The pre-Pieri rules

Darij Grinberg

October 6, 2021

Abstract. Let R be a commutative ring and $n \geq 1$ and $p \geq 0$ two integers. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Let $h_{k,\ i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$. For any $\alpha \in \mathbb{Z}^n$, we define

$$t_\alpha := \det \begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+2,\ 1} & \cdots & h_{\alpha_1+n,\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+2,\ 2} & \cdots & h_{\alpha_2+n,\ 2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\alpha_n+1,\ n} & h_{\alpha_n+2,\ n} & \cdots & h_{\alpha_n+n,\ n} \end{pmatrix} \in R$$

(where α_i denotes the i-th entry of α). Then, we have the identity

$$\sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \det \begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+2,\ 1} & \cdots & h_{\alpha_1+(n-1),\ 1} & h_{\alpha_1+(n+p),\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+2,\ 2} & \cdots & h_{\alpha_2+(n-1),\ 2} & h_{\alpha_2+(n+p),\ 2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ h_{\alpha_n+1,\ n} & h_{\alpha_n+2,\ n} & \cdots & h_{\alpha_n+(n-1),\ n} & h_{\alpha_n+(n+p),\ n} \end{pmatrix}$$

(where $\alpha + \beta$ denotes the entrywise sum of the tuples α and β). (The matrix on the right hand side here is the one from the definition of t_α, except for its last column, where the "$+n$"s have been replaced by "$+ (n+p)$"s.) Furthermore, if $p \leq n$, then

$$\sum_{\substack{\beta \in \{0,1\}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \det \begin{pmatrix} h_{\alpha_1+\xi_1,\ 1} & h_{\alpha_1+\xi_2,\ 1} & \cdots & h_{\alpha_1+\xi_n,\ 1} \\ h_{\alpha_2+\xi_1,\ 2} & h_{\alpha_2+\xi_2,\ 2} & \cdots & h_{\alpha_2+\xi_n,\ 2} \\ \vdots & \vdots & \ddots & \vdots \\ h_{\alpha_n+\xi_1,\ n} & h_{\alpha_n+\xi_2,\ n} & \cdots & h_{\alpha_n+\xi_n,\ n} \end{pmatrix},$$

where $\xi = (1, 2, \ldots, n-p, n-p+2, n-p+3, \ldots, n+1)$. We prove these two identities (in a slightly more general setting, where R is not assumed commutative) and use them to derive some variants of the Pieri rule found in the literature.

Keywords: determinantal identities, determinant, Pieri rules, symmetric functions, Schur functions, immaculate functions.

MSC classes (2020): 15A15, 05E05, 15A24.

Contents

Introduction

The Pieri rules in the theory of symmetric functions (see, e.g., [Macdon95, Chapter I, (5.16) and (5.17)] or [Stanle01, Theorem 7.15.7 and discussion before Corollary 7.15.9]) give simple formulas for multiplying a Schur function by a complete homogeneous or elementary symmetric function. One of the simplest ways to state them (sidestepping the combinatorial background and the geometric motivation) is as follows: We define the ring Λ of symmetric functions as a polynomial ring in countably many indeterminates $h_1, h_2, h_3, \ldots$ over some commutative ring (say, over $\mathbb{Z}$); we furthermore set $h_0 := 1$ and $h_i := 0$ for all negative i. If $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ is an integer partition[1], then the corresponding *Schur function* $s_\lambda \in \Lambda$ can be defined by the formula

$$s_\lambda = \det\left(\left(h_{\lambda_i - i + j}\right)_{i,j\in[n]}\right), \tag{1}$$

where n is a nonnegative integer satisfying $\lambda_{n+1} = \lambda_{n+2} = \lambda_{n+3} = \cdots = 0$ (note that there are infinitely many possible values for n, but they all give the same s_λ). If the partition λ has the form $(1, 1, \ldots, 1, 0, 0, 0, \ldots)$ with k many 1's, then the corresponding Schur function s_λ is called e_n. (Of course, these are not the usual definitions of s_λ and e_n; see [Macdon95, Chapter I, (3.4) and (3.9)] for their equivalence to more standard definitions.) Now, the *first Pieri rule* ([Macdon95,

[1] An *integer partition* (or, for short, just *partition*) means a weakly decreasing sequence $(\lambda_1, \lambda_2, \lambda_3, \ldots)$ of nonnegative integers such that all but finitely many $i > 0$ satisfy $\lambda_i = 0$.

Chapter I, (5.16)]) states that for each $p \in \mathbb{N}$ and each partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, we have

$$s_\lambda h_p = \sum_\mu s_\mu, \tag{2}$$

where the sum ranges over all partitions $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ such that μ / λ is a *horizontal p-strip*[2]. For example,

$$s_{(2,1)} h_2 = s_{(4,1)} + s_{(3,2)} + s_{(3,1,1)} + s_{(2,2,1)},$$

where we are using the standard convention of omitting zeroes from a partition (i.e., we identify a partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$ with the n-tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ when $\lambda_{n+1} = \lambda_{n+2} = \lambda_{n+3} = \cdots = 0$). The *second Pieri rule* ([Macdon95, Chapter I, (5.17)]) states that for each $p \in \mathbb{N}$ and each partition $\lambda = (\lambda_1, \lambda_2, \lambda_3, \ldots)$, we have

$$s_\lambda e_p = \sum_\mu s_\mu, \tag{3}$$

where the sum ranges over all partitions $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$ such that μ / λ is a *vertical p-strip*[3].

There is a less-known variant of the first Pieri rule (2), which sometimes appears as a stepping stone to its proof (e.g., in [Tamvak13, §2.4]). In order to state it, we agree to define a "Schur function" s_λ by the equality (1) not just whenever λ is a partition, but also whenever $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is any n-tuple of integers (not necessarily nonnegative, not necessarily weakly decreasing). This does not significantly extend the notion of a "Schur function", since any such s_λ equals either 0 or $\pm s_\mu$ for an (honest) partition μ. (This follows easily from basic properties of determinants.) Now, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is any integer partition with $\lambda_n = 0$, and if $p \in \mathbb{N}$ is arbitrary, then

$$s_\lambda h_p = \sum_\mu s_\mu, \tag{4}$$

where the sum now ranges over all n-tuples $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n$ such that

$$(\mu_i \geq \lambda_i \text{ for each } i) \text{ and } \sum_{i=1}^{n} (\mu_i - \lambda_i) = p.$$

[2] The notion of a "horizontal p-strip" and the notation μ / λ are best explained in terms of Young diagrams. However, for our purposes, we can define them algebraically: We say that "μ / λ is a horizontal p-strip" if and only if

$$\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq \mu_3 \geq \lambda_3 \geq \cdots \qquad \text{and} \sum_{i \geq 1} (\mu_i - \lambda_i) = p.$$

[3] The notion of a "vertical p-strip" and the notation μ / λ are best explained in terms of Young diagrams. However, for our purposes, we can define them algebraically: We say that "μ / λ is a vertical p-strip" if and only if

$$(\mu_i - \lambda_i \in \{0, 1\} \text{ for each } i \geq 1) \qquad \text{and} \sum_{i \geq 1} (\mu_i - \lambda_i) = p.$$

In other words, if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is any integer partition with $\lambda_n = 0$, and if $p \in \mathbb{N}$ is arbitrary, then

$$s_\lambda h_p = \sum_{\substack{\beta=(\beta_1,\beta_2,\ldots,\beta_n)\in\mathbb{N}^n;\\ \beta_1+\beta_2+\cdots+\beta_n=p}} s_{\lambda+\beta}, \tag{5}$$

where $\lambda + \beta$ denotes the entrywise sum of the n-tuples λ and β.

Note that each addend in the sum in (2) is also contained in the sum in (4), but (usually) not the other way around: An n-tuple μ appearing on the right hand side of (4) might fail to be a partition, and even if it is one, it may violate the "horizontal p-strip" condition (by failing to satisfy $\lambda_i \geq \mu_{i+1}$ for some i). Some of these extraneous addends in (4) are 0, while others cancel each other out. This cancellation argument appears (in some slightly different contexts) in [Tamvak13, Lemma 2] and [LakTho07, (2.1) vs. (2.2)].

The "alternative first Pieri rule" (4) aka (5) is itself not hard to prove. In this note, we shall generalize it in multiple directions. The ultimate generalization – our Theorem 2.1 – we call the *first pre-Pieri rule*; it is an identity for row-determinants of matrices over noncommutative rings. We will give an elementary proof of Theorem 2.1 (using simple combinatorics and manipulation of sums[4]) and derive several corollaries, which include not only (5), but also a noncommutative "right-Pieri rule" for immaculate functions due to Berg, Bergeron, Saliola, Serrano and Zabrocki [BBSSZ13, Theorem 3.5] as well as a Pieri-like rule for Macdonald's 9th-variation Schur functions [Fun12, Proposition 3.9].[5]

We will also show a *second pre-Pieri rule*: an analogue of the first pre-Pieri rule with a parallel retinue of corollaries. One such corollary is an analogue of (5) for e_p instead of h_p; it says that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is any partition, and if $p \in \{0, 1, \ldots, n\}$ satisfies $\lambda_{n-p+1} = \lambda_{n-p+2} = \cdots = \lambda_n = 0$, then

$$s_\lambda e_p = \sum_{\substack{\beta=(\beta_1,\beta_2,\ldots,\beta_n)\in\{0,1\}^n;\\ \beta_1+\beta_2+\cdots+\beta_n=p}} s_{\lambda+\beta}. \tag{6}$$

This can be viewed as an alternative version of the second Pieri rule (3), and indeed it is possible to obtain (3) from (6) by removing vanishing addends. (Unlike for the first Pieri rule, cancellations are not required.)

Finally, we shall speculate on the existence of a "pre-LR rule", which might include both pre-Pieri rules as particular cases.

[4] Our proof could be viewed as a sequence of sign-reversing involutions, although we do not state it in such a form.

[5] It should also be possible to derive the "uncancelled Pieri rule" [Grinbe19, Theorem 11.7] from our first pre-Pieri rule, but this will likely require more effort than it is worth. (Note that [Grinbe19, Theorem 11.7] is not about symmetric functions, but about symmetric polynomials in k variables; on the other hand, unlike (5), there is no $\lambda_n = 0$ requirement in [Grinbe19, Theorem 11.7]. These differences are fairly substantial, and it is not immediately obvious how to bridge them.)

1. Notations

Let us first introduce the notations that will be used throughout this note.

- Let R be a ring (unital and associative, but not necessarily commutative).
- Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ and $\mathbb{P} := \{1, 2, 3, \ldots\}$.
- For any $n \in \mathbb{N}$, we let $[n]$ denote the n-element set $\{1, 2, \ldots, n\}$.
- If α is an n-tuple (for some $n \in \mathbb{N}$), and if $i \in [n]$, then we let α_i denote the i-th entry of α (so that $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$).
- If α is an n-tuple of integers (for some $n \in \mathbb{N}$), then we define $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_n$.
- For any $n \in \mathbb{N}$, we let S_n denote the n-th symmetric group (i.e., the group of all permutations of the set $[n]$).
- If $n \in \mathbb{N}$ and $\sigma \in S_n$, then we let $(-1)^\sigma$ denote the sign of the permutation σ.
- If $n \in \mathbb{N}$, and if we are given an element $a_{i,j} \in R$ for each pair $(i,j) \in [n] \times [n]$, then we let $\left(a_{i,j}\right)_{i,j \in [n]}$ denote the $n \times n$-matrix whose (i,j)-th entry is $a_{i,j}$ for each $(i,j) \in [n] \times [n]$. That is, we let

$$\left(a_{i,j}\right)_{i,j \in [n]} := \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{pmatrix} \in R^{n \times n}.$$

- If $A = \left(a_{i,j}\right)_{i,j \in [n]}$ is any $n \times n$-matrix over R (for some $n \in \mathbb{N}$), then we define an element $\operatorname{rowdet} A \in R$ by

$$\operatorname{rowdet} A := \sum_{\sigma \in S_n} (-1)^\sigma a_{1,\sigma(1)} a_{2,\sigma(2)} \cdots a_{n,\sigma(n)}.$$

  This element is called the *row-determinant* of A. When the ring R is commutative, this row-determinant $\operatorname{rowdet} A$ is just the usual determinant of A.

- We regard the set $\mathbb{Z}^n$ as a $\mathbb{Z}$-module in the usual way: i.e., we have

$$\alpha + \beta = (\alpha_1 + \beta_1, \ \alpha_2 + \beta_2, \ \ldots, \ \alpha_n + \beta_n) \qquad \text{and}$$
$$\alpha - \beta = (\alpha_1 - \beta_1, \ \alpha_2 - \beta_2, \ \ldots, \ \alpha_n - \beta_n)$$

  for any $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}^n$. Thus, $\alpha + \beta$ and $\alpha - \beta$ are defined for $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^n$ as well (since $\mathbb{N}^n$ is a subset of $\mathbb{Z}^n$).

2. The first pre-Pieri rule

2.1. The theorem

We can now state our "first pre-Pieri rule" in full generality:

Theorem 2.1 (first pre-Pieri rule)**.** Let $n \in \mathbb{P}$ and $p \in \mathbb{N}$. Let $h_{k,\ i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$.

For any $\alpha \in \mathbb{Z}^n$, we define

$$t_\alpha := \operatorname{rowdet}\left(\left(h_{\alpha_i+j,\ i}\right)_{i,j\in[n]}\right) \in R.$$

Let η be the n-tuple

$$(1,2,\ldots,n) + \left(\underbrace{0,0,\ldots,0}_{n-1 \text{ zeroes}}, p\right) = (1,2,\ldots,n-1,n+p) \in \mathbb{Z}^n.$$

Let $\alpha \in \mathbb{Z}^n$. Then,

$$\sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i+\eta_j,\ i}\right)_{i,j\in[n]}\right). \tag{7}$$

Example 2.2. For this example, set $n = 2$ and $p = 2$, and let $\alpha \in \mathbb{Z}^2$ be arbitrary. Fix arbitrary elements $h_{k,\ i} \in R$ for all $k \in \mathbb{Z}$ and $i \in [n]$. Then, the n-tuple η defined in Theorem 2.1 is $(1,2) + (0,2) = (1,4)$. Hence, (7) says that

$$\sum_{\substack{\beta\in\mathbb{N}^2;\\ |\beta|=2}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i+\eta_j,\ i}\right)_{i,j\in[2]}\right) = \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+4,\ 2} \end{pmatrix}.$$

The left hand side of this equality can be rewritten as

$$\begin{aligned}
&\sum_{\substack{\beta\in\mathbb{N}^2;\\ |\beta|=2}} \underbrace{t_{\alpha+\beta}}_{\substack{=\operatorname{rowdet}\left(\left(h_{(\alpha+\beta)_i+j,\ i}\right)_{i,j\in[2]}\right)\\ \text{(by the definition of } t_{\alpha+\beta})}} \\
&= \sum_{\substack{\beta\in\mathbb{N}^2;\\ |\beta|=2}} \operatorname{rowdet}\left(\left(h_{(\alpha+\beta)_i+j,\ i}\right)_{i,j\in[2]}\right) \\
&= \sum_{\substack{\beta\in\mathbb{N}^2;\\ |\beta|=2}} \operatorname{rowdet}\left(\left(h_{\alpha_i+\beta_i+j,\ i}\right)_{i,j\in[2]}\right) \\
&\qquad\qquad (\text{since } (\alpha+\beta)_i = \alpha_i+\beta_i \text{ for all } i\in[2]) \\
&= \sum_{\substack{\beta\in\mathbb{N}^2;\\ |\beta|=2}} \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+\beta_1+1,\ 1} & h_{\alpha_1+\beta_1+2,\ 1} \\ h_{\alpha_2+\beta_2+1,\ 2} & h_{\alpha_2+\beta_2+2,\ 2} \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
&= \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+2+1,\ 1} & h_{\alpha_1+2+2,\ 1} \\ h_{\alpha_2+0+1,\ 2} & h_{\alpha_2+0+2,\ 2} \end{pmatrix} + \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1+1,\ 1} & h_{\alpha_1+1+2,\ 1} \\ h_{\alpha_2+1+1,\ 2} & h_{\alpha_2+1+2,\ 2} \end{pmatrix} \\
&\qquad + \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+0+1,\ 1} & h_{\alpha_1+0+2,\ 1} \\ h_{\alpha_2+2+1,\ 2} & h_{\alpha_2+2+2,\ 2} \end{pmatrix} \\
&\qquad\qquad \left(\begin{array}{c}\text{since there are exactly three 2-tuples } \beta\in\mathbb{N}^2 \\ \text{satisfying } |\beta|=2\text{, namely } (2,0),\ (1,1) \text{ and } (0,2)\end{array}\right) \\
&= \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+3,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+2,\ 2} \end{pmatrix} + \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+2,\ 1} & h_{\alpha_1+3,\ 1} \\ h_{\alpha_2+2,\ 2} & h_{\alpha_2+3,\ 2} \end{pmatrix} \\
&\qquad + \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+2,\ 1} \\ h_{\alpha_2+3,\ 2} & h_{\alpha_2+4,\ 2} \end{pmatrix}.
\end{aligned}$$

Therefore, (7) rewrites as

$$\begin{aligned}
&\operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+3,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+2,\ 2} \end{pmatrix} + \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+2,\ 1} & h_{\alpha_1+3,\ 1} \\ h_{\alpha_2+2,\ 2} & h_{\alpha_2+3,\ 2} \end{pmatrix} \\
&\qquad + \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+2,\ 1} \\ h_{\alpha_2+3,\ 2} & h_{\alpha_2+4,\ 2} \end{pmatrix} \\
&= \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+4,\ 2} \end{pmatrix}.
\end{aligned}$$

This is easy to check directly by expanding all four row-determinants.

2.2. The proof

We shall now prepare for the proof of Theorem 2.1 by introducing some notations.

First, we introduce a right action of the symmetric group S_n on the set $\mathbb{Z}^n$ of all n-tuples of integers:

Definition 2.3. Let $n \in \mathbb{N}$. Let $\eta \in \mathbb{Z}^n$ and $\sigma \in S_n$. Then, we define $\eta \circ \sigma$ to be the n-tuple $\left(\eta_{\sigma(1)}, \eta_{\sigma(2)}, \ldots, \eta_{\sigma(n)}\right) \in \mathbb{Z}^n$.

Thus, the n-tuple $\eta \circ \sigma$ is obtained from η by permuting the entries using the permutation σ.

The following two properties of this right action are near-obvious:

Proposition 2.4. Let $n \in \mathbb{N}$. Let $\eta \in \mathbb{Z}^n$ and $\sigma \in S_n$. Then,

$$|\eta \circ \sigma| = |\eta| .$$

Proposition 2.5. Let $n \in \mathbb{N}$. Let $\eta \in \mathbb{Z}^n$. Assume that the n numbers $\eta_1, \eta_2, \ldots, \eta_n$ are distinct. Let $\sigma \in S_n$ and $\pi \in S_n$ be two distinct permutations. Then, $\eta \circ \sigma \neq \eta \circ \pi$.

Finally, we will use the *Iverson bracket notation*:

Definition 2.6. If $\mathcal{A}$ is a logical statement, then $[\mathcal{A}]$ means the *truth value* of $\mathcal{A}$; this is the integer $\begin{cases} 1, & \text{if } \mathcal{A} \text{ is true;} \\ 0, & \text{if } \mathcal{A} \text{ is false.} \end{cases}$

For example, $[2+2=4] = 1$ and $[2+2=5] = 0$. The following easy property of truth values can serve as a warm-up:

Lemma 2.7. Let $u \in \mathbb{Z}$ and $k \in \mathbb{Z}$ satisfy $u \neq k$. Then, $[u \geq k] = [u \geq k+1]$.

Besides the above generalities, our proof of Theorem 2.1 will rely on some more specific lemmas. The first is an easy exercise on the pigeonhole principle:

Lemma 2.8. Let $n \in \mathbb{P}$. Let $\nu \in \mathbb{Z}^n$ and $\eta \in \mathbb{Z}^n$ satisfy $|\nu| = |\eta|$ and

$$\{\eta_1, \eta_2, \ldots, \eta_{n-1}\} \subseteq \{\nu_1, \nu_2, \ldots, \nu_n\} \qquad \text{and} \qquad |\{\eta_1, \eta_2, \ldots, \eta_{n-1}\}| = n-1.$$

Then, there exists some permutation $\pi \in S_n$ satisfying $\nu = \eta \circ \pi$.

Proof of Lemma 2.8. The claim that we must prove can be restated as "the n-tuple ν is a permutation of η" (that is, "the n-tuple ν can be obtained from η by permuting the entries"). Thus, we can permute the entries of ν without loss of generality (since neither the truth of this claim, nor the integer $|\nu|$, nor the set $\{\nu_1, \nu_2, \ldots, \nu_n\}$ change when we permute the entries of ν).

The $n-1$ numbers $\eta_1, \eta_2, \ldots, \eta_{n-1}$ are distinct (since $|\{\eta_1, \eta_2, \ldots, \eta_{n-1}\}| = n-1$). Furthermore, each of these $n-1$ numbers appears as an entry in the n-tuple ν (since $\{\eta_1, \eta_2, \ldots, \eta_{n-1}\} \subseteq \{\nu_1, \nu_2, \ldots, \nu_n\}$). Since these $n-1$ numbers are distinct, they must therefore appear as **distinct** entries in ν (that is, no two of the $n-1$ numbers $\eta_1, \eta_2, \ldots, \eta_{n-1}$ can appear in the same position of ν). By permuting the entries of ν, we can therefore ensure that these $n-1$ numbers $\eta_1, \eta_2, \ldots, \eta_{n-1}$ are the **first** $n-1$ entries of ν in this very order; i.e., that we have

$$\eta_i = \nu_i \qquad \text{for each } i \in [n-1]. \tag{8}$$

Thus, let us WLOG assume that (8) holds (since we can permute the entries of ν without loss of generality). Now, summing up the equalities (8) over all $i \in [n-1]$, we obtain $\eta_1 + \eta_2 + \cdots + \eta_{n-1} = \nu_1 + \nu_2 + \cdots + \nu_{n-1}$. However,

$$\begin{aligned} |\nu| = |\eta| &= \eta_1 + \eta_2 + \cdots + \eta_n = \underbrace{(\eta_1 + \eta_2 + \cdots + \eta_{n-1})}_{=\nu_1+\nu_2+\cdots+\nu_{n-1}} + \eta_n \\ &= (\nu_1 + \nu_2 + \cdots + \nu_{n-1}) + \eta_n. \end{aligned}$$

Comparing this with

$$|\nu| = \nu_1 + \nu_2 + \cdots + \nu_n = (\nu_1 + \nu_2 + \cdots + \nu_{n-1}) + \nu_n,$$

we obtain $(\nu_1 + \nu_2 + \cdots + \nu_{n-1}) + \eta_n = (\nu_1 + \nu_2 + \cdots + \nu_{n-1}) + \nu_n$. Cancelling $\nu_1 + \nu_2 + \cdots + \nu_{n-1}$, we obtain $\eta_n = \nu_n$. Thus, the equality (8) holds not only for each $i \in [n-1]$, but also for $i = n$. Hence, this equality holds for all $i \in [n]$. In other words, we have $\eta = \nu$. Thus, the n-tuple ν is a permutation of η. But this is precisely what we needed to show. Thus, Lemma 2.8 is proven. □

Our second lemma is about an integer determinant:

Lemma 2.9. Let $n \in \mathbb{P}$ and $p \in \mathbb{N}$. Let η be the n-tuple

$$(1, 2, \ldots, n) + \left(\underbrace{0, 0, \ldots, 0}_{n-1 \text{ zeroes}}, p\right) = (1, 2, \ldots, n-1, n+p) \in \mathbb{Z}^n.$$

Let $\nu \in \mathbb{Z}^n$ be an n-tuple satisfying $|\nu| = |\eta|$. Then,

$$\det\left(([\nu_i \geq j])_{i,j \in [n]}\right) = \sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}} (-1)^\sigma. \tag{9}$$

Note that the matrix $([\nu_i \geq j])_{i,j \in [n]}$ in (9) is a matrix with integer entries; thus, its determinant is a well-defined integer.

Before we prove Lemma 2.9, a remark is in order:

Remark 2.10. The sum $\sum_{\substack{\sigma\in S_n;\\ \nu=\eta\circ\sigma}} (-1)^\sigma$ on the right hand side of (9) always has either no addends or only one addend. (Indeed, it is easy to see that the n-tuples $\eta\circ\sigma$ for different $\sigma \in S_n$ are distinct; thus, no more than one of these n-tuples can equal ν.) Thus, this sum can be rewritten as

$$\begin{cases} (-1)^\sigma, & \text{if } \nu = \eta\circ\sigma \text{ for some } \sigma\in S_n; \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Lemma 2.9. The definition of η yields

$$\eta = (1,2,\ldots,n) + \left(\underbrace{0,0,\ldots,0}_{n-1 \text{ zeroes}}, p\right) = (1,2,\ldots,n-1,n+p).$$

Thus,

$$(\eta_1,\eta_2,\ldots,\eta_n) = \eta = (1,2,\ldots,n-1,n+p). \tag{10}$$

In other words, we have

$$\eta_i = i \qquad \text{for each } i\in[n-1] \tag{11}$$

and

$$\eta_n = n+p. \tag{12}$$

It follows easily that the n numbers $\eta_1,\eta_2,\ldots,\eta_n$ are distinct (since $p\in\mathbb{N}$).

We are in one of the following two cases:

Case 1: We have $[n-1] \not\subseteq \{\nu_1,\nu_2,\ldots,\nu_n\}$.

Case 2: We have $[n-1] \subseteq \{\nu_1,\nu_2,\ldots,\nu_n\}$.

Let us first consider Case 1. In this case, we have $[n-1] \not\subseteq \{\nu_1,\nu_2,\ldots,\nu_n\}$. In other words, there exists some $k\in[n-1]$ such that $k \notin \{\nu_1,\nu_2,\ldots,\nu_n\}$. Consider this k. For each $i\in[n]$, we have $\nu_i \neq k$ (since $k\notin\{\nu_1,\nu_2,\ldots,\nu_n\}$) and therefore

$$[\nu_i \geq k] = [\nu_i \geq k+1]$$

(by Lemma 2.7, applied to $u=\nu_i$). This shows that the k-th and the $(k+1)$-st columns of the matrix $([\nu_i\geq j])_{i,j\in[n]}$ are equal. Hence, this matrix $([\nu_i\geq j])_{i,j\in[n]}$ has two equal columns; therefore, the determinant of this matrix is 0. In other words,

$$\det\left(([\nu_i\geq j])_{i,j\in[n]}\right) = 0. \tag{13}$$

On the other hand, $k\in[n-1]$ entails $\eta_k = k$ (by (11)). Thus, the n-tuple η contains the entry k. However, the n-tuple ν does not (since $k\notin\{\nu_1,\nu_2,\ldots,\nu_n\}$).

Thus, the n-tuple ν is not a permutation of the n-tuple η. In other words, there exists no $\sigma \in S_n$ satisfying $\nu = \eta \circ \sigma$. Hence,

$$\sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}} (-1)^{\sigma} = (\text{empty sum}) = 0.$$

Comparing this with (13), we obtain $\det\left(([\nu_i \geq j])_{i,j \in [n]}\right) = \sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}} (-1)^{\sigma}$. Thus, Lemma 2.9 is proved in Case 1.

Let us now consider Case 2. In this case, we have $[n-1] \subseteq \{\nu_1, \nu_2, \ldots, \nu_n\}$. However, (11) shows that

$$\{\eta_1, \eta_2, \ldots, \eta_{n-1}\} = \{1, 2, \ldots, n-1\} = [n-1] \subseteq \{\nu_1, \nu_2, \ldots, \nu_n\}.$$

Moreover, from $\{\eta_1, \eta_2, \ldots, \eta_{n-1}\} = \{1, 2, \ldots, n-1\}$, we obtain $|\{\eta_1, \eta_2, \ldots, \eta_{n-1}\}| = |\{1, 2, \ldots, n-1\}| = n-1$. Hence, Lemma 2.8 yields that there exists some permutation $\pi \in S_n$ satisfying $\nu = \eta \circ \pi$. Consider this π.

Thus, π is a permutation $\sigma \in S_n$ satisfying $\nu = \eta \circ \sigma$. Furthermore, it is easy to see that π is the **only** such permutation σ (because the n numbers $\eta_1, \eta_2, \ldots, \eta_n$ are distinct)[6]. Hence, the sum $\sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}} (-1)^{\sigma}$ has only one addend, namely the addend for $\sigma = \pi$. Thus,

$$\sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}} (-1)^{\sigma} = (-1)^{\pi}. \tag{14}$$

We have $\nu = \eta \circ \pi = \left(\eta_{\pi(1)}, \eta_{\pi(2)}, \ldots, \eta_{\pi(n)}\right)$ (by Definition 2.3). Thus, for each $i \in [n]$, we have

$$\nu_i = \left(\eta_{\pi(1)}, \eta_{\pi(2)}, \ldots, \eta_{\pi(n)}\right)_i = \eta_{\pi(i)}. \tag{15}$$

On the other hand, it is well-known (see, e.g., [Grinbe21, Corollary 6.4.15] or [Grinbe15, Lemma 6.17 **(a)**]) that when the rows of a matrix are permuted, then the determinant of this matrix gets multiplied by $(-1)^{\tau}$, where τ is the permutation used to permute the rows. In other words: If $\left(a_{i,j}\right)_{i,j \in [n]}$ is a square matrix (with integer entries), and if $\tau \in S_n$ is a permutation, then

$$\det\left(\left(a_{\tau(i),j}\right)_{i,j \in [n]}\right) = (-1)^{\tau} \cdot \det\left(\left(a_{i,j}\right)_{i,j \in [n]}\right).$$

Applying this to $a_{i,j} = [\eta_i \geq j]$ and $\tau = \pi$, we obtain

$$\det\left(\left(\left[\eta_{\pi(i)} \geq j\right]\right)_{i,j \in [n]}\right) = (-1)^{\pi} \cdot \det\left(([\eta_i \geq j])_{i,j \in [n]}\right).$$

[6]Here is the argument in more detail: Recall that the n numbers $\eta_1, \eta_2, \ldots, \eta_n$ are distinct. Therefore, if $\sigma \in S_n$ is a permutation distinct from π, then Proposition 2.5 shows that $\eta \circ \sigma \neq \eta \circ \pi = \nu$, so that $\nu \neq \eta \circ \sigma$. Hence, the only permutation $\sigma \in S_n$ satisfying $\nu = \eta \circ \sigma$ is π.

In view of (15), we can rewrite this as

$$\det\left(\left([\nu_i \geq j]\right)_{i,j\in[n]}\right) = (-1)^{\pi} \cdot \det\left(\left([\eta_i \geq j]\right)_{i,j\in[n]}\right). \tag{16}$$

However, from the definition of η, we can easily see that every $i, j \in [n]$ satisfying $i < j$ satisfy $\eta_i < j$ and therefore $[\eta_i \geq j] = 0$. In other words, the matrix $\left([\eta_i \geq j]\right)_{i,j\in[n]}$ is lower-triangular. Therefore, its determinant is the product $\prod_{i=1}^{n} [\eta_i \geq i]$ of its diagonal entries. In other words,

$$\det\left(\left([\eta_i \geq j]\right)_{i,j\in[n]}\right) = \prod_{i=1}^{n} \underbrace{[\eta_i \geq i]}_{\substack{=1 \\ \text{(since it is easy} \\ \text{to see that } \eta_i \geq i\text{)}}} = \prod_{i=1}^{n} 1 = 1.$$

Thus, (16) becomes

$$\det\left(\left([\nu_i \geq j]\right)_{i,j\in[n]}\right) = (-1)^{\pi} \cdot \underbrace{\det\left(\left([\eta_i \geq j]\right)_{i,j\in[n]}\right)}_{=1} = (-1)^{\pi} = \sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}} (-1)^{\sigma}$$

(by (14)). Thus, Lemma 2.9 is proved in Case 2.

We have now proved Lemma 2.9 in both Cases 1 and 2. This completes the proof of Lemma 2.9. □

Our third lemma is another expression for the same determinant as in Lemma 2.9:

Lemma 2.11. Let $n \in \mathbb{N}$. For any permutation $\sigma \in S_n$, we let $\overline{\sigma}$ denote the n-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathbb{Z}^n$.

Let $\nu \in \mathbb{Z}^n$ be an n-tuple. Then,

$$\det\left(\left([\nu_i \geq j]\right)_{i,j\in[n]}\right) = \sum_{\substack{\sigma \in S_n; \\ \nu - \overline{\sigma} \in \mathbb{N}^n}} (-1)^{\sigma}.$$

Proof of Lemma 2.11. If $\mathcal{A}$ and $\mathcal{B}$ are two equivalent logical statements, then $[\mathcal{A}] =$

$[\mathcal{B}]$. Thus, for each $\sigma \in S_n$, we have

$$\begin{aligned}
&[\nu - \overline{\sigma} \in \mathbb{N}^n] \\
&= [(\nu - \overline{\sigma})_i \in \mathbb{N} \text{ for each } i \in [n]] \\
&= [\nu_i - \sigma(i) \in \mathbb{N} \text{ for each } i \in [n]] \\
&\qquad \left(\begin{array}{c} \text{since each } i \in [n] \text{ satisfies } (\nu - \overline{\sigma})_i = \nu_i - \overline{\sigma}_i = \nu_i - \sigma(i) \\ \text{(because the definition of } \overline{\sigma} \text{ yields } \overline{\sigma}_i = \sigma(i)\text{)} \end{array} \right) \\
&= [\nu_i \geq \sigma(i) \text{ for each } i \in [n]] \\
&\qquad \left(\begin{array}{c} \text{since two integers } a \text{ and } b \text{ satisfy } a - b \in \mathbb{N} \\ \text{if and only if } a \geq b \end{array} \right) \\
&= [\nu_1 \geq \sigma(1) \text{ and } \nu_2 \geq \sigma(2) \text{ and } \cdots \text{ and } \nu_n \geq \sigma(n)] \\
&= \prod_{i=1}^{n} [\nu_i \geq \sigma(i)] \qquad (17)
\end{aligned}$$

(because for any n logical statements $\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n$, we have $[\mathcal{A}_1 \text{ and } \mathcal{A}_2 \text{ and } \cdots \text{ and } \mathcal{A}_n] = \prod_{i=1}^{n} [\mathcal{A}_i]$).

Now, the definition of the determinant of a matrix yields

$$\det\left(([\nu_i \geq j])_{i,j \in [n]}\right) = \sum_{\sigma \in S_n} (-1)^{\sigma} \underbrace{\prod_{i=1}^{n} [\nu_i \geq \sigma(i)]}_{\substack{=[\nu - \overline{\sigma} \in \mathbb{N}^n] \\ \text{(by (17))}}} = \sum_{\sigma \in S_n} (-1)^{\sigma} [\nu - \overline{\sigma} \in \mathbb{N}^n]. \qquad (18)$$

Recall that a truth value $[\mathcal{A}]$ is always either 1 or 0, depending on whether the statement $\mathcal{A}$ is true or false. Hence, each addend of the sum on the right hand side of (18) is either of the form $(-1)^{\sigma} \cdot 1$ or of the form $(-1)^{\sigma} \cdot 0$, depending on whether the statement "$\nu - \overline{\sigma} \in \mathbb{N}^n$" is true or false. The addends of the form $(-1)^{\sigma} \cdot 1$ can be simplified to $(-1)^{\sigma}$, whereas the addends of the form $(-1)^{\sigma} \cdot 0$ can be discarded (since they are 0). Thus, the sum simplifies as follows:

$$\sum_{\sigma \in S_n} (-1)^{\sigma} [\nu - \overline{\sigma} \in \mathbb{N}^n] = \sum_{\substack{\sigma \in S_n; \\ \nu - \overline{\sigma} \in \mathbb{N}^n}} (-1)^{\sigma}.$$

Combining this with (18), we obtain

$$\det\left(([\nu_i \geq j])_{i,j \in [n]}\right) = \sum_{\substack{\sigma \in S_n; \\ \nu - \overline{\sigma} \in \mathbb{N}^n}} (-1)^{\sigma}.$$

This proves Lemma 2.11. □

Our fourth and last lemma is a trivial property of the $\mathbb{Z}$-module $\mathbb{Z}^n$:

Lemma 2.12. Let $\alpha \in \mathbb{Z}^n$ and $\beta \in \mathbb{Z}^n$. Then, $|\alpha + \beta| = |\alpha| + |\beta|$.

We are now ready to prove Theorem 2.1:

Proof of Theorem 2.1. For any permutation $\sigma \in S_n$, we let $\overline{\sigma}$ denote the n-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathbb{Z}^n$. This n-tuple $\overline{\sigma}$ satisfies

$$|\overline{\sigma}| = \sigma(1) + \sigma(2) + \cdots + \sigma(n) = 1 + 2 + \cdots + n$$

and therefore

$$|\overline{\sigma}| + p = (1 + 2 + \cdots + n) + p = |\eta| \tag{19}$$

(since the definition of η yields
$|\eta| = 1 + 2 + \cdots + (n-1) + (n+p) = (1 + 2 + \cdots + n) + p$).

Let us now set $\prod_{i=1}^{n} a_i := a_1 a_2 \cdots a_n$ for any $a_1, a_2, \ldots, a_n \in R$. (This is, of course, the usual meaning of the notation $\prod_{i=1}^{n} a_i$ when the ring R is commutative; however, we are now extending it to the case of arbitrary R.)

Using this notation, we can rewrite the definition of a row-determinant as follows: If $\left(a_{i,j}\right)_{i,j \in [n]} \in R^{n \times n}$ is any $n \times n$-matrix over R, then

$$\operatorname{rowdet}\left(\left(a_{i,j}\right)_{i,j \in [n]}\right) = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} a_{i,\sigma(i)}. \tag{20}$$

For each $\beta \in \mathbb{N}^n$, we have

$$\begin{aligned} t_{\alpha+\beta} &= \operatorname{rowdet}\left(\left(h_{(\alpha+\beta)_i + j,\ i}\right)_{i,j \in [n]}\right) \qquad \left(\text{by the definition of } t_{\alpha+\beta}\right) \\ &= \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} h_{(\alpha+\beta)_i + \sigma(i),\ i} \end{aligned} \tag{21}$$

(by (20), applied to $a_{i,j} = h_{(\alpha+\beta)_i + j,\ i}$). However, for each $\beta \in \mathbb{N}^n$, each $\sigma \in S_n$ and each $i \in [n]$, we have

$$\underbrace{(\alpha+\beta)_i}_{=\alpha_i + \beta_i} + \underbrace{\sigma(i)}_{\substack{=\overline{\sigma}_i \\ \text{(by the definition of } \overline{\sigma})}} = \alpha_i + \underbrace{\beta_i + \overline{\sigma}_i}_{=(\beta+\overline{\sigma})_i} = \alpha_i + (\beta + \overline{\sigma})_i .$$

This allows us to rewrite (21) as follows:

$$t_{\alpha+\beta} = \sum_{\sigma \in S_n} (-1)^{\sigma} \prod_{i=1}^{n} h_{\alpha_i + (\beta+\overline{\sigma})_i,\ i} \qquad \text{for each } \beta \in \mathbb{N}^n.$$

Summing these equalities over all $\beta \in \mathbb{N}^n$ satisfying $|\beta| = p$, we obtain

$$\sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} t_{\alpha+\beta} = \sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i}$$
$$= \sum_{\sigma\in S_n} (-1)^\sigma \sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} \prod_{i=1}^n h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i}. \qquad (22)$$

Now, fix a permutation $\sigma \in S_n$. We shall rewrite the sum $\sum\limits_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} \prod\limits_{i=1}^n h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i}$ in terms of $\beta + \overline{\sigma}$. Indeed, we have the following equality of summation signs:

$$\sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} = \sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|+|\overline{\sigma}|=p+|\overline{\sigma}|}} \qquad \left(\begin{array}{c}\text{since the condition " } |\beta| = p \text{" is clearly}\\ \text{equivalent to " } |\beta| + |\overline{\sigma}| = p + |\overline{\sigma}| \text{ "}\end{array}\right)$$
$$= \sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta+\overline{\sigma}|=|\eta|}} \qquad \left(\begin{array}{c}\text{since Lemma 2.12 yields } |\beta| + |\overline{\sigma}| = |\beta+\overline{\sigma}|\text{,}\\ \text{and since } p + |\overline{\sigma}| = |\overline{\sigma}| + p = |\eta| \text{ (by (19))}\end{array}\right).$$

Hence,

$$\sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} \prod_{i=1}^n h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i} = \sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta+\overline{\sigma}|=|\eta|}} \prod_{i=1}^n h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i}.$$

However, $\mathbb{Z}^n$ is a group (under addition). Hence, when β runs over $\mathbb{Z}^n$, the sum $\beta + \overline{\sigma}$ also runs over $\mathbb{Z}^n$. (Formally speaking, this is saying that the map

$$\mathbb{Z}^n \to \mathbb{Z}^n,$$
$$\beta \mapsto \beta + \overline{\sigma}$$

is a bijection.) Thus, when β runs over $\mathbb{N}^n$, the sum $\beta + \overline{\sigma}$ runs over the set of all $\nu \in \mathbb{Z}^n$ that satisfy $\nu - \overline{\sigma} \in \mathbb{N}^n$. Therefore, we can substitute ν for $\beta + \overline{\sigma}$ in the sum $\sum\limits_{\substack{\beta\in\mathbb{N}^n;\\ |\beta+\overline{\sigma}|=|\eta|}} \prod\limits_{i=1}^n h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i}$. We thus obtain

$$\sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta+\overline{\sigma}|=|\eta|}} \prod_{i=1}^n h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i} = \sum_{\substack{\nu\in\mathbb{Z}^n;\\ \nu-\overline{\sigma}\in\mathbb{N}^n;\\ |\nu|=|\eta|}} \prod_{i=1}^n h_{\alpha_i+\nu_i,\ i}.$$

Hence, our above computation becomes

$$\sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} \prod_{i=1}^{n} h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i} = \sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta+\overline{\sigma}|=|\eta|}} \prod_{i=1}^{n} h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i} = \sum_{\substack{\nu\in\mathbb{Z}^n;\\ \nu-\overline{\sigma}\in\mathbb{N}^n;\\ |\nu|=|\eta|}} \prod_{i=1}^{n} h_{\alpha_i+\nu_i,\ i}. \tag{23}$$

Forget that we fixed σ. We thus have proved (23) for each permutation $\sigma \in S_n$.

Now, (22) becomes

$$\begin{aligned}
\sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} t_{\alpha+\beta} &= \sum_{\sigma\in S_n} (-1)^\sigma \sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} \prod_{i=1}^n h_{\alpha_i+(\beta+\overline{\sigma})_i,\ i} \\
&= \sum_{\sigma\in S_n} (-1)^\sigma \sum_{\substack{\nu\in\mathbb{Z}^n;\\ \nu-\overline{\sigma}\in\mathbb{N}^n;\\ |\nu|=|\eta|}} \prod_{i=1}^n h_{\alpha_i+\nu_i,\ i} \qquad \text{(by (23))} \\
&= \underbrace{\sum_{\sigma\in S_n} \sum_{\substack{\nu\in\mathbb{Z}^n;\\ \nu-\overline{\sigma}\in\mathbb{N}^n;\\ |\nu|=|\eta|}}}_{=\sum_{\substack{\nu\in\mathbb{Z}^n;\\ |\nu|=|\eta|}} \sum_{\substack{\sigma\in S_n;\\ \nu-\overline{\sigma}\in\mathbb{N}^n}}} (-1)^\sigma \prod_{i=1}^n h_{\alpha_i+\nu_i,\ i} \\
&= \sum_{\substack{\nu\in\mathbb{Z}^n;\\ |\nu|=|\eta|}} \sum_{\substack{\sigma\in S_n;\\ \nu-\overline{\sigma}\in\mathbb{N}^n}} (-1)^\sigma \prod_{i=1}^n h_{\alpha_i+\nu_i,\ i} \\
&= \sum_{\substack{\nu\in\mathbb{Z}^n;\\ |\nu|=|\eta|}} \underbrace{\left(\sum_{\substack{\sigma\in S_n;\\ \nu-\overline{\sigma}\in\mathbb{N}^n}} (-1)^\sigma\right)}_{\substack{=\det\left(([\nu_i\geq j])_{i,j\in[n]}\right)\\ \text{(by Lemma 2.11)}}} \prod_{i=1}^n h_{\alpha_i+\nu_i,\ i} \\
&= \sum_{\substack{\nu\in\mathbb{Z}^n;\\ |\nu|=|\eta|}} \underbrace{\det\left(([\nu_i\geq j])_{i,j\in[n]}\right)}_{\substack{=\sum_{\substack{\sigma\in S_n;\\ \nu=\eta\circ\sigma}} (-1)^\sigma\\ \text{(by Lemma 2.9)}}} \cdot \prod_{i=1}^n h_{\alpha_i+\nu_i,\ i} \\
&= \sum_{\substack{\nu\in\mathbb{Z}^n;\\ |\nu|=|\eta|}} \left(\sum_{\substack{\sigma\in S_n;\\ \nu=\eta\circ\sigma}} (-1)^\sigma\right) \prod_{i=1}^n h_{\alpha_i+\nu_i,\ i} \\
&= \sum_{\substack{\nu\in\mathbb{Z}^n;\\ |\nu|=|\eta|}} \sum_{\substack{\sigma\in S_n;\\ \nu=\eta\circ\sigma}} (-1)^\sigma \prod_{i=1}^n h_{\alpha_i+\nu_i,\ i}. \qquad (24)
\end{aligned}$$

On the other hand,

$$\operatorname{rowdet}\left(\left(h_{\alpha_i+\eta_j,\ i}\right)_{i,j\in[n]}\right) = \sum_{\sigma\in S_n} (-1)^\sigma \prod_{i=1}^n h_{\alpha_i+\eta_{\sigma(i)},\ i} \qquad (25)$$

(by (20), applied to $a_{i,j} = h_{\alpha_i+\eta_j,\ i}$). However, for each $\sigma\in S_n$, the n-tuple $\eta\circ\sigma$ belongs to $\mathbb{Z}^n$ and satisfies $|\eta\circ\sigma| = |\eta|$ (by Proposition 2.4). In other words, for each $\sigma \in S_n$, the n-tuple $\eta \circ \sigma$ is a $\nu \in \mathbb{Z}^n$ satisfying $|\nu| = |\eta|$. Hence, any sum ranging over all $\sigma \in S_n$ can be split according to the value of $\eta \circ \sigma$. In other words, we have the following equality of summation signs:

$$\sum_{\sigma \in S_n} = \sum_{\substack{\nu \in \mathbb{Z}^n; \\ |\nu| = |\eta|}} \sum_{\substack{\sigma \in S_n; \\ \eta \circ \sigma = \nu}} = \sum_{\substack{\nu \in \mathbb{Z}^n; \\ |\nu| = |\eta|}} \sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}} .$$

Thus, (25) becomes

$$\begin{aligned}
\operatorname{rowdet}\left(\left(h_{\alpha_i + \eta_j,\ i} \right)_{i,j \in [n]} \right) &= \underbrace{\sum_{\sigma \in S_n}}_{= \sum_{\substack{\nu \in \mathbb{Z}^n; \\ |\nu| = |\eta|}} \sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}}} (-1)^\sigma \prod_{i=1}^n h_{\alpha_i + \eta_{\sigma(i)},\ i} \\
&= \sum_{\substack{\nu \in \mathbb{Z}^n; \\ |\nu| = |\eta|}} \sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}} (-1)^\sigma \prod_{i=1}^n \underbrace{h_{\alpha_i + \eta_{\sigma(i)},\ i}}_{\substack{= h_{\alpha_i + \nu_i,\ i} \\ \text{(since } \nu = \eta \circ \sigma \text{ entails } \nu_i = \eta_{\sigma(i)}, \\ \text{so that } h_{\alpha_i + \nu_i,\ i} = h_{\alpha_i + \eta_{\sigma(i)},\ i})}} \\
&= \sum_{\substack{\nu \in \mathbb{Z}^n; \\ |\nu| = |\eta|}} \sum_{\substack{\sigma \in S_n; \\ \nu = \eta \circ \sigma}} (-1)^\sigma \prod_{i=1}^n h_{\alpha_i + \nu_i,\ i}.
\end{aligned}$$

Comparing this with (24), we obtain

$$\sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} t_{\alpha + \beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i + \eta_j,\ i} \right)_{i,j \in [n]} \right).$$

This proves Theorem 2.1. □

3. Corollaries

3.1. The $h_{k,\ n} = 0$ case

We shall now derive some corollaries from Theorem 2.1 by imposing some conditions on R or on the $h_{k,\ i}$. We begin with the most basic one, in which we force $h_{k,\ n}$ to be 0 for all negative k:

Corollary 3.1. Let $n \in \mathbb{P}$ and $p \in \mathbb{Z}$. Let $h_{k,\ i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$. Assume that

$$h_{k,\ n} = 0 \qquad \text{for all } k < 0. \tag{26}$$

For any $\alpha \in \mathbb{Z}^n$, we define

$$t_\alpha := \operatorname{rowdet}\left(\left(h_{\alpha_i+j,\ i}\right)_{i,j\in[n]}\right) \in R.$$

Let $\alpha \in \mathbb{Z}^n$ be such that $\alpha_n \leq -n$. Then,

$$\sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i+j,\ i}\right)_{i,j\in[n-1]}\right) \cdot h_{\alpha_n+n+p,\ n}.$$

In order to derive this from Theorem 2.1, we shall need the following formula for the row-determinant of a matrix whose last row is 0 except for its rightmost entry:

Lemma 3.2. Let n be a positive integer. Let $A = \left(a_{i,j}\right)_{i,j\in[n]} \in R^{n\times n}$ be an $n \times n$-matrix. Assume that

$$a_{n,j} = 0 \qquad \text{for every } j \in \{1, 2, \ldots, n-1\}. \tag{27}$$

Then,

$$\operatorname{rowdet} A = \operatorname{rowdet}\left(\left(a_{i,j}\right)_{i,j\in[n-1]}\right) \cdot a_{n,n}.$$

Example 3.3. For $n = 3$ and $A = \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix}$, the claim of Lemma 3.2 says that

$$\operatorname{rowdet}\begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & g \end{pmatrix} = \operatorname{rowdet}\begin{pmatrix} a & b \\ d & e \end{pmatrix} \cdot g.$$

(The two zeroes in the third row of A are necessary for Lemma 3.2 to be applicable, as they guarantee that the condition (27) is satisfied.)

Proof of Lemma 3.2. This is a generalization of [Grinbe15, Theorem 6.43] to the case of arbitrary R (not necessarily commutative). The proof given in [Grinbe15] still works for this generalization, as long as we keep in mind that the products are noncommutative. □

Proof of Corollary 3.1. If $p < 0$, then Corollary 3.1 is easily seen to hold[7]. Thus, for the rest of this proof, we WLOG assume that we don't have $p < 0$. Hence, we have $p \geq 0$. Hence, $p \in \mathbb{N}$ (since $p \in \mathbb{Z}$).

[7] *Proof.* Assume that $p < 0$. Then, the sum $\sum_{\substack{\beta\in\mathbb{N}^n;\\ |\beta|=p}} t_{\alpha+\beta}$ is empty and thus equals 0. However, we

Define an n-tuple $\eta \in \mathbb{Z}^n$ as in Theorem 2.1. Thus,

$$(\eta_1, \eta_2, \ldots, \eta_n) = \eta = (1, 2, \ldots, n-1, n+p).$$

In other words, we have

$$\eta_j = j \qquad \text{for each } j \in [n-1] \tag{28}$$

and

$$\eta_n = n + p. \tag{29}$$

Now, Theorem 2.1 yields

$$\sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i + \eta_j,\ i} \right)_{i,j \in [n]} \right). \tag{30}$$

However, it is easy to see that $h_{\alpha_n + \eta_j,\ n} = 0$ for every $j \in \{1, 2, \ldots, n-1\}$ [8]. Hence, Lemma 3.2 (applied to $\left(h_{\alpha_i + \eta_j,\ i} \right)_{i,j \in [n]}$ and $h_{\alpha_i + \eta_j,\ i}$ instead of A and $a_{i,j}$) yields

$$\begin{aligned} \operatorname{rowdet}\left(\left(h_{\alpha_i + \eta_j,\ i} \right)_{i,j \in [n]} \right) &= \operatorname{rowdet}\left(\left(h_{\alpha_i + \eta_j,\ i} \right)_{i,j \in [n-1]} \right) \cdot h_{\alpha_n + \eta_n,\ n} \\ &= \operatorname{rowdet}\left(\left(h_{\alpha_i + j,\ i} \right)_{i,j \in [n-1]} \right) \cdot h_{\alpha_n + n + p,\ n} \end{aligned}$$

(by (28) and (29)). Hence, (30) can be rewritten as

$$\sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i + j,\ i} \right)_{i,j \in [n-1]} \right) \cdot h_{\alpha_n + n + p,\ n}.$$

Thus, Corollary 3.1 is proved. □

3.2. The commutative case

have $\underbrace{\alpha_n}_{\leq -n} + n + p \leq (-n) + n + p = p < 0$, and therefore (26) (applied to $k = \alpha_n + n + p$) yields $h_{\alpha_n + n + p,\ n} = 0$. Thus, the equality

$$\sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i + j,\ i} \right)_{i,j \in [n-1]} \right) \cdot h_{\alpha_n + n + p,\ n}$$

holds due to both of its sides being 0. Thus, we have shown that Corollary 3.1 holds under the assumption that $p < 0$.

[8] *Proof.* Let $j \in \{1, 2, \ldots, n-1\}$. We must show that $h_{\alpha_n + \eta_j,\ n} = 0$.

We have $j \in \{1, 2, \ldots, n-1\} = [n-1]$ and therefore $\eta_j = j$ (by (28)). From $j \in \{1, 2, \ldots, n-1\}$, we also obtain $j \leq n - 1 < n$, so that $\eta_j = j < n$ and thus $\alpha_n + \eta_j < \alpha_n + n \leq 0$ (since $\alpha_n \leq -n$). Therefore, (26) (applied to $k = \alpha_n + \eta_j$) yields $h_{\alpha_n + \eta_j,\ n} = 0$, qed.

Corollary 3.4. Assume that R is commutative. Let $n \in \mathbb{P}$ and $p \in \mathbb{Z}$. Let $h_{k,\ i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$. Assume that

$$h_{k,\ n} = 0 \qquad \text{for all } k < 0.$$

For any $\alpha \in \mathbb{Z}^n$, we define

$$t_\alpha := \det \left(\left(h_{\alpha_i + j,\ i} \right)_{i,j \in [n]} \right) \in R.$$

Let $\alpha \in \mathbb{Z}^n$ be such that $\alpha_n \leq -n$. Then,

$$\sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \det \left(\left(h_{\alpha_i + j,\ i} \right)_{i,j \in [n-1]} \right) \cdot h_{\alpha_n + n + p,\ n}.$$

Proof of Corollary 3.4. We have assumed that the ring R is commutative. Thus, every square matrix over R has a well-defined determinant. Furthermore, the row-determinant of any square matrix over R is the same as the determinant of this matrix (since our above definition of the row-determinant clearly generalizes the standard definition of a determinant). In other words, if A is any square matrix over R, then

$$\operatorname{rowdet} A = \det A. \tag{31}$$

Hence, we can apply Corollary 3.1, replacing "rowdet" by "det" throughout the statement. As a result, we obtain precisely the claim of Corollary 3.4. □

3.3. A Schur-like reindexing

Here are some more consequences of the first pre-Pieri rule:

Corollary 3.5. Let $n \in \mathbb{P}$ and $p \in \mathbb{Z}$. Let $h_{k,\ i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$. Assume that

$$h_{k,\ n} = 0 \qquad \text{for all } k < 0.$$

For any $m \in \{0, 1, \ldots, n\}$ and any $\lambda \in \mathbb{Z}^m$, we define

$$s_\lambda := \operatorname{rowdet} \left(\left(h_{\lambda_i + j - i,\ i} \right)_{i,j \in [m]} \right) \in R.$$

Fix an n-tuple $\mu \in \mathbb{Z}^n$ with $\mu_n = 0$. Let $\overline{\mu} = (\mu_1, \mu_2, \ldots, \mu_{n-1})$. Then,

$$s_{\overline{\mu}} \cdot h_{p,\ n} = \sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} s_{\mu+\beta}.$$

Proof of Corollary 3.5. Define $t_\alpha \in R$ for each $\alpha \in \mathbb{Z}^n$ as in Corollary 3.1.

Define an n-tuple $\alpha \in \mathbb{Z}^n$ by

$$\alpha = (\mu_1 - 1,\ \mu_2 - 2,\ \ldots,\ \mu_n - n).$$

Thus,

$$\alpha_i = \mu_i - i \qquad \text{for each } i \in [n]. \tag{32}$$

Applying this to $i = n$, we obtain $\alpha_n = \underbrace{\mu_n}_{=0} - n = -n \leq -n$. Hence, Corollary 3.1 yields

$$\sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i + j,\ i} \right)_{i,j \in [n-1]} \right) \cdot h_{\alpha_n + n + p,\ n}. \tag{33}$$

However, using the definitions of $s_{\overline{\mu}}$ and α, it is easy to see that

$$\operatorname{rowdet}\left(\left(h_{\alpha_i + j,\ i} \right)_{i,j \in [n-1]} \right) = s_{\overline{\mu}}.$$

Furthermore, again using the definition of α, we can easily check that

$$t_{\alpha+\beta} = s_{\mu+\beta} \qquad \text{for each } \beta \in \mathbb{N}^n$$

(indeed, both $t_{\alpha+\beta}$ and $s_{\mu+\beta}$ are defined as row-determinants of certain matrices, and a simple calculation of indices shows that these two matrices are identical). Finally, we have

$$\alpha_n + n + p = p \qquad (\text{since } \alpha_n = -n).$$

Using these three equalities, we can rewrite (33) as

$$\sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} s_{\mu+\beta} = s_{\overline{\mu}} \cdot h_{p,\ n}.$$

This proves Corollary 3.5. □

Corollary 3.6. Assume that R is commutative. Let $n \in \mathbb{P}$ and $p \in \mathbb{Z}$. Let $h_{k,\ i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$. Assume that

$$h_{k,\ n} = 0 \qquad \text{for all } k < 0.$$

For any $m \in \{0, 1, \ldots, n\}$ and any $\lambda \in \mathbb{Z}^m$, we define

$$s_\lambda := \det\left(\left(h_{\lambda_i + j - i,\ i} \right)_{i,j \in [m]} \right) \in R.$$

Fix an n-tuple $\mu \in \mathbb{Z}^n$ with $\mu_n = 0$. Let $\overline{\mu} = (\mu_1, \mu_2, \ldots, \mu_{n-1})$. Then,

$$h_{p,\ n} \cdot s_{\overline{\mu}} = \sum_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}} s_{\mu+\beta}. \tag{34}$$

Proof of Corollary 3.6. This can be derived from Corollary 3.5 in the same way as we derived Corollary 3.4 from Corollary 3.1. □

We note that (5) is the particular case of Corollary 3.6 for $R = \Lambda$, $h_{k,\ i} = h_k$ and $\mu = \lambda$.

3.4. Recovering Fun's rule

Let us now explain how [Fun12, Proposition 3.9] follows from Corollary 3.6. Here is the claim of [Fun12, Proposition 3.9] with slightly modified notations:

Proposition 3.7. Assume that R is commutative. Let $\ell \in \mathbb{N}$, $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$. Let $g_{k,\ j}$ be an element of R for all $k, j \in \mathbb{Z}$. Assume that

$$g_{k,\ j} = 0 \qquad \text{for all } k < 0 \text{ and } j \in \mathbb{Z}. \tag{35}$$

For any $m \in \mathbb{N}$ and any $\mu \in \mathbb{Z}^m$ and $\beta \in \mathbb{Z}^m$, we define

$$s_{\mu,\ \beta} := \det\left(\left(g_{\mu_i+j-i,\ \beta_i+j-i}\right)_{i,j\in[m]}\right) \in R.$$

Let $\mu \in \mathbb{Z}^{\ell+1}$ be an $(\ell+1)$-tuple satisfying $\mu_{\ell+1} = 0$. Let $\overline{\mu}$ be the ℓ-tuple $(\mu_1, \mu_2, \ldots, \mu_\ell) \in \mathbb{Z}^\ell$.

Let $\beta \in \mathbb{Z}^\ell$ be an ℓ-tuple, and let $\beta' \in \mathbb{Z}^{\ell+1}$ be the $(\ell+1)$-tuple $(\beta_1, \beta_2, \ldots, \beta_\ell, q-p)$.

Then,

$$g_{p,\ q} \cdot s_{\overline{\mu},\ \beta} = \sum_{\substack{\delta\in\mathbb{N}^{\ell+1};\\ |\delta|=p}} s_{\mu+\delta,\ \beta'+\delta}.$$

To be precise, Proposition 3.7 does not only differ from [Fun12, Proposition 3.9] in the notations (what we call q and $g_{k,\ j}$ corresponds to β_e and $h_{k,\ j}$ in [Fun12, Proposition 3.9]; furthermore, the summation index δ in our sum is called σ in [Fun12, Proposition 3.9]), but is also slightly more general (our μ can be any $(\ell+1)$-tuple in $\mathbb{Z}^{\ell+1}$ satisfying $\mu_{\ell+1} = 0$ rather than just a partition of length ℓ; our $g_{k,\ j}$ are not required to satisfy $g_{0,\ j} = 1$; our p is not assumed to be positive). Note that our definition of $s_{\mu,\ \beta}$ in Proposition 3.7 is equivalent to the one in [Fun12], because of [Fun12, (3.3)].

As promised, we can now easily derive Proposition 3.7 from Corollary 3.6:

Proof of Proposition 3.7. Set $n := \ell + 1$. Set

$$h_{k,\ i} := g_{k,\ \beta'_i-\mu_i+k} \qquad \text{for every } k \in \mathbb{Z} \text{ and } i \in [n].$$

Then, (35) entails that $h_{k,\ n} = 0$ for all $k < 0$. Furthermore, using $\mu_n = \mu_{\ell+1} = 0$ and $\beta'_n = \beta'_{\ell+1} = q - p$, we obtain $h_{p,\ n} = g_{p,\ (q-p)-0+p} = g_{p,\ q}$.

For any $m \in \{0, 1, \ldots, n\}$ and any $\lambda \in \mathbb{Z}^m$, we define $s_\lambda \in R$ as in Corollary 3.6. Thus, (34) (with the summation index β renamed as δ) yields

$$h_{p,\ n} \cdot s_{\overline{\mu}} = \sum_{\substack{\delta \in \mathbb{N}^n; \\ |\delta| = p}} s_{\mu+\delta}. \tag{36}$$

However, let us recall that $h_{p,\ n} = g_{p,\ q}$ and $n = \ell + 1$; furthermore, it is easy to see that $s_{\overline{\mu}} = s_{\overline{\mu},\ \beta}$ and

$$s_{\mu+\delta} = s_{\mu+\delta,\ \beta'+\delta} \qquad \text{for each } \delta \in \mathbb{N}^{\ell+1}.$$

Thus, (36) rewrites as

$$g_{p,\ q} \cdot s_{\overline{\mu},\ \beta} = \sum_{\substack{\delta \in \mathbb{N}^{\ell+1}; \\ |\delta| = p}} s_{\mu+\delta,\ \beta'+\delta}.$$

This proves Proposition 3.7. □

3.5. Recovering the immaculate Pieri rule

Next, we shall exhibit [BBSSZ13, Theorem 3.5] as a consequence of Corollary 3.5. To this aim, we will briefly introduce the relevant parts of the scene of [BBSSZ13, Theorem 3.5]. We fix a commutative ring $\mathbf{k}$, and we let NSym be the algebra of noncommutative polynomials in countably many variables $H_1, H_2, H_3, \ldots$ over $\mathbf{k}$. We also set $H_0 := 1$ and $H_k := 0$ for all $k < 0$. For every $m \in \mathbb{N}$ and every $\alpha \in \mathbb{Z}^m$, we set

$$H_\alpha := H_{\alpha_1} H_{\alpha_2} \cdots H_{\alpha_m} \in \mathrm{NSym}.$$

For every $m \in \mathbb{N}$ and every $\alpha \in \mathbb{Z}^m$, we set

$$\mathfrak{S}_\alpha := \sum_{\sigma \in S_m} (-1)^\sigma H_{(\alpha_1+\sigma(1)-1,\ \alpha_2+\sigma(2)-2,\ \ldots,\ \alpha_m+\sigma(m)-m)} \in \mathrm{NSym}.$$

(This is not how $\mathfrak{S}_\alpha$ is defined in [BBSSZ13], but it is an equivalent definition, because [BBSSZ13, Theorem 3.27] shows that the $\mathfrak{S}_\alpha$ from [BBSSZ13] is given by the same formula.) Now, [BBSSZ13, Theorem 3.5] (the "right-Pieri rule for multiplication by H_s" in the terminology of [BBSSZ13]) states the following:

Proposition 3.8. Let $n \in \mathbb{N}$ and $\alpha \in \mathbb{Z}^n$. Let $s \in \mathbb{Z}$. Then,

$$\mathfrak{S}_\alpha H_s = \sum_{\substack{\beta \in \mathbb{Z}^{n+1}; \\ |\beta| = |\alpha| + s; \\ \alpha_i \leq \beta_i \text{ for all } i \in [n]; \\ 0 \leq \beta_{n+1}}} \mathfrak{S}_\beta.$$

(The conditions under the summation sign here are essentially what is denoted by "$\alpha \subset_s \beta$" in [BBSSZ13].)

To be fully precise, Proposition 3.8 is a bit more general than [BBSSZ13, Theorem 3.5], since α is assumed to be a composition (i.e., a tuple of positive integers) in [BBSSZ13, Theorem 3.5], while we are only assuming that $\alpha \in \mathbb{Z}^n$.

Proof of Proposition 3.8. Let R be the ring NSym. Set

$$h_{k,\ i} := H_k \in R \qquad \text{for each } k \in \mathbb{Z} \text{ and } i \in \mathbb{Z}.$$

Thus, $h_{s,\ n+1} = H_s$.

For any $m \in \{0, 1, \ldots, n\}$ and any $\lambda \in \mathbb{Z}^m$, we define $s_\lambda \in R$ as in Corollary 3.5. (This s_λ has nothing to do with the integer s.)

Let $\mu \in \mathbb{Z}^{n+1}$ be the $(n+1)$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n, 0)$. Thus, $\mu_{n+1} = 0$ and $|\mu| = |\alpha|$.

Let $\overline{\mu} = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{Z}^n$. Thus, $\overline{\mu} = \alpha$.

For each $k < 0$, we have

$$\begin{aligned} h_{k,\ n+1} &= H_k && (\text{by the definition of } h_{k,\ n+1}) \\ &= 0 && (\text{since } k < 0). \end{aligned}$$

Thus, we have shown that $h_{k,\ n+1} = 0$ for all $k < 0$. Hence, Corollary 3.5 (applied to $n+1$ and s instead of n and p) yields

$$s_{\overline{\mu}} \cdot h_{s,\ n+1} = \sum_{\substack{\beta \in \mathbb{N}^{n+1}; \\ |\beta| = s}} s_{\mu+\beta}.$$

In view of $h_{s,\ n+1} = H_s$ and $\overline{\mu} = \alpha$, we can rewrite this as

$$s_\alpha \cdot H_s = \sum_{\substack{\beta \in \mathbb{N}^{n+1}; \\ |\beta| = s}} s_{\mu+\beta}. \tag{37}$$

It is easy to see (by comparing the definitions of $\mathfrak{S}_\lambda$ and s_λ) that

$$\mathfrak{S}_\lambda = s_\lambda \qquad \text{for any } m \in \mathbb{N} \text{ and any } \lambda \in \mathbb{Z}^m. \tag{38}$$

Hence, (37) rewrites as

$$\mathfrak{S}_\alpha \cdot H_s = \sum_{\substack{\beta \in \mathbb{N}^{n+1}; \\ |\beta| = s}} \mathfrak{S}_{\mu+\beta}. \tag{39}$$

Thus,

$$\mathfrak{S}_\alpha \cdot H_s = \sum_{\substack{\beta \in \mathbb{N}^{n+1}; \\ |\beta| = s}} \mathfrak{S}_{\mu+\beta} = \sum_{\substack{\gamma \in \mathbb{Z}^{n+1}; \\ |\gamma| = |\mu| + s; \\ \mu_i \leq \gamma_i \text{ for all } i \in [n+1]}} \mathfrak{S}_\gamma$$

(here, we have substituted γ for $\mu + \beta$ in the sum, noticing that the conditions under the new summation sign are precisely the conditions that guarantee that γ has the form $\mu + \beta$ for some $\beta \in \mathbb{N}^{n+1}$ satisfying $|\beta| = s$). Hence,

$$\mathfrak{S}_\alpha \cdot H_s = \sum_{\substack{\gamma \in \mathbb{Z}^{n+1}; \\ |\gamma| = |\mu| + s; \\ \mu_i \leq \gamma_i \text{ for all } i \in [n+1]}} \mathfrak{S}_\gamma = \sum_{\substack{\gamma \in \mathbb{Z}^{n+1}; \\ |\gamma| = |\alpha| + s; \\ \alpha_i \leq \gamma_i \text{ for all } i \in [n]; \\ 0 \leq \gamma_{n+1}}} \mathfrak{S}_\gamma$$

(here, we have rewritten the conditions under the summation sign in an equivalent fashion, using the facts that $\mu = (\alpha_1, \alpha_2, \ldots, \alpha_n, 0)$ and $|\mu| = |\alpha|$). Renaming the summation index γ as β, we obtain precisely the claim of Proposition 3.8. $\square$

4. The second pre-Pieri rule

The "second pre-Pieri rule" structurally resembles the first, but involves a sum over (a subset of) $\{0,1\}^n$ instead of $\mathbb{N}^n$. This is, of course, analogous to the relationship between the elementary symmetric functions and the complete homogeneous symmetric functions, or the relationship between sets and multisets, or various other "combinatorial reciprocities".

4.1. The theorem

Before we state the second pre-Pieri rule, we observe that $\{0,1\}^n \subseteq \mathbb{Z}^n$ for each $n \in \mathbb{N}$.

Theorem 4.1 (second pre-Pieri rule)**.** Let $n \in \mathbb{N}$ and $p \in \{0, 1, \ldots, n\}$. Let $h_{k,\, i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$.

For any $\alpha \in \mathbb{Z}^n$, we define

$$t_\alpha := \operatorname{rowdet}\left(\left(h_{\alpha_i + j,\, i} \right)_{i,j \in [n]} \right) \in R.$$

Let ξ be the n-tuple

$$(1, 2, \ldots, n) + \Bigg(\underbrace{0, 0, \ldots, 0}_{n-p \text{ zeroes}}, \underbrace{1, 1, \ldots, 1}_{p \text{ ones}} \Bigg)$$
$$= (1, 2, \ldots, n-p, n-p+2, n-p+3, \ldots, n+1) \in \mathbb{Z}^n.$$

Let $\alpha \in \mathbb{Z}^n$. Then,

$$\sum_{\substack{\beta \in \{0,1\}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i + \xi_j,\, i} \right)_{i,j \in [n]} \right). \tag{40}$$

Example 4.2. For this example, set $n = 3$ and $p = 2$, and let $\alpha \in \mathbb{Z}^3$ be arbitrary. Fix arbitrary elements $h_{k,\ i} \in R$ for all $k \in \mathbb{Z}$ and $i \in [n]$. Then, the n-tuple ξ defined in Theorem 4.1 is $(1, 2, 3) + (0, 1, 1) = (1, 3, 4)$. Hence, (40) says that

$$\sum_{\substack{\beta \in \{0,1\}^3; \\ |\beta|=2}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i+\xi_j,\ i}\right)_{i,j\in[3]}\right) = \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+3,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+3,\ 2} & h_{\alpha_2+4,\ 2} \\ h_{\alpha_3+1,\ 2} & h_{\alpha_3+3,\ 2} & h_{\alpha_3+4,\ 2} \end{pmatrix}.$$

The left hand side of this equality can be rewritten as

$$\begin{aligned}
&\sum_{\substack{\beta \in \{0,1\}^3; \\ |\beta|=2}} \underbrace{t_{\alpha+\beta}}_{\substack{=\operatorname{rowdet}\left(\left(h_{(\alpha+\beta)_i+j,\ i}\right)_{i,j\in[3]}\right) \\ \text{(by the definition of } t_{\alpha+\beta}\text{)}}} \\
&= \sum_{\substack{\beta \in \{0,1\}^3; \\ |\beta|=2}} \operatorname{rowdet}\left(\left(h_{(\alpha+\beta)_i+j,\ i}\right)_{i,j\in[3]}\right) \\
&= \sum_{\substack{\beta \in \{0,1\}^3; \\ |\beta|=2}} \operatorname{rowdet}\left(\left(h_{\alpha_i+\beta_i+j,\ i}\right)_{i,j\in[3]}\right) \\
&\qquad \left(\text{since } (\alpha+\beta)_i = \alpha_i + \beta_i \text{ for all } i \in [3]\right) \\
&= \sum_{\substack{\beta \in \{0,1\}^3; \\ |\beta|=2}} \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+\beta_1+1,\ 1} & h_{\alpha_1+\beta_1+2,\ 1} & h_{\alpha_1+\beta_1+3,\ 1} \\ h_{\alpha_2+\beta_2+1,\ 2} & h_{\alpha_2+\beta_2+2,\ 2} & h_{\alpha_2+\beta_2+3,\ 2} \\ h_{\alpha_3+\beta_3+1,\ 3} & h_{\alpha_3+\beta_3+2,\ 3} & h_{\alpha_3+\beta_3+3,\ 3} \end{pmatrix}
\end{aligned}$$

$$= \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1+1,\ 1} & h_{\alpha_1+1+2,\ 1} & h_{\alpha_1+1+3,\ 1} \\ h_{\alpha_2+1+1,\ 2} & h_{\alpha_2+1+2,\ 2} & h_{\alpha_2+1+3,\ 2} \\ h_{\alpha_3+0+1,\ 3} & h_{\alpha_3+0+2,\ 3} & h_{\alpha_3+0+3,\ 3} \end{pmatrix}$$

$$+ \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1+1,\ 1} & h_{\alpha_1+1+2,\ 1} & h_{\alpha_1+1+3,\ 1} \\ h_{\alpha_2+0+1,\ 2} & h_{\alpha_2+0+2,\ 2} & h_{\alpha_2+0+3,\ 2} \\ h_{\alpha_3+1+1,\ 3} & h_{\alpha_3+1+2,\ 3} & h_{\alpha_3+1+3,\ 3} \end{pmatrix}$$

$$+ \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+0+1,\ 1} & h_{\alpha_1+0+2,\ 1} & h_{\alpha_1+0+3,\ 1} \\ h_{\alpha_2+1+1,\ 2} & h_{\alpha_2+1+2,\ 2} & h_{\alpha_2+1+3,\ 2} \\ h_{\alpha_3+1+1,\ 3} & h_{\alpha_3+1+2,\ 3} & h_{\alpha_3+1+3,\ 3} \end{pmatrix}$$

$$\left(\begin{array}{c}\text{since there are exactly three 2-tuples } \beta \in \{0,1\}^3 \\ \text{satisfying } |\beta| = 2\text{, namely } (1,1,0),\ (1,0,1) \text{ and } (0,1,1)\end{array}\right)$$

$$= \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+2,\ 1} & h_{\alpha_1+3,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+2,\ 2} & h_{\alpha_2+3,\ 2} & h_{\alpha_2+4,\ 2} \\ h_{\alpha_3+1,\ 3} & h_{\alpha_3+2,\ 3} & h_{\alpha_3+3,\ 3} \end{pmatrix}$$

$$+ \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+2,\ 1} & h_{\alpha_1+3,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+2,\ 2} & h_{\alpha_2+3,\ 2} \\ h_{\alpha_3+2,\ 3} & h_{\alpha_3+3,\ 3} & h_{\alpha_3+4,\ 3} \end{pmatrix}$$

$$+ \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+2,\ 1} & h_{\alpha_1+3,\ 1} \\ h_{\alpha_2+2,\ 2} & h_{\alpha_2+3,\ 2} & h_{\alpha_2+4,\ 2} \\ h_{\alpha_3+2,\ 3} & h_{\alpha_3+3,\ 3} & h_{\alpha_3+4,\ 3} \end{pmatrix}.$$

Therefore, (7) rewrites as

$$\operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+2,\ 1} & h_{\alpha_1+3,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+2,\ 2} & h_{\alpha_2+3,\ 2} & h_{\alpha_2+4,\ 2} \\ h_{\alpha_3+1,\ 3} & h_{\alpha_3+2,\ 3} & h_{\alpha_3+3,\ 3} \end{pmatrix}$$

$$+ \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+2,\ 1} & h_{\alpha_1+3,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+2,\ 2} & h_{\alpha_2+3,\ 2} \\ h_{\alpha_3+2,\ 3} & h_{\alpha_3+3,\ 3} & h_{\alpha_3+4,\ 3} \end{pmatrix}$$

$$+ \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+2,\ 1} & h_{\alpha_1+3,\ 1} \\ h_{\alpha_2+2,\ 2} & h_{\alpha_2+3,\ 2} & h_{\alpha_2+4,\ 2} \\ h_{\alpha_3+2,\ 3} & h_{\alpha_3+3,\ 3} & h_{\alpha_3+4,\ 3} \end{pmatrix}$$

$$= \operatorname{rowdet}\begin{pmatrix} h_{\alpha_1+1,\ 1} & h_{\alpha_1+3,\ 1} & h_{\alpha_1+4,\ 1} \\ h_{\alpha_2+1,\ 2} & h_{\alpha_2+3,\ 2} & h_{\alpha_2+4,\ 2} \\ h_{\alpha_3+1,\ 2} & h_{\alpha_3+3,\ 2} & h_{\alpha_3+4,\ 2} \end{pmatrix}.$$

This is easy to check directly by expanding all four row-determinants.

4.2. The proof

Our proof of the second pre-Pieri rule will be similar to that of the first.

Again, we will use Definition 2.3 and Definition 2.6. Again, several lemmas will be used. The first is an analogue of Lemma 2.8:

Lemma 4.3. Let $n \in \mathbb{N}$. Let $q \in \mathbb{Z}$ and $\nu \in \mathbb{Z}^n$ and $\xi \in \mathbb{Z}^n$ satisfy $|\nu| = |\xi|$ and

$$\{\nu_1, \nu_2, \ldots, \nu_n\} \subseteq \{\xi_1, \xi_2, \ldots, \xi_n\} \cup \{q\} \qquad \text{and} \qquad |\{\nu_1, \nu_2, \ldots, \nu_n\}| = n.$$

Then, there exists some permutation $\pi \in S_n$ satisfying $\nu = \xi \circ \pi$.

Proof of Lemma 4.3. Set $\xi_{n+1} := q$. Thus, $\{\nu_1, \nu_2, \ldots, \nu_n\} \subseteq \{\xi_1, \xi_2, \ldots, \xi_n\} \cup \{q\}$ rewrites as

$$\{\nu_1, \nu_2, \ldots, \nu_n\} \subseteq \{\xi_1, \xi_2, \ldots, \xi_{n+1}\}. \tag{41}$$

The n numbers $\nu_1, \nu_2, \ldots, \nu_n$ are distinct (since $|\{\nu_1, \nu_2, \ldots, \nu_n\}| = n$). Furthermore, each of these n numbers appears in the $(n+1)$-tuple $(\xi_1, \xi_2, \ldots, \xi_{n+1})$ (by (41)). Since these n numbers are distinct, they must therefore appear as n **distinct** entries in this $(n+1)$-tuple. Thus, they must be the entries $\xi_1, \xi_2, \ldots, \xi_{p-1}, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{n+1}$ in some order, where p is some element of $[n+1]$. Consider this p.

Now,

$$|\nu| = |\xi| = \xi_1 + \xi_2 + \cdots + \xi_n = (\xi_1 + \xi_2 + \cdots + \xi_{n+1}) - \xi_{n+1}.$$

Comparing this with

$$\begin{aligned} |\nu| &= \nu_1 + \nu_2 + \cdots + \nu_n = \xi_1 + \xi_2 + \cdots + \xi_{p-1} + \xi_{p+1} + \xi_{p+2} + \cdots + \xi_{n+1} \\ &\qquad \left(\begin{array}{c} \text{since the } n \text{ numbers } \nu_1, \nu_2, \ldots, \nu_n \text{ are} \\ \text{the numbers } \xi_1, \xi_2, \ldots, \xi_{p-1}, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{n+1} \text{ in some order} \end{array} \right) \\ &= (\xi_1 + \xi_2 + \cdots + \xi_{n+1}) - \xi_p, \end{aligned}$$

we obtain $(\xi_1 + \xi_2 + \cdots + \xi_{n+1}) - \xi_{n+1} = (\xi_1 + \xi_2 + \cdots + \xi_{n+1}) - \xi_p$. In other words, $\xi_p = \xi_{n+1}$. Hence, the numbers $\xi_1, \xi_2, \ldots, \xi_{p-1}, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{n+1}$ are precisely the numbers $\xi_1, \xi_2, \ldots, \xi_n$ (up to order). Since the n numbers $\nu_1, \nu_2, \ldots, \nu_n$ are the numbers $\xi_1, \xi_2, \ldots, \xi_{p-1}, \xi_{p+1}, \xi_{p+2}, \ldots, \xi_{n+1}$ in some order, we thus conclude that the n numbers $\nu_1, \nu_2, \ldots, \nu_n$ are the numbers $\xi_1, \xi_2, \ldots, \xi_n$ in some order. In other words, the n-tuple ν is obtained from ξ by permuting the entries. This proves Lemma 4.3. □

We shall furthermore use the following notation:

Definition 4.4. Let u and v be two integers. We write "$u \trianglerighteq v$" if and only if $u - v \in \{0, 1\}$.

Thus, for example, $2 \trianglerighteq 2$ and $2 \trianglerighteq 1$, but we don't have $2 \trianglerighteq 0$.

We need the following simple lemma:

Lemma 4.5. Let $n \in \mathbb{N}$ and $p \in \{0, 1, \ldots, n\}$. Let ξ be the n-tuple

$$(1, 2, \ldots, n) + \left(\underbrace{0, 0, \ldots, 0}_{n-p \text{ zeroes}}, \underbrace{1, 1, \ldots, 1}_{p \text{ ones}} \right) = (1, 2, \ldots, n-p, n-p+2, n-p+3, \ldots, n+1) \in \mathbb{Z}^n.$$

Let $\sigma \in S_n$ be a permutation such that $\sigma \neq \mathrm{id}$ (where id denotes the identity map $[n] \to [n]$). Then,

$$\prod_{i=1}^{n} [\xi_i \trianglerighteq \sigma(i)] = 0.$$

Proof of Lemma 4.5. The definition of ξ shows that

$$\xi_i = i \qquad \text{for each } i \in \{1, 2, \ldots, n-p\} \tag{42}$$

and

$$\xi_i = i + 1 \qquad \text{for each } i \in \{n-p+1, n-p+2, \ldots, n\}. \tag{43}$$

We assumed that $\sigma \neq \mathrm{id}$. Hence, there exists some $i \in [n]$ such that $\sigma(i) \neq i$. Let a be the **smallest** such i, and let b be the **largest** such i. Then, $\sigma(a) > a$ [9] and $\sigma(b) < b$ [10] and $a \leq b$ [11].

Now, we are in one of the following two cases:

Case 1: We have $a \leq n - p$.

Case 2: We have $a > n - p$.

Let us first consider Case 1. In this case, we have $a \leq n - p$. Thus, $a \in \{1, 2, \ldots, n-p\}$. Hence, (42) (applied to $i = a$) yields $\xi_a = a < \sigma(a)$ (since $\sigma(a) > a$), so that $\xi_a - \sigma(a) < 0$. Hence, we do not have $\xi_a \trianglerighteq \sigma(a)$ (since $\xi_a \trianglerighteq \sigma(a)$ would mean that $\xi_a - \sigma(a) \in \{0, 1\}$, which would contradict $\xi_a - \sigma(a) < 0$). In other words, we have $[\xi_a \trianglerighteq \sigma(a)] = 0$.

Therefore, $\prod_{i=1}^{n} [\xi_i \trianglerighteq \sigma(i)] = 0$ (since $[\xi_a \trianglerighteq \sigma(a)]$ is one of the factors of the product $\prod_{i=1}^{n} [\xi_i \trianglerighteq \sigma(i)]$). Thus, Lemma 4.5 is proved in Case 1.

Let us next consider Case 2. In this case, we have $a > n - p$. Thus, $b \geq a > n - p$, so that $b \in \{n-p+1, n-p+2, \ldots, n\}$. Hence, (43) (applied to $i = b$) yields $\xi_b = b + 1$. However, recall that $\sigma(b) < b$. Thus, $\underbrace{\xi_b}_{=b+1} - \underbrace{\sigma(b)}_{<b} > (b+1) - b = 1$. Therefore, we do not have $\xi_b \trianglerighteq \sigma(b)$ (because $\xi_b \trianglerighteq \sigma(b)$ would mean that

[9] *Proof.* We have $\sigma(a) \neq a$ (since a is an $i \in [n]$ such that $\sigma(i) \neq i$). Thus, $\sigma(\sigma(a)) \neq \sigma(a)$ (since σ is a permutation and therefore injective). Hence, $\sigma(a)$ is an $i \in [n]$ such that $\sigma(i) \neq i$. Since a is the **smallest** such i, we thus conclude that $\sigma(a) \geq a$. Hence, $\sigma(a) > a$ (since $\sigma(a) \neq a$).

[10] The proof of this is similar to the proof we just gave for $\sigma(a) > a$.

[11] since a is the **smallest** $i \in [n]$ such that $\sigma(i) \neq i$, while b is the **largest** such i

$\xi_b - \sigma(b) \in \{0,1\}$, which would contradict $\xi_b - \sigma(b) > 1$). In other words, we have $[\xi_b \trianglerighteq \sigma(b)] = 0$.

Therefore, $\prod_{i=1}^{n} [\xi_i \trianglerighteq \sigma(i)] = 0$ (since $[\xi_b \trianglerighteq \sigma(b)]$ is one of the factors of the product $\prod_{i=1}^{n} [\xi_i \trianglerighteq \sigma(i)]$). Thus, Lemma 4.5 is proved in Case 2.

We have now proved Lemma 4.5 in both Cases 1 and 2. Hence, the proof of Lemma 4.5 is complete. □

The following lemma is an analogue of Lemma 2.9:

Lemma 4.6. Let $n \in \mathbb{N}$ and $p \in \{0,1,\ldots,n\}$. Let ξ be the n-tuple

$$(1,2,\ldots,n) + \left(\underbrace{0,0,\ldots,0}_{n-p \text{ zeroes}}, \underbrace{1,1,\ldots,1}_{p \text{ ones}}\right)$$
$$= (1,2,\ldots,n-p,n-p+2,n-p+3,\ldots,n+1) \in \mathbb{Z}^n.$$

Let $\nu \in \mathbb{Z}^n$ be an n-tuple satisfying $|\nu| = |\xi|$. Then,

$$\det\left(([\nu_i \trianglerighteq j])_{i,j\in[n]}\right) = \sum_{\substack{\sigma\in S_n;\\ \nu=\xi\circ\sigma}} (-1)^{\sigma}. \tag{44}$$

Note that the matrix $([\nu_i \trianglerighteq j])_{i,j\in[n]}$ in (44) is a matrix with integer entries; thus, its determinant is a well-defined integer.

Before we prove Lemma 4.6, a remark is in order:

Remark 4.7. The sum $\sum_{\substack{\sigma\in S_n;\\ \nu=\xi\circ\sigma}} (-1)^{\sigma}$ on the right hand side of (44) always has either no addends or only one addend. (Indeed, it is easy to see that the n-tuples $\xi \circ \sigma$ for different $\sigma \in S_n$ are distinct; thus, no more than one of these n-tuples can equal ν.) Thus, this sum can be rewritten as

$$\begin{cases} (-1)^{\sigma}, & \text{if } \nu = \xi \circ \sigma \text{ for some } \sigma \in S_n; \\ 0, & \text{otherwise.} \end{cases}$$

Proof of Lemma 4.6. The definition of ξ yields

$$\xi = (1,2,\ldots,n) + \left(\underbrace{0,0,\ldots,0}_{n-p \text{ zeroes}}, \underbrace{1,1,\ldots,1}_{p \text{ ones}}\right)$$
$$= (1,2,\ldots,n-p,n-p+2,n-p+3,\ldots,n+1).$$

Thus,

$$\begin{aligned}&(\xi_1, \xi_2, \ldots, \xi_n)\\&= \xi = (1, 2, \ldots, n-p, n-p+2, n-p+3, \ldots, n+1). \end{aligned} \tag{45}$$

In other words, we have

$$\xi_i = i \qquad \text{for each } i \in \{1, 2, \ldots, n-p\} \tag{46}$$

and

$$\xi_i = i+1 \qquad \text{for each } i \in \{n-p+1, n-p+2, \ldots, n\}. \tag{47}$$

It follows easily that the n numbers $\xi_1, \xi_2, \ldots, \xi_n$ are distinct. Therefore, we have $|\{\xi_1, \xi_2, \ldots, \xi_n\}| = n$.

We are in one of the following three cases:

Case 1: We have $\{\nu_1, \nu_2, \ldots, \nu_n\} \not\subseteq [n+1]$.

Case 2: We have $|\{\nu_1, \nu_2, \ldots, \nu_n\}| \neq n$.

Case 3: We have neither $\{\nu_1, \nu_2, \ldots, \nu_n\} \not\subseteq [n+1]$ nor $|\{\nu_1, \nu_2, \ldots, \nu_n\}| \neq n$.

Let us first consider Case 1. In this case, we have $\{\nu_1, \nu_2, \ldots, \nu_n\} \not\subseteq [n+1]$. In other words, there exists some $k \in [n]$ such that $\nu_k \notin [n+1]$. Consider this k. Then, $[\nu_k \trianglerighteq j] = 0$ for each $j \in [n]$ (since $\nu_k \trianglerighteq j$ would entail $\nu_k \in \{j, j+1\} \subseteq [n+1]$, contradicting $\nu_k \notin [n+1]$). Hence, the matrix $([\nu_i \trianglerighteq j])_{i,j\in[n]}$ has a zero row (namely, the k-th row); therefore, the determinant of this matrix is 0. In other words,

$$\det\left(([\nu_i \trianglerighteq j])_{i,j\in[n]}\right) = 0. \tag{48}$$

On the other hand, the n-tuple ν contains the entry ν_k (obviously), whereas the n-tuple ξ does not (since $\nu_k \notin [n+1]$, but all entries of ξ belong to $[n+1]$). Thus, the n-tuple ν is not a permutation of the n-tuple ξ. In other words, there exists no $\sigma \in S_n$ satisfying $\nu = \xi \circ \sigma$. Hence,

$$\sum_{\substack{\sigma\in S_n;\\ \nu=\xi\circ\sigma}} (-1)^\sigma = (\text{empty sum}) = 0.$$

Comparing this with (48), we obtain $\det\left(([\nu_i \trianglerighteq j])_{i,j\in[n]}\right) = \sum_{\substack{\sigma\in S_n;\\ \nu=\xi\circ\sigma}} (-1)^\sigma$. Thus, Lemma 4.6 is proved in Case 1.

Let us next consider Case 2. In this case, we have $|\{\nu_1, \nu_2, \ldots, \nu_n\}| \neq n$. Thus, two of the numbers $\nu_1, \nu_2, \ldots, \nu_n$ are equal. Hence, the corresponding two rows of the matrix $([\nu_i \trianglerighteq j])_{i,j\in[n]}$ are equal. Thus, this matrix $([\nu_i \trianglerighteq j])_{i,j\in[n]}$ has two equal rows, and therefore its determinant is 0. In other words,

$$\det\left(([\nu_i \trianglerighteq j])_{i,j\in[n]}\right) = 0. \tag{49}$$

On the other hand, the n-tuple ν contains two equal entries (since two of the numbers $\nu_1, \nu_2, \ldots, \nu_n$ are equal), whereas the n-tuple ξ does not (since $\xi_1, \xi_2, \ldots, \xi_n$ are distinct). Thus, the n-tuple ν is not a permutation of the n-tuple ξ. In other words, there exists no $\sigma \in S_n$ satisfying $\nu = \xi \circ \sigma$. Hence,

$$\sum_{\substack{\sigma \in S_n; \\ \nu = \xi \circ \sigma}} (-1)^{\sigma} = (\text{empty sum}) = 0.$$

Comparing this with (49), we obtain $\det\left(([\nu_i \trianglerighteq j])_{i,j\in[n]}\right) = \sum_{\substack{\sigma \in S_n; \\ \nu = \xi \circ \sigma}} (-1)^{\sigma}$. Thus, Lemma 4.6 is proved in Case 2.

Finally, let us consider Case 3. In this case, we have neither $\{\nu_1, \nu_2, \ldots, \nu_n\} \not\subseteq [n+1]$ nor $|\{\nu_1, \nu_2, \ldots, \nu_n\}| \neq n$. In other words, we have $\{\nu_1, \nu_2, \ldots, \nu_n\} \subseteq [n+1]$ and $|\{\nu_1, \nu_2, \ldots, \nu_n\}| = n$.

Note that

$$\{\nu_1, \nu_2, \ldots, \nu_n\} \subseteq [n+1] = \{\xi_1, \xi_2, \ldots, \xi_n\} \cup \{n - p + 1\}$$

(since (45) shows that $\xi_1, \xi_2, \ldots, \xi_n$ are precisely the numbers $1, 2, \ldots, n+1$ except for $n - p + 1$). Thus, Lemma 4.3 (applied to $q = n - p + 1$) yields that there exists some permutation $\pi \in S_n$ satisfying $\nu = \xi \circ \pi$. Consider this π.

Thus, π is a permutation $\sigma \in S_n$ satisfying $\nu = \xi \circ \sigma$. Furthermore, it is easy to see that π is the **only** such permutation σ (because the n numbers $\xi_1, \xi_2, \ldots, \xi_n$ are distinct)[12]. Hence, the sum $\sum_{\substack{\sigma \in S_n; \\ \nu = \xi \circ \sigma}} (-1)^{\sigma}$ has only one addend, namely the addend for $\sigma = \pi$. Thus,

$$\sum_{\substack{\sigma \in S_n; \\ \nu = \xi \circ \sigma}} (-1)^{\sigma} = (-1)^{\pi}. \tag{50}$$

From $\nu = \xi \circ \pi$, we easily obtain

$$\det\left(([\nu_i \trianglerighteq j])_{i,j\in[n]}\right) = (-1)^{\pi} \cdot \det\left(([\xi_i \trianglerighteq j])_{i,j\in[n]}\right). \tag{51}$$

(Indeed, this can be proved just as we showed (16), except that η and the $\geq$ sign are replaced by ξ and the $\trianglerighteq$ sign.)

[12]Here is the argument in more detail: Recall that the n numbers $\xi_1, \xi_2, \ldots, \xi_n$ are distinct. Therefore, if $\sigma \in S_n$ is a permutation distinct from π, then Proposition 2.5 (applied to $\eta = \xi$) shows that $\xi \circ \sigma \neq \xi \circ \pi = \nu$, so that $\nu \neq \xi \circ \sigma$. Hence, the only permutation $\sigma \in S_n$ satisfying $\nu = \xi \circ \sigma$ is π.

However, the definition of the determinant of a matrix yields

$$\begin{aligned}
\det\left(([\xi_i \trianglerighteq j])_{i,j\in[n]}\right) &= \sum_{\sigma\in S_n} (-1)^{\sigma} \prod_{i=1}^{n} [\xi_i \trianglerighteq \sigma(i)] \\
&= \underbrace{(-1)^{\operatorname{id}}}_{=1} \prod_{i=1}^{n} \left[\xi_i \trianglerighteq \underbrace{\operatorname{id}(i)}_{=i}\right] + \sum_{\substack{\sigma\in S_n;\\ \sigma\neq\operatorname{id}}} (-1)^{\sigma} \underbrace{\prod_{i=1}^{n} [\xi_i \trianglerighteq \sigma(i)]}_{\substack{=0 \\ \text{(by Lemma 4.5)}}} \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the addend} \\ \text{for } \sigma=\operatorname{id} \text{ from the sum (since } \operatorname{id}\in S_n\text{)}\end{array}\right) \\
&= \prod_{i=1}^{n} \underbrace{[\xi_i \trianglerighteq i]}_{\substack{=1 \\ \text{(this follows easily} \\ \text{from (46) and (47))}}} + \underbrace{\sum_{\substack{\sigma\in S_n;\\ \sigma\neq\operatorname{id}}} (-1)^{\sigma} 0}_{=0} = \prod_{i=1}^{n} 1 = 1.
\end{aligned}$$

Thus, (51) becomes

$$\det\left(([\nu_i \trianglerighteq j])_{i,j\in[n]}\right) = (-1)^{\pi} \cdot \underbrace{\det\left(([\xi_i \trianglerighteq j])_{i,j\in[n]}\right)}_{=1} = (-1)^{\pi} = \sum_{\substack{\sigma\in S_n;\\ \nu=\xi\circ\sigma}} (-1)^{\sigma}$$

(by (50)). Thus, Lemma 4.6 is proved in Case 3.

We have now proved Lemma 4.6 in all three Cases 1, 2 and 3. This completes the proof of Lemma 4.6. □

Our next lemma is an analogue to Lemma 2.11:

Lemma 4.8. Let $n \in \mathbb{N}$. For any permutation $\sigma \in S_n$, we let $\overline{\sigma}$ denote the n-tuple $(\sigma(1), \sigma(2), \ldots, \sigma(n)) \in \mathbb{Z}^n$.

Let $\nu \in \mathbb{Z}^n$ be an n-tuple. Then,

$$\det\left(([\nu_i \trianglerighteq j])_{i,j\in[n]}\right) = \sum_{\substack{\sigma\in S_n;\\ \nu-\overline{\sigma}\in\{0,1\}^n}} (-1)^{\sigma}.$$

Proof of Lemma 4.8. This proof is similar to the above proof of Lemma 2.11; we leave the necessary changes to the reader. □

We can now prove Theorem 4.1:

Proof of Theorem 4.1. The first step is to check that $|\xi| = (1+2+\cdots+n)+p$. The rest of the proof is a straightforward modification of the above proof of Theorem 2.1 (Lemmas 4.6 and 4.8 have to be applied instead of Lemmas 2.9 and 2.11). We leave the details to the reader. □

4.3. Corollaries

Several corollaries can be obtained from Theorem 4.1 in the same way as we did above with Theorem 2.1. Here is an analogue of Corollary 3.1:

Corollary 4.9. Let $n \in \mathbb{N}$ and $p \in \{0, 1, \ldots, n\}$. Let $q = n - p$. Let $h_{k,\ i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$. Assume that

$$h_{k,\ i} = 0 \qquad \text{for all } k < 0 \text{ and } i > q. \tag{52}$$

For any $\alpha \in \mathbb{Z}^n$, we define

$$t_\alpha := \operatorname{rowdet}\left(\left(h_{\alpha_i + j,\ i} \right)_{i,j \in [n]} \right) \in R.$$

Let $\alpha \in \mathbb{Z}^n$. Assume that

$$\alpha_i < -q \qquad \text{for each } i > q. \tag{53}$$

Then,

$$\sum_{\substack{\beta \in \{0,1\}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i + j,\ i} \right)_{i,j \in [q]} \right) \cdot \operatorname{rowdet}\left(\left(h_{\alpha_{q+i} + q + j + 1,\ q+i} \right)_{i,j \in [p]} \right).$$

This can be derived from Theorem 4.1 using the following lemma (which generalizes both our Lemma 3.2 and [Grinbe15, Exercise 6.29], although it is stated in a rather different way):

Lemma 4.10. Let $n \in \mathbb{N}$. Let $A = \left(a_{i,j} \right)_{i,j \in [n]} \in R^{n \times n}$ be an $n \times n$-matrix. Let $k \in \{0, 1, \ldots, n\}$. Assume that

$$a_{i,j} = 0 \qquad \text{for every } i \in \{k+1, k+2, \ldots, n\} \text{ and } j \in \{1, 2, \ldots, k\}.$$

Then,

$$\operatorname{rowdet} A = \operatorname{rowdet}\left(\left(a_{i,j} \right)_{i,j \in [k]} \right) \cdot \operatorname{rowdet}\left(\left(a_{k+i,k+j} \right)_{i,j \in [n-k]} \right).$$

Example 4.11. For $n = 4$ and $k = 2$, Lemma 4.10 is saying that

$$\operatorname{rowdet} \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ a_{2,1} & a_{2,2} & a_{2,3} & a_{2,4} \\ 0 & 0 & a_{3,3} & a_{3,4} \\ 0 & 0 & a_{4,3} & a_{4,4} \end{pmatrix} = \operatorname{rowdet} \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \cdot \operatorname{rowdet} \begin{pmatrix} a_{3,3} & a_{3,4} \\ a_{4,3} & a_{4,4} \end{pmatrix}.$$

Proof of Lemma 4.10. This is a straightforward generalization of [Grinbe15, Exercise 6.29], and can be proved in the same way (as long as the requisite attention is paid to the order of factors in products, since the ring R is not required to be commutative[13]). We leave the details to the reader. $\square$

We can now easily derive Corollary 4.9 from Theorem 4.1:

Proof of Corollary 4.9. We have $q = n - p \in \{0, 1, \ldots, n\}$ (since $p \in \{0, 1, \ldots, n\}$). From $q = n - p$, we obtain $n - q = p$ and $q + p = n$.

Define an n-tuple $\xi \in \mathbb{Z}^n$ as in Theorem 4.1. Then, Theorem 4.1 yields

$$\sum_{\substack{\beta \in \{0,1\}^n; \\ |\beta| = p}} t_{\alpha+\beta} = \operatorname{rowdet}\left(\left(h_{\alpha_i + \xi_j,\ i} \right)_{i,j \in [n]} \right). \tag{54}$$

However, the definition of ξ yields

$$\begin{aligned} \xi &= (1, 2, \ldots, n-p, n-p+2, n-p+3, \ldots, n+1) \\ &= (1, 2, \ldots, q, q+2, q+3, \ldots, n+1) \qquad (\text{since } n - p = q). \end{aligned}$$

Hence,

$$\xi_k = k \qquad \text{for each } k \in \{1, 2, \ldots, q\} \tag{55}$$

and

$$\xi_k = k + 1 \qquad \text{for each } k \in \{q+1, q+2, \ldots, n\}. \tag{56}$$

Now, it is easy to see that

$$h_{\alpha_i + \xi_j,\ i} = 0 \qquad \text{for every } i \in \{q+1, q+2, \ldots, n\} \text{ and } j \in \{1, 2, \ldots, q\}$$

(because if $i \in \{q+1, q+2, \ldots, n\}$ and $j \in \{1, 2, \ldots, q\}$, then (55) yields $\xi_j = j \leq q$, whereas (53) yields $\alpha_i < -q$, so that $\underbrace{\alpha_i}_{<-q} + \underbrace{\xi_j}_{\leq q} < -q + q = 0$, and therefore (52)

[13] This means, in particular, that some products in [Grinbe15, solution to Exercise 6.29] need to be reordered (and the finite product notation $\prod_{i \in I} a_i$ needs to be understood as the product of the a_i in the order of increasing i). Furthermore, instead of using [Grinbe15, Theorem 6.82 **(a)**], we need to use the fact that any $n \times n$-matrix $A = \left(a_{i,j} \right)_{i,j \in [n]}$ with $n > 0$ satisfies

$$\operatorname{rowdet} A = \sum_{q=1}^{n} (-1)^{n+q} \operatorname{rowdet}\left(A_{\sim n, \sim q} \right) \cdot a_{n,q}.$$

(This generalizes the $p = n$ case of [Grinbe15, Theorem 6.82 **(a)**] to the case of noncommutative R. The general case of [Grinbe15, Theorem 6.82 **(a)**] cannot be generalized to noncommutative R, but fortunately we only need the $p = n$ case in our proof.)

(applied to $k = \alpha_i + \xi_j$) yields $h_{\alpha_i+\xi_j,\ i} = 0$). Therefore, Lemma 4.10 (applied to $a_{i,j} = h_{\alpha_i+\xi_j,\ i}$ and $A = \left(h_{\alpha_i+\xi_j,\ i}\right)_{i,j\in[n]}$ and $k = q$) yields

$$\begin{aligned}
&\operatorname{rowdet}\left(\left(h_{\alpha_i+\xi_j,\ i}\right)_{i,j\in[n]}\right) \\
&= \operatorname{rowdet}\underbrace{\left(\left(h_{\alpha_i+\xi_j,\ i}\right)_{i,j\in[q]}\right)}_{\substack{=\left(h_{\alpha_i+j,\ i}\right)_{i,j\in[q]} \\ \text{(since (55) yields } \xi_j=j \\ \text{for } j\in[q])}} \cdot \operatorname{rowdet}\underbrace{\left(\left(h_{\alpha_{q+i}+\xi_{q+j},\ q+i}\right)_{i,j\in[n-q]}\right)}_{\substack{=\left(h_{\alpha_{q+i}+q+j+1,\ q+i}\right)_{i,j\in[n-q]} \\ \text{(since (56) yields } \xi_{q+j}=q+j+1 \\ \text{for } j\in[n-q])}} \\
&= \operatorname{rowdet}\left(\left(h_{\alpha_i+j,\ i}\right)_{i,j\in[q]}\right) \cdot \operatorname{rowdet}\left(\left(h_{\alpha_{q+i}+q+j+1,\ q+i}\right)_{i,j\in[n-q]}\right) \\
&= \operatorname{rowdet}\left(\left(h_{\alpha_i+j,\ i}\right)_{i,j\in[q]}\right) \cdot \operatorname{rowdet}\left(\left(h_{\alpha_{q+i}+q+j+1,\ q+i}\right)_{i,j\in[p]}\right)
\end{aligned}$$

(since $n - q = p$). Combining this with (54), we obtain

$$\begin{aligned}
\sum_{\substack{\beta\in\{0,1\}^n; \\ |\beta|=p}} t_{\alpha+\beta} &= \operatorname{rowdet}\left(\left(h_{\alpha_i+\xi_j,\ i}\right)_{i,j\in[n]}\right) \\
&= \operatorname{rowdet}\left(\left(h_{\alpha_i+j,\ i}\right)_{i,j\in[q]}\right) \cdot \operatorname{rowdet}\left(\left(h_{\alpha_{q+i}+q+j+1,\ q+i}\right)_{i,j\in[p]}\right).
\end{aligned}$$

This proves Corollary 4.9. □

Next, let us state a counterpart to Corollary 3.5:

Corollary 4.12. Let $n \in \mathbb{N}$ and $p \in \{0, 1, \ldots, n\}$. Let $q = n - p$. Let $h_{k,\ i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$. Assume that

$$h_{k,\ n} = 0 \qquad \text{for all } k < 0.$$

For any $m \in \{0, 1, \ldots, n\}$ and any $\lambda \in \mathbb{Z}^m$, we define

$$s_\lambda := \operatorname{rowdet}\left(\left(h_{\lambda_i+j-i,\ i}\right)_{i,j\in[m]}\right) \in R.$$

Set

$$e_{p,\ q} := \operatorname{rowdet}\left(\left(h_{1+j-i,\ q+i}\right)_{i,j\in[p]}\right) \in R.$$

Fix an n-tuple $\mu \in \mathbb{Z}^n$. Assume that

$$\mu_i = 0 \qquad \text{for all } i > q. \tag{57}$$

Let $\overline{\mu} = (\mu_1, \mu_2, \ldots, \mu_q)$. Then,

$$s_{\overline{\mu}} \cdot e_{p,\ q} = \sum_{\substack{\beta\in\{0,1\}^n; \\ |\beta|=p}} s_{\mu+\beta}.$$

Proof of Corollary 4.12. This follows from Corollary 4.9 in the same way as Corollary 3.5 follows from Corollary 3.1 (i.e., by setting $\alpha = (\mu_1 - 1,\ \mu_2 - 2,\ \ldots,\ \mu_n - n)$ and rewriting all determinants involved). □

Counterparts to Corollary 3.4 and Corollary 3.6 can be stated as well, but we omit them to save space. (They are trivial consequences of Corollary 4.9 and Corollary 4.12.)

We note that (6) is the particular case of Corollary 4.12 for $R = \Lambda$, $h_{k,\ i} = h_k$ and $\mu = \lambda$.

5. A pre-LR rule?

We have now proved two fairly general determinantal identities – Theorem 2.1 and Theorem 4.1 – and seen some of their consequences. Anyone familiar with symmetric functions will likely view these two identities as two "antipodal" statements, in the sense in which the complete homogeneous symmetric functions are "antipodal" to the elementary symmetric functions.[14] The latter "antipodality" can be understood particularly well by viewing both families of symmetric functions as corner cases of *Schur functions* (see, e.g., [Stanle01, Chapter 7] or [Macdon95, §I.3]). It is thus natural to ask whether our determinantal identities can be viewed as corner cases of something more general, too:

Question 5.1. Is there a common generalization of Theorem 2.1 and Theorem 4.1?

Such a generalization might resemble (perhaps even generalize) the "immaculate Littlewood–Richardson rule" of Berg, Bergeron, Saliola, Serrano and Zabrocki ([BBSSZ15, Theorem 7.3]). Indeed, as we have seen above (in our proof of Proposition 3.8), the "right-Pieri rule" [BBSSZ13, Theorem 3.5] is a particular case of our Theorem 2.1; one can likewise derive a "second right-Pieri rule" (with $E_s = \mathfrak{S}_{(1^s)}$ taking the place of H_s) from our Theorem 4.1. Both of these "right-Pieri rules" are particular cases of the "immaculate Littlewood–Richardson rule". Thus, it is not too outlandish to suspect that the latter rule can, too, be viewed as a particular case of a (noncommutative) determinantal identity.

A step in the general direction of such a generalization appears to be the following proposition:

[14] The analogy strikes the eye from many directions: The summation signs $\sum\limits_{\substack{\beta \in \mathbb{N}^n; \\ |\beta| = p}}$ and $\sum\limits_{\substack{\beta \in \{0,1\}^n; \\ |\beta| = p}}$ in Theorem 2.1 and Theorem 4.1 are precisely the ones that appear in the definitions of the respective symmetric functions; the determinant $e_{p,\ q}$ in Corollary 4.12 is a rather straightforward generalization of the Jacobi–Trudi determinant for the p-th elementary symmetric function; and so on.

Proposition 5.2. Let $n \in \mathbb{N}$. Let $h_{k,\ i}$ be an element of R for all $k \in \mathbb{Z}$ and $i \in [n]$.

Let B be a finite set of n-tuples $\beta \in \mathbb{Z}^n$. Assume that this set B is invariant under the right S_n-action on $\mathbb{Z}^n$. (This S_n-action was introduced in Definition 2.3.)

For any $\alpha \in \mathbb{Z}^n$, we define

$$t_\alpha := \operatorname{rowdet}\left(\left(h_{\alpha_i+j,\ i}\right)_{i,j\in[n]}\right) \in R.$$

Let $\alpha \in \mathbb{Z}^n$. Then, there exists a family $\left(\lambda_\gamma\right)_{\gamma\in\mathbb{Z}^n}$ of coefficients $\lambda_\gamma \in \mathbb{Z}$ such that all but finitely many $\gamma \in \mathbb{Z}^n$ satisfy $\lambda_\gamma = 0$, and such that

$$\sum_{\beta\in B} t_{\alpha+\beta} = \sum_{\gamma\in\mathbb{Z}^n} \lambda_\gamma \operatorname{rowdet}\left(\left(h_{\alpha_i+\gamma_j,\ i}\right)_{i,j\in[n]}\right). \tag{58}$$

Proof of Proposition 5.2 (sketched). Define the ring

$$N := \mathbb{Z}\left\langle X_{k,\ i} \ \mid\ k \in \mathbb{Z} \text{ and } i \in [n]\right\rangle.$$

This is the ring of noncommutative polynomials over $\mathbb{Z}$ in the variables $X_{k,\ i}$ for all $k \in \mathbb{Z}$ and $i \in [n]$.

A noncommutative monomial in N will be called *multilinear* if it has the form $X_{p_1,\ 1}X_{p_2,\ 2}\cdots X_{p_n,\ n}$ for some $p \in \mathbb{Z}^n$. Let N^{mult} denote the $\mathbb{Z}$-linear span of all multilinear monomials in N. The symmetric group S_n acts $\mathbb{Z}$-linearly on this $\mathbb{Z}$-submodule N^{mult} from the right according to the rule

$$\left(X_{p_1,\ 1}X_{p_2,\ 2}\cdots X_{p_n,\ n}\right)\cdot\tau = X_{p_{\tau(1)},\ 1}X_{p_{\tau(2)},\ 2}\cdots X_{p_{\tau(n)},\ n} \tag{59}$$

(for all multilinear monomials $X_{p_1,\ 1}X_{p_2,\ 2}\cdots X_{p_n,\ n}$ and all $\tau \in S_n$).

An element $p \in N^{\text{mult}}$ will be called *antisymmetric* if each $\tau \in S_n$ satisfies $p\cdot\tau = (-1)^\tau p$. Let N^{sign} denote the set of all antisymmetric elements $p \in N^{\text{mult}}$; this is a $\mathbb{Z}$-submodule of N^{mult}.

For each $\alpha \in \mathbb{Z}^n$, we define

$$T_\alpha := \operatorname{rowdet}\left(\left(X_{\alpha_i+j,\ i}\right)_{i,j\in[n]}\right) \in N.$$

It is easy to see that $T_\alpha \in N^{\text{mult}}$ for each $\alpha \in \mathbb{Z}^n$. Hence, $\sum\limits_{\beta\in B} T_\beta \in N^{\text{mult}}$. Set

$$T_B := \sum_{\beta\in B} T_\beta. \tag{60}$$

We shall now show that $T_B \in N^{\text{sign}}$.

[*Proof:* We have $T_B = \sum\limits_{\beta\in B} T_\beta \in N^{\text{mult}}$. It thus remains to show that T_B is antisymmetric, i.e., that each $\tau \in S_n$ satisfies $T_B\cdot\tau = (-1)^\tau T_B$.

Let $\tau \in S_n$ be arbitrary. For each $\beta \in \mathbb{Z}^n$, we have

$$\begin{aligned} T_\beta &= \operatorname{rowdet}\left(\left(X_{\beta_i+j,\ i}\right)_{i,j\in[n]}\right) \qquad \left(\text{by the definition of } T_\beta\right) \\ &= \sum_{\sigma\in S_n} (-1)^\sigma X_{\beta_1+\sigma(1),\ 1} X_{\beta_2+\sigma(2),\ 2} \cdots X_{\beta_n+\sigma(n),\ n} \end{aligned} \tag{61}$$

and thus

$$\begin{aligned} & T_\beta \cdot \tau \\ &= \left(\sum_{\sigma\in S_n} (-1)^\sigma X_{\beta_1+\sigma(1),\ 1} X_{\beta_2+\sigma(2),\ 2} \cdots X_{\beta_n+\sigma(n),\ n}\right) \cdot \tau \\ &= \sum_{\sigma\in S_n} (-1)^\sigma \underbrace{\left(X_{\beta_1+\sigma(1),\ 1} X_{\beta_2+\sigma(2),\ 2} \cdots X_{\beta_n+\sigma(n),\ n}\right) \cdot \tau}_{\substack{= X_{\beta_{\tau(1)}+\sigma(\tau(1)),\ 1} X_{\beta_{\tau(2)}+\sigma(\tau(2)),\ 2} \cdots X_{\beta_{\tau(n)}+\sigma(\tau(n)),\ n} \\ \text{(by (59))}}} \\ &= \sum_{\sigma\in S_n} (-1)^\sigma X_{\beta_{\tau(1)}+\sigma(\tau(1)),\ 1} X_{\beta_{\tau(2)}+\sigma(\tau(2)),\ 2} \cdots X_{\beta_{\tau(n)}+\sigma(\tau(n)),\ n} \\ &= \sum_{\sigma\in S_n} (-1)^\sigma X_{(\beta\cdot\tau)_1+(\sigma\circ\tau)(1),\ 1} X_{(\beta\cdot\tau)_2+(\sigma\circ\tau)(2),\ 2} \cdots X_{(\beta\cdot\tau)_n+(\sigma\circ\tau)(n),\ n} \\ &\qquad \left(\text{since } \beta_{\tau(i)} = (\beta\cdot\tau)_i \text{ and } \sigma(\tau(i)) = (\sigma\circ\tau)(i) \text{ for each } i\right) \\ &= \sum_{\sigma\in S_n} \underbrace{(-1)^{\sigma\circ\tau^{-1}}}_{=(-1)^\sigma(-1)^\tau} X_{(\beta\cdot\tau)_1+\sigma(1),\ 1} X_{(\beta\cdot\tau)_2+\sigma(2),\ 2} \cdots X_{(\beta\cdot\tau)_n+\sigma(n),\ n} \\ &\qquad (\text{here, we have substituted } \sigma \text{ for } \sigma\circ\tau \text{ in the sum}) \\ &= (-1)^\tau \underbrace{\sum_{\sigma\in S_n} (-1)^\sigma X_{(\beta\cdot\tau)_1+\sigma(1),\ 1} X_{(\beta\cdot\tau)_2+\sigma(2),\ 2} \cdots X_{(\beta\cdot\tau)_n+\sigma(n),\ n}}_{\substack{=T_{\beta\cdot\tau} \\ \text{(by (61), applied to } \beta\cdot\tau \text{ instead of } \beta)}} \\ &= (-1)^\tau \cdot T_{\beta\cdot\tau}. \end{aligned} \tag{62}$$

Now, from (60), we obtain

$$\begin{aligned} T_B \cdot \tau &= \left(\sum_{\beta\in B} T_\beta\right) \cdot \tau = \sum_{\beta\in B} \underbrace{T_\beta \cdot \tau}_{\substack{=(-1)^\tau \cdot T_{\beta\cdot\tau} \\ \text{(by (62))}}} = \sum_{\beta\in B} (-1)^\tau \cdot T_{\beta\cdot\tau} = (-1)^\tau \cdot \sum_{\beta\in B} T_{\beta\cdot\tau} \\ &= (-1)^\tau \cdot \underbrace{\sum_{\beta\in B} T_\beta}_{=T_B} \qquad \left(\begin{array}{c} \text{here, we have substituted } \beta \text{ for } \beta\cdot\tau \text{ in the sum,} \\ \text{since the map } B \to B,\ \beta \mapsto \beta\cdot\tau \text{ is a bijection} \\ \text{(because } B \text{ is invariant under the } S_n\text{-action)} \end{array}\right) \\ &= (-1)^\tau T_B. \end{aligned}$$

This completes our proof of $T_B \in N^{\text{sign}}$.]

On the other hand, we claim the following:

Claim 1: Each $p \in N^{\text{sign}}$ is a $\mathbb{Z}$-linear combination of the row-determinants $\operatorname{rowdet}\left(\left(X_{\gamma_j,\, i}\right)_{i,j\in[n]}\right)$ with $\gamma \in \mathbb{Z}^n$.

Before we prove this, let us note that all these row-determinants $\operatorname{rowdet}\left(\left(X_{\gamma_j,\, i}\right)_{i,j\in[n]}\right)$ actually belong to N^{sign} (since each $\gamma \in \mathbb{Z}^n$ satisfies

$$\operatorname{rowdet}\left(\left(X_{\gamma_j,\, i}\right)_{i,j\in[n]}\right) = \sum_{\sigma\in S_n} (-1)^\sigma \underbrace{X_{\gamma_{\sigma(1)},\, 1} X_{\gamma_{\sigma(2)},\, 2} \cdots X_{\gamma_{\sigma(n)},\, n}}_{\substack{=\left(X_{\gamma_1,\, 1}X_{\gamma_2,\, 2}\cdots X_{\gamma_n,\, n}\right)\cdot\sigma \\ \text{(by (59))}}}$$
$$= \sum_{\sigma\in S_n} (-1)^\sigma \left(X_{\gamma_1,\, 1} X_{\gamma_2,\, 2} \cdots X_{\gamma_n,\, n}\right) \cdot \sigma,$$

which easily yields that $\operatorname{rowdet}\left(\left(X_{\gamma_j,\, i}\right)_{i,j\in[n]}\right) \in N^{\text{sign}}$); thus, Claim 1 shows that these row-determinants span the $\mathbb{Z}$-module N^{sign}. However, we will not need this.

[*Proof of Claim 1:* Let $p \in N^{\text{sign}}$. We shall show that p is a $\mathbb{Z}$-linear combination of the row-determinants $\operatorname{rowdet}\left(\left(X_{\gamma_j,\, i}\right)_{i,j\in[n]}\right)$.

We know that p is antisymmetric (since $p \in N^{\text{sign}}$). In other words, each $\sigma \in S_n$ satisfies

$$p \cdot \sigma = (-1)^\sigma p. \tag{63}$$

On the other hand, $p \in N^{\text{sign}} \subseteq N^{\text{mult}}$. Hence, we can write p as a $\mathbb{Z}$-linear combination of multilinear monomials (by the definition of N^{mult}). In other words,

$$p = \sum_{\alpha\in\mathbb{Z}^n} \mu_\alpha X_{\alpha_1,\, 1} X_{\alpha_2,\, 2} \cdots X_{\alpha_n,\, n} \tag{64}$$

for some scalars $\mu_\alpha \in \mathbb{Z}$ (almost all of them 0). These scalars μ_α must furthermore satisfy

$$\mu_{\alpha\cdot\sigma} = (-1)^\sigma \cdot \mu_\alpha \qquad \text{for all } \alpha \in \mathbb{Z}^n \text{ and all } \sigma \in S_n \tag{65}$$

(by comparing coefficients in (63)). Hence, we have $\mu_\alpha = 0$ for any n-tuple $\alpha \in \mathbb{Z}^n$ that has at least two equal entries (because for any such α, there exists some transposition $\sigma \in S_n$ such that $\alpha \cdot \sigma = \alpha$, so that the equality (65) simplifies to

$\mu_\alpha = \underbrace{(-1)^\sigma}_{=-1} \cdot \mu_\alpha = -\mu_\alpha$, and therefore we have $\mu_\alpha = 0$). Therefore, (64) simplifies to

$$\begin{aligned}
p &= \sum_{\substack{\alpha\in\mathbb{Z}^n;\\ \text{all entries of } \alpha \text{ are distinct}}} \mu_\alpha X_{\alpha_1,\,1} X_{\alpha_2,\,2} \cdots X_{\alpha_n,\,n} \\
&= \sum_{\substack{\gamma\in\mathbb{Z}^n;\\ \gamma_1<\gamma_2<\cdots<\gamma_n}} \sum_{\sigma\in S_n} \underbrace{\mu_{\gamma\cdot\sigma}}_{\substack{=(-1)^\sigma \mu_\gamma\\ \text{(by (65))}}} X_{\gamma_{\sigma(1)},\,1} X_{\gamma_{\sigma(2)},\,2} \cdots X_{\gamma_{\sigma(n)},\,n} \\
&\qquad \left(\begin{array}{c}\text{here, we have split the sum according to the } n\text{-tuple } \gamma \\ \text{obtained by sorting the } n\text{-tuple } \alpha \text{ in increasing order}\end{array}\right) \\
&= \sum_{\substack{\gamma\in\mathbb{Z}^n;\\ \gamma_1<\gamma_2<\cdots<\gamma_n}} \sum_{\sigma\in S_n} (-1)^\sigma \mu_\gamma X_{\gamma_{\sigma(1)},\,1} X_{\gamma_{\sigma(2)},\,2} \cdots X_{\gamma_{\sigma(n)},\,n} \\
&= \sum_{\substack{\gamma\in\mathbb{Z}^n;\\ \gamma_1<\gamma_2<\cdots<\gamma_n}} \mu_\gamma \underbrace{\sum_{\sigma\in S_n} (-1)^\sigma X_{\gamma_{\sigma(1)},\,1} X_{\gamma_{\sigma(2)},\,2} \cdots X_{\gamma_{\sigma(n)},\,n}}_{=\operatorname{rowdet}\left(\left(X_{\gamma_j,\,i}\right)_{i,j\in[n]}\right)} \\
&= \sum_{\substack{\gamma\in\mathbb{Z}^n;\\ \gamma_1<\gamma_2<\cdots<\gamma_n}} \mu_\gamma \operatorname{rowdet}\left(\left(X_{\gamma_j,\,i}\right)_{i,j\in[n]}\right).
\end{aligned}$$

This equality shows that p is a $\mathbb{Z}$-linear combination of the row-determinants $\operatorname{rowdet}\left(\left(X_{\gamma_j,\,i}\right)_{i,j\in[n]}\right)$. This completes our proof of Claim 1.]

Now, recall that $T_B \in N^{\text{sign}}$. Hence, Claim 1 shows that T_B is a $\mathbb{Z}$-linear combination of the row-determinants $\operatorname{rowdet}\left(\left(X_{\gamma_j,\,i}\right)_{i,j\in[n]}\right)$ with $\gamma \in \mathbb{Z}^n$. In other words, there exists a family $(\lambda_\gamma)_{\gamma\in\mathbb{Z}^n}$ of coefficients $\lambda_\gamma \in \mathbb{Z}$ such that all but finitely many $\gamma \in \mathbb{Z}^n$ satisfy $\lambda_\gamma = 0$, and such that

$$T_B = \sum_{\gamma\in\mathbb{Z}^n} \lambda_\gamma \operatorname{rowdet}\left(\left(X_{\gamma_j,\,i}\right)_{i,j\in[n]}\right). \tag{66}$$

Consider this family $(\lambda_\gamma)_{\gamma\in\mathbb{Z}^n}$. We shall now show that this family also satisfies (58).

Indeed, consider the ring homomorphism $f : N \to R$ that sends each $X_{k,\,i}$ to $h_{\alpha_i+k,\,i}$. (This clearly exists by the universal property of the free $\mathbb{Z}$-algebra N.) For each $\beta \in \mathbb{Z}^n$, we have

$$\begin{aligned}
f\left(T_\beta\right) &= f\left(\operatorname{rowdet}\left(\left(X_{\beta_i+j,\ i}\right)_{i,j\in[n]}\right)\right) && \text{(by the definition of } T_\beta\text{)} \\
&= \operatorname{rowdet}\left(\left(f\left(X_{\beta_i+j,\ i}\right)\right)_{i,j\in[n]}\right) && \text{(since } f \text{ is a ring homomorphism)} \\
&= \operatorname{rowdet}\left(\left(h_{(\alpha+\beta)_i+j,\ i}\right)_{i,j\in[n]}\right) \\
& \qquad \left(\begin{array}{c}\text{because we have } f\left(X_{\beta_i+j,\ i}\right) = h_{\alpha_i+\beta_i+j,\ i} = h_{(\alpha+\beta)_i+j,\ i} \\ \text{for all } i \in [n] \text{ and } j \in [n]\end{array}\right) \\
&= t_{\alpha+\beta} && (67)
\end{aligned}$$

(by the definition of $t_{\alpha+\beta}$). Now, applying the homomorphism f to both sides of the equality (60), we obtain

$$\begin{aligned}
f\left(T_B\right) &= f\left(\sum_{\beta\in B} T_\beta\right) = \sum_{\beta\in B} \underbrace{f\left(T_\beta\right)}_{\substack{=t_{\alpha+\beta} \\ \text{(by (67))}}} \qquad \text{(since } f \text{ is a ring homomorphism)} \\
&= \sum_{\beta\in B} t_{\alpha+\beta}.
\end{aligned}$$

Hence, applying the homomorphism f to both sides of (66), we obtain

$$\begin{aligned}
\sum_{\beta\in B} t_{\alpha+\beta} &= \sum_{\gamma\in\mathbb{Z}^n} \lambda_\gamma \underbrace{f\left(\operatorname{rowdet}\left(\left(X_{\gamma_j,\ i}\right)_{i,j\in[n]}\right)\right)}_{\substack{=\operatorname{rowdet}\left(\left(f\left(X_{\gamma_j,\ i}\right)\right)_{i,j\in[n]}\right) \\ \text{(since } f \text{ is a ring homomorphism)}}} \\
&= \sum_{\gamma\in\mathbb{Z}^n} \lambda_\gamma \operatorname{rowdet}\left(\left(\underbrace{f\left(X_{\gamma_j,\ i}\right)}_{\substack{=h_{\alpha_i+\gamma_j,\ i} \\ \text{(by the definition of } f\text{)}}}\right)_{i,j\in[n]}\right) \\
&= \sum_{\gamma\in\mathbb{Z}^n} \lambda_\gamma \operatorname{rowdet}\left(\left(h_{\alpha_i+\gamma_j,\ i}\right)_{i,j\in[n]}\right).
\end{aligned}$$

In other words, (58) holds.

Thus, we have found a family $(\lambda_\gamma)_{\gamma\in\mathbb{Z}^n}$ of coefficients $\lambda_\gamma \in \mathbb{Z}$ such that all but finitely many $\gamma \in \mathbb{Z}^n$ satisfy $\lambda_\gamma = 0$, and such that (58) holds. The proof of Proposition 5.2 is thus complete. □

References

[BBSSZ13] Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, *A Lift of the Schur and Hall–Littlewood Bases to Non-commutative Symmetric Functions*, Canadian Journal of Mathematics **66** (2014), Issue 3, pp. 525–565.

[BBSSZ15] Chris Berg, Nantel Bergeron, Franco Saliola, Luis Serrano, Mike Zabrocki, *Multiplicative structures of the immaculate basis of non-commutative symmetric functions*, Journal of Combinatorial Theory, Series A **152** (2017), pp. 10–44.
See arXiv:1305.4700v3 for a preprint.

[Fun12] Alex Fun, *Raising operators, recurrences, and the Littlewood–Richardson polynomials*, PhD thesis, University of Sydney 2012.

[Grinbe15] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 25 May 2021.
`http://www.cip.ifi.lmu.de/~grinberg/primes2015/sols.pdf`
The numbering of theorems and formulas in this link might shift when the project gets updated; for a "frozen" version whose numbering is guaranteed to match that in the citations above, see arXiv:2008.09862v2 or `https://github.com/darijgr/detnotes/releases/tag/2019-01-10` .

[Grinbe19] Darij Grinberg, *A basis for a quotient of symmetric polynomials (draft)*, 22 September 2021, arXiv:1910.00207v2.
`https://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf`

[Grinbe21] Darij Grinberg, *An Introduction to Algebraic Combinatorics [Math 701, Spring 2021 lecture notes]*, 10 September 2021.
`https://www.cip.ifi.lmu.de/~grinberg/t/21s/lecs.pdf`

[LakTho07] Dan Laksov, Anders Thorup, *A Determinantal Formula for the Exterior Powers of the Polynomial Ring*, Indiana University Mathematics Journal **56**, No. 2 (2007), pp. 825–845.

[Macdon95] Ian G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Mathematical Monographs, 2nd edition, Oxford Science Publications 1995.

[Stanle01] Richard P. Stanley, *Enumerative Combinatorics, volume 2*, First edition, Cambridge University Press 2001.
See `http://math.mit.edu/~rstan/ec/` for errata.

[Tamvak13] Harry Tamvakis, *The theory of Schur polynomials revisited*, Enseign. Math. **58** (2012), pp. 147–163, arXiv:1008.3094v2.