For any positive integer $k$ and nonnegative integer $m$, we consider the symmetric function $G(k, m)$ defined as the sum of all monomials of degree $m$ that involve only exponents smaller than $k$. We call $G(k, m)$ a *Petrie symmetric function* in honor
of Flinders Petrie, as the coefficients in its expansion in the Schur basis are determinants of Petrie matrices (and thus belong to \{0, 1, -1\} by a classical result of Gordon and Wilkinson). More generally, we prove a Pieri-like rule for expanding a product of the form \( G(k, m) \cdot s_\mu \) in the Schur basis whenever \( \mu \) is a partition; all coefficients in this expansion belong to \{0, 1, -1\}. We show a further formula for \( G(k, m) \) and prove that \( G(k, 1), G(k, 2), G(k, 3), \ldots \) form an algebraically independent generating set for the symmetric functions when \( 1 - k \) is invertible in the base ring. We prove a conjecture of Liu and Polo in [LiuPol19, Remark 1.4.5] about the expansion of \( G(k, 2k - 1) \) in the Schur basis.

This paper (not counting Section 1, which introduces notations) consists of two independent parts. The first part is Section 2 which is devoted to proving the conjecture of Liu and Polo. The second part covers all remaining sections, and is devoted to general properties of the \( G(k, m) \).

Some proofs below are just outlined, but I will eventually expand upon them.

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Remarks

1. While finishing this work, I have become aware of the preprint [FuMei20] by Houshan Fu and Zhousheng Mei, which also introduces the Petrie symmetric functions \( G(k, m) \) and refers to them as truncated homogeneous symmetric functions \( h_{m}^{[k-1]} \). Some results below are also independently obtained in [FuMei20]. In particular, Theorem 3.8 is a formula in [FuMei20, §2], and Theorem 3.13 is equivalent to [FuMei20, Proposition 2.9]. The particular case of Theorem 6.9 when \( k = Q \) is part of [FuMei20, Theorem 2.7].

2. The Petrie symmetric functions have been added to Per Alexandersson’s collection of symmetric functions at https://www.math.upenn.edu/~peal/polynomials/petrie.htm.
1. Notations

We will use the following notations (most of which are also used in [GriRei18, §2.1]):

- We let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- We fix a commutative ring $k$; we will use this $k$ as the base ring in what follows.
- A weak composition means an infinite sequence of nonnegative integers that contains only finitely many nonzero entries (i.e., a sequence $(a_1, a_2, a_3, \ldots) \in \mathbb{N}^\infty$ such that all but finitely many $i \in \{1, 2, 3, \ldots\}$ satisfy $a_i = 0$).
- We let $WC$ denote the set of all weak compositions.
- For any weak composition $\alpha$ and any positive integer $i$, we let $\alpha_i$ denote the $i$-th entry of $\alpha$ (so that $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$). Furthermore, the size $|\alpha|$ of a weak composition $\alpha$ is defined to be $\alpha_1 + \alpha_2 + \alpha_3 + \cdots \in \mathbb{N}$.
- A partition means a weak composition whose entries weakly decrease (i.e., a weak composition $\alpha$ satisfying $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$).
- If $n \in \mathbb{Z}$, then a partition of $n$ means a partition $\alpha$ having size $n$ (that is, satisfying $|\alpha| = n$).
- We let $\text{Par}$ denote the set of all partitions. For each $n \in \mathbb{Z}$, we let $\text{Par}_n$ denote the set of all partitions of $n$.
- We will sometimes omit trailing zeroes from partitions: i.e., a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ will be identified with the $k$-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ whenever $k$ satisfies $\lambda_{k+1} = \lambda_{k+2} = \lambda_{k+3} = \cdots = 0$. For example, $(3, 2, 1, 0, 0, 0, \ldots) = (3, 2, 1) = (3, 2, 1, 0)$.
- The partition $(0, 0, 0, \ldots) = ()$ is called the empty partition and denoted by $\emptyset$.
- A part of a partition $\lambda$ means a nonzero entry of $\lambda$. For example, the parts of the partition $(3, 1, 1) = (3, 1, 1, 0, 0, 0, \ldots)$ are $3, 1, 1$.
- We will use the notation $1^k$ for “$1, 1, \ldots, 1$” in partitions. (For example, $(2, 1^4) = (2, 1, 1, 1, 1)$. This notation is a particular case of the more general notation $m^k$ for “$m, m, \ldots, m$” in partitions, used, e.g., in [GriRei18] Definition 2.2.1].)
• We let $\Lambda$ denote the ring of symmetric functions in infinitely many variables $x_1, x_2, x_3, \ldots$ over $k$. This is a subring of the ring $k[[x_1, x_2, x_3, \ldots]]$ of formal power series. To be more specific, $\Lambda$ consists of all power series in $k[[x_1, x_2, x_3, \ldots]]$ that are symmetric (i.e., invariant under permutations of the variables) and of bounded degree (see [GriRei18, §2.1] for the precise meaning of this).

• A monomial shall mean a formal expression of the form $x_1^{a_1}x_2^{a_2}x_3^{a_3} \cdots$ with $\alpha \in \text{WC}$. Formal power series are formal $k$-linear combinations of such monomials.

• For any weak composition $\alpha$, we let $x^\alpha$ denote the monomial $x_1^{a_1}x_2^{a_2}x_3^{a_3} \cdots$.

• The degree of a monomial $x^\alpha$ is defined to be $|\alpha|$.

• A formal power series is said to be homogeneous of degree $n$ (for some $n \in \mathbb{N}$) if all monomials appearing in it (with nonzero coefficient) have degree $n$. In particular, the power series 0 is homogeneous of any degree.

• The $k$-algebra $\Lambda$ is graded: i.e., any symmetric function $f$ can be uniquely written as a sum $\sum_{i \in \mathbb{N}} f_i$, where each $f_i$ is a homogeneous symmetric function of degree $i$, and where all but finitely many $i \in \mathbb{N}$ satisfy $f_i = 0$.

We shall use the symmetric functions $m_\lambda, h_n, e_n, p_n, s_\lambda$ in $\Lambda$ as defined in [GriRei18, Sections 2.1 and 2.2]. Let us briefly recall how they are defined:

• For any partition $\lambda$, we define the monomial symmetric function $m_\lambda \in \Lambda$ by

$$m_\lambda = \sum x^\alpha,$$

where the sum ranges over all weak compositions $\alpha \in \text{WC}$ that can be obtained from $\lambda$ by permuting entries. For example,

$$m_{(2,2,1)} = \sum_{i<j<k} x_i^2x_j^2x_k + \sum_{i<j<k} x_i^2x_jx_k^2 + \sum_{i<j<k} x_i^2x_k^2.$$

The family $(m_\lambda)_{\lambda \in \text{Par}}$ (that is, the family of the symmetric functions $m_\lambda$ as $\lambda$ ranges over all partitions) is a basis of the $k$-module $\Lambda$.

• For each $n \in \mathbb{Z}$, we define the complete homogeneous symmetric function $h_n$ by

$$h_n = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1}x_{i_2} \cdots x_{i_n} = \sum_{\alpha \in \text{WC}} x^\alpha = \sum_{\lambda \in \text{Par}_n} m_\lambda.$$

(Thus, $h_0 = 1$ and $h_n = 0$ for all $n < 0$.)

---

1Here, we understand $\lambda$ to be an infinite sequence, not a finite tuple, so the entries being permuted include infinitely many 0’s.
We know (e.g., from [GriRei18, Proposition 2.4.1]) that the family \((h_n)_{n \geq 1} = (h_1, h_2, h_3, \ldots)\) is algebraically independent and generates \(\Lambda\) as a \(k\)-algebra. In other words, \(\Lambda\) is freely generated by \(h_1, h_2, h_3, \ldots\) as a commutative \(k\)-algebra.

- For each \(n \in \mathbb{Z}\), we define the \textit{elementary symmetric function} \(e_n\) by

\[
e_n = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} = \sum_{\alpha \in WC \cap \{0,1\}^\infty; \ |\alpha| = n} x^\alpha.
\]

(Thus, \(e_0 = 1\) and \(e_n = 0\) for all \(n < 0\). If \(n \geq 0\), then \(e_n = m_{(1^n)}\), where, as we have agreed above, the notation \((1^n)\) stands for the \(n\)-tuple \((1,1,\ldots,1)\).)

- For each positive integer \(n\), we define the \textit{power-sum symmetric function} \(p_n\) by

\[
p_n = x_1^n + x_2^n + x_3^n + \cdots = m_{(n)}.
\]

- For each partition \(\lambda\), we define the \textit{Schur function} \(s_\lambda\) by

\[
s_\lambda = \sum x_T,
\]

where the sum ranges over all semistandard tableaux \(T\) of shape \(\lambda\), and where \(x_T\) denotes the monomial obtained by multiplying the \(x_i\) for all entries \(i\) of \(T\). We refer the reader to [GriRei18, Definition 2.2.1] or to [Stanle01, §7.10] for the details of this definition and further descriptions of the Schur functions.

One of the most important properties of Schur functions (see, e.g., [GriRei18, (2.4.9) for \(\mu = \emptyset\)] or [Stanle01, Theorem 7.16.1 for \(\mu = \emptyset\)]) is the fact that

\[
s_\lambda = \det \left( (h_{\lambda_i-i+j})_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right)
\]

for any partition \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)\). This is known as the \textit{(first, straight-shape) Jacobi–Trudi formula}.

The family \((s_\lambda)_{\lambda \in \text{Par}}\) is a basis of the \(k\)-module \(\Lambda\), and is known as the \textit{Schur basis}.

Finally, we will sometimes use the \textit{Hall inner product} \((\cdot, \cdot) : \Lambda \times \Lambda \to k\) as defined in [GriRei18, Definition 2.5.12]. This is the \(k\)-bilinear form on \(\Lambda\) that is defined by the requirement that

\[
(s_\lambda, s_\mu) = \delta_{\lambda,\mu} \quad \text{for any } \lambda, \mu \in \text{Par}
\]

(\(\delta_{\lambda,\mu}\) denotes the Kronecker delta). It is easy to see that the Hall inner product \((\cdot, \cdot)\) is graded: i.e., we have

\[
(f, g) = 0
\]

if \(f\) and \(g\) are two homogeneous symmetric functions of different degrees.
2. The Schur expansion of $\sum m_\lambda$ over all $\lambda \in \text{Par}_{2n-1}$ satisfying $(n - 1, n - 1, 1) \triangleright \lambda$

Let us recall a well-known partial order on the set of partitions of a given $n \in \mathbb{N}$:

**Definition 2.1.** Let $n \in \mathbb{N}$. We define a binary relation $\triangleright$ on the set $\text{Par}_n$ as follows: Two partitions $\lambda, \mu \in \text{Par}_n$ shall satisfy $\lambda \triangleright \mu$ if and only if we have

$$\lambda_1 + \lambda_2 + \cdots + \lambda_k \geq \mu_1 + \mu_2 + \cdots + \mu_k$$

for each $k \in \{1, 2, \ldots, n\}$.

This relation $\triangleright$ is the greater-or-equal relation of a partial order on $\text{Par}_n$, which is known as the *dominance order* (or the *majorization order*).

This definition is precisely [GriRei18, Definition 2.2.7]. Note that if we replace “for each $k \in \{1, 2, \ldots, n\}$” by “for each $k \in \{1, 2, 3, \ldots\}$” in this definition, then the relation $\triangleright$ does not change.

Our first goal is to prove the conjecture made in [LiuPol19, Remark 1.4.5]. We state this conjecture as follows:

**Theorem 2.2.** Let $n$ be an integer such that $n > 1$. Then,

$$\sum_{\lambda \in \text{Par}_{2n-1}; (n-1,n-1,1) \triangleright \lambda} m_\lambda = \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})}.$$

**Example 2.3.** For this example, let $n = 3$. Then, $n - 1 = 2$ and $2n - 1 = 5$. Hence, the left hand side of the equality in Theorem 2.2 is

$$\sum_{\lambda \in \text{Par}_{5}; (n-1,n-1,1) \triangleright \lambda} m_\lambda = m_{(2,2,1)} + m_{(2,1,1,1)} + m_{(1,1,1,1,1)}.$$ 

Meanwhile, the right hand side of the equality in Theorem 2.2 is

$$\sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})} = \sum_{i=0}^{1} (-1)^i s_{(2,2-i,1^{i+1})} = s_{(2,2,1)} - s_{(2,1,1,1)}.$$ 

Thus, Theorem 2.2 claims that $m_{(2,2,1)} + m_{(2,1,1,1)} + m_{(1,1,1,1,1)} = s_{(2,2,1)} - s_{(2,1,1,1)}$ in this case.

We will pave our way to the proof of this theorem by many little lemmas. We begin with a particularly simple one:

\[^2\text{Note that } (n - 1, n - 1, 1) \text{ is a partition whenever } n > 1 \text{ is an integer.}\]
**Lemma 2.4.** Let \( n \) be an integer such that \( n > 1 \). Let \( \lambda \in \text{Par}_{2n-1} \). Then, \((n-1, n-1, 1) \triangleright \lambda\) if and only if all positive integers \( i \) satisfy \( \lambda_i < n \).

**Proof.** \( \implies \): Assume that \((n-1, n-1, 1) \triangleright \lambda\). Thus, \( n-1 \geq \lambda_1 \) (by the definition of dominance). Hence, \( \lambda_1 \leq n-1 < n \). But \( \lambda \) is a partition; thus, \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \). Hence, all positive integers \( i \) satisfy \( \lambda_i \leq \lambda_1 < n \). This proves the “\( \implies \)” direction of Lemma 2.4.

\( \impliedby \): Assume that all positive integers \( i \) satisfy \( \lambda_i < n \). Thus, all positive integers \( i \) satisfy \( \lambda_i \leq n-1 \) (since \( \lambda_i \) and \( n \) are integers). Hence, in particular, \( \lambda_1 \leq n-1 \) and \( \lambda_2 \leq n-1 \).

Define a partition \( \mu \) by \( \mu = (n-1, n-1, 1) \); thus, \(|\mu| = (n-1) + (n-1) + 1 = 2n-1 \), so that \( \mu \in \text{Par}_{2n-1} \). Also, \( \lambda \in \text{Par}_{2n-1} \) (as we know). Thus, \( \mu \triangleright \lambda \) holds if and only if each \( k \in \{1, 2, \ldots, 2n-1\} \) satisfies

\[
\mu_1 + \mu_2 + \cdots + \mu_k \geq \lambda_1 + \lambda_2 + \cdots + \lambda_k
\]

(by the definition of dominance).

But each \( k \in \{1, 2, \ldots, 2n-1\} \) satisfies (3).

*Proof:* Let \( k \in \{1, 2, \ldots, 2n-1\} \). We must prove (3). If \( k \geq 3 \), then

\[
\mu_1 + \mu_2 + \cdots + \mu_k \geq \mu_1 + \mu_2 + \mu_3 = (n-1) + (n-1) + 1 \quad \text{(since } \mu = (n-1, n-1, 1)\text{)}
\]

\[
= 2n-1 = |\lambda| \quad \text{(since } \lambda \in \text{Par}_{2n-1} \text{)}
\]

\[
= \lambda_1 + \lambda_2 + \lambda_3 + \cdots \geq \lambda_1 + \lambda_2 + \cdots + \lambda_k,
\]

and thus (3) is proven in this case. Hence, it remains to prove (3) for \( k \leq 2 \). But \( \mu = (n-1, n-1, 1) \), and thus \( \mu_1 = n-1 \geq \lambda_1 \) and \( \mu_2 = n-1 \geq \lambda_2 \). Hence, \( \mu_1 \geq \lambda_1 \) and \( \mu_2 \geq \lambda_2 \). In other words, (3) is proven for \( k \leq 2 \). As we have said, this concludes the proof of (3).

Thus, we have \( \mu \triangleright \lambda \) (since \( \mu \triangleright \lambda \) holds if and only if each \( k \in \{1, 2, \ldots, 2n-1\} \) satisfies (3)). In other words, \((n-1, n-1, 1) \triangleright \lambda \) holds (since \( \mu = (n-1, n-1, 1) \)). This proves the “\( \impliedby \)” direction of Lemma 2.4. \( \square \)

**Corollary 2.5.** Let \( n \) be an integer such that \( n > 1 \). Then,

\[
\sum_{\lambda \in \text{Par}_{2n-1}; \ (n-1, n-1, 1) \triangleright \lambda} m_{\lambda} = \sum_{a \in \text{WC}; \ |a| = 2n-1; \ a_i < n \text{ for all } i} x^a.
\]

**Proof.** By the definition of the monomial symmetric functions \( m_{\lambda} \), we have

\[
m_{\lambda} = \sum_{a \in S_{(\infty)}^L} x^a
\]
for each partition $\lambda$ (where the group $\mathfrak{S}_{(\infty)}$ and its action on the set $WC$ are defined as in [GriRei18, §2.1]). Now, Lemma 2.4 shows that

$$
\sum_{\lambda \in \text{Par}_{2n-1}; (n-1,n-1,1) \triangleright \lambda} m_\lambda = \sum_{\lambda \in \text{Par}_{2n-1}; \lambda_i < n \text{ for all } i} m_\lambda = \sum_{\alpha \in \mathfrak{S}_{(\infty)}\lambda} x^\alpha
$$

$$
= \sum_{\alpha \in WC; |\alpha| = 2n-1; \alpha_i < n \text{ for all } i} x^\alpha
$$

(since each weak composition is a permutation of a unique partition, and since the entries of the latter partition are the entries of the former composition)

**Proposition 2.6.** Let $n$ be a positive integer. Let $k \in \{0, 1, \ldots, n-1\}$. Then,

$$
\sum_{\alpha \in WC; |\alpha| = n+k; \alpha_i < n \text{ for all } i} x^\alpha = h_{n+k} - h_k p_n.
$$

**Proof.** From $k \in \{0, 1, \ldots, n-1\}$, we obtain $k < n$ and thus $n+k < n+n$. Thus we conclude:

**Observation 1:** A monomial of degree $n+k$ cannot have more than one variable appear in it with exponent $\geq n$ (since this would require it to have degree $\geq n+n > n+k$).

Let $M_k$ be the set of all monomials of degree $k$. The definition of $h_k$ shows that $h_k$ is the sum of all monomials of degree $k$. In other words,

$$
h_k = \sum_{m \in M_k} m.
$$

(4)

Let $M_{n+k}$ be the set of all monomials of degree $n+k$. The definition of $h_{n+k}$ shows that $h_{n+k}$ is the sum of all monomials of degree $n+k$. In other words,

$$
h_{n+k} = \sum_{n \in M_{n+k}} n.
$$

(5)

Let $M$ be the set of all monomials of degree $n+k$ in which all variables appear with exponents $< n$. These monomials are exactly the $x^\alpha$ for $\alpha \in WC$ satisfying
\[|\alpha| = n + k \text{ and } (\alpha_i < n \text{ for all } i). \text{ Hence,}\]
\[
\sum_{n \in \mathcal{N}} n = \sum_{\alpha \in \mathcal{W}_C: |\alpha| = n + k; \alpha_i < n \text{ for all } i} x^\alpha. \tag{6}
\]

Clearly, the set \( \mathcal{N} \) is a subset of \( \mathcal{M}_{n+k} \), and furthermore its complement \( \mathcal{M}_{n+k} \setminus \mathcal{N} \) is the set of all monomials of degree \( n + k \) in which at least one variable appears with exponent \( \geq n \). Hence, the map
\[
\mathcal{M}_k \times \{1, 2, 3, \ldots\} \to \mathcal{M}_{n+k} \setminus \mathcal{N},
\]
\[(m, i) \mapsto m \cdot x_i^n\]
is well-defined (because if \( m \) is a monomial of degree \( k \), then \( m \cdot x_i^n \) is a monomial of degree \( k + n = n + k \), and the variable \( x_i \) appears in it with exponent \( \geq n \)). This map is furthermore surjective (for easy reasons) and injective (in fact, if \( n \in \mathcal{N} \), then \( n \) is a monomial of degree \( n + k \), and thus Observation 1 yields that there is at most one variable \( x_i \) that appears in \( n \) with exponent \( \geq n \); but this means that the only preimage of \( n \) under our map is \( (n, x_i^n, i) \)). Hence, this map is a bijection. We can thus use it to substitute \( m \cdot x_i^n \) for \( n \) in the sum \( \sum_{n \in \mathcal{M}_{n+k} \setminus \mathcal{N}} n \). We thus obtain
\[
\sum_{n \in \mathcal{M}_{n+k} \setminus \mathcal{N}} n = \sum_{(m, i) \in \mathcal{M}_k \times \{1, 2, 3, \ldots\}} m \cdot x_i^n = \left( \sum_{m \in \mathcal{M}_k} m \right) \cdot \left( \sum_{i \in \{1, 2, 3, \ldots\}} x_i^n \right) = h_k p_n \tag{7}
\]

But (5) becomes
\[
h_{n+k} = \sum_{n \in \mathcal{M}_{n+k}} n = \sum_{n \in \mathcal{N}} n + \sum_{n \in \mathcal{M}_{n+k} \setminus \mathcal{N}} n \quad \text{(since } \mathcal{N} \subseteq \mathcal{M}_{n+k})
\]
\[
= \sum_{\alpha \in \mathcal{W}_C: |\alpha| = n + k; \alpha_i < n \text{ for all } i} x^\alpha \quad \text{(by (4))}
\]
\[
= \sum_{\alpha \in \mathcal{W}_C: |\alpha| = n + k; \alpha_i < n \text{ for all } i} x^\alpha + h_k p_n. \quad \text{(by (6))}
\]

In other words,
\[
\sum_{\alpha \in \mathcal{W}_C: |\alpha| = n + k; \alpha_i < n \text{ for all } i} x^\alpha = h_{n+k} - h_k p_n.
\]

\[\square\]
Corollary 2.7. Let \( n \) be an integer such that \( n > 1 \). Then,

\[
\sum_{\lambda \in \text{Par}_{2n-1}; (n-1,n-1,1) \triangleright \lambda} m_{\lambda} = h_{2n-1} - h_{n-1} p_{n}.
\]

Proof. Corollary 2.5 yields

\[
\sum_{\lambda \in \text{Par}_{2n-1}; (n-1,n-1,1) \triangleright \lambda} m_{\lambda} = \sum_{\alpha \in \text{WC}; |\alpha| = 2n-1; \alpha_i < n \text{ for all } i} x_{\alpha} = \sum_{\alpha \in \text{WC}; |\alpha| = n+(n-1); \alpha_i < n \text{ for all } i} x_{\alpha} \quad \text{(since } 2n-1 = n + (n-1))
\]

\[
= h_{n+(n-1)} - h_{n-1} p_{n} \quad \text{(by Proposition 2.6, applied to } k = n - 1)
\]

\[
= h_{2n-1} - h_{n-1} p_{n}.
\]

Next, we need a formula for power-sum symmetric functions:

Proposition 2.8. Let \( n \) be a positive integer. Then,

\[
p_{n} = \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1^i)}.
\]

Proof. This is [GriRei18, Exercise 5.4.12(g)] or [MenRem15, Exercise 2.2]. Alternatively, this is an easy consequence of the Murnaghan-Nakayama rule (see [Sam17, Theorem 4.4.2] or [Stanle01, Theorem 7.17.3]), applied to the product \( p_{n}s_{\emptyset} \) (since \( s_{\emptyset} = 1 \)).

Corollary 2.9. Let \( n \) be a positive integer. Then,

\[
h_{n} - p_{n} = \sum_{i=0}^{n-2} (-1)^i s_{(n-1-i,1^{i+1})}.
\]

Proof. Proposition 2.8 yields

\[
p_{n} = \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1^i)} = (-1)^0 s_{(n-0,1^0)} + \sum_{i=1}^{n-1} (-1)^i s_{(0,1^i)} = h_{n} + \sum_{i=1}^{n-1} (-1)^i s_{(n-i,1^i)}.
\]
so that

\[
    h_n - p_n = - \sum_{i=1}^{n-1} (-1)^i s_{(n-i,1^i)} = \sum_{i=1}^{n-1} \left( -(-1)^i \right) s_{(n-i,1^i)} = \sum_{i=1}^{n-1} (-1)^{i-1} s_{(n-i,1^i)}
\]

\[
    = \sum_{i=0}^{n-2} (-1)^i s_{(n-1-i,1^{i+1})}
\]

(here, we have substituted \( i + 1 \) for \( i \) in the sum).

Next, we need an evaluation of the Hall inner product:

**Proposition 2.10.** Let \( n \) be a positive integer. Then, \((e_n, p_n) = (-1)^{n-1}\).

**Proof.** This is [GriRei18, Exercise 2.8.6(a)] in the next version of [GriRei18]. But here is a self-contained proof: Proposition 2.8 yields

\[
    (e_n, p_n) = \left( e_n, \sum_{i=0}^{n-1} (-1)^i s_{(n-i,1^i)} \right) = \sum_{i=0}^{n-1} (-1)^i \left( e_n, s_{(n-i,1^i)} \right) = \sum_{i=0}^{n-1} (-1)^i \left( 1, \text{ if } i = n-1; \right.
\]

\[
    \left. 0, \text{ if } i \neq n-1 \right) = \sum_{i=0}^{n-1} (-1)^i \left( s_{(1^n)}, s_{(n-i,1^i)} \right) = \sum_{i=0}^{n-1} (-1)^i \left\{ \begin{array}{ll}
    1, & \text{if } i = n-1; \\
    0, & \text{if } i \neq n-1
\end{array} \right.
\]

(since the basis \((s_\lambda)_{\lambda \in \text{Par}}\) of \( \Lambda \) is orthonormal with respect to the Hall inner product, and since we have \((1^n) = \left( n-i,1^i \right) \) if and only if \( i = n-1 \))

\[
    = (-1)^{n-1}.
\]

Let us now introduce some further features of symmetric functions from [GriRei18, Chapter 2].

We will use the skewing operations \( f^\perp : \Lambda \to \Lambda \) for all \( f \in \Lambda \) as defined in [GriRei18 §2.8].

Now, for any \( m \in \mathbb{N} \), we define a map \( B_m : \Lambda \to \Lambda \) by setting

\[
    B_m(f) = \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^\perp f \quad \text{for all } f \in \Lambda.
\]
It is known (GriRei18 Exercise 2.9.1(a)) that this map $B_m$ is well-defined and $k$-linear. Moreover, if $\lambda$ is any partition and if $m \in \mathbb{Z}$ satisfies $m \geq \lambda_1$, then
\[
\sum_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^\perp s_\lambda = s_{(m, \lambda_1, \lambda_2, \lambda_3, \ldots)}.
\] (8)
(This is GriRei18 Exercise 2.9.1(b).) Thus, if $\lambda$ is any partition and if $m \in \mathbb{Z}$ satisfies $m \geq \lambda_1$, then
\[
B_m (s_\lambda) = \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^\perp s_\lambda \quad \text{(by the definition of $B_m$)}
\]
\[
= s_{(m, \lambda_1, \lambda_2, \lambda_3, \ldots)} \quad \text{(by (8)).} \tag{9}\]

**Lemma 2.11.** Let $n$ be a positive integer. Let $m \in \{0, 1, \ldots, n\}$. Then, $B_m (h_n) = h_m h_n - h_{m+1} h_{n-1}$.

**Proof of Lemma 2.11.** We have $e_0 = 1$ and thus $e_0^\perp = 1^\perp = \text{id}$. Hence, $e_0^\perp (h_n) = h_n$.

We shall use the notion of skew Schur functions $s_{\lambda/\mu}$ (see GriRei18 Definition 2.3.1).

Recall that any two partitions $\lambda$ and $\mu$ satisfy
\[
s_{\mu}^\perp (s_\lambda) = s_{\lambda/\mu}; \tag{10}\]
here, $s_{\lambda/\mu}$ means 0 when $\mu \not\subseteq \lambda$.

From $e_1 = s_{(1)}$ and $h_n = s_{(n)}$, we obtain
\[
e_1^\perp (h_n) = s_{(1)}^\perp (s_{(n)}) = s_{(n)/(1)} \quad \text{(by (10))}
\]
\[
= s_{(n-1)} \quad \text{(since the partitions $(n)/(1)$ and $(n-1)$ have the same diagram, up to parallel shift)}
\]
\[
= h_{n-1}.
\]

For each integer $i > 1$, we have
\[
e_i^\perp (h_n) = s_{(1^i)}^\perp (s_{(n)}) \quad \text{(since $e_i = s_{(1^i)}$ and $h_n = s_{(n)}$)}
\]
\[
= s_{(n)/(1^i)} \quad \text{(by (10))}
\]
\[
= 0 \quad \text{(since $(1^i) \not\subseteq (n)$ (because $i > 1$)).} \tag{11}\]

Now, the definition of $B_m$ yields
\[
B_m (h_n) = \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} e_i^\perp (h_n)
\]
\[
= (-1)^0 h_{m+0} e_0^\perp (h_n) + (-1)^1 h_{m+1} e_1^\perp (h_n) + \sum_{i \geq 2} (-1)^i h_{m+i} e_i^\perp (h_n) \quad \text{(by (11))}
\]
\[
= h_m h_n - h_{m+1} h_{n-1}.
\]

\[\square\]
**Corollary 2.12.** Let $n$ be a positive integer. Then, $B_{n-1}(h_n) = 0$.

*Proof.* Apply Lemma 2.11 to $m = n - 1$ and simplify. □

**Lemma 2.13.** Let $m \in \mathbb{N}$. Let $n$ be a positive integer. Then, $B_m(p_n) = h_m p_n - h_{m+n}$.

*Proof.* This is [GriRei18, Exercise 2.9.1(f)] in the next version of [GriRei18]. But here is a self-contained proof: We will use the comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$ of the Hopf algebra $\Lambda$ (see [GriRei18, §2.3]). Here and in the following, the “$\otimes$” sign denotes $\otimes_k$. The power sum $p_n$ is primitive (see [GriRei18, Proposition 2.3.6]); thus,

$$\Delta(p_n) = 1 \otimes p_n + p_n \otimes 1.$$  

Hence, for each $i \in \mathbb{N}$, the definition of $e_i^\perp$ (see [GriRei18, Definition 2.8.1]) yields

$$e_i^\perp(p_n) = (e_i,1)p_n + (e_i, p_n)1.$$  

(12)

Now, the definition of $B_m$ yields

$$B_m(p_n) = \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} \cdot e_i^\perp(p_n)$$

$$= \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} \cdot (e_i,1)p_n + (e_i, p_n)1$$

(by (12))

$$= \sum_{i \in \mathbb{N}} (-1)^i h_{m+i} \cdot (e_i,1)p_n + (e_i, p_n)1$$

(by Proposition 2.10)

$$= h_m p_n + (-1)^n h_{m+n} \cdot (e_n, p_n)1$$

(by Proposition 2.10)

$$= h_m p_n - h_{m+n}.$$

□

**Lemma 2.14.** Let $n$ be a positive integer. Then,

$$B_{n-1}(h_n - p_n) = h_{2n-1} - h_{n-1}p_n.$$
Proof. We have

\[
B_{n-1} (h_n - p_n) = \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})}.
\]

Lemma 2.15. Let \( n \) be a positive integer. Then,

\[
B_{n-1} (h_n - p_n) = \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})}.
\]

Proof of Lemma 2.15. We have

\[
B_{n-1} (h_n - p_n) = B_{n-1} \left( \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})} \right) \quad \text{(by Corollary 2.9)}
\]

\[
= \sum_{i=0}^{n-2} (-1)^i \underbrace{B_{n-1} \left( s_{(n-1,n-1-i,1^{i+1})} \right)}_{=s_{(n-1,n-1-i,1^{i+1})}} \quad \text{by (9), applied to } m=n-1 \text{ and } \lambda=(n-1,n-1-i,1^{i+1}) \quad \text{(since } n-i \geq n-1-i)\]

\[
= \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})}.
\]

Now the proof of Theorem 2.2 is a trivial concatenation of equalities:

Proof of Theorem 2.2. Corollary 2.7 yields

\[
= \sum_{\lambda \in \text{Par}_{2n-1}} m_\lambda
\]

\[
= h_{2n-1} - h_{n-1} p_n = B_{n-1} (h_n - p_n) \quad \text{(by Lemma 2.14)}
\]

\[
= \sum_{i=0}^{n-2} (-1)^i s_{(n-1,n-1-i,1^{i+1})} \quad \text{(by Lemma 2.15)}.
\]
3. Petrie symmetric functions

3.1. Definitions

Our next goal is to study symmetric functions generalizing the \( \sum_{\alpha \in WC; |\alpha| = n+k; a_i < n \text{ for all } i} x^\alpha \) we have studied above:

**Definition 3.1.** (a) For any positive integer \( k \), we let
\[
G(k) = \sum_{\alpha \in WC; \alpha_i < k \text{ for all } i} x^\alpha.
\]
This is a symmetric formal power series in \( k \left[ [x_1, x_2, x_3, \ldots] \right] \) (but does not belong to \( \Lambda \) in general).

(b) For any positive integer \( k \) and any \( m \in \mathbb{N} \), we let
\[
G(k, m) = \sum_{\alpha \in WC; |\alpha| = m; a_i < k \text{ for all } i} x^\alpha \in \Lambda.
\]

We suggest the name \( k \)-Petrie symmetric function for \( G(k) \) and the name \( (k, m) \)-Petrie symmetric function for \( G(k, m) \). The reason for this naming is that the coefficients of these functions in the Schur basis of \( \Lambda \) are determinants of Petrie matrices, as we will see in Theorem 3.8. (See below for what Petrie matrices are.)

Corollary 2.5 shows that
\[
\sum_{\lambda \in \operatorname{Par}_{2n-1}; (n-1, n-1, 1) > \lambda} m_\lambda = G(n, 2n-1)
\]
for each integer \( n > 1 \) (where the \( \triangleright \) sign stands for the dominance order, as in Section 2). Thus, we are interested in studying \( G(k, m) \) in general.

3.2. Basic properties

We begin with simple properties:

**Proposition 3.2.** Let \( k \) be a positive integer.

(a) The symmetric function \( G(k, m) \) is the \( m \)-th degree homogeneous component of \( G(k) \) for each \( m \in \mathbb{N} \).

(b) We have
\[
G(k) = \sum_{\alpha \in WC; \alpha_i < k \text{ for all } i} x^\alpha = \sum_{\lambda \in \operatorname{Par}; \lambda_i < k \text{ for all } i} m_\lambda = \prod_{i=1}^{\infty} \left( x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right).
\]
(c) We have
\[
G(k, m) = \sum_{\alpha \in WC; \quad |\alpha| = m; \quad \alpha_i < k \text{ for all } i} x^\alpha = \sum_{\lambda \in \text{Par}; \quad |\lambda| = m; \quad \lambda_i < k \text{ for all } i} m_\lambda
\]
for each \( m \in \mathbb{N} \).

(d) If \( m \in \mathbb{N} \) satisfies \( k > m \), then \( G(k, m) = h_m \).

(e) If \( m \in \mathbb{N} \) and \( k = 2 \), then \( G(k, m) = e_m \).

Proof of Proposition 3.2. The proof is straightforward and only given here for the sake of completeness.

(a) Let \( m \in \mathbb{N} \). Consider the sum on the right hand side of the equality (13). The addends of this sum are monomials \( x^\alpha \); each such monomial \( x^\alpha \) is homogeneous of degree \( |\alpha| \). Thus, we can obtain the \( m \)-th degree homogeneous component of this sum by restricting the sum only to those \( \alpha \) that satisfy \( |\alpha| = m \). In other words,

\[
\left( \text{the } m \text{-th degree homogeneous component of } \sum_{\alpha \in WC; \quad \alpha_i < k \text{ for all } i} x^\alpha \right) = \sum_{\alpha \in WC; \quad \alpha_i < k \text{ for all } i; \quad |\alpha| = m} x^\alpha.
\]

But (14) yields
\[
G(k, m) = \sum_{\alpha \in WC; \quad |\alpha| = m; \quad \alpha_i < k \text{ for all } i} x^\alpha = \sum_{\alpha \in WC; \quad |\alpha| = m} x^\alpha.
\]
Comparing these two equalities, we find
\[
G(k, m) = \left( \text{the } m \text{-th degree homogeneous component of } \sum_{\alpha \in WC; \quad \alpha_i < k \text{ for all } i} x^\alpha \right) = \sum_{\alpha \in WC; \quad |\alpha| = m} x^\alpha.
\]

This proves Proposition 3.2 (a).

(b) Let us define the group \( \mathcal{S}_{(\infty)} \) and its action on the set \( WC \) as in [GriRei18, §2.1]. Then, by the definition of the monomial symmetric functions \( m_\lambda \), we have
\[
m_\lambda = \sum_{\alpha \in \mathcal{S}_{(\infty)} \lambda} x^\alpha \tag{15}
\]
for each partition $\lambda$. Thus,

\[
\sum_{\lambda \in \text{Par}; \lambda_i < k \text{ for all } i} \frac{m_\lambda}{\lambda_i < k \text{ for all } i} x^\lambda = \sum_{a \in \mathcal{S}(\infty)} \sum_{\lambda \in \mathcal{S}(\infty)} \frac{m_\lambda}{\lambda_i < k \text{ for all } i} x^\lambda = \sum_{a \in \mathcal{W}_C; a_i < k \text{ for all } i} x^a.
\]

(see each weak composition is a permutation of a unique partition, and since the entries of the latter partition are the entries of the former composition)

Comparing this with (13), we obtain

\[
G(k) = \sum_{\lambda \in \text{Par}; \lambda_i < k \text{ for all } i} m_\lambda.
\]

(16)

Comparing (13) with

\[
\prod_{i=1}^\infty \left( x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right) = \sum_{u \in \{0,1,\ldots,k-1\}} x_i^u
\]

(by the product rule)

\[
= \prod_{i=1}^\infty \sum_{u \in \{0,1,\ldots,k-1\}} x_i^u = \sum_{(u_1,u_2,u_3,\ldots) \in \{0,1,\ldots,k-1\}^\infty \cap \mathcal{W}_C} x_1^{u_1} x_2^{u_2} x_3^{u_3} \cdots
\]

\[
= \sum_{(u_1,u_2,u_3,\ldots) \in \mathcal{W}_C; u_i < k \text{ for all } i} x_1^{u_1} x_2^{u_2} x_3^{u_3} \cdots
\]

\[
= \sum_{(a_1,a_2,a_3,\ldots) \in \mathcal{W}_C; a_i < k \text{ for all } i} x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots
\]

(here, we have renamed the summation index $(u_1,u_2,u_3,\ldots)$ as $(a_1,a_2,a_3,\ldots)$)

\[
= \sum_{(a_1,a_2,a_3,\ldots) \in \mathcal{W}_C; a_i < k \text{ for all } i} x^{(a_1,a_2,a_3,\ldots)}
\]

(here, we have renamed the summation index $(a_1,a_2,a_3,\ldots)$ as $a$),

we obtain

\[
G(k) = \prod_{i=1}^\infty \left( x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right).
\]

Combining this equality with (16) and (13), we obtain

\[
G(k) = \sum_{a \in \mathcal{W}_C; a_i < k \text{ for all } i} x^a = \sum_{\lambda \in \text{Par}; \lambda_i < k \text{ for all } i} m_\lambda = \prod_{i=1}^\infty \left( x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right).
\]
This proves Proposition 3.2 (b).

(c) Let \( m \in \mathbb{N} \). Let us define the group \( \mathfrak{S}_{(\infty)} \) and its action on the set \( WC \) as in [GriRei18 §2.1]. Then, by the definition of the monomial symmetric functions \( m_\lambda \), we have

\[
m_\lambda = \sum_{\alpha \in \mathfrak{S}_{(\infty)} \lambda} x^\alpha
\]

for each partition \( \lambda \). Thus,

\[
\sum_{\lambda \in \text{Par}; \ |\lambda| = m; \ \lambda_i < k \text{ for all } i} m_\lambda = \sum_{\lambda \in \text{Par}; \ |\lambda| = m; \ \lambda_i < k \text{ for all } i} \sum_{\alpha \in \mathfrak{S}_{(\infty)} \lambda} x^\alpha = \sum_{\alpha \in WC; \ |\alpha| = m; \ \alpha_i < k \text{ for all } i} x^\alpha.
\]

(3)\footnote{Proof. Let \( \alpha \in WC \) satisfy \( |\alpha| = m \). We must prove that \( \alpha_i < k \) for all \( i \).

Indeed, let \( i \in \{1, 2, 3, \ldots\} \). We must prove that \( \alpha_i < k \).

The definition of \( |\alpha| \) yields

\[
|\alpha| = a_1 + a_2 + a_3 + \cdots = \sum_{j \geq 1} a_j = a_i + \sum_{j \geq 1; \ j \neq i} a_j > 0 \quad \text{(since } \alpha \in WC \subseteq \mathbb{N}^\infty)\]

(here, we have split off the addend for \( j = i \) from the sum)

\[
\geq a_i + \sum_{j \geq 1; \ j \neq i} 0 = a_i,
\]

so that \( a_i \leq |\alpha| = m < k \) (since \( k > m \)). Thus, we have proved that \( \alpha_i < k \). Qed.}

Comparing this with (14), we obtain

\[
G(k, m) = \sum_{\lambda \in \text{Par}; \ |\lambda| = m; \ \lambda_i < k \text{ for all } i} m_\lambda.
\]

Comparing this equality with (14), we obtain

\[
G(k, m) = \sum_{\alpha \in WC; \ |\alpha| = m; \ \alpha_i < k \text{ for all } i} x^\alpha = \sum_{\lambda \in \text{Par}; \ |\lambda| = m; \ \lambda_i < k \text{ for all } i} m_\lambda.
\]

This proves Proposition 3.2 (c).

(d) Let \( m \in \mathbb{N} \) satisfy \( k > m \). Then, each \( \alpha \in WC \) satisfying \( |\alpha| = m \) must automatically satisfy \((\alpha_i < k \text{ for all } i)\) \footnote{Hence, the condition “\( \alpha_i < k \text{ for all } i \)”}

\[
G(k, m) = \sum_{\alpha \in WC; \ |\alpha| = m; \ \alpha_i < k \text{ for all } i} x^\alpha = \sum_{\lambda \in \text{Par}; \ |\lambda| = m; \ \lambda_i < k \text{ for all } i} m_\lambda.
\]
under the summation sign \( \sum_{\alpha \in \mathcal{WC}; |\alpha| = m; \alpha_i < k \text{ for all } i} \) is redundant and can be removed. In other words, we have the following equality between summation signs:

\[
\sum_{\alpha \in \mathcal{WC}; |\alpha| = m; \alpha_i < k \text{ for all } i} = \sum_{\alpha \in \mathcal{WC}; |\alpha| = m} .
\]

Now, (14) yields

\[
G(k, m) = \sum_{\alpha \in \mathcal{WC}; |\alpha| = m; \alpha_i < k \text{ for all } i} x^\alpha = \sum_{\alpha \in \mathcal{WC}; |\alpha| = m} x^\alpha = h_m
\]

(since the definition of \( h_m \) yields \( h_m = \sum_{\alpha \in \mathcal{WC}; |\alpha| = m} x^\alpha \)). This proves Proposition 3.2 (d).

(e) Let \( m \in \mathbb{N} \), and assume that \( k = 2 \). Then, an \( \alpha \in \mathcal{WC} \) satisfies \( (\alpha_i < k \text{ for all } i) \) if and only if it satisfies \( \alpha \in \{0, 1\}^\infty \) (because we have the chain of logical equivalences

\[
(\alpha_i < k \text{ for all } i) \iff (\alpha_i < 2 \text{ for all } i) \quad \text{(since } k = 2) \\
\iff (\alpha_i \in \{0, 1\} \text{ for all } i) \quad \text{(since } \alpha_i \in \mathbb{N} \text{ for all } i) \\
\iff (\alpha \in \{0, 1\}^\infty)
\]

for each \( \alpha \in \mathcal{WC} \). Therefore, the condition \( \alpha_i < k \text{ for all } i \) under the summation sign \( \sum_{\alpha \in \mathcal{WC}; |\alpha| = m; \alpha_i < k \text{ for all } i} \) can be replaced by \( \alpha \in \{0, 1\}^\infty \). Thus, we obtain the following equality between summation signs:

\[
\sum_{\alpha \in \mathcal{WC}; |\alpha| = m; \alpha_i < k \text{ for all } i} = \sum_{\alpha \in \mathcal{WC}; |\alpha| = m} = \sum_{\alpha \in \mathcal{WC}; \alpha \in \{0, 1\}^\infty; |\alpha| = m} = \sum_{\alpha \in \mathcal{WC} \cap \{0, 1\}^\infty; |\alpha| = m} .
\]

Now, (14) yields

\[
G(k, m) = \sum_{\alpha \in \mathcal{WC}; \alpha \in \{0, 1\}^\infty; |\alpha| = m; \alpha_i < k \text{ for all } i} x^\alpha = \sum_{\alpha \in \mathcal{WC} \cap \{0, 1\}^\infty; |\alpha| = m} x^\alpha = e_m
\]
(since the definition of \( e_m \) yields \( e_m = \sum_{a \in WC \cap \{0,1\}^\infty; |a|=m} x^a \)). This proves Proposition 3.2 (e).

Parts (d) and (e) of Proposition 3.2 show that the Petrie symmetric functions \( G(k,m) \) can be seen as interpolating between the \( h_m \) and the \( e_m \). Another easily proved identity is \( G(m,m) = h_m - p_m \) for each positive integer \( m \).

### 3.3. The Schur expansion

See [Stanle01, Exercise 7.3] for an expansion of \( G(3) \) in terms of the elementary symmetric functions. We are here interested in expanding \( G(k) \) in terms of Schur functions, however. For this, we need to define some notations.

**Convention 3.3.** We shall use the Iverson bracket notation: i.e., if \( A \) is a logical statement, then \([A]\) shall denote the truth value of \( A \) (that is, the integer \( \begin{cases} 1, & \text{if } A \text{ is true;} \\ 0, & \text{if } A \text{ is false} \end{cases} \)).

We shall furthermore use the notation \( (a_{i,j})_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \) for the \( \ell \times \ell \)-matrix whose \((i,j)\)-th entry is \( a_{i,j} \) for each \( i,j \in \{1,2,\ldots,\ell\} \).

**Definition 3.4.** Let \( \lambda = (\lambda_1,\lambda_2,\ldots,\lambda_\ell) \in \text{Par} \), and let \( k \) be a positive integer. Then, the \( k\)-Petrie number \( \text{pet}_k(\lambda) \) of \( \lambda \) is the integer defined by

\[
\text{pet}_k(\lambda) = \det \left( ([0 \leq \lambda_i - i + j < k])_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right).
\]

Note that this integer does not depend on the choice of \( \ell \) (in the sense that it does not change if we enlarge \( \ell \) by adding trailing zeroes to the representation of \( \lambda \)); this follows from Lemma 3.6 below.

**Example 3.5.** Let \( \lambda \) be the partition \((3,1,1) \in \text{Par} \), let \( \ell = 3 \), and let \( k \) be a positive integer. Then, the definition of \( \text{pet}_k(\lambda) \) yields

\[
\text{pet}_k(\lambda) = \det \left( ([0 \leq \lambda_i - i + j < k])_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right)
= \det \begin{pmatrix}
[0 \leq \lambda_1 < k] & [0 \leq \lambda_1 + 1 < k] & [0 \leq \lambda_1 + 2 < k] \\
[0 \leq \lambda_2 - 1 < k] & [0 \leq \lambda_2 < k] & [0 \leq \lambda_2 + 1 < k] \\
[0 \leq \lambda_3 - 2 < k] & [0 \leq \lambda_3 - 1 < k] & [0 \leq \lambda_3 < k]
\end{pmatrix}
= \det \begin{pmatrix}
[0 \leq 3 < k] & [0 \leq 4 < k] & [0 \leq 5 < k] \\
[0 \leq 0 < k] & [0 \leq 1 < k] & [0 \leq 2 < k] \\
[0 \leq -1 < k] & [0 \leq 0 < k] & [0 \leq 1 < k]
\end{pmatrix}
\]
(since $\lambda_1 = 3$ and $\lambda_2 = 1$ and $\lambda_3 = 1$). Thus, taking $k = 4$, we obtain

$$\text{pet}_4(\lambda) = \det \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1
\end{pmatrix} = 0.$$  

On the other hand, taking $k = 5$, we obtain

$$\text{pet}_5(\lambda) = \det \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix} = -1.$$

**Lemma 3.6.** Let $\lambda \in \text{Par}$, and let $k$ be a positive integer. Let $\ell \in \mathbb{N}$ be such that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Then, the determinant $\det \begin{pmatrix} [0 \leq \lambda_i - i + j < k] \end{pmatrix}_{1 \leq i \leq \ell, 1 \leq j \leq \ell}$ does not depend on the choice of $\ell$.

**Proof of Lemma 3.6.** It suffices to show that

$$\det \begin{pmatrix} [0 \leq \lambda_i - i + j < k] \end{pmatrix}_{1 \leq i \leq \ell, 1 \leq j \leq \ell} = \det \begin{pmatrix} [0 \leq \lambda_i - i + j < k] \end{pmatrix}_{1 \leq i \leq \ell+1, 1 \leq j \leq \ell+1}.$$  

So let us prove this.

From $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, we obtain $\lambda_{\ell+1} = 0$. Hence, each $j \in \{1, 2, \ldots, \ell\}$ satisfies

$$[0 \leq \lambda_{\ell+1} - (\ell + 1) + j < k] = 0 \quad (17)$$  

In other words, the first $\ell$ entries of the last row of the matrix $([0 \leq \lambda_i - i + j < k])_{1 \leq i \leq \ell+1, 1 \leq j \leq \ell+1}$ are 0. Hence, if we expand the determinant of this matrix along this last row, then we obtain a sum having only one (potentially)

---

4Proof of (17): Let $j \in \{1, 2, \ldots, \ell\}$. Then, $\lambda_{\ell+1} - (\ell + 1) + j < 0 - \ell + \ell = 0$. Hence, $0 \leq \lambda_{\ell+1} - (\ell + 1) + j < k$ cannot hold. Thus, $[0 \leq \lambda_{\ell+1} - (\ell + 1) + j < k] = 0$. Qed.
nonzero addend, namely
\[
0 \leq \lambda_{\ell+1} - (\ell + 1) + (\ell + 1) < k
\]
(since \(\lambda_{\ell+1} - (\ell + 1) + (\ell + 1) = \lambda_{\ell+1} = 0\))
\[
= \sum_{0 \leq 0 < k} \det \left( \left[ 0 \leq \lambda_i - i + j < k \right]_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right)
\]
\[
= \det \left( \left[ 0 \leq \lambda_i - i + j < k \right]_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right).
\]
Thus,
\[
\det \left( \left[ 0 \leq \lambda_i - i + j < k \right]_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right) = \det \left( \left[ 0 \leq \lambda_i - i + j < k \right]_{1 \leq i \leq \ell+1, 1 \leq j \leq \ell+1} \right).
\]
This completes our proof of Lemma 3.6.

**Proposition 3.7.** Let \(\lambda \in \text{Par}\), and let \(k\) be a positive integer. Then, \(\text{pet}_k(\lambda) \in \{-1, 0, 1\}\).

**Proof of Proposition 3.7.** Write \(\lambda\) in the form \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)\).

We will use the concept of Petrie matrices (see [GorWil74, Theorem 1]). Namely, a **Petrie matrix** is a matrix whose entries all belong to \(\{0, 1\}\) and such that the 1’s in each column occur consecutively. In other words, a Petrie matrix is a matrix whose each column has the form

\[
\begin{pmatrix}
0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0 \\
\text{a zeroes}
\end{pmatrix}
\begin{pmatrix}
0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0 \\
\text{b ones}
\end{pmatrix}
\begin{pmatrix}
0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0 \\
\text{c zeroes}
\end{pmatrix}
\]

for some nonnegative integers \(a, b, c\) (where any of \(a, b, c\) can be 0).

From [GorWil74, Theorem 1], it follows that if a square matrix \(A\) is a Petrie matrix, then

\[
\det A \in \{-1, 0, 1\}.
\]

Hence, if a square matrix \(A\) is the transpose of a Petrie matrix, then

\[
\det A \in \{-1, 0, 1\}
\]

as well (since \(\det(B^T) = \det B\) for any square matrix \(B\)).

Each row of the matrix \(\left[ 0 \leq \lambda_i - i + j < k \right]_{1 \leq i \leq \ell, 1 \leq j \leq \ell}\) has the form

\[
\begin{pmatrix}
0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0 \\
\text{a zeroes}
\end{pmatrix}
\begin{pmatrix}
0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0 \\
\text{b ones}
\end{pmatrix}
\begin{pmatrix}
0, 0, \ldots, 0, 1, 1, \ldots, 1, 0, 0, \ldots, 0 \\
\text{c zeroes}
\end{pmatrix}
\]
for some nonnegative integers \(a, b, c\) (where any of \(a, b, c\) can be 0). Thus, the matrix \((0 \leq \lambda_i - i + j < k)\) is the transpose of a Petrie matrix. Hence, its determinant is \(\in \{−1, 0, 1\}\) (by (18)). In other words, \(\text{pet}_k(\lambda) \in \{−1, 0, 1\}\) (since \(\text{pet}_k(\lambda)\) is defined to be the determinant of this matrix).

Is there a more explicit description of the \(k\)-Petrie numbers \(\text{pet}_k(\lambda)\)? Yes, and we will see such later (in Subsection 3.4).

We can now expand the Petrie symmetric functions \(G(k)\) in the basis \((s_\lambda)_{\lambda \in \text{Par}}\) of \(\Lambda:\)

**Theorem 3.8.** Let \(k\) be a positive integer. Then,

\[
G(k) = \sum_{\lambda \in \text{Par}} \text{pet}_k(\lambda) s_\lambda.
\]

**Proof of Theorem 3.8.** Consider the ring \(k[[x, y]] : = k[[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]]\) of formal power series. We shall use the notations \(x\) and \(y\) for the sequences \((x_1, x_2, x_3, \ldots)\) and \((y_1, y_2, y_3, \ldots)\) of indeterminates. If \(f \in k[[x]] : = k[[x_1, x_2, x_3, \ldots]]\) is any formal power series, then \(f(y)\) shall mean the result of substituting \(y_1, y_2, y_3, \ldots\) for the variables \(x_1, x_2, x_3, \ldots\) in \(f\). (This will be a formal power series in \(k[[y_1, y_2, y_3, \ldots]]\).)

For the sake of symmetry, we also use the analogous notation \(f(x)\) for the result of substituting \(x_1, x_2, x_3, \ldots\) for \(x_1, x_2, x_3, \ldots\) in \(f\); of course, this \(f(x)\) is just \(f\).

Recall that the commutative \(k\)-algebra \(\Lambda\) is freely generated by \(h_1, h_2, h_3, \ldots\). Thus, there exists a \(k\)-algebra homomorphism \(\alpha_k : \Lambda \rightarrow k\) that sends \(h_i\) to \([i < k]\) for all positive integers \(i\). Consider this \(\alpha_k\). Thus,

\[
\alpha_k(h_i) = [i < k] \quad \text{for all } i \in \mathbb{N}. \tag{19}
\]

(Indeed, this follows from the definition of \(\alpha_k\) when \(i > 0\), and otherwise follows from \(h_0 = 1\).) More generally,

\[
\alpha_k(h_i) = [0 \leq i < k] \quad \text{for all } i \in \mathbb{Z}. \tag{20}
\]

(Indeed, this follows from the previous equality when \(i \geq 0\), and otherwise follows from \(h_0 = 0\).)

If \(\lambda\) is an arbitrary partition, then we define a symmetric function \(h_\lambda \in \Lambda\) by

\[
h_\lambda = h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots.
\]

This infinite product \(h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots\) is well-defined, since every sufficiently high positive integer \(i\) satisfies \(\lambda_i = 0\) and thus \(h_{\lambda_i} = h_0 = 1\). Moreover, if we write a partition \(\lambda \in \text{Par}\) in the form \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)\) for some \(\ell \in \mathbb{N}\) (so that all \(i > \ell\) satisfy \(\lambda_i = 0\)), then

\[
h_\lambda = h_{\lambda_1} h_{\lambda_2} h_{\lambda_3} \cdots = (h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}) \quad \left(\frac{h_{\lambda_{\ell+1}} h_{\lambda_{\ell+2}} h_{\lambda_{\ell+3}} \cdots}{=1,1,1,\ldots}\right)
\]

(since all \(i > \ell\) satisfy \(\lambda_i = 0\) and thus \(h_{\lambda_i} = 1\))

\[
= h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_\ell}.
\]
Thus, our definition of $h_\lambda$ agrees with the definition given in [GriRei18, Definition 2.2.1].

From [GriRei18, Theorem 2.5.1], we obtain

$$\sum_{\lambda \in \text{Par}} s_\lambda (x) s_\lambda (y) = \prod_{i,j=1}^\infty (1 - x_i y_j)^{-1} = \sum_{\lambda \in \text{Par}} h_\lambda (x) m_\lambda (y)$$

(according to [GriRei18] proof of Proposition 2.5.15). Consider this as an equality in the ring $\Lambda[[y]] = \Lambda[[y_1, y_2, y_3, \ldots]]$. Apply the $k$-algebra homomorphism $\alpha_k$, or more precisely the induced $k[[y]]$-algebra homomorphism $\alpha_k[[]]: \Lambda[[y]] \rightarrow k[[y]]$, to both sides of this equality. We obtain

$$\sum_{\lambda \in \text{Par}} \alpha_k(s_\lambda (x)) s_\lambda (y) = \sum_{\lambda \in \text{Par}} \alpha_k(h_\lambda (x)) m_\lambda (y).$$  \hspace{1cm} (21)

But every partition $\lambda$ satisfies

$$\alpha_k(h_\lambda (x)) = \alpha_k\left( \prod_{i \geq 1} h_{\lambda_i} (x) \right) \quad \text{(since $h_\lambda (x) = \prod_{i \geq 1} h_{\lambda_i} (x)$)} \quad \text{and} \quad \alpha_k(h_{\lambda_i}) = \prod_{i \geq 1} \lambda_i < k \quad \text{(by (19))}$$

$$= \prod_{i \geq 1} \lambda_i < k$$

$$= [\lambda_i < k \text{ for all } i].$$  \hspace{1cm} (22)

Moreover, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$ is any partition, then

$$s_\lambda (x) = s_\lambda = \det \left( (h_{\lambda_i-i+j})_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right)$$

(by (1)) and thus

$$\alpha_k(s_\lambda (x)) = \alpha_k\left( \det \left( (h_{\lambda_i-i+j})_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right) \right) = \det \left( \alpha_k\left( h_{\lambda_i-i+j} \right) \right)_{1 \leq i \leq \ell, 1 \leq j \leq \ell}$$

\hspace{1cm} (since $\alpha_k$ is a $k$-algebra homomorphism and thus respects determinants)

$$= \det \left( [0 \leq \lambda_i - i + j < k]_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right)$$

$$= \text{pet}_k(\lambda)$$
(by the definition of $\text{pet}_k(\lambda)$). Thus, every partition $\lambda$ satisfies
\[ \alpha_k(s_\lambda(x)) = \text{pet}_k(\lambda). \quad (23) \]

In view of (22) and (23), the equality (21) rewrites as
\[ \sum_{\lambda \in \text{Par}} \text{pet}_k(\lambda) s_\lambda(y) = \sum_{\lambda \in \text{Par}, \lambda_i < k \text{ for all } i} m_\lambda(y) = \sum_{\lambda \in \text{Par}, \lambda_i < k \text{ for all } i} m_\lambda(y). \]

Renaming the indeterminates $y$ as $x$ on both sides of this equality, we obtain
\[ \sum_{\lambda \in \text{Par}} \text{pet}_k(\lambda) s_\lambda(x) = \sum_{\lambda \in \text{Par}, \lambda_i < k \text{ for all } i} m_\lambda(x) = \sum_{\lambda \in \text{Par}, \lambda_i < k \text{ for all } i} m_\lambda = G(k) \]

(by Proposition 3.2(b)). Thus,
\[ G(k) = \sum_{\lambda \in \text{Par}} \text{pet}_k(\lambda) s_\lambda(x) = \sum_{\lambda \in \text{Par}} \text{pet}_k(\lambda) s_\lambda. \]

\[ \square \]

**Corollary 3.9.** Let $k$ be a positive integer. Let $m \in \mathbb{N}$. Then,
\[ G(k,m) = \sum_{\lambda \in \text{Par}_m} \text{pet}_k(\lambda) s_\lambda. \]

**Proof.** Combine Theorem 3.8 with Proposition 3.2(a). \[ \square \]

### 3.4. An explicit description of the $k$-Petrie numbers

Can the $k$-Petrie numbers $\text{pet}_k(\lambda)$ from Definition 3.4 be described more explicitly than as determinants? To be somewhat pedantic, the answer to this question depends on one’s notion of “explicit”, as determinants are not hard to compute, and another algorithm for calculating $\text{pet}_k(\lambda)$ can be extracted from our proof of Proposition 3.7 (when combined with [GorWil74, proof of Theorem 1]). Nevertheless, there is a description which (at least to my standards) appears simpler. This description will be stated in Theorem 3.13 further below.

First, let us get a simple case out of the way:

**Proposition 3.10.** Let $\lambda \in \text{Par}$, and let $k$ be a positive integer such that $\lambda_1 \geq k$. Then, $\text{pet}_k(\lambda) = 0$.

**Proof of Proposition 3.10** Write $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$. Then, each $j \in \{1, 2, \ldots, \ell\}$ satisfies $|0 \leq \lambda_i - i + j < k| = 0$ (since $\lambda_1 - 1 + j \geq \lambda_1 - 1 + 1 = \lambda_1 \geq k$). In other words, the $\ell \times \ell$-matrix $([0 \leq \lambda_i - i + j < k])_{1 \leq i \leq \ell, 1 \leq j \leq \ell}$ has first row $(0, 0, \ldots, 0)$. Therefore, its determinant is 0. In other words, $\text{pet}_k(\lambda) = 0$ (since $\text{pet}_k(\lambda)$ is defined to be its determinant). This proves Proposition 3.10 \[ \square \]
Stating Theorem 3.13 will require one further piece of notation:

**Convention 3.11.** For any $\lambda \in \text{Par}$, we define the transpose of $\lambda$ to be the partition $\lambda^t \in \text{Par}$ determined by

$$(\lambda^t)_i = |\{ j \in \{1,2,3,\ldots \} \mid \lambda_j \geq i \}| \quad \text{for each } i \geq 1.$$  

This partition $\lambda^t$ is also known as the conjugate of $\lambda$, and is perhaps easiest to understand in terms of Young diagrams (to wit, the Young diagram of $\lambda^t$ is obtained from that of $\lambda$ by a flip across the main diagonal).

One important use of transpose partitions is the following fact (see, e.g., [GriRei18, (2.4.10) for $\mu = \varnothing$] or [Stanle01, Theorem 7.16.2 applied to $\lambda^t$ and $\varnothing$ instead of $\lambda$ and $\mu$] for proofs): We have

$$s_{\lambda^t} = \det \left( e_{\lambda_i - i + j} \right)_{1 \leq i \leq \ell, \ 1 \leq j \leq \ell}$$  

for any partition $\lambda = (\lambda_1,\lambda_2,\ldots,\lambda_\ell)$. This is known as the (second, straight-shape) Jacobi–Trudi formula.

Another piece of notation we will use is a notation for quotients and remainders:

**Convention 3.12.** Let $k$ be a positive integer. Let $n \in \mathbb{Z}$. Then, $n \% k$ shall denote the remainder of $n$ divided by $k$, whereas $n \div k$ shall denote the quotient of this division (an integer). Thus, $n \div k$ and $n \% k$ are uniquely determined by the three requirements that $n \div k \in \mathbb{Z}$ and $n \% k \in \{0,1,\ldots,k-1\}$ and $n = (n \div k) \cdot k + (n \% k)$.

The “$\div$” and “$\%$” signs bind more strongly than the “$+$” and “$-$” signs. That is, for example, the expression “$a + b \% k$” shall be understood to mean “$a + (b \% k)$” rather than “$(a + b) \% k$”.

Now, we can state our “formula” for $k$-Petrie numbers.

**Theorem 3.13.** Let $\lambda \in \text{Par}$, and let $k$ be a positive integer. Let $\mu = \lambda^t$.

(a) If $\mu_k \neq 0$, then pet$_k(\lambda) = 0$.

From now on, let us assume that $\mu_k = 0$.

Define a $(k-1)$-tuple $(\beta_1,\beta_2,\ldots,\beta_{k-1}) \in \mathbb{Z}^{k-1}$ by setting

$$\beta_i = \mu_i - i \quad \text{for each } i \in \{1,2,\ldots,k-1\}.$$  

Define a $(k-1)$-tuple $(\gamma_1,\gamma_2,\ldots,\gamma_{k-1}) \in \{1,2,\ldots,k\}^{k-1}$ by setting

$$\gamma_i = 1 + (\beta_i - 1) \% k \quad \text{for each } i \in \{1,2,\ldots,k-1\}.$$  

(b) If the $k-1$ numbers $\gamma_1,\gamma_2,\ldots,\gamma_{k-1}$ are not distinct, then pet$_k(\lambda) = 0$.

(c) Assume that the $k-1$ numbers $\gamma_1,\gamma_2,\ldots,\gamma_{k-1}$ are distinct. Let

$$g = \left| \{(i,j) \in \{1,2,\ldots,k-1\}^2 \mid i < j \text{ and } \gamma_i < \gamma_j \} \right|.$$  

Then, pet$_k(\lambda) = (-1)^{(\beta_1+\beta_2+\cdots+\beta_{k-1})+g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})}$. 
Our proof of this theorem will depend on two lemmas about determinants:

**Lemma 3.14.** Let \( m \in \mathbb{N} \). Let \( R \) be a commutative ring. Let \( (a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq m} \in \mathbb{R}^{m \times m} \) be an \( m \times m \)-matrix.

(a) If \( \tau \) is any permutation of \( \{1, 2, \ldots, m\} \), then
\[
\det \left( (a_{\tau(i),j})_{1 \leq i \leq m, 1 \leq j \leq m} \right) = (-1)^{\tau} \cdot \det \left( (a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq m} \right).
\]
Here, \((-1)^{\tau}\) denotes the sign of the permutation \( \tau \).

(b) Let \( u_1, u_2, \ldots, u_m \) be \( m \) elements of \( R \). Let \( v_1, v_2, \ldots, v_m \) be \( m \) elements of \( R \). Then,
\[
\det \left( (u_i v_j a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq m} \right) = \left( \prod_{i=1}^{m} (u_i v_i) \right) \cdot \det \left( (a_{i,j})_{1 \leq i \leq m, 1 \leq j \leq m} \right).
\]

**Proof of Lemma 3.14.** (a) This is just the well-known fact that if the rows of a square matrix are permuted using a permutation \( \tau \), then the determinant of this matrix gets multiplied by \((-1)^{\tau}\).

(b) This follows easily from the definition of the determinant. \( \square \)

**Lemma 3.15.** Let \( k \) be a positive integer. Let \( \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \) be \( k-1 \) elements of the set \( \{1, 2, \ldots, k\} \).

Let \( G \) be the \((k-1) \times (k-1)\)-matrix
\[
\left( (-1)^{(\gamma_i+j) \% k} \left[ (\gamma_i+j) \% k \in \{0, 1\} \right] \right)_{1 \leq i < k-1, 1 \leq j < k-1}.
\]

(a) If the \( k-1 \) numbers \( \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \) are not distinct, then
\[
\det G = 0.
\]

(b) If \( \gamma_1 > \gamma_2 > \cdots > \gamma_{k-1} \), then
\[
\det G = (-1)^{(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.
\]

(c) Assume that the \( k-1 \) numbers \( \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \) are distinct. Let
\[
g = \left\{ (i,j) \in \{1, 2, \ldots, k-1\}^2 \mid i < j \text{ and } \gamma_i < \gamma_j \right\}.
\]
Then,
\[
\det G = (-1)^{g+(\gamma_1+\gamma_2+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}.
\]

**Proof of Lemma 3.15.** (a) Assume that the \( k-1 \) numbers \( \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \) are not distinct. In other words, there exist two elements \( u \) and \( v \) of \( \{1, 2, \ldots, k-1\} \) such that
For $u < v$ and $\gamma_u = \gamma_v$. Consider these $u$ and $v$. Now, from $\gamma_u = \gamma_v$, we conclude that the $u$-th and the $v$-th rows of the matrix $G$ are equal (by the construction of $G$). Hence, the matrix $G$ has two equal rows (since $u < v$). Thus, $\det G = 0$. This proves Lemma 3.15(a).

(b) Assume that $\gamma_1 > \gamma_2 > \cdots > \gamma_{k-1}$. Thus, $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are distinct. Hence, $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ is a $(k-1)$-element set. But $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ is a subset of $\{1, 2, \ldots, k\}$ (since $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are elements of $\{1, 2, \ldots, k\}$). Hence, $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ is a $(k-1)$-element subset of $\{1, 2, \ldots, k\}$.

But $\{1, 2, \ldots, k\}$ is a $k$-element set. Hence, each $(k-1)$-element subset of $\{1, 2, \ldots, k\}$ has the form $\{1, 2, \ldots, k\} \setminus \{u\}$ for some $u \in \{1, 2, \ldots, k\}$. Thus, in particular, $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ has this form (since $\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\}$ is a $(k-1)$-element subset of $\{1, 2, \ldots, k\}$). In other words,

$$\{\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\} = \{1, 2, \ldots, k\} \setminus \{u\}$$

(27)

for some $u \in \{1, 2, \ldots, k\}$. Consider this $u$. From (27), we conclude that $\gamma_1, \gamma_2, \ldots, \gamma_{k-1}$ are the $k-1$ elements of the set $\{1, 2, \ldots, k\} \setminus \{u\}$, listed in decreasing order (since $\gamma_1 > \gamma_2 > \cdots > \gamma_{k-1}$). In other words,

$$(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}) = (k, k-1, \ldots, \hat{u}, \ldots, 2, 1),$$

(28)

where the “hat” over the $u$ signifies that $u$ is omitted from the list (i.e., the expression “$(k, k-1, \ldots, \hat{u}, \ldots, 2, 1)$” is understood to mean the $(k-1)$-element list $(k, k-1, \ldots, u+1, u-1, \ldots, 2, 1)$, which contains all $k$ integers from 1 to $k$ in decreasing order except for $u$). Thus,

$$(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}) = (k, k-1, \ldots, u+1)$$

and

$$(\gamma_{k-u+1}, \gamma_{k-u+2}, \ldots, \gamma_{k-1}) = (u-1, u-2, \ldots, 1).$$

(29) (30)

Now, we claim that

$$(-1)^{(\gamma_i + j) \% k} [(\gamma_i + j) \% k \in \{0, 1\}] = (-1)^{\gamma_i + j - k} [\gamma_i + j \in \{k, k+1\}]$$

(31)

for any $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$.

[Proof of (31): Let $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$. We must prove the equality (31).

From $i \in \{1, 2, \ldots, k-1\}$, we obtain $1 \leq i \leq k-1$ and thus $k-1 \geq 1$. Thus, $k > k-1 \geq 1$. Hence, $k+1 < 2k$, so that $(k+1) \% k = 1$.

If we don’t have $(\gamma_i + j) \% k \in \{0, 1\}$, then we cannot have $\gamma_i + j \in \{k, k+1\}$ either (because $\gamma_i + j \in \{k, k+1\}$ would entail $(\gamma_i + j) \% k \in \begin{cases} k \% k, & \text{if } k \% k \in \{0, 1\} \\ 0, & \text{if } 0 \leq k \% k \leq 1 \end{cases}$).

Thus, if we don’t have $(\gamma_i + j) \% k \in \{0, 1\}$, then both truth values $[(\gamma_i + j) \% k \in \{0, 1\}]$ and $[\gamma_i + j \in \{k, k+1\}]$ are 0, and therefore the equality (31) simplifies to $(-1)^{(\gamma_i + j) \% k} 0 = (-1)^{\gamma_i + j - k} 0$ in this case, which is obviously true.
Hence, for the rest of this proof, we WLOG assume that we do have \((\gamma_i + j) \% k \in \{0, 1\}\).

But \(\gamma_i \in \{1, 2, \ldots, k\}\), so that \(1 \leq \gamma_i \leq k\). Also, \(j \in \{1, 2, \ldots, k - 1\}\), so that \(1 \leq j \leq k - 1\). Hence, \(\gamma_i + j \geq 1 + 1 = 2\) and \(\gamma_i + j \leq k + (k - 1) = 2k - 1\).

Altogether, we thus obtain \(2 \leq \gamma_i + j \leq 2k - 1\), so that \(\gamma_i + j \in \{2, 3, \ldots, 2k - 1\}\).

The remainders of the numbers \(2, 3, \ldots, 2k - 1\) upon division by \(k\) are \(2, 3, \ldots, k - 1, 0, 1, \ldots, k - 1\) (in this order). Thus, the only numbers \(p \in \{2, 3, \ldots, 2k - 1\}\) that satisfy \(p \% k \in \{0, 1\}\) are \(k\) and \(k + 1\). In other words, for any number \(p \in \{2, 3, \ldots, 2k - 1\}\) satisfying \(p \% k \in \{0, 1\}\), we have \(p \in \{k, k + 1\}\). Applying this to \(p = \gamma_i + j\), we obtain \(\gamma_i + j \in \{k, k + 1\}\) (since \(\gamma_i + j \in \{2, 3, \ldots, 2k - 1\}\) and \((\gamma_i + j) \% k \in \{0, 1\}\)). Hence, \(k \leq \gamma_i + j \leq k + 1\), so that \(k \leq \gamma_i + j < 2k\) (since \(k + 1 < 2k\)). Thus, \((\gamma_i + j) / k = 1\). But every integer \(p\) satisfies \(p = (p / k) k + (p \% k)\). Applying this to \(p = \gamma_i + j\), we obtain \(\gamma_i + j = (\gamma_i + j) / k k + ((\gamma_i + j) \% k) = k + ((\gamma_i + j) \% k)\). Hence, \((\gamma_i + j) \% k = \gamma_i + j - k\). Thus,

\[
\frac{(-1)^{(\gamma_i+j)\%k}}{\frac{(-1)^{(\gamma_i+j)\%k}}{\frac{((\gamma_i + j) \% k \in \{0, 1\})}{(\text{since } (\gamma_i+j)\%k=\gamma_i+j-k)}}} = (-1)^{\gamma_i+j-k}. \]

Comparing this with

\[
(-1)^{\gamma_i+j-k} \left[\gamma_i + j \in \{k, k + 1\}\right] = (-1)^{\gamma_i+j-k},
\]

we obtain

\[
(-1)^{(\gamma_i+j)\%k} \left[(\gamma_i + j) \% k \in \{0, 1\}\right] = (-1)^{\gamma_i+j-k} \left[\gamma_i + j \in \{k, k + 1\}\right].
\]

This proves (31).

Now, \(G\) is a \((k-1) \times (k-1)\)-matrix. For each \(i \in \{1, 2, \ldots, k - 1\}\) and \(j \in \{1, 2, \ldots, k - 1\}\), we have

\[
\text{(the } (i,j) \text{-th entry of } G) = (-1)^{\gamma_i+j\%k} \left[(\gamma_i + j) \% k \in \{0, 1\}\right] \quad \text{(by the definition of } G) = (-1)^{\gamma_i+j-k} \left[\gamma_i + j \in \{k, k + 1\}\right] \quad \text{(by (31))} = \begin{cases} 1, & \text{if } \gamma_i + j = k; \\ -1, & \text{if } \gamma_i + j = k + 1; \\ 0, & \text{otherwise} \end{cases} \quad \text{if } j = k - \gamma_i; \]

\[
= \begin{cases} 1, & \text{if } j = k - \gamma_i; \\ -1, & \text{if } j = k - \gamma_i + 1; \\ 0, & \text{otherwise} \end{cases} \quad \text{if } j = k - \gamma_i + 1;.
\]

Thus, we can explicitly describe the matrix \(G\) as follows: For each \(i \in \{1, 2, \ldots, k - 1\}\), the \(i\)-th row of \(G\) has an entry equal to 1 in position \(k - \gamma_i\) if \(k - \gamma_i > 0\), and an
entry equal to $-1$ in position $k - \gamma_i + 1$ if $k - \gamma_i + 1 < k$; all remaining entries of this row are 0. Recalling (29) and (30), we thus see that $G$ has the following form:

$$
G = \begin{pmatrix}
-1 & 1 & -1 & 1 & \cdots & 1 \\
1 & -1 & 1 & -1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\
1 & -1 & 1 & -1 & 1 & \cdots & 1 \\
1 & -1 & 1 & -1 & \ldots & \ldots & 1 \\
\end{pmatrix},
$$

where the horizontal bar separates the $(k - u)$-th row from the $(k - u + 1)$-st row, while the vertical bar separates the $(k - u)$-th column from the $(k - u + 1)$-st column. In other words, $G$ can be written as a block matrix

$$
G = \begin{pmatrix}
A & 0_{(k-u)\times(u-1)} \\
0_{(u-1)\times(k-u)} & B \\
\end{pmatrix},
$$

where $A$ is the $(k - u) \times (k - u)$-matrix whose diagonal entries are $-1$ and whose entries immediately below the diagonal are 1, while all its other entries are 0), and where $B$ is the $(u - 1) \times (u - 1)$-matrix whose diagonal entries are 1 and whose entries immediately above the diagonal are $-1$, while all its other entries are 0). Thus, $G$ (as written in (32)) is a block-diagonal matrix (since $A$ and $B$ are square matrices). Since the determinant of a block-diagonal matrix equals the product of the determinants of its diagonal

\footnote{Empty cells are understood to have entry 0.}
blocks, we thus conclude that

\[
\det G = \frac{\det A}{(-1)^{k-u}} = \frac{\det B}{1} = (-1)^{k-u}.
\]

But (28) yields

\[
\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1} = k + (k-1) + \cdots + \hat{u} + \cdots + 2 + 1
\]

\[
= \frac{(k + (k-1) + \cdots + 2 + 1) - u}{1+2+\cdots+(k-1)} + k
\]

\[
= (1 + 2 + \cdots + (k-1)) + k - u.
\]

Solving this for \(k-u\), we find

\[
k - u = (\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1}) - (1 + 2 + \cdots + (k-1)).
\]

Hence, (33) rewrites as

\[
\det G = (-1)^{k-u}(\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1}) - (1 + 2 + \cdots + (k-1)).
\]

This proves Lemma 3.15 (b).

(c) Assume that the \(k-1\) numbers \(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\) are distinct. Then, there exists a unique permutation \(\sigma\) of \(\{1, 2, \ldots, k-1\}\) such that \(\gamma_{\sigma(1)} > \gamma_{\sigma(2)} > \cdots > \gamma_{\sigma(k-1)}\) (indeed, this is simply saying that the \((k-1)\)-tuple \((\gamma_1, \gamma_2, \ldots, \gamma_{k-1})\) can be sorted into decreasing order by a unique permutation). Consider this \(\sigma\).

Let \(\tau\) denote the permutation \(\sigma^{-1}\). Thus, \(\tau\) is a permutation of \(\{1, 2, \ldots, k-1\}\) and satisfies \(\sigma \circ \tau = \text{id}\).

Let \(\delta_1, \delta_2, \ldots, \delta_{k-1}\) denote the \(k-1\) elements \(\gamma_{\sigma(1)}, \gamma_{\sigma(2)}, \ldots, \gamma_{\sigma(k-1)}\) of \(\{1, 2, \ldots, k\}\). Thus, for each \(j \in \{1, 2, \ldots, k-1\}\), we have

\[
\delta_j = \gamma_{\sigma(j)}.
\]

Hence, the chain of inequalities \(\gamma_{\sigma(1)} > \gamma_{\sigma(2)} > \cdots > \gamma_{\sigma(k-1)}\) (which is true) can be rewritten as \(\delta_1 > \delta_2 > \cdots > \delta_{k-1}\).

Moreover, from (34), we obtain

\[
\delta_1 + \delta_2 + \cdots + \delta_{k-1} = \gamma_{\sigma(1)} + \gamma_{\sigma(2)} + \cdots + \gamma_{\sigma(k-1)}
\]

\[
= \gamma_1 + \gamma_2 + \cdots + \gamma_{k-1}
\]

(35) (since \(\sigma\) is a permutation of \(\{1, 2, \ldots, k-1\}\)).
Moreover, for each \( i \in \{1, 2, \ldots, k-1\} \), we have

\[
\delta_{\tau(i)} = \gamma_{\sigma(\tau(i))} \quad \text{(by (34), applied to } j = \tau(i))
\]

\[
= \gamma_i \quad \text{(since } \sigma(\tau(i)) = (\sigma \circ \tau)(i) = i) \quad \text{. (36)}
\]

Recall that an inversion of the permutation \( \tau \) is defined to be a pair \((i, j)\) of elements of \(\{1, 2, \ldots, k-1\}\) satisfying \(i < j\) and \(\tau(i) > \tau(j)\). Hence,

\[
\{\text{the inversions of } \tau\} = \left\{ (i, j) \in \{1, 2, \ldots, k-1\}^2 \mid i < j \text{ and } \tau(i) > \tau(j) \right\}
\]

This is equivalent to \((\delta_{\tau(i)} < \delta_{\tau(j)})\) (since \(\delta_1 > \delta_2 > \cdots > \delta_{k-1}\)).

\[
= \left\{ (i, j) \in \{1, 2, \ldots, k-1\}^2 \mid i < j \text{ and } \delta_{\tau(i)}(\gamma_i) < \delta_{\tau(j)}(\gamma_j) \right\} \quad \text{(by (36))}
\]

\[
= \left\{ (i, j) \in \{1, 2, \ldots, k-1\}^2 \mid i < j \text{ and } \gamma_i < \gamma_j \right\} \quad \text{. (37)}
\]

Recall that the length \(\ell(\tau)\) of the permutation \(\tau\) is defined to be the number of inversions of \(\tau\). Thus,

\[
\ell(\tau) = (\text{the number of inversions of } \tau)
\]

\[
= \left| \{\text{the inversions of } \tau\} \right|
\]

\[
= \left| \left\{ (i, j) \in \{1, 2, \ldots, k-1\}^2 \mid i < j \text{ and } \gamma_i < \gamma_j \right\} \right| \quad \text{(by (37))}
\]

\[
= g \quad \text{ (by the definition of } g) .
\]

Recall that the sign \((-1)^{\tau}\) of the permutation \(\tau\) is defined by \((-1)^{\tau} = (-1)^{\ell(\tau)}\). Hence, \((-1)^{\tau} = (-1)^{\ell(\tau)} = (-1)^g\) (since \(\ell(\tau) = g\)).

Let \(H\) be the \((k-1) \times (k-1)\)-matrix

\[
\left( (-1)^{(\delta_{i+j})\%k} \left[ (\delta_{i} + j) \%k \in \{0, 1\} \right] \right)_{1 \leq i \leq k-1, 1 \leq j \leq k-1}.
\]

Then, we can apply Lemma 3.15 (b) to \(\delta_{i}\) and \(H\) instead of \(\gamma_{i}\) and \(G\) (since \(\delta_{1}, \delta_{2}, \ldots, \delta_{k-1}\) are \(k-1\) elements of \(\{1, 2, \ldots, k\}\) and satisfy \(\delta_{1} > \delta_{2} > \cdots > \delta_{k-1}\)). We thus obtain

\[
\det H = (-1)^{(\delta_{1}+\delta_{2}+\cdots+\delta_{k-1})-(1+2+\cdots+(k-1))} = (-1)^{(\gamma_{1}+\gamma_{2}+\cdots+\gamma_{k-1})-(1+2+\cdots+(k-1))}
\]
Proposition 3.10 yields pet

In other words, the set

Hence,

\[ G = \begin{pmatrix}
(-1)^{(\gamma_i + j) \mod k} [(\gamma_i + j) \mod k \in \{0, 1\}] \\
= (-1)^{\delta_{(i) + j} \mod k} [(\delta_{(i) + j} \mod k \in \{0, 1\}] \\
1 \leq i \leq k - 1, 1 \leq j \leq k - 1
\end{pmatrix}
\]

Hence,

\[ \det G = \det \left( (-1)^{\delta_{(i) + j} \mod k} [(\delta_{(i) + j} \mod k \in \{0, 1\}] \right)_{1 \leq i \leq k - 1, 1 \leq j \leq k - 1}
\]

by Lemma 3.14 (a), applied to \( m = k - 1 \)

and \( R = k \) and \( a_{ij} = (-1)^{\delta_{i+j} \mod k} [(\delta_{i+j} \mod k \in \{0, 1\}] \)

\[ = (-1)^{\gamma_1 + \gamma_2 + \cdots + \gamma_{k-1} - (1+2+\cdots+(k-1))}
\]

This proves Lemma 3.15 (c).

Proof of Theorem 3.13 (a) Assume that \( \mu_k \neq 0 \). But \( \mu = \lambda' \), whence

\[ \mu_k = (\lambda')_k = |\{ j \in \{1, 2, 3, \ldots \} \mid \lambda_j \geq k \}| \] (by the definition of \( \lambda' \)).

Hence,

\[ |\{ j \in \{1, 2, 3, \ldots \} \mid \lambda_j \geq k \}| = \mu_k \neq 0. \]

In other words, the set \( \{ j \in \{1, 2, 3, \ldots \} \mid \lambda_j \geq k \} \) is nonempty. Hence, there exists some \( j \in \{1, 2, 3, \ldots \} \) satisfying \( \lambda_j \geq k \). Consider this \( j \). We have \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \) (since \( \lambda \in \text{Par} \)) and thus \( \lambda_1 \geq \lambda_j \) (since \( 1 \leq j \)). Hence, \( \lambda_1 \geq \lambda_j \geq k \). Thus, Proposition 3.10 yields \( \text{pet}_k (\lambda) = 0 \). This proves Theorem 3.13 (a).

Now, let us prepare for the proof of parts (b) and (c).
Consider the $k$-algebra homomorphism $\alpha_k : \Lambda \to k$ defined in the proof of Theorem 3.8 above. Also, recall Convention 3.3. Next, we claim that if $k > 1$, then

$$\alpha_k(e_r) = (-1)^r + r\%k \quad [r\%k \in \{0, 1\}]$$

(38)

for each integer $r > -k + 1$.

[Proof of (38): Assume that $k > 1$. Consider the ring $(k[[x_1, x_2, x_3, \ldots]])[[t]]$ of formal power series in one indeterminate $t$ over $k[[x_1, x_2, x_3, \ldots]]$. In this ring, define the two power series

$$H(t) = \prod_{i=1}^{\infty} (1 - x_i t)^{-1} \in (k[[x_1, x_2, x_3, \ldots]])[[t]]$$

and

$$E(t) = \prod_{i=1}^{\infty} (1 + x_i t) \in (k[[x_1, x_2, x_3, \ldots]])[[t]].$$

Then, from [GriRei18], we know the identities

$$H(t) = \sum_{n \geq 0} h_n t^n \quad \text{and} \quad E(t) = \sum_{n \geq 0} e_n t^n.$$  

(Indeed, the first of these two identities is [GriRei18 (2.4.1)], whereas the second is [GriRei18 (2.4.2)].) Thus,

$$\prod_{i=1}^{\infty} (1 + x_i t) = E(t) = \sum_{n \geq 0} e_n t^n.$$  

Substituting $-t$ for $t$ in this equality, we obtain

$$\prod_{i=1}^{\infty} (1 - x_i t) = \sum_{n \geq 0} e_n (-t)^n = \sum_{n \geq 0} (-1)^n e_n t^n.$$  

On the other hand,

$$\prod_{i=1}^{\infty} (1 - x_i t)^{-1} = H(t) = \sum_{n \geq 0} h_n t^n.$$  

Multiplying these two equalities, we find

$$\left( \prod_{i=1}^{\infty} (1 - x_i t) \right) \left( \prod_{i=1}^{\infty} (1 - x_i t)^{-1} \right) = \left( \sum_{n \geq 0} (-1)^n e_n t^n \right) \left( \sum_{n \geq 0} h_n t^n \right).$$  

This simplifies to

$$1 = \left( \sum_{n \geq 0} (-1)^n e_n t^n \right) \left( \sum_{n \geq 0} h_n t^n \right).$$

(39)
The map $\alpha_k : \Lambda \rightarrow k$ is a $k$-algebra homomorphism. Hence, it induces a continuous $k[[t]]$-algebra homomorphism

$$\alpha_k [t] : \Lambda [[t]] \rightarrow k [[t]]$$

that sends each formal power series $\sum_{n \geq 0} a_n t^n \in \Lambda [[t]]$ (with $a_n \in \Lambda$) to $\sum_{n \geq 0} \alpha_k (a_n) t^n$.

Consider this $k[[t]]$-algebra homomorphism $\alpha_k [t]$. In particular, it satisfies

$$(\alpha_k [t])(t^i) = t^i$$

for each $i \in \mathbb{N}$.

Applying the map $\alpha_k [t]$ to both sides of the equality (39), we obtain

$$(\alpha_k [t])(1) = (\alpha_k [t]) \left( \sum_{n \geq 0} (-1)^n e_n t^n \right) \left( \sum_{n \geq 0} h_n t^n \right)$$

$$= (\alpha_k [t]) \sum_{n \geq 0} (-1)^n e_n t^n \cdot (\alpha_k [t]) \sum_{n \geq 0} h_n t^n$$

(by the definition of $\alpha_k [t])$ (by the definition of $\alpha_k [t]$)

(since $\alpha_k [t]$ is a $k$-algebra homomorphism)

$$= \sum_{n \geq 0} \alpha_k((-1)^n e_n) t^n \cdot \sum_{n \geq 0} \alpha_k(h_n) t^n$$

(by $\alpha_k$ is $k$-linear)

$$= \sum_{n \geq 0} (-1)^n \alpha_k(e_n) t^n \cdot \sum_{n \geq 0} [n < k] t^n$$

(by (19))

$$= (\sum_{n \geq 0} (-1)^n \alpha_k(e_n) t^n) \cdot \frac{1 - t^k}{1 - t}$$

Comparing this with

$$(\alpha_k [t])(1) = 1$$

(since $\alpha_k [t]$ is a $k$-algebra homomorphism),

we obtain

$$\left( \sum_{n \geq 0} (-1)^n \alpha_k(e_n) t^n \right) \cdot \frac{1 - t^k}{1 - t} = 1.$$
Hence,
\[
\sum_{n \geq 0} (-1)^n \alpha_k (e_n) t^n = \frac{1 - t}{1 - t^k} \cdot \frac{1}{1 - t^{k^2}} = (1 - t) \cdot \left(1 + t^k + t^{2k} + t^{3k} + \ldots \right)
\]
\[
= (1 - t) \cdot \left(1 + t^k + t^{2k} + t^{3k} + \ldots \right)
\]
\[
= 1 - t + t^k - t^{k+1} + t^{2k} - t^{2k+1} + t^{3k} - t^{3k+1} + \ldots
\]
\[
= \sum_{n \geq 0} (-1)^n \cdot k \cdot [n \% k \in \{0, 1\}] t^n
\]
(here, we have used that \(k > 1\), since for \(k = 1\) there would be cancellations in the sum \(1 - t + t^k - t^{k+1} + t^{2k} - t^{2k+1} + t^{3k} - t^{3k+1} + \ldots\)). Comparing coefficients before \(t^r\) on both sides of this equality, we obtain
\[
(-1)^r \alpha_k (e_r) = (-1)^{r+k} [r \% k \in \{0, 1\}]
\]
\[(40)\]
for each \(r \in \mathbb{N}\).

Now, let \(r\) be an integer such that \(r > -k + 1\). We must prove that
\[
\alpha_k (e_r) = (-1)^{r+k} [r \% k \in \{0, 1\}].
\]

If \(r \in \mathbb{N}\), then this follows by multiplying both sides of (40) by \((-1)^r\) (and sim-
pifying the result using \((-1)^r (-1)^r = 1\) and \((-1)^r (-1)^{r+k} = (-1)^{r+k}\)). Hence,
for the rest of this proof, we WLOG assume that \(r \not\in \mathbb{N}\). Thus, \(r\) is negative (since \(r\) is an integer). In view of \(r > -k + 1\), this yields \(r \in \{-k+2, -k+3, \ldots, -1\}\). Hence, \(r \% k \in \{2, 3, \ldots, k-1\}\). Thus, \(r \% k \not\in \{0, 1\}\). Hence, \([r \% k \in \{0, 1\}] = 0\).

Also, \(e_r = 0\) (since \(r\) is negative) and thus \(\alpha_k (e_r) = \alpha_k (0) = 0\) (since the map \(\alpha_k\) is \(k\)-linear). Comparing this with \((-1)^{r+k} [r \% k \in \{0, 1\}] = 0\), we obtain
\[
\alpha_k (e_r) = (-1)^{r+k} [r \% k \in \{0, 1\}] = 0.
\]

This concludes the proof of (38).]

For each \(i \in \{1, 2, \ldots, k-1\}\), we have
\[
\gamma_i = 1 + (\beta_i - 1) \cdot k \\
\in \{1, 2, \ldots, k\} \quad \text{(by (26))}
\]
\[
\in \{1, 2, \ldots, k\} \quad \text{(since } (\beta_i - 1) \cdot k \in \{0, 1, \ldots, k-1\}).
\]

Hence, the \((k-1)\)-tuple \((\gamma_1, \gamma_2, \ldots, \gamma_{k-1})\) really belongs to \(\{1, 2, \ldots, k\}^{k-1}\). In other words, \(\gamma_1, \gamma_2, \ldots, \gamma_{k-1}\) are \(k-1\) elements of the set \(\{1, 2, \ldots, k\}\).

Assume that \(\mu_k = 0\). But \(\mu \in \Par\) and thus \(\mu_1 \geq \mu_2 \geq \mu_3 \geq \ldots\). Hence, from \(\mu_k = 0\), we obtain \(\mu_1 = \mu_k+1 = \mu_k+2 = \cdots = 0\). Thus, \(\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})\).

It is known that taking the transpose of the transpose of a partition returns the original partition. Thus, \((\lambda^t)^t = \lambda\). In view of \(\mu = \lambda^t\), this rewrites as \(\mu^t = \lambda\). But
\[
s_\lambda (x) = s_\lambda = s_\mu^t \quad \text{(since } \lambda = \mu^t\)
\]
\[
= \det \left( (e_{\mu_i - i+j})_{1 \leq i \leq k-1, 1 \leq j \leq k-1} \right)
\]
Petrie symmetric functions

(by (24), applied to $\mu$ and $k - 1$ instead of $\lambda$ and $\ell$), because $\mu = (\mu_1, \mu_2, \ldots, \mu_{k-1})$. Applying the map $\alpha_k$ to both sides of this equality, we find

$$\alpha_k(s_\lambda(x)) = \alpha_k\left(\det\left(\begin{cases} e_{\mu_i-i+j} & \text{(since (25) yields $\mu_i-i=\beta_i$)} \\ \text{for } 1 \leq i \leq k-1, 1 \leq j \leq k-1 \end{cases}\right)\right)$$

$$= \alpha_k\left(\det\left(\begin{cases} e_{\beta_i+j} & \text{for } 1 \leq i \leq k-1, 1 \leq j \leq k-1 \end{cases}\right)\right)$$

$$= \det\left(\alpha_k(e_{\beta_i+j})\right)_{1 \leq i \leq k-1, 1 \leq j \leq k-1}$$

(since $\alpha_k$ is a $k$-algebra homomorphism, and thus commutes with taking determinants of matrices). Comparing this with (23), we obtain

$$\text{pet}_k(\lambda) = \det\left(\alpha_k(e_{\beta_i+j})\right)_{1 \leq i \leq k-1, 1 \leq j \leq k-1}. \quad (41)$$

But each $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$ satisfy $k > 1$ and

$$\beta_i + j = \frac{\mu_i - i}{\text{by (25)}} \geq 0 \quad \frac{i}{\leq k-1} + j > 0 \quad (k-1) + 0 = -k + 1 \quad \text{and thus}$$

$$\alpha_k(e_{\beta_i+j}) = (-1)^{(\beta_i+j)+\beta_i+j}\%k \quad \{(\beta_i+j) \%k \in \{0,1\}\} \quad (42)$$

(by (38), applied to $r = \beta_i+j$).

Furthermore, each $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$ satisfy

$$(-1)^{(\beta_i+j)+\beta_i+j}\%k \quad \{(\beta_i+j) \%k \in \{0,1\}\}$$

$$= (-1)^\beta_i(-1)^j(-1)^{\%k} \quad \{(\gamma_i+j) \%k \in \{0,1\}\}. \quad (43)$$

[Proof of (43): Let $i \in \{1, 2, \ldots, k-1\}$ and $j \in \{1, 2, \ldots, k-1\}$. The definition of $\gamma_i$ yields

$$\gamma_i = 1 + \underbrace{(\beta_i - 1)\%k}_{\equiv \beta_i - 1 \mod k} \equiv 1 + (\beta_i - 1) = \beta_i \mod k.$$

Hence, $\gamma_i + j \equiv \beta_i + j \mod k$. But if two integers are congruent modulo $k$, then they must leave the same remainder upon division by $k$. In other words, if $u \in \mathbb{Z}$

Indeed, if $i \in \{1, 2, \ldots, k-1\}$, then $1 \leq i \leq k-1$ and thus $k-1 \geq 1 > 0$, so that $k > 1$.}
and \( v \in \mathbb{Z} \) satisfy \( u \equiv v \mod k \), then \( u \% k = v \% k \). Applying this to \( u = \gamma_i + j \) and \( v = \beta_i + j \), we obtain \((\gamma_i + j) \% k = (\beta_i + j) \% k\). Hence,

\[
(-1)^{\beta_i} (-1)^j (-1)^{(\gamma_i+j) \% k} [(\gamma_i + j) \% k \in \{0, 1\}] = (-1)^{\beta_i} (-1)^j (-1)^{(\beta_i+j) \% k} [(\beta_i + j) \% k \in \{0, 1\}] = (-1)^{(\beta_i+j)+(\beta_i+j) \% k} [(\beta_i + j) \% k \in \{0, 1\}].
\]

This proves (43).

Now, (41) becomes

\[
\text{pet}_k(\lambda) = \det \left( \left( a_k^{(\epsilon_{\mu_i-i+j})} \right) \right)_{1 \leq i \leq k-1, 1 \leq j \leq k-1} = \det \left( (-1)^{\beta_i} (-1)^j (-1)^{(\gamma_i+j) \% k} [(\gamma_i + j) \% k \in \{0, 1\}] \right)_{1 \leq i \leq k-1, 1 \leq j \leq k-1} = \det \left( (-1)^{(\gamma_i+j) \% k} [(\gamma_i + j) \% k \in \{0, 1\}] \right)_{1 \leq i \leq k-1, 1 \leq j \leq k-1} = \left( \prod_{i=1}^{k-1} (-1)^{\beta_i} (-1)^j \right) \cdot \det \left( (-1)^{(\gamma_i+j) \% k} [(\gamma_i + j) \% k \in \{0, 1\}] \right)_{1 \leq i \leq k-1, 1 \leq j \leq k-1} = \left( \prod_{i=1}^{k-1} (-1)^{\beta_i} (-1)^j \right) \cdot \det G.
\]

(by Lemma 3.14 (b), applied to \( m = k-1 \) and \( a_{i,j} = (-1)^{(\gamma_i+j) \% k} [(\gamma_i + j) \% k \in \{0, 1\}] \) and \( u_i = (-1)^{\beta_i} \) and \( v_j = (-1)^j \).

Define a \((k-1) \times (k-1)\)-matrix \( G \) as in Lemma 3.15 Then, this becomes

\[
\text{pet}_k(\lambda) = \left( \prod_{i=1}^{k-1} (-1)^{\beta_i} (-1)^j \right) \cdot \det G.
\]

Now, we can readily prove parts (b) and (c) of Theorem 3.13:

(b) Assume that the \( k-1 \) numbers \( \gamma_1, \gamma_2, \ldots, \gamma_{k-1} \) are not distinct. Then, Lemma 3.15 (a) yields \( \det G = 0 \). Hence, (44) yields

\[
\text{pet}_k(\lambda) = \left( \prod_{i=1}^{k-1} (-1)^{\beta_i} (-1)^j \right) \cdot \det G = 0.
\]
This proves Theorem 3.13 (b).  
(c) The equality (44) becomes

\[
\text{pet}_k(\lambda) = \left( \prod_{i=1}^{k-1} (-1)^{\beta_i} (1+i)^i \right) = (-1)^g + (g + \gamma_1 + \gamma_2 + \cdots + \gamma_k - 1 - (1+2+\cdots+(k-1)))
\]

(by Lemma 3.15 (c))

This proves Theorem 3.13 (c).

\[\square\]

It is possible to restate part of Theorem 3.13 without using \(\lambda^1\):

**Proposition 3.16.** Let \(\lambda \in \text{Par}\), and let \(k\) be a positive integer. Assume that \(\lambda_1 < k\). Define a subset \(B\) of \(\mathbb{Z}\) by

\[B = \{ \lambda_i - i \mid i \in \{1,2,3,\ldots\} \} .\]

Let \(0,1,\ldots,k-1\) be the residue classes of the integers 0, 1, \ldots, \(k-1\) modulo \(k\) (considered as subsets of \(\mathbb{Z}\)). Let \(W\) be the set of all integers smaller than \(k-1\). Then, \(\text{pet}_k(\lambda) \neq 0\) if and only if exactly \(k-1\) of the \(k\) numbers \(i \in \{0,1,\ldots,k-1\}\) satisfy \(|(\tilde{t} \cap W) \setminus B| = 1\) while the remaining \(i\) satisfies \(|(\tilde{t} \cap W) \setminus B| = 0\).

**Proof of Proposition 3.16 (sketched).** Let \(\mu = \lambda^1\). Then, from \(\lambda_1 < k\), we can easily obtain \(\mu_k = 0\). Let us define a \((k-1)\)-tuple \((\beta_1, \beta_2, \ldots, \beta_{k-1}) \in \mathbb{Z}^{k-1}\) as in Theorem 3.13. Note that \(\beta_1 > \beta_2 > \cdots > \beta_{k-1}\) and \(\lambda_1 - 1 > \lambda_2 - 2 > \lambda_3 - 3 > \cdots\). Using [Macdon95, Chapter I, (1.7)], it is easy to see that \(W\) is the union of the two disjoint sets \(B\) and \(\{ \lambda - i \mid j \in \{1,2,\ldots,k-1\} \}\). Hence, Proposition 3.16 can easily be deduced from Theorem 3.13.

\[\square\]

The sets \(B\) and \(\tilde{t} \cap W \setminus B\) in Proposition 3.16 are related to the \(k\)-modular structure of the partition \(\lambda\), such as the \(\beta\)-set, the \(k\)-abacus, the Maya diagram, the \(k\)-core and the \(k\)-quotient (see [Olsson93, \S\S 1–3] for some of these concepts). Essentially equivalent concepts include the ”Dirac sea” representation of \(\lambda\), the Maya diagram of \(\lambda\), and the first column hook lengths of \(\lambda\).
4. A “Pieri” rule

Theorem 3.8 can be generalized. For that, we need to define a “relative” version of Petrie numbers:

**Definition 4.1.** Let \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \text{Par} \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \in \text{Par} \), and let \( k \) be a positive integer. Then, the \( k \)-Petrie number \( \text{pet}_k(\lambda, \mu) \) of \( \lambda \) and \( \mu \) is the integer defined by

\[
\text{pet}_k(\lambda, \mu) = \det \left( \left( \begin{array}{c}
0 \leq \lambda_i - \mu_j - i + j < k \\
1 \leq i \leq \ell, 1 \leq j \leq \ell
\end{array} \right) \right).
\]

Note that this integer does not depend on the choice of \( \ell \) (in the sense that it does not change if we enlarge \( \ell \) by adding trailing zeroes to the representations of \( \lambda \) and \( \mu \)); this follows from Lemma 4.2 below.

**Lemma 4.2.** Let \( \lambda \in \text{Par} \) and \( \mu \in \text{Par} \), and let \( k \) be a positive integer. Let \( \ell \in \mathbb{N} \) be such that \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \) and \( \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \). Then, the determinant

\[
\det \left( \left( \begin{array}{c}
0 \leq \lambda_i - \mu_j - i + j < k \\
1 \leq i \leq \ell, 1 \leq j \leq \ell
\end{array} \right) \right)
\]

does not depend on the choice of \( \ell \).

**Proof of Lemma 4.2** It suffices to show that

\[
\det \left( \left( \begin{array}{c}
0 \leq \lambda_i - \mu_j - i + j < k \\
1 \leq i \leq \ell, 1 \leq j \leq \ell
\end{array} \right) \right) = \det \left( \left( \begin{array}{c}
0 \leq \lambda_i - \mu_j - i + j < k \\
1 \leq i \leq \ell + 1, 1 \leq j \leq \ell + 1
\end{array} \right) \right).
\]

So let us prove this.

From \( \mu = (\mu_1, \mu_2, \ldots, \mu_\ell) \), we obtain \( \mu_{\ell+1} = 0 \). From \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \), we obtain \( \lambda_{\ell+1} = 0 \). Hence, each \( j \in \{1, 2, \ldots, \ell\} \) satisfies

\[
0 \leq \lambda_{\ell+1} - \mu_j - (\ell + 1) + j < k = 0
\]

(45)

In other words, the first \( \ell \) entries of the last row of the matrix

\[
\left( \begin{array}{c}
0 \leq \lambda_i - \mu_j - i + j < k \\
1 \leq i \leq \ell + 1, 1 \leq j \leq \ell + 1
\end{array} \right)
\]

are 0. Hence, if we expand the determinant of this matrix along this last row, then we obtain a sum having only one

---

\( \text{Proof of (45):} \) Let \( j \in \{1, 2, \ldots, \ell\} \). Then, \( \lambda_{\ell+1} - \mu_j - (\ell + 1) + j < 0 - 0 - \ell + \ell = 0 \). Hence,

\[
0 \leq \lambda_{\ell+1} - \mu_j - (\ell + 1) + j < k \text{ cannot hold. Thus, } [0 \leq \lambda_{\ell+1} - \mu_j - (\ell + 1) + j < k] = 0. \text{ Qed.}
\]
(potentially) nonzero addend, namely

\[
\begin{bmatrix}
0 & \leq & \lambda_{\ell+1} - \mu_{\ell+1} - (\ell + 1) + (\ell + 1) < k \\
\end{bmatrix}
\cdot 
\det \left( \left( \begin{array}{ccc}
[0 & \leq & \lambda_i - \mu_j - i + j < k] \\
1 \leq i \leq \ell, 1 \leq j \leq \ell
\end{array} \right) \right)
\]

\[
= \left( \begin{array}{cc}
0 & \leq \lambda_{\ell+1} - \mu_{\ell+1} - (\ell + 1) + (\ell + 1) < k \\
\end{array} \right)
\cdot 
\det \left( \left( \begin{array}{ccc}
[0 & \leq \lambda_i - \mu_j - i + j < k] \\
1 \leq i \leq \ell, 1 \leq j \leq \ell
\end{array} \right) \right)
\]

\[
= \det \left( \left( \begin{array}{ccc}
[0 & \leq \lambda_i - \mu_j - i + j < k] \\
1 \leq i \leq \ell, 1 \leq j \leq \ell
\end{array} \right) \right).
\]

This completes our proof of Lemma 4.2. \[ \square \]

**Proposition 4.3.** Let \( \lambda \in \text{Par} \) and \( \mu \in \text{Par} \), and let \( k \) be a positive integer. Then, \( \text{pet}_k (\lambda, \mu) \in \{-1, 0, 1\} \).

**Proof.** Analogous to the proof of Proposition 3.7. \[ \square \]

Now, the following generalization of Theorem 3.8 holds:

**Theorem 4.4.** Let \( k \) be a positive integer. Let \( \mu \in \text{Par} \). Then,

\[
G(k) \cdot s_{\mu} = \sum_{\lambda \in \text{Par}} \text{pet}_k (\lambda, \mu) s_{\lambda}.
\]

Applying Theorem 4.4 to \( \mu = \emptyset \), we obtain Theorem 3.8 since it is easy to see that \( \text{pet}_k (\lambda, \emptyset) = \text{pet}_k (\lambda) \) for each \( \lambda \in \text{Par} \).

We have two proofs of Theorem 4.4. One proof is similar to the above proof of Theorem 3.8, but uses the identity

\[
\sum_{\lambda \in \text{Par}} s_{\lambda} (x) s_{\lambda/\mu} (y) = s_{\mu} (x) \cdot \prod_{i,j=1}^{\infty} \left( 1 - x_i y_j \right)^{-1} \quad \text{(for any } \mu \in \text{Par})
\]

instead of the Cauchy identity. But here is a different proof, which relies on [GriRei18 §2.6] and specifically on the notion of alternants:

\[9\] See [Macdon95 §I.5, example 26] or [GriRei18 Exercise 2.5.11(a)] for a proof of this identity.
Proof of Theorem 4.4. Fix an \( \ell \in \mathbb{N} \) such that \( \ell (\mu) \leq \ell \). (Here, \( \ell (\mu) \) denotes the length of the partition \( \mu \); it is defined as the unique \( i \in \mathbb{N} \) such that \( \mu_1, \mu_2, \ldots, \mu_i \) are positive but \( \mu_{i+1}, \mu_{i+2}, \mu_{i+3}, \ldots \) are zero.)

Let \( P_\ell \) denote the set of all partitions with at most \( \ell \) parts. We shall show that

\[
(G (k)) (x_1, x_2, \ldots, x_\ell) \cdot s_\mu (x_1, x_2, \ldots, x_\ell) \quad = \quad \sum_{\lambda \in P_\ell} \text{pet}_k (\lambda, \mu) s_\lambda (x_1, x_2, \ldots, x_\ell) .
\]

Once this is done, the usual “let \( \ell \) tend to \( \infty \)” argument (analogous to [GriRei18, proof of Corollary 2.6.9]) will yield the validity of Theorem 4.4.

Any partition \( \lambda \in P_\ell \) satisfies \( \lambda_{\ell+1} = \lambda_{\ell+2} = \lambda_{\ell+3} = \cdots = 0 \) and thus \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \). Thus, any partition \( \lambda \in P_\ell \) can be regarded as an \( \ell \)-tuple of non-negative integers. More precisely, the partitions \( \lambda \in P_\ell \) are precisely the \( \ell \)-tuples \( (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{N}^\ell \) satisfying \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \).

Define the \( \ell \)-tuple \( \rho = (\ell - 1, \ell - 2, \ldots, 0) \).

For any \( \ell \)-tuple \( \alpha \in \mathbb{N}^\ell \) and each \( i \in \{1, 2, \ldots, \ell\} \), we shall write \( \alpha_i \) for the \( i \)-th entry of \( \alpha \).

For any \( \ell \)-tuple \( \alpha \in \mathbb{N}^\ell \), we let \( x^\alpha \) denote the monomial

\[
x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_\ell^{\alpha_\ell} .
\]

Let \( S_\ell \) denote the symmetric group of the set \( \{1, 2, \ldots, \ell\} \). This group \( S_\ell \) acts by \( k \)-algebra homomorphisms on the polynomial ring \( k \left[ x_1, x_2, \ldots, x_\ell \right] \).

If \( \alpha \in \mathbb{N}^\ell \) is any \( \ell \)-tuple, then we define the polynomial \( a_\alpha \in k \left[ x_1, x_2, \ldots, x_\ell \right] \) by

\[
a_\alpha = \sum_{\sigma \in S_\ell} (-1)^\sigma \sigma (x^\alpha) = \det \left( x_\sigma_i^j \right)_{1 \leq i \leq \ell, 1 \leq j \leq \ell} .
\]

This is called the \( \alpha \)-alternant.

We define addition of \( \ell \)-tuples \( \alpha \in \mathbb{N}^\ell \) entrywise (so that \( (\alpha + \beta)_i = \alpha_i + \beta_i \) for every \( \alpha, \beta \in \mathbb{N}^\ell \) and \( i \in \{1, 2, \ldots, \ell\} \)).

It is known ([GriRei18, Corollary 2.6.6]) that

\[
s_\lambda (x_1, x_2, \ldots, x_\ell) = \frac{a_{\lambda + \rho}}{a_\rho} \quad \text{for every} \quad \lambda \in P_\ell .
\]

for every \( \lambda \in P_\ell \).

Now, define \( \alpha \in \mathbb{N}^\ell \) by \( \alpha = \mu + \rho \). Proposition 3.2 (b) yields

\[
G (k) = \prod_{i=1}^\infty \left( x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right) .
\]
Substituting 0, 0, 0, . . . for \( x_{\ell+1}, x_{\ell+2}, x_{\ell+3}, \ldots \) in this equality, we obtain

\[
(G (k)) (x_1, x_2, \ldots, x_\ell)
\]

\[
= \left( \prod_{i=1}^{\ell} (x_i^0 + x_i^1 + \cdots + x_i^{k-1}) \right) \cdot \left( \prod_{i=\ell+1}^{\infty} \left( 0^0 + 0^1 + \cdots + 0^{k-1} \right) \right)
\]

\[
= \prod_{i=1}^{\ell} \left( x_i^0 + x_i^1 + \cdots + x_i^{k-1} \right) = \prod_{i=1}^{\ell} \sum_{j=0}^{k-1} x_i^j
\]

\[
= \sum_{(j_1, j_2, \ldots, j_\ell) \in \{0,1,\ldots,k-1\}^\ell} \frac{x_1^{j_1} x_2^{j_2} \cdots x_\ell^{j_\ell}}{x_\ell^{(j_1, j_2, \ldots, j_\ell)}} \quad \text{(by the product rule)}
\]

\[
= \sum_{(j_1, j_2, \ldots, j_\ell) \in \{0,1,\ldots,k-1\}^\ell} x^{(j_1, j_2, \ldots, j_\ell)}
\]

\[
= \sum_{\beta \in \mathbb{N}^\ell; \beta_i < k \text{ for all } i} x^\beta
\]

(here, we have substituted \( \beta \) for \( (j_1, j_2, \ldots, j_\ell) \) in the sum).

Now, \( (G (k)) (x_1, x_2, \ldots, x_\ell) \in k [x_1, x_2, \ldots, x_\ell] \) is a symmetric polynomial (since
\( G(k) \) is a symmetric power series). But from \( a_\alpha = \sum_{\sigma \in S_\ell} (-1)^\sigma \sigma(x^\alpha) \), we obtain

\[
(G(k))(x_1, x_2, \ldots, x_\ell) \cdot a_\alpha = (G(k))(x_1, x_2, \ldots, x_\ell) \cdot \sum_{\sigma \in S_\ell} (-1)^\sigma \sigma(x^\alpha)
\]

\[
= \sum_{\sigma \in S_\ell} (-1)^\sigma (G(k))(x_1, x_2, \ldots, x_\ell) \cdot \sigma(x^\alpha)
\]

\[
= \sum_{\sigma \in S_\ell} (-1)^\sigma \sigma(G(k))(x_1, x_2, \ldots, x_\ell) \cdot \sigma(x^\alpha)
\]

\[
= \sum_{\sigma \in S_\ell} (-1)^\sigma \sigma(G(k))(x_1, x_2, \ldots, x_\ell) \cdot \sigma(x^\alpha)
\]

\[
= \sum_{\sigma \in S_\ell} (-1)^\sigma \sigma \left( \sum_{\beta \in \mathbb{N}_\ell; \beta_i < k \text{ for all } i} x^\beta \cdot x^\alpha \right)
\]

\[
= \sum_{\sigma \in S_\ell} (-1)^\sigma \sigma \left( \sum_{\beta \in \mathbb{N}_\ell; \beta_i < k \text{ for all } i} x^{\alpha+\beta} \right)
\]

\[
= \sum_{\gamma \in \mathbb{N}_\ell; 0 \leq \gamma_i - \alpha_i < k \text{ for all } i} a_\gamma
\]

(here, we have substituted \( \gamma \) for \( \alpha + \beta \) in the sum).

But the symmetric group \( S_\ell \) acts on the set \( \mathbb{N}_\ell \) by permuting the entries of an \( \ell \)-tuple. This action has the property that

\[
x^{\sigma \cdot \beta} = \sigma(x^\beta) \quad \text{for any } \sigma \in S_\ell \text{ and } \beta \in \mathbb{N}_\ell.
\]

It is well-known (and easy to check using the properties of determinants\(^{10}\)) that if an \( \ell \)-tuple \( \gamma \in \mathbb{N}_\ell \) has two equal entries, then

\[
a_\gamma = 0.
\]

\(^{10}\) Specifically: using the fact that a square matrix with two equal rows always has determinant 0.
Moreover, any $\ell$-tuple $\gamma \in \mathbb{N}^\ell$ and any $\sigma \in S_\ell$ satisfy

$$a_{\sigma, \gamma} = (-1)^\sigma \cdot a_\gamma.$$  \hspace{1cm} (51)

(This, too, follows from the properties of determinants\(^{11}\).)

Let $SP_\ell$ denote the set of all $\ell$-tuples $\delta \in \mathbb{N}^\ell$ such that $\delta_1 > \delta_2 > \cdots > \delta_\ell$. Then, the map

$$P_\ell \rightarrow SP_\ell,$$

$$\lambda \mapsto \lambda + \rho$$  \hspace{1cm} (52)

is a bijection.

If an $\ell$-tuple $\gamma \in \mathbb{N}^\ell$ has no two equal entries, then $\gamma$ can be uniquely written in the form $\sigma \cdot \delta$ for some $\sigma \in S_\ell$ and some $\delta \in SP_\ell$ (indeed, $\delta$ is the result of sorting $\gamma$ into decreasing order, while $\sigma$ is the permutation that achieves this sorting). In other words, the map

$$S_\ell \times SP_\ell \rightarrow \left\{ \gamma \in \mathbb{N}^\ell \mid \text{the } \ell\text{-tuple } \gamma \text{ has no two equal entries} \right\},$$

$$(\sigma, \delta) \mapsto \sigma \cdot \delta$$  \hspace{1cm} (53)

is a bijection.

\(^{11}\)Specifically: using the fact that permuting the rows of a square matrix results in its determinant getting multiplied by the sign of the permutation
Now, (48) becomes
\[
(G (k)) (x_1, x_2, \ldots, x_\ell) \cdot a_\lambda
= \sum_{0 \leq \gamma_i - a_i < k \text{ for all } i} a_\gamma
+ \sum_{0 \leq \gamma_i - a_i < k \text{ for all } i; \text{ the } \ell\text{-tuple } \gamma \text{ has no two equal entries}} a_\gamma
+ \sum_{0 \leq \gamma_i - a_i < k \text{ for all } i; \text{ the } \ell\text{-tuple } \gamma \text{ has two equal entries}} a_\gamma
\]
(by (40))
\[
= \sum_{\gamma \in \mathbb{N}^\ell; 0 \leq \gamma_i - a_i < k \text{ for all } i; \text{ the } \ell\text{-tuple } \gamma \text{ has no two equal entries}} a_\gamma
\]
\[
= \sum_{\gamma \in \mathbb{N}^\ell; 0 \leq \gamma_i - a_i < k \text{ for all } i; \text{ the } \ell\text{-tuple } \gamma \text{ has no two equal entries}} [0 \leq \gamma_i - a_i < k \text{ for all } i] \cdot a_\gamma
\]
\[
= \sum_{(\sigma, \delta) \in \mathfrak{S}_\ell \times \mathfrak{S}_\ell} \left[ \prod_{i=1}^\ell \left[ 0 \leq (\sigma \cdot \delta)_i - a_i < k \right] \right] \cdot a_{\sigma, \delta}
\]
(by (51))
\[
= \prod_{i=1}^\ell \left[ 0 \leq \delta_{\sigma^{-1}(i)} - a_i < k \right]
\]
(here, we have substituted \(\sigma \cdot \delta\) for \(\gamma\) in the sum, since the map (53) is a bijection)
\[
= \prod_{i=1}^\ell \left[ 0 \leq \delta_i - a_{\sigma(i)} < k \right]
\]
(here, we have substituted \(\sigma(i)\) for \(i\) in the product, since \(\sigma\) is a bijection)
\[
= \sum_{\delta \in \mathfrak{S}_\ell} \left[ \prod_{i=1}^\ell \left[ 0 \leq \delta_i - a_{\sigma(i)} < k \right] \right] \cdot (-1)^\sigma a_\delta
\]
\[
= \sum_{\delta \in \mathfrak{S}_\ell} \det \left( \left[ 0 \leq \delta_i - a_j < k \right] \right)_{1 \leq i \leq \ell, 1 \leq j \leq \ell}
\]
(by the definition of a determinant)
\[
= \sum_{\lambda \in \mathfrak{P}_\ell} \det \left( \left[ 0 \leq (\lambda + \rho)_i - a_j < k \right] \right)_{1 \leq i \leq \ell, 1 \leq j \leq \ell}
\]
\[
a_{\lambda + \rho}
\]
(here, we have substituted \(\lambda + \rho\) for \(\delta\) in the sum, since the map (52) is a bijection).
Every $\lambda \in P_\ell$ and every $i, j \in \{1, 2, \ldots, \ell\}$ satisfy

$$(\lambda + \rho)_j - a_j = \lambda_i + \ell - i - (\mu + \rho)_j = \lambda_i + \ell - i - (\mu + \ell - j)$$

(since the definition of $\rho$ yields $\rho_i = \ell - i$)

$$= \lambda_i + \ell - i - (\mu + \rho)_j = \lambda_i + \ell - i - (\mu + \ell - j)$$

(since the definition of $\rho$ yields $\rho_j = \ell - j$)

$$= \lambda_i - \mu_j - i + j. \quad (55)$$

Now, (47) (applied to $\lambda = \mu$) yields

$$s_\mu (x_1, x_2, \ldots, x_\ell) = \frac{a_{\mu + \rho}}{a_\rho} = \frac{a_\rho}{a_\rho}$$

(since $\mu + \rho = \alpha$). Multiplying this equality by $(G (k)) (x_1, x_2, \ldots, x_\ell)$, we find

$$\begin{align*}
(G (k)) (x_1, x_2, \ldots, x_\ell) \cdot s_\mu (x_1, x_2, \ldots, x_\ell) &= (G (k)) (x_1, x_2, \ldots, x_\ell) \cdot \frac{a_\alpha}{a_\rho} \\
&= \frac{1}{a_\rho} \cdot (G (k)) (x_1, x_2, \ldots, x_\ell) \cdot \frac{a_\rho}{a_\rho} \\
&= \sum_{\lambda \in P_\ell} \det \left( \left( \begin{array}{c} 0 \\ \begin{array}{c} \downarrow \downarrow \\ \begin{array}{c} 0 \leq (\lambda + \rho)_j - a_j < k \\
\downarrow \downarrow \\
= \lambda_i - \mu_j - i + j \\
(\text{by (55)})
\end{array}
\end{array} \right)_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right) \frac{a_\lambda + \rho}{a_\rho} \left( \begin{array}{c} \mathbf{s}_\lambda (x_1, x_2, \ldots, x_\ell) \\
(\text{by (47)})
\end{array} \right)
\end{align*}$$

$$= \sum_{\lambda \in P_\ell} \det \left( \left( \begin{array}{c} 0 \leq (\lambda + \rho)_j - a_j < k \\
\downarrow \downarrow \\
= \lambda_i - \mu_j - i + j \\
(\text{by (55)})
\end{array} \right)_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right) \frac{a_\lambda + \rho}{a_\rho} \left( \begin{array}{c} \mathbf{s}_\lambda (x_1, x_2, \ldots, x_\ell) \\
(\text{by (47)})
\end{array} \right)$$

$$= \sum_{\lambda \in P_\ell} \det \left( \left( \begin{array}{c} 0 \leq \lambda_i - \mu_j - i + j < k \\
\downarrow \downarrow \\
= \mathbf{pet}_k (\lambda, \mu) \\
(\text{by the definition of } \mathbf{pet}_k (\lambda, \mu))
\end{array} \right)_{1 \leq i \leq \ell, 1 \leq j \leq \ell} \right) \mathbf{s}_\lambda (x_1, x_2, \ldots, x_\ell)$$

This proves (46). As mentioned above, this completes the proof of Theorem 4.4. \qed

We can also generalize Corollary 3.9 to obtain a Pieri-like rule for multiplication by $G (k, m)$:
Corollary 4.5. Let $k$ be a positive integer. Let $m \in \mathbb{N}$. Let $\mu \in \text{Par}$. Then,

$$
G(k, m) \cdot s_\mu = \sum_{\lambda \in \text{Par}_{m+|\mu|}} \text{pet}_k(\lambda, \mu) s_\lambda.
$$

Proof. Follows from Theorem 4.4 by projecting onto the $(m + |\mu|)$-th graded component of $\Lambda$. \hfill \Box

5. Coproducts of Petrie functions

The $k$-algebra $\Lambda$ is a Hopf algebra due to the presence of a comultiplication $\Delta : \Lambda \to \Lambda \otimes \Lambda$ \footnote{Here and in the following, the “$\otimes$” sign denotes $\otimes_k$.} We recall (from [GriRei18, §2.1]) one way to define this comultiplication:

Consider the rings

$$
\mathbb{k}[[x]] := \mathbb{k}[[x_1, x_2, x_3, \ldots]] \quad \text{and} \quad \mathbb{k}[[x,y]] := \mathbb{k}[[x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots]]
$$

of formal power series. We shall use the notations $x$ and $y$ for the sequences $(x_1, x_2, x_3, \ldots)$ and $(y_1, y_2, y_3, \ldots)$ of indeterminates. If $f \in \mathbb{k}[[x]]$ is any formal power series, then $f(y)$ shall mean the result of substituting $y, y, y, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ in $f$. (This will be a formal power series in $\mathbb{k}[[y_1, y_2, y_3, \ldots]]$.) For the sake of symmetry, we also use the analogous notation $f(x)$ for the result of substituting $x_1, x_2, x_3, \ldots$ for $x_1, x_2, x_3, \ldots$ in $f$; of course, this $f(x)$ is just $f$. Finally, if the power series $f \in \mathbb{k}[[x]]$ is symmetric, then we use the notation $f(x,y)$ for the result of substituting the variables $x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots$ for the variables $x_1, x_2, x_3, \ldots$ in $f$ (that is, choosing some bijection $\phi : \{x_1, x_2, x_3, \ldots\} \to \{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$ \footnote{Such bijections clearly exist, since the sets $\{x_1, x_2, x_3, \ldots\}$ and $\{x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}$ have the same cardinality (namely, $\aleph_0$). This is one of several observations commonly illustrated by the story of “Hilbert’s hotel”.} and substituting $\phi(x_i)$ for each $x_i$ in $f$). This result does not depend on the order in which the former variables are substituted for the latter (i.e., on the choice of bijection $\phi$) because $f$ is symmetric.

Now, the comultiplication of $\Lambda$ is the map $\Delta : \Lambda \to \Lambda \otimes \Lambda$ determined as follows: For a symmetric function $f \in \Lambda$, we have

$$
\Delta(f) = \sum_{i \in I} f_{1,i} \otimes f_{2,i}, \quad (56)
$$

where $f_{1,i}, f_{2,i} \in \Lambda$ are such that

$$
f(x,y) = \sum_{i \in I} f_{1,i}(x) f_{2,i}(y). \quad (57)
$$
More precisely, if \( f \in \Lambda \), if \( I \) is a finite set, and if \( (f_{i,j})_{i \in I} \in \Lambda^I \) and \( (f_{2,i})_{i \in I} \in \Lambda^I \) are two families satisfying (57), then \( \Delta (f) \) is given by (56).14

For example, for any \( n \in \mathbb{N} \), it is easy to see that

\[
e_n(x,y) = \sum_{i=0}^{n} e_i(x) e_{n-i}(y),
\]

and thus

\[
\Delta (e_n) = \sum_{i=0}^{n} e_i \otimes e_{n-i}.
\]

**Theorem 5.1.** Let \( k \) be a positive integer. Let \( m \in \mathbb{N} \). Then,

\[
\Delta (G(k,m)) = \sum_{i=0}^{m} G(k,i) \otimes G(k,m-i).
\]

**Proof of Theorem 5.1.** In this proof, the word “monomial” may refer to a monomial in any set of variables (not necessarily in \( x_1, x_2, x_3, \ldots \)).

In the following, an \( i \)-monomial (where \( i \in \mathbb{N} \)) shall mean a monomial of degree \( i \).

We shall say that a monomial is \( k \)-bounded if all exponents in this monomial are \(< k \). In other words, a monomial is \( k \)-bounded if it can be written in the form \( z_1^{a_1} z_2^{a_2} \cdots z_s^{a_s} \), where \( z_1, z_2, \ldots, z_s \) are distinct variables and \( a_1, a_2, \ldots, a_s \) are nonnegative integers \(< k \). Thus, the \( k \)-bounded monomials in the variables \( x_1, x_2, x_3, \ldots \) are precisely the monomials of the form \( x^\alpha \) for \( \alpha \in \mathbb{N}^k \) satisfying \( |\alpha| = m \) and \( \alpha_i < k \) for all \( i \).

Hence, the \( k \)-bounded \( m \)-monomials in the variables \( x_1, x_2, x_3, \ldots \) are precisely the monomials of the form \( x^\alpha \) for \( \alpha \in \mathbb{N}^k \) satisfying \( |\alpha| = m \) and \( \alpha_i < k \) for all \( i \).

Now, the definition of \( G(k,m) \) yields

\[
G(k,m) = \sum_{\alpha \in \mathbb{N}^k; \atop |\alpha| = m; \atop \alpha_i < k \text{ for all } i} x^\alpha
\]

(58)

(since the \( k \)-bounded \( m \)-monomials in the variables \( x_1, x_2, x_3, \ldots \) are precisely the monomials of the form \( x^\alpha \) for \( \alpha \in \mathbb{N}^k \) satisfying \( |\alpha| = m \) and \( \alpha_i < k \) for all \( i \)).

Renaming the variables \( x_1, x_2, x_3, \ldots \) as \( x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots \) (in some order) in this equality, we obtain

\[
(G(k,m))(x,y) = \text{(the sum of all } k \text{-bounded } m \text{-monomials in the variables } x_1, x_2, x_3, \ldots \text{)}
\]

Now, the language of [GriRei18 §2.1], this can be restated as \( \Delta (f) = f(x,y) \), because \( \Lambda \otimes \Lambda \) is identified with a certain subring of \( k[[x,y]] \) in [GriRei18 §2.1] (via the injection \( \Lambda \otimes \Lambda \to k[[x,y]] \)) that sends any \( u \otimes v \in \Lambda \otimes \Lambda \) to \( u(x) v(y) \in k[[x,y]] \).
(since \((G(k,m))(x,y)\) is obtained from \(G(k,m)\) by renaming the variables \(x_1, x_2, x_3, \ldots\) as \(x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\) in some order, but obviously, renaming the variables preserves the set of \(k\)-bounded \(m\)-monomials).

But any monomial \(m\) in the variables \(x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\) can be uniquely written as a product \(np\), where \(n\) is a monomial in the variables \(x_1, x_2, x_3, \ldots\) and where \(p\) is a monomial in the variables \(y_1, y_2, y_3, \ldots\). Moreover, if \(m\) is written in this form, then:

- the degree of \(m\) equals the sum of the degrees of \(n\) and \(p\);
- thus, \(m\) is an \(m\)-monomial if and only if there exists some \(i \in \mathbb{N}\) such that \(n\) is an \(i\)-monomial and \(p\) is an \((m - i)\)-monomial;
- furthermore, \(m\) is \(k\)-bounded if and only if both \(n\) and \(p\) are \(k\)-bounded.

Thus, any \(k\)-bounded \(m\)-monomial \(m\) in the variables \(x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\) can be uniquely written as a product \(np\), where \(i \in \mathbb{N}\), where \(n\) is a \(k\)-bounded \(i\)-monomial in the variables \(x_1, x_2, x_3, \ldots\) and where \(p\) is a \(k\)-bounded \((m - i)\)-monomial in the variables \(y_1, y_2, y_3, \ldots\). Conversely, every such product is a \(k\)-bounded \(m\)-monomial \(m\) in the variables \(x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\). Thus, we obtain a bijection

\[
\bigcup_{i=0}^{m} \left( \{k\text{-bounded } i\text{-monomials in the variables } x_1, x_2, x_3, \ldots\} \right)
\times \left\{k\text{-bounded } (m - i)\text{-monomials in the variables } y_1, y_2, y_3, \ldots\right\}
\rightarrow \{k\text{-bounded } m\text{-monomials in the variables } x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots\}
\]

that sends each pair \((n, p)\) to \(np\). Thus, the product rule yields

\[
\text{(the sum of all } k\text{-bounded } m\text{-monomials in the variables } x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots) = \sum_{i=0}^{m} \left( \text{(the sum of all } k\text{-bounded } i\text{-monomials in the variables } x_1, x_2, x_3, \ldots) \cdot \text{(the sum of all } k\text{-bounded } (m - i)\text{-monomials in the variables } y_1, y_2, y_3, \ldots) \right).
\]

But the same reasoning that gave us \((58)\) can be applied to \(i\) instead of \(m\). Thus we obtain

\[
G(k,i) = \text{(the sum of all } k\text{-bounded } i\text{-monomials in the variables } x_1, x_2, x_3, \ldots).
\]

(61)

Also, the same reasoning that gave us \((58)\) can be applied to \(m - i\) instead of \(m\). Thus we obtain

\[
G(k,m - i) = \text{(the sum of all } k\text{-bounded } (m - i)\text{-monomials in the variables } x_1, x_2, x_3, \ldots).
\]
Renaming the variables \(x_1, x_2, x_3, \ldots\) as \(y_1, y_2, y_3, \ldots\) in this equality, we obtain
\[
(G(k, m - i))(y)
= (\text{the sum of all } k\text{-bounded } (m - i)\text{-monomials in the variables } y_1, y_2, y_3, \ldots).
\] (62)

Now, (59) becomes
\[
(G(k, m))(x, y)
= (\text{the sum of all } k\text{-bounded } m\text{-monomials in the variables } x_1, x_2, x_3, \ldots, y_1, y_2, y_3, \ldots)
\]
\[
= \sum_{i=0}^{m} (\text{the sum of all } k\text{-bounded } i\text{-monomials in the variables } x_1, x_2, x_3, \ldots)
\cdot (\text{the sum of all } k\text{-bounded } (m - i)\text{-monomials in the variables } y_1, y_2, y_3, \ldots)
\]
\[
= \sum_{i=0}^{m} (G(k,i))(x) \cdot (G(k,m-i))(y)
\] (by (60))
\[
= \sum_{i=0}^{m} G(k,i) \otimes G(k,m-i).
\]

Hence, (57) holds for \(f = G(k, m), I = \{0, 1, \ldots, m\}, (f_1,i)_{i \in I} = (G(k,i))_{i \in \{0,1,\ldots,m\}}\) and \((f_2,i)_{i \in I} = (G(k,m-i))_{i \in \{0,1,\ldots,m\}}\). Therefore, (56) (applied to these \(f, I, (f_1,i)_{i \in I}\) and \((f_2,i)_{i \in I}\)) yields
\[
\Delta(G(k,m)) = \sum_{i=0}^{m} G(k,i) \otimes G(k,m-i).
\]

This proves Theorem 5.1 \(\Box\)

6. Petrie functions and Frobenius maps

6.1. The Frobenius formula for Petrie functions

We shall next derive another formula for the Petrie symmetric functions \(G(k,m)\). For this formula, we need the following definition ([GriRei18, Exercise 2.9.9]):

**Definition 6.1.** Let \(n \in \{1, 2, 3, \ldots\}\). We define a map \(f_n : \Lambda \to \Lambda\) by
\[
(f_n(a)) = a(x_1^n, x_2^n, x_3^n, \ldots) \quad \text{for each } a \in \Lambda.
\]

This map \(f_n\) is called the \(n\)-th Frobenius endomorphism of \(\Lambda\).
It is known (from [GriRei18, Exercise 2.9.9(d)]) that this map $f_n : \Lambda \to \Lambda$ is a Hopf algebra endomorphism of $\Lambda$.

**Theorem 6.2.** Let $k$ be a positive integer. Let $m \in \mathbb{N}$. Then,

$$G(k, m) = \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot f_k(e_i).$$

(The sum on the right hand side of this equality is well-defined, since all sufficiently high $i \in \mathbb{N}$ satisfy $m - ki < 0$ and thus $h_{m-ki} = 0$.)

Proposition 2.6 can be viewed as a particular case of Theorem 6.2 (applied to $n$ and $n + k$ instead of $k$ and $m$), after realizing that in the sum on the right hand side, only the first two addends will (potentially) be nonzero in this case.

**Proof of Theorem 6.2.** Consider the ring $(k[[x_1, x_2, x_3, \ldots]])[[t]]$ of formal power series in one indeterminate $t$ over $k[[x_1, x_2, x_3, \ldots]]$. In this ring, define the two power series

$$H(t) = \prod_{i=1}^{\infty} (1 - x_i t)^{-1} \in (k[[x_1, x_2, x_3, \ldots]])[[t]]$$

and

$$E(t) = \prod_{i=1}^{\infty} (1 + x_i t) \in (k[[x_1, x_2, x_3, \ldots]])[[t]].$$

Then, from [GriRei18], we know the identities

$$H(t) = \sum_{n \geq 0} h_n t^n$$

and

$$E(t) = \sum_{n \geq 0} e_n t^n.$$

(Indeed, the first of these two identities is [GriRei18 (2.4.1)], whereas the second is [GriRei18 (2.4.2)].)

Now, consider the map

$$F_k : k[[x_1, x_2, x_3, \ldots]] \to k[[x_1, x_2, x_3, \ldots]],$$

$$a \mapsto a\left( x_1^k, x_2^k, x_3^k, \ldots \right).$$

This map $F_k$ is a $k$-algebra homomorphism (since it is an evaluation homomorphism). Hence, it induces a continuous $^{15}k[[t]]$-algebra homomorphism $F_k[[t]] : (k[[x_1, x_2, x_3, \ldots]])[[t]] \to (k[[x_1, x_2, x_3, \ldots]])[[t]]$ that sends each formal power series $\sum_{n \geq 0} a_n t^n \in (k[[x_1, x_2, x_3, \ldots]])[[t]]$ (with $a_n \in k[[x_1, x_2, x_3, \ldots]]$) to $\sum_{n \geq 0} F_k(a_n) t^n$. Consider this $k[[t]]$-algebra homomorphism $F_k[[t]]$. In particular, it satisfies

$$\left( F_k[[t]] \right)(t^i) = t^i$$

for each $i \in \mathbb{N}$.

$^{15}$Continuity is defined with respect to the usual topology on $(k[[x_1, x_2, x_3, \ldots]])[[t]]$. 
The definition of $F_k$ yields
\[ F_k(x_i) = x_i^k \quad \text{for each } i \in \{1, 2, 3, \ldots\}. \tag{63} \]
Also, for each $a \in \Lambda$, we have
\[ F_k(a) = a \left( x_1^k, x_2^k, x_3^k, \ldots \right) \quad \text{(by the definition of } F_k) \]
\[ = f_k(a) \tag{64} \]
(since the definition of $f_k$ yields $f_k(a) = a \left( x_1^k, x_2^k, x_3^k, \ldots \right)$).

Applying the map $F_k[[t]]$ to both sides of the equality $E(t) = \sum_{n \geq 0} e_n t^n$, we obtain
\[ (F_k[[t]]) (E(t)) = (F_k[[t]]) \left( \sum_{n \geq 0} e_n t^n \right) = \sum_{n \geq 0} F_k(e_n) t^n \tag{65} \]
(by the definition of $F_k[[t]]$). On the other hand, applying $F_k[[t]]$ to both sides of the equality $E(t) = \prod_{i=1}^{\infty} (1 + x_i t)$, we obtain
\[ (F_k[[t]]) (E(t)) = (F_k[[t]]) \left( \prod_{i=1}^{\infty} (1 + x_i t) \right) = \prod_{i=1}^{\infty} (1 + x_i^k t) \]
(by the definition of $F_k[[t]]$).

Comparing this equality with (65), we find
\[ \sum_{n \geq 0} F_k(e_n) t^n = \prod_{i=1}^{\infty} \left( 1 + x_i^k t \right). \]
Substituting $-t^k$ for $t$ in this equality, we find
\[ \sum_{n \geq 0} F_k(e_n) \left( -t^k \right)^n = \prod_{i=1}^{\infty} \left( 1 + x_i^k \left( -t^k \right) \right) = \prod_{i=1}^{\infty} \left( 1 - (x_i t)^k \right) = (1 - x_i t) \left( (x_i t)^0 + (x_i t)^1 + \cdots + (x_i t)^{k-1} \right) \]
\[ = \prod_{i=1}^{\infty} \left( (1 - x_i t) \left( (x_i t)^0 + (x_i t)^1 + \cdots + (x_i t)^{k-1} \right) \right) \]
\[ = \left( \prod_{i=1}^{\infty} (1 - x_i t) \right) \left( \prod_{i=1}^{\infty} \left( (x_i t)^0 + (x_i t)^1 + \cdots + (x_i t)^{k-1} \right) \right). \]
We can divide both sides of this equality by \( \prod_{i=1}^{\infty} (1 - x_i t) \) (since the formal power series \( \prod_{i=1}^{\infty} (1 - x_i t) \) has constant term 1 and thus is invertible), and thus obtain

\[
\frac{\sum_{n \geq 0} F_k(e_n)(-t^k)^n}{\prod_{i=1}^{\infty} (1 - x_i t)} = \sum_{a \in \{0, 1, \ldots, k-1\}^\infty \text{ is a weak composition}} \frac{(x_1 t)^a_1 (x_2 t)^a_2 (x_3 t)^a_3 \ldots}{(x_1^{a_1} x_2^{a_2} x_3^{a_3} \ldots) (t^{a_1} t^{a_2} t^{a_3} \ldots)} = \sum_{a \in WC; \ a_i < k \text{ for all } i} x^a t^{|a|}
\]

(here, we have expanded the product)

\[
= \sum_{a \in WC; \ a_i < k \text{ for all } i} x^a t^{|a|}.
\]

Hence,

\[
\sum_{n \geq 0} \frac{F_k(e_n)(-t^k)^n}{\prod_{i=1}^{\infty} (1 - x_i t)} = \left( \sum_{n \geq 0} F_k(e_n)(-t^k)^n \right) \cdot \prod_{i=1}^{\infty} (1 - x_i t)^{-1} = H(t)^{-1} = H(t)
\]

(by the definition of \( H(t) \))

\[
= \left( \sum_{n \geq 0} F_k(e_n)(-1)^n t^{kn} \right) \cdot \frac{H(t)}{\sum_{n \geq 0} h_{n+1}^n = \sum_{j \geq 0} h_{j+1}^j} = \sum_{j \geq 0} f_k(e_n)(-1)^n h_j t^j = \sum_{j \geq 0} f_k(e_n)(-1)^n h_j t^{kn+j}.
\]

This is an equality between two power series in (\( k [[x_1, x_2, x_3, \ldots]] [t] \)). If we compare the coefficients of \( t^m \) on both sides of it (where \( x_1, x_2, x_3, \ldots \) are considered scalars, not monomials), we obtain

\[
\sum_{a \in WC; \ a_i < k \text{ for all } i; \ |a| = m} x^a = \sum_{(n, j) \in \mathbb{N}^2; \ kn+j = m} f_k(e_n)(-1)^n h_j = \sum_{n \in \mathbb{N}} f_k(e_n)(-1)^n \cdot \sum_{j \in \mathbb{N}; \ kn+j = m} h_j.
\]
Now, the definition of $G(k, m)$ yields

$$G(k, m) = \sum_{\alpha \in WC; |\alpha| = m; \alpha_i < k \text{ for all } i} x^\alpha = \sum_{\alpha \in WC; |\alpha| = m; \alpha_i < k \text{ for all } i} x^\alpha$$

$$= \sum_{n \in \mathbb{N}} f_k(e_n) (-1)^n \cdot \sum_{j \in \mathbb{N}; kn+j=m} h_j. \tag{66}$$

But for each $n \in \mathbb{N}$, we have

$$\sum_{j \in \mathbb{N}; kn+j=m} h_j = h_{m-kn}. \tag{67}$$

[Proof of (67): Let $n \in \mathbb{N}$. We must prove the equality (67). If $m - kn < 0$, then $h_{m-kn} = 0$ and $\sum_{j \in \mathbb{N}; kn+j=m} h_j = (\text{empty sum}) = 0$. Thus, if $m - kn < 0$, then $\sum_{j \in \mathbb{N}; kn+j=m} h_j = 0 = h_{m-kn}$ is proven. Therefore, for the rest of the proof of (67), we WLOG assume that $m - kn \geq 0$. Hence, the sum $\sum_{j \in \mathbb{N}; kn+j=m} h_j$ has exactly one addend, namely the addend for $j = m - kn$. Therefore,

$$\sum_{j \in \mathbb{N}; kn+j=m} h_j = h_{m-kn}. \text{ This proves (67).}$$

Now, (66) becomes

$$G(k, m) = \sum_{n \in \mathbb{N}} f_k(e_n) (-1)^n \cdot \sum_{j \in \mathbb{N}; kn+j=m} h_j = \underbrace{\sum_{n \in \mathbb{N}} (-1)^n h_{m-kn} \cdot f_k(e_n) = \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot f_k(e_i)}_{\text{by (67)}}$$

(here, we have renamed the summation index $n$ as $i$). This proves Theorem 6.2. □

### 6.2. Computing the Hall inner products $(p_m, G(k, m))$

Now we recall another definition (see [GriRei18, Exercise 2.9.10]):

**Definition 6.3.** Let $n \in \{1, 2, 3, \ldots\}$. We define a $k$-algebra homomorphism $v_n : \Lambda \to \Lambda$ by

$$v_n(h_m) = \begin{cases} h_{m/n}, & \text{if } n \mid m; \\
0, & \text{if } n \nmid m \end{cases} \text{ for each } m > 0.$$
(This is well-defined, since the sequence \((h_1, h_2, h_3, \ldots)\) is an algebraically independent generating set of the commutative \(k\)-algebra \(\Lambda\).)

This map \(v_n\) is called the \(n\)-th Verschiebung endomorphism of \(\Lambda\).

Again, it is known ([GriRei18, Exercise 2.9.10(e)]) that this map \(v_n: \Lambda \to \Lambda\) is a Hopf algebra endomorphism of \(\Lambda\). Moreover, the following holds ([GriRei18, Exercise 2.9.10(f)]):

**Proposition 6.4.** Let \(n \in \{1, 2, 3, \ldots\}\). Then, the maps \(f_n: \Lambda \to \Lambda\) and \(v_n: \Lambda \to \Lambda\) are adjoint with respect to the Hall inner product on \(\Lambda\). That is, any \(a \in \Lambda\) and \(b \in \Lambda\) satisfy

\[
(a, f_n(b)) = (v_n(a), b).
\]

Furthermore, any positive integers \(n\) and \(m\) satisfy

\[
v_n(p_m) = \begin{cases} np_{m/n}, & \text{if } n \mid m; \\
0, & \text{if } n \not| m. \end{cases}
\]  

(68)

(This is [GriRei18, Exercise 2.9.10(a)].)

We shall use this to compute some Hall inner products:

**Lemma 6.5.** Let \(k\) and \(m\) be positive integers. Let \(j \in \mathbb{N}\). Then, \((p_m, f_k(e_j)) = (-1)^{j-1} \left[ m = kj \right] k\).

Here, we are again using the Iverson bracket notation.

**Proof of Lemma 6.5** Applying (68) to \(n = k\), we obtain

\[
v_k(p_m) = \begin{cases} kp_{m/k}, & \text{if } k \mid m; \\
0, & \text{if } k \not| m. \end{cases}
\]  

(69)

Applying Proposition 6.4 to \(n = k\), \(a = p_m\) and \(b = e_j\), we obtain

\[
(p_m, f_k(e_j)) = (v_k(p_m), e_j).
\]  

(70)

Now, we are in one of the following three cases:

*Case 1:* We have \(m = kj\).

*Case 2:* We have \(k \not| m\).

*Case 3:* We have neither \(m = kj\) nor \(k \not| m\).

Let us first consider Case 1. In this case, we have \(m = kj\). Thus, \(k \mid m\) and \(m/k = j\). Hence, (69) becomes

\[
v_k(p_m) = \begin{cases} kp_{m/k}, & \text{if } k \mid m; \\
0, & \text{if } k \not| m = kp_{m/k} \quad \text{(since } k \mid m) \\
= kp_j & \text{(since } m/k = j). \end{cases}
\]
Thus, (70) becomes

\[
(p_m, f_k(e_j)) = \left( v_k(p_m), e_j \right) = (kp_j, e_j)
\]

\[
= (e_j, kp_j) \quad \text{(since the Hall inner product is symmetric)}
\]

\[
= k \left( e_j, p_j \right) = k (-1)^{j-1} = (-1)^{j-1} k.
\]

(by Proposition 2.10 applied to \( n=j \))

Comparing this with

\[
(-1)^{j-1} \left[ m = kj \right] k = (-1)^{j-1} k,
\]

we obtain \( (p_m, f_k(e_j)) = (-1)^{j-1} [m = kj] k. \) Thus, Lemma 6.5 is proven in Case 1.

Let us next consider Case 2. In this case, we have \( k \nmid m. \) Hence, \( m \neq kj \) (since otherwise, we would have \( m = kj, \) thus \( k \mid m, \) contradicting \( k \nmid m \)). Now, (69) becomes

\[
v_k(p_m) = \begin{cases} kp_{m/k}, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m = 0 \quad \text{(since } k \nmid m). \end{cases}
\]

Thus, (70) becomes

\[
(p_m, f_k(e_j)) = \left( v_k(p_m), e_j \right) = (0, e_j) = 0.
\]

Comparing this with

\[
(-1)^{j-1} \left[ m = kj \right] k = 0,
\]

we obtain \( (p_m, f_k(e_j)) = (-1)^{j-1} [m = kj] k. \) Thus, Lemma 6.5 is proven in Case 2.

Let us finally consider Case 3. In this case, we have neither \( m = kj \) nor \( k \nmid m. \) In other words, we have \( m \neq kj \) and \( k \mid m. \) From \( m \neq kj, \) we obtain \( m/k \neq j. \) Thus, the symmetric functions \( p_{m/k} \) and \( e_j \) are homogeneous of different degrees, and therefore satisfy \( (p_{m/k}, e_j) = 0 \) (since the Hall inner product is graded).

Now, (69) becomes

\[
v_k(p_m) = \begin{cases} kp_{m/k}, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m = kp_{m/k} \quad \text{(since } k \mid m). \end{cases}
\]

\[\text{16since } p_{m/k} \text{ is homogeneous of degree } m/k, \text{ whereas } e_j \text{ is homogeneous of degree } j.\]
Thus, (70) becomes

\[ (p_m, f_k(e_j)) = \left( \frac{v_k(p_m), e_j}{= kp_m/k} \right) = (kp_m/k, e_j) = k \left( \frac{p_m/k, e_j}{=0} \right) = 0. \]

Comparing this with

\[ (-1)^{l-1} \begin{array}{c} m = kj \\ = 0 \end{array} k = 0, \]

we obtain \((p_m, f_k(e_j)) = (-1)^{l-1}[m = kj]k.\) Thus, Lemma 6.5 is proven in Case 3.

We have thus proven Lemma 6.5 in all three Cases 1, 2 and 3. \( \square \)

**Lemma 6.6.** Let \(m, \alpha\) and \(\beta\) be positive integers. Let \(a\) be a homogeneous symmetric function of degree \(\alpha\). Let \(b\) be a homogeneous symmetric function of degree \(\beta\). Then, \((p_m, ab) = 0.\)

**Proof of Lemma 6.6** The symmetric function \(a\) is homogeneous of degree \(\alpha\), whereas the symmetric function 1 is homogeneous of degree 0. Since \(\alpha \neq 0\) (because \(\alpha\) is positive), this shows that \((1, a) = 0\) (since the Hall inner product is graded). Similarly, \((1, b) = 0.\)

Let \(\tilde{p}_m\) be the map \(\Lambda \to k, g \mapsto (p_m, g)\). This is a \(k\)-linear map.

The power sum \(p_m\) is primitive as an element of the Hopf algebra \(\Lambda\) (see [GriRei18 Proposition 2.3.6]). Now, consider the graded dual \(\Lambda^\circ\) of the Hopf algebra \(\Lambda\) (as defined in [GriRei18 §1.6]). The map \(\Phi : \Lambda \to \Lambda^\circ\) that sends each \(f \in \Lambda\) to the \(k\)-linear map \(\Lambda \to k, g \mapsto (f, g)\) is a Hopf algebra isomorphism (by [GriRei18 Corollary 2.5.14]). Thus, this map \(\Phi\) sends primitive elements of \(\Lambda\) to primitive elements of \(\Lambda^\circ\). Hence, in particular, \(\Phi(p_m) \in \Lambda^\circ\) is primitive (since \(p_m \in \Lambda\) is primitive). In other words, \(\tilde{p}_m = \tilde{p}_m^\circ\) is primitive (since the definitions of \(\Phi\) and of \(\tilde{p}_m\) quickly reveal that \(\Phi(p_m) = \tilde{p}_m\)). In other words, \(\Delta_{\Lambda^\circ} (\tilde{p}_m) = 1 \otimes \tilde{p}_m + \tilde{p}_m \otimes 1.\)

Now, the definition of \(\tilde{p}_m\) yields \(\tilde{p}_m(ab) = (p_m, ab),\) so that

\[ (p_m, ab) = \tilde{p}_m(ab) = (\tilde{p}_m, ab) = \left( \frac{\Delta_{\Lambda^\circ} (\tilde{p}_m), a \otimes b}{= 1 \otimes \tilde{p}_m + \tilde{p}_m \otimes 1} \right) \quad \text{(by the definition of } \Delta_{\Lambda^\circ}) \]

\[ = (1 \otimes \tilde{p}_m + \tilde{p}_m \otimes 1, a \otimes b) = (1 \otimes \tilde{p}_m, a \otimes b) + (\tilde{p}_m \otimes 1, a \otimes b) \]

\[ = (1, a) (\tilde{p}_m, b) + (\tilde{p}_m, a) (1, b) = 0. \]

This proves Lemma 6.6 \(\square\)

Next, we need an analogue of Proposition 2.10.
Proposition 6.7. Let \( n \) be a positive integer. Then, \((h_n, p_n) = 1\).

Proof of Proposition 6.7. There are myriad ways to prove this. Here is perhaps the simplest one: Let us use the notations of [GriRei18, §2.5], and let us also use the notation \( h_\lambda \) as defined in the proof of Theorem 3.8. From [GriRei18, Corollary 2.5.17(a)], we see that the families \((h_\lambda)_\lambda \in \text{Par}\) and \((m_\lambda)_\lambda \in \text{Par}\) are dual bases with respect to the Hall inner product. Thus,

\[
(h_\lambda, m_\mu) = \delta_{\lambda,\mu} \quad \text{for any } \lambda \in \text{Par} \text{ and } \mu \in \text{Par}.
\]

Applying this to \( \lambda = (n) \) and \( \mu = (n) \), we obtain

\[
(h_{(n)}, m_{(n)}) = \delta_{(n),(n)} = 1.
\]

In view of \( h_\lambda = h_\lambda \) = \( h_n \) and \( m_\lambda = m_\lambda \) = \( p_n \), this rewrites as \((h_n, p_n) = 1\). This proves Proposition 6.7.

Proposition 6.8. Let \( k \) and \( m \) be positive integers. Then, \((p_m, G(k, m)) = 1 - \lfloor k \mid m \rfloor k\).

Proof of Proposition 6.8. Theorem 6.2 yields

\[
G(k, m) = \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot f_k(e_i).
\]

Hence,

\[
(p_m, G(k, m)) = \left( p_m, \sum_{i \in \mathbb{N}} (-1)^i h_{m-ki} \cdot f_k(e_i) \right)
\]

\[
= \sum_{i \in \mathbb{N}} (-1)^i (p_m, h_{m-ki} \cdot f_k(e_i)) \quad (71)
\]

(since the Hall inner product is \( k \)-bilinear).

Now, we claim that every \( i \in \mathbb{N} \setminus \{0, m/k\} \) satisfies

\[
(p_m, h_{m-ki} \cdot f_k(e_i)) = 0. \quad (72)
\]

[Proof of (72): Let \( i \in \mathbb{N} \setminus \{0, m/k\} \). Thus, \( i \neq 0 \) and \( i \neq m/k \). From \( i \neq m/k \), we obtain \( ki \neq m \), so that \( m-ki \neq 0 \).

We must prove the equality (72). If \( m-ki < 0 \), then \( h_{m-ki} = 0 \), and therefore

\[
\left( p_m, h_{m-ki} \cdot f_k(e_i) \right) = (p_m, 0) = 0.
\]

Hence, the equality (72) is proven if \( m-ki < 0 \). Thus, for the rest of this proof, we WLOG assume that \( m-ki \geq 0 \). Combining this with \( m-ki \neq 0 \), we obtain \( m-ki > 0 \). Thus, \( m-ki \) is a positive integer. Also, \( i \) is a positive integer (since \( i \in \mathbb{N} \) and \( i \neq 0 \)), and thus \( ki \) is a positive integer (since \( k \) is a positive integer).

If \( g \in \Lambda \) is any homogeneous symmetric function of some degree \( \gamma \), then \( f_k(g) \) is a homogeneous symmetric function of degree \( k\gamma \) (indeed, this follows easily from the definition of \( f_k \), since \( f_k \) replaces each \( x_i \) to \( x_i^k \) in a symmetric function).
Applying this to $g = e_i$ and $\gamma = i$, we conclude that $f_k(e_i)$ is a homogeneous symmetric function of degree $ki$ (since $e_i$ is a homogeneous symmetric function of degree $i$). Also, $h_{m-ki}$ is a homogeneous symmetric function of degree $m - ki$.

Hence, Lemma 6.6 (applied to $\alpha = m - ki$, $a = h_{m-ki}$, $\beta = ki$ and $b = f_k(e_i)$) yields $(p_m, h_{m-ki} \cdot f_k(e_i)) = 0$. This proves (72).

Note that $e_0 = 1$ and thus $f_k(e_0) = f_k(1) = 1$ (by the definition of $f_k$).

Note that $m/k > 0$ (since $m$ and $k$ are positive). Hence, $m/k \neq 0$. Now, we are in one of the following two cases:

Case 1: We have $k \mid m$.

Case 2: We have $k \nmid m$.

Let us consider Case 1 first. In this case, we have $k \mid m$. Hence, $m/k$ is a positive integer (since $m$ and $k$ are positive integers). Thus, 0 and $m/k$ are two distinct elements of $\mathbb{N}$ (indeed, they are distinct because $m/k \neq 0$). Lemma 6.5 (applied to $j = m/k$) yields

$$(p_m, f_k(e_{m/k})) = \langle -1 \rangle^{m/k-1} \left\lceil m = k \left(\frac{m}{k}\right) \right\rceil k = \langle -1 \rangle^{m/k-1} k.$$ 

(since $m = k(m/k)$)
Now, (71) becomes

\[
(p_m, G(k, m)) = \sum_{i \in \mathbb{N}} (-1)^i (p_m, h_{m-ki} \cdot f_k(e_i))
\]

\[
= (-1)^0 \begin{pmatrix} p_m, h_{m-0} \cdot f_k(e_0) \end{pmatrix} + (-1)^{m/k} \begin{pmatrix} p_m, h_{m-k \cdot m/k} \cdot f_k(e_{m/k}) \end{pmatrix}
\]

\[
= (-1)^0 \begin{pmatrix} p_m, h_{m-0} \cdot f_k(e_0) \end{pmatrix} + (-1)^{m/k} \begin{pmatrix} p_m, h_{m-k \cdot m/k} \cdot f_k(e_{m/k}) \end{pmatrix}
\]

\[
= \begin{pmatrix} p_m, h_m \end{pmatrix} + (-1)^{m/k} \begin{pmatrix} p_m, h_0 \cdot f_k(e_{m/k}) \end{pmatrix} + \sum_{i \in \mathbb{N} \setminus \{0, m/k\}} (-1)^i 0
\]

\[
= \begin{pmatrix} p_m, h_m \end{pmatrix} + (-1)^{m/k} \begin{pmatrix} p_m, f_k(e_{m/k}) \end{pmatrix}
\]

\[
= 1 + (-1)^{m/k} (-1)^{m/k-1} k = 1 - k.
\]

Comparing this with

\[
1 - \left\lfloor \frac{k}{m} \right\rfloor k = 1 - k,
\]

we obtain \((p_m, G(k, m)) = 1 - \left\lfloor \frac{k}{m} \right\rfloor k\). Hence, Proposition 6.8 is proven in Case 1.

Let us now consider Case 2. In this case, we have \(k \nmid m\). Hence, \(m/k \notin \mathbb{Z}\), so that
Thus, \( N \setminus \{0\} = N \setminus \{0, m/k\} \). Now, (71) becomes

\[
(p_m, G(k, m)) = \sum_{i \in \mathbb{N}} (-1)^i (p_m, h_{m-ki} \cdot f_k(e_i))
\]

\[
= (-1)^0 (p_m, h_{m-0} \cdot f_k(e_0)) + \sum_{i \in \mathbb{N} \setminus \{0\}} (-1)^i (p_m, h_{m-ki} \cdot f_k(e_i))
\]

\[
= (h_m, p_m) + \sum_{i \in \mathbb{N} \setminus \{0, m/k\}} (-1)^i (p_m, h_{m-ki} \cdot f_k(e_i))
\]

(here, we have split off the addend for \( i = 0 \) from the sum)

\[
= (h_m, p_m) + \sum_{i \in \mathbb{N} \setminus \{0, m/k\}} (-1)^i 0 = 1.
\]

Comparing this with

\[
1 - [k \mid m] k = 1,
\]

we obtain \( (p_m, G(k, m)) = 1 - [k \mid m] k \). Hence, Proposition 6.8 is proven in Case 2.

We have now proven Proposition 6.8 both in Case 1 and in Case 2. Hence, Proposition 6.8 always holds.

### 6.3. The Petrie functions as polynomial generators of \( \Lambda \)

We now claim the following:

**Theorem 6.9.** Fix a positive integer \( k \). Assume that \( 1 - k \) is invertible in \( k \).

Then, the family \( (G(k, m))_{m \geq 1} = (G(k, 1), G(k, 2), G(k, 3), \ldots) \) is an algebraically independent generating set of the commutative \( k \)-algebra \( \Lambda \). (In other words, the canonical \( k \)-algebra homomorphism

\[
k[u_1, u_2, u_3, \ldots] \to \Lambda,
\]

\[
u_m \mapsto G(k, m)
\]

is an isomorphism.)

This will follow from Proposition 6.8 using the following general criterion for generating sets of \( \Lambda \):
Proposition 6.10. For each positive integer \( m \), let \( v_m \in \Lambda \) be a homogeneous symmetric function of degree \( m \).

Assume that \((p_m, v_m)\) is an invertible element of \( k \) for each positive integer \( m \). Then, the family \((v_m)_{m \geq 1} = (v_1, v_2, v_3, \ldots)\) is an algebraically independent generating set of the commutative \( k \)-algebra \( \Lambda \).

Proof of Proposition 6.10 Proposition 6.10 is \cite[Exercise 2.5.24]{GriRei18} (in the next version of \cite{GriRei18}). □

Proof of Theorem 6.9. Let \( m \) be a positive integer. Proposition 6.8 yields that

\[
(p_m, G(k,m)) = 1 - \left\lfloor \frac{k}{m} \right\rfloor k = 1 - \begin{cases} 1, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases} \cdot k
\]

\[
= \begin{cases} 1, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases}
\]

\[
= \begin{cases} 1 - 1 \cdot k, & \text{if } k \mid m; \\ 1 - 0 \cdot k, & \text{if } k \nmid m \end{cases} = \begin{cases} 1 - k, & \text{if } k \mid m; \\ 1, & \text{if } k \nmid m \end{cases}.
\]

Hence, \((p_m, G(k,m))\) is an invertible element of \( k \) (because both \( 1 - k \) and \( 1 \) are invertible elements of \( k \)).

Forget that we fixed \( m \). We thus have showed that \((p_m, v_m)\) is an invertible element of \( k \) for each positive integer \( m \). Also, clearly, for each positive integer \( m \), the element \( G(k,m) \in \Lambda \) is a homogeneous symmetric function of degree \( m \). Thus, Proposition 6.10 (applied to \( v_m = G(k,m) \)) shows that the family \((G(k,m))_{m \geq 1} = (G(k,1), G(k,2), G(k,3), \ldots)\) is an algebraically independent generating set of the commutative \( k \)-algebra \( \Lambda \). This proves Theorem 6.9. □

6.4. The Hopf endomorphisms \( U_k \) and \( V_k \)

We shall next discuss another way to obtain the Petrie symmetric functions \( G(k,m) \).

Convention 6.11. As already mentioned, \( \Lambda \) is a connected graded Hopf algebra. We let \( S \) denote its antipode.

Definition 6.12. If \( C \) is a \( k \)-coalgebra and \( A \) is a \( k \)-algebra, and if \( f, g : C \to A \) are two \( k \)-linear maps, then the convolution \( f \ast g \) of \( f \) and \( g \) is defined to be the \( k \)-linear map \( m_A \circ (f \otimes g) \circ \Delta_C : C \to A \), where \( \Delta_C : C \to C \otimes C \) is the comultiplication of the \( k \)-coalgebra \( C \), and where \( m_A : A \otimes A \to A \) is the \( k \)-linear map sending each pure tensor \( a \otimes b \in A \otimes A \) to \( ab \in A \).

We also recall Definition 6.3 and Definition 6.1. Now, our goal is the following.
Theorem 6.13. Fix a positive integer $k$. Let $U_k$ be the map $f_k \circ S \circ v_k : \Lambda \to \Lambda$. Let $V_k$ be the map $\text{id}_\Lambda \star U_k : \Lambda \to \Lambda$. (This is well-defined by Definition 6.12 since $\Lambda$ is both a $k$-coalgebra and a $k$-algebra.) Then:

(a) The map $U_k$ is a $k$-Hopf algebra homomorphism.
(b) The map $V_k$ is a $k$-Hopf algebra homomorphism.
(c) We have $G(k,m) = V_k(h_m)$ for each $m \in \mathbb{N}$.

Proof of Theorem 6.13. (All references to [GriRei18] in this proof are references to its coming version, not to arXiv:1409.8356v5. That said, the numbering should normally not differ too much between these two versions.)

The $k$-Hopf algebra $\Lambda$ is both commutative and cocommutative (by [GriRei18, Exercise 2.3.7(a)]). Thus, its antipode $S$ is a $k$-Hopf algebra homomorphism\footnote{Proof. The antipode of a Hopf algebra is an algebra anti-endomorphism (by [GriRei18, Proposition 1.4.8]). Thus, $S$ is an algebra anti-endomorphism (since $S$ is the antipode of the Hopf algebra $\Lambda$). But since $\Lambda$ is commutative, an algebra anti-endomorphism of $\Lambda$ is the same thing as an algebra endomorphism of $\Lambda$. Hence, $S$ is an algebra endomorphism of $\Lambda$ as well.}

(a) The map $f_k$ is a $k$-Hopf algebra homomorphism (by [GriRei18, Exercise 2.9.9(d)], applied to $n = k$). The map $v_k$ is a $k$-Hopf algebra homomorphism (by [GriRei18, Exercise 2.9.10(e)], applied to $n = k$). Thus, we have shown that all three maps $f_k, S$ and $v_k$ are $k$-Hopf algebra homomorphisms. Hence, their composition $f_k \circ S \circ v_k$ is a $k$-Hopf algebra homomorphism as well. In other words, $U_k$ is a $k$-Hopf algebra homomorphism (since $U_k = f_k \circ S \circ v_k$). This proves Theorem 6.13 (a).

(b) Recall (from [GriRei18] Exercise 1.5.9(a)) the following fact:

Claim 1: If $H$ is a $k$-bialgebra and $A$ is a commutative $k$-bialgebra, then the convolution $f \star g$ of any two $k$-algebra homomorphisms $f, g : H \to A$ is again a $k$-algebra homomorphism.

The following fact is dual to Claim 1:

Claim 2: If $H$ is a $k$-bialgebra and $C$ is a cocommutative $k$-coalgebra, then the convolution $f \star g$ of any two $k$-coalgebra homomorphisms $f, g : C \to H$ is again a $k$-coalgebra homomorphism.

(See [GriRei18] solution to Exercise 1.5.9(h)] for why exactly Claim 2 is dual to Claim 1, and how it can be proved.)

Proof. The antipode of a Hopf algebra is an algebra anti-endomorphism (by [GriRei18, Proposition 1.4.8]). Thus, $S$ is an algebra anti-endomorphism (since $S$ is the antipode of the Hopf algebra $\Lambda$). But since $\Lambda$ is commutative, an algebra anti-endomorphism of $\Lambda$ is the same thing as an algebra endomorphism of $\Lambda$. Hence, $S$ is an algebra endomorphism of $\Lambda$ as well.

The antipode of a Hopf algebra is a coalgebra anti-endomorphism (by [GriRei18, Exercise 1.4.25]). Thus, $S$ is a coalgebra anti-endomorphism (since $S$ is the antipode of the Hopf algebra $\Lambda$). But since $\Lambda$ is cocommutative, a coalgebra anti-endomorphism of $\Lambda$ is the same thing as a coalgebra endomorphism of $\Lambda$. Hence, $S$ is a coalgebra endomorphism of $\Lambda$ as well.

We now know that $S$ is an algebra endomorphism of $\Lambda$ and a coalgebra endomorphism of $\Lambda$ at the same time. In other words, $S$ is a bialgebra endomorphism of $\Lambda$. Hence, $S$ is a $k$-Hopf algebra endomorphism of $\Lambda$. In other words, $S$ is a $k$-Hopf algebra homomorphism.
Theorem 6.13 (a) yields that the map $U_k$ is a $k$-Hopf algebra homomorphism. Hence, $U_k$ is both a $k$-algebra homomorphism and a $k$-coalgebra homomorphism.

Now, recall that $\Lambda$ is commutative, and that $id_\Lambda$ and $U_k$ are two $k$-algebra homomorphisms from $\Lambda$ to $\Lambda$. Hence, Claim 1 (applied to $H = \Lambda$, $A = \Lambda$, $f = id_\Lambda$ and $g = U_k$) shows that the convolution $id_\Lambda \ast U_k$ is a $k$-algebra homomorphism. In other words, $\nu_k$ is a $k$-algebra homomorphism (since $\nu_k = id_\Lambda \ast U_k$).

Next, recall that $\Lambda$ is cocommutative, and that $id_\Lambda$ and $U_k$ are two $k$-coalgebra homomorphisms from $\Lambda$ to $\Lambda$. Hence, Claim 2 (applied to $H = \Lambda$, $C = \Lambda$, $f = id_\Lambda$ and $g = U_k$) shows that the convolution $id_\Lambda \ast U_k$ is a $k$-coalgebra homomorphism. In other words, $\nu_k$ is a $k$-coalgebra homomorphism (since $\nu_k = id_\Lambda \ast U_k$).

So we know that the map $\nu_k$ is both a $k$-algebra homomorphism and a $k$-coalgebra homomorphism. Thus, $\nu_k$ is a $k$-bialgebra homomorphism, thus a $k$-Hopf algebra homomorphism. This proves Theorem 6.13 (b).

(c) The map $\nu_k$ is a $k$-algebra homomorphism; thus, $\nu_k(1) = 1$. Now, we have

$$\nu_k(h_m) = \begin{cases} h_{m/k}, & \text{if } k \mid m; \\ 0, & \text{if } k \nmid m \end{cases}$$

for each $m \in \mathbb{N}$. (Indeed, if $m > 0$, then this follows from the definition of $\nu_k$. But if $m = 0$, then this follows from $\nu_k(1) = 1$.)

We have

$$S(h_n) = (-1)^n e_n$$

for each $n \in \mathbb{N}$. (This follows from [GriRei18, Proposition 2.4.1(iii)].)

Now, if $j \in \mathbb{N}$ satisfies $k \mid j$, then

$$\nu_k(h_j) = \begin{cases} h_{j/k}, & \text{if } k \mid j; \\ 0, & \text{if } k \nmid j \end{cases}$$

(by (73), applied to $m = j$)

and

$$U_k(h_j) = (f_k \circ S \circ \nu_k)(h_j)$$

(since $U_k = f_k \circ S \circ \nu_k$)

$$= f_k \left( S \left( \nu_k(h_j) \right) \right) = f_k \left( S \left( h_{j/k} \right) \right) = f_k \left( (-1)^{j/k} e_{j/k} \right)$$

(by (74)

$$= (-1)^{j/k} f_k(e_{j/k})$$

(since the map $f_k$ is $k$-linear).

On the other hand, if $j \in \mathbb{N}$ satisfies $k \nmid j$, then

$$\nu_k(h_j) = \begin{cases} h_{j/k}, & \text{if } k \mid j; \\ 0, & \text{if } k \nmid j \end{cases}$$

(by (73), applied to $m = j$)

and

$$U_k(h_j) = (f_k \circ S \circ \nu_k)(h_j)$$

(since $U_k = f_k \circ S \circ \nu_k$)

$$= f_k \left( S \left( \nu_k(h_j) \right) \right) = f_k \left( S \left( h_{j/k} \right) \right) = f_k \left( (-1)^{j/k} e_{j/k} \right)$$

(by (74)

$$= 0$$

(since $k \nmid j$)
and
\[ U_k (h_j) = (f_k \circ S \circ v_k) (h_j) \]  
(since \( U_k = f_k \circ S \circ v_k \))
\[ = (f_k \circ S) \left( \begin{array}{c} v_k (h_j) \\ 0 \end{array} \right) = (f_k \circ S) (0) \]  
(by \( \text{(27)} \))
\[ = 0 \]  
(78)

(since the map \( f_k \circ S \) is \( k \)-linear).

Let \( \Delta_\Lambda \) be the comultiplication \( \Delta : \Lambda \to \Lambda \otimes \Lambda \) of the \( k \)-coalgebra \( \Lambda \). Let \( m_\Lambda : \Lambda \otimes \Lambda \to \Lambda \) be the \( k \)-linear map sending each pure tensor \( a \otimes b \in \Lambda \otimes \Lambda \) to \( ab \in \Lambda \). Definition 6.12 then yields
\[ id_\Lambda \ast U_k = m_\Lambda \circ (id_\Lambda \otimes U_k) \circ \Delta_\Lambda. \]  
Thus,
\[ V_k = id_\Lambda \ast U_k = m_\Lambda \circ (id_\Lambda \otimes U_k) \circ \Delta_\Lambda \]  
\[ = m_\Lambda \circ (id_\Lambda \otimes U_k) \circ \Delta. \]  
(79)

Let \( m \in \mathbb{N} \) (not to be mistaken for the map \( m_\Lambda \)). Then, [GriRei18, Proposition 2.3.6(iii)] (applied to \( n = m \)) yields
\[ \Delta (h_m) = \sum_{i+j=m} h_i \otimes h_j \]  
(where the sum ranges over all pairs \( (i,j) \in \mathbb{N} \times \mathbb{N} \) with \( i + j = m \))
\[ = \sum_{j=0}^m h_{m-j} \otimes h_j \]
(here, we have substituted \( (m-j, j) \) for \( (i,j) \) in the sum). Applying the map \( id_\Lambda \otimes U_k \) to both sides of this equality, we obtain
\[ (id_\Lambda \otimes U_k) (\Delta (h_m)) = (id_\Lambda \otimes U_k) \left( \sum_{j=0}^m h_{m-j} \otimes h_j \right) \]  
\[ = \sum_{j=0}^m \underbrace{id_\Lambda (h_{m-j}) \otimes U_k (h_j)}_{= h_{m-j} \otimes U_k (h_j)} = \sum_{j=0}^m h_{m-j} \otimes U_k (h_j). \]

Applying the map \( m_\Lambda \) to both sides of this equality, we find
\[ m_\Lambda ((id_\Lambda \otimes U_k) (\Delta (h_m))) \]  
\[ = m_\Lambda \left( \sum_{j=0}^m h_{m-j} \otimes U_k (h_j) \right) = \sum_{j=0}^m m_\Lambda (h_{m-j} \otimes U_k (h_j)) = \sum_{j=0}^m h_{m-j} U_k (h_j) \]  
(by the definition of \( m_\Lambda \))
\[ = \sum_{j \in \mathbb{N}} h_{m-j} U_k (h_j) \]
(since)

\[
\sum_{j \in \mathbb{N}} h_{m-j} U_k (h_j) = \sum_{j=0}^{m} h_{m-j} U_k (h_j) + \sum_{j=m+1}^{\infty} h_{m-j} U_k (h_j)
\]

\[
= \sum_{j=0}^{m} h_{m-j} U_k (h_j) + \sum_{j=m+1}^{\infty} 0 U_k (h_j) = \sum_{j=0}^{m} h_{m-j} U_k (h_j)
\]

\[
(\text{since } m-j<0 \text{ (because } j \geq m+1 > m))
\]

\[
\sum_{j \in \mathbb{N}} h_{m-j} U_k (h_j) = \sum_{j=0}^{m} h_{m-j} U_k (h_j) + \sum_{j=m+1}^{\infty} 0 U_k (h_j) = \sum_{j=0}^{m} h_{m-j} U_k (h_j)
\]

). Therefore,

\[
m_\Lambda ((\text{id}_\Lambda \otimes U_k) (\Delta (h_m)))
\]

\[
= \sum_{j \in \mathbb{N}} h_{m-j} U_k (h_j) = \sum_{j \in \mathbb{N}} h_{m-j} U_k (h_j) + \sum_{j \in \mathbb{N}} h_{m-j} U_k (h_j)
\]

\[
= \sum_{j \in \mathbb{N}} h_{m-j} (\frac{-1}{h^{m-j}} f_k (e_{j/k}) + \sum_{j \in \mathbb{N}} h_{m-j} 0 = \sum_{j \in \mathbb{N}} \sum_{j \in \mathbb{N}} h_{m-j} \cdot f_k (e_{j/k})
\]

\[
= \sum_{i \in \mathbb{N}} (\frac{-1}{h^{m-ki}} f_k (e_i))
\]

(here, we have substituted \(ki\) for \(j\) in the sum). Comparing this with

\[
G (k, m) = \sum_{i \in \mathbb{N}} \frac{-1}{h^{m-ki}} f_k (e_i) \quad \text{(by Theorem 6.2)},
\]

we obtain

\[
G (k, m) = m_\Lambda ((\text{id}_\Lambda \otimes U_k) (\Delta (h_m))) = \frac{m_\Lambda \circ (\text{id}_\Lambda \otimes U_k) \circ \Delta}{} (h_m) = V_k (h_m).
\]

This proves Theorem 6.13 (c).}

Using Theorem 6.13, we can give a new proof for Theorem 5.1:

Second proof of Theorem 5.1. Theorem 6.13 (b) shows that the map \(V_k\) is a \(k\)-Hopf algebra homomorphism. Thus, in particular, \(V_k\) is a \(k\)-coalgebra homomorphism. In other words, we have

\[
(V_k \otimes V_k) \circ \Delta = \Delta \circ V_k \quad \text{and} \quad \epsilon = \epsilon \circ V_k
\]
(where \( \varepsilon \) denotes the counit of the \( k \)-coalgebra \( \Lambda \)). But we have
\[
G (k, n) = V_k (h_n) \quad \text{for each } n \in \mathbb{N} \tag{80}
\]
(by Theorem 6.13(c), applied to \( n \) instead of \( m \)). Applying this to \( n = m \), we obtain
\[
G (k, m) = V_k (h_m).
\]
Applying the map \( \Delta \) to both sides of this equality, we find
\[
\Delta (G (k, m)) = \Delta (V_k (h_m)) = (\Delta \circ V_k) (h_m) = ((V_k \otimes V_k) \circ \Delta) (h_m)
\]
\[
= (V_k \otimes V_k) (\Delta (h_m)). \tag{81}
\]
But [GriRei18, Proposition 2.3.6(iii)] (applied to \( n = m \)) yields
\[
\Delta (h_m) = \sum_{i+j=m} h_i \otimes h_j
\]
(\( \sum \) ranges over all pairs \((i, j) \in \mathbb{N} \times \mathbb{N} \) with \( i + j = m \))
\[
= \sum_{i=0}^{m} h_i \otimes h_{m-i}
\]
(here, we have substituted \((i, m-i)\) for \((i, j)\) in the sum). Hence, \( \Delta (G (k, m)) \) becomes
\[
\Delta (G (k, m)) = (V_k \otimes V_k) \left( \sum_{i=0}^{m} h_i \otimes h_{m-i} \right)
\]
\[
= \sum_{i=0}^{m} V_k (h_i) \otimes V_k (h_{m-i}).
\]
Comparing this with
\[
\sum_{i=0}^{m} G (k, i) \otimes G (k, m-i) = \sum_{i=0}^{m} V_k (h_i) \otimes V_k (h_{m-i}), \tag{80}
\]
we obtain
\[
\Delta (G (k, m)) = \sum_{i=0}^{m} G (k, i) \otimes G (k, m-i).
\]
Thus, Theorem 5.1 is proved again. \( \Box \)

References


See also [http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf](http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf) for a version that gets updated.


