Rep#1: Deformations of a bimodule algebra

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[not completed, not proofread]

The purpose of this short note is to generalize Problem 2.24 in [1]. First a couple of definitions:

Definition 1. In the following, a *ring* will always mean a (not necessarily commutative) ring with unity. Ring homomorphisms are always assumed to respect the unity. For every ring R, we denote the unity of R by 1_R.

Furthermore, if A is a ring, an A-*algebra* will mean a (not necessarily commutative) ring R along with a ring homomorphism $\rho : A \to R$. In the case of such an A-algebra R, we will denote the product $\rho(a) r$ by ar and the product $r\rho(a)$ by ra for any $a \in A$ and any $r \in R$.

An A-algebra R is said to be *symmetric*[1] if $ar = ra$ for every $a \in A$ and $r \in R$.

Definition 2. Let A be a ring. An A-*bimodule algebra* will be defined as a ring B along with an A-left module structure on B and an A-right module structure on B which satisfy the following three axioms:

$$(ab) b' = a (bb') \qquad \text{for any } a \in A,\ b \in B \text{ and } b' \in B;$$
$$b (b'a) = (bb') a \qquad \text{for any } a \in A,\ b \in B \text{ and } b' \in B;$$
$$(ab) a' = a (ba') \qquad \text{for any } a \in A,\ b \in B \text{ and } a' \in A.$$

Definition 3. Let B be a ring. Then, we denote by $B[[t]]$ the ring of formal power series over B in the indeterminate t, where t is supposed to commute with every element of B. Formally, this means that we define $B[[t]]$ as the ring of all sequences $(b_0, b_1, b_2, ...) \in B^{\mathbb{N}}$ (where $\mathbb{N}$ means the set $\{0, 1, 2, ...\}$), with addition defined by

$$(b_0, b_1, b_2, ...) + (b'_0, b'_1, b'_2, ...) = (b_0 + b'_0, b_1 + b'_1, b_2 + b'_2, ...)$$

and multiplication defined by

$$(b_0, b_1, b_2, ...) \cdot (b'_0, b'_1, b'_2, ...) = \left(\sum_{\substack{(i,j)\in\mathbb{N}^2; \\ i+j=0}} b_i b'_j, \sum_{\substack{(i,j)\in\mathbb{N}^2; \\ i+j=1}} b_i b'_j, \sum_{\substack{(i,j)\in\mathbb{N}^2; \\ i+j=2}} b_i b'_j, ... \right),$$

and we denote a sequence $(b_0, b_1, b_2, ...)$ by $\sum\limits_{i=0}^{\infty} b_i t^i$. For every $m \in \mathbb{N}$, the element $b_m \in B$ is called the *coefficient of the power series* $(b_0, b_1, b_2, ...) = \sum\limits_{i=0}^{\infty} b_i t^i$ *before* t^m. The element $b_0 \in B$ is also called the *constant term* of the power series $(b_0, b_1, b_2, ...) = \sum\limits_{i=0}^{\infty} b_i t^i$.

[1] What we call "symmetric A-algebra" happens to be what most authors call "A-algebra".

Definition 4. Let B be a ring, and let $(b_m, b_{m+1}, ..., b_n)$ be a sequence of elements of B. Then, we denote by $\overleftarrow{\prod_{i=m}^{n}} b_i$ the product $b_n b_{n-1} ... b_m$ (this product is supposed to mean 1 if $m > n$).

Now comes the generalization of Problem 2.24 (a) in [1][2]:

Theorem 1. Let K be a commutative ring. Let A be a symmetric K-algebra. Let B be an A-bimodule algebra such that B is a symmetric K-algebra (where the K-algebra structure on B is given by the ring homomorphism $K \to B$, $k \mapsto (k \cdot 1_A) \cdot 1_B$).

Assume that

$$\left(\begin{array}{c} \text{for every } K\text{-linear map } f : A \to B \text{ which satisfies} \\ \left(f(aa') = af(a') + f(a)a' \text{ for all } a \in A \text{ and } a' \in A\right), \\ \text{there exists an element } s \in B \text{ such that} \\ \left(f(a) = as - sa \text{ for all } a \in A\right). \end{array}\right) \qquad (1)$$

Let $B[[t]]$ be the ring of formal power series over B in the indeterminate t, where t is supposed to commute with every element of B.

Here and in the following, let 1 denote the unity 1_B of the ring B.

Let $\overline{\rho} : A \to B[[t]]$ be a K-linear homomorphism such that any $a \in A$ and any $a' \in A$ satisfy $\overline{\rho}(aa') = \overline{\rho}(a)\overline{\rho}(a')$, and such that for every $a \in A$, the constant term of the power series $\overline{\rho}(a)$ equals $a \cdot 1$. (Note that $a \cdot 1$ is simply the canonical image of a in the A-algebra B).

Then, there exists a power series $b \in B[[t]]$ such that for every $a \in A$, the power series $b\overline{\rho}(a)b^{-1} \in B[[t]]$ equals the (constant) power series $a \cdot 1$.

Proof of Theorem 1. First, we endow the ring $B[[t]]$ with the (t)-adic topology. This topology is defined in such a way that for every $p \in B[[t]]$, the family $(p + t^0 B[[t]], p + t^1 B[[t]], p + t^2 B[[t]], ...)$ is a basis of open neighbourhoods of p. This topology makes $B[[t]]$ a topological ring, since $t^i B[[t]]$ is a two-sided ideal of $B[[t]]$ for every $i \in \mathbb{N}$.

For any k elements u_1, u_2, ..., u_k of B, and for every $a \in A$, let us denote by $\overline{\rho}_{u_1,u_2,...,u_k}(a)$ the element

$$\overleftarrow{\prod_{i=1}^{k}} \left(1 - u_i t^i\right) \cdot \overline{\rho}(a) \cdot \left(\overleftarrow{\prod_{i=1}^{k}} \left(1 - u_i t^i\right)\right)^{-1} \in B[[t]].$$

Clearly, $\overline{\rho}_{u_1,u_2,...,u_k} : A \to B[[t]]$ is a K-linear map for any k elements u_1, u_2, ..., u_k of B.

Now, we are going to recursively construct a sequence $(u_1, u_2, u_3, ...) \in B^{\{1,2,3,...\}}$ of elements of B such that every $n \in \mathbb{N}$ satisfies

$$\left(\overline{\rho}_{u_1,u_2,...,u_n}(a) \equiv a \cdot 1 \mod t^{n+1} B[[t]] \qquad \text{for every } a \in A\right). \qquad (2)$$

[2] Problem 2.24 in [1] is recovered from this generalization by setting $B = \operatorname{End} V$.

In fact, we first notice that the equation (2) is satisfied for $n = 0$ (note that the product $\overleftarrow{\prod\limits_{i=1}^{n}} (1 - u_i t^i)$ is an empty product when $n = 0$), because in the case $n = 0$, we have $\overline{\rho}_{u_1,u_2,...,u_n}(a) = (\text{empty product}) \cdot \overline{\rho}(a) \cdot (\text{empty product})^{-1} = \overline{\rho}(a) \equiv a \cdot 1 \mod tB[[t]]$ (since the constant term of the power series $\overline{\rho}(a)$ equals $a \cdot 1$). Now, we are going to construct our sequence $(u_1, u_2, u_3, ...) \in B^{\{1,2,3,...\}}$ by induction: Let $m \in \mathbb{N}$ be such that $m > 0$. Assume that we have constructed some elements u_1, u_2, ..., u_{m-1} of B such that (2) holds for $n = m - 1$. Then, we are going to construct a new element u_m of B such that (2) holds for $n = m$.

In fact, applying (2) to $n = m - 1$ (we can do this since we have *assumed* that (2) holds for $n = m - 1$), we obtain

$$\overline{\rho}_{u_1,u_2,...,u_{m-1}}(a) \equiv a \cdot 1 \mod t^m B[[t]] \qquad \text{for every } a \in A.$$

In other words, every $a \in A$ satisfies

$$\overline{\rho}_{u_1,u_2,...,u_{m-1}}(a) - a \cdot 1 \in t^m B[[t]].$$

Denoting the power series $\overline{\rho}_{u_1,u_2,...,u_{m-1}}(a) - a \cdot 1$ by $p(a)$, we thus have $p(a) \in t^m B[[t]]$. Hence, $p_0(a) = p_1(a) = ... = p_{m-1}(a) = 0$, where $p_i(a)$ denotes the coefficient of the power series $p(a)$ before t^i for every $i \in \mathbb{N}$. Thus,

$$p(a) = \sum_{i=0}^{\infty} p_i(a) t^i = \sum_{i=0}^{m-1} \underbrace{p_i(a)}_{\substack{=0 \text{ (since} \\ p_0(a)=p_1(a)=...=p_{m-1}(a)=0)}} t^i + p_m(a) t^m + \sum_{i=m+1}^{\infty} \underbrace{p_i(a) t^i}_{\substack{\equiv 0 \bmod t^{m+1}B[[t]] \\ \text{(since } i \geq m+1 \text{ yields} \\ t^i \equiv 0 \bmod t^{m+1}B[[t]])}}$$

$$\equiv \sum_{i=0}^{m-1} 0 t^i + p_m(a) t^m + \sum_{i=m+1}^{\infty} 0 = p_m(a) t^m \bmod t^{m+1} B[[t]].$$

Hence,

$$\overline{\rho}_{u_1,u_2,...,u_{m-1}}(a) = \underbrace{\left(\overline{\rho}_{u_1,u_2,...,u_{m-1}}(a) - a \cdot 1\right)}_{=p(a) \equiv p_m(a) t^m \bmod t^{m+1}B[[t]]} + a \cdot 1 \equiv p_m(a) t^m + a \cdot 1 \bmod t^{m+1} B[[t]]. \tag{3}$$

Let us notice that the map $p : A \to B[[t]]$ is K-linear (by its definition, since the map $\overline{\rho}_{u_1,u_2,...,u_{m-1}} : A \to B[[t]]$ is K-linear), and thus the map $p_m : A \to B$ is K-linear as well (since $p_m = \operatorname{coeff}_m \circ p$, where $\operatorname{coeff}_m : B[[t]] \to B$ is the map that takes every power series to its coefficient before t^m, and thus p_m is K-linear because both coeff_m and p are K-linear).

Now, any $a \in A$ and $a' \in A$ satisfy

$$\begin{aligned}
&\overline{\rho}_{u_1,u_2,...,u_{m-1}}(a) \cdot \overline{\rho}_{u_1,u_2,...,u_{m-1}}(a') \\
&= \left(\overleftarrow{\prod_{i=1}^{m-1}} \left(1 - u_i t^i\right) \cdot \overline{\rho}(a) \cdot \left(\overleftarrow{\prod_{i=1}^{m-1}} \left(1 - u_i t^i\right) \right)^{-1} \right) \cdot \left(\overleftarrow{\prod_{i=1}^{m-1}} \left(1 - u_i t^i\right) \cdot \overline{\rho}(a') \cdot \left(\overleftarrow{\prod_{i=1}^{m-1}} \left(1 - u_i t^i\right) \right)^{-1} \right) \\
&= \overleftarrow{\prod_{i=1}^{m-1}} \left(1 - u_i t^i\right) \cdot \underbrace{\overline{\rho}(a) \cdot \overline{\rho}(a')}_{\substack{= \overline{\rho}(aa') \text{ (by a condition} \\ \text{of Theorem 1)}}} \cdot \left(\overleftarrow{\prod_{i=1}^{m-1}} \left(1 - u_i t^i\right) \right)^{-1} = \overleftarrow{\prod_{i=1}^{m-1}} \left(1 - u_i t^i\right) \cdot \overline{\rho}(aa') \cdot \left(\overleftarrow{\prod_{i=1}^{m-1}} \left(1 - u_i t^i\right) \right)^{-1} \\
&= \overline{\rho}_{u_1,u_2,...,u_{m-1}}(aa').
\end{aligned}$$

Since

$$\underbrace{\overline{\rho}_{u_1,u_2,...,u_{m-1}}(a)}_{\substack{\equiv p_m(a)t^m+a\cdot 1 \bmod t^{m+1}B[[t]] \\ \text{(by (3))}}} \cdot \underbrace{\overline{\rho}_{u_1,u_2,...,u_{m-1}}(a')}_{\substack{\equiv p_m(a')t^m+a'\cdot 1 \bmod t^{m+1}B[[t]] \\ \text{(by (3), applied to } a' \text{ instead of } a)}}$$

$$\equiv (p_m(a)t^m + a\cdot 1)\cdot(p_m(a')t^m + a'\cdot 1) = \underbrace{p_m(a)p_m(a')t^{2m}}_{\substack{\equiv 0 \bmod t^{m+1}B[[t]] \text{ (since } 2m\geq m+1 \\ \text{yields } t^{2m}\equiv 0 \bmod t^{m+1}B[[t]])}} + \underbrace{p_m(a)a't^m + ap_m(a')t^m}_{=(p_m(a)a'+ap_m(a'))t^m} + aa'\cdot 1$$

$$\equiv 0 + (p_m(a)a' + ap_m(a'))t^m + aa'\cdot 1 = (p_m(a)a' + ap_m(a'))t^m + aa'\cdot 1 \bmod t^{m+1}B[[t]]$$

and

$$\overline{\rho}_{u_1,u_2,...,u_{m-1}}(aa') \equiv p_m(aa')t^m + aa'\cdot 1 \bmod t^{m+1}B[[t]] \qquad \text{(by (3))},$$

this equation yields

$$(p_m(a)a' + ap_m(a'))t^m + aa'\cdot 1 \equiv p_m(aa')t^m + aa'\cdot 1 \bmod t^{m+1}B[[t]].$$

In other words,

$$(p_m(a)a' + ap_m(a'))t^m \equiv p_m(aa')t^m \bmod t^{m+1}B[[t]].$$

Hence, for every $i \in \{0, 1, ..., m\}$, the coefficient of the power series $(p_m(a)a' + ap_m(a'))t^m$ before t^i equals the coefficient of the power series $p_m(aa')t^m$ before t^i. Applying this to $i = m$, we see that the coefficient of the power series $(p_m(a)a' + ap_m(a'))t^m$ before t^m equals the coefficient of the power series $p_m(aa')t^m$ before t^m. But the coefficient of the power series $(p_m(a)a' + ap_m(a'))t^m$ is $p_m(a)a' + ap_m(a')$, and the coefficient of the power series $p_m(aa')t^m$ before t^m is $p_m(aa')$. Hence, $p_m(a)a' + ap_m(a')$ equals $p_m(aa')$. In other words, $p_m(aa') = p_m(a)a' + ap_m(a') = ap_m(a') + p_m(a)a'$. Since p_m is a K-linear map, the condition (1) (applied to $f = p_m$) yields that there exists an element $s \in B$ such that

$$(p_m(a) = as - sa \text{ for all } a \in A).$$

Now, let u_m be the element $-s$. Then, we conclude that

$$p_m(a) = u_m a - a u_m \text{ for all } a \in A \tag{4}$$

(since $u_m = -s$ yields $s = -u_m$ and thus $p_m(a) = as - sa = a(-u_m) - (-u_m)a = u_m a - a u_m$). Now, we must prove that (2) holds for $n = m$. In fact, every $a \in A$

satisfies

$$\overline{\rho}_{u_1,u_2,\ldots,u_m}(a)$$

$$= \underbrace{\overleftarrow{\prod_{i=1}^{m}}\left(1-u_it^i\right)}_{=(1-u_mt^m)\cdot\overleftarrow{\prod}_{i=1}^{m-1}(1-u_it^i)} \cdot\overline{\rho}(a)\cdot\left(\underbrace{\overleftarrow{\prod_{i=1}^{m}}\left(1-u_it^i\right)}_{=(1-u_mt^m)\cdot\overleftarrow{\prod_{i=1}^{m-1}}(1-u_it^i)}\right)^{-1}$$

$$= (1-u_mt^m)\cdot\underbrace{\overleftarrow{\prod_{i=1}^{m-1}}\left(1-u_it^i\right)\cdot\overline{\rho}(a)\cdot\left(\overleftarrow{\prod_{i=1}^{m-1}}\left(1-u_it^i\right)\right)^{-1}}_{\substack{=\overline{\rho}_{u_1,u_2,\ldots,u_{m-1}}(a)\equiv p_m(a)t^m+a\cdot 1 \bmod t^{m+1}B[[t]] \\ \text{(by (3))}}}\cdot\underbrace{(1-u_mt^m)^{-1}}_{\substack{\equiv 1+u_mt^m \bmod t^{m+1}B[[t]] \\ \text{(since } (1-u_mt^m)(1+u_mt^m) \\ =1-u_m^2t^{2m}\equiv 1 \bmod t^{m+1}B[[t]])}}$$

$$\equiv (1-u_mt^m)(p_m(a)t^m+a\cdot 1)(1+u_mt^m)$$

$$= p_m(a)t^m+p_m(a)u_mt^{2m}+a\cdot 1+au_mt^m-u_mp_m(a)t^{2m}-u_mp_m(a)u_mt^{3m}-u_mat^m-u_mau_mt^{2m}$$

$$\equiv p_m(a)t^m+a\cdot 1+au_mt^m-u_mat^m$$

$$\left(\begin{array}{c}\text{here we have removed all addends where } t^{2m} \text{ or } t^{3m} \text{ occurs, because } 2m\geq m+1 \text{ yields} \\ t^{2m}\equiv 0 \bmod t^{m+1}B[[t]] \text{ and because } 3m\geq m+1 \text{ yields } t^{3m}\equiv 0 \bmod t^{m+1}B[[t]]\end{array}\right)$$

$$= (u_ma-au_m)t^m+a\cdot 1+au_mt^m-u_mat^m \qquad \text{(by (4))}$$

$$= a\cdot 1 \bmod t^{m+1}B[[t]].$$

Hence, (2) holds for $n=m$.

Thus we have shown that, if we have constructed some elements u_1, u_2, ..., u_{m-1} of B such that (2) holds for $n=m-1$, then we can define a new element u_m of B in a way such that (2) holds for $n=m$. This way, we can recursively construct elements u_1, u_2, u_3, ... of B which satisfy the equation (2) for every $n\in\mathbb{N}$. Now, define a power series $b\in B[[t]]$ by $b=\lim\limits_{n\to\infty}\overleftarrow{\prod\limits_{i=1}^{n}}(1-u_it^i)$ (this power series b is well-defined since the sequence $\left(\overleftarrow{\prod\limits_{i=1}^{n}}(1-u_it^i)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to the (t)-adic topology on the ring $B[[t]]$ [3] and therefore converges). Then, every $a \in A$ satisfies

$$b\overline{\rho}(a)b^{-1} = \lim_{n\to\infty} \overleftarrow{\prod_{i=1}^{n}} \left(1 - u_i t^i\right) \cdot \overline{\rho}(a) \cdot \left(\lim_{n\to\infty} \overleftarrow{\prod_{i=1}^{n}} \left(1 - u_i t^i\right)\right)^{-1}$$

$$= \lim_{n\to\infty} \left(\overleftarrow{\prod_{i=1}^{n}} \left(1 - u_i t^i\right) \cdot \overline{\rho}(a) \cdot \left(\overleftarrow{\prod_{i=1}^{n}} \left(1 - u_i t^i\right)\right)^{-1}\right) = \lim_{n\to\infty} \overline{\rho}_{u_1,u_2,\dots,u_n}(a) = a \cdot 1$$

(because of (2) [4]). This proves Theorem 1.

References

[1] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Elena Udovina and Dmitry Vaintrob, *Introduction to representation theory*, July 13, 2010.
http://math.mit.edu/~etingof/replect.pdf

[3]This is because for every $k \in \mathbb{N}$, there exists some $j \in \mathbb{N}$ such that

$$\left(\overleftarrow{\prod_{i=1}^{n}} \left(1 - u_i t^i\right) \equiv \overleftarrow{\prod_{i=1}^{m}} \left(1 - u_i t^i\right) \bmod t^k B[[t]] \text{ for every } n \in \mathbb{N} \text{ and } m \in \mathbb{N} \text{ satisfying } n \geq j \text{ and } m \geq j\right)$$

(namely, take $j = k$; then, any $n \geq j$ satisfies

$$\overleftarrow{\prod_{i=1}^{n}} \left(1 - u_i t^i\right) = \overleftarrow{\prod_{i=j+1}^{n}} \underbrace{\left(1 - u_i t^i\right)}_{\substack{\equiv 1 \bmod t^k B[[t]], \\ \text{since } i \geq j+1 = k+1 > k \\ \text{yields } t^i \equiv 0 \bmod t^k B[[t]]}} \cdot \overleftarrow{\prod_{i=1}^{j}} \left(1 - u_i t^i\right) \equiv \overleftarrow{\prod_{i=j+1}^{n}} 1 \cdot \overleftarrow{\prod_{i=1}^{j}} \left(1 - u_i t^i\right) = \overleftarrow{\prod_{i=1}^{j}} \left(1 - u_i t^i\right) \bmod t^k B[[t]],$$

and similarly any $m \geq j$ satisfies

$$\overleftarrow{\prod_{i=1}^{m}} \left(1 - u_i t^i\right) \equiv \overleftarrow{\prod_{i=1}^{j}} \left(1 - u_i t^i\right) \bmod t^k B[[t]],$$

so that any $n \geq j$ and $m \geq j$ satisfy $\overleftarrow{\prod_{i=1}^{n}} \left(1 - u_i t^i\right) \equiv \overleftarrow{\prod_{i=1}^{m}} \left(1 - u_i t^i\right) \bmod t^k B[[t]]$).

[4]In fact, for every $i \in \mathbb{N}$, there exists some $k \in \mathbb{N}$ such that every $n \in \mathbb{N}$ satisfying $n \geq k$ satisfies $\overline{\rho}_{u_1,u_2,\dots,u_n}(a) \equiv a \cdot 1 \bmod t^i B[[t]]$ (namely, set $k = i - 1$; then, (2) yields $\overline{\rho}_{u_1,u_2,\dots,u_n}(a) \equiv a \cdot 1 \bmod t^{n+1} B[[t]]$ and thus also $\overline{\rho}_{u_1,u_2,\dots,u_n}(a) \equiv a \cdot 1 \bmod t^i B[[t]]$ because $t^{n+1} B[[t]] \subseteq t^i B[[t]]$ (since $n \geq k$ yields $n + 1 \geq k + 1 = (i - 1) + 1 = i$).