such that $\zeta = \alpha_i$. Consider this i. We then have $\zeta = \alpha_i = \lambda_i - i$ (by the definition of α_i).

Also, $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \eta_3, \ldots\} \subseteq \{\eta_1, \eta_2, \eta_3, \ldots\}$; in other words, there exists some $j \in \{1, 2, 3, \ldots\}$ such that $\zeta = \eta_j$. Consider this j. We then have $\zeta = \eta_j = -1 - \beta_j$ (by the definition of η_j). But the definition of β_j yields $\beta_j = \mu_j - j$. Hence, $\zeta = -1 - \underbrace{\beta_j}_{=\mu_j - j} = -1 - (\mu_j - j) = -1 - \mu_j + j$. Comparing this with $\zeta = \lambda_i - i$, we obtain $\lambda_i - i = -1 - \mu_j + j$. In other words, $\lambda_i + \mu_j - i - j = -1$. But Proposition 3.18 **(c)** yields $\lambda_i + \mu_j - i - j \neq -1$. This contradicts $\lambda_i + \mu_j - i - j = -1$.

Forget that we fixed ζ. We thus have found a contradiction for each $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \eta_3, \ldots\}$. Thus, there exists no $\zeta \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \eta_3, \ldots\}$. In other words, the set $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cap \{\eta_1, \eta_2, \eta_3, \ldots\}$ is empty. In other words, the sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \eta_3, \ldots\}$ are disjoint.

[57]*Proof.* Combining $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \subseteq \mathbb{Z}$ with $\{\eta_1, \eta_2, \eta_3, \ldots\} \subseteq \mathbb{Z}$, we obtain $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \eta_3, \ldots\} \subseteq \mathbb{Z}$. We shall now prove the reverse inclusion.

Fix $m \in \mathbb{Z}$.

Let $p = \max\{\lambda_1, m+1\}$. Thus, $p = \max\{\lambda_1, m+1\} \geq \lambda_1$ and $p = \max\{\lambda_1, m+1\} \geq m+1 > m$. From $p > m$, we obtain $m < p$. Thus, m is a $k \in \mathbb{Z}$ satisfying $k < p$ (since $m \in \mathbb{Z}$ and $m < p$). In other words, $m \in \{k \in \mathbb{Z} \mid k < p\}$.

But Proposition 3.18 **(e)** yields that the two sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \ldots, \eta_p\}$ are disjoint, and their union is

$$\{\ldots, p-3, p-2, p-1\} = \{k \in \mathbb{Z} \mid k < p\}.$$

Thus, in particular, their union is $\{\ldots, p-3, p-2, p-1\} = \{k \in \mathbb{Z} \mid k < p\}$. In other words,

$$\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \ldots, \eta_p\} = \{\ldots, p-3, p-2, p-1\} = \{k \in \mathbb{Z} \mid k < p\}.$$

Hence,

$$m \in \{k \in \mathbb{Z} \mid k < p\} = \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \underbrace{\{\eta_1, \eta_2, \ldots, \eta_p\}}_{\subseteq \{\eta_1, \eta_2, \eta_3, \ldots\}} \subseteq \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \eta_3, \ldots\}.$$

and $\{\eta_1, \eta_2, \eta_3, \ldots\}$ is $\mathbb{Z}$.

Thus, we have shown that the two sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \eta_3, \ldots\}$ are disjoint, and their union is $\mathbb{Z}$. This proves Proposition 3.18 **(d)**. $\square$

Proof of Proposition 2.16 (sketched). Let $\mu = \lambda^t$. Then, the number of parts of μ is λ_1. Hence, from $\lambda_1 < k$, we conclude that μ has fewer than k parts. Thus, $\mu_k = 0$.

For each positive integer i, set $\alpha_i = \lambda_i - i$. Hence,

$$\{\alpha_1, \alpha_2, \alpha_3, \ldots\} = \left\{ \underbrace{\alpha_i}_{=\lambda_i - i} \mid i \in \{1, 2, 3, \ldots\} \right\} = \{\lambda_i - i \mid i \in \{1, 2, 3, \ldots\}\}$$
$$= B \qquad \text{(by the definition of } B\text{)}.$$

For each positive integer j, set $\beta_j = \mu_j - j$ and $\eta_j = -1 - \beta_j$. Note that $(\beta_1, \beta_2, \ldots, \beta_{k-1}) \in \mathbb{Z}^{k-1}$ is thus the same $(k-1)$-tuple that was called $(\beta_1, \beta_2, \ldots, \beta_{k-1})$ in Theorem 2.15. It is easy to see that $\beta_1 > \beta_2 > \cdots > \beta_{k-1}$ and $\lambda_1 - 1 > \lambda_2 - 2 > \lambda_3 - 3 > \cdots$.

From $\lambda_1 < k$, we obtain $k - 1 \geq \lambda_1$. Hence, Proposition 3.18 **(e)** (applied to $p = k - 1$) yields that the two sets $\{\alpha_1, \alpha_2, \alpha_3, \ldots\}$ and $\{\eta_1, \eta_2, \ldots, \eta_{k-1}\}$ are disjoint, and their union is

$$\{\ldots, (k-1) - 3, (k-1) - 2, (k-1) - 1\} = \{\text{all integers smaller than } k - 1\} = W.$$

Since $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} = B$, we can restate this as follows: The two sets B and $\{\eta_1, \eta_2, \ldots, \eta_{k-1}\}$ are disjoint, and their union is W. Hence, $\{\eta_1, \eta_2, \ldots, \eta_{k-1}\} = W \setminus B$.

It is also easy to see that $\beta_1 > \beta_2 > \cdots > \beta_{k-1}$, so that $\eta_1 < \eta_2 < \cdots < \eta_{k-1}$. Hence, $\eta_1, \eta_2, \ldots, \eta_{k-1}$ are the elements of the set $\{\eta_1, \eta_2, \ldots, \eta_{k-1}\}$ listed in increasing order (with no repetition).

Let us define a $(k-1)$-tuple $(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}) \in \{1, 2, \ldots, k\}^{k-1}$ as in Theorem

Forget that we fixed m. We thus have shown that $m \in \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \eta_3, \ldots\}$ for each $m \in \mathbb{Z}$. In other words, $\mathbb{Z} \subseteq \{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \eta_3, \ldots\}$. Combining this with $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \eta_3, \ldots\} \subseteq \mathbb{Z}$, we obtain $\{\alpha_1, \alpha_2, \alpha_3, \ldots\} \cup \{\eta_1, \eta_2, \eta_3, \ldots\} = \mathbb{Z}$. Qed.

2.15. Now, we have the following chain of logical equivalences:

$$\begin{aligned}
&(\operatorname{pet}_k(\lambda, \varnothing) \neq 0) \\
\iff &(\text{the } k-1 \text{ numbers } \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \text{ are distinct}) \\
&\qquad (\text{by parts } \textbf{(b)} \text{ and } \textbf{(c)} \text{ of Theorem 2.15}) \\
\iff &(\text{the } k-1 \text{ numbers } (\beta_1 - 1)\,\%k, (\beta_2 - 1)\,\%k, \ldots, (\beta_{k-1} - 1)\,\%k \text{ are distinct}) \\
&\qquad (\text{since } \gamma_i = 1 + (\beta_i - 1)\,\%k \text{ for each } i) \\
\iff &(\text{no two of the } k-1 \text{ numbers } \beta_1 - 1, \beta_2 - 1, \ldots, \beta_{k-1} - 1 \text{ are congruent modulo } k) \\
\iff &(\text{no two of the } k-1 \text{ numbers } \beta_1, \beta_2, \ldots, \beta_{k-1} \text{ are congruent modulo } k) \\
\iff &(\text{no two of the } k-1 \text{ numbers } -1-\beta_1, -1-\beta_2, \ldots, -1-\beta_{k-1} \\
&\text{are congruent modulo } k) \\
\iff &(\text{no two of the } k-1 \text{ numbers } \eta_1, \eta_2, \ldots, \eta_{k-1} \text{ are congruent modulo } k) \\
&\qquad \left(\text{since } \eta_j = -1 - \beta_j \text{ for each } j\right) \\
\iff &(\text{no two of the } k-1 \text{ elements of } \{\eta_1, \eta_2, \ldots, \eta_{k-1}\} \text{ are congruent modulo } k) \\
&\left(\begin{array}{c}\text{since } \eta_1, \eta_2, \ldots, \eta_{k-1} \text{ are the elements of the set } \{\eta_1, \eta_2, \ldots, \eta_{k-1}\} \\ \text{listed in increasing order (with no repetition)}\end{array}\right) \\
\iff &(\text{no two of the } k-1 \text{ elements of } W \setminus B \text{ are congruent modulo } k) \\
&\qquad (\text{since } \{\eta_1, \eta_2, \ldots, \eta_{k-1}\} = W \setminus B) \\
\iff &\left(\text{each congruence class } \bar{i} \text{ has at most 1 element in common with } W \setminus B\right) \\
\iff &\left(\text{each } i \in \{0, 1, \ldots, k-1\} \text{ satisfies } \left|\bar{i} \cap (W \setminus B)\right| \leq 1\right) \\
\iff &\left(\text{each } i \in \{0, 1, \ldots, k-1\} \text{ satisfies } \left|\left(\bar{i} \cap W\right) \setminus B\right| \leq 1\right)
\end{aligned}$$

(since $\bar{i} \cap (W \setminus B) = \left(\bar{i} \cap W\right) \setminus B$ for each i). This proves Proposition 2.16. □

3.14. Proof of Theorem 2.19

Proof of Theorem 2.19. In this proof, the word "monomial" may refer to a monomial in any set of variables (not necessarily in $x_1, x_2, x_3, \ldots$).

In the following, an *i-monomial* (where $i \in \mathbb{N}$) shall mean a monomial of degree i.

We shall say that a monomial is *k-bounded* if all exponents in this monomial are $< k$. In other words, a monomial is *k-bounded* if it can be written in the form $z_1^{a_1} z_2^{a_2} \cdots z_s^{a_s}$, where $z_1, z_2, \ldots, z_s$ are distinct variables and $a_1, a_2, \ldots, a_s$ are nonnegative integers $< k$. Thus, the k-bounded monomials in the variables $x_1, x_2, x_3, \ldots$ are precisely the monomials of the form $\mathbf{x}^\alpha$ for $\alpha \in \mathrm{WC}$ satisfying ($\alpha_i < k$ for all i). Hence, the k-bounded m-monomials in the variables $x_1, x_2, x_3, \ldots$ are precisely the monomials of the form $\mathbf{x}^\alpha$ for $\alpha \in \mathrm{WC}$ satisfying $|\alpha| = m$ and ($\alpha_i < k$ for all i).

Now, the definition of $G(k,m)$ yields

$$G(k,m) = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha$$
$$= (\text{the sum of all } k\text{-bounded } m\text{-monomials in the variables } x_1, x_2, x_3, \ldots) \qquad (89)$$

(since the k-bounded m-monomials in the variables $x_1, x_2, x_3, \ldots$ are precisely the monomials of the form $\mathbf{x}^\alpha$ for $\alpha \in \mathrm{WC}$ satisfying $|\alpha| = m$ and ($\alpha_i < k$ for all i)).

Let us now substitute the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ on both sides of the equality (89). (This means that we choose some bijection $\phi : \{x_1, x_2, x_3, \ldots\} \to \{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$, and substitute $\phi(x_i)$ for each x_i on both sides of (89).) The left hand side of (89) turns into $(G(k,m))(\mathbf{x}, \mathbf{y})$ upon this substitution[58], whereas the right hand side turns into

$$(\text{the sum of all } k\text{-bounded } m\text{-monomials in the variables } x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots)$$

[59]. Thus, our substitution transforms the equality (89) into

$$(G(k,m))(\mathbf{x}, \mathbf{y})$$
$$= (\text{the sum of all } k\text{-bounded } m\text{-monomials in the variables } x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots). \qquad (90)$$

But any monomial $\mathfrak{m}$ in the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ can be uniquely written as a product $\mathfrak{n}\mathfrak{p}$, where $\mathfrak{n}$ is a monomial in the variables $x_1, x_2, x_3, \ldots$ and where $\mathfrak{p}$ is a monomial in the variables $y_1, y_2, y_3, \ldots$. Moreover, if $\mathfrak{m}$ is written in this form, then:

- the degree of $\mathfrak{m}$ equals the sum of the degrees of $\mathfrak{n}$ and $\mathfrak{p}$;
- thus, $\mathfrak{m}$ is an m-monomial if and only if there exists some $i \in \{0, 1, \ldots, m\}$ such that $\mathfrak{n}$ is an i-monomial and $\mathfrak{p}$ is an $(m-i)$-monomial;
- furthermore, $\mathfrak{m}$ is k-bounded if and only if both $\mathfrak{n}$ and $\mathfrak{p}$ are k-bounded.

Thus, any k-bounded m-monomial $\mathfrak{m}$ in the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ can be uniquely written as a product $\mathfrak{n}\mathfrak{p}$, where $i \in \{0, 1, \ldots, m\}$, where $\mathfrak{n}$ is a k-bounded i-monomial in the variables $x_1, x_2, x_3, \ldots$ and where $\mathfrak{p}$ is a k-bounded $(m-i)$-monomial in the variables $y_1, y_2, y_3, \ldots$. Conversely, every such product $\mathfrak{n}\mathfrak{p}$

[58] because this is how $(G(k,m))(\mathbf{x}, \mathbf{y})$ was defined

[59] Indeed, the substitution can be regarded as simply renaming the variables $x_1, x_2, x_3, \ldots$ as $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ (in some order). Thus, it turns the k-bounded m-monomials in the variables $x_1, x_2, x_3, \ldots$ into the k-bounded m-monomials in the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$.

is a k-bounded m-monomial $\mathfrak{m}$ in the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$. Thus, we obtain a bijection

$$\begin{aligned}
&\bigsqcup_{i\in\{0,1,\ldots,m\}} \Big(\{k\text{-bounded } i\text{-monomials in the variables } x_1, x_2, x_3, \ldots\} \\
&\qquad\qquad \times \{k\text{-bounded } (m-i)\text{-monomials in the variables } y_1, y_2, y_3, \ldots\} \Big) \\
&\quad \to \{k\text{-bounded } m\text{-monomials in the variables } x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}
\end{aligned}$$

that sends each pair $(\mathfrak{n}, \mathfrak{p})$ to $\mathfrak{n}\mathfrak{p}$. Hence,

$$\begin{aligned}
&(\text{the sum of all } k\text{-bounded } m\text{-monomials in the variables } x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots) \\
&= \sum_{i\in\{0,1,\ldots,m\}} \sum_{\substack{\mathfrak{n} \text{ is a } k\text{-bounded} \\ i\text{-monomial in the} \\ \text{variables } x_1, x_2, x_3, \ldots}} \sum_{\substack{\mathfrak{p} \text{ is a } k\text{-bounded} \\ (m-i)\text{-monomial in the} \\ \text{variables } y_1, y_2, y_3, \ldots}} \mathfrak{n}\mathfrak{p} \\
&= \sum_{i\in\{0,1,\ldots,m\}} \left(\sum_{\substack{\mathfrak{n} \text{ is a } k\text{-bounded} \\ i\text{-monomial in the} \\ \text{variables } x_1, x_2, x_3, \ldots}} \mathfrak{n} \right) \left(\sum_{\substack{\mathfrak{p} \text{ is a } k\text{-bounded} \\ (m-i)\text{-monomial in the} \\ \text{variables } y_1, y_2, y_3, \ldots}} \mathfrak{p} \right) \\
&= \sum_{i\in\{0,1,\ldots,m\}} (\text{the sum of all } k\text{-bounded } i\text{-monomials in the variables } x_1, x_2, x_3, \ldots) \\
&\qquad \cdot (\text{the sum of all } k\text{-bounded } (m-i)\text{-monomials in the variables } y_1, y_2, y_3, \ldots) .
\end{aligned} \tag{91}$$

Now, let $i \in \{0, 1, \ldots, m\}$. The same reasoning that gave us (89) can be applied to i instead of m. Thus we obtain

$$G(k, i) = (\text{the sum of all } k\text{-bounded } i\text{-monomials in the variables } x_1, x_2, x_3, \ldots) . \tag{92}$$

Also, $i \in \{0, 1, \ldots, m\}$, so that $m - i \in \{0, 1, \ldots, m\} \subseteq \mathbb{N}$. Hence, the same reasoning that gave us (89) can be applied to $m - i$ instead of m. Thus we obtain

$$\begin{aligned}
&G(k, m-i) \\
&= (\text{the sum of all } k\text{-bounded } (m-i)\text{-monomials in the variables } x_1, x_2, x_3, \ldots) .
\end{aligned}$$

Renaming the variables $x_1, x_2, x_3, \ldots$ as $y_1, y_2, y_3, \ldots$ in this equality, we obtain

$$\begin{aligned}
&(G(k, m-i))(\mathbf{y}) \\
&= (\text{the sum of all } k\text{-bounded } (m-i)\text{-monomials in the variables } y_1, y_2, y_3, \ldots) .
\end{aligned} \tag{93}$$

Forget that we fixed i. We thus have proved (92) and (93) for each $i \in \{0, 1, \ldots, m\}$.

Now, (90) becomes

$$\begin{aligned}
&(G(k,m))(\mathbf{x},\mathbf{y}) \\
&= (\text{the sum of all } k\text{-bounded } m\text{-monomials in the variables } x_1,x_2,x_3,\ldots,y_1,y_2,y_3,\ldots) \\
&= \sum_{i\in\{0,1,\ldots,m\}} \underbrace{(\text{the sum of all } k\text{-bounded } i\text{-monomials in the variables } x_1,x_2,x_3,\ldots)}_{\substack{=G(k,i) \\ \text{(by (92))}}} \\
&\qquad \cdot \underbrace{(\text{the sum of all } k\text{-bounded } (m-i)\text{-monomials in the variables } y_1,y_2,y_3,\ldots)}_{\substack{=(G(k,m-i))(\mathbf{y}) \\ \text{(by (93))}}} \\
&= \sum_{i\in\{0,1,\ldots,m\}} \underbrace{G(k,i)}_{\substack{=(G(k,i))(\mathbf{x})}}^{\text{(by (91))}} \cdot (G(k,m-i))(\mathbf{y}) = \sum_{i\in\{0,1,\ldots,m\}} (G(k,i))(\mathbf{x})\cdot(G(k,m-i))(\mathbf{y}).
\end{aligned}$$

Hence, (10) holds for $f = G(k,m)$, $I = \{0,1,\ldots,m\}$, $(f_{1,i})_{i\in I} = (G(k,i))_{i\in\{0,1,\ldots,m\}}$ and $(f_{2,i})_{i\in I} = (G(k,m-i))_{i\in\{0,1,\ldots,m\}}$. Therefore, (9) (applied to these f, I, $(f_{1,i})_{i\in I}$ and $(f_{2,i})_{i\in I}$) yields

$$\Delta(G(k,m)) = \sum_{i\in\{0,1,\ldots,m\}} G(k,i)\otimes G(k,m-i) = \sum_{i=0}^{m} G(k,i)\otimes G(k,m-i).$$

This proves Theorem 2.19. □

3.15. Proof of Theorem 2.21

Proof of Theorem 2.21. Consider the ring $(\mathbf{k}[[x_1,x_2,x_3,\ldots]])[[t]]$ of formal power series in one indeterminate t over $\mathbf{k}[[x_1,x_2,x_3,\ldots]]$. We equip this ring with the topology that is obtained by identifying it with $\mathbf{k}[[x_1,x_2,x_3,\ldots,t]]$ (or, equivalently, which is obtained by considering $\mathbf{k}[[x_1,x_2,x_3,\ldots]]$ itself as equipped with the standard topology on a ring of formal power series, and then adjoining the extra indeterminate t).

Now, consider the map

$$\begin{aligned}
\mathbf{F}_k : \mathbf{k}[[x_1,x_2,x_3,\ldots]] &\to \mathbf{k}[[x_1,x_2,x_3,\ldots]], \\
a &\mapsto a\left(x_1^k,x_2^k,x_3^k,\ldots\right).
\end{aligned}$$

This map $\mathbf{F}_k$ is a continuous $\mathbf{k}$-algebra homomorphism (since it is an evaluation homomorphism)[60]. Hence, it induces a continuous[61] $\mathbf{k}[[t]]$-algebra homomorphism

$$\mathbf{F}_k[[t]] : (\mathbf{k}[[x_1,x_2,x_3,\ldots]])[[t]] \to (\mathbf{k}[[x_1,x_2,x_3,\ldots]])[[t]]$$

[60]It is well-defined, since k is positive.

[61]Continuity is defined with respect to the topology that we defined on $(\mathbf{k}[[x_1,x_2,x_3,\ldots]])[[t]]$.

that sends each formal power series $\sum_{n\geq 0} a_n t^n \in (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]]$ (with $a_n \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$) to $\sum_{n\geq 0} \mathbf{F}_k(a_n) t^n$. Consider this $\mathbf{k}[[t]]$-algebra homomorphism $\mathbf{F}_k[[t]]$. In particular, it satisfies

$$(\mathbf{F}_k[[t]])\left(t^i\right) = t^i \qquad \text{for each } i \in \mathbb{N}.$$

The definition of $\mathbf{F}_k$ yields

$$\mathbf{F}_k(x_i) = x_i^k \qquad \text{for each } i \in \{1, 2, 3, \ldots\}. \tag{94}$$

Also, for each $a \in \Lambda$, we have

$$\begin{aligned}\mathbf{F}_k(a) &= a\left(x_1^k, x_2^k, x_3^k, \ldots\right) \qquad \text{(by the definition of } \mathbf{F}_k\text{)} \\ &= \mathbf{f}_k(a) \end{aligned} \tag{95}$$

(since the definition of $\mathbf{f}_k$ yields $\mathbf{f}_k(a) = a\left(x_1^k, x_2^k, x_3^k, \ldots\right)$). Thus, in particular, each $n \in \mathbb{N}$ satisfies

$$\mathbf{F}_k(e_n) = \mathbf{f}_k(e_n) \tag{96}$$

(by (95), applied to $a = e_n$).

Applying the map $\mathbf{F}_k[[t]]$ to both sides of the equality (79), we obtain

$$(\mathbf{F}_k[[t]])\left(\prod_{i=1}^{\infty}(1 + x_i t)\right) = (\mathbf{F}_k[[t]])\left(\sum_{n\geq 0} e_n t^n\right) = \sum_{n\geq 0} \mathbf{F}_k(e_n) t^n$$

(by the definition of $\mathbf{F}_k[[t]]$). Hence,

$$\begin{aligned}\sum_{n\geq 0} \mathbf{F}_k(e_n) t^n &= (\mathbf{F}_k[[t]])\left(\prod_{i=1}^{\infty}(1 + x_i t)\right) = \prod_{i=1}^{\infty} \underbrace{(\mathbf{F}_k[[t]])(1 + x_i t)}_{\substack{=1+\mathbf{F}_k(x_i)t \\ \text{(by the definition of } \mathbf{F}_k[[t]])}} \\ &\qquad \left(\begin{array}{c}\text{since } \mathbf{F}_k[[t]] \text{ is a continuous } \mathbf{k}[[t]]\text{-algebra homomorphism,} \\ \text{and thus respects infinite products}\end{array}\right) \\ &= \prod_{i=1}^{\infty}\left(1 + \underbrace{\mathbf{F}_k(x_i)}_{\substack{=x_i^k \\ \text{(by (94))}}} t\right) = \prod_{i=1}^{\infty}\left(1 + x_i^k t\right).\end{aligned}$$

Substituting $-t^k$ for t in this equality, we find

$$\sum_{n\geq 0} \mathbf{F}_k(e_n)\left(-t^k\right)^n = \prod_{i=1}^{\infty} \underbrace{\left(1 + x_i^k\left(-t^k\right)\right)}_{\substack{=1-(x_it)^k \\ =(1-x_it)\left((x_it)^0+(x_it)^1+\cdots+(x_it)^{k-1}\right) \\ \text{(since } 1-u^k=(1-u)\left(u^0+u^1+\cdots+u^{k-1}\right) \\ \text{for any element } u \text{ of any ring)}}}$$

$$= \prod_{i=1}^{\infty}\left((1-x_it)\left((x_it)^0+(x_it)^1+\cdots+(x_it)^{k-1}\right)\right)$$

$$= \left(\prod_{i=1}^{\infty}(1-x_it)\right)\left(\prod_{i=1}^{\infty}\left((x_it)^0+(x_it)^1+\cdots+(x_it)^{k-1}\right)\right).$$

We can divide both sides of this equality by $\prod\limits_{i=1}^{\infty}(1-x_it)$ (since the formal power series $\prod\limits_{i=1}^{\infty}(1-x_it)$ has constant term 1 and thus is invertible), and thus obtain

$$\frac{\sum\limits_{n\geq 0} \mathbf{F}_k(e_n)\left(-t^k\right)^n}{\prod\limits_{i=1}^{\infty}(1-x_it)} = \prod_{i=1}^{\infty} \underbrace{\left((x_it)^0+(x_it)^1+\cdots+(x_it)^{k-1}\right)}_{=\sum\limits_{u=0}^{k-1}(x_it)^u} = \prod_{i=1}^{\infty}\sum_{u=0}^{k-1}(x_it)^u$$

$$= \underbrace{\sum_{\substack{\alpha=(\alpha_1,\alpha_2,\alpha_3,\ldots)\in\{0,1,\ldots,k-1\}^\infty; \\ \alpha_i=0 \text{ for all but finitely many } i}}}_{\substack{= \sum\limits_{\substack{\alpha\in\{0,1,\ldots,k-1\}^\infty \\ \text{is a weak composition}}} \\ = \sum\limits_{\substack{\alpha\in\mathrm{WC}; \\ \alpha_i<k \text{ for all } i}}}} \underbrace{(x_1t)^{\alpha_1}(x_2t)^{\alpha_2}(x_3t)^{\alpha_3}\cdots}_{=\left(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\cdots\right)\left(t^{\alpha_1}t^{\alpha_2}t^{\alpha_3}\cdots\right)}$$

$$\text{(here, we have expanded the product)}$$

$$= \sum_{\substack{\alpha\in\mathrm{WC}; \\ \alpha_i<k \text{ for all } i}} \underbrace{\left(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\cdots\right)}_{\substack{=\mathbf{x}^\alpha \\ \text{(by the definition of } \mathbf{x}^\alpha\text{)}}} \underbrace{\left(t^{\alpha_1}t^{\alpha_2}t^{\alpha_3}\cdots\right)}_{\substack{=t^{\alpha_1+\alpha_2+\alpha_3+\cdots}=t^{|\alpha|} \\ \text{(since } \alpha_1+\alpha_2+\alpha_3+\cdots=|\alpha|\text{)}}}$$

$$= \sum_{\substack{\alpha\in\mathrm{WC}; \\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha t^{|\alpha|}.$$

Hence,

$$\begin{aligned}
&\sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha t^{|\alpha|} \\
&= \frac{\sum_{n\geq 0} \mathbf{F}_k\left(e_n\right)\left(-t^k\right)^n}{\prod_{i=1}^{\infty}\left(1-x_i t\right)} = \left(\sum_{n\geq 0} \underbrace{\mathbf{F}_k\left(e_n\right)}_{\substack{=\mathbf{f}_k(e_n)\\ \text{(by (96))}}} \underbrace{\left(-t^k\right)^n}_{=(-1)^n t^{kn}}\right) \cdot \underbrace{\prod_{i=1}^{\infty}\left(1-x_i t\right)^{-1}}_{\substack{=\sum_{n\geq 0} h_n t^n\\ \text{(by (78))}}} \\
&= \left(\sum_{n\geq 0} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n t^{kn}\right) \cdot \underbrace{\left(\sum_{n\geq 0} h_n t^n\right)}_{=\sum_{j\geq 0} h_j t^j} = \left(\sum_{n\geq 0} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n t^{kn}\right) \cdot \left(\sum_{j\geq 0} h_j t^j\right) \\
&= \underbrace{\sum_{n\geq 0}\sum_{j\geq 0}}_{=\sum_{(n,j)\in\mathbb{N}^2}} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n \underbrace{t^{kn} h_j t^j}_{=h_j t^{kn+j}} = \sum_{(n,j)\in\mathbb{N}^2} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n h_j t^{kn+j}.
\end{aligned}$$

This is an equality between two power series in $\left(\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]\right)\left[\left[t\right]\right]$. If we compare the coefficients of t^m on both sides of it (where $x_1, x_2, x_3, \ldots$ are considered scalars, not monomials), we obtain

$$\begin{aligned}
\sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i;\\ |\alpha|=m}} \mathbf{x}^\alpha &= \underbrace{\sum_{\substack{(n,j)\in\mathbb{N}^2;\\ kn+j=m}}}_{=\sum_{n\in\mathbb{N}}\sum_{\substack{j\in\mathbb{N};\\ kn+j=m}}} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n h_j \\
&= \sum_{n\in\mathbb{N}} \sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n h_j = \sum_{n\in\mathbb{N}} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n \cdot \sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j.
\end{aligned}$$

Now, the definition of $G\left(k,m\right)$ yields

$$\begin{aligned}
G\left(k,m\right) &= \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=m;\\ \alpha_i<k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\alpha\in\mathrm{WC};\\ \alpha_i<k \text{ for all } i;\\ |\alpha|=m}} \mathbf{x}^\alpha \\
&= \sum_{n\in\mathbb{N}} \mathbf{f}_k\left(e_n\right)\left(-1\right)^n \cdot \sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j. \qquad (97)
\end{aligned}$$

But the right hand side of this equality can be simplified. Namely, for each $n\in\mathbb{N}$, we have

$$\sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j = h_{m-kn}. \qquad (98)$$

[*Proof of (98):* Let $n \in \mathbb{N}$. We must prove the equality (98). If $m - kn < 0$, then

$$\begin{aligned}\sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j &= (\text{empty sum}) && \left(\begin{array}{c}\text{since there exists no } j \in \mathbb{N} \text{ such that } kn+j=m\\ \text{(because } m < kn \text{ (since } m-kn<0))\end{array}\right)\\ &= 0 = h_{m-kn} && (\text{since } h_{m-kn} = 0 \text{ (because } m-kn<0))\end{aligned}$$

Hence, if $m - kn < 0$, then (98) is proven. Therefore, for the rest of the proof of (98), we WLOG assume that $m - kn \geq 0$. Thus, $m - kn \in \mathbb{N}$. Hence, there exists exactly one $j \in \mathbb{N}$ satisfying $kn + j = m$, namely $j = m - kn$. Thus, the sum $\sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j$ has exactly one addend, namely the addend for $j = m - kn$. Therefore, $\sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j = h_{m-kn}$. This proves (98).]

Now, (97) becomes

$$\begin{aligned}G(k,m) &= \sum_{n\in\mathbb{N}} \mathbf{f}_k(e_n)(-1)^n \cdot \underbrace{\sum_{\substack{j\in\mathbb{N};\\ kn+j=m}} h_j}_{\substack{=h_{m-kn}\\ \text{(by (98))}}} = \sum_{n\in\mathbb{N}} \mathbf{f}_k(e_n)(-1)^n \cdot h_{m-kn}\\ &= \sum_{n\in\mathbb{N}} (-1)^n h_{m-kn} \cdot \mathbf{f}_k(e_n) = \sum_{i\in\mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k(e_i)\end{aligned}$$

(here, we have renamed the summation index n as i). This proves Theorem 2.21. □

Another proof of Theorem 2.21 is sketched in a footnote in Section 4 below.

3.16. Proofs of the results from Section 2.8

We shall now prove the results from Section 2.8. We begin with Lemma 2.23. This will rely on the Verschiebung endomorphisms $\mathbf{v}_n$ introduced in Definition 2.25, and on Proposition 2.26 and the equality (11).

Proof of Lemma 2.23. Applying (11) to $n = k$, we obtain

$$\mathbf{v}_k(p_m) = \begin{cases} kp_{m/k}, & \text{if } k \mid m;\\ 0, & \text{if } k \nmid m\end{cases}. \tag{99}$$

Applying Proposition 2.26 to $n = k$, $a = p_m$ and $b = e_j$, we obtain

$$\left\langle p_m, \mathbf{f}_k(e_j)\right\rangle = \left\langle \mathbf{v}_k(p_m), e_j\right\rangle. \tag{100}$$

Now, we are in one of the following three cases:

Case 1: We have $m = kj$.

Case 2: We have $k \nmid m$.

Case 3: We have neither $m = kj$ nor $k \nmid m$.

Let us first consider Case 1. In this case, we have $m = kj$. Thus, $k \mid m$ (since $j \in \mathbb{N} \subseteq \mathbb{Z}$) and $m/k = j$. Hence, $j = m/k$, so that the integer j is positive (since m and k are positive). But (99) becomes

$$\begin{aligned} \mathbf{v}_k(p_m) &= \begin{cases} kp_{m/k}, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases} = kp_{m/k} \qquad (\text{since } k \mid m) \\ &= kp_j \qquad (\text{since } m/k = j). \end{aligned}$$

Thus, (100) becomes

$$\begin{aligned} \langle p_m, \mathbf{f}_k(e_j) \rangle &= \left\langle \underbrace{\mathbf{v}_k(p_m)}_{=kp_j}, e_j \right\rangle = \langle kp_j, e_j \rangle \\ &= \langle e_j, kp_j \rangle \qquad (\text{since the Hall inner product is symmetric}) \\ &= k \underbrace{\langle e_j, p_j \rangle}_{\substack{=(-1)^{j-1} \\ (\text{by Proposition 1.3, applied to } n=j)}} = k(-1)^{j-1} = (-1)^{j-1} k. \end{aligned}$$

Comparing this with

$$(-1)^{j-1} \underbrace{[m = kj]}_{\substack{=1 \\ (\text{since } m=kj)}} k = (-1)^{j-1} k,$$

we obtain $\langle p_m, \mathbf{f}_k(e_j) \rangle = (-1)^{j-1} [m = kj] k$. Thus, Lemma 2.23 is proven in Case 1.

Let us next consider Case 2. In this case, we have $k \nmid m$. Hence, $m \neq kj$ (since otherwise, we would have $m = kj$, thus $k \mid m$ (since $j \in \mathbb{N}$), contradicting $k \nmid m$). Now, (99) becomes

$$\mathbf{v}_k(p_m) = \begin{cases} kp_{m/k}, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases} = 0 \qquad (\text{since } k \nmid m).$$

Thus, (100) becomes

$$\langle p_m, \mathbf{f}_k(e_j) \rangle = \left\langle \underbrace{\mathbf{v}_k(p_m)}_{=0}, e_j \right\rangle = \langle 0, e_j \rangle = 0.$$

Comparing this with

$$(-1)^{j-1} \underbrace{[m = kj]}_{\substack{=0 \\ (\text{since } m \neq kj)}} k = 0,$$

we obtain $\left\langle p_m, \mathbf{f}_k\left(e_j\right)\right\rangle = (-1)^{j-1}[m = kj]\,k$. Thus, Lemma 2.23 is proven in Case 2.

Let us finally consider Case 3. In this case, we have neither $m = kj$ nor $k \nmid m$. In other words, we have $m \neq kj$ and $k \mid m$. From $k \mid m$, we conclude that m/k is a positive integer[62]. From $m \neq kj$, we obtain $m/k \neq j$. Thus, the symmetric functions $p_{m/k}$ and e_j are homogeneous of different degrees[63], and therefore satisfy $\left\langle p_{m/k}, e_j\right\rangle = 0$ (by (2), applied to $f = p_{m/k}$ and $g = e_j$).

Now, (99) becomes

$$\mathbf{v}_k\left(p_m\right) = \begin{cases} kp_{m/k}, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases} = kp_{m/k} \qquad (\text{since } k \mid m)\,.$$

Thus, (100) becomes

$$\left\langle p_m, \mathbf{f}_k\left(e_j\right)\right\rangle = \left\langle \underbrace{\mathbf{v}_k\left(p_m\right)}_{=kp_{m/k}}, e_j \right\rangle = \left\langle kp_{m/k}, e_j\right\rangle = k\underbrace{\left\langle p_{m/k}, e_j\right\rangle}_{=0} = 0.$$

Comparing this with

$$(-1)^{j-1}\underbrace{[m = kj]}_{\substack{=0 \\ (\text{since } m \neq kj)}}k = 0,$$

we obtain $\left\langle p_m, \mathbf{f}_k\left(e_j\right)\right\rangle = (-1)^{j-1}[m = kj]\,k$. Thus, Lemma 2.23 is proven in Case 3.

We have thus proven Lemma 2.23 in all three Cases 1, 2 and 3. Thus, Lemma 2.23 always holds. ☐

Next, let us prove a simple property of Hall inner products:

Lemma 3.19. Let m, α and β be positive integers. Let a be a homogeneous symmetric function of degree α. Let b be a homogeneous symmetric function of degree β. Then, $\left\langle p_m, ab\right\rangle = 0$.

We shall give two proofs of this lemma: one using (20), and one using Hopf-algebraic machinery.

First proof of Lemma 3.19 (sketched). For each $n \in \mathbb{N}$, let Λ_n denote the n-th homogeneous component of the graded $\mathbf{k}$-algebra Λ. Thus, $a \in \Lambda_\alpha$ and $b \in \Lambda_\beta$ (since a and b are homogeneous symmetric functions of degrees α and β).

But it is known that the family $\left(h_\lambda\right)_{\lambda \in \operatorname{Par}}$ is a graded basis of the graded $\mathbf{k}$-module Λ; this means that for each $n \in \mathbb{N}$, its subfamily $\left(h_\lambda\right)_{\lambda \in \operatorname{Par}_n}$ is a basis of the $\mathbf{k}$-module Λ_n. [64] Applying this to $n = \alpha$, we conclude that the subfamily $\left(h_\lambda\right)_{\lambda \in \operatorname{Par}_\alpha}$

[62] Indeed, it is positive since m and k are positive.

[63] since $p_{m/k}$ is homogeneous of degree m/k, whereas e_j is homogeneous of degree j

[64] This fact appears, e.g., in [GriRei20, Proposition 2.4.3(j)].

is a basis of Λ_α. Hence, a is a $\mathbf{k}$-linear combination of this family $(h_\lambda)_{\lambda\in\operatorname{Par}_\alpha}$ (since $a \in \Lambda_\alpha$).

We must prove the equality $\langle p_m, ab\rangle = 0$. Both sides of this equality depend $\mathbf{k}$-linearly on a. Thus, in proving it, we can WLOG assume that a belongs to the family $(h_\lambda)_{\lambda\in\operatorname{Par}_\alpha}$ (because we know that a is a $\mathbf{k}$-linear combination of this family). In other words, we can WLOG assume that $a = h_\lambda$ for some $\lambda \in \operatorname{Par}_\alpha$. Assume this. For similar reasons, we can WLOG assume that $b = h_\mu$ for some $\mu \in \operatorname{Par}_\beta$. Assume this, too. Consider these two partitions λ and μ.

We have $\lambda \in \operatorname{Par}_\alpha$ and thus $|\lambda| = \alpha > 0$, so that $\lambda \neq \varnothing$. Hence, the partition λ has at least one part. Likewise, the partition μ has at least one part.

Now, let $\lambda \sqcup \mu$ be the partition obtained by listing all parts of λ and of μ and sorting the resulting list in weakly decreasing order.[65] Using Definition 3.4, we can easily see that $h_{\lambda\sqcup\mu} = h_\lambda h_\mu$. Comparing this with $\underbrace{a}_{=h_\lambda}\underbrace{b}_{=h_\mu} = h_\lambda h_\mu$, we obtain $ab = h_{\lambda\sqcup\mu}$.

But the partition $\lambda \sqcup \mu$ has as many parts as λ and μ have combined. Thus, the partition $\lambda \sqcup \mu$ has at least 2 parts (since λ has at least one part, and μ has at least one part). Therefore, $\lambda \sqcup \mu \neq (m)$ (since the partition $\lambda \sqcup \mu$ has at least 2 parts, while the partition (m) has only 1 part). Now, recall that $p_m = m_{(m)}$ (where, of course, the two "m"s in "$m_{(m)}$" mean completely unrelated things). Thus,

$$\begin{aligned}
\left\langle \underbrace{p_m}_{=m_{(m)}}, \underbrace{ab}_{=h_{\lambda\sqcup\mu}} \right\rangle &= \left\langle m_{(m)}, h_{\lambda\sqcup\mu}\right\rangle = \left\langle h_{\lambda\sqcup\mu}, m_{(m)}\right\rangle \\
&\qquad \left(\text{since the Hall inner product is symmetric}\right) \\
&= \delta_{\lambda\sqcup\mu,(m)} \qquad \left(\begin{array}{c}\text{by (20), applied to } \lambda\sqcup\mu \text{ and } (m) \\ \text{instead of } \lambda \text{ and } \mu\end{array}\right) \\
&= 0 \qquad \left(\text{since } \lambda\sqcup\mu \neq (m)\right).
\end{aligned}$$

This proves Lemma 3.19. □

Second proof of Lemma 3.19. Let $\widetilde{p_m}$ be the map $\Lambda \to \mathbf{k}$, $g \mapsto \langle p_m, g\rangle$. This is a $\mathbf{k}$-linear map.

The power-sum symmetric function p_m is primitive as an element of the Hopf algebra Λ (see [GriRei20, Proposition 2.3.6(i)]). Now, consider the graded dual Λ° of the Hopf algebra Λ (as defined in [GriRei20, §1.6]). The map $\Phi : \Lambda \to \Lambda^\circ$ that sends each $f \in \Lambda$ to the $\mathbf{k}$-linear map $\Lambda \to \mathbf{k}$, $g \mapsto \langle f, g\rangle$ is a Hopf algebra isomorphism (by [GriRei20, Corollary 2.5.14]). Thus, this map Φ sends primitive elements of Λ to primitive elements of Λ°. Hence, in particular, $\Phi(p_m) \in \Lambda^\circ$ is primitive (since $p_m \in \Lambda$ is primitive). In other words, $\widetilde{p_m} \in \Lambda^\circ$ is primitive (since the definitions of Φ and of $\widetilde{p_m}$ quickly reveal that $\Phi(p_m) = \widetilde{p_m}$). In other words, $\Delta_{\Lambda^\circ}(\widetilde{p_m}) = 1_{\Lambda^\circ} \otimes \widetilde{p_m} + \widetilde{p_m} \otimes 1_{\Lambda^\circ}$.

[65]For example: If $\lambda = (5,3,2)$ and $\mu = (6,4,3,1,1)$, then $\lambda \sqcup \mu = (6,5,4,3,3,2,1,1)$.

Consider the $\mathbf{k}$-bilinear pairing $\langle \cdot, \cdot \rangle : \Lambda^\circ \times \Lambda \to \mathbf{k}$ that sends each pair $(f, a) \in \Lambda^\circ \times \Lambda$ to $f(a) \in \mathbf{k}$. It induces a pairing $\langle \cdot, \cdot \rangle : (\Lambda^\circ \otimes \Lambda^\circ) \times (\Lambda \times \Lambda) \to \mathbf{k}$ that sends each pair $(f \otimes g, a \otimes b)$ to $\langle f, a \rangle \cdot \langle g, b \rangle = f(a) \cdot g(b) \in \mathbf{k}$. We now have defined three $\mathbf{k}$-bilinear forms, all of which we denote by $\langle \cdot, \cdot \rangle$; they will be distinguished by what is inside the parentheses.

The definition of the graded dual Λ° yields that 1_{Λ° is a homogeneous element of Λ° of degree 0. Thus, 1_{Λ° annihilates all homogeneous components of Λ except for the 0-th component. In other words, if $f \in \Lambda$ is homogeneous of degree γ, where $\gamma \in \mathbb{N}$ is distinct from 0, then $\langle 1_{\Lambda^\circ}, f \rangle = 0$. Applying this to $f = a$ and $\gamma = \alpha$, we obtain $\langle 1_{\Lambda^\circ}, a \rangle = 0$ (since α is distinct from 0 (because α is positive)). Similarly, $\langle 1_{\Lambda^\circ}, b \rangle = 0$.

Now, the definition of $\widetilde{p_m}$ yields $\widetilde{p_m}(ab) = \langle p_m, ab \rangle$, so that

$$\begin{aligned}
\langle p_m, ab \rangle &= \widetilde{p_m}(ab) = \langle \widetilde{p_m}, ab \rangle \\
&= \left\langle \underbrace{\Delta_{\Lambda^\circ}(\widetilde{p_m})}_{=1_{\Lambda^\circ} \otimes \widetilde{p_m} + \widetilde{p_m} \otimes 1_{\Lambda^\circ}}, a \otimes b \right\rangle \qquad \text{(by the definition of } \Delta_{\Lambda^\circ}) \\
&= \langle 1_{\Lambda^\circ} \otimes \widetilde{p_m} + \widetilde{p_m} \otimes 1_{\Lambda^\circ}, a \otimes b \rangle = \underbrace{\langle 1_{\Lambda^\circ} \otimes \widetilde{p_m}, a \otimes b \rangle}_{=\langle 1_{\Lambda^\circ}, a \rangle \cdot \langle \widetilde{p_m}, b \rangle} + \underbrace{\langle \widetilde{p_m} \otimes 1_{\Lambda^\circ}, a \otimes b \rangle}_{=\langle \widetilde{p_m}, a \rangle \cdot \langle 1_{\Lambda^\circ}, b \rangle} \\
&= \underbrace{\langle 1_{\Lambda^\circ}, a \rangle}_{=0} \cdot \langle \widetilde{p_m}, b \rangle + \langle \widetilde{p_m}, a \rangle \cdot \underbrace{\langle 1_{\Lambda^\circ}, b \rangle}_{=0} = 0.
\end{aligned}$$

This proves Lemma 3.19. □

We can now prove Proposition 2.24:

Proof of Proposition 2.24. Theorem 2.21 yields $G(k, m) = \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k(e_i)$. Hence,

$$\begin{aligned}
\langle p_m, G(k, m) \rangle &= \left\langle p_m, \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k(e_i) \right\rangle \\
&= \sum_{i \in \mathbb{N}} (-1)^i \langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i) \rangle \qquad (101)
\end{aligned}$$

(since the Hall inner product is $\mathbf{k}$-bilinear).

Now, we claim that every $i \in \mathbb{N} \setminus \{0, m/k\}$ satisfies

$$\langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i) \rangle = 0. \qquad (102)$$

[*Proof of (102):* Let $i \in \mathbb{N} \setminus \{0, m/k\}$. Thus, $i \in \mathbb{N}$ and $i \notin \{0, m/k\}$. From $i \notin \{0, m/k\}$, we obtain $i \neq 0$ and $i \neq m/k$. From $i \neq m/k$, we obtain $ki \neq m$, so that $m - ki \neq 0$.

We must prove the equality (102). If $m - ki < 0$, then $h_{m-ki} = 0$, and therefore

$$\left\langle p_m, \underbrace{h_{m-ki}}_{=0} \cdot \mathbf{f}_k(e_i) \right\rangle = \langle p_m, 0 \rangle = 0.$$

Hence, the equality (102) is proven if $m - ki < 0$. Thus, for the rest of this proof, we WLOG assume that $m - ki \geq 0$. Combining this with $m - ki \neq 0$, we obtain $m - ki > 0$. Thus, $m - ki$ is a positive integer. Also, i is a positive integer (since $i \in \mathbb{N}$ and $i \neq 0$), and thus ki is a positive integer (since k is a positive integer).

The map $\mathbf{f}_k : \Lambda \to \Lambda$ operates by replacing each x_i by x_i^k in a symmetric function (by the definition of $\mathbf{f}_k$). Thus, if $g \in \Lambda$ is any homogeneous symmetric function of some degree γ, then $\mathbf{f}_k(g)$ is a homogeneous symmetric function of degree $k\gamma$. Applying this to $g = e_i$ and $\gamma = i$, we conclude that $\mathbf{f}_k(e_i)$ is a homogeneous symmetric function of degree ki (since e_i is a homogeneous symmetric function of degree i). Also, h_{m-ki} is a homogeneous symmetric function of degree $m - ki$.

Hence, Lemma 3.19 (applied to $\alpha = m - ki$, $a = h_{m-ki}$, $\beta = ki$ and $b = \mathbf{f}_k(e_i)$) yields $\langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i) \rangle = 0$. This proves (102).]

Note that $e_0 = 1$ and thus $\mathbf{f}_k(e_0) = \mathbf{f}_k(1) = 1$ (by the definition of $\mathbf{f}_k$).

Note that $m/k > 0$ (since m and k are positive). Hence, $m/k \neq 0$. Now, we are in one of the following two cases:

Case 1: We have $k \mid m$.

Case 2: We have $k \nmid m$.

Let us consider Case 1 first. In this case, we have $k \mid m$. Hence, m/k is a positive integer (since m and k are positive integers). Thus, 0 and m/k are two distinct elements of $\mathbb{N}$ (indeed, they are distinct because $m/k \neq 0$). Lemma 2.23 (applied to $j = m/k$) yields

$$\langle p_m, \mathbf{f}_k(e_{m/k}) \rangle = (-1)^{m/k-1} \underbrace{[m = k(m/k)]}_{\substack{=1 \\ (\text{since } m = k(m/k))}} k = (-1)^{m/k-1} k.$$

Now, (101) becomes

$$\begin{aligned}
&\left\langle p_m, G\left(k,m\right)\right\rangle \\
&= \sum_{i\in\mathbb{N}} \left(-1\right)^i \left\langle p_m, h_{m-ki}\cdot \mathbf{f}_k\left(e_i\right)\right\rangle \\
&= \underbrace{\left(-1\right)^0}_{=1} \left\langle p_m, \underbrace{h_{m-k\cdot 0}}_{=h_m}\cdot \underbrace{\mathbf{f}_k\left(e_0\right)}_{=1}\right\rangle + \left(-1\right)^{m/k} \left\langle p_m, \underbrace{h_{m-k\cdot m/k}}_{\substack{=h_0 \\ \text{(since } m-k\cdot m/k=0\text{)}}} \cdot \mathbf{f}_k\left(e_{m/k}\right)\right\rangle \\
&\qquad + \sum_{i\in\mathbb{N}\setminus\{0,m/k\}} \left(-1\right)^i \underbrace{\left\langle p_m, h_{m-ki}\cdot \mathbf{f}_k\left(e_i\right)\right\rangle}_{\substack{=0 \\ \text{(by (102))}}} \\
&\qquad\qquad \left(\begin{array}{c}\text{here, we have split off the addends for } i=0 \text{ and for } i=m/k \\ \text{from the sum (since 0 and } m/k \text{ are two distinct elements of } \mathbb{N}\text{)}\end{array}\right) \\
&= \underbrace{\left\langle p_m, h_m\right\rangle}_{\substack{=\left\langle h_m, p_m\right\rangle \\ \text{(since the Hall inner} \\ \text{product is symmetric)}}} + \left(-1\right)^{m/k} \left\langle p_m, \underbrace{h_0}_{=1}\cdot \mathbf{f}_k\left(e_{m/k}\right)\right\rangle + \underbrace{\sum_{i\in\mathbb{N}\setminus\{0,m/k\}} \left(-1\right)^i 0}_{=0} \\
&= \underbrace{\left\langle h_m, p_m\right\rangle}_{\substack{=1 \\ \text{(by Proposition 1.2,} \\ \text{applied to } n=m\text{)}}} + \left(-1\right)^{m/k} \underbrace{\left\langle p_m, \mathbf{f}_k\left(e_{m/k}\right)\right\rangle}_{=\left(-1\right)^{m/k-1}k} = 1 + \underbrace{\left(-1\right)^{m/k}\left(-1\right)^{m/k-1}}_{=-1} k = 1-k.
\end{aligned}$$

Comparing this with

$$1 - \underbrace{\left[k \mid m\right]}_{\substack{=1 \\ \text{(since } k\mid m\text{)}}} k = 1-k,$$

we obtain $\left\langle p_m, G\left(k,m\right)\right\rangle = 1 - \left[k \mid m\right] k$. Hence, Proposition 2.24 is proven in Case 1.

Let us now consider Case 2. In this case, we have $k \nmid m$. Hence, $m/k \notin \mathbb{Z}$, so that

$m/k \notin \mathbb{N}$. Thus, $\mathbb{N} \setminus \{0\} = \mathbb{N} \setminus \{0, m/k\}$. Now, (101) becomes

$$\begin{aligned}
&\langle p_m, G(k,m)\rangle \\
&= \sum_{i\in\mathbb{N}} (-1)^i \langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i)\rangle \\
&= \underbrace{(-1)^0}_{=1} \left\langle p_m, \underbrace{h_{m-k\cdot 0}}_{=h_m} \cdot \underbrace{\mathbf{f}_k(e_0)}_{=1} \right\rangle + \underbrace{\sum_{i\in\mathbb{N}\setminus\{0\}}}_{\substack{=\sum_{i\in\mathbb{N}\setminus\{0,m/k\}} \\ (\text{since } \mathbb{N}\setminus\{0\}=\mathbb{N}\setminus\{0,m/k\})}} (-1)^i \langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i)\rangle \\
&\qquad (\text{here, we have split off the addend for } i=0 \text{ from the sum}) \\
&= \underbrace{\langle p_m, h_m\rangle}_{\substack{=\langle h_m, p_m\rangle \\ (\text{since the Hall inner} \\ \text{product is symmetric})}} + \sum_{i\in\mathbb{N}\setminus\{0,m/k\}} (-1)^i \underbrace{\langle p_m, h_{m-ki} \cdot \mathbf{f}_k(e_i)\rangle}_{\substack{=0 \\ (\text{by (102)})}} \\
&= \underbrace{\langle h_m, p_m\rangle}_{\substack{=1 \\ (\text{by Proposition 1.2,} \\ \text{applied to } n=m)}} + \underbrace{\sum_{i\in\mathbb{N}\setminus\{0,m/k\}} (-1)^i 0}_{=0} = 1.
\end{aligned}$$

Comparing this with

$$1 - \underbrace{[k \mid m]}_{\substack{=0 \\ (\text{since } k\nmid m)}} k = 1,$$

we obtain $\langle p_m, G(k,m)\rangle = 1 - [k \mid m]\, k$. Hence, Proposition 2.24 is proven in Case 2.

We have now proven Proposition 2.24 both in Case 1 and in Case 2. Hence, Proposition 2.24 always holds. □

Theorem 2.22 will follow from Proposition 2.24 using the following general criterion for generating sets of Λ:

Proposition 3.20. For each positive integer m, let $v_m \in \Lambda$ be a homogeneous symmetric function of degree m.

Assume that $\langle p_m, v_m\rangle$ is an invertible element of $\mathbf{k}$ for each positive integer m.

Then, the family $(v_m)_{m\geq 1} = (v_1, v_2, v_3, \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra Λ.

Proof of Proposition 3.20. Proposition 3.20 is [GriRei20, Exercise 2.5.24]. □

Proof of Theorem 2.22. Let m be a positive integer. Proposition 2.24 yields that

$$\begin{aligned} \langle p_m, G(k,m)\rangle &= 1 - \underbrace{[k \mid m]}_{=\begin{cases} 1, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases}} k = 1 - \begin{cases} 1, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases} \cdot k \\ &= \begin{cases} 1 - 1 \cdot k, & \text{if } k \mid m; \\ 1 - 0 \cdot k, & \text{if } k \nmid m \end{cases} = \begin{cases} 1 - k, & \text{if } k \mid m; \\ 1, & \text{if } k \nmid m \end{cases}. \end{aligned}$$

Hence, $\langle p_m, G(k,m)\rangle$ is an invertible element of $\mathbf{k}$ (because both $1-k$ and 1 are invertible elements of $\mathbf{k}$).

Forget that we fixed m. We thus have showed that $\langle p_m, G(k,m)\rangle$ is an invertible element of $\mathbf{k}$ for each positive integer m. Also, clearly, for each positive integer m, the element $G(k,m) \in \Lambda$ is a homogeneous symmetric function of degree m. Thus, Proposition 3.20 (applied to $v_m = G(k,m)$) shows that the family $(G(k,m))_{m \geq 1} = (G(k,1), G(k,2), G(k,3), \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra Λ. This proves Theorem 2.22. □

3.17. Proof of Theorem 2.29

Proof of Theorem 2.29. The $\mathbf{k}$-Hopf algebra Λ is both commutative and cocommutative (by [GriRei20, Exercise 2.3.7(a)]).

Thus, its antipode S is a $\mathbf{k}$-Hopf algebra homomorphism[66].

(a) The map $\mathbf{f}_k$ is a $\mathbf{k}$-Hopf algebra homomorphism (by [GriRei20, Exercise 2.9.9(d)], applied to $n = k$). The map $\mathbf{v}_k$ is a $\mathbf{k}$-Hopf algebra homomorphism (by [GriRei20, Exercise 2.9.10(e)], applied to $n = k$). Thus, we have shown that all three maps $\mathbf{f}_k$, S and $\mathbf{v}_k$ are $\mathbf{k}$-Hopf algebra homomorphisms. Hence, their composition $\mathbf{f}_k \circ S \circ \mathbf{v}_k$ is a $\mathbf{k}$-Hopf algebra homomorphism as well. In other words, U_k is a $\mathbf{k}$-Hopf algebra homomorphism (since $U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k$). This proves Theorem 2.29 **(a)**.

[66] *Proof.* This is actually the claim of [GriRei20, Proposition 2.4.3(g)], but let us also give a self-contained proof here:

The antipode of a Hopf algebra is an algebra anti-endomorphism (by [GriRei20, Proposition 1.4.10]). Thus, S is an algebra anti-endomorphism (since S is the antipode of the Hopf algebra Λ). But since Λ is commutative, an algebra anti-endomorphism of Λ is the same thing as an algebra endomorphism of Λ (by [GriRei20, Exercise 1.5.8(a)]). Hence, S is an algebra endomorphism of Λ (since S is an algebra anti-endomorphism of Λ).

The antipode of a Hopf algebra is a coalgebra anti-endomorphism (by [GriRei20, Exercise 1.4.28]). Thus, S is a coalgebra anti-endomorphism (since S is the antipode of the Hopf algebra Λ). But since Λ is cocommutative, a coalgebra anti-endomorphism of Λ is the same thing as a coalgebra endomorphism of Λ (by [GriRei20, Exercise 1.5.8(b)]). Hence, S is a coalgebra endomorphism of Λ (since S is a coalgebra anti-endomorphism of Λ).

We now know that S is an algebra endomorphism of Λ and a coalgebra endomorphism of Λ at the same time. In other words, S is a bialgebra endomorphism of Λ. Hence, S is a $\mathbf{k}$-Hopf algebra endomorphism of Λ. In other words, S is a $\mathbf{k}$-Hopf algebra homomorphism.

(b) Recall (from [GriRei20, Exercise 1.5.11(a)]) the following fact:

> *Claim 1:* If H is a $\mathbf{k}$-bialgebra and A is a commutative $\mathbf{k}$-algebra, then the convolution $f \star g$ of any two $\mathbf{k}$-algebra homomorphisms $f, g : H \to A$ is again a $\mathbf{k}$-algebra homomorphism.

The following fact is dual to Claim 1:

> *Claim 2:* If H is a $\mathbf{k}$-bialgebra and C is a cocommutative $\mathbf{k}$-coalgebra, then the convolution $f \star g$ of any two $\mathbf{k}$-coalgebra homomorphisms $f, g : C \to H$ is again a $\mathbf{k}$-coalgebra homomorphism.

(See [GriRei20, solution to Exercise 1.5.11(h)] for why exactly Claim 2 is dual to Claim 1, and how it can be proved.)

Theorem 2.29 **(a)** yields that the map U_k is a $\mathbf{k}$-Hopf algebra homomorphism. Hence, U_k is both a $\mathbf{k}$-algebra homomorphism and a $\mathbf{k}$-coalgebra homomorphism.

Now, recall that Λ is commutative, and that $\operatorname{id}_\Lambda$ and U_k are two $\mathbf{k}$-algebra homomorphisms from Λ to Λ. Hence, Claim 1 (applied to $H = \Lambda$, $A = \Lambda$, $f = \operatorname{id}_\Lambda$ and $g = U_k$) shows that the convolution $\operatorname{id}_\Lambda \star U_k$ is a $\mathbf{k}$-algebra homomorphism. In other words, V_k is a $\mathbf{k}$-algebra homomorphism (since $V_k = \operatorname{id}_\Lambda \star U_k$).

Next, recall that Λ is cocommutative, and that $\operatorname{id}_\Lambda$ and U_k are two $\mathbf{k}$-coalgebra homomorphisms from Λ to Λ. Hence, Claim 2 (applied to $H = \Lambda$, $C = \Lambda$, $f = \operatorname{id}_\Lambda$ and $g = U_k$) shows that the convolution $\operatorname{id}_\Lambda \star U_k$ is a $\mathbf{k}$-coalgebra homomorphism. In other words, V_k is a $\mathbf{k}$-coalgebra homomorphism (since $V_k = \operatorname{id}_\Lambda \star U_k$).

So we know that the map V_k is both a $\mathbf{k}$-algebra homomorphism and a $\mathbf{k}$-coalgebra homomorphism. Thus, V_k is a $\mathbf{k}$-bialgebra homomorphism, thus a $\mathbf{k}$-Hopf algebra homomorphism[67]. This proves Theorem 2.29 **(b)**.

(c) The map $\mathbf{v}_k$ is a $\mathbf{k}$-algebra homomorphism; thus, $\mathbf{v}_k(1) = 1$. Now, we have

$$\mathbf{v}_k(h_m) = \begin{cases} h_{m/k}, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases} \tag{103}$$

for each $m \in \mathbb{N}$. (Indeed, if $m > 0$, then this follows from the definition of $\mathbf{v}_k$. But if $m = 0$, then this follows from $\mathbf{v}_k(1) = 1$, since $h_0 = 1$.)

We have

$$S(h_n) = (-1)^n e_n \qquad \text{for each } n \in \mathbb{N}. \tag{104}$$

(This follows from [GriRei20, Proposition 2.4.1(iii)].)

Each $i \in \mathbb{N}$ satisfies

$$\begin{aligned} \mathbf{v}_k(h_{ki}) &= \begin{cases} h_{ki/k}, & \text{if } k \mid ki; \\ 0, & \text{if } k \nmid ki \end{cases} \qquad \text{(by (103), applied to } m = ki) \\ &= h_{ki/k} \qquad (\text{since } k \mid ki) \\ &= h_i \qquad (\text{since } ki/k = i) \end{aligned} \tag{105}$$

[67] since any $\mathbf{k}$-bialgebra homomorphism between two $\mathbf{k}$-Hopf algebras is automatically a $\mathbf{k}$-Hopf algebra homomorphism

and

$$\begin{aligned}
U_k(h_{ki}) &= (\mathbf{f}_k \circ S \circ \mathbf{v}_k)(h_{ki}) \qquad (\text{since } U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k) \\
&= \mathbf{f}_k \left(S \left(\underbrace{\mathbf{v}_k(h_{ki})}_{\substack{=h_i \\ (\text{by (105)})}} \right) \right) = \mathbf{f}_k \left(\underbrace{S(h_i)}_{\substack{=(-1)^i e_i \\ (\text{by (104)})}} \right) = \mathbf{f}_k \left((-1)^i e_i \right) \\
&= (-1)^i \mathbf{f}_k(e_i)
\end{aligned} \tag{106}$$

(since the map $\mathbf{f}_k$ is $\mathbf{k}$-linear).

On the other hand, if $j \in \mathbb{N}$ satisfies $k \nmid j$, then

$$\begin{aligned}
\mathbf{v}_k(h_j) &= \begin{cases} h_{j/k}, & \text{if } k \mid j; \\ 0, & \text{if } k \nmid j \end{cases} \qquad (\text{by (103), applied to } m = j) \\
&= 0 \qquad (\text{since } k \nmid j)
\end{aligned} \tag{107}$$

and

$$\begin{aligned}
U_k(h_j) &= (\mathbf{f}_k \circ S \circ \mathbf{v}_k)(h_j) \qquad (\text{since } U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k) \\
&= (\mathbf{f}_k \circ S) \left(\underbrace{\mathbf{v}_k(h_j)}_{\substack{=0 \\ (\text{by (107)})}} \right) = (\mathbf{f}_k \circ S)(0) \\
&= 0
\end{aligned} \tag{108}$$

(since the map $\mathbf{f}_k \circ S$ is $\mathbf{k}$-linear).

Let Δ_Λ be the comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$ of the $\mathbf{k}$-coalgebra Λ. Let $m_\Lambda : \Lambda \otimes \Lambda \to \Lambda$ be the $\mathbf{k}$-linear map sending each pure tensor $a \otimes b \in \Lambda \otimes \Lambda$ to $ab \in \Lambda$. Definition 2.28 then yields $\mathrm{id}_\Lambda \star U_k = m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \Delta_\Lambda$. Thus,

$$\begin{aligned}
V_k &= \mathrm{id}_\Lambda \star U_k = m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \underbrace{\Delta_\Lambda}_{=\Delta} \\
&= m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \Delta.
\end{aligned} \tag{109}$$

Let $m \in \mathbb{N}$ (not to be mistaken for the map m_Λ). Then, [GriRei20, Proposition 2.3.6(iii)] (applied to $n = m$) yields

$$\begin{aligned}
\Delta(h_m) &= \sum_{i+j=m} h_i \otimes h_j \\
&\qquad (\text{where the sum ranges over all pairs } (i,j) \in \mathbb{N} \times \mathbb{N} \text{ with } i + j = m) \\
&= \sum_{j=0}^{m} h_{m-j} \otimes h_j
\end{aligned}$$

(here, we have substituted $(m-j,j)$ for (i,j) in the sum, since the map $\{0,1,\ldots,m\} \to \{(i,j) \in \mathbb{N}\times\mathbb{N} \mid i+j=m\}$ that sends each j to $(m-j,j)$ is a bijection). Applying the map $\operatorname{id}_\Lambda \otimes U_k$ to both sides of this equality, we obtain

$$\begin{aligned}(\operatorname{id}_\Lambda \otimes U_k)(\Delta(h_m)) &= (\operatorname{id}_\Lambda \otimes U_k)\left(\sum_{j=0}^{m} h_{m-j}\otimes h_j\right) \\ &= \sum_{j=0}^{m} \underbrace{\operatorname{id}_\Lambda(h_{m-j})}_{=h_{m-j}} \otimes U_k(h_j) = \sum_{j=0}^{m} h_{m-j}\otimes U_k(h_j).\end{aligned}$$

Applying the map m_Λ to both sides of this equality, we find

$$\begin{aligned}&m_\Lambda((\operatorname{id}_\Lambda \otimes U_k)(\Delta(h_m))) \\ &= m_\Lambda\left(\sum_{j=0}^{m} h_{m-j}\otimes U_k(h_j)\right) = \sum_{j=0}^{m} \underbrace{m_\Lambda(h_{m-j}\otimes U_k(h_j))}_{\substack{=h_{m-j}U_k(h_j) \\ (\text{by the definition of } m_\Lambda)}} = \sum_{j=0}^{m} h_{m-j}U_k(h_j) \\ &= \sum_{j\in\mathbb{N}} h_{m-j}U_k(h_j)\end{aligned}$$

(since

$$\begin{aligned}\sum_{j\in\mathbb{N}} h_{m-j}U_k(h_j) &= \sum_{j=0}^{m} h_{m-j}U_k(h_j) + \sum_{j=m+1}^{\infty} \underbrace{h_{m-j}}_{\substack{=0 \\ (\text{since } m-j<0 \\ (\text{because } j\geq m+1>m))}} U_k(h_j) \\ &= \sum_{j=0}^{m} h_{m-j}U_k(h_j) + \underbrace{\sum_{j=m+1}^{\infty} 0U_k(h_j)}_{=0} = \sum_{j=0}^{m} h_{m-j}U_k(h_j)\end{aligned}$$

). Therefore,

$$\begin{aligned}&m_\Lambda((\operatorname{id}_\Lambda \otimes U_k)(\Delta(h_m))) \\ &= \sum_{j\in\mathbb{N}} h_{m-j}U_k(h_j) = \sum_{\substack{j\in\mathbb{N}; \\ k\mid j}} h_{m-j}U_k(h_j) + \sum_{\substack{j\in\mathbb{N}; \\ k\nmid j}} h_{m-j}\underbrace{U_k(h_j)}_{\substack{=0 \\ (\text{by (108)})}} \\ &\qquad (\text{since each } j\in\mathbb{N} \text{ satisfies either } k\mid j \text{ or } k\nmid j \text{ (but not both)}) \\ &= \sum_{\substack{j\in\mathbb{N}; \\ k\mid j}} h_{m-j}U_k(h_j) + \underbrace{\sum_{\substack{j\in\mathbb{N}; \\ k\nmid j}} h_{m-j}0}_{=0} = \sum_{\substack{j\in\mathbb{N}; \\ k\mid j}} h_{m-j}U_k(h_j) = \sum_{i\in\mathbb{N}} h_{m-ki}\underbrace{U_k(h_{ki})}_{\substack{=(-1)^i\mathbf{f}_k(e_i) \\ (\text{by (106)})}} \\ &\qquad (\text{here, we have substituted } ki \text{ for } j \text{ in the sum}) \\ &= \sum_{i\in\mathbb{N}} \underbrace{h_{m-ki}(-1)^i}_{=(-1)^i h_{m-ki}} \mathbf{f}_k(e_i) = \sum_{i\in\mathbb{N}} (-1)^i h_{m-ki}\cdot \mathbf{f}_k(e_i).\end{aligned}$$

Comparing this with

$$G(k,m)=\sum_{i\in\mathbb{N}}(-1)^i h_{m-ki}\cdot\mathbf{f}_k(e_i)\qquad\text{(by Theorem 2.21)},$$

we obtain

$$G(k,m)=m_\Lambda\left(\left(\mathrm{id}_\Lambda\otimes U_k\right)\left(\Delta\left(h_m\right)\right)\right)=\underbrace{\left(m_\Lambda\circ\left(\mathrm{id}_\Lambda\otimes U_k\right)\circ\Delta\right)}_{\substack{=V_k\\ \text{(by (109))}}}\left(h_m\right)=V_k\left(h_m\right).$$

This proves Theorem 2.29 **(c)**.

(d) Let us recall a few facts from [GriRei20].

From [GriRei20, Exercise 2.9.10(a)], we know that every positive integers n and m satisfy

$$\mathbf{v}_n\left(p_m\right)=\begin{cases}np_{m/n}, & \text{if } n\mid m;\\ 0, & \text{if } n\nmid m\end{cases}.\tag{110}$$

On the other hand, it is easy to see (directly using the definition of $\mathbf{f}_n$) that every positive integers n and m satisfy

$$\mathbf{f}_n\left(p_m\right)=p_{nm}.\tag{111}$$

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Finally, [GriRei20, Proposition 2.4.1(i)] yields that every positive integer n satisfies

$$S\left(p_n\right)=-p_n.\tag{112}$$

Now, let n be a positive integer. We first claim the following:

Claim 1: We have $U_k\left(p_n\right)=-\left[k\mid n\right]kp_n$.

[*Proof of Claim 1:* We are in one of the following two cases:

Case 1: We have $k\mid n$.

Case 2: We have $k\nmid n$.

[68] *Proof.* Let n and m be two positive integers. Then, the definition of p_{nm} yields $p_{nm}=x_1^{nm}+x_2^{nm}+x_3^{nm}+\cdots=\sum_{i\geq1}x_i^{nm}$. But the definition of p_m yields $p_m=x_1^m+x_2^m+x_3^m+\cdots=\sum_{i\geq1}x_i^m$. Now, the definition of $\mathbf{f}_n$ yields

$$\begin{aligned}\mathbf{f}_n\left(p_m\right)&=p_m\left(x_1^n,x_2^n,x_3^n,\ldots\right)=\sum_{i\geq1}\underbrace{\left(x_i^n\right)^m}_{=x_i^{nm}}\qquad\left(\text{since } p_m=\sum_{i\geq1}x_i^m\right)\\&=\sum_{i\geq1}x_i^{nm}=p_{nm}\qquad\left(\text{since } p_{nm}=\sum_{i\geq1}x_i^{nm}\right).\end{aligned}$$

This proves (111).

Let us first consider Case 1. In this case, we have $k \mid n$. Hence, n/k is a positive integer. Now, (110) (applied to k and n instead of n and m) yields

$$\mathbf{v}_k\left(p_n\right) = \begin{cases} kp_{n/k}, & \text{if } k \mid n; \\ 0, & \text{if } k \nmid n \end{cases} = kp_{n/k} \qquad \left(\text{since } k \mid n\right).$$

Applying the map S to both sides of this equality, we find

$$S\left(\mathbf{v}_k\left(p_n\right)\right) = S\left(kp_{n/k}\right) = k \underbrace{S\left(p_{n/k}\right)}_{\substack{=-p_{n/k} \\ \text{(by (112),} \\ \text{applied to } n/k \text{ instead of } n)}} \qquad \left(\text{since the map } S \text{ is } \mathbf{k}\text{-linear}\right)$$
$$= k\left(-p_{n/k}\right) = -kp_{n/k}.$$

Applying the map $\mathbf{f}_k$ to both sides of this equality, we find

$$\mathbf{f}_k\left(S\left(\mathbf{v}_k\left(p_n\right)\right)\right) = \mathbf{f}_k\left(-kp_{n/k}\right) = -k \underbrace{\mathbf{f}_k\left(p_{n/k}\right)}_{\substack{=p_{k(n/k)} \\ \text{(by (111),} \\ \text{applied to } k \text{ and } n/k \\ \text{instead of } n \text{ and } m)}} \qquad \left(\text{since the map } \mathbf{f}_k \text{ is } \mathbf{k}\text{-linear}\right)$$
$$= -kp_{k(n/k)} = -kp_n \qquad \left(\text{since } k\left(n/k\right) = n\right).$$

Now, the definition of U_k yields $U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k$. Hence,

$$U_k\left(p_n\right) = \left(\mathbf{f}_k \circ S \circ \mathbf{v}_k\right)\left(p_n\right) = \mathbf{f}_k\left(S\left(\mathbf{v}_k\left(p_n\right)\right)\right) = -kp_n.$$

Comparing this with

$$-\underbrace{\left[k \mid n\right]}_{\substack{=1 \\ (\text{since } k \mid n)}} kp_n = -kp_n,$$

we obtain $U_k\left(p_n\right) = -\left[k \mid n\right] kp_n$. Hence, Claim 1 is proved in Case 1.

Let us now consider Case 2. In this case, we have $k \nmid n$. But (110) (applied to k and n instead of n and m) yields

$$\mathbf{v}_k\left(p_n\right) = \begin{cases} kp_{n/k}, & \text{if } k \mid n; \\ 0, & \text{if } k \nmid n \end{cases} = 0 \qquad \left(\text{since } k \nmid n\right).$$

But the definition of U_k yields $U_k = \mathbf{f}_k \circ S \circ \mathbf{v}_k$. Hence,

$$U_k\left(p_n\right) = \left(\mathbf{f}_k \circ S \circ \mathbf{v}_k\right)\left(p_n\right) = \left(\mathbf{f}_k \circ S\right)\left(\underbrace{\mathbf{v}_k\left(p_n\right)}_{=0}\right) = \left(\mathbf{f}_k \circ S\right)\left(0\right) = 0$$

(since the map $\mathbf{f}_k \circ S$ is $\mathbf{k}$-linear). Comparing this with

$$-\underbrace{\left[k \mid n\right]}_{\substack{=0 \\ (\text{since } k \nmid n)}} kp_n = 0,$$

we obtain $U_k(p_n) = -[k \mid n]\, kp_n$. Hence, Claim 1 is proved in Case 2.

We have now proved Claim 1 in both Cases 1 and 2. Thus, Claim 1 always holds.]

Theorem 2.29 **(a)** shows that the map U_k is a $\mathbf{k}$-Hopf algebra homomorphism. Hence, U_k is a $\mathbf{k}$-algebra homomorphism. Thus, $U_k(1) = 1$.

Now, let Δ_Λ be the comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$ of the $\mathbf{k}$-coalgebra Λ. Let $m_\Lambda : \Lambda \otimes \Lambda \to \Lambda$ be the $\mathbf{k}$-linear map sending each pure tensor $a \otimes b \in \Lambda \otimes \Lambda$ to $ab \in \Lambda$. Then, the definition of V_k yields $V_k = \mathrm{id}_\Lambda \star U_k = m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \Delta_\Lambda$ (by the definition of convolution).

But [GriRei20, Proposition 2.3.6(i)] yields $\Delta_\Lambda(p_n) = 1 \otimes p_n + p_n \otimes 1$. Now,

$$\begin{aligned}
&\underbrace{V_k}_{= m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \Delta_\Lambda} (p_n) \\
&= (m_\Lambda \circ (\mathrm{id}_\Lambda \otimes U_k) \circ \Delta_\Lambda)(p_n) \\
&= m_\Lambda \left((\mathrm{id}_\Lambda \otimes U_k) \left(\underbrace{\Delta_\Lambda(p_n)}_{= 1 \otimes p_n + p_n \otimes 1} \right) \right) \\
&= m_\Lambda \left(\underbrace{(\mathrm{id}_\Lambda \otimes U_k)(1 \otimes p_n + p_n \otimes 1)}_{= \mathrm{id}_\Lambda(1) \otimes U_k(p_n) + \mathrm{id}_\Lambda(p_n) \otimes U_k(1)} \right) \\
&= m_\Lambda(\mathrm{id}_\Lambda(1) \otimes U_k(p_n) + \mathrm{id}_\Lambda(p_n) \otimes U_k(1)) \\
&= \underbrace{\mathrm{id}_\Lambda(1)}_{=1} \cdot \underbrace{U_k(p_n)}_{\substack{= -[k \mid n] kp_n \\ \text{(by Claim 1)}}} + \underbrace{\mathrm{id}_\Lambda(p_n)}_{= p_n} \cdot \underbrace{U_k(1)}_{=1} \qquad \text{(by the definition of } m_\Lambda\text{)} \\
&= -[k \mid n]\, kp_n + p_n = (1 - [k \mid n]\, k)\, p_n.
\end{aligned}$$

This proves Theorem 2.29 **(d)**. ☐

3.18. Second proof of Theorem 2.19

Let us reprove Theorem 2.19 using Theorem 2.29:

Second proof of Theorem 2.19. Theorem 2.29 **(b)** shows that the map V_k is a $\mathbf{k}$-Hopf algebra homomorphism. Thus, in particular, V_k is a $\mathbf{k}$-coalgebra homomorphism. In other words, we have

$$(V_k \otimes V_k) \circ \Delta = \Delta \circ V_k \qquad \text{and} \qquad \varepsilon = \varepsilon \circ V_k$$

(where ε denotes the counit of the $\mathbf{k}$-coalgebra Λ). But we have

$$V_k(h_n) = G(k, n) \qquad \text{for each } n \in \mathbb{N} \tag{113}$$

(by Theorem 2.29 **(c)**, applied to n instead of m). Applying this to $n = m$, we obtain $V_k(h_m) = G(k, m)$, so that $G(k, m) = V_k(h_m)$. Applying the map Δ to both sides of this equality, we find

$$\Delta\left(G\left(k,m\right)\right)=\Delta\left(V_k\left(h_m\right)\right)=\underbrace{\left(\Delta\circ V_k\right)}_{=\left(V_k\otimes V_k\right)\circ\Delta}\left(h_m\right)=\left(\left(V_k\otimes V_k\right)\circ\Delta\right)\left(h_m\right)$$
$$=\left(V_k\otimes V_k\right)\left(\Delta\left(h_m\right)\right). \tag{114}$$

But [GriRei20, Proposition 2.3.6(iii)] (applied to $n=m$) yields

$$\Delta\left(h_m\right)=\sum_{i+j=m}h_i\otimes h_j$$
(where the sum ranges over all pairs $(i,j)\in\mathbb{N}\times\mathbb{N}$ with $i+j=m$)
$$=\sum_{i=0}^{m}h_i\otimes h_{m-i}$$

(here, we have substituted $(i,m-i)$ for (i,j) in the sum, since the map $\{0,1,\ldots,m\}\to\{(i,j)\in\mathbb{N}\times\mathbb{N}\ \mid\ i+j=m\}$ that sends each i to $(i,m-i)$ is a bijection). Hence, (114) becomes

$$\Delta\left(G\left(k,m\right)\right)=\left(V_k\otimes V_k\right)\left(\underbrace{\Delta\left(h_m\right)}_{=\sum_{i=0}^{m}h_i\otimes h_{m-i}}\right)=\left(V_k\otimes V_k\right)\left(\sum_{i=0}^{m}h_i\otimes h_{m-i}\right)$$
$$=\sum_{i=0}^{m}\underbrace{V_k\left(h_i\right)}_{\substack{=G(k,i)\\ \text{(by (113))}}}\otimes\underbrace{V_k\left(h_{m-i}\right)}_{\substack{=G(k,m-i)\\ \text{(by (113))}}}=\sum_{i=0}^{m}G\left(k,i\right)\otimes G\left(k,m-i\right).$$

Thus, Theorem 2.19 is proved again. □

3.19. Proof of Corollary 2.30

Proof of Corollary 2.30. Recall that the family $(h_n)_{n\geq1}=(h_1,h_2,h_3,\ldots)$ generates Λ as a $\mathbf{k}$-algebra. Hence, each $g\in\Lambda$ can be written as a polynomial in $h_1,h_2,h_3,\ldots$. Applying this to $g=p_n$, we conclude that p_n can be written as a polynomial in $h_1,h_2,h_3,\ldots$. In other words, there exists a polynomial $f\in\mathbf{k}[x_1,x_2,x_3,\ldots]$ such that

$$p_n=f\left(h_1,h_2,h_3,\ldots\right). \tag{115}$$

Consider this f. We shall show that this f satisfies (12). This will clearly prove Corollary 2.30.

Consider the map V_k defined in Theorem 2.29. Theorem 2.29 **(c)** yields that $V_k\left(h_m\right)=G\left(k,m\right)$ for each positive integer m. In other words,

$$\left(V_k\left(h_1\right),V_k\left(h_2\right),V_k\left(h_3\right),\ldots\right)=\left(G\left(k,1\right),G\left(k,2\right),G\left(k,3\right),\ldots\right). \tag{116}$$

The map V_k is a $\mathbf{k}$-Hopf algebra homomorphism (by Theorem 2.29 **(b)**), and thus is a $\mathbf{k}$-algebra homomorphism. Hence, it commutes with polynomials over $\mathbf{k}$. Thus,

$$\begin{aligned} V_k\left(f\left(h_1, h_2, h_3, \ldots\right)\right) &= f\left(V_k\left(h_1\right), V_k\left(h_2\right), V_k\left(h_3\right), \ldots\right) \\ &= f\left(G\left(k,1\right), G\left(k,2\right), G\left(k,3\right), \ldots\right) \qquad \text{(by (116))}. \end{aligned}$$

Now, applying the map V_k to both sides of the equality (115), we obtain

$$V_k\left(p_n\right) = V_k\left(f\left(h_1, h_2, h_3, \ldots\right)\right) = f\left(G\left(k,1\right), G\left(k,2\right), G\left(k,3\right), \ldots\right).$$

Comparing this with

$$V_k\left(p_n\right) = \left(1 - \left[k \mid n\right] k\right) p_n \qquad \text{(by Theorem 2.29 } \textbf{(d)}\text{)},$$

we obtain

$$\left(1 - \left[k \mid n\right] k\right) p_n = f\left(G\left(k,1\right), G\left(k,2\right), G\left(k,3\right), \ldots\right).$$

Thus, we have shown that our f satisfies (12). As we said, this proves Corollary 2.30. $\square$

4. Proof of the Liu–Polo conjecture

Let us recall a well-known partial order on the set of partitions of a given $n \in \mathbb{N}$:

Definition 4.1. Let $n \in \mathbb{N}$. We define a binary relation $\triangleright$ on the set Par_n as follows: Two partitions $\lambda, \mu \in \operatorname{Par}_n$ shall satisfy $\lambda \triangleright \mu$ if and only if we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k \qquad \text{for each } k \in \{1, 2, \ldots, n\}.$$

This relation $\triangleright$ is the greater-or-equal relation of a partial order on Par_n, which is known as the *dominance order* (or the *majorization order*).

This definition is precisely [GriRei20, Definition 2.2.7]. Note that if we replace "for each $k \in \{1, 2, \ldots, n\}$" by "for each $k \in \{1, 2, 3, \ldots\}$" in this definition, then the relation $\triangleright$ does not change.

Our goal in this section is to prove the conjecture made in [LiuPol19, Remark 1.4.5]. We state this conjecture as follows:[69]

Theorem 4.2. Let n be an integer such that $n > 1$. Then:

(a) We have

$$\sum_{\substack{\lambda \in \operatorname{Par}_n; \\ (n-1,1) \triangleright \lambda}} m_\lambda = \sum_{i=0}^{n-2} (-1)^i s_{\left(n-1-i, 1^{i+1}\right)}.$$

[69]Note that $(n-1, n-1, 1)$ is a partition whenever $n > 1$ is an integer.

(b) We have

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\triangleright\lambda}} m_\lambda = \sum_{i=0}^{n-2} (-1)^i s_{\left(n-1,n-1-i,1^{i+1}\right)}.$$

Example 4.3. For this example, let $n = 3$. Then, $n-1 = 2$ and $2n-1 = 5$. Hence, the left hand side of the equality in Theorem 4.2 **(b)** is

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\triangleright\lambda}} m_\lambda = \sum_{\substack{\lambda\in\operatorname{Par}_5;\\ (2,2,1)\triangleright\lambda}} m_\lambda = m_{(2,2,1)} + m_{(2,1,1,1)} + m_{(1,1,1,1,1)}.$$

Meanwhile, the right hand side of the equality in Theorem 4.2 **(b)** is

$$\sum_{i=0}^{n-2} (-1)^i s_{\left(n-1,n-1-i,1^{i+1}\right)} = \sum_{i=0}^{1} (-1)^i s_{\left(2,2-i,1^{i+1}\right)} = s_{(2,2,1)} - s_{(2,1,1,1)}.$$

Thus, Theorem 4.2 **(b)** claims that $m_{(2,2,1)} + m_{(2,1,1,1)} + m_{(1,1,1,1,1)} = s_{(2,2,1)} - s_{(2,1,1,1)}$ in this case.

We will pave our way to the proof of Theorem 4.2 by several lemmas. We begin with a particularly simple one:

Lemma 4.4. Let n be an integer such that $n > 1$. Let $\lambda \in \operatorname{Par}_{2n-1}$. Then, $(n-1, n-1, 1) \triangleright \lambda$ if and only if all positive integers i satisfy $\lambda_i < n$.

Proof of Lemma 4.4. $\Longrightarrow$: Assume that $(n-1, n-1, 1) \triangleright \lambda$. Thus, $n-1 \geq \lambda_1$ (by Definition 4.1). Hence, $\lambda_1 \leq n-1 < n$. But λ is a partition; thus, $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$. Hence, all positive integers i satisfy $\lambda_i \leq \lambda_1 < n$. This proves the "$\Longrightarrow$" direction of Lemma 4.4.

$\Longleftarrow$: Assume that all positive integers i satisfy $\lambda_i < n$. Thus, all positive integers i satisfy $\lambda_i \leq n-1$ (since λ_i and n are integers). Hence, in particular, $\lambda_1 \leq n-1$ and $\lambda_2 \leq n-1$.

Define a partition μ by $\mu = (n-1, n-1, 1)$; thus, $|\mu| = (n-1) + (n-1) + 1 = 2n-1$, so that $\mu \in \operatorname{Par}_{2n-1}$. Also, $\lambda \in \operatorname{Par}_{2n-1}$ (as we know). Thus, $\mu \triangleright \lambda$ holds if and only if each $k \in \{1, 2, \ldots, 2n-1\}$ satisfies

$$\mu_1 + \mu_2 + \cdots + \mu_k \geq \lambda_1 + \lambda_2 + \cdots + \lambda_k \tag{117}$$

(by Definition 4.1).

But each $k \in \{1, 2, \ldots, 2n-1\}$ satisfies (117).

[*Proof of (117):* Let $k \in \{1, 2, \ldots, 2n-1\}$. We must prove (117).

If $k \geq 3$, then

$$\begin{aligned}\mu_1 + \mu_2 + \cdots + \mu_k &\geq \mu_1 + \mu_2 + \mu_3 \\ &= (n-1) + (n-1) + 1 \qquad (\text{since } \mu = (n-1, n-1, 1)) \\ &= 2n - 1 = |\lambda| \qquad (\text{since } \lambda \in \operatorname{Par}_{2n-1}) \\ &= \lambda_1 + \lambda_2 + \lambda_3 + \cdots \geq \lambda_1 + \lambda_2 + \cdots + \lambda_k,\end{aligned}$$

and thus (117) is proven in this case. Hence, it remains to prove (117) for $k \leq 2$. But $\mu = (n-1, n-1, 1)$, and thus $\mu_1 = n - 1 \geq \lambda_1$ and $\mu_2 = n - 1 \geq \lambda_2$. Hence, $\mu_1 \geq \lambda_1$ and $\underbrace{\mu_1}_{\geq \lambda_1} + \underbrace{\mu_2}_{\geq \lambda_2} \geq \lambda_1 + \lambda_2$. In other words, (117) is proven for $k \leq 2$. As we have said, this concludes the proof of (117).]

Thus, we have $\mu \trianglerighteq \lambda$ (since $\mu \trianglerighteq \lambda$ holds if and only if each $k \in \{1, 2, \ldots, 2n-1\}$ satisfies (117)). In other words, $(n-1, n-1, 1) \trianglerighteq \lambda$ holds (since $\mu = (n-1, n-1, 1)$). This proves the "$\Longleftarrow$" direction of Lemma 4.4. ☐

Lemma 4.5. Let n be an integer such that $n > 1$. Let $\lambda \in \operatorname{Par}_n$. Then, $(n-1, 1) \trianglerighteq \lambda$ if and only if all positive integers i satisfy $\lambda_i < n$.

Proof of Lemma 4.5. This is analogous to the proof of Lemma 4.4. ☐

The next lemma identifies the left hand side of Theorem 4.2 **(a)** as the Petrie symmetric function $G(n, n)$, and the left hand side of Theorem 4.2 **(b)** as the Petrie symmetric function $G(n, 2n-1)$:

Corollary 4.6. Let n be an integer such that $n > 1$. Then:

(a) We have

$$\sum_{\substack{\lambda \in \operatorname{Par}_n; \\ (n-1,1) \trianglerighteq \lambda}} m_\lambda = G(n, n).$$

(b) We have

$$\sum_{\substack{\lambda \in \operatorname{Par}_{2n-1}; \\ (n-1,n-1,1) \trianglerighteq \lambda}} m_\lambda = G(n, 2n-1).$$

Proof. **(b)** Proposition 2.3 **(c)** (applied to $k = n$ and $m = 2n - 1$) yields

$$G(n, 2n-1) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = 2n-1; \\ \alpha_i < n \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = 2n-1; \\ \lambda_i < n \text{ for all } i}} m_\lambda. \tag{118}$$

But Lemma 4.4 yields the following equality of summation signs:

$$\sum_{\substack{\lambda \in \operatorname{Par}_{2n-1}; \\ (n-1,n-1,1) \trianglerighteq \lambda}} = \sum_{\substack{\lambda \in \operatorname{Par}_{2n-1}; \\ \lambda_i < n \text{ for all } i}} = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = 2n-1; \\ \lambda_i < n \text{ for all } i}}.$$

Hence,

$$\sum_{\substack{\lambda \in \operatorname{Par}_{2n-1}; \\ (n-1,n-1,1) \rhd \lambda}} m_\lambda = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = 2n-1; \\ \lambda_i < n \text{ for all } i}} m_\lambda .$$

Comparing this with (118), we obtain

$$\sum_{\substack{\lambda \in \operatorname{Par}_{2n-1}; \\ (n-1,n-1,1) \rhd \lambda}} m_\lambda = G(n, 2n-1).$$

This proves Corollary 4.6 **(b)**.

(a) This is analogous to Corollary 4.6 **(b)**, but uses Lemma 4.5 instead of Lemma 4.4. □

It was Corollary 4.6 that led the author to introduce and study the Petrie symmetric functions $G(k, m)$ in general, even if little of their general properties has proven relevant to Theorem 4.2.

The next proposition gives a simple formula for certain kinds of Petrie symmetric functions:

Proposition 4.7. Let n be a positive integer. Let $k \in \{0, 1, \ldots, n-1\}$. Then,

$$G(n, n+k) = h_{n+k} - h_k p_n.$$

Proposition 4.7 can be viewed as a particular case of Theorem 2.21 (applied to n and $n+k$ instead of k and m), after realizing that in the sum on the right hand side of Theorem 2.21, only the first two addends will (potentially) be nonzero in this case. However, let us give an independent proof of the proposition.

Proof of Proposition 4.7. From $k \in \{0, 1, \ldots, n-1\}$, we obtain $k < n$ and thus $n + k < n + n$. Thus we conclude:

> *Observation 1:* A monomial of degree $n+k$ cannot have more than one variable appear in it with exponent $\geq n$ (since this would require it to have degree $\geq n + n > n + k$).

Let $\mathfrak{M}_k$ be the set of all monomials of degree k. The definition of h_k shows that h_k is the sum of all monomials of degree k. In other words,

$$h_k = \sum_{\mathfrak{m} \in \mathfrak{M}_k} \mathfrak{m}. \tag{119}$$

Let $\mathfrak{M}_{n+k}$ be the set of all monomials of degree $n+k$. The definition of h_{n+k} shows that h_{n+k} is the sum of all monomials of degree $n+k$. In other words,

$$h_{n+k} = \sum_{\mathfrak{n} \in \mathfrak{M}_{n+k}} \mathfrak{n}. \tag{120}$$

Let $\mathfrak{N}$ be the set of all monomials of degree $n+k$ in which all exponents are $< n$. These monomials are exactly the $\mathbf{x}^\alpha$ for $\alpha \in \mathrm{WC}$ satisfying $|\alpha| = n+k$ and ($\alpha_i < n$ for all i). Hence,

$$\sum_{\mathfrak{n} \in \mathfrak{N}} \mathfrak{n} = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = n+k;\\ \alpha_i < n \text{ for all } i}} \mathbf{x}^\alpha. \tag{121}$$

But Proposition 2.3 **(c)** (applied to n and $n+k$ instead of k and m) yields

$$G(n, n+k) = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = n+k;\\ \alpha_i < n \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \mathrm{Par};\\ |\lambda| = n+k;\\ \lambda_i < n \text{ for all } i}} m_\lambda.$$

Hence,

$$G(n, n+k) = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = n+k;\\ \alpha_i < n \text{ for all } i}} \mathbf{x}^\alpha = \sum_{\mathfrak{n} \in \mathfrak{N}} \mathfrak{n} \tag{122}$$

(by (121)).

Clearly, the set $\mathfrak{N}$ is a subset of $\mathfrak{M}_{n+k}$, and furthermore its complement $\mathfrak{M}_{n+k} \setminus \mathfrak{N}$ is the set of all monomials of degree $n+k$ in which at least one exponent is $\geq n$. Hence, the map

$$\begin{aligned} \mathfrak{M}_k \times \{1, 2, 3, \ldots\} &\to \mathfrak{M}_{n+k} \setminus \mathfrak{N}, \\ (\mathfrak{m}, i) &\mapsto \mathfrak{m} \cdot x_i^n \end{aligned}$$

is well-defined (because if $\mathfrak{m}$ is a monomial of degree k, and if $i \in \{1, 2, 3, \ldots\}$, then $\mathfrak{m} \cdot x_i^n$ is a monomial of degree $k + n = n + k$, and the variable x_i appears in it with exponent $\geq n$). This map is furthermore surjective (for simple reasons) and injective (in fact, if $\mathfrak{n} \in \mathfrak{M}_{n+k} \setminus \mathfrak{N}$, then $\mathfrak{n}$ is a monomial of degree $n+k$, and thus Observation 1 yields that there is **at most** one variable x_i that appears in $\mathfrak{n}$ with exponent $\geq n$; but this means that the only preimage of $\mathfrak{n}$ under our map is $\left(\dfrac{\mathfrak{n}}{x_i^n}, i \right)$). Hence, this map is a bijection. We can thus use it to substitute $\mathfrak{m} \cdot x_i^n$ for $\mathfrak{n}$ in the sum $\sum\limits_{\mathfrak{n} \in \mathfrak{M}_{n+k} \setminus \mathfrak{N}} \mathfrak{n}$. We thus obtain

$$\begin{aligned} \sum_{\mathfrak{n} \in \mathfrak{M}_{n+k} \setminus \mathfrak{N}} \mathfrak{n} &= \sum_{(\mathfrak{m}, i) \in \mathfrak{M}_k \times \{1,2,3,\ldots\}} \mathfrak{m} \cdot x_i^n = \underbrace{\left(\sum_{\mathfrak{m} \in \mathfrak{M}_k} \mathfrak{m} \right)}_{\substack{= h_k \\ \text{(by (119))}}} \cdot \underbrace{\sum_{i \in \{1,2,3,\ldots\}} x_i^n}_{= p_n} \\ &= h_k p_n. \end{aligned} \tag{123}$$

But (120) becomes

$$h_{n+k} = \sum_{\mathfrak{n} \in \mathfrak{M}_{n+k}} \mathfrak{n} = \underbrace{\sum_{\mathfrak{n} \in \mathfrak{N}} \mathfrak{n}}_{\substack{=G(n,n+k) \\ \text{(by (122))}}} + \underbrace{\sum_{\mathfrak{n} \in \mathfrak{M}_{n+k} \setminus \mathfrak{N}} \mathfrak{n}}_{\substack{=h_k p_n \\ \text{(by (123))}}} \qquad (\text{since } \mathfrak{N} \subseteq \mathfrak{M}_{n+k})$$
$$= G(n, n+k) + h_k p_n.$$

In other words,

$$G(n, n+k) = h_{n+k} - h_k p_n.$$

This proves Proposition 4.7. □

We note in passing that the idea used in the above proof of Proposition 4.7 can be generalized to yield a second proof of Theorem 2.21, using an inclusion/exclusion argument.[70]

Corollary 4.8. Let n be an integer such that $n > 1$. Then:
(a) We have

$$\sum_{\substack{\lambda \in \operatorname{Par}_n; \\ (n-1,1) \triangleright \lambda}} m_\lambda = h_n - p_n.$$

---

[70]Here is an outline of this second proof: For any positive integer k and any $m \in \mathbb{N}$, we have

$$G(k,m) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha = \sum_{I \subseteq \{1,2,3,\ldots\}} (-1)^{|I|} \underbrace{\sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m; \\ \alpha_i \geq k \text{ for all } i \in I}} \mathbf{x}^\alpha}_{= \left(\prod_{i \in I} x_i^k\right) \cdot \sum_{\substack{\beta \in \mathrm{WC}; \\ |\beta| = m - k|I|}} \mathbf{x}^\beta}$$

(by an infinite-set version of the inclusion-exclusion principle)

$$= \underbrace{\sum_{I \subseteq \{1,2,3,\ldots\}}}_{= \sum_{p \in \mathbb{N}} \sum_{\substack{I \subseteq \{1,2,3,\ldots\}; \\ |I| = p}}} (-1)^{|I|} \left(\prod_{i \in I} x_i^k\right) \cdot \underbrace{\sum_{\substack{\beta \in \mathrm{WC}; \\ |\beta| = m - k|I|}} \mathbf{x}^\beta}_{= h_{m-k|I|}}$$

$$= \sum_{p \in \mathbb{N}} \sum_{\substack{I \subseteq \{1,2,3,\ldots\}; \\ |I| = p}} \underbrace{(-1)^{|I|}}_{=(-1)^p} \left(\prod_{i \in I} x_i^k\right) \cdot \underbrace{h_{m-k|I|}}_{\substack{= h_{m-kp} \\ (\text{since } |I| = p)}}$$

$$= \sum_{p \in \mathbb{N}} (-1)^p h_{m-kp} \underbrace{\sum_{\substack{I \subseteq \{1,2,3,\ldots\}; \\ |I| = p}} \prod_{i \in I} x_i^k}_{\substack{= \mathbf{f}_k(e_p) \\ \text{(this is easy to check)}}} = \sum_{p \in \mathbb{N}} (-1)^p h_{m-kp} \cdot \mathbf{f}_k(e_p)$$

$$= \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot \mathbf{f}_k(e_i).$$

(b) We have

$$\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\rhd\lambda}} m_\lambda = h_{2n-1} - h_{n-1}p_n.$$

Proof. **(b)** Corollary 4.6 **(b)** yields

$$\begin{aligned}
\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1};\\ (n-1,n-1,1)\rhd\lambda}} m_\lambda &= G\left(n,2n-1\right) = G\left(n,n+\left(n-1\right)\right) && \left(\text{since } 2n-1 = n+\left(n-1\right)\right)\\
&= h_{n+(n-1)} - h_{n-1}p_n && \left(\text{by Proposition 4.7, applied to } k=n-1\right)\\
&= h_{2n-1} - h_{n-1}p_n.
\end{aligned}$$

This proves Corollary 4.8 **(b)**.

(a) Corollary 4.6 **(a)** yields

$$\begin{aligned}
\sum_{\substack{\lambda\in\operatorname{Par}_{n};\\ (n-1,1)\rhd\lambda}} m_\lambda &= G\left(n,n\right) = G\left(n,n+0\right)\\
&= \underbrace{h_{n+0}}_{=h_n} - \underbrace{h_0}_{=1} p_n && \left(\text{by Proposition 4.7, applied to } k=0\right)\\
&= h_n - p_n.
\end{aligned}$$

This proves Corollary 4.8 **(a)**. □

Our next claim is an easy consequence of Proposition 1.1:

Corollary 4.9. Let n be a positive integer. Then,

$$h_n - p_n = \sum_{i=0}^{n-2} \left(-1\right)^i s_{\left(n-1-i,1^{i+1}\right)}.$$

Proof. Proposition 1.1 yields

$$\begin{aligned}
p_n &= \sum_{i=0}^{n-1} \left(-1\right)^i s_{\left(n-i,1^i\right)} = \underbrace{\left(-1\right)^0}_{=1} \underbrace{s_{\left(n-0,1^0\right)}}_{=s_{(n-0)}=s_{(n)}=h_n} + \sum_{i=1}^{n-1} \left(-1\right)^i s_{\left(n-i,1^i\right)}\\
&= h_n + \sum_{i=1}^{n-1} \left(-1\right)^i s_{\left(n-i,1^i\right)},
\end{aligned}$$

so that

$$\begin{aligned}
h_n - p_n &= -\sum_{i=1}^{n-1} \left(-1\right)^i s_{\left(n-i,1^i\right)} = \sum_{i=1}^{n-1} \underbrace{\left(-\left(-1\right)^i\right)}_{=\left(-1\right)^{i-1}} s_{\left(n-i,1^i\right)} = \sum_{i=1}^{n-1} \left(-1\right)^{i-1} s_{\left(n-i,1^i\right)}\\
&= \sum_{i=0}^{n-2} \left(-1\right)^i s_{\left(n-1-i,1^{i+1}\right)}
\end{aligned}$$

(here, we have substituted $i+1$ for i in the sum). □

We can now immediately prove Theorem 4.2 **(a)**:

Proof of Theorem 4.2 ***(a)****.* Corollary 4.8 **(a)** yields

$$\sum_{\substack{\lambda\in\operatorname{Par}_n;\\ (n-1,1)\rhd\lambda}} m_\lambda = h_n - p_n = \sum_{i=0}^{n-2} (-1)^i s_{\left(n-1-i,1^{i+1}\right)} \qquad \text{(by Corollary 4.9)}.$$

This proves Theorem 4.2 **(a)**. □

We shall use the *skewing operators* $f^\perp : \Lambda \to \Lambda$ for all $f \in \Lambda$ as defined in [GriRei20, §2.8] or in [Macdon95, Chapter I, Section 5, Example 3]. The easiest way to define them (following [Macdon95, Chapter I, Section 5, Example 3]) is as follows: For each $f \in \Lambda$, we let $f^\perp : \Lambda \to \Lambda$ be the $\mathbf{k}$-linear map adjoint to the map $L_f : \Lambda \to \Lambda,\ g \mapsto fg$ (that is, to the map that multiplies every element of Λ by f) with respect to the Hall inner product. That is, $f^\perp$ is the $\mathbf{k}$-linear map from Λ to Λ that satisfies

$$\left\langle g, f^\perp(a)\right\rangle = \langle fg, a\rangle \qquad \text{for all } a \in \Lambda \text{ and } g \in \Lambda.$$

It is not hard to show that such an operator $f^\perp$ exists[71]. The definition of $f^\perp$ in [GriRei20, §2.8] is different but equivalent (because of [GriRei20, Proposition 2.8.2(i)]). One of the most important properties of skewing operators is the following fact ([GriRei20, (2.8.2)]):

Lemma 4.10. Let λ and μ be any two partitions. Then,

$$s_\mu^\perp(s_\lambda) = s_{\lambda/\mu}. \tag{124}$$

(Here, $s_{\lambda/\mu}$ is a skew Schur function, defined in Subsection 3.5.)

Using skewing operators, we can define another helpful family of operators on Λ:

Definition 4.11. For any $m \in \mathbb{Z}$, we define a map $\mathbf{B}_m : \Lambda \to \Lambda$ by setting

$$\mathbf{B}_m(f) = \sum_{i\in\mathbb{N}} (-1)^i h_{m+i} e_i^\perp f \qquad \text{for all } f \in \Lambda.$$

It is known ([GriRei20, Exercise 2.9.1(a)]) that this map $\mathbf{B}_m$ is well-defined and $\mathbf{k}$-linear.

[71] This is not completely automatic: Not every $\mathbf{k}$-linear map from Λ to Λ has an adjoint with respect to the Hall inner product! (For example, the $\mathbf{k}$-linear map $\Lambda \to \Lambda$ that sends each Schur function s_λ to 1 has none.) The reason why the map $L_f : \Lambda \to \Lambda,\ g \mapsto fg$ has an adjoint is that when f is homogeneous of degree k, this map L_f sends each graded component Λ_m of Λ to Λ_{m+k}, and both of these graded components Λ_m and Λ_{m+k} are $\mathbf{k}$-modules with **finite** bases. (The case when f is not homogeneous can be reduced to the case when f is homogeneous, since each $f \in \Lambda$ is a sum of finitely many homogeneous elements.)

(Actually, the well-definedness of $\mathbf{B}_m$ is easy to check: If $f \in \Lambda$ has degree d, then all integers $i > d$ satisfy $e_i^\perp f = 0$ for degree reasons, and thus the sum $\sum\limits_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^\perp f$ has only finitely many nonzero addends. The $\mathbf{k}$-linearity of $\mathbf{B}_m$ is even clearer.)

The operators $\mathbf{B}_m$ for $m \in \mathbb{Z}$ have first appeared in Zelevinsky's [Zelevi81, §4.20] (in the different-looking but secretly equivalent setting of a PSH-algebra), where they are credited to J. N. Bernstein. They have since been dubbed the *Bernstein creation operators* and proved useful in various contexts (e.g., the definition of the "dual immaculate functions" in [BBSSZ13] takes them for inspiration). One of their most fundamental properties is the following fact (which originates in [Zelevi81, 4.20, (∗∗)] and appears implicitly in [Macdon95, Chapter I, Section 5, Example 29]):

Proposition 4.12. Let λ be any partition. Let $m \in \mathbb{Z}$ satisfy $m \geq \lambda_1$. Then,

$$\sum_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^\perp s_\lambda = s_{(m, \lambda_1, \lambda_2, \lambda_3, \ldots)}. \tag{125}$$

See [GriRei20, Exercise 2.9.1(b)] for a proof of Proposition 4.12. Thus, if λ is any partition, and if $m \in \mathbb{Z}$ satisfies $m \geq \lambda_1$, then

$$\begin{aligned} \mathbf{B}_m (s_\lambda) &= \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^\perp s_\lambda && \text{(by the definition of } \mathbf{B}_m\text{)} \\ &= s_{(m, \lambda_1, \lambda_2, \lambda_3, \ldots)} && \text{(by (125))} . \end{aligned} \tag{126}$$

Lemma 4.13. Let n be a positive integer. Let $m \in \mathbb{N}$. Then, $\mathbf{B}_m (h_n) = h_m h_n - h_{m+1} h_{n-1}$.

Proof of Lemma 4.13. We have $e_0 = 1$ and thus $e_0^\perp = 1^\perp = \mathrm{id}$. Hence, $e_0^\perp (h_n) = h_n$.

We shall use the notion of skew Schur functions $s_{\lambda/\mu}$ (as in Subsection 3.5). Recall that $s_{\lambda/\mu} = 0$ when $\mu \not\subseteq \lambda$.

From $e_1 = s_{(1)}$ and $h_n = s_{(n)}$, we obtain

$$e_1^\perp (h_n) = s_{(1)}^\perp \left(s_{(n)} \right) = s_{(n)/(1)} \qquad \text{(by (124))} .$$

But it is easy to see that $s_{(n)/(1)} = s_{(n-1)}$. (Indeed, this follows from the combinatorial definition of skew Schur functions, since the skew Ferrers diagram of $(n) / (1)$ can be obtained from the Ferrers diagram of $(n-1)$ by parallel shift[72]. Alternatively, this follows easily from Theorem 3.9, because $s_{(n-1)} = h_{n-1}$.)

Thus, we obtain

$$e_1^\perp (h_n) = s_{(n)/(1)} = s_{(n-1)} = h_{n-1}.$$

[72] See [GriRei20, §2.3] for the notions we are using here.

For each integer $i > 1$, we have

$$\begin{aligned}
e_i^{\perp}(h_n) &= s_{(1^i)}^{\perp}\left(s_{(n)}\right) && \left(\text{since } e_i = s_{(1^i)} \text{ and } h_n = s_{(n)}\right) \\
&= s_{(n)/(1^i)} && (\text{by (124)}) \\
&= 0 && \left(\text{since } \left(1^i\right) \not\subseteq (n) \text{ (because } i > 1)\right).
\end{aligned} \tag{127}$$

Now, the definition of $\mathbf{B}_m$ yields

$$\begin{aligned}
\mathbf{B}_m(h_n) &= \sum_{i\in\mathbb{N}} (-1)^i h_{m+i} e_i^{\perp}(h_n) \\
&= \underbrace{(-1)^0}_{=1} \underbrace{h_{m+0}}_{=h_m} \underbrace{e_0^{\perp}(h_n)}_{=h_n} + \underbrace{(-1)^1}_{=-1} h_{m+1} \underbrace{e_1^{\perp}(h_n)}_{=h_{n-1}} + \sum_{i\geq 2} (-1)^i h_{m+i} \underbrace{e_i^{\perp}(h_n)}_{\substack{=0 \\ \text{(by (127))}}} \\
&= h_m h_n - h_{m+1} h_{n-1}.
\end{aligned}$$

□

Corollary 4.14. Let n be a positive integer. Then, $\mathbf{B}_{n-1}(h_n) = 0$.

Proof. Apply Lemma 4.13 to $m = n - 1$ and simplify. □

Lemma 4.15. Let $m \in \mathbb{N}$. Let n be a positive integer. Then, $\mathbf{B}_m(p_n) = h_m p_n - h_{m+n}$.

Proof. This is [GriRei20, Exercise 2.9.1(f)]. But here is a more direct proof: We will use the comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$ of the Hopf algebra Λ (see [GriRei20, §2.3]). Here and in the following, the "$\otimes$" sign denotes $\otimes_{\mathbf{k}}$. The power-sum symmetric function p_n is primitive[73] (see [GriRei20, Proposition 2.3.6(i)]); thus,

$$\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1.$$

Hence, for each $i \in \mathbb{N}$, the definition of $e_i^{\perp}$ given in [GriRei20, Definition 2.8.1] (not the equivalent definition we gave above) yields

$$e_i^{\perp}(p_n) = \langle e_i, 1\rangle p_n + \langle e_i, p_n\rangle 1. \tag{128}$$

[73] Recall that an element x of a Hopf algebra H is said to be *primitive* if the comultiplication Δ_H of H satisfies $\Delta_H(x) = 1 \otimes x + x \otimes 1$.

Now, the definition of $\mathbf{B}_m$ yields

$$\begin{aligned}
\mathbf{B}_m(p_n) &= \sum_{i\in\mathbb{N}} (-1)^i h_{m+i} \underbrace{e_i^{\perp}(p_n)}_{\substack{=\langle e_i,1\rangle p_n + \langle e_i, p_n\rangle 1 \\ \text{(by (128))}}} \\
&= \sum_{i\in\mathbb{N}} (-1)^i h_{m+i} \left(\langle e_i, 1\rangle p_n + \langle e_i, p_n\rangle 1\right) \\
&= \underbrace{\sum_{i\in\mathbb{N}} (-1)^i h_{m+i}\cdot \langle e_i,1\rangle p_n}_{\substack{=(-1)^0 h_{m+0}\cdot\langle e_0,1\rangle p_n \\ \text{(because the Hall inner product } \langle e_i,1\rangle \\ \text{equals 0 whenever } i\neq 0 \text{ (by (2)),} \\ \text{and thus the only nonzero addend of this} \\ \text{sum is the addend for } i=0)}} + \underbrace{\sum_{i\in\mathbb{N}} (-1)^i h_{m+i}\cdot\langle e_i,p_n\rangle 1}_{\substack{=(-1)^n h_{m+n}\cdot\langle e_n,p_n\rangle 1 \\ \text{(because the Hall inner product } \langle e_i,p_n\rangle \\ \text{equals 0 whenever } i\neq n \text{ (by (2)),} \\ \text{and thus the only nonzero addend of this} \\ \text{sum is the addend for } i=n)}} \\
&= \underbrace{(-1)^0}_{=1} \underbrace{h_{m+0}}_{=h_m} \cdot \underbrace{\langle e_0, 1\rangle}_{=\langle 1,1\rangle = 1} p_n + (-1)^n h_{m+n} \cdot \underbrace{\langle e_n, p_n\rangle}_{\substack{=(-1)^{n-1} \\ \text{(by Proposition 1.3)}}} 1 \\
&= h_m p_n + \underbrace{(-1)^n h_{m+n}\cdot(-1)^{n-1} 1}_{=-h_{m+n}} = h_m p_n - h_{m+n}.
\end{aligned}$$

□

Lemma 4.16. Let n be a positive integer. Then,

$$\mathbf{B}_{n-1}(h_n - p_n) = h_{2n-1} - h_{n-1}p_n.$$

Proof. The map $\mathbf{B}_{n-1}$ is $\mathbf{k}$-linear. Thus,

$$\begin{aligned}
\mathbf{B}_{n-1}(h_n - p_n) &= \underbrace{\mathbf{B}_{n-1}(h_n)}_{\substack{=0 \\ \text{(by Corollary 4.14)}}} - \underbrace{\mathbf{B}_{n-1}(p_n)}_{\substack{=h_{n-1}p_n - h_{(n-1)+n} \\ \text{(by Lemma 4.15,} \\ \text{applied to } m=n-1)}} \\
&= -\left(h_{n-1}p_n - h_{(n-1)+n}\right) = \underbrace{h_{(n-1)+n}}_{=h_{2n-1}} - h_{n-1}p_n = h_{2n-1} - h_{n-1}p_n.
\end{aligned}$$

□

Lemma 4.17. Let n be a positive integer. Then,

$$\mathbf{B}_{n-1}(h_n - p_n) = \sum_{i=0}^{n-2} (-1)^i s_{\left(n-1,n-1-i,1^{i+1}\right)}.$$

Proof of Lemma 4.17. We have

$$\begin{aligned}
\mathbf{B}_{n-1}\left(h_n - p_n\right) &= \mathbf{B}_{n-1}\left(\sum_{i=0}^{n-2}(-1)^i s_{\left(n-1-i,1^{i+1}\right)}\right) && \text{(by Corollary 4.9)} \\
&= \sum_{i=0}^{n-2}(-1)^i \underbrace{\mathbf{B}_{n-1}\left(s_{\left(n-1-i,1^{i+1}\right)}\right)}_{\substack{=s_{\left(n-1,n-1-i,1^{i+1}\right)} \\ \text{(by (126), applied to } m=n-1 \\ \text{and } \lambda=\left(n-1-i,1^{i+1}\right) \\ \text{(since } n-1\geq n-1-i\text{))}}} && \text{(since } \mathbf{B}_{n-1} \text{ is } \mathbf{k}\text{-linear)} \\
&= \sum_{i=0}^{n-2}(-1)^i s_{\left(n-1,n-1-i,1^{i+1}\right)}.
\end{aligned}$$

□

Now the proof of Theorem 4.2 **(b)** is a trivial concatenation of equalities:

*Proof of Theorem 4.2 **(b)**.* Corollary 4.8 **(b)** yields

$$\begin{aligned}
\sum_{\substack{\lambda\in\operatorname{Par}_{2n-1}; \\ (n-1,n-1,1)\rhd\lambda}} m_\lambda = h_{2n-1} - h_{n-1}p_n &= \mathbf{B}_{n-1}\left(h_n - p_n\right) && \text{(by Lemma 4.16)} \\
&= \sum_{i=0}^{n-2}(-1)^i s_{\left(n-1,n-1-i,1^{i+1}\right)} && \text{(by Lemma 4.17)}.
\end{aligned}$$

□

5. Final remarks

5.1. SageMath code

The SageMath computer algebra system [SageMath] does not (yet) natively know the Petrie symmetric functions $G(k,m)$; but they can be easily constructed in it. For example, the code that follows computes $G(k,n)$ expanded in the Schur basis:

```
Sym = SymmetricFunctions(QQ) # Replace QQ by your favorite base ring.
m = Sym.m() # monomial symmetric functions
s = Sym.s() # Schur functions

def G(k, n): # a Petrie function
  return s(m.sum(m[lam] for lam in Partitions(n, max_part=k-1)))
```

5.2. Understanding the Petrie numbers

Combining Corollary 2.10 with Theorem 2.15 yields an explicit expression of all coefficients in the expansion of a Petrie symmetric function $G(k,m)$ in the Schur basis. It would stand to reason if the identity in Theorem 4.2 **(b)** (whose left hand side is $G(n, 2n-1)$) could be obtained from this expression. Surprisingly, we have been unable to do so, which suggests that the description of $\operatorname{pet}_k(\lambda, \varnothing)$ in Theorem 2.15 might not be optimal.

As to $\operatorname{pet}_k(\lambda,\mu)$, we do not have an explicit description at all, unless we count the recursive one that can be extracted from the proof in [GorWil74].

5.3. MNable symmetric functions

Combining Theorem 2.17 with Proposition 2.8, we conclude that for any $k > 0$ and $m \in \mathbb{N}$, the symmetric function $G(k,m) \in \Lambda$ has the following property: For any $\mu \in \operatorname{Par}$, its product $G(k,m) \cdot s_\mu$ with s_μ can be written in the form $\sum\limits_{\lambda \in \operatorname{Par}} u_\lambda s_\lambda$ with $u_\lambda \in \{-1, 0, 1\}$ for all $\lambda \in \operatorname{Par}$. It has this property in common with the symmetric functions h_m and e_m (according to the Pieri rules) and p_m (according to the Murnaghan–Nakayama rule) as well as several others. The study of symmetric functions having this property – which we call *MNable symmetric functions* (in honor of Murnaghan and Nakayama) – has been initiated in [Grinbe20a, §8], but there is much to be done.

5.4. A conjecture of Per Alexandersson

In February 2020, Per Alexandersson suggested the following conjecture:

Conjecture 5.1. Let k be a positive integer, and $m \in \mathbb{N}$. Then, $G(k,m) \cdot p_2 \in \Lambda$ can be written in the form $\sum\limits_{\lambda \in \operatorname{Par}} u_\lambda s_\lambda$ with $u_\lambda \in \{-1,0,1\}$ for all $\lambda \in \operatorname{Par}$.

For example,

$$G(3,4) \cdot p_2 = s_{(1,1,1,1,1,1)} + s_{(2,2,2)} - s_{(3,1,1,1)} - s_{(3,3)} + s_{(4,2)}.$$

Conjecture 5.1 has been verified for all k and m satisfying $k + m \leq 30$.

Note that Conjecture 5.1 becomes false if p_2 is replaced by p_3. For example,

$$G(3,4) \cdot p_3 = -s_{(1,1,1,1,1,1,1)} + s_{(2,2,1,1,1)} - 2s_{(2,2,2,1)} + s_{(3,2,1,1)} - s_{(4,1,1,1)} - s_{(4,3)} + s_{(5,2)}.$$

5.5. A conjecture of François Bergeron

An even more mysterious conjecture was suggested by François Bergeron in April 2020:

Conjecture 5.2. Let k and n be positive integers, and $m \in \mathbb{N}$. Let ∇ be the *nabla operator* as defined (e.g.) in [Berger19, §3.2.1]. Then, there exists a sign $\sigma_{n,k,m} \in \{1, -1\}$ such that $\sigma_{n,k,m} \nabla^n \left(G\left(k, m\right)\right)$ is an $\mathbb{N}\left[q, t\right]$-linear combination of Schur functions.

Using SageMath, this conjecture has been checked for $n = 1$ and all $k, m \in \{0, 1, \ldots, 9\}$; the signs $\sigma_{1,k,m}$ are given by the following table:

	1	2	3	4	5	6	7	8	9
2	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	$+$
3	$+$	$-$	$-$	$-$	$+$	$+$	$+$	$-$	$-$
4	$+$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$+$
5	$+$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$+$
6	$+$	$-$	$+$	$-$	$+$	$+$	$+$	$-$	$+$
7	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$+$
8	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$+$	$+$
9	$+$	$-$	$+$	$-$	$+$	$-$	$+$	$-$	$-$

(where the entry in the row indexed k and the column indexed m is the sign $\sigma_{1,k,m}$, represented by a "$+$" sign if it is 1 and by a "$-$" sign if it is -1). I am not aware of a pattern in these signs, apart from the fact that $\sigma_{1,2,m} = 1$ for all $m \in \mathbb{N}$ (a consequence of Haiman's famous interpretation of $\nabla\left(e_m\right)$ as a character), and that $\sigma_{1,k,m}$ appears to be $\left(-1\right)^{m-1}$ whenever $1 \leq m < k$ (which would follow from the conjecture that $\left(-1\right)^{m-1} \nabla\left(h_m\right)$ is an $\mathbb{N}\left[q, t\right]$-linear combination of Schur functions for any $m \geq 1$).

5.6. "Petriefication" of Schur functions

Theorem 2.29 shows the existence of a Hopf algebra homomorphism $V_k : \Lambda \to \Lambda$ that sends the complete homogeneous symmetric functions $h_1, h_2, h_3, \ldots$ to the Petrie symmetric functions $G\left(k, 1\right), G\left(k, 2\right), G\left(k, 3\right), \ldots$. It thus is natural to consider the images of all Schur functions s_λ under this homomorphism V_k. Experiments with small λ's may suggest that these images $V_k\left(s_\lambda\right)$ all can be written in the form $\sum\limits_{\lambda \in \operatorname{Par}} u_\lambda s_\lambda$ with $u_\lambda \in \{-1, 0, 1\}$. But this is not generally the case; counterexamples include $V_3\left(s_{(4,4,4)}\right)$, $V_4\left(s_{(4,4)}\right)$ and $V_4\left(s_{(5,1,1,1,1)}\right)$. (Of course, it is true when λ is a single row, because of $V_k\left(s_{(m)}\right) = V_k\left(h_m\right) = G\left(k, m\right)$; and it is also true when λ is a single column, because the Hopf algebra homomorphism V_k commutes with the antipode S that sends $h_m \mapsto \left(-1\right)^m e_m$ and $s_\lambda \mapsto \left(-1\right)^{|\lambda|} s_{\lambda^t}$.)

Note that these images $V_k\left(s_\lambda\right)$ are precisely the *modular Schur functions* s'_λ studied in [Walker94].

5.7. Postnikov's generalization

At the MIT Algebraic Combinatorics preseminar roundtable (2020), Alexander Postnikov has suggested a generalization of the Petrie symmetric functions that preserves some of their more elementary properties. In this subsection, we shall survey this generalization.

Convention 5.3. We fix a formal power series $F \in \mathbf{k}[[t]]$ whose constant term is 1. (We will keep this F fixed throughout the present subsection.)

The notations in the following definition will also be used throughout this subsection:

Definition 5.4. **(a)** Let $f_0, f_1, f_2, \ldots$ be the coefficients of the formal power series F, so that $F = \sum_{n \in \mathbb{N}} f_n t^n$. Thus, f_0 is the constant term of F; hence, $f_0 = 1$ (since the constant term of F is 1).

(b) We set $f_i = 0$ for each negative integer i.

(c) For any weak composition α, we define an element $f_\alpha \in \mathbf{k}$ by

$$f_\alpha = f_{\alpha_1} f_{\alpha_2} f_{\alpha_3} \cdots .$$

(Here, the infinite product $f_{\alpha_1} f_{\alpha_2} f_{\alpha_3} \cdots$ is well-defined, since every sufficiently high positive integer i satisfies $\alpha_i = 0$ and thus $f_{\alpha_i} = f_0 = 1$.)

(d) We define the power series

$$G_F = \sum_{\alpha \in \mathrm{WC}} f_\alpha \mathbf{x}^\alpha. \tag{129}$$

This is a formal power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

(e) For any $m \in \mathbb{N}$, we define the power series

$$G_{F,m} = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m}} f_\alpha \mathbf{x}^\alpha. \tag{130}$$

This is a formal power series in $\mathbf{k}[[x_1, x_2, x_3, \ldots]]$.

Example 5.5. Let us see how these power series G_F and $G_{F,m}$ look for specific values of F.

(a) Let $F = \dfrac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$. Then, $f_i = 1$ for each $i \in \mathbb{N}$. Hence, $f_\alpha = \underbrace{1 \cdot 1 \cdot 1 \cdot \cdots}_{=1} = 1$ for any weak composition α. Thus,

$$G_F = \sum_{\alpha \in \mathrm{WC}} \underbrace{f_\alpha}_{=1} \mathbf{x}^\alpha = \sum_{\alpha \in \mathrm{WC}} \mathbf{x}^\alpha$$

and

$$G_{F,m} = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m}} \underbrace{f_\alpha}_{=1} \mathbf{x}^\alpha = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = m}} \mathbf{x}^\alpha = h_m \qquad \text{for each } m \in \mathbb{N}.$$

(b) Now, let $F = 1$. Then, $f_i = [i = 0]$ for each $i \in \mathbb{N}$ (where we are using the Iverson bracket notation from Convention 2.4). Hence, $f_\alpha = [\alpha = \varnothing]$ for any weak composition α. Thus, it is easy to see that $G_F = 1$ and $G_{F,m} = [m = 0]$ for each $m \in \mathbb{N}$.

(c) Now, fix a positive integer k, and set $F = 1 + t + t^2 + \cdots + t^{k-1}$. Then, $f_i = [i < k]$ for each $i \in \mathbb{N}$. Hence, $f_\alpha = \prod_{i \geq 1} [\alpha_i < k] = [\alpha_i < k \text{ for all } i]$ for any weak composition α. Thus,

$$\begin{aligned} G_F &= \sum_{\alpha \in \mathrm{WC}} \underbrace{f_\alpha}_{=[\alpha_i < k \text{ for all } i]} \mathbf{x}^\alpha = \sum_{\alpha \in \mathrm{WC}} [\alpha_i < k \text{ for all } i] \, \mathbf{x}^\alpha \\ &= \sum_{\substack{\alpha \in \mathrm{WC}; \\ \alpha_i < k \text{ for all } i}} \mathbf{x}^\alpha \qquad \left(\begin{array}{c} \text{since the } [\alpha_i < k \text{ for all } i] \text{ factor} \\ \text{makes all addends that don't} \\ \text{satisfy "} \alpha_i < k \text{ for all } i \text{" vanish} \end{array} \right) \\ &= G(k). \end{aligned}$$

Likewise, we can see that $G_{F,m} = G(k,m)$ for each $m \in \mathbb{N}$. This shows that the G_F and the $G_{F,m}$ are generalizations of the Petrie symmetric series $G(k)$ and the Petrie symmetric functions $G(k,m)$, respectively.

The next proposition generalizes parts **(a)**, **(b)** and **(c)** of Proposition 2.3:

Proposition 5.6. **(a)** The formal power series $G_{F,m}$ is the m-th degree homogeneous component of G_F for each $m \in \mathbb{N}$.

(b) We have

$$G_F = \sum_{\alpha \in \mathrm{WC}} f_\alpha \mathbf{x}^\alpha = \sum_{\lambda \in \mathrm{Par}} f_\lambda m_\lambda = \prod_{i=1}^{\infty} F(x_i).$$

(c) We have

$$G_{F,m} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m}} f_\alpha \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = m}} f_\lambda m_\lambda \in \Lambda$$

for each $m \in \mathbb{N}$.

(d) The formal power series G_F is symmetric.

(e) We have $G_{F,0} = 1$.

Our proof of Proposition 5.6 will use the group $\mathfrak{S}_{(\infty)}$ and its action on the set WC defined in Subsection 3.1. This action has the following property:

Lemma 5.7. Let $\lambda \in \mathrm{Par}$. Let $\alpha \in \mathfrak{S}_{(\infty)} \lambda$. Then, $f_\alpha = f_\lambda$.

Proof of Lemma 5.7. We have $\alpha \in \mathfrak{S}_{(\infty)} \lambda$. In other words, there exists some $\sigma \in \mathfrak{S}_{(\infty)}$ such that $\alpha = \sigma \cdot \lambda$. Consider this σ. In our proof of Lemma 3.7, we have seen that σ^{-1} is a bijection from $\{1, 2, 3, \ldots\}$ to $\{1, 2, 3, \ldots\}$. In that same proof, we have shown that $(\alpha_1, \alpha_2, \alpha_3, \ldots) = \left(\lambda_{\sigma^{-1}(1)}, \lambda_{\sigma^{-1}(2)}, \lambda_{\sigma^{-1}(3)}, \ldots\right)$. In other words,

$$\alpha_i = \lambda_{\sigma^{-1}(i)} \qquad \text{for each } i \in \{1, 2, 3, \ldots\}. \tag{131}$$

Now, the definition of f_α yields

$$f_\alpha = f_{\alpha_1} f_{\alpha_2} f_{\alpha_3} \cdots = \prod_{i \in \{1,2,3,\ldots\}} \underbrace{f_{\alpha_i}}_{\substack{= f_{\lambda_{\sigma^{-1}(i)}} \\ \text{(since } \alpha_i = \lambda_{\sigma^{-1}(i)} \\ \text{(by (131)))}}} = \prod_{i \in \{1,2,3,\ldots\}} f_{\lambda_{\sigma^{-1}(i)}}$$
$$= \prod_{i \in \{1,2,3,\ldots\}} f_{\lambda_i} \qquad \left(\begin{array}{c}\text{here, we have substituted } i \text{ for } \sigma^{-1}(i) \text{ in the product,} \\ \text{since } \sigma^{-1} \text{ is a bijection from } \{1,2,3,\ldots\} \text{ to } \{1,2,3,\ldots\}\end{array}\right)$$
$$= f_{\lambda_1} f_{\lambda_2} f_{\lambda_3} \cdots = f_\lambda$$

(since the definition of f_λ yields $f_\lambda = f_{\lambda_1} f_{\lambda_2} f_{\lambda_3} \cdots$). This proves Lemma 5.7. $\square$

Proof of Proposition 5.6. **(a)** It is easy to see that for any $m \in \mathbb{N}$, the formal power series $G_{F,m}$ is homogeneous of degree m [74]. Moreover, (129) yields

$$G_F = \underbrace{\sum_{\alpha \in \mathrm{WC}}}_{\substack{= \sum\limits_{m \in \mathbb{N}} \sum\limits_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m}} \\ \text{(since } |\alpha| \in \mathbb{N} \text{ for each } \alpha \in \mathrm{WC})}} f_\alpha \mathbf{x}^\alpha = \sum_{m \in \mathbb{N}} \underbrace{\sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m}} f_\alpha \mathbf{x}^\alpha}_{\substack{= G_{F,m} \\ \text{(by (130))}}}$$
$$= \sum_{m \in \mathbb{N}} G_{F,m}. \tag{132}$$

Thus, the family $(G_{F,m})_{m \in \mathbb{N}}$ is the homogeneous decomposition of G_F (since each $G_{F,m}$ is homogeneous of degree m). Hence, for each $m \in \mathbb{N}$, the power series $G_{F,m}$ is the m-th degree homogeneous component of G_F. This proves Proposition 5.6 **(a)**.

(b) Let us define the group $\mathfrak{S}_{(\infty)}$ and its action on the set WC as in Subsection

[74] *Proof.* Let $m \in \mathbb{N}$. For any $\alpha \in \mathrm{WC}$, the monomial $\mathbf{x}^\alpha$ is a monomial of degree $|\alpha|$. Thus, if $\alpha \in \mathrm{WC}$ satisfies $|\alpha| = m$, then $\mathbf{x}^\alpha$ is a monomial of degree m (since $|\alpha| = m$). Hence, $\sum\limits_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m}} f_\alpha \mathbf{x}^\alpha$ is a $\mathbf{k}$-linear combination of monomials of degree m (since $f_\alpha \in \mathbf{k}$ for each $\alpha \in \mathrm{WC}$). In view of

$$G_{F,m} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = m}} f_\alpha \mathbf{x}^\alpha \qquad \text{(by (130))},$$

we can restate this as follows: $G_{F,m}$ is a $\mathbf{k}$-linear combination of monomials of degree m. Thus, the formal power series $G_{F,m}$ is homogeneous of degree m. Qed.

3.1. Then,

$$\begin{aligned}
\sum_{\lambda\in\operatorname{Par}} f_\lambda \underbrace{m_\lambda}_{\substack{=\sum\limits_{\alpha\in\mathfrak{S}_{(\infty)}\lambda}\mathbf{x}^\alpha \\ \text{(by (19))}}} &= \sum_{\lambda\in\operatorname{Par}} f_\lambda \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda}\mathbf{x}^\alpha = \sum_{\lambda\in\operatorname{Par}} \underbrace{\sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda}}_{\substack{=\sum\limits_{\substack{\alpha\in\operatorname{WC};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} \\ \text{(since } \mathfrak{S}_{(\infty)}\lambda\subseteq\operatorname{WC})}} \underbrace{f_\lambda}_{\substack{=f_\alpha \\ \text{(since Lemma 5.7} \\ \text{yields } f_\alpha=f_\lambda)}} \mathbf{x}^\alpha \\
&= \underbrace{\sum_{\lambda\in\operatorname{Par}} \sum_{\substack{\alpha\in\operatorname{WC};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}}}_{=\sum\limits_{\alpha\in\operatorname{WC}}\sum\limits_{\substack{\lambda\in\operatorname{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}}} f_\alpha\mathbf{x}^\alpha \\
&= \sum_{\alpha\in\operatorname{WC}} \sum_{\substack{\lambda\in\operatorname{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} f_\alpha\mathbf{x}^\alpha. \qquad (133)
\end{aligned}$$

Now, fix some $\alpha \in \operatorname{WC}$. Then, Lemma 3.6 yields that there exists a unique partition $\lambda \in \operatorname{Par}$ such that $\alpha \in \mathfrak{S}_{(\infty)}\lambda$. Thus, the sum $\sum\limits_{\substack{\lambda\in\operatorname{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} f_\alpha\mathbf{x}^\alpha$ has exactly one addend. Hence, this sum simplifies as follows:

$$\sum_{\substack{\lambda\in\operatorname{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} f_\alpha\mathbf{x}^\alpha = f_\alpha\mathbf{x}^\alpha. \qquad (134)$$

Forget that we fixed α. We thus have proved (134) for each $\alpha \in \operatorname{WC}$. Thus, (133) becomes

$$\sum_{\lambda\in\operatorname{Par}} f_\lambda m_\lambda = \sum_{\alpha\in\operatorname{WC}} \underbrace{\sum_{\substack{\lambda\in\operatorname{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} f_\alpha\mathbf{x}^\alpha}_{\substack{=f_\alpha\mathbf{x}^\alpha \\ \text{(by (134))}}} = \sum_{\alpha\in\operatorname{WC}} f_\alpha\mathbf{x}^\alpha.$$

Comparing this with (129), we obtain

$$G_F = \sum_{\lambda\in\operatorname{Par}} f_\lambda m_\lambda. \qquad (135)$$

On the other hand, $F = \sum\limits_{n\in\mathbb{N}} f_n t^n$ (as we have noticed in Definition 5.4). Thus, for each $i \in \{1,2,3,\ldots\}$, we have

$$F(x_i) = \sum_{n\in\mathbb{N}} f_n x_i^n = \sum_{u\in\mathbb{N}} f_u x_i^u$$

(here, we have renamed the summation index n as u). Multiplying these equalities over all $i \in \{1, 2, 3, \ldots\}$, we obtain

$$\begin{aligned}
&\prod_{i=1}^{\infty} F(x_i) \\
&= \prod_{i=1}^{\infty} \sum_{u \in \mathbb{N}} f_u x_i^u \\
&= \underbrace{\sum_{\substack{(u_1,u_2,u_3,\ldots)\in\mathbb{N}^\infty;\\ \text{all but finitely many } i \text{ satisfy } u_i=0}}}_{\substack{= \sum_{(u_1,u_2,u_3,\ldots)\in\mathrm{WC}} \\ \text{(since a sequence } (u_1,u_2,u_3,\ldots) \text{ of nonnegative integers} \\ \text{satisfies the statement ``all but finitely many } i \text{ satisfy } u_i=0\text{''} \\ \text{if and only if it satisfies } (u_1,u_2,u_3,\ldots)\in\mathrm{WC})}} \underbrace{\left(f_{u_1}x_1^{u_1}\right)\left(f_{u_2}x_2^{u_2}\right)\left(f_{u_3}x_3^{u_3}\right)\cdots}_{=\left(f_{u_1}f_{u_2}f_{u_3}\cdots\right)\left(x_1^{u_1}x_2^{u_2}x_3^{u_3}\cdots\right)} \\
&\qquad \text{(by the product rule)} \\
&= \sum_{(u_1,u_2,u_3,\ldots)\in\mathrm{WC}} \left(f_{u_1}f_{u_2}f_{u_3}\cdots\right)\left(x_1^{u_1}x_2^{u_2}x_3^{u_3}\cdots\right) \\
&= \sum_{(\alpha_1,\alpha_2,\alpha_3,\ldots)\in\mathrm{WC}} \underbrace{\left(f_{\alpha_1}f_{\alpha_2}f_{\alpha_3}\cdots\right)}_{\substack{=f_{(\alpha_1,\alpha_2,\alpha_3,\ldots)} \\ \text{(by the definition of } f_{(\alpha_1,\alpha_2,\alpha_3,\ldots)})}} \underbrace{\left(x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3}\cdots\right)}_{\substack{=\mathbf{x}^{(\alpha_1,\alpha_2,\alpha_3,\ldots)} \\ \text{(by the definition of } \mathbf{x}^{(\alpha_1,\alpha_2,\alpha_3,\ldots)})}} \\
&\qquad \left(\begin{array}{c}\text{here, we have renamed the} \\ \text{summation index } (u_1,u_2,u_3,\ldots) \text{ as } (\alpha_1,\alpha_2,\alpha_3,\ldots)\end{array}\right) \\
&= \sum_{(\alpha_1,\alpha_2,\alpha_3,\ldots)\in\mathrm{WC}} f_{(\alpha_1,\alpha_2,\alpha_3,\ldots)}\mathbf{x}^{(\alpha_1,\alpha_2,\alpha_3,\ldots)} = \sum_{\alpha\in\mathrm{WC}} f_\alpha \mathbf{x}^\alpha \\
&\qquad \left(\begin{array}{c}\text{here, we have renamed the} \\ \text{summation index } (\alpha_1,\alpha_2,\alpha_3,\ldots) \text{ as } \alpha\end{array}\right).
\end{aligned}$$

Comparing this with (129), we obtain

$$G_F = \prod_{i=1}^{\infty} F(x_i).$$

Combining this equality with (135) and (129), we obtain

$$G_F = \sum_{\alpha\in\mathrm{WC}} f_\alpha \mathbf{x}^\alpha = \sum_{\lambda\in\mathrm{Par}} f_\lambda m_\lambda = \prod_{i=1}^{\infty} F(x_i).$$

This proves Proposition 5.6 **(b)**.

(c) Let $m \in \mathbb{N}$. Let us define the group $\mathfrak{S}_{(\infty)}$ and its action on the set WC as in

Subsection 3.1. Then,

$$\sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} f_\lambda \underbrace{m_\lambda}_{\substack{=\sum\limits_{\alpha\in\mathfrak{S}_{(\infty)}\lambda} \mathbf{x}^\alpha\\ \text{(by (19))}}} = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} f_\lambda \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda} \mathbf{x}^\alpha = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda} \underbrace{f_\lambda}_{\substack{=f_\alpha\\ \text{(since Lemma 5.7}\\ \text{yields } f_\alpha=f_\lambda)}} \mathbf{x}^\alpha$$
$$= \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda} f_\alpha \mathbf{x}^\alpha. \qquad (136)$$

Now, we have the following equality of summation signs:

$$\sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda} = \sum_{\lambda\in\operatorname{Par}} \underbrace{\sum_{\substack{\alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ |\lambda|=m}}}_{\substack{=\sum\limits_{\substack{\alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ |\alpha|=m}}\\ \text{(because for each } \alpha\in\mathfrak{S}_{(\infty)}\lambda,\\ \text{we have } |\lambda|=|\alpha|\\ \text{(by Lemma 3.7 (a)))}}} = \sum_{\lambda\in\operatorname{Par}} \underbrace{\sum_{\substack{\alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ |\alpha|=m}}}_{\substack{=\sum\limits_{\substack{\alpha\in\operatorname{WC};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ |\alpha|=m}}\\ \text{(since } \mathfrak{S}_{(\infty)}\lambda\subseteq\operatorname{WC})}}$$
$$= \sum_{\lambda\in\operatorname{Par}} \sum_{\substack{\alpha\in\operatorname{WC};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda;\\ |\alpha|=m}} = \sum_{\substack{\alpha\in\operatorname{WC};\\ |\alpha|=m}} \sum_{\substack{\lambda\in\operatorname{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} .$$

Hence, (136) becomes

$$\sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} f_\lambda m_\lambda = \underbrace{\sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} \sum_{\alpha\in\mathfrak{S}_{(\infty)}\lambda}}_{=\sum\limits_{\substack{\alpha\in\operatorname{WC};\\ |\alpha|=m}} \sum\limits_{\substack{\lambda\in\operatorname{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}}} f_\alpha \mathbf{x}^\alpha = \sum_{\substack{\alpha\in\operatorname{WC};\\ |\alpha|=m}} \underbrace{\sum_{\substack{\lambda\in\operatorname{Par};\\ \alpha\in\mathfrak{S}_{(\infty)}\lambda}} f_\alpha \mathbf{x}^\alpha}_{\substack{=f_\alpha\mathbf{x}^\alpha\\ \text{(by (134))}}}$$
$$= \sum_{\substack{\alpha\in\operatorname{WC};\\ |\alpha|=m}} f_\alpha \mathbf{x}^\alpha. \qquad (137)$$

Now, (130) becomes

$$G_{F,m} = \sum_{\substack{\alpha\in\operatorname{WC};\\ |\alpha|=m}} f_\alpha \mathbf{x}^\alpha = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} f_\lambda \underbrace{m_\lambda}_{\in\Lambda} \qquad \text{(by (137))}$$
$$\in \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} f_\lambda \Lambda \subseteq \Lambda \qquad \text{(since } \Lambda \text{ is a } \mathbf{k}\text{-module)}.$$

This proves Proposition 5.6 **(c)**.

(d) Proposition 5.6 **(b)** yields $G_F = \prod\limits_{i=1}^{\infty} F(x_i)$. Thus, the power series G_F is symmetric (since $\prod\limits_{i=1}^{\infty} F(x_i)$ is obviously symmetric). This proves Proposition 5.6 **(d)**.

(e) The definition of $G_{F,0}$ yields

$$G_{F,0} = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha|=0}} f_\alpha \mathbf{x}^\alpha = f_{(0,0,0,\ldots)} \underbrace{\mathbf{x}^{(0,0,0,\ldots)}}_{=1} \qquad \left(\begin{array}{c}\text{since the only } \alpha \in \mathrm{WC} \text{ satisfying } |\alpha| = 0 \\ \text{is the weak composition } (0,0,0,\ldots)\end{array}\right)$$

$$= f_{(0,0,0,\ldots)} = f_0 f_0 f_0 \cdots \qquad \left(\text{by the definition of } f_{(0,0,0,\ldots)}\right)$$
$$= 1 \cdot 1 \cdot 1 \cdot \cdots \qquad (\text{since } f_0 = 1)$$
$$= 1.$$

This proves Proposition 5.6 **(e)**. □

Next, let us generalize Definition 2.5:

Definition 5.8. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathrm{Par}$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \mathrm{Par}$. Then, the *F-Petrie number* $\mathrm{pet}_F(\lambda, \mu)$ of λ and μ is the element of $\mathbf{k}$ defined by

$$\mathrm{pet}_F(\lambda, \mu) = \det\left(\left(f_{\lambda_i - \mu_j - i + j}\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right). \tag{138}$$

Note that this integer does not depend on the choice of ℓ (in the sense that it does not change if we enlarge ℓ by adding trailing zeroes to the representations of λ and μ); this follows from Lemma 5.10 below.

Example 5.9. For $\ell = 3$, the equality (138) rewrites as

$$\mathrm{pet}_F(\lambda, \mu) = \det\begin{pmatrix} f_{\lambda_1 - \mu_1} & f_{\lambda_1 - \mu_2 + 1} & f_{\lambda_1 - \mu_3 + 2} \\ f_{\lambda_2 - \mu_1 - 1} & f_{\lambda_2 - \mu_2} & f_{\lambda_2 - \mu_3 + 1} \\ f_{\lambda_3 - \mu_1 - 2} & f_{\lambda_3 - \mu_2 - 1} & f_{\lambda_3 - \mu_3} \end{pmatrix}.$$

We can now state the generalization of Lemma 2.7 that is needed to justify Definition 5.8:

Lemma 5.10. Let $\lambda \in \mathrm{Par}$ and $\mu \in \mathrm{Par}$. Let $\ell \in \mathbb{N}$ be such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$. Then, the determinant $\det\left(\left(f_{\lambda_i - \mu_j - i + j}\right)_{1 \leq i \leq \ell,\ 1 \leq j \leq \ell}\right)$ does not depend on the choice of ℓ.

The slickest way to prove Lemma 5.10 is using a $\mathbf{k}$-algebra homomorphism $\alpha_F : \Lambda \to \mathbf{k}$ that generalizes the homomorphism α_k from Definition 3.11. Let us introduce this homomorphism α_F (which will also be used in other proofs). We recall the h-universal property of Λ, which was stated in Subsection 3.7.

Definition 5.11. The h-universal property of Λ shows that there is a unique **k**-algebra homomorphism $\alpha_F : \Lambda \to \mathbf{k}$ that sends h_i to f_i for all positive integers i. Consider this α_F.

We will use this homomorphism α_F several times in what follows; let us thus begin by stating some elementary properties of α_F.

Lemma 5.12. **(a)** We have

$$\alpha_F(h_i) = f_i \qquad \text{for all } i \in \mathbb{N}. \tag{139}$$

(b) We have

$$\alpha_F(h_i) = f_i \qquad \text{for all } i \in \mathbb{Z}. \tag{140}$$

(c) Let λ be a partition. Define h_λ as in Definition 3.4. Then,

$$\alpha_F(h_\lambda) = f_\lambda. \tag{141}$$

Proof of Lemma 5.12. **(a)** Let $i \in \mathbb{N}$. We must prove that $\alpha_F(h_i) = f_i$. If $i > 0$, then this follows from the definition of α_F. Thus, we WLOG assume that we don't have $i > 0$. Hence, $i = 0$ (since $i \in \mathbb{N}$). Therefore, $h_i = h_0 = 1$, so that $\alpha_F(h_i) = \alpha_F(1) = 1$ (since α_F is a **k**-algebra homomorphism). On the other hand, $i = 0$, so that $f_i = f_0 = 1$. Comparing this with $\alpha_F(h_i) = 1$, we obtain $\alpha_F(h_i) = f_i$. This proves Lemma 5.12 **(a)**.

(b) Let $i \in \mathbb{Z}$. We must prove that $\alpha_F(h_i) = f_i$. If $i < 0$, then this easily follows from $0 = 0$ [75]. Hence, we WLOG assume that we don't have $i < 0$. Therefore, $i \geq 0$, so that $i \in \mathbb{N}$. Hence, Lemma 5.12 **(a)** yields $\alpha_F(h_i) = f_i$. This proves Lemma 5.12 **(b)**.

(c) Write the partition λ in the form $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, where $\lambda_1, \lambda_2, \ldots, \lambda_\ell$ are positive integers. Then, the definition of h_λ yields

$$h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell} = \prod_{i=1}^{\ell} h_{\lambda_i}.$$

Applying the map α_F to both sides of this equality, we find

$$\begin{aligned}
\alpha_F(h_\lambda) = \alpha_F\left(\prod_{i=1}^{\ell} h_{\lambda_i}\right) &= \prod_{i=1}^{\ell} \underbrace{\alpha_F(h_{\lambda_i})}_{\substack{= f_{\lambda_i} \\ \text{(by (139), applied to } \lambda_i \text{ instead of } i)}} \\
&\qquad (\text{since } \alpha_F \text{ is a } \mathbf{k}\text{-algebra homomorphism}) \\
&= \prod_{i=1}^{\ell} f_{\lambda_i}.
\end{aligned} \tag{142}$$

[75] *Proof.* Assume that $i < 0$. Thus, $h_i = 0$, so that $\alpha_F(h_i) = \alpha_F(0) = 0$ (since α_F is a **k**-algebra homomorphism). But Definition 5.4 **(b)** yields $f_i = 0$ (since i is negative). Comparing this with $\alpha_F(h_i) = 0$, we obtain $\alpha_F(h_i) = f_i$, qed.

But we have $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and thus $\lambda_{\ell+1} = \lambda_{\ell+2} = \lambda_{\ell+3} = \cdots = 0$. In other words, we have $\lambda_i = 0$ for all $i \in \{\ell+1, \ell+2, \ell+3, \ldots\}$. Hence, we have $f_{\lambda_i} = f_0 = 1$ for all $i \in \{\ell+1, \ell+2, \ell+3, \ldots\}$. Multiplying these equalities over all $i \in \{\ell+1, \ell+2, \ell+3, \ldots\}$, we obtain $\prod\limits_{i=\ell+1}^{\infty} f_{\lambda_i} = \prod\limits_{i=\ell+1}^{\infty} 1 = 1$.

Now, the definition of f_λ yields

$$f_\lambda = f_{\lambda_1} f_{\lambda_2} f_{\lambda_3} \cdots = \prod_{i=1}^{\infty} f_{\lambda_i} = \left(\prod_{i=1}^{\ell} f_{\lambda_i}\right) \underbrace{\left(\prod_{i=\ell+1}^{\infty} f_{\lambda_i}\right)}_{=1} = \prod_{i=1}^{\ell} f_{\lambda_i} = \alpha_F(h_\lambda)$$

(by (142)). Thus, Lemma 5.12 **(c)** is proved. □

The following proof of Lemma 5.10 is a straightforward adaptation of our first proof of Lemma 2.7 (note that the second proof can be adapted just as easily).

Proof of Lemma 5.10. Recall that $\alpha_F : \Lambda \to \mathbf{k}$ is a $\mathbf{k}$-algebra homomorphism. Thus, α_F respects determinants; i.e., if $(a_{i,j})_{1 \le i \le m,\ 1 \le j \le m} \in \Lambda^{m \times m}$ is an $m \times m$-matrix over Λ (for some $m \in \mathbb{N}$), then

$$\begin{aligned} &\alpha_F\left(\det\left((a_{i,j})_{1 \le i \le m,\ 1 \le j \le m}\right)\right) \\ &= \det\left((\alpha_F(a_{i,j}))_{1 \le i \le m,\ 1 \le j \le m}\right). \end{aligned} \tag{143}$$

Applying α_F to both sides of the equality (32), we obtain

$$\begin{aligned} \alpha_F(s_{\lambda/\mu}) &= \alpha_F\left(\det\left(\left(h_{\lambda_i - \mu_j - i + j}\right)_{1 \le i \le \ell,\ 1 \le j \le \ell}\right)\right) \\ &= \det\left(\left(\underbrace{\alpha_F\left(h_{\lambda_i - \mu_j - i + j}\right)}_{\substack{= f_{\lambda_i - \mu_j - i + j} \\ \text{(by (140), applied to } \lambda_i - \mu_j - i + j \\ \text{instead of } i)}}\right)_{1 \le i \le \ell,\ 1 \le j \le \ell}\right) \\ &\qquad \left(\text{by (143), applied to } m = \ell \text{ and } a_{i,j} = h_{\lambda_i - \mu_j - i + j}\right) \\ &= \det\left(\left(f_{\lambda_i - \mu_j - i + j}\right)_{1 \le i \le \ell,\ 1 \le j \le \ell}\right). \end{aligned} \tag{144}$$

Clearly, the element $\alpha_F(s_{\lambda/\mu})$ does not depend on the choice of ℓ. In view of the equality (144), we can rewrite this as follows: The element $\det\left(\left(f_{\lambda_i - \mu_j - i + j}\right)_{1 \le i \le \ell,\ 1 \le j \le \ell}\right)$ does not depend on the choice of ℓ. This proves Lemma 5.10. □

Just as with Lemma 2.7, our proof of Lemma 5.10 leads to a useful consequence (analogous to Lemma 3.13):

Lemma 5.13. Let λ and μ be two partitions. Then, the homomorphism $\alpha_F : \Lambda \to \mathbf{k}$ from Definition 5.11 satisfies

$$\alpha_F\left(s_{\lambda/\mu}\right) = \operatorname{pet}_F(\lambda, \mu). \tag{145}$$

Proof of Lemma 5.13. Write the partitions λ and μ in the forms $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_\ell)$ for some $\ell \in \mathbb{N}$ [76]. Then, the equality (144) (which we showed in our proof of Lemma 5.10) yields

$$\alpha_F\left(s_{\lambda/\mu}\right) = \det\left(\left(f_{\lambda_i-\mu_j-i+j}\right)_{1\leq i\leq \ell,\ 1\leq j\leq \ell}\right) = \operatorname{pet}_F(\lambda, \mu)$$

(by the definition of $\operatorname{pet}_F(\lambda, \mu)$). This proves Lemma 5.13. □

We now come to more substantive properties of G_F and $G_{F,m}$.

First, we shall state them; the proofs will come later.

The following theorem generalizes Theorem 2.9:

Theorem 5.14. We have

$$G_F = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_F(\lambda, \varnothing)\, s_\lambda.$$

(Recall that $\varnothing$ denotes the empty partition $() = (0, 0, 0, \ldots)$.)

The following corollary (which already appeared in [Stanle01, Exercise 7.91 (d)]) generalizes Corollary 2.10:

Corollary 5.15. Let $m \in \mathbb{N}$. Then,

$$G_{F,m} = \sum_{\lambda \in \operatorname{Par}_m} \operatorname{pet}_F(\lambda, \varnothing)\, s_\lambda.$$

The following theorem generalizes Theorem 2.17:

Theorem 5.16. Let $\mu \in \operatorname{Par}$. Then,

$$G_F \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}} \operatorname{pet}_F(\lambda, \mu)\, s_\lambda.$$

The following corollary generalizes Corollary 2.18:

[76] Such an ℓ can always be found, since each of λ and μ has only finitely many nonzero entries.

Corollary 5.17. Let $m \in \mathbb{N}$. Let $\mu \in \operatorname{Par}$. Then,

$$G_{F,m} \cdot s_\mu = \sum_{\lambda \in \operatorname{Par}_{m+|\mu|}} \operatorname{pet}_F(\lambda, \mu) \, s_\lambda.$$

Let us prove these four results. We begin with Theorem 5.16, which (just as its particular case the Theorem 2.17) can be proved in two ways. We shall only give the first proof:

Proof of Theorem 5.16. We shall use the notations $\mathbf{k}[[\mathbf{x}]]$, $\mathbf{k}[[\mathbf{x},\mathbf{y}]]$, $\mathbf{x}$, $\mathbf{y}$, $f(\mathbf{x})$ and $f(\mathbf{y})$ introduced in Subsection 2.6. If R is any commutative ring, then $R[[\mathbf{y}]]$ shall denote the ring $R[[y_1, y_2, y_3, \ldots]]$ of formal power series in the indeterminates $y_1, y_2, y_3, \ldots$ over the ring R. We will identify the ring $\mathbf{k}[[\mathbf{x},\mathbf{y}]]$ with the ring $(\mathbf{k}[[\mathbf{x}]])[[\mathbf{y}]] = (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[y_1, y_2, y_3, \ldots]]$. Note that $\Lambda \subseteq \mathbf{k}[[\mathbf{x}]]$ and thus $\Lambda[[\mathbf{y}]] \subseteq (\mathbf{k}[[\mathbf{x}]])[[\mathbf{y}]] = \mathbf{k}[[\mathbf{x},\mathbf{y}]]$. We equip the rings $\mathbf{k}[[\mathbf{y}]]$, $\Lambda[[\mathbf{y}]]$ and $\mathbf{k}[[\mathbf{x},\mathbf{y}]]$ with the usual topologies that are defined on rings of power series, where Λ itself is equipped with the discrete topology. This has the somewhat confusing consequence that $\Lambda[[\mathbf{y}]] \subseteq \mathbf{k}[[\mathbf{x},\mathbf{y}]]$ is an inclusion of rings but not of topological spaces; however, this will not cause us any trouble, since all infinite sums in $\Lambda[[\mathbf{y}]]$ we will consider (such as $\sum\limits_{\lambda \in \operatorname{Par}} s_{\lambda/\mu}(\mathbf{x}) \, s_\lambda(\mathbf{y})$ and $\sum\limits_{\lambda \in \operatorname{Par}} h_\lambda(\mathbf{x}) \, m_\lambda(\mathbf{y})$) will converge to the same value in either topology.

We consider both $\mathbf{k}[[\mathbf{y}]]$ and Λ as subrings of $\Lambda[[\mathbf{y}]]$ (indeed, $\mathbf{k}[[\mathbf{y}]]$ embeds into $\Lambda[[\mathbf{y}]]$ because $\mathbf{k}$ is a subring of Λ, whereas Λ embeds into $\Lambda[[\mathbf{y}]]$ because $\Lambda[[\mathbf{y}]]$ is a ring of power series over Λ).

In this proof, the word "monomial" may refer to a monomial in any set of variables (not necessarily in $x_1, x_2, x_3, \ldots$).

Recall the $\mathbf{k}$-algebra homomorphism $\alpha_F : \Lambda \to \mathbf{k}$ from Definition 5.11. This $\mathbf{k}$-algebra homomorphism $\alpha_F : \Lambda \to \mathbf{k}$ induces a $\mathbf{k}[[\mathbf{y}]]$-algebra homomorphism $\alpha_F[[\mathbf{y}]] : \Lambda[[\mathbf{y}]] \to \mathbf{k}[[\mathbf{y}]]$, which is given by the formula

$$(\alpha_F[[\mathbf{y}]]) \left(\sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} f_{\mathfrak{n}} \mathfrak{n} \right) = \sum_{\substack{\mathfrak{n} \text{ is a monomial} \\ \text{in } y_1, y_2, y_3, \ldots}} \alpha_F(f_{\mathfrak{n}}) \, \mathfrak{n}$$

for any family $(f_{\mathfrak{n}})_{\mathfrak{n} \text{ is a monomial in } y_1, y_2, y_3, \ldots}$ of elements of Λ. This induced $\mathbf{k}[[\mathbf{y}]]$-algebra homomorphism $\alpha_F[[\mathbf{y}]]$ is $\mathbf{k}[[\mathbf{y}]]$-linear and continuous (with respect to the usual topologies on the power series rings $\Lambda[[\mathbf{y}]]$ and $\mathbf{k}[[\mathbf{y}]]$), and thus preserves infinite $\mathbf{k}[[\mathbf{y}]]$-linear combinations. Moreover, it extends α_F (that is, for any $f \in \Lambda$, we have $(\alpha_F[[\mathbf{y}]])(f) = \alpha_F(f)$).

Recall the skew Schur functions $s_{\lambda/\mu}$ defined in Subsection 3.5. Also, recall the symmetric functions h_λ defined in Definition 3.4. In the First proof of Theorem 2.17, we have proved the equality

$$\sum_{\lambda \in \operatorname{Par}} s_\lambda(\mathbf{y}) \, s_{\lambda/\mu} = \sum_{\lambda \in \operatorname{Par}} s_\mu(\mathbf{y}) \, m_\lambda(\mathbf{y}) \, h_\lambda.$$

Consider this as an equality in the ring $\Lambda[[\mathbf{y}]] = \Lambda[[y_1, y_2, y_3, \ldots]]$. Apply the map $\alpha_F[[\mathbf{y}]] : \Lambda[[\mathbf{y}]] \to \mathbf{k}[[\mathbf{y}]]$ to both sides of this equality. We obtain

$$(\alpha_F[[\mathbf{y}]])\left(\sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y})\, s_{\lambda/\mu}\right) = (\alpha_F[[\mathbf{y}]])\left(\sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y})\, h_\lambda\right).$$

Comparing this with

$$\begin{aligned}
&(\alpha_F[[\mathbf{y}]])\left(\sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y})\, s_{\lambda/\mu}\right) \\
&= \sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y}) \cdot \underbrace{(\alpha_F[[\mathbf{y}]])\left(s_{\lambda/\mu}\right)}_{\substack{=\alpha_F\left(s_{\lambda/\mu}\right) \\ \text{(since } \alpha_F[[\mathbf{y}]] \text{ extends } \alpha_F)}} \\
&\qquad \left(\text{since the map } \alpha_F[[\mathbf{y}]] \text{ preserves infinite } \mathbf{k}[[\mathbf{y}]]\text{-linear combinations}\right) \\
&= \sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y}) \cdot \underbrace{\alpha_F\left(s_{\lambda/\mu}\right)}_{\substack{=\operatorname{pet}_F(\lambda,\mu) \\ \text{(by (145))}}} = \sum_{\lambda\in\operatorname{Par}} s_\lambda(\mathbf{y}) \cdot \operatorname{pet}_F(\lambda,\mu) = \sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_F(\lambda,\mu) \cdot s_\lambda(\mathbf{y}),
\end{aligned}$$

we obtain

$$\begin{aligned}
&\sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_F(\lambda,\mu) \cdot s_\lambda(\mathbf{y}) \\
&= (\alpha_F[[\mathbf{y}]])\left(\sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y})\, h_\lambda\right) = \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y}) \underbrace{(\alpha_F[[\mathbf{y}]])(h_\lambda)}_{\substack{=\alpha_F(h_\lambda) \\ \text{(since } \alpha_F[[\mathbf{y}]] \text{ extends } \alpha_F)}} \\
&\qquad \left(\text{since the map } \alpha_F[[\mathbf{y}]] \text{ preserves infinite } \mathbf{k}[[\mathbf{y}]]\text{-linear combinations}\right) \\
&= \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y}) \underbrace{\alpha_F(h_\lambda)}_{\substack{=f_\lambda \\ \text{(by (141))}}} = \sum_{\lambda\in\operatorname{Par}} s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y}) \cdot f_\lambda = \sum_{\lambda\in\operatorname{Par}} f_\lambda \cdot s_\mu(\mathbf{y})\, m_\lambda(\mathbf{y}).
\end{aligned}$$

Renaming the indeterminates $\mathbf{y} = (y_1, y_2, y_3, \ldots)$ as $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ on both sides of this equality, we obtain

$$\sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_F(\lambda,\mu) \cdot s_\lambda(\mathbf{x}) = \sum_{\lambda\in\operatorname{Par}} f_\lambda \cdot \underbrace{s_\mu(\mathbf{x})}_{=s_\mu} \underbrace{m_\lambda(\mathbf{x})}_{=m_\lambda} = \sum_{\lambda\in\operatorname{Par}} \underbrace{f_\lambda \cdot s_\mu}_{=s_\mu f_\lambda} m_\lambda = \sum_{\lambda\in\operatorname{Par}} s_\mu f_\lambda m_\lambda.$$

Comparing this with

$$G_F \cdot s_\mu = s_\mu \cdot \underbrace{G_F}_{\substack{=\sum\limits_{\lambda\in\operatorname{Par}} f_\lambda m_\lambda \\ \text{(by Proposition 5.6 \textbf{(b)})}}} = s_\mu \cdot \sum_{\lambda\in\operatorname{Par}} f_\lambda m_\lambda = \sum_{\lambda\in\operatorname{Par}} s_\mu f_\lambda m_\lambda,$$

we obtain
$$G_F \cdot s_\mu = \sum_{\lambda \in \mathrm{Par}} \mathrm{pet}_F(\lambda, \mu) \cdot \underbrace{s_\lambda(\mathbf{x})}_{= s_\lambda} = \sum_{\lambda \in \mathrm{Par}} \mathrm{pet}_F(\lambda, \mu)\, s_\lambda.$$

This proves Theorem 5.16. $\square$

Proof of Corollary 5.17. Forget that we fixed m. If $n \in \mathbb{N}$, then the power series $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is homogeneous of degree n [77].

[77] *Proof.* Let $n \in \mathbb{N}$. We must prove that the power series $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n.

We are in one of the following two cases:

Case 1: We have $n \geq |\mu|$.

Case 2: We have $n < |\mu|$.

Let us first consider Case 1. In this case, we have $n \geq |\mu|$. Hence, $n - |\mu| \in \mathbb{N}$. But Proposition 5.6 **(a)** (applied to $m = n - |\mu|$) yields that the power series $G_{F,n-|\mu|}$ is the $(n - |\mu|)$-th degree homogeneous component of G_F. Hence, $G_{F,n-|\mu|}$ is homogeneous of degree $n - |\mu|$.

On the other hand, recall that for any $\lambda \in \mathrm{Par}$, the Schur function s_λ is homogeneous of degree $|\lambda|$. Applying this to $\lambda = \mu$, we conclude that the Schur function s_μ is homogeneous of degree $|\mu|$.

So we know that $G_{F,n-|\mu|}$ is homogeneous of degree $n - |\mu|$, whereas s_μ is homogeneous of degree $|\mu|$. Since Λ is a graded algebra, this entails that the power series $G_{F,n-|\mu|} \cdot s_\mu$ (being the product of $G_{F,n-|\mu|}$ and s_μ) is homogeneous of degree $(n - |\mu|) + |\mu|$. In other words, the power series $G_{F,n-|\mu|} \cdot s_\mu$ is homogeneous of degree n (since $(n - |\mu|) + |\mu| = n$). In other words, the power series $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n (since

$$\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases} = G_{F,n-|\mu|} \cdot s_\mu \qquad (\text{because } n \geq |\mu|)$$

). Thus, we have proved in Case 1 that the power series $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n.

Let us now consider Case 2. In this case, we have $n < |\mu|$. The power series 0 is homogeneous of degree n (since 0 is homogeneous of any degree). In other words, the power series $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n (since

$$\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases} = 0 \qquad (\text{because } n < |\mu|)$$

). Thus, we have proved in Case 2 that the power series $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n.

Thus, our claim (namely, that the power series $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu|; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of

In the proof of Proposition 5.6 **(a)**, we have shown that $G_F = \sum_{m\in\mathbb{N}} G_{F,m}$. Multiplying both sides of this equality by s_μ, we find

$$G_F \cdot s_\mu = \left(\sum_{m\in\mathbb{N}} G_{F,m} \right) \cdot s_\mu = \sum_{m\in\mathbb{N}} G_{F,m} \cdot s_\mu .$$

Comparing this with

$$\begin{aligned}
&\sum_{n\in\mathbb{N}} \begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu| ; \\ 0, & \text{if } n < |\mu| \end{cases} \\
&= \sum_{\substack{n\in\mathbb{N};\\ n\geq|\mu|}} \underbrace{\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu| ; \\ 0, & \text{if } n < |\mu| \end{cases}}_{\substack{=G_{F,n-|\mu|}\cdot s_\mu \\ \text{(since } n\geq|\mu|)}} + \sum_{\substack{n\in\mathbb{N};\\ n<|\mu|}} \underbrace{\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu| ; \\ 0, & \text{if } n < |\mu| \end{cases}}_{\substack{=0 \\ \text{(since } n<|\mu|)}} \\
&\qquad \left(\begin{array}{c} \text{since each } n \in \mathbb{N} \text{ satisfies either } n \geq |\mu| \text{ or } n < |\mu| \\ \text{(but not both at the same time)} \end{array} \right) \\
&= \sum_{\substack{n\in\mathbb{N};\\ n\geq|\mu|}} G_{F,n-|\mu|} \cdot s_\mu + \underbrace{\sum_{\substack{n\in\mathbb{N};\\ n<|\mu|}} 0}_{=0} = \sum_{\substack{n\in\mathbb{N};\\ n\geq|\mu|}} G_{F,n-|\mu|} \cdot s_\mu \\
&= \sum_{m\in\mathbb{N}} G_{F,m} \cdot s_\mu \qquad \left(\begin{array}{c} \text{here, we have substituted } m \\ \text{for } n - |\mu| \text{ in the sum} \end{array} \right),
\end{aligned}$$

we obtain

$$G_F \cdot s_\mu = \sum_{n\in\mathbb{N}} \begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu| ; \\ 0, & \text{if } n < |\mu| \end{cases} . \tag{146}$$

But recall that each $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu| ; \\ 0, & \text{if } n < |\mu| \end{cases}$ is homogeneous of degree n. Thus, the equality (146) reveals that the family

$$\left(\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu| ; \\ 0, & \text{if } n < |\mu| \end{cases} \right)_{n\in\mathbb{N}}$$

is the homogeneous decomposition of $G_F \cdot s_\mu$ (by the definition of a homogeneous decomposition). Therefore, for each $n \in \mathbb{N}$, the power series $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu| ; \\ 0, & \text{if } n < |\mu| \end{cases}$ is the n-th degree homogeneous component of $G_F \cdot s_\mu$.

degree n) has been proven in both Cases 1 and 2. Since these cases cover all possibilities, we thus conclude that our claim always holds. Qed.

Now, let $m \in \mathbb{N}$. We have just shown that for each $n \in \mathbb{N}$, the power series $\begin{cases} G_{F,n-|\mu|} \cdot s_\mu, & \text{if } n \geq |\mu| ; \\ 0, & \text{if } n < |\mu| \end{cases}$ is the n-th degree homogeneous component of $G_F \cdot s_\mu$. Applying this to $n = m + |\mu|$, we conclude that the power series $\begin{cases} G_{F,(m+|\mu|)-|\mu|} \cdot s_\mu, & \text{if } m+|\mu| \geq |\mu| ; \\ 0, & \text{if } m+|\mu| < |\mu| \end{cases}$ is the $(m+|\mu|)$-th degree homogeneous component of $G_F \cdot s_\mu$. Since

$$\begin{aligned} &\begin{cases} G_{F,(m+|\mu|)-|\mu|} \cdot s_\mu, & \text{if } m+|\mu| \geq |\mu| ; \\ 0, & \text{if } m+|\mu| < |\mu| \end{cases} \\ &= G_{F,(m+|\mu|)-|\mu|} \cdot s_\mu \qquad (\text{since } m+|\mu| \geq |\mu| \text{ (because } m \geq 0)) \\ &= G_{F,m} \cdot s_\mu \qquad (\text{since } (m+|\mu|) - |\mu| = m), \end{aligned}$$

we can rewrite this as follows: The power series $G_{F,m} \cdot s_\mu$ is the $(m+|\mu|)$-th degree homogeneous component of $G_F \cdot s_\mu$. In other words,

$$\begin{aligned} &G_{F,m} \cdot s_\mu \\ &= \left(\text{the } (m+|\mu|) \text{-th degree homogeneous component of } G_F \cdot s_\mu\right). \end{aligned} \tag{147}$$

On the other hand, Theorem 5.16 yields

$$\begin{aligned} G_F \cdot s_\mu &= \underbrace{\sum_{\lambda \in \mathrm{Par}}}_{\substack{= \sum_{n\in\mathbb{N}} \sum_{\lambda \in \mathrm{Par}; \, |\lambda|=n} \\ \text{(since each } \lambda \in \mathrm{Par} \\ \text{satisfies } |\lambda| \in \mathbb{N})}} \mathrm{pet}_F(\lambda, \mu) \, s_\lambda \\ &= \sum_{n\in\mathbb{N}} \sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_F(\lambda, \mu) \, s_\lambda. \end{aligned} \tag{148}$$

For each $n \in \mathbb{N}$, the formal power series $\sum\limits_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_F(\lambda, \mu) \, s_\lambda$ is homogeneous of degree n [78]. Thus, the equality (148) reveals that

$$\left(\sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_F(\lambda, \mu) \, s_\lambda \right)_{n\in\mathbb{N}}$$

[78] *Proof.* Let $n \in \mathbb{N}$. Recall that for any $\lambda \in \mathrm{Par}$, the Schur function s_λ is homogeneous of degree $|\lambda|$. Hence, if $\lambda \in \mathrm{Par}$ satisfies $|\lambda| = n$, then the Schur function s_λ is homogeneous of degree n (since $|\lambda| = n$). Thus, $\sum\limits_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_F(\lambda, \mu) \, s_\lambda$ is a $\mathbf{k}$-linear combination of Schur functions that are homogeneous of degree n. Therefore, $\sum\limits_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda|=n}} \mathrm{pet}_F(\lambda, \mu) \, s_\lambda$ is homogeneous of degree n. Qed.

is the homogeneous decomposition of $G_F \cdot s_\mu$. Therefore, for each $n \in \mathbb{N}$, the power series $\sum\limits_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=n}} \operatorname{pet}_F(\lambda,\mu) s_\lambda$ is the n-th degree homogeneous component of $G_F \cdot s_\mu$. Applying this to $n = m + |\mu|$, we conclude that the power series $\sum\limits_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m+|\mu|}} \operatorname{pet}_F(\lambda,\mu) s_\lambda$ is the $(m+|\mu|)$-th degree homogeneous component of $G_F \cdot s_\mu$. In other words,

$$\sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m+|\mu|}} \operatorname{pet}_F(\lambda,\mu) s_\lambda = \left(\text{the } (m+|\mu|)\text{-th degree homogeneous component of } G_F \cdot s_\mu\right).$$

Comparing this with (147), we find

$$G_{F,m} \cdot s_\mu = \underbrace{\sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m+|\mu|}}}_{\substack{=\sum\limits_{\lambda\in\operatorname{Par}_{m+|\mu|}} \\ \text{(since } \operatorname{Par}_{m+|\mu|} \text{ is defined as the} \\ \text{set of all } \lambda\in\operatorname{Par} \text{ satisfying } |\lambda|=m+|\mu|)}} \operatorname{pet}_F(\lambda,\mu) s_\lambda = \sum_{\lambda\in\operatorname{Par}_{m+|\mu|}} \operatorname{pet}_F(\lambda,\mu) s_\lambda.$$

This proves Corollary 5.17. $\square$

Proof of Theorem 5.14. Theorem 5.16 (applied to $\mu = \varnothing$) yields

$$G_F \cdot s_\varnothing = \sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_F(\lambda,\varnothing) s_\lambda.$$

Comparing this with $G_F \cdot \underbrace{s_\varnothing}_{=1} = G_F$, we obtain

$$G_F = \sum_{\lambda\in\operatorname{Par}} \operatorname{pet}_F(\lambda,\varnothing) s_\lambda.$$

This proves Theorem 5.14. $\square$

Proof of Corollary 5.15. Corollary 5.17 (applied to $\mu = \varnothing$) yields

$$G_{F,m} \cdot s_\varnothing = \sum_{\lambda\in\operatorname{Par}_{m+|\varnothing|}} \operatorname{pet}_F(\lambda,\varnothing) s_\lambda.$$

In view of $G_{F,m} \cdot \underbrace{s_\varnothing}_{=1} = G_{F,m}$ and $m + \underbrace{|\varnothing|}_{=0} = m$, we can rewrite this as

$$G_{F,m} = \sum_{\lambda\in\operatorname{Par}_m} \operatorname{pet}_F(\lambda,\varnothing) s_\lambda.$$

This proves Corollary 5.15. $\square$

Proposition 5.6 **(c)** shows that $G_{F,m} \in \Lambda$ for each $m \in \mathbb{N}$. Hence, we can apply the comultiplication Δ of the Hopf algebra Λ to $G_{F,m}$. The next theorem (which generalizes Theorem 2.19) gives a simple expression for the result of this:

Theorem 5.18. Let $m \in \mathbb{N}$. Then,

$$\Delta\left(G_{F,m}\right) = \sum_{i=0}^{m} G_{F,i} \otimes G_{F,m-i}.$$

Proof of Theorem 5.18. Forget that we fixed m. Recall that

$$G_F = \sum_{m\in\mathbb{N}} G_{F,m} \tag{149}$$

(indeed, we have proved this in our proof of Proposition 5.6 **(a)**). On the other hand, Proposition 5.6 **(d)** says that the power series G_F is symmetric. Furthermore, Proposition 5.6 **(b)** tells us that

$$G_F = \prod_{i=1}^{\infty} F\left(x_i\right). \tag{150}$$

Comparing this with (149), we obtain

$$\sum_{m\in\mathbb{N}} G_{F,m} = \prod_{i=1}^{\infty} F\left(x_i\right). \tag{151}$$

Substituting the variables $y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ in this equality, we obtain

$$\sum_{m\in\mathbb{N}} G_{F,m}\left(\mathbf{y}\right) = \prod_{i=1}^{\infty} F\left(y_i\right) \tag{152}$$

(since the left hand side of (151) turns into $\sum\limits_{m\in\mathbb{N}} G_{F,m}\left(\mathbf{y}\right)$ upon this substitution[79], whereas the right hand side turns into $\prod\limits_{i=1}^{\infty} F\left(y_i\right)$).

On the other hand, let us substitute the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ on both sides of the equality (150). (This means that we choose some bijection $\phi : \{x_1, x_2, x_3, \ldots\} \to \{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$, and substitute $\phi\left(x_i\right)$ for each x_i on both sides of (150).) Thus, we readily obtain

$$G_F\left(\mathbf{x}, \mathbf{y}\right) = \left(\prod_{i=1}^{\infty} F\left(x_i\right)\right)\left(\prod_{i=1}^{\infty} F\left(y_i\right)\right). \tag{153}$$

[Here is a detailed *proof of (153):* Choose some bijection $\phi : \{x_1, x_2, x_3, \ldots\} \to \{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$. (Such a bijection clearly exists, since both $\{x_1, x_2, x_3, \ldots\}$

[79] because each $G_{F,m}$ turns into $G_{F,m}\left(\mathbf{y}\right)$ upon this substitution

and $\{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$ are countably infinite sets.) Then, $G_F(\mathbf{x}, \mathbf{y})$ is the result of substituting $\phi(x_i)$ for each x_i in G_F (by the definition of $G_F(\mathbf{x}, \mathbf{y})$, since the power series G_F is symmetric). In other words,

$$G_F(\mathbf{x}, \mathbf{y}) = G_F(\phi(x_1), \phi(x_2), \phi(x_3), \ldots) = \prod_{i=1}^{\infty} F(\phi(x_i)) \tag{154}$$

(here, we have substituted $\phi(x_i)$ for each x_i on both sides of the equality (150)). Now, recall that the map ϕ is a bijection from $\{x_1, x_2, x_3, \ldots\}$ to $\{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$. Thus, its values $\phi(x_1), \phi(x_2), \phi(x_3), \ldots$ are precisely the indeterminates $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ (in some order). Hence,

$$\begin{aligned}
&F(\phi(x_1)) \cdot F(\phi(x_2)) \cdot F(\phi(x_3)) \cdot \cdots \\
&= \prod_{u \in \{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}} F(u) \\
&= \underbrace{(F(x_1) \cdot F(x_2) \cdot F(x_3) \cdot \cdots)}_{= \prod_{i=1}^{\infty} F(x_i)} \cdot \underbrace{(F(y_1) \cdot F(y_2) \cdot F(y_3) \cdot \cdots)}_{= \prod_{i=1}^{\infty} F(y_i)} \\
&= \left(\prod_{i=1}^{\infty} F(x_i)\right) \left(\prod_{i=1}^{\infty} F(y_i)\right).
\end{aligned}$$

Hence, (154) becomes

$$\begin{aligned}
G_F(\mathbf{x}, \mathbf{y}) &= \prod_{i=1}^{\infty} F(\phi(x_i)) = F(\phi(x_1)) \cdot F(\phi(x_2)) \cdot F(\phi(x_3)) \cdot \cdots \\
&= \left(\prod_{i=1}^{\infty} F(x_i)\right) \left(\prod_{i=1}^{\infty} F(y_i)\right).
\end{aligned}$$

This proves (153).]

Now, (153) becomes

$$\begin{aligned}
G_F(\mathbf{x},\mathbf{y}) &= \underbrace{\left(\prod_{i=1}^{\infty} F(x_i)\right)}_{\substack{=\sum\limits_{m\in\mathbb{N}} G_{F,m} \\ \text{(by (151))}}} \underbrace{\left(\prod_{i=1}^{\infty} F(y_i)\right)}_{\substack{=\sum\limits_{m\in\mathbb{N}} G_{F,m}(\mathbf{y}) \\ \text{(by (152))}}} \\
&= \left(\sum_{m\in\mathbb{N}} \underbrace{G_{F,m}}_{=G_{F,m}(\mathbf{x})}\right)\left(\sum_{m\in\mathbb{N}} G_{F,m}(\mathbf{y})\right) \\
&= \underbrace{\left(\sum_{m\in\mathbb{N}} G_{F,m}(\mathbf{x})\right)}_{\substack{=\sum\limits_{i\in\mathbb{N}} G_{F,i}(\mathbf{x}) \\ \text{(here, we have renamed} \\ \text{the summation index } m \text{ as } i\text{)}}} \underbrace{\left(\sum_{m\in\mathbb{N}} G_{F,m}(\mathbf{y})\right)}_{\substack{=\sum\limits_{j\in\mathbb{N}} G_{F,j}(\mathbf{y}) \\ \text{(here, we have renamed} \\ \text{the summation index } m \text{ as } j\text{)}}} \\
&= \left(\sum_{i\in\mathbb{N}} G_{F,i}(\mathbf{x})\right)\left(\sum_{j\in\mathbb{N}} G_{F,j}(\mathbf{y})\right) = \underbrace{\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}}_{\substack{=\sum\limits_{(i,j)\in\mathbb{N}\times\mathbb{N}} \\ =\sum\limits_{n\in\mathbb{N}}\ \sum\limits_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}; \\ i+j=n}}}} G_{F,i}(\mathbf{x})\, G_{F,j}(\mathbf{y}) \\
&= \sum_{n\in\mathbb{N}}\ \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}; \\ i+j=n}} G_{F,i}(\mathbf{x})\, G_{F,j}(\mathbf{y}) \\
&= \sum_{n\in\mathbb{N}} \left(\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}; \\ i+j=n}} G_{F,i}(\mathbf{x})\, G_{F,j}(\mathbf{y})\right). \qquad (155)
\end{aligned}$$

If $n \in \mathbb{N}$, then the power series $\sum\limits_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N}; \\ i+j=n}} G_{F,i}(\mathbf{x})\, G_{F,j}(\mathbf{y}) \in \mathbf{k}[[\mathbf{x},\mathbf{y}]]$ is homogeneous of degree n [80]. Thus, the equality (155) reveals that the family

$$\left(\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{F,i}\left(\mathbf{x}\right) G_{F,j}\left(\mathbf{y}\right)\right)_{n\in\mathbb{N}}$$

is the homogeneous decomposition of $G_F\left(\mathbf{x},\mathbf{y}\right)$ (by the definition of a homogeneous decomposition).

On the other hand, we have

$$G_F\left(\mathbf{x},\mathbf{y}\right) = \sum_{m\in\mathbb{N}} G_{F,m}\left(\mathbf{x},\mathbf{y}\right). \tag{156}$$

[*Proof of (156):* If we substitute the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ on both sides of the equality (149), then we obtain

$$G_F\left(\mathbf{x},\mathbf{y}\right) = \sum_{m\in\mathbb{N}} G_{F,m}\left(\mathbf{x},\mathbf{y}\right)$$

(because this substitution transforms G_F into $G_F\left(\mathbf{x},\mathbf{y}\right)$ and transforms $G_{F,m}$ into $G_{F,m}\left(\mathbf{x},\mathbf{y}\right)$ for each $m\in\mathbb{N}$). This proves (156).]

Now, if $n\in\mathbb{N}$, then the power series $G_{F,n}\left(\mathbf{x},\mathbf{y}\right)\in\mathbf{k}\left[\left[\mathbf{x},\mathbf{y}\right]\right]$ is homogeneous of

[80] *Proof.* Let $n\in\mathbb{N}$. We must prove that the power series $\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{F,i}\left(\mathbf{x}\right) G_{F,j}\left(\mathbf{y}\right)$ is homogeneous of degree n.

Let $(i,j)\in\mathbb{N}\times\mathbb{N}$ be such that $i+j=n$. Then, Proposition 5.6 **(a)** (applied to $m=i$) shows that the formal power series $G_{F,i}$ is the i-th degree homogeneous component of G_F. Hence, this formal power series $G_{F,i}$ is homogeneous of degree i. In other words, $G_{F,i}\left(\mathbf{x}\right)$ is homogeneous of degree i (since $G_{F,i}\left(\mathbf{x}\right)=G_{F,i}$).

Moreover, Proposition 5.6 **(a)** (applied to $m=j$) shows that the formal power series $G_{F,j}$ is the j-th degree homogeneous component of G_F. Hence, this formal power series $G_{F,j}$ is homogeneous of degree j. Hence, the power series $G_{F,j}\left(\mathbf{y}\right)$ is homogeneous of degree j as well (since this $G_{F,j}\left(\mathbf{y}\right)$ is obtained by substituting $y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ in $G_{F,j}$; but this substitution clearly preserves homogeneity and degree).

Now we have shown that the two power series $G_{F,i}\left(\mathbf{x}\right)$ and $G_{F,j}\left(\mathbf{y}\right)$ are homogeneous of degrees i and j, respectively. Thus, their product $G_{F,i}\left(\mathbf{x}\right) G_{F,j}\left(\mathbf{y}\right)$ is homogeneous of degree $i+j$. In other words, $G_{F,i}\left(\mathbf{x}\right) G_{F,j}\left(\mathbf{y}\right)$ is homogeneous of degree n (since $i+j=n$).

Forget that we fixed (i,j). We thus have shown that $G_{F,i}\left(\mathbf{x}\right) G_{F,j}\left(\mathbf{y}\right)$ is homogeneous of degree n whenever $(i,j)\in\mathbb{N}\times\mathbb{N}$ satisfies $i+j=n$. In other words, each addend of the sum $\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{F,i}\left(\mathbf{x}\right) G_{F,j}\left(\mathbf{y}\right)$ is homogeneous of degree n. Hence, the entire sum $\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{F,i}\left(\mathbf{x}\right) G_{F,j}\left(\mathbf{y}\right)$ is homogeneous of degree n as well. This completes our proof.

degree n [81]. Thus, the equality

$$G_F(\mathbf{x},\mathbf{y}) = \sum_{m\in\mathbb{N}} G_{F,m}(\mathbf{x},\mathbf{y}) = \sum_{n\in\mathbb{N}} G_{F,n}(\mathbf{x},\mathbf{y})$$

(here, we have renamed the summation index m as n)

reveals that $(G_{F,n}(\mathbf{x},\mathbf{y}))_{n\in\mathbb{N}}$ is the homogeneous decomposition of $G_F(\mathbf{x},\mathbf{y})$ (by the definition of a homogeneous decomposition).

We have now shown that each of the two families

$$(G_{F,n}(\mathbf{x},\mathbf{y}))_{n\in\mathbb{N}} \qquad \text{and} \qquad \left(\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{F,i}(\mathbf{x})\, G_{F,j}(\mathbf{y})\right)_{n\in\mathbb{N}}$$

is the homogeneous decomposition of $G_F(\mathbf{x},\mathbf{y})$. Since the homogeneous decomposition of $G_F(\mathbf{x},\mathbf{y})$ is unique, this entails that these two families are equal. In other words, we have

$$(G_{F,n}(\mathbf{x},\mathbf{y}))_{n\in\mathbb{N}} = \left(\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{F,i}(\mathbf{x})\, G_{F,j}(\mathbf{y})\right)_{n\in\mathbb{N}}.$$

In other words, we have

$$G_{F,n}(\mathbf{x},\mathbf{y}) = \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{F,i}(\mathbf{x})\, G_{F,j}(\mathbf{y}) \tag{157}$$

for each $n\in\mathbb{N}$.

Now, let $m\in\mathbb{N}$. Then, (157) (applied to $n=m$) yields

$$\begin{aligned} &G_{F,m}(\mathbf{x},\mathbf{y}) \\ &= \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=m}} G_{F,i}(\mathbf{x})\, G_{F,j}(\mathbf{y}) = \sum_{i\in\{0,1,\ldots,m\}} G_{F,i}(\mathbf{x})\, G_{F,m-i}(\mathbf{y}) \\ &\qquad \left(\begin{array}{c}\text{here, we have substituted } (i,m-i) \text{ for } (i,j) \text{ in the sum,}\\ \text{since the map } \{0,1,\ldots,m\}\to\{(i,j)\in\mathbb{N}\times\mathbb{N} \mid i+j=m\}\\ \text{that sends each } i \text{ to } (i,m-i) \text{ is a bijection}\end{array}\right). \end{aligned}$$

[81] *Proof.* Let $n\in\mathbb{N}$. We must prove that the power series $G_{F,n}(\mathbf{x},\mathbf{y})$ is homogeneous of degree n.

Proposition 5.6 **(a)** (applied to $m=n$) shows that the formal power series $G_{F,n}$ is the n-th degree homogeneous component of G_F. Hence, this formal power series $G_{F,n}$ is homogeneous of degree n. Hence, the power series $G_{F,n}(\mathbf{x},\mathbf{y})$ is homogeneous of degree n as well (since this $G_{F,n}(\mathbf{x},\mathbf{y})$ is obtained by substituting the variables $x_1,x_2,x_3,\ldots,y_1,y_2,y_3,\ldots$ for the variables $x_1,x_2,x_3,\ldots$ in $G_{F,n}$; but this substitution clearly preserves homogeneity and degree). This completes our proof.

Hence, (10) holds for $f = G_{F,m}$, $I = \{0, 1, \ldots, m\}$, $(f_{1,i})_{i \in I} = (G_{F,i})_{i \in \{0,1,\ldots,m\}}$ and $(f_{2,i})_{i \in I} = (G_{F,m-i})_{i \in \{0,1,\ldots,m\}}$. Therefore, (9) (applied to these f, I, $(f_{1,i})_{i \in I}$ and $(f_{2,i})_{i \in I}$) yields

$$\Delta (G_{F,m}) = \sum_{i \in \{0,1,\ldots,m\}} G_{F,i} \otimes G_{F,m-i} = \sum_{i=0}^{m} G_{F,i} \otimes G_{F,m-i}.$$

This proves Theorem 5.18. □

The next few results we will state rely on the following definition:

Definition 5.19. Let F' be the derivative of the formal power series $F \in \mathbf{k}[[t]]$. Let us write the formal power series $\dfrac{F'}{F} \in \mathbf{k}[[t]]$ (which is well-defined, since F has constant term 1) in the form $\dfrac{F'}{F} = \sum\limits_{n \in \mathbb{N}} \gamma_n t^n$ for some $\gamma_0, \gamma_1, \gamma_2, \ldots \in \mathbf{k}$.

Example 5.20. Let us see how F' and γ_n look for specific values of F.

(a) Let $F = \dfrac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots$. Then, $F' = \dfrac{1}{(1-t)^2}$, so that

$$\frac{F'}{F} = \frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots = \sum_{n \in \mathbb{N}} t^n.$$

Therefore, $\gamma_n = 1$ for each $n \in \mathbb{N}$.

(b) Now, let $F = 1$. Then, $F' = 0$, so that $\dfrac{F'}{F} = 0 = \sum\limits_{n \in \mathbb{N}} 0 t^n$. Therefore, $\gamma_n = 0$ for each $n \in \mathbb{N}$.

(c) Now, fix a positive integer k, and set $F = 1 + t + t^2 + \cdots + t^{k-1}$. Then, $F = \dfrac{1 - t^k}{1-t}$, and thus a simple calculation using the quotient rule shows that $F' = \dfrac{1 + (k-1)\, t^k - k t^{k-1}}{(1-t)^2}$. Hence,

$$\begin{aligned}
\frac{F'}{F} &= \frac{1 + (k-1)\, t^k - k t^{k-1}}{(1-t)\left(1 - t^k\right)} = \underbrace{\frac{1}{1-t}}_{= \sum_{n \in \mathbb{N}} t^n} - k t^{k-1} \cdot \underbrace{\frac{1}{1-t^k}}_{= \sum_{n \in \mathbb{N}} \left(t^k\right)^n} \\
&= \sum_{n \in \mathbb{N}} t^n - \underbrace{k t^{k-1} \cdot \sum_{n \in \mathbb{N}} \left(t^k\right)^n}_{\substack{= \sum_{n \in \mathbb{N}} k t^{nk+k-1} \\ = \sum_{n \in \mathbb{N};\ k \mid n+1} k t^n}} = \sum_{n \in \mathbb{N}} t^n - \sum_{\substack{n \in \mathbb{N}; \\ k \mid n+1}} k t^n \\
&= \sum_{n \in \mathbb{N}} \left(1 - [k \mid n+1]\, k\right) t^n.
\end{aligned}$$

Therefore, $\gamma_n = 1 - [k \mid n+1]\, k$ for each $n \in \mathbb{N}$.

The next proposition is easily seen to generalize Proposition 2.24:

Proposition 5.21. Let m be a positive integer. Then, $\langle p_m, G_{F,m} \rangle = \gamma_{m-1}$.

The proof of this proposition relies on the following property of the $\mathbf{k}$-algebra homomorphism $\alpha_F : \Lambda \to \mathbf{k}$ from Definition 5.11:

Lemma 5.22. We have $\alpha_F(p_m) = \gamma_{m-1}$ for each positive integer m.

Proof of Lemma 5.22. Consider the ring $\Lambda[[t]]$ of formal power series in one indeterminate t over Λ. Consider also the analogous ring $\mathbf{k}[[t]]$ over $\mathbf{k}$.

The map $\alpha_F : \Lambda \to \mathbf{k}$ is a $\mathbf{k}$-algebra homomorphism. Hence, it induces a continuous[82] $\mathbf{k}[[t]]$-algebra homomorphism

$$\alpha_F[[t]] : \Lambda[[t]] \to \mathbf{k}[[t]]$$

that sends each formal power series $\sum\limits_{n \geq 0} a_n t^n \in \Lambda[[t]]$ (with $a_n \in \Lambda$) to $\sum\limits_{n \geq 0} \alpha_F(a_n)\, t^n$. Consider this $\mathbf{k}[[t]]$-algebra homomorphism $\alpha_F[[t]]$.

Define the formal power series

$$H(t) = \prod_{i=1}^{\infty} (1 - x_i t)^{-1} \in (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]].$$

Then, from [GriRei20, (2.4.1)], we know that

$$H(t) = \sum_{n \geq 0} \underbrace{h_n(\mathbf{x})}_{=h_n} t^n = \sum_{n \geq 0} h_n t^n \in \Lambda[[t]].$$

Hence, $(\alpha_F[[t]])(H(t))$ is well-defined. Moreover, applying the map $\alpha_F[[t]]$ to both sides of the equality $H(t) = \sum\limits_{n \geq 0} h_n t^n$, we obtain

$$\begin{aligned}
(\alpha_F[[t]])(H(t)) = (\alpha_F[[t]])\left(\sum_{n \geq 0} h_n t^n\right) &= \underbrace{\sum_{n \geq 0}}_{= \sum_{n \in \mathbb{N}}} \underbrace{\alpha_F(h_n)}_{\substack{=f_n \\ \text{(by Lemma 5.12 (b)} \\ \text{(applied to } i=n))}} t^n \\
&\qquad \text{(by the definition of } \alpha_F[[t]]) \\
&= \sum_{n \in \mathbb{N}} f_n t^n = F \qquad \left(\text{since } F = \sum_{n \in \mathbb{N}} f_n t^n\right).
\end{aligned}$$

[82] Continuity is defined with respect to the usual topologies on $\Lambda[[t]]$ and $\mathbf{k}[[t]]$, where we equip both Λ and $\mathbf{k}$ with the discrete topologies.

Moreover, consider the derivative $H'(t)$ of the power series $H(t) \in \Lambda[[t]]$. This derivative again belongs to $\Lambda[[t]]$, so that $(\alpha_F[[t]])(H'(t))$ is well-defined. Moreover, from $H(t) = \sum_{n\geq 0} h_n t^n$, we obtain $H'(t) = \sum_{n\geq 1} n h_n t^{n-1}$ (by the definition of a derivative), so that

$$H'(t) = \sum_{n\geq 1} n h_n t^{n-1} = \sum_{n\geq 0} (n+1) h_{n+1} t^n$$

(here, we have substituted $n+1$ for n in the sum). Applying the map $\alpha_F[[t]]$ to both sides of this equality, we find

$$\begin{aligned}
(\alpha_F[[t]])\left(H'(t)\right) &= (\alpha_F[[t]])\left(\sum_{n\geq 0} (n+1) h_{n+1} t^n\right) = \sum_{n\geq 0} \underbrace{\alpha_F((n+1) h_{n+1})}_{\substack{=(n+1)\alpha_F(h_{n+1}) \\ \text{(since } \alpha_F \text{ is } \mathbf{k}\text{-linear)}}} t^n \\
&\qquad \text{(by the definition of } \alpha_F[[t]]) \\
&= \sum_{n\geq 0} (n+1) \underbrace{\alpha_F(h_{n+1})}_{\substack{=f_{n+1} \\ \text{(by Lemma 5.12 } \mathbf{(b)} \\ \text{(applied to } i=n+1))}} t^n \\
&= \sum_{n\geq 0} (n+1) f_{n+1} t^n. \qquad (158)
\end{aligned}$$

On the other hand, from $F = \sum_{n\in\mathbb{N}} f_n t^n$, we obtain

$$\begin{aligned}
F' &= \sum_{n\geq 1} n f_n t^{n-1} \qquad \text{(by the definition of the derivative of a power series)} \\
&= \sum_{n\geq 0} (n+1) f_{n+1} t^n
\end{aligned}$$

(here, we have substituted $n+1$ for n in the sum). Comparing this with (158), we obtain

$$(\alpha_F[[t]])\left(H'(t)\right) = F'.$$

From [GriRei20, Exercise 2.5.21], we know that

$$\sum_{m\geq 0} p_{m+1} t^m = \frac{H'(t)}{H(t)}.$$

Hence,

$$\frac{H'(t)}{H(t)} = \sum_{m\geq 0} p_{m+1} t^m = \sum_{n\geq 0} p_{n+1} t^n$$

(here, we have renamed the summation index m as n). Applying the map $\alpha_F[[t]]$ to both sides of this equality, we find

$$(\alpha_F[[t]])\left(\frac{H'(t)}{H(t)}\right) = (\alpha_F[[t]])\left(\sum_{n\geq 0} p_{n+1}t^n\right) = \underbrace{\sum_{n\geq 0}}_{=\sum_{n\in\mathbb{N}}} \alpha_F(p_{n+1})\,t^n$$
$$\text{(by the definition of } \alpha_F[[t]]\text{)}$$
$$= \sum_{n\in\mathbb{N}} \alpha_F(p_{n+1})\,t^n. \tag{159}$$

Now, the map $\alpha_F[[t]]$ is a **k**-algebra homomorphism, and thus respects quotients. Hence,

$$(\alpha_F[[t]])\left(\frac{H'(t)}{H(t)}\right) = \frac{(\alpha_F[[t]])(H'(t))}{(\alpha_F[[t]])(H(t))} = \frac{F'}{F}$$
$$\left(\text{since } (\alpha_F[[t]])\left(H'(t)\right) = F' \text{ and } (\alpha_F[[t]])(H(t)) = F\right)$$
$$= \sum_{n\in\mathbb{N}} \gamma_n t^n.$$

Comparing this with (159), we obtain

$$\sum_{n\in\mathbb{N}} \alpha_F(p_{n+1})\,t^n = \sum_{n\in\mathbb{N}} \gamma_n t^n.$$

Comparing coefficients on both sides of this equality, we find

$$\alpha_F(p_{n+1}) = \gamma_n \qquad \text{for each } n\in\mathbb{N}. \tag{160}$$

Now, let m be a positive integer. Thus, $m-1\in\mathbb{N}$. Hence, (160) (applied to $n=m-1$) yields $\alpha_F\left(p_{(m-1)+1}\right) = \gamma_{m-1}$. In other words, $\alpha_F(p_m) = \gamma_{m-1}$ (since $(m-1)+1 = m$). This proves Lemma 5.22. □

Proof of Proposition 5.21. Proposition 5.6 **(c)** yields $G_{F,m} = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} f_\lambda m_\lambda$. Hence,

$$\langle p_m, G_{F,m}\rangle = \left\langle p_m, \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} f_\lambda m_\lambda \right\rangle = \sum_{\substack{\lambda\in\operatorname{Par};\\ |\lambda|=m}} f_\lambda \langle p_m, m_\lambda\rangle \tag{161}$$

(since the Hall inner product is **k**-bilinear).

Recall the following fundamental fact from linear algebra: If A is a **k**-module, if $\langle\cdot,\cdot\rangle : A\times A\to\mathbf{k}$ is a symmetric bilinear form on A, and if $(u_\lambda)_{\lambda\in L}$ and $(v_\lambda)_{\lambda\in L}$ are two bases of the **k**-module A that are dual to each other with respect to the form $\langle\cdot,\cdot\rangle$ (where L is some indexing set), then every $a\in A$ satisfies

$$a = \sum_{\lambda\in L} \langle u_\lambda, a\rangle\, v_\lambda.$$

We can apply this fact to $A = \Lambda$, $L = \operatorname{Par}$, $(u_\lambda)_{\lambda \in L} = (m_\lambda)_{\lambda \in \operatorname{Par}}$ and $(v_\lambda)_{\lambda \in L} = (h_\lambda)_{\lambda \in \operatorname{Par}}$ (since the bases $(m_\lambda)_{\lambda \in \operatorname{Par}}$ and $(h_\lambda)_{\lambda \in \operatorname{Par}}$ of Λ are dual to each other with respect to the Hall inner product $\langle \cdot, \cdot \rangle$). We thus conclude that every $a \in \Lambda$ satisfies

$$a = \sum_{\lambda \in \operatorname{Par}} \langle m_\lambda, a \rangle \, h_\lambda.$$

Applying this to $a = p_m$, we obtain

$$p_m = \sum_{\lambda \in \operatorname{Par}} \underbrace{\langle m_\lambda, p_m \rangle}_{\substack{= \langle p_m, m_\lambda \rangle \\ \text{(since the Hall inner} \\ \text{product is symmetric)}}} h_\lambda = \sum_{\lambda \in \operatorname{Par}} \langle p_m, m_\lambda \rangle \, h_\lambda. \tag{162}$$

Now, it is easy to see that if $\lambda \in \operatorname{Par}$ satisfies $|\lambda| \neq m$, then

$$\langle p_m, m_\lambda \rangle = 0. \tag{163}$$

[*Proof of (163):* Let $\lambda \in \operatorname{Par}$ satisfy $|\lambda| \neq m$. Thus, $m \neq |\lambda|$. The two symmetric functions p_m and m_λ are homogeneous of degrees m and $|\lambda|$, respectively. Thus, these two symmetric functions p_m and m_λ are homogeneous of different degrees (since $m \neq |\lambda|$). Hence, (2) (applied to $f = p_m$ and $g = m_\lambda$) yields $\langle p_m, m_\lambda \rangle = 0$. This proves (163).]

Now, (162) becomes

$$\begin{aligned} p_m &= \sum_{\lambda \in \operatorname{Par}} \langle p_m, m_\lambda \rangle \, h_\lambda = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m}} \langle p_m, m_\lambda \rangle \, h_\lambda + \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| \neq m}} \underbrace{\langle p_m, m_\lambda \rangle}_{\substack{=0 \\ \text{(by (163))}}} h_\lambda \\ &\qquad \left(\begin{array}{c} \text{since each } \lambda \in \operatorname{Par} \text{ satisfies either } |\lambda| = m \\ \text{or } |\lambda| \neq m \text{, but not both at the same time} \end{array} \right) \\ &= \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m}} \langle p_m, m_\lambda \rangle \, h_\lambda + \underbrace{\sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| \neq m}} 0 h_\lambda}_{=0} = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m}} \langle p_m, m_\lambda \rangle \, h_\lambda. \end{aligned}$$

Applying the map α_F to both sides of this equality, we find

$$\begin{aligned} \alpha_F(p_m) &= \alpha_F \left(\sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m}} \langle p_m, m_\lambda \rangle \, h_\lambda \right) = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m}} \langle p_m, m_\lambda \rangle \underbrace{\alpha_F(h_\lambda)}_{\substack{= f_\lambda \\ \text{(by Lemma 5.12 } \textbf{(c)})}} \\ &\qquad \text{(since the map } \alpha_F \text{ is } \mathbf{k}\text{-linear)} \\ &= \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m}} \underbrace{\langle p_m, m_\lambda \rangle f_\lambda}_{= f_\lambda \langle p_m, m_\lambda \rangle} = \sum_{\substack{\lambda \in \operatorname{Par}; \\ |\lambda| = m}} f_\lambda \langle p_m, m_\lambda \rangle = \langle p_m, G_{F,m} \rangle \end{aligned}$$

(by (161)). Hence,

$$\langle p_m, G_{F,m} \rangle = \alpha_F(p_m) = \gamma_{m-1} \qquad \text{(by Lemma 5.22)}.$$

This proves Proposition 5.21. $\square$

We can now generalize Theorem 2.22:

Theorem 5.23. Assume that all the elements $\gamma_0, \gamma_1, \gamma_2, \ldots$ are invertible in $\mathbf{k}$.

Then, the family $(G_{F,m})_{m\geq 1} = (G_{F,1}, G_{F,2}, G_{F,3}, \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra Λ. (In other words, the canonical $\mathbf{k}$-algebra homomorphism

$$\mathbf{k}[u_1, u_2, u_3, \ldots] \to \Lambda,$$
$$u_m \mapsto G_{F,m}$$

is an isomorphism.)

Proof of Theorem 5.23. For each positive integer m, the power series $G_{F,m}$ belongs to Λ (by Proposition 5.6 **(c)**), and thus is a symmetric function. Moreover, this symmetric function $G_{F,m}$ is homogeneous of degree m (by Proposition 5.6 **(a)**). Hence, for each positive integer m, the element $G_{F,m} \in \Lambda$ is a homogeneous symmetric function of degree m.

Let m be a positive integer. From Proposition 5.21, we obtain $\langle p_m, G_{F,m}\rangle = \gamma_{m-1}$. Hence, $\langle p_m, G_{F,m}\rangle$ is an invertible element of $\mathbf{k}$ (because γ_{m-1} is an invertible element of $\mathbf{k}$ (since all the elements $\gamma_0, \gamma_1, \gamma_2, \ldots$ are invertible in $\mathbf{k}$)).

Forget that we fixed m. We thus have showed that $\langle p_m, G_{F,m}\rangle$ is an invertible element of $\mathbf{k}$ for each positive integer m. Also, as we know, for each positive integer m, the element $G_{F,m} \in \Lambda$ is a homogeneous symmetric function of degree m. Thus, Proposition 3.20 (applied to $v_m = G_{F,m}$) shows that the family $(G_{F,m})_{m\geq 1} = (G_{F,1}, G_{F,2}, G_{F,3}, \ldots)$ is an algebraically independent generating set of the commutative $\mathbf{k}$-algebra Λ. This proves Theorem 5.23. $\square$

Remark 5.24. It is not hard to verify that the converse of Theorem 5.23 also holds: If the family $(G_{F,m})_{m\geq 1} = (G_{F,1}, G_{F,2}, G_{F,3}, \ldots)$ generates the $\mathbf{k}$-algebra Λ, then all the elements $\gamma_0, \gamma_1, \gamma_2, \ldots$ are invertible in $\mathbf{k}$. We omit the proof of this.

The next theorem generalizes parts of Theorem 2.29 (specifically, it generalizes the properties of the map V_k stated in Theorem 2.29, even though it defines this map differently):[83]

Theorem 5.25. The h-universal property of Λ shows that there is a unique $\mathbf{k}$-algebra homomorphism $V_F : \Lambda \to \Lambda$ that sends h_i to $G_{F,i}$ for all positive integers i (since $G_{F,i} \in \Lambda$ for each positive integer i). Consider this V_F.

(a) This map V_F is a $\mathbf{k}$-Hopf algebra homomorphism.

(b) We have $V_F(h_m) = G_{F,m}$ for each $m \in \mathbb{N}$.

(c) We have $V_F(p_n) = \gamma_{n-1} p_n$ for each positive integer n. (See Definition 5.19 for the meaning of γ_{n-1}.)

Our proof of this theorem (specifically, of its part **(c)**) will use the notion of the logarithmic derivative of a formal power series. We first recall its definition:

[83]We recall the "h-universal property of Λ", which we stated in Subsection 3.7.

Definition 5.26. Let R be a commutative ring. Let $F \in R[[t]]$ be a formal power series whose constant term is 1. Thus, F is invertible (since F has constant term 1).

The *logarithmic derivative* of F is defined to be the formal power series $\frac{F'}{F} \in R[[t]]$ (this is well-defined, since F is invertible). This logarithmic derivative is denoted by $\operatorname{lder} F$.

It is easy to see that $\operatorname{lder} F$ is the derivative of $\log F$ if R is a commutative $\mathbb{Q}$-algebra[84]. However, if R is not a $\mathbb{Q}$-algebra, then $\log F$ is not defined, so that $\operatorname{lder} F$ can only be defined via Definition 5.26.

We shall now state (and, for the sake of completeness, prove) a few well-known properties of logarithmic derivatives:

Proposition 5.27. Let R be a commutative ring. Let $u, v \in R[[t]]$ be two formal power series whose constant terms are 1. Then, $\operatorname{lder}(uv) = \operatorname{lder} u + \operatorname{lder} v$.

Proof of Proposition 5.27. The two power series u and v have constant term 1. Hence, their product uv has constant term 1 as well (since the constant term of the product of two power series equals the product of their constant terms). Thus, $\operatorname{lder}(uv)$ is well-defined.

The Leibniz rule yields $(uv)' = u'v + uv'$. However, the definition of $\operatorname{lder} u$ yields $\operatorname{lder} u = \frac{u'}{u}$. Likewise, $\operatorname{lder} v = \frac{v'}{v}$. Adding these two equalities, we obtain

$$\operatorname{lder} u + \operatorname{lder} v = \frac{u'}{u} + \frac{v'}{v} = \frac{u'v + uv'}{uv}.$$

On the other hand, the definition of $\operatorname{lder}(uv)$ yields

$$\operatorname{lder}(uv) = \frac{(uv)'}{uv} = \frac{u'v + uv'}{uv} \qquad \left(\text{since } (uv)' = u'v + uv'\right).$$

Comparing these two equalities, we find $\operatorname{lder}(uv) = \operatorname{lder} u + \operatorname{lder} v$. This proves Proposition 5.27. $\square$

Proposition 5.28. Let R be a commutative topological ring. Let $(u_n)_{n\in\mathbb{N}} = (u_0, u_1, u_2, \ldots) \in R[[t]]^{\mathbb{N}}$ be a sequence of formal power series whose constant terms are 1. Let $u \in R[[t]]$ be a formal power series whose constant term is 1. Assume that $\lim\limits_{n\to\infty} u_n = u$ (with respect to the standard topology on $R[[t]]$ induced by the topology on R). Then, $\lim\limits_{n\to\infty} (\operatorname{lder} u_n) = \operatorname{lder} u$ (with respect to the same topology on $R[[t]]$).

[84] This explains the name "logarithmic derivative".

Proof of Proposition 5.28. It is well-known that the map

$$R[[t]] \to R[[t]], \qquad v \mapsto v'$$

(that is, the map that sends each power series $v \in R[[t]]$ to its derivative v') is continuous[85]. Hence, from $\lim_{n\to\infty} u_n = u$, we obtain $\lim_{n\to\infty} u_n' = u'$.

Let $R[[t]]_1$ be the subset $\{v \in R[[t]] \mid \text{ the constant term of } v \text{ is } 1\}$ of $R[[t]]$. Then, the formal power series $u_0, u_1, u_2, \ldots$ and u all belong to $R[[t]]_1$ (since their constant terms are 1). It is well-known that the map

$$R[[t]]_1 \to R[[t]], \qquad v \mapsto \frac{1}{v}$$

is continuous[86]. Thus, from $\lim_{n\to\infty} u_n = u$, we obtain $\lim_{n\to\infty} \frac{1}{u_n} = \frac{1}{u}$ (since the formal power series $u_0, u_1, u_2, \ldots$ and u all belong to $R[[t]]_1$).

Finally, it is well-known that the map

$$R[[t]] \times R[[t]] \to R[[t]], \qquad (v, w) \mapsto vw$$

is continuous[87]. Hence, from $\lim_{n\to\infty} u_n' = u'$ and $\lim_{n\to\infty} \frac{1}{u_n} = \frac{1}{u}$, we obtain

$$\lim_{n\to\infty} \left(u_n' \cdot \frac{1}{u_n} \right) = u' \cdot \frac{1}{u} = \frac{u'}{u} = \operatorname{lder} u \tag{164}$$

---

[85] This follows from the fact that for each $n \in \mathbb{N}$, the n-th coefficient of the derivative v' of a power series $v \in R[[t]]$ is a continuous function of the first $n+2$ coefficients of v (indeed, it equals $n+1$ times the $(n+1)$-st coefficient of v).

[86] This follows from the fact that for each $n \in \mathbb{N}$, the n-th coefficient of the power series $\frac{1}{v}$ (where $v \in R[[t]]_1$) is a continuous function of the first $n+1$ coefficients of v (indeed, if we let v_i and $\left(\frac{1}{v}\right)_i$ denote the i-th coefficients of the power series v and $\frac{1}{v}$ for all $i \in \mathbb{N}$, then the coefficients of $\frac{1}{v}$ can be computed recursively from the coefficients of v using the formulas

$$\left(\frac{1}{v}\right)_0 = 1 \qquad \text{and}$$
$$\left(\frac{1}{v}\right)_n = -\sum_{i=1}^{n} v_i \left(\frac{1}{v}\right)_{n-i} \qquad \text{for each } n > 0;$$

these formulas rely only on addition, subtraction and multiplication of elements of R, and therefore define continuous maps).

[87] This follows from the fact that for each $n \in \mathbb{N}$, the n-th coefficient of the power series vw (where $(v, w) \in R[[t]] \times R[[t]]$) is a continuous function of the first $n+1$ coefficients of v and of w (indeed, it is equal to $\sum_{k=0}^{n} v_k w_{n-k}$, where v_i and w_i denote the i-th coefficients of v and w).

(since $\operatorname{lder} u$ is defined to be $\dfrac{u'}{u}$). However, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
\operatorname{lder} u_n &= \frac{u_n'}{u_n} \qquad \text{(by the definition of } \operatorname{lder} u_n\text{)} \\
&= u_n' \cdot \frac{1}{u_n}.
\end{aligned}$$

Thus, (164) rewrites as $\lim\limits_{n\to\infty} (\operatorname{lder} u_n) = \operatorname{lder} u$. This proves Proposition 5.28. □

Proposition 5.29. Let R be a commutative ring. Let $u_1, u_2, \ldots, u_n \in R[[t]]$ be finitely many formal power series whose constant terms are 1. Then,

$$\operatorname{lder}\left(\prod_{i=1}^{n} u_i\right) = \sum_{i=1}^{n} \operatorname{lder} u_i.$$

Proof of Proposition 5.29. We shall show that

$$\operatorname{lder}\left(\prod_{i=1}^{m} u_i\right) = \sum_{i=1}^{m} \operatorname{lder} u_i \tag{165}$$

for each $m \in \{0, 1, \ldots, n\}$.

Indeed, let us prove (165) by induction on m:

Induction base: We have $\prod\limits_{i=1}^{0} u_i = (\text{empty product}) = 1$ and thus

$$\begin{aligned}
\operatorname{lder}\left(\prod_{i=1}^{0} u_i\right) &= \operatorname{lder} 1 = \frac{1'}{1} \qquad \text{(by the definition of } \operatorname{lder} 1\text{)} \\
&= 1' = 0 = \sum_{i=1}^{0} \operatorname{lder} u_i
\end{aligned}$$

(since $\sum\limits_{i=1}^{0} \operatorname{lder} u_i = (\text{empty sum}) = 0$). In other words, (165) holds for $m = 0$.

Induction step: Let $k \in \{0, 1, \ldots, n-1\}$. Assume that (165) holds for $m = k$. We must prove that (165) holds for $m = k + 1$.

The power series $u_1, u_2, \ldots, u_k$ have constant term 1. Hence, their product $u_1 u_2 \cdots u_k$ has constant term 1 as well (since the constant term of the product of some power series equals the product of their constant terms[88]).

We have assumed that (165) holds for $m = k$. In other words, we have

$$\operatorname{lder}\left(\prod_{i=1}^{k} u_i\right) = \sum_{i=1}^{k} \operatorname{lder} u_i.$$

[88] This is a consequence of the fact that the map from $\mathbf{k}[[t]]$ to $\mathbf{k}$ that sends each power series to its constant term is a $\mathbf{k}$-algebra homomorphism.

Now,

$$\begin{aligned}
&\operatorname{lder} \underbrace{\left(\prod_{i=1}^{k+1} u_i\right)}_{=\left(\prod_{i=1}^{k} u_i\right)\cdot u_{k+1}} \\
&= \operatorname{lder}\left(\left(\prod_{i=1}^{k} u_i\right)\cdot u_{k+1}\right) = \underbrace{\operatorname{lder}\left(\prod_{i=1}^{k} u_i\right)}_{=\sum_{i=1}^{k}\operatorname{lder} u_i} + \operatorname{lder} u_{k+1} \\
&\qquad\qquad \left(\text{by Proposition 5.27, applied to } u = \prod_{i=1}^{k} u_i \text{ and } v = u_{k+1}\right) \\
&= \sum_{i=1}^{k}\operatorname{lder} u_i + \operatorname{lder} u_{k+1} = \sum_{i=1}^{k+1}\operatorname{lder} u_i.
\end{aligned}$$

In other words, (165) holds for $m = k+1$. This completes the induction step. Thus, (165) is proved by induction.

Now, applying (165) to $m = n$, we obtain

$$\operatorname{lder}\left(\prod_{i=1}^{n} u_i\right) = \sum_{i=1}^{n}\operatorname{lder} u_i.$$

This proves Proposition 5.29. □

Proposition 5.30. Let R be a commutative topological ring. Let $u_1, u_2, u_3, \ldots \in R[[t]]$ be infinitely many formal power series whose constant terms are 1. Assume that the infinite product $\prod_{i=1}^{\infty} u_i$ converges (with respect to the standard topology on $R[[t]]$ induced by the topology on R). Then, the infinite sum $\sum_{i=1}^{\infty}\operatorname{lder} u_i$ converges as well, and we have

$$\operatorname{lder}\left(\prod_{i=1}^{\infty} u_i\right) = \sum_{i=1}^{\infty}\operatorname{lder} u_i. \tag{166}$$

Proof of Proposition 5.30. The power series $u_1, u_2, u_3, \ldots$ have constant term 1. Hence, their product $\prod_{i=1}^{\infty} u_i$ has constant term 1 as well (since the constant term of the product of some power series equals the product of their constant terms). Moreover, all the partial products $\prod_{i=1}^{n} u_i$ of this infinite product also have constant term 1 (for the same reason).

The infinite product $\prod\limits_{i=1}^{\infty} u_i$ converges. Thus, the sequence $\left(\prod\limits_{i=1}^{n} u_i\right)_{n\in\mathbb{N}} \in R\left[\left[t\right]\right]^{\mathbb{N}}$ converges, and its limit is $\lim\limits_{n\to\infty}\left(\prod\limits_{i=1}^{n} u_i\right) = \prod\limits_{i=1}^{\infty} u_i$. Hence, Proposition 5.28 (applied to $\prod\limits_{i=1}^{n} u_i$ and $\prod\limits_{i=1}^{\infty} u_i$ instead of u_n and u) yields

$$\lim_{n\to\infty}\left(\operatorname{lder}\left(\prod_{i=1}^{n} u_i\right)\right) = \operatorname{lder}\left(\prod_{i=1}^{\infty} u_i\right). \tag{167}$$

However,

$$\lim_{n\to\infty}\underbrace{\left(\operatorname{lder}\left(\prod_{i=1}^{n} u_i\right)\right)}_{\substack{=\sum\limits_{i=1}^{n}\operatorname{lder} u_i \\ \text{(by Proposition 5.29)}}} = \lim_{n\to\infty}\sum_{i=1}^{n}\operatorname{lder} u_i.$$

Thus, (167) rewrites as

$$\lim_{n\to\infty}\sum_{i=1}^{n}\operatorname{lder} u_i = \operatorname{lder}\left(\prod_{i=1}^{\infty} u_i\right).$$

Thus, the sequence $\left(\sum\limits_{i=1}^{n}\operatorname{lder} u_i\right)_{n\in\mathbb{N}}$ converges. In other words, the infinite sum $\sum\limits_{i=1}^{\infty}\operatorname{lder} u_i$ converges. Moreover, the value of this sum is

$$\sum_{i=1}^{\infty}\operatorname{lder} u_i = \lim_{n\to\infty}\sum_{i=1}^{n}\operatorname{lder} u_i = \operatorname{lder}\left(\prod_{i=1}^{\infty} u_i\right).$$

This proves (166). Thus, the proof of Proposition 5.30 is complete. ☐

Proposition 5.31. Let R be a commutative ring. Let $u \in R\left[\left[t\right]\right]$ be a formal power series whose constant term is 1. Let $\lambda \in R$. Then,

$$\operatorname{lder}\left(u\left(\lambda t\right)\right) = \lambda\cdot\left(\operatorname{lder} u\right)\left(\lambda t\right).$$

Proof of Proposition 5.31. We first claim that the derivative of the power series $u\left(\lambda t\right)$ is

$$\left(u\left(\lambda t\right)\right)' = \lambda\cdot u'\left(\lambda t\right). \tag{168}$$

[*Proof of (168):* This is easy to see using the chain rule, but let us show this directly: Write the power series $u \in R\left[\left[t\right]\right]$ in the form $u = \sum\limits_{n\geq 0} u_n t^n$ for some $u_0, u_1, u_2, \ldots \in R$. Thus,

$$u\left(\lambda t\right) = \sum_{n\geq 0} u_n \underbrace{\left(\lambda t\right)^n}_{=\lambda^n t^n} = \sum_{n\geq 0} u_n \lambda^n t^n.$$

Hence, the definition of a derivative yields

$$\begin{aligned}(u(\lambda t))' &= \sum_{n\geq 1} nu_n \underbrace{\lambda^n}_{\substack{=\lambda\cdot\lambda^{n-1} \\ (\text{since } n\geq 1)}} t^{n-1} = \sum_{n\geq 1} nu_n\lambda \cdot \underbrace{\lambda^{n-1}t^{n-1}}_{=(\lambda t)^{n-1}} \\ &= \sum_{n\geq 1} nu_n\lambda\cdot(\lambda t)^{n-1}. \qquad (169)\end{aligned}$$

On the other hand, from $u = \sum_{n\geq 0} u_n t^n$, we obtain $u' = \sum_{n\geq 1} nu_n t^{n-1}$. Substituting λt for t on both sides of this equality, we find

$$u'(\lambda t) = \sum_{n\geq 1} nu_n (\lambda t)^{n-1}.$$

Multiplying this equality by λ, we find

$$\lambda\cdot u'(\lambda t) = \lambda\cdot\sum_{n\geq 1} nu_n (\lambda t)^{n-1} = \sum_{n\geq 1} nu_n\lambda\cdot(\lambda t)^{n-1}.$$

Comparing this with (169), we obtain $(u(\lambda t))' = \lambda\cdot u'(\lambda t)$. This proves (168).]

Now, the definition of $\operatorname{lder} u$ yields $\operatorname{lder} u = \dfrac{u'}{u}$. Substituting λt for t on both sides of this equality, we obtain

$$(\operatorname{lder} u)(\lambda t) = \frac{u'(\lambda t)}{u(\lambda t)}.$$

Multiplying this equality by λ, we find

$$\lambda\cdot(\operatorname{lder} u)(\lambda t) = \lambda\cdot\frac{u'(\lambda t)}{u(\lambda t)} = \frac{\lambda\cdot u'(\lambda t)}{u(\lambda t)}.$$

On the other hand, the definition of $\operatorname{lder}(u(\lambda t))$ yields

$$\operatorname{lder}(u(\lambda t)) = \frac{(u(\lambda t))'}{u(\lambda t)} = \frac{\lambda\cdot u'(\lambda t)}{u(\lambda t)} \qquad (\text{by (168)}).$$

Comparing these two equalities, we obtain $\operatorname{lder}(u(\lambda t)) = \lambda\cdot(\operatorname{lder} u)(\lambda t)$. This proves Proposition 5.31. □

Proposition 5.32. Let R and S be two commutative $\mathbf{k}$-algebras. Let $\alpha : R \to S$ be a $\mathbf{k}$-algebra homomorphism. As we know, α induces a continuous $\mathbf{k}[[t]]$-algebra homomorphism

$$\alpha[[t]] : R[[t]] \to S[[t]]$$

that sends each formal power series $\sum_{n\geq 0} a_n t^n \in R[[t]]$ (with $a_n \in R$) to $\sum_{n\geq 0} \alpha(a_n) t^n \in S[[t]]$.

Let $u \in R[[t]]$ be a formal power series whose constant term is 1. Then, the constant term of the power series $(\alpha[[t]])(u)$ is 1, and we have

$$\operatorname{lder}((\alpha[[t]])(u)) = (\alpha[[t]])(\operatorname{lder} u).$$

Proof of Proposition 5.32. Write the power series $u \in R[[t]]$ in the form $u = \sum_{n\geq 0} u_n t^n$ with $u_0, u_1, u_2, \ldots \in R$. Then, the definition of $\alpha[[t]]$ yields $(\alpha[[t]])(u) = \sum_{n\geq 0} \alpha(u_n) t^n$. However, u_0 is the constant term of u (since $u = \sum_{n\geq 0} u_n t^n$), and thus is 1 (since we know that the constant term of u is 1). In other words, $u_0 = 1$. Hence, $\alpha(u_0) = \alpha(1) = 1$ (since α is a $\mathbf{k}$-algebra homomorphism). However, the constant term of the power series $(\alpha[[t]])(u)$ is $\alpha(u_0)$ (since $(\alpha[[t]])(u) = \sum_{n\geq 0} \alpha(u_n) t^n$). In other words, the constant term of the power series $(\alpha[[t]])(u)$ is 1 (since $\alpha(u_0) = 1$). Hence, $\operatorname{lder}((\alpha[[t]])(u))$ is well-defined. The definition of $\operatorname{lder}((\alpha[[t]])(u))$ yields

$$\operatorname{lder}((\alpha[[t]])(u)) = \frac{((\alpha[[t]])(u))'}{(\alpha[[t]])(u)}. \tag{170}$$

Now, recall that $(\alpha[[t]])(u) = \sum_{n\geq 0} \alpha(u_n) t^n$ with $\alpha(u_0), \alpha(u_1), \alpha(u_2), \ldots \in S$. Hence, the definition of the derivative of a power series yields

$$\begin{aligned} ((\alpha[[t]])(u))' &= \sum_{n\geq 1} n\alpha(u_n) t^{n-1} \\ &= \sum_{n\geq 0} (n+1)\alpha(u_{n+1}) t^n \end{aligned} \tag{171}$$

(here, we have substituted $n+1$ for n in the sum). On the other hand, from $u = \sum_{n\geq 0} u_n t^n$, we obtain

$$\begin{aligned} u' &= \sum_{n\geq 1} n u_n t^{n-1} \qquad \text{(by the definition of the derivative)} \\ &= \sum_{n\geq 0} (n+1) u_{n+1} t^n \end{aligned}$$

(here, we have substituted $n+1$ for n in the sum). Applying the map $\alpha[[t]]$ to both sides of this equality, we obtain

$$\begin{aligned} (\alpha[[t]])(u') &= (\alpha[[t]])\left(\sum_{n\geq 0} (n+1) u_{n+1} t^n\right) = \sum_{n\geq 0} \underbrace{\alpha((n+1) u_{n+1})}_{\substack{=(n+1)\alpha(u_{n+1}) \\ \text{(since the map } \alpha \\ \text{is } \mathbf{k}\text{-linear)}}} t^n \\ &\qquad \text{(by the definition of } \alpha[[t]]) \\ &= \sum_{n\geq 0} (n+1)\alpha(u_{n+1}) t^n. \end{aligned}$$

Comparing this with (171), we obtain

$$((\alpha [[t]]) (u))' = (\alpha [[t]]) (u') .$$

Hence, (170) rewrites as

$$\operatorname{lder} ((\alpha [[t]]) (u)) = \frac{(\alpha [[t]]) (u')}{(\alpha [[t]]) (u)}. \tag{172}$$

On the other hand, the definition of $\operatorname{lder} u$ yields $\operatorname{lder} u = \dfrac{u'}{u}$. Applying the map $\alpha [[t]]$ to both sides of this equality, we obtain

$$(\alpha [[t]]) (\operatorname{lder} u) = (\alpha [[t]]) \left(\frac{u'}{u}\right) = \frac{(\alpha [[t]]) (u')}{(\alpha [[t]]) (u)}$$

(since the map $\alpha [[t]]$ is a $\mathbf{k} [[t]]$-algebra homomorphism and thus respects quotients). Comparing this with (172), we obtain $\operatorname{lder} ((\alpha [[t]]) (u)) = (\alpha [[t]]) (\operatorname{lder} u)$. This completes the proof of Proposition 5.32. $\square$

Next, we shall prove a simple property of homogeneous power series:

Lemma 5.33. Consider the ring $(\mathbf{k} [[x_1, x_2, x_3, \ldots]]) [[t]]$ of formal power series in one indeterminate t over $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$. Let $n \in \mathbb{N}$. Let $u \in \mathbf{k} [[x_1, x_2, x_3, \ldots]]$ be any power series that is homogeneous of degree n. Then,

$$u (tx_1, tx_2, tx_3, \ldots) = t^n \cdot u.$$

Proof of Lemma 5.33. The power series u is homogeneous of degree n. In other words, it can be written as an infinite $\mathbf{k}$-linear combination of monomials of degree n. In other words, it can be written in the form

$$u = \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{of degree } n}} u_{\mathfrak{m}} \mathfrak{m} \tag{173}$$

for some coefficients $u_{\mathfrak{m}} \in \mathbf{k}$. Consider these coefficients $u_{\mathfrak{m}}$. Substituting $tx_1, tx_2, tx_3, \ldots$ for $x_1, x_2, x_3, \ldots$ on both sides of the equality (173), we obtain

$$u (tx_1, tx_2, tx_3, \ldots) = \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{of degree } n}} u_{\mathfrak{m}} \mathfrak{m} (tx_1, tx_2, tx_3, \ldots) . \tag{174}$$

However, if $\mathfrak{m}$ is a monomial of degree n, then

$$\mathfrak{m} (tx_1, tx_2, tx_3, \ldots) = t^n \cdot \mathfrak{m}. \tag{175}$$

[*Proof of (175):* Let $\mathfrak{m}$ be a monomial of degree n. Thus, $\mathfrak{m}$ is a product of n indeterminates. In other words, $\mathfrak{m} = x_{i_1} x_{i_2} \cdots x_{i_n}$ for some $i_1, i_2, \ldots, i_n \in \{1, 2, 3, \ldots\}$.

Consider these $i_1, i_2, \ldots, i_n$. Substituting $tx_1, tx_2, tx_3, \ldots$ for $x_1, x_2, x_3, \ldots$ on both sides of the equality $\mathfrak{m} = x_{i_1} x_{i_2} \cdots x_{i_n}$, we obtain

$$\mathfrak{m}\left(tx_1, tx_2, tx_3, \ldots\right) = \left(tx_{i_1}\right)\left(tx_{i_2}\right)\cdots\left(tx_{i_n}\right) = t^n \cdot \underbrace{x_{i_1} x_{i_2} \cdots x_{i_n}}_{=\mathfrak{m}} = t^n \cdot \mathfrak{m}.$$

This proves (175).]

Hence, (174) becomes

$$u\left(tx_1, tx_2, tx_3, \ldots\right) = \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{of degree } n}} u_{\mathfrak{m}} \underbrace{\mathfrak{m}\left(tx_1, tx_2, tx_3, \ldots\right)}_{\substack{=t^n \cdot \mathfrak{m} \\ \text{(by (175))}}} = \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{of degree } n}} u_{\mathfrak{m}} t^n \cdot \mathfrak{m}.$$

Comparing this with

$$t^n \cdot \underbrace{u}_{= \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{of degree } n}} u_{\mathfrak{m}} \mathfrak{m}} = t^n \cdot \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{of degree } n}} u_{\mathfrak{m}} \mathfrak{m} = \sum_{\substack{\mathfrak{m} \text{ is a monomial} \\ \text{of degree } n}} u_{\mathfrak{m}} t^n \cdot \mathfrak{m},$$

we obtain $u\left(tx_1, tx_2, tx_3, \ldots\right) = t^n \cdot u$. This proves Lemma 5.33. □

Proof of Theorem 5.25. **(b)** Let $m \in \mathbb{N}$. We must prove that $V_F\left(h_m\right) = G_{F,m}$.

We have $h_0 = 1$ and thus $V_F\left(h_0\right) = V_F\left(1\right) = 1$ (since V_F is a $\mathbf{k}$-algebra homomorphism). Comparing this with $G_{F,0} = 1$ (which was proved in Proposition 5.6 **(e)**), we obtain $V_F\left(h_0\right) = G_{F,0}$. Hence, $V_F\left(h_m\right) = G_{F,m}$ is proved for $m = 0$. Thus, for the rest of this proof, we WLOG assume that $m \neq 0$.

Now, m is a positive integer (since $m \in \mathbb{N}$ and $m \neq 0$). However, the definition of V_F says that $V_F\left(h_i\right) = G_{F,i}$ for all positive integers i. We can apply this to $i = m$ (since m is a positive integer), and thus obtain $V_F\left(h_m\right) = G_{F,m}$. Thus, Theorem 5.25 **(b)** is proven.

(a) We recall that the family $\left(h_n\right)_{n\geq 1}$ generates Λ as a $\mathbf{k}$-algebra. Hence, any two $\mathbf{k}$-algebra homomorphisms with domain Λ that agree on this family $\left(h_n\right)_{n\geq 1}$ must be identical. In other words, if A is any $\mathbf{k}$-algebra, and if $f : \Lambda \to A$ and $g : \Lambda \to A$ are two $\mathbf{k}$-algebra homomorphisms such that

$$\left(f\left(h_n\right) = g\left(h_n\right) \qquad \text{for each positive integer } n\right),$$

then

$$f = g. \tag{176}$$

The map V_F is a $\mathbf{k}$-algebra homomorphism. Hence, the map $V_F \otimes V_F$ is a $\mathbf{k}$-algebra homomorphism as well (since the tensor product of two $\mathbf{k}$-algebra homomorphisms is always a $\mathbf{k}$-algebra homomorphism).

Let Δ and ε be the comultiplication and the counit of the Hopf algebra Λ. Both of these maps Δ and ε are $\mathbf{k}$-algebra homomorphisms (since Λ is a bialgebra). Hence, the three maps $\Delta \circ V_F$ and $\left(V_F \otimes V_F\right) \circ \Delta$ and $\varepsilon \circ V_F$ are $\mathbf{k}$-algebra homomorphisms as well (since these three maps are compositions of some of the **k**-algebra homomorphisms Δ and ε and V_F and $V_F \otimes V_F$).

Now, let n be a positive integer. Then, [GriRei20, Proposition 2.3.6(iii)] yields

$$\begin{aligned}
\Delta\left(h_n\right) &= \sum_{i+j=n} h_i \otimes h_j \\
&\qquad \left(\text{where the sum ranges over all pairs } (i,j) \in \mathbb{N} \times \mathbb{N} \text{ with } i+j=n\right) \\
&= \sum_{i \in \{0,1,\ldots,n\}} h_i \otimes h_{n-i}
\end{aligned}$$

(here, we have substituted $(i, n-i)$ for (i,j) in the sum, since the map $\{0,1,\ldots,n\} \to \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid i+j=n\}$ that sends each i to $(i, n-i)$ is a bijection). Applying the map $V_F \otimes V_F$ to both sides of this equality, we find

$$\begin{aligned}
\left(V_F \otimes V_F\right)\left(\Delta\left(h_n\right)\right) &= \left(V_F \otimes V_F\right)\left(\sum_{i \in \{0,1,\ldots,n\}} h_i \otimes h_{n-i}\right) = \sum_{i \in \{0,1,\ldots,n\}} \underbrace{\left(V_F \otimes V_F\right)\left(h_i \otimes h_{n-i}\right)}_{=V_F(h_i) \otimes V_F(h_{n-i})} \\
&\qquad \left(\text{since the map } V_F \otimes V_F \text{ is } \mathbf{k}\text{-linear}\right) \\
&= \underbrace{\sum_{i \in \{0,1,\ldots,n\}}}_{=\sum_{i=0}^{n}} \underbrace{V_F\left(h_i\right)}_{\substack{=G_{F,i} \\ \text{(by Theorem 5.25 (b),} \\ \text{applied to } m=i)}} \otimes \underbrace{V_F\left(h_{n-i}\right)}_{\substack{=G_{F,n-i} \\ \text{(by Theorem 5.25 (b),} \\ \text{applied to } m=n-i)}} \\
&= \sum_{i=0}^{n} G_{F,i} \otimes G_{F,n-i}.
\end{aligned}$$

Comparing this with

$$\begin{aligned}
\left(\Delta \circ V_F\right)\left(h_n\right) &= \Delta\left(\underbrace{V_F\left(h_n\right)}_{\substack{=G_{F,n} \\ \text{(by Theorem 5.25 (b),} \\ \text{applied to } m=n)}}\right) = \Delta\left(G_{F,n}\right) \\
&= \sum_{i=0}^{n} G_{F,i} \otimes G_{F,n-i} \qquad \left(\text{by Theorem 5.18, applied to } m=n\right),
\end{aligned}$$

we obtain

$$\left(\Delta \circ V_F\right)\left(h_n\right) = \left(V_F \otimes V_F\right)\left(\Delta\left(h_n\right)\right) = \left(\left(V_F \otimes V_F\right) \circ \Delta\right)\left(h_n\right).$$

Now, forget that we fixed n. We thus have shown that $\left(\Delta \circ V_F\right)\left(h_n\right) = \left(\left(V_F \otimes V_F\right) \circ \Delta\right)\left(h_n\right)$ for each positive integer n. Since $\Delta \circ V_F$ and $\left(V_F \otimes V_F\right) \circ \Delta$ are **k**-algebra homomorphisms, we thus conclude that

$$\Delta \circ V_F = \left(V_F \otimes V_F\right) \circ \Delta \tag{177}$$

(by (176), applied to $A = \Lambda \otimes \Lambda$ and $f = \Delta \circ V_F$ and $g = (V_F \otimes V_F) \circ \Delta$).

Again, let n be a positive integer. Recall that the counit ε of Λ sends every homogeneous symmetric function of positive degree to 0. In other words, if $u \in \Lambda$ is homogeneous of positive degree, then

$$\varepsilon (u) = 0. \tag{178}$$

The complete homogeneous symmetric function h_n is homogeneous of degree n, thus homogeneous of positive degree (since n is positive). Hence, (178) (applied to $u = h_n$) yields $\varepsilon (h_n) = 0$.

Theorem 5.25 **(b)** (applied to $m = n$) yields $V_F (h_n) = G_{F,n}$. Proposition 5.6 **(a)** (applied to $m = n$) shows that the formal power series $G_{F,n}$ is the n-th degree homogeneous component of G_F. Hence, this $G_{F,n}$ is homogeneous of degree n. Thus, $G_{F,n}$ is homogeneous of positive degree (since n is positive). In other words, $V_F (h_n)$ is homogeneous of positive degree (since $V_F (h_n) = G_{F,n}$). Since $V_F (h_n)$ is clearly a symmetric function, we thus conclude that $\varepsilon (V_F (h_n)) = 0$ (by (178), applied to $u = V_F (h_n)$). Thus, $(\varepsilon \circ V_F) (h_n) = \varepsilon (V_F (h_n)) = 0$. Comparing this with $\varepsilon (h_n) = 0$, we find $(\varepsilon \circ V_F) (h_n) = \varepsilon (h_n)$.

Now, forget that we fixed n. We thus have shown that $(\varepsilon \circ V_F) (h_n) = \varepsilon (h_n)$ for each positive integer n. Since $\varepsilon \circ V_F$ and ε are $\mathbf{k}$-algebra homomorphisms, we thus conclude that

$$\varepsilon \circ V_F = \varepsilon \tag{179}$$

(by (176), applied to $A = \mathbf{k}$ and $f = \varepsilon \circ V_F$ and $g = \varepsilon$).

The two equalities (177) and (179) show that V_F is a $\mathbf{k}$-coalgebra homomorphism (since V_F is $\mathbf{k}$-linear). Since we also know that V_F is a $\mathbf{k}$-algebra homomorphism, we thus conclude that this map V_F is a $\mathbf{k}$-bialgebra homomorphism. Hence, V_F is a $\mathbf{k}$-Hopf algebra homomorphism[89]. This proves Theorem 5.25 **(a)**.

(c) Let $m \in \mathbb{N}$. Then, Proposition 5.6 **(a)** shows that the formal power series $G_{F,m}$ is the m-th degree homogeneous component of G_F. Hence, this $G_{F,m}$ is homogeneous of degree m.

Forget that we fixed m. We thus have shown that $G_{F,m}$ is homogeneous of degree m for each $m \in \mathbb{N}$.

Now, consider the ring $(\mathbf{k} [[x_1, x_2, x_3, \ldots]]) [[t]]$ of formal power series in one indeterminate t over $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$. This ring has a subring $\Lambda [[t]]$ that consists of those formal power series whose coefficients belong to Λ. We consider $(\mathbf{k} [[x_1, x_2, x_3, \ldots]]) [[t]]$ as a topological ring, where the topology is the standard one induced by the standard topology on $\mathbf{k} [[x_1, x_2, x_3, \ldots]]$ (not the discrete topology!). This topological ring $(\mathbf{k} [[x_1, x_2, x_3, \ldots]]) [[t]]$ is, of course, isomorphic to $\mathbf{k} [[x_1, x_2, x_3, \ldots, t]]$.

Now, for each $m \in \mathbb{N}$, we know that $G_{F,m}$ is homogeneous of degree m, and therefore we conclude that

$$G_{F,m} (tx_1, tx_2, tx_3, \ldots) = t^m \cdot G_{F,m} \tag{180}$$

[89] since any $\mathbf{k}$-bialgebra homomorphism between two $\mathbf{k}$-Hopf algebras is automatically a $\mathbf{k}$-Hopf algebra homomorphism

(by Lemma 5.33, applied to $u = G_{F,m}$ and $n = m$).

On the other hand,

$$G_F = \sum_{m \in \mathbb{N}} G_{F,m}$$

(as we have seen in our proof of Proposition 5.6 **(a)**). Comparing this with

$$G_F = \prod_{i=1}^{\infty} F(x_i) \qquad \text{(by Proposition 5.6 } \textbf{(b)}\text{)},$$

we obtain

$$\prod_{i=1}^{\infty} F(x_i) = \sum_{m \in \mathbb{N}} G_{F,m}.$$

Substituting $tx_1, tx_2, tx_3, \ldots$ for $x_1, x_2, x_3, \ldots$ on both sides of this equality, we obtain

$$\prod_{i=1}^{\infty} F(tx_i) = \sum_{m \in \mathbb{N}} \underbrace{G_{F,m}(tx_1, tx_2, tx_3, \ldots)}_{\substack{= t^m \cdot G_{F,m} \\ \text{(by (180))}}} = \sum_{m \in \mathbb{N}} t^m \cdot G_{F,m}. \tag{181}$$

The map $V_F : \Lambda \to \Lambda$ is a $\mathbf{k}$-algebra homomorphism. Hence, it induces a $\mathbf{k}[[t]]$-algebra homomorphism

$$V_F[[t]] : \Lambda[[t]] \to \Lambda[[t]]$$

that sends each formal power series $\sum_{n \geq 0} a_n t^n \in \Lambda[[t]]$ (with $a_n \in \Lambda$) to $\sum_{n \geq 0} V_F(a_n) t^n$. Consider this $\mathbf{k}[[t]]$-algebra homomorphism $V_F[[t]]$. (Note that $V_F[[t]]$ is continuous with respect to an appropriate topology on $\Lambda[[t]]$, but we shall not use this fact.)

Define the formal power series

$$H(t) = \prod_{i=1}^{\infty} (1 - x_i t)^{-1} \in (\mathbf{k}[[x_1, x_2, x_3, \ldots]])[[t]].$$

Then, from [GriRei20, (2.4.1)], we know that

$$H(t) = \sum_{n \geq 0} \underbrace{h_n(\mathbf{x})}_{= h_n} t^n = \sum_{n \geq 0} h_n t^n \in \Lambda[[t]].$$

Hence, $(V_F[[t]])(H(t))$ is well-defined. Moreover, $H(t) = \sum_{n \geq 0} h_n t^n$ shows that the constant term of $H(t)$ is $h_0 = 1$. Thus, $\operatorname{lder}(H(t))$ is well-defined.

Applying the map $V_F[[t]]$ to both sides of the equality $H(t) = \sum_{n\geq 0} h_n t^n$, we obtain

$$\begin{aligned}
(V_F[[t]])(H(t)) &= (V_F[[t]])\left(\sum_{n\geq 0} h_n t^n\right) = \underbrace{\sum_{n\geq 0}}_{=\sum_{n\in\mathbb{N}}} \underbrace{V_F(h_n)}_{\substack{=G_{F,n} \\ \text{(by Theorem 5.25 (b),} \\ \text{applied to } m=n)}} t^n \qquad \text{(by the definition of } V_F[[t]]) \\
&= \sum_{n\in\mathbb{N}} \underbrace{G_{F,n}t^n}_{=t^n\cdot G_{F,n}} = \sum_{n\in\mathbb{N}} t^n \cdot G_{F,n} = \sum_{m\in\mathbb{N}} t^m \cdot G_{F,m}
\end{aligned}$$

(here, we have renamed the summation index n as m). Comparing this with (181), we find

$$(V_F[[t]])(H(t)) = \prod_{i=1}^{\infty} F(tx_i). \tag{182}$$

From [GriRei20, Exercise 2.5.21], we know that

$$\sum_{m\geq 0} p_{m+1}t^m = \frac{H'(t)}{H(t)}.$$

On the other hand, the definition of $\operatorname{lder}(H(t))$ yields

$$\operatorname{lder}(H(t)) = \frac{H'(t)}{H(t)}.$$

Comparing these two equalities, we obtain

$$\operatorname{lder}(H(t)) = \sum_{m\geq 0} p_{m+1}t^m = \sum_{n\geq 0} p_{n+1}t^n$$

(here, we have renamed the summation index m as n). Applying the map $V_F[[t]]$ to both sides of this equality, we find

$$\begin{aligned}
(V_F[[t]])(\operatorname{lder}(H(t))) &= (V_F[[t]])\left(\sum_{n\geq 0} p_{n+1}t^n\right) = \underbrace{\sum_{n\geq 0}}_{=\sum_{n\in\mathbb{N}}} V_F(p_{n+1})\,t^n \qquad \text{(by the definition of } V_F[[t]]) \\
&= \sum_{n\in\mathbb{N}} V_F(p_{n+1})\,t^n.
\end{aligned} \tag{183}$$

On the other hand, the constant term of $H(t)$ is 1 (as we have shown above). Hence, Proposition 5.32 (applied to $R = \Lambda$ and $S = \Lambda$ and $\alpha = V_F$ and $u = H(t)$) shows that the constant term of the power series $(V_F[[t]])(H(t))$ is 1, and that we have

$$\operatorname{lder}((V_F[[t]])(H(t))) = (V_F[[t]])(\operatorname{lder}(H(t))). \tag{184}$$

Now, (183) yields

$$\sum_{n\in\mathbb{N}} V_F\left(p_{n+1}\right)t^n = \left(V_F\left[[t]\right]\right)\left(\operatorname{lder}\left(H\left(t\right)\right)\right)$$
$$= \operatorname{lder}\left(\underbrace{\left(V_F\left[[t]\right]\right)\left(H\left(t\right)\right)}_{\substack{=\prod_{i=1}^{\infty} F(tx_i)\\ \text{(by (182))}}}\right) \qquad \text{(by (184))}$$
$$= \operatorname{lder}\left(\prod_{i=1}^{\infty} F\left(tx_i\right)\right). \tag{185}$$

Now, $F\left(tx_1\right), F\left(tx_2\right), F\left(tx_3\right), \ldots$ are infinitely many formal power series in t over the ring $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ whose constant terms are 1 [90]. The infinite product $\prod_{i=1}^{\infty} F\left(tx_i\right)$ converges (as we know from (181)). Hence, Proposition 5.30 (applied to $R = \mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ and $u_i = F\left(tx_i\right)$) yields that the infinite sum $\sum_{i=1}^{\infty} \operatorname{lder}\left(F\left(tx_i\right)\right)$ converges as well, and that we have

$$\operatorname{lder}\left(\prod_{i=1}^{\infty} F\left(tx_i\right)\right) = \sum_{i=1}^{\infty} \operatorname{lder}\left(F\left(tx_i\right)\right).$$

Hence, (185) becomes

$$\sum_{n\in\mathbb{N}} V_F\left(p_{n+1}\right)t^n = \operatorname{lder}\left(\prod_{i=1}^{\infty} F\left(tx_i\right)\right) = \sum_{i=1}^{\infty} \operatorname{lder}\left(F\left(\underbrace{tx_i}_{=x_i t}\right)\right)$$
$$= \sum_{i=1}^{\infty} \underbrace{\operatorname{lder}\left(F\left(x_i t\right)\right)}_{\substack{=x_i\cdot(\operatorname{lder} F)(x_i t)\\ \text{(by Proposition 5.31,}\\ \text{applied to } R=\mathbf{k}[[x_1,x_2,x_3,\ldots]] \text{ and } u=F \text{ and } \lambda=x_i)}}$$
$$= \sum_{i=1}^{\infty} x_i \cdot \left(\operatorname{lder} F\right)\left(x_i t\right). \tag{186}$$

[90] *Proof.* Let $i \in \{1, 2, 3, \ldots\}$. We must prove that $F\left(tx_i\right)$ is a formal power series in t over the ring $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ whose constant term is 1.

Recall that $F = \sum_{n\in\mathbb{N}} f_n t^n$. Substituting tx_i for t in this equality, we find $F\left(tx_i\right) = \sum_{n\in\mathbb{N}} f_n \underbrace{\left(tx_i\right)^n}_{=t^n x_i^n = x_i^n t^n} = \sum_{n\in\mathbb{N}} f_n x_i^n t^n$. Hence, $F\left(tx_i\right)$ is a formal power series in t over the ring $\mathbf{k}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ whose constant term is $f_0 \underbrace{x_i^0}_{=1} = f_0 = 1$. This is exactly what we wanted to prove. Thus, our proof is complete.

Now, let $i \in \{1,2,3,\ldots\}$ be arbitrary. The definition of $\operatorname{lder} F$ yields

$$\operatorname{lder} F = \frac{F'}{F} = \sum_{n\in\mathbb{N}} \gamma_n t^n.$$

Substituting $x_i t$ for t on both sides of this equality, we obtain

$$(\operatorname{lder} F)(x_i t) = \sum_{n\in\mathbb{N}} \gamma_n \underbrace{(x_i t)^n}_{=x_i^n t^n} = \sum_{n\in\mathbb{N}} \gamma_n x_i^n t^n. \tag{187}$$

Forget that we fixed i. We thus have proved (187) for each $i \in \{1,2,3,\ldots\}$. Now, (186) becomes

$$\sum_{n\in\mathbb{N}} V_F(p_{n+1}) t^n = \sum_{i=1}^{\infty} x_i \cdot \underbrace{(\operatorname{lder} F)(x_i t)}_{\substack{=\sum\limits_{n\in\mathbb{N}} \gamma_n x_i^n t^n \\ \text{(by (187))}}} = \sum_{i=1}^{\infty} x_i \cdot \sum_{n\in\mathbb{N}} \gamma_n x_i^n t^n = \underbrace{\sum_{i=1}^{\infty} \sum_{n\in\mathbb{N}}}_{=\sum\limits_{n\in\mathbb{N}} \sum\limits_{i=1}^{\infty}} \underbrace{x_i \gamma_n x_i^n}_{=\gamma_n x_i^{n+1}} t^n$$

$$= \sum_{n\in\mathbb{N}} \sum_{i=1}^{\infty} \gamma_n x_i^{n+1} t^n = \sum_{n\in\mathbb{N}} \gamma_n \underbrace{\left(\sum_{i=1}^{\infty} x_i^{n+1}\right)}_{\substack{=x_1^{n+1}+x_2^{n+1}+x_3^{n+1}+\cdots \\ =p_{n+1} \\ \text{(by the definition of } p_{n+1})}} t^n = \sum_{n\in\mathbb{N}} \gamma_n p_{n+1} t^n.$$

Comparing coefficients before t^n in this equality, we conclude that

$$V_F(p_{n+1}) = \gamma_n p_{n+1} \qquad \text{for each } n \in \mathbb{N}. \tag{188}$$

Now, let n be a positive integer. Then, $n-1 \in \mathbb{N}$. Hence, (188) (applied to $n-1$ instead of n) yields

$$V_F\left(p_{(n-1)+1}\right) = \gamma_{n-1} p_{(n-1)+1}.$$

In other words, $V_F(p_n) = \gamma_{n-1} p_n$ (since $(n-1)+1 = 1$). This proves Theorem 5.25 **(c)**. ☐

Our next (and last) few results are not generalizations of any properties of Petrie functions. To state them, we take a somewhat more high-level point of view. We forget that we fixed the power series F. Instead, for **every** power series $F \in \mathbf{k}[[t]]$ whose constant term is 1, we define a power series G_F according to Definition 5.4 **(d)**. Moreover, for **every** power series $F \in \mathbf{k}[[t]]$ whose constant term is 1, and for **every** $m \in \mathbb{N}$, we define a power series $G_{F,m}$ according to Definition 5.4 **(e)**. We then have the following:

Proposition 5.34. Let A and B be two power series in $\mathbf{k}[[t]]$ whose constant terms are 1. Then:

(a) We have $G_{AB} = G_A G_B$.

(b) Let $n \in \mathbb{N}$. We have $G_{AB,n} = \sum\limits_{i=0}^{n} G_{A,i} G_{B,n-i}$.

Proof of Proposition 5.34. The constant terms of the power series A and B are 1 (by assumption). Hence, the constant term of the power series AB is 1 as well (since the constant term of the product of two power series equals the product of their constant terms). Thus, G_{AB} is well-defined.

(a) Proposition 5.6 **(b)** yields that

$$G_F = \prod_{i=1}^{\infty} F(x_i) \tag{189}$$

for any power series $F \in \mathbf{k}[[t]]$ whose constant term is 1.

Recall that the constant term of the power series AB is 1. Hence, (189) (applied to $F = AB$) yields

$$G_{AB} = \prod_{i=1}^{\infty} \underbrace{(AB)(x_i)}_{=A(x_i)B(x_i)} = \prod_{i=1}^{\infty} (A(x_i) B(x_i)). \tag{190}$$

On the other hand, (189) (applied to $F = A$) yields

$$G_A = \prod_{i=1}^{\infty} A(x_i).$$

Moreover, (189) (applied to $F = B$) yields

$$G_B = \prod_{i=1}^{\infty} B(x_i).$$

Multiplying these two equalities, we find

$$G_A G_B = \left(\prod_{i=1}^{\infty} A(x_i)\right)\left(\prod_{i=1}^{\infty} B(x_i)\right) = \prod_{i=1}^{\infty} (A(x_i) B(x_i)).$$

Comparing this with (190), we find $G_{AB} = G_A G_B$. This proves Proposition 5.34 **(a)**.

(b) Forget that we fixed n.

In the proof of Proposition 5.6 **(a)**, we have shown that

$$G_F = \sum_{m \in \mathbb{N}} G_{F,m} \tag{191}$$

for any power series $F \in \mathbf{k}[[t]]$ whose constant term is 1. (Indeed, this is precisely the equality (132).)

Recall that the constant term of the power series A is 1. Hence, (191) (applied to $F = A$) yields

$$G_A = \sum_{m \in \mathbb{N}} G_{A,m} = \sum_{i \in \mathbb{N}} G_{A,i} \tag{192}$$

(here, we have renamed the summation index m as i).

Furthermore, recall that the constant term of the power series B is 1. Hence, (191) (applied to $F = B$) yields

$$G_B = \sum_{m\in\mathbb{N}} G_{B,m} = \sum_{j\in\mathbb{N}} G_{B,j} \tag{193}$$

(here, we have renamed the summation index m as j).

Now, Proposition 5.34 **(a)** yields

$$\begin{aligned} G_{AB} &= G_A G_B = \left(\sum_{i\in\mathbb{N}} G_{A,i}\right)\left(\sum_{j\in\mathbb{N}} G_{B,j}\right) \qquad \text{(by (192) and (193))} \\ &= \underbrace{\underbrace{\sum_{i\in\mathbb{N}}\sum_{j\in\mathbb{N}}}_{=\sum_{(i,j)\in\mathbb{N}\times\mathbb{N}}}}_{=\sum_{n\in\mathbb{N}}\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}}} G_{A,i}G_{B,j} = \sum_{n\in\mathbb{N}}\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{A,i}G_{B,j}. \end{aligned} \tag{194}$$

If $n \in \mathbb{N}$, then the power series $\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{A,i}G_{B,j} \in \mathbf{k}[[x_1, x_2, x_3, \ldots]]$ is homogeneous of degree n [91]. Thus, the equality (194) reveals that the family

$$\left(\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{A,i}G_{B,j}\right)_{n\in\mathbb{N}}$$

is the homogeneous decomposition of G_{AB} (by the definition of a homogeneous

[91] *Proof.* Let $n \in \mathbb{N}$. We must prove that the power series $\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{A,i}G_{B,j}$ is homogeneous of degree n.

Let $(i,j) \in \mathbb{N}\times\mathbb{N}$ be such that $i + j = n$. Then, Proposition 5.6 **(a)** (applied to $m = i$ and $F = A$) shows that the formal power series $G_{A,i}$ is the i-th degree homogeneous component of G_A. Hence, this formal power series $G_{A,i}$ is homogeneous of degree i.

Moreover, Proposition 5.6 **(a)** (applied to $m = j$ and $F = B$) shows that the formal power series $G_{B,j}$ is the j-th degree homogeneous component of G_B. Hence, this formal power series $G_{B,j}$ is homogeneous of degree j.

Now we have shown that the two power series $G_{A,i}$ and $G_{B,j}$ are homogeneous of degrees i and j, respectively. Thus, their product $G_{A,i}G_{B,j}$ is homogeneous of degree $i + j$. In other words, $G_{A,i}G_{B,j}$ is homogeneous of degree n (since $i + j = n$).

Forget that we fixed (i,j). We thus have shown that $G_{A,i}G_{B,j}$ is homogeneous of degree n whenever $(i,j) \in \mathbb{N}\times\mathbb{N}$ satisfies $i + j = n$. In other words, each addend of the sum $\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{A,i}G_{B,j}$ is homogeneous of degree n. Hence, the entire sum $\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{A,i}G_{B,j}$ is homogeneous of degree n as well. This completes our proof.

decomposition). Hence, for each $n \in \mathbb{N}$, we have

$$\sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{A,i}G_{B,j} = (\text{the } n\text{-th degree homogeneous component of } G_{AB}). \tag{195}$$

Now, let $n \in \mathbb{N}$. Recall that the constant term of the power series AB is 1. Hence, Proposition 5.6 **(a)** (applied to $F = AB$ and $m = n$) yields that the formal power series $G_{AB,n}$ is the n-th degree homogeneous component of G_{AB}. In other words,

$$G_{AB,n} = (\text{the } n\text{-th degree homogeneous component of } G_{AB}).$$

Comparing this with (195), we obtain

$$\begin{aligned} G_{AB,n} &= \sum_{\substack{(i,j)\in\mathbb{N}\times\mathbb{N};\\ i+j=n}} G_{A,i}G_{B,j} = \sum_{i\in\{0,1,\ldots,n\}} G_{A,i}G_{B,n-i} \\ &\qquad \left(\begin{array}{c}\text{here, we have substituted } (i,n-i) \text{ for } (i,j) \text{ in the sum,}\\ \text{since the map } \{0,1,\ldots,n\} \to \{(i,j)\in\mathbb{N}\times\mathbb{N} \mid i+j=n\}\\ \text{that sends each } i \text{ to } (i,n-i) \text{ is a bijection}\end{array}\right) \\ &= \sum_{i=0}^{n} G_{A,i}G_{B,n-i}. \end{aligned}$$

This proves Proposition 5.34 **(b)**. □

Finally, we can express the image of the symmetric function $G_{F,n}$ under the antipode of Λ (a result suggested by Sasha Postnikov):

Theorem 5.35. Let S be the antipode of the Hopf algebra Λ. Let $F \in \mathbf{k}[[t]]$ be a formal power series whose constant term is 1. Then, for each $n \in \mathbb{N}$, we have

$$S(G_{F,n}) = G_{F^{-1},n}. \tag{196}$$

Our proof of this theorem will rely on the following simple lemma[92]:

Lemma 5.36. For any $n \in \mathbb{N}$, we have $G_{1,n} = [n = 0]$. (Here, the "1" in "$G_{1,n}$" means the constant power series $1 \in \mathbf{k}[[t]]$.)

Proof of Lemma 5.36. Let F be the constant power series $1 \in \mathbf{k}[[t]]$. Then, the constant term of F is 1; thus, $G_{F,n}$ is well-defined for each $n \in \mathbb{N}$.

[92] We are using Convention 2.4 again.

We shall use the notations introduced in Definition 5.4. Thus, $F = \sum_{n\in\mathbb{N}} f_n t^n$. On the other hand,

$$\sum_{n\in\mathbb{N}} [n=0]\, t^n = \underbrace{[0=0]}_{\substack{=1\\ \text{(since } 0=0)}} \underbrace{t^0}_{=1} + \sum_{\substack{n\in\mathbb{N};\\ n\neq 0}} \underbrace{[n=0]}_{\substack{=0\\ \text{(since } n\neq 0)}} t^n = 1 + \underbrace{\sum_{\substack{n\in\mathbb{N};\\ n\neq 0}} 0t^n}_{=0} = 1 = F$$

(since $F=1$), so that

$$\sum_{n\in\mathbb{N}} [n=0]\, t^n = F = \sum_{n\in\mathbb{N}} f_n t^n.$$

This is an equality of power series. Comparing coefficients in front of t^n in this equality, we thus obtain

$$[n=0] = f_n \qquad \text{for each } n\in\mathbb{N}. \tag{197}$$

Now, let $n\in\mathbb{N}$. We must prove that $G_{1,n} = [n=0]$. If $n=0$, then this is obvious[93]. Thus, for the rest of this proof, we WLOG assume that $n\neq 0$. Hence, we don't have $n=0$. Thus, we have $[n=0]=0$. On the other hand, we have

$$f_\alpha = 0 \qquad \text{for any } \alpha\in\mathrm{WC} \text{ satisfying } |\alpha| = n \tag{198}$$

[94]. Now, the definition of $G_{F,n}$ yields

$$G_{F,n} = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n}} \underbrace{f_\alpha}_{\substack{=0\\ \text{(by (198))}}} \mathbf{x}^\alpha = \sum_{\substack{\alpha\in\mathrm{WC};\\ |\alpha|=n}} 0\mathbf{x}^\alpha = 0.$$

In view of $F=1$, this rewrites as $G_{1,n}=0$. Comparing this with $[n=0]=0$, we obtain $G_{1,n}=[n=0]$. Thus, Lemma 5.36 is proven. □

[93] *Proof.* Assume that $n=0$. Thus, $G_{1,n} = G_{1,0} = 1$ (by Proposition 5.6 **(e)**, applied to 1 instead of F). However, from $n=0$, we obtain $[n=0]=1$. Comparing this with $G_{1,n}=1$, we obtain $G_{1,n}=[n=0]$. Thus, we have proved that $G_{1,n}=[n=0]$ under the assumption that $n=0$.

[94] *Proof of (198):* Let $\alpha\in\mathrm{WC}$ satisfy $|\alpha|=n$.

If we had $\alpha=\varnothing$, then we would have $|\alpha| = |\varnothing| = 0$, which would contradict $|\alpha| = n\neq 0$. Hence, we cannot have $\alpha=\varnothing$. Thus, we have $\alpha\neq\varnothing$. Now, $\alpha = (\alpha_1,\alpha_2,\alpha_3,\ldots)$, so that

$$(\alpha_1,\alpha_2,\alpha_3,\ldots) = \alpha \neq \varnothing = (0,0,0,\ldots).$$

In other words, there exists some $i\in\{1,2,3,\ldots\}$ such that $\alpha_i\neq 0$. Consider this i.

We have $\alpha_i\neq 0$. Thus, we don't have $\alpha_i=0$. Hence, we have $[\alpha_i=0]=0$. Now, (197) (applied to α_i instead of n) yields $[\alpha_i=0]=f_{\alpha_i}$. Hence, $f_{\alpha_i} = [\alpha_i=0] = 0$. Thus, we have shown that f_{α_i} is equal to 0.

Now, the definition of f_α yields

$$\begin{aligned} f_\alpha &= f_{\alpha_1} f_{\alpha_2} f_{\alpha_3}\cdots = \underbrace{\left(f_{\alpha_1} f_{\alpha_2}\cdots f_{\alpha_i}\right)}_{=\left(f_{\alpha_1} f_{\alpha_2}\cdots f_{\alpha_{i-1}}\right) f_{\alpha_i}} \left(f_{\alpha_{i+1}} f_{\alpha_{i+2}} f_{\alpha_{i+3}}\cdots\right) \\ &= \left(f_{\alpha_1} f_{\alpha_2}\cdots f_{\alpha_{i-1}}\right) \underbrace{f_{\alpha_i}}_{=0} \left(f_{\alpha_{i+1}} f_{\alpha_{i+2}} f_{\alpha_{i+3}}\cdots\right) = \left(f_{\alpha_1} f_{\alpha_2}\cdots f_{\alpha_{i-1}}\right) 0 \left(f_{\alpha_{i+1}} f_{\alpha_{i+2}} f_{\alpha_{i+3}}\cdots\right) = 0. \end{aligned}$$

This proves (198).

Proof of Theorem 5.35. We shall use the convolution $\star$ introduced in Definition 2.28.

Let Δ and ε be the comultiplication and the counit of the Hopf algebra Λ. Let $\eta : \mathbf{k} \to \Lambda$ be the map that sends each $u \in \mathbf{k}$ to $u \cdot 1_\Lambda \in \Lambda$. Let $m_\Lambda : \Lambda \otimes \Lambda \to \Lambda$ be the $\mathbf{k}$-linear map sending each pure tensor $a \otimes b \in \Lambda \otimes \Lambda$ to $ab \in \Lambda$. Definition 2.28 then yields

$$S \star \mathrm{id}_\Lambda = m_\Lambda \circ (S \otimes \mathrm{id}_\Lambda) \circ \Delta. \tag{199}$$

It is easy to see that each positive integer n satisfies

$$\varepsilon (G_{F,n}) = 0. \tag{200}$$

[*Proof of (200):* Let n be a positive integer. Proposition 5.6 **(a)** (applied to $m = n$) yields that the formal power series $G_{F,n}$ is the n-th degree homogeneous component of G_F. Hence, this power series $G_{F,n}$ is homogeneous of degree n. Thus, $G_{F,n}$ is homogeneous of positive degree (since n is positive). Also, $G_{F,n} \in \Lambda$ (by Proposition 5.6 **(c)**, applied to $m = n$).

Recall that the counit ε of Λ sends every homogeneous symmetric function of positive degree to 0. In other words, if $u \in \Lambda$ is homogeneous of positive degree, then $\varepsilon (u) = 0$. We can apply this to $u = G_{F,n}$ (since $G_{F,n}$ is homogeneous of positive degree), and thus obtain $\varepsilon (G_{F,n}) = 0$. This proves (200).]

Recall that the antipode of a Hopf algebra is defined to be the $\star$-inverse of its identity map (i.e., to be the inverse of its identity map with respect to the convolution $\star$). Thus, the antipode S of Λ is the $\star$-inverse of the map $\mathrm{id}_\Lambda : \Lambda \to \Lambda$. In other words,

$$S \star \mathrm{id}_\Lambda = \mathrm{id}_\Lambda \star S = \eta \circ \varepsilon \tag{201}$$

(since $\eta \circ \varepsilon : \Lambda \to \Lambda$ is the neutral element with respect to $\star$). We also have $S (1) = 1$ (by one of the fundamental properties of the antipode of a Hopf algebra[95]).

Now, each positive integer n satisfies

$$S (G_{F,n}) = - \sum_{i=0}^{n-1} S (G_{F,i}) \cdot G_{F,n-i}. \tag{202}$$

[95]See, e.g., [GriRei20, Proposition 1.4.10] for this property.

[*Proof of (202):* Let n be a positive integer. Then,

$$\underbrace{(S \star \mathrm{id}_\Lambda)}_{\substack{= m_\Lambda \circ (S \otimes \mathrm{id}_\Lambda) \circ \Delta \\ \text{(by (199))}}} (G_{F,n})$$

$$= (m_\Lambda \circ (S \otimes \mathrm{id}_\Lambda) \circ \Delta)(G_{F,n}) = m_\Lambda \left((S \otimes \mathrm{id}_\Lambda) \left(\underbrace{\Delta(G_{F,n})}_{\substack{= \sum_{i=0}^{n} G_{F,i} \otimes G_{F,n-i} \\ \text{(by Theorem 5.18,} \\ \text{applied to } m=n)}} \right) \right)$$

$$= m_\Lambda \left(\underbrace{(S \otimes \mathrm{id}_\Lambda) \left(\sum_{i=0}^{n} G_{F,i} \otimes G_{F,n-i} \right)}_{\substack{= \sum_{i=0}^{n} (S \otimes \mathrm{id}_\Lambda)(G_{F,i} \otimes G_{F,n-i}) \\ \text{(since the map } S \otimes \mathrm{id}_\Lambda \text{ is } \mathbf{k}\text{-linear)}}} \right) = m_\Lambda \left(\sum_{i=0}^{n} \underbrace{(S \otimes \mathrm{id}_\Lambda)(G_{F,i} \otimes G_{F,n-i})}_{= S(G_{F,i}) \otimes \mathrm{id}_\Lambda (G_{F,n-i})} \right)$$

$$= m_\Lambda \left(\sum_{i=0}^{n} S(G_{F,i}) \otimes \underbrace{\mathrm{id}_\Lambda (G_{F,n-i})}_{= G_{F,n-i}} \right) = m_\Lambda \left(\sum_{i=0}^{n} S(G_{F,i}) \otimes G_{F,n-i} \right)$$

$$= \sum_{i=0}^{n} \underbrace{m_\Lambda (S(G_{F,i}) \otimes G_{F,n-i})}_{\substack{= S(G_{F,i}) \cdot G_{F,n-i} \\ \text{(by the definition of the map } m_\Lambda)}} \qquad \text{(since the map } m_\Lambda \text{ is } \mathbf{k}\text{-linear)}$$

$$= \sum_{i=0}^{n} S(G_{F,i}) \cdot G_{F,n-i} = \sum_{i=0}^{n-1} S(G_{F,i}) \cdot G_{F,n-i} + S(G_{F,n}) \cdot \underbrace{G_{F,n-n}}_{\substack{= G_{F,0} = 1 \\ \text{(by Proposition 5.6 } \textbf{(e)}\text{)}}}$$

(here, we have split off the addend for $i = n$ from the sum)

$$= \sum_{i=0}^{n-1} S(G_{F,i}) \cdot G_{F,n-i} + S(G_{F,n}).$$

Thus,

$$\begin{aligned}
&\sum_{i=0}^{n-1} S\left(G_{F,i}\right) \cdot G_{F,n-i} + S\left(G_{F,n}\right) \\
&= \underbrace{\left(S \star \mathrm{id}_\Lambda\right)}_{\substack{=\eta\circ\varepsilon \\ \text{(by (201))}}} \left(G_{F,n}\right) = \left(\eta \circ \varepsilon\right)\left(G_{F,n}\right) = \eta\left(\varepsilon\left(G_{F,n}\right)\right) \\
&= \underbrace{\varepsilon\left(G_{F,n}\right)}_{\substack{=0 \\ \text{(by (200))}}} \cdot 1_\Lambda \qquad \text{(by the definition of } \eta\text{)} \\
&= 0,
\end{aligned}$$

so that

$$S\left(G_{F,n}\right) = -\sum_{i=0}^{n-1} S\left(G_{F,i}\right) \cdot G_{F,n-i}.$$

This proves (202).]

The map from $\mathbf{k}[[t]]$ to $\mathbf{k}$ that sends each power series to its constant term is a $\mathbf{k}$-algebra homomorphism. Thus, the constant term of the power series F^{-1} is the reciprocal of the constant term of F. Since the constant term of F is 1, we thus conclude that the constant term of the power series F^{-1} is the reciprocal of 1. In other words, the constant term of the power series F^{-1} is 1.

We must prove (196) for each $n \in \mathbb{N}$. We shall do this by strong induction on n:

Induction step: Let $m \in \mathbb{N}$. Assume (as the induction hypothesis) that (196) holds for all $n < m$. We must now prove that (196) holds for $n = m$. In other words, we must prove that $S\left(G_{F,m}\right) = G_{F^{-1},m}$. If $m = 0$, then this is obvious[96]. Thus, for the rest of this induction step, we WLOG assume that $m \neq 0$. Hence, m is a positive integer (since $m \in \mathbb{N}$).

We have assumed that (196) holds for all $n < m$. In other words, for all $n \in \mathbb{N}$ satisfying $n < m$, we have

$$S\left(G_{F,n}\right) = G_{F^{-1},n}. \tag{203}$$

Now, let $i \in \{0, 1, \ldots, m-1\}$. Thus, $i \leq m - 1 < m$. Therefore, (203) (applied to $n = i$) yields

$$S\left(G_{F,i}\right) = G_{F^{-1},i}. \tag{204}$$

Forget that we fixed i. We thus have proved (204) for each $i \in \{0, 1, \ldots, m-1\}$.

[96] *Proof.* Assume that $m = 0$. Thus, $G_{F,m} = G_{F,0} = 1$ (by Proposition 5.6 **(e)**) and $G_{F^{-1},m} = G_{F^{-1},0} = 1$ (by Proposition 5.6 **(e)**, applied to F^{-1} instead of F). Now, applying the map S to both sides of the equality $G_{F,m} = 1$, we obtain $S\left(G_{F,m}\right) = S(1) = 1$. Comparing this with $G_{F^{-1},m} = 1$, we obtain $S\left(G_{F,m}\right) = G_{F^{-1},m}$. Thus, we have proved that $S\left(G_{F,m}\right) = G_{F^{-1},m}$ under the assumption that $m = 0$.

Now, (202) (applied to $n = m$) yields

$$S\left(G_{F,m}\right) = -\sum_{i=0}^{m-1} \underbrace{S\left(G_{F,i}\right)}_{\substack{=G_{F^{-1},i} \\ \text{(by (204))}}} \cdot G_{F,m-i} = -\sum_{i=0}^{m-1} G_{F^{-1},i} G_{F,m-i}. \tag{205}$$

On the other hand, Proposition 5.34 **(b)** (applied to $A = F^{-1}$ and $B = F$ and $n = m$) yields

$$\begin{aligned} G_{F^{-1}F,m} &= \sum_{i=0}^{m} G_{F^{-1},i} G_{F,m-i} \\ &= \sum_{i=0}^{m-1} G_{F^{-1},i} G_{F,m-i} + G_{F^{-1},m} \underbrace{G_{F,m-m}}_{\substack{=G_{F,0}=1 \\ \text{(by Proposition 5.6 (e))}}} \\ &\qquad \text{(here, we have split off the addend for } i = m \text{ from the sum)} \\ &= \sum_{i=0}^{m-1} G_{F^{-1},i} G_{F,m-i} + G_{F^{-1},m}. \end{aligned}$$

Hence,

$$\begin{aligned} \sum_{i=0}^{m-1} G_{F^{-1},i} G_{F,m-i} + G_{F^{-1},m} &= G_{F^{-1}F,m} = G_{1,m} \qquad \left(\text{since } F^{-1}F = 1\right) \\ &= [m = 0] \qquad \text{(by Lemma 5.36, applied to } n = m) \\ &= 0 \qquad \text{(since we don't have } m = 0 \text{ (because } m \neq 0)). \end{aligned}$$

Thus,

$$G_{F^{-1},m} = -\sum_{i=0}^{m-1} G_{F^{-1},i} G_{F,m-i}.$$

Comparing this with (205), we obtain $S\left(G_{F,m}\right) = G_{F^{-1},m}$. In other words, (196) holds for $n = m$. This completes the induction step. Thus, (196) is proved by strong induction. This completes the proof of Theorem 5.35. □

As a consequence of Theorem 5.35, we obtain a formula for the antipode of a Petrie symmetric function:

Corollary 5.37. Let k be a positive integer such that $k > 1$. A weak composition α will be called *k-friendly* if each $i \in \{1, 2, 3, \ldots\}$ satisfies $\alpha_i \equiv 0 \bmod k$ or $\alpha_i \equiv 1 \bmod k$. If α is a weak composition, then $w\left(\alpha\right)$ shall denote the number of all $i \in \{1, 2, 3, \ldots\}$ satisfying $\alpha_i \equiv 1 \bmod k$.

Let S be the antipode of the Hopf algebra Λ. Then, for each $n \in \mathbb{N}$, we have

$$S\left(G\left(k, n\right)\right) = \sum_{\substack{\alpha \in \mathrm{WC}; \\ |\alpha| = n; \\ \alpha \text{ is } k\text{-friendly}}} \left(-1\right)^{w(\alpha)} \mathbf{x}^{\alpha} = \sum_{\substack{\lambda \in \mathrm{Par}; \\ |\lambda| = n; \\ \lambda \text{ is } k\text{-friendly}}} \left(-1\right)^{w(\lambda)} m_{\lambda}.$$

Proof of Corollary 5.37 (sketched). Let $F = 1 + t + t^2 + \cdots + t^{k-1} \in \mathbf{k}[[t]]$. Then, F is a power series whose constant term is 1. Hence, its reciprocal F^{-1} is well-defined and again is a power series whose constant term is 1. Let us denote this reciprocal F^{-1} by Q; thus, $Q = F^{-1}$.

Let $q_0, q_1, q_2, \ldots$ be the coefficients of the formal power series Q, so that $Q = \sum_{n\in\mathbb{N}} q_n t^n$. Thus, q_0 is the constant term of Q; hence, $q_0 = 1$ (since the constant term of Q is 1).

On the other hand,

$$\begin{aligned}
Q = F^{-1} &= \left(\frac{1-t^k}{1-t}\right)^{-1} \qquad \left(\text{since } F = 1 + t + t^2 + \cdots + t^{k-1} = \frac{1-t^k}{1-t}\right) \\
&= \frac{1-t}{1-t^k} = (1-t)\cdot \underbrace{\left(1-t^k\right)^{-1}}_{\substack{=\sum_{m\in\mathbb{N}} (t^k)^m = \sum_{m\in\mathbb{N}} t^{mk} \\ = t^0 + t^k + t^{2k} + t^{3k} + \cdots}} = (1-t)\cdot\left(t^0 + t^k + t^{2k} + t^{3k} + \cdots\right) \\
&= \underbrace{\left(t^0 + t^k + t^{2k} + t^{3k} + \cdots\right)}_{\substack{= \sum_{n\in\mathbb{N};\ n\equiv 0 \bmod k} t^n \\ = \sum_{n\in\mathbb{N}} [n\equiv 0 \bmod k] t^n}} - \underbrace{t\cdot\left(t^0 + t^k + t^{2k} + t^{3k} + \cdots\right)}_{\substack{= t^1 + t^{k+1} + t^{2k+1} + t^{3k+1} + \cdots \\ = \sum_{n\in\mathbb{N};\ n\equiv 1 \bmod k} t^n \\ = \sum_{n\in\mathbb{N}} [n\equiv 1 \bmod k] t^n}} \\
&= \sum_{n\in\mathbb{N}} [n \equiv 0 \bmod k]\, t^n - \sum_{n\in\mathbb{N}} [n \equiv 1 \bmod k]\, t^n \\
&= \sum_{n\in\mathbb{N}} \left([n \equiv 0 \bmod k] - [n \equiv 1 \bmod k]\right) t^n.
\end{aligned}$$

Comparing this with $Q = \sum_{n\in\mathbb{N}} q_n t^n$, we obtain

$$\sum_{n\in\mathbb{N}} q_n t^n = \sum_{n\in\mathbb{N}} \left([n \equiv 0 \bmod k] - [n \equiv 1 \bmod k]\right) t^n.$$

Comparing coefficients on both sides of this equality, we find

$$q_n = [n \equiv 0 \bmod k] - [n \equiv 1 \bmod k] \qquad \text{for each } n \in \mathbb{N}. \tag{206}$$

For any weak composition α, we define an element $q_\alpha \in \mathbf{k}$ by

$$q_\alpha = q_{\alpha_1} q_{\alpha_2} q_{\alpha_3} \cdots .$$

(Here, the infinite product $q_{\alpha_1} q_{\alpha_2} q_{\alpha_3} \cdots$ is well-defined, since every sufficiently high positive integer i satisfies $\alpha_i = 0$ and thus $q_{\alpha_i} = q_0 = 1$.)

It is now easy to see (using (206)) that

$$q_\alpha = [\alpha \text{ is } k\text{-friendly}] \cdot (-1)^{w(\alpha)} \tag{207}$$

for any weak composition α.

[*Proof of (207):* Let α be a weak composition. We must prove (207).

It is easy to see that (207) holds if α is not k-friendly[97]. Hence, for the rest of this proof of (207), we WLOG assume that α is k-friendly. In other words, each $i \in \{1, 2, 3, \ldots\}$ satisfies

$$\alpha_i \equiv 0 \operatorname{mod} k \qquad \text{or} \qquad \alpha_i \equiv 1 \operatorname{mod} k. \tag{208}$$

Now, if $i \in \{1, 2, 3, \ldots\}$ satisfies $\alpha_i \equiv 1 \operatorname{mod} k$, then

$$q_{\alpha_i} = -1 \tag{209}$$

[98].

On the other hand, if $i \in \{1, 2, 3, \ldots\}$ satisfies $\alpha_i \not\equiv 1 \operatorname{mod} k$, then

$$q_{\alpha_i} = 1 \tag{210}$$

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[97] *Proof.* Assume that α is not k-friendly. Thus, **not** each $i \in \{1, 2, 3, \ldots\}$ satisfies $\alpha_i \equiv 0 \operatorname{mod} k$ or $\alpha_i \equiv 1 \operatorname{mod} k$. In other words, there exists some $i \in \{1, 2, 3, \ldots\}$ that satisfies neither $\alpha_i \equiv 0 \operatorname{mod} k$ nor $\alpha_i \equiv 1 \operatorname{mod} k$. Consider this i.

We have $[\alpha_i \equiv 0 \operatorname{mod} k] = 0$ (since i does not satisfy $\alpha_i \equiv 0 \operatorname{mod} k$) and $[\alpha_i \equiv 1 \operatorname{mod} k] = 0$ (since i does not satisfy $\alpha_i \equiv 1 \operatorname{mod} k$). Now, (206) (applied to $n = \alpha_i$) yields

$$q_{\alpha_i} = \underbrace{[\alpha_i \equiv 0 \operatorname{mod} k]}_{=0} - \underbrace{[\alpha_i \equiv 1 \operatorname{mod} k]}_{=0} = 0 - 0 = 0.$$

Now, the definition of q_α yields

$$\begin{aligned} q_\alpha &= q_{\alpha_1} q_{\alpha_2} q_{\alpha_3} \cdots = \underbrace{\left(q_{\alpha_1} q_{\alpha_2} \cdots q_{\alpha_i}\right)}_{=\left(q_{\alpha_1} q_{\alpha_2} \cdots q_{\alpha_{i-1}}\right) q_{\alpha_i}} \left(q_{\alpha_{i+1}} q_{\alpha_{i+2}} q_{\alpha_{i+3}} \cdots\right) \\ &= \left(q_{\alpha_1} q_{\alpha_2} \cdots q_{\alpha_{i-1}}\right) \underbrace{q_{\alpha_i}}_{=0} \left(q_{\alpha_{i+1}} q_{\alpha_{i+2}} q_{\alpha_{i+3}} \cdots\right) = 0. \end{aligned}$$

Comparing this with

$$\underbrace{[\alpha \text{ is } k\text{-friendly}]}_{\substack{=0 \\ (\text{since } \alpha \text{ is not } k\text{-friendly})}} \cdot (-1)^{w(\alpha)} = 0,$$

we obtain $q_\alpha = [\alpha \text{ is } k\text{-friendly}] \cdot (-1)^{w(\alpha)}$. Thus, we have proved (207) under the assumption that α is not k-friendly.

[98] *Proof of (209):* Let $i \in \{1, 2, 3, \ldots\}$ satisfy $\alpha_i \equiv 1 \operatorname{mod} k$. Then, i cannot satisfy $\alpha_i \equiv 0 \operatorname{mod} k$ (because $\alpha_i \equiv 0 \operatorname{mod} k$ would entail $0 \equiv \alpha_i \equiv 1 \operatorname{mod} k$ and therefore $k \mid 0 - 1 = -1$, which would contradict $k > 1$). Hence, $[\alpha_i \equiv 0 \operatorname{mod} k] = 0$. Also, $[\alpha_i \equiv 1 \operatorname{mod} k] = 1$ (since $\alpha_i \equiv 1 \operatorname{mod} k$). Now, (206) (applied to $n = \alpha_i$) yields

$$q_{\alpha_i} = \underbrace{[\alpha_i \equiv 0 \operatorname{mod} k]}_{=0} - \underbrace{[\alpha_i \equiv 1 \operatorname{mod} k]}_{=1} = 0 - 1 = -1.$$

This proves (209).

[99].

Now, the definition of $q_\alpha = q_{\alpha_1} q_{\alpha_2} q_{\alpha_3} \cdots$ yields

$$\begin{aligned}
q_\alpha &= q_{\alpha_1} q_{\alpha_2} q_{\alpha_3} \cdots = \prod_{i \in \{1,2,3,\ldots\}} q_{\alpha_i} = \left(\prod_{\substack{i \in \{1,2,3,\ldots\};\\ \alpha_i \equiv 1 \bmod k}} \underbrace{q_{\alpha_i}}_{\substack{=-1\\ \text{(by (209))}}} \right) \cdot \left(\prod_{\substack{i \in \{1,2,3,\ldots\};\\ \alpha_i \not\equiv 1 \bmod k}} \underbrace{q_{\alpha_i}}_{\substack{=1\\ \text{(by (210))}}} \right) \\
&\qquad \left(\begin{array}{c} \text{since each } i \in \{1,2,3,\ldots\} \text{ satisfies either } \alpha_i \equiv 1 \bmod k \\ \text{or } \alpha_i \not\equiv 1 \bmod k \text{ (but not both at the same time)} \end{array} \right) \\
&= \left(\prod_{\substack{i \in \{1,2,3,\ldots\};\\ \alpha_i \equiv 1 \bmod k}} (-1) \right) \cdot \underbrace{\left(\prod_{\substack{i \in \{1,2,3,\ldots\};\\ \alpha_i \not\equiv 1 \bmod k}} 1 \right)}_{=1} = \prod_{\substack{i \in \{1,2,3,\ldots\};\\ \alpha_i \equiv 1 \bmod k}} (-1) \\
&= (-1)^{(\text{the number of all } i \in \{1,2,3,\ldots\} \text{ satisfying } \alpha_i \equiv 1 \bmod k)} = (-1)^{w(\alpha)}
\end{aligned}$$

(since $(\text{the number of all } i \in \{1,2,3,\ldots\} \text{ satisfying } \alpha_i \equiv 1 \bmod k) = w(\alpha)$ (by the definition of $w(\alpha)$)). Comparing this with

$$\underbrace{[\alpha \text{ is } k\text{-friendly}]}_{\substack{=1\\ \text{(since } \alpha \text{ is } k\text{-friendly)}}} \cdot (-1)^{w(\alpha)} = (-1)^{w(\alpha)},$$

we obtain $q_\alpha = [\alpha \text{ is } k\text{-friendly}] \cdot (-1)^{w(\alpha)}$. Hence, (207) is proved.]

Now, let $n \in \mathbb{N}$. Recall that our scalars q_i and q_α were defined in the exact same way as the scalars f_i and f_α were defined in Definition 5.4, but using the power series Q instead of F. Hence, Proposition 5.6 **(c)** (applied to Q, q_i, q_α and n instead of F, f_i, f_α and m) yields that

$$G_{Q,n} = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha| = n}} q_\alpha \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \mathrm{Par};\\ |\lambda| = n}} q_\lambda m_\lambda \in \Lambda.$$

[99] *Proof of (210):* Let $i \in \{1,2,3,\ldots\}$ satisfy $\alpha_i \not\equiv 1 \bmod k$. Then, i cannot satisfy $\alpha_i \equiv 1 \bmod k$. Hence, $[\alpha_i \equiv 1 \bmod k] = 0$. However, (208) shows that we have $\alpha_i \equiv 0 \bmod k$ or $\alpha_i \equiv 1 \bmod k$. Hence, we have $\alpha_i \equiv 0 \bmod k$ (since i cannot satisfy $\alpha_i \equiv 1 \bmod k$). Thus, $[\alpha_i \equiv 0 \bmod k] = 1$. Now, (206) (applied to $n = \alpha_i$) yields

$$q_{\alpha_i} = \underbrace{[\alpha_i \equiv 0 \bmod k]}_{=1} - \underbrace{[\alpha_i \equiv 1 \bmod k]}_{=0} = 1 - 0 = 1.$$

This proves (210).

Hence,

$$G_{Q,n} = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha|=n}} \underbrace{q_\alpha}_{\substack{=[\alpha \text{ is } k\text{-friendly}]\cdot(-1)^{w(\alpha)}\\ \text{(by (207))}}} \mathbf{x}^\alpha = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha|=n}} [\alpha \text{ is } k\text{-friendly}] \cdot (-1)^{w(\alpha)} \mathbf{x}^\alpha$$

$$= \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha|=n;\\ \alpha \text{ is } k\text{-friendly}}} (-1)^{w(\alpha)} \mathbf{x}^\alpha \qquad (211)$$

(since the factor $[\alpha \text{ is } k\text{-friendly}]$ inside the sum makes all the addends vanish except for those that satisfy "α is k-friendly") and

$$G_{Q,n} = \sum_{\substack{\lambda \in \mathrm{Par};\\ |\lambda|=n}} \underbrace{q_\lambda}_{\substack{=[\lambda \text{ is } k\text{-friendly}]\cdot(-1)^{w(\lambda)}\\ \text{(by (207),}\\ \text{applied to } \lambda\\ \text{instead of } \alpha)}} m_\lambda = \sum_{\substack{\lambda \in \mathrm{Par};\\ |\lambda|=n}} [\lambda \text{ is } k\text{-friendly}] \cdot (-1)^{w(\lambda)} m_\lambda$$

$$= \sum_{\substack{\lambda \in \mathrm{Par};\\ |\lambda|=n;\\ \lambda \text{ is } k\text{-friendly}}} (-1)^{w(\lambda)} m_\lambda \qquad (212)$$

(since the factor $[\lambda \text{ is } k\text{-friendly}]$ inside the sum makes all the addends vanish except for those that satisfy "λ is k-friendly").

However, in Example 5.5 **(c)**, we have seen that $G_{F,m} = G(k,m)$ for each $m \in \mathbb{N}$. Applying this to $m = n$, we obtain $G_{F,n} = G(k,n)$. Thus, $G(k,n) = G_{F,n}$, so that

$$\begin{aligned} S(G(k,n)) = S(G_{F,n}) &= G_{F^{-1},n} && \text{(by Theorem 5.35)} \\ &= G_{Q,n} && \left(\text{since } F^{-1} = Q\right) \\ &= \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha|=n;\\ \alpha \text{ is } k\text{-friendly}}} (-1)^{w(\alpha)} \mathbf{x}^\alpha && \text{(by (211))}. \end{aligned}$$

Combining this with

$$S(G(k,n)) = G_{Q,n} = \sum_{\substack{\lambda \in \mathrm{Par};\\ |\lambda|=n;\\ \lambda \text{ is } k\text{-friendly}}} (-1)^{w(\lambda)} m_\lambda \qquad \text{(by (212))},$$

we obtain

$$S(G(k,n)) = \sum_{\substack{\alpha \in \mathrm{WC};\\ |\alpha|=n;\\ \alpha \text{ is } k\text{-friendly}}} (-1)^{w(\alpha)} \mathbf{x}^\alpha = \sum_{\substack{\lambda \in \mathrm{Par};\\ |\lambda|=n;\\ \lambda \text{ is } k\text{-friendly}}} (-1)^{w(\lambda)} m_\lambda.$$

This proves Corollary 5.37. $\square$

One last property of $G_{F,n}$ shall be noted in passing:

Proposition 5.38. For any power series $F \in \mathbf{k}[[t]]$ whose constant term is 1, we define a $\mathbf{k}$-algebra homomorphism $V_F : \Lambda \to \Lambda$ as in Theorem 5.25. Then:

(a) If A and B are two power series in $\mathbf{k}[[t]]$ whose constant terms are 1, then $V_{AB} = V_A \star V_B$.

(b) We have $V_1 = \eta \circ \varepsilon$.

(c) For any power series $F \in \mathbf{k}[[t]]$ whose constant term is 1, we have $V_{F^{-1}} = V_F \circ S$, where S is the antipode of Λ.

We leave the proof of Proposition 5.38 (which follows easily from Proposition 5.34) to the reader.

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