Rook sums in the symmetric group algebra

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Abstract. Let $\mathcal{A}$ be the group algebra $\mathbf{k}[S_n]$ of the n-th symmetric group S_n over a commutative ring $\mathbf{k}$. For any two subsets A and B of $[n]$, we define the elements

$$\nabla_{B,A} := \sum_{\substack{w \in S_n;\\ w(A)=B}} w \qquad \text{and} \qquad \widetilde{\nabla}_{B,A} := \sum_{\substack{w \in S_n;\\ w(A)\subseteq B}} w$$

of $\mathcal{A}$. We study these elements, showing in particular that their minimal polynomials factor into linear factors (with integer coefficients). We express the product $\nabla_{D,C}\nabla_{B,A}$ as a $\mathbb{Z}$-linear combination of $\nabla_{U,V}$'s.

More generally, for any two set compositions (i.e., ordered set partitions) $\mathbf{A}$ and $\mathbf{B}$ of $\{1,2,\ldots,n\}$, we define $\nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{A}$ to be the sum of all permutations $w \in S_n$ that send each block of $\mathbf{A}$ to the corresponding block of $\mathbf{B}$. This generalizes $\nabla_{B,A}$. The factorization property of minimal polynomials does not extend to the $\nabla_{\mathbf{B},\mathbf{A}}$, but we describe the ideal spanned by the $\nabla_{\mathbf{B},\mathbf{A}}$ and its complement.

In this note, we explore some easily definable elements in the group algebra of a symmetric group.

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1. Rook sums in the symmetric group algebra

1.1. Definitions

Let n be a nonnegative integer. Let $[n] := \{1,2,\ldots,n\}$.

Fix a commutative ring $\mathbf{k}$.

Let S_n be the n-th symmetric group, and let $\mathcal{A} := \mathbf{k}[S_n]$ be its group algebra over $\mathbf{k}$.

The *antipode* of the group algebra $\mathcal{A}$ is the $\mathbf{k}$-linear map $\mathcal{A} \to \mathcal{A}$ that sends each permutation $w \in S_n$ to w^{-1}. We will denote this map by S. It is well-known that S is a $\mathbf{k}$-algebra anti-automorphism and an involution (i.e., satisfies $S \circ S = \mathrm{id}$).

For any two subsets A and B of $[n]$, we define the elements

$$\nabla_{B,A} := \sum_{\substack{w \in S_n; \\ w(A)=B}} w \qquad \text{and} \qquad \widetilde{\nabla}_{B,A} := \sum_{\substack{w \in S_n; \\ w(A)\subseteq B}} w$$

of $\mathcal{A}$. We shall refer to these elements as *rectangular rook sums*.

The following proposition collects some easy properties of these elements:

Proposition 1.1.1. Let A and B be two subsets of $[n]$. Then:

- **(a)** We have $\nabla_{B,A} = 0$ if $|A| \neq |B|$.
- **(b)** We have $\widetilde{\nabla}_{B,A} = 0$ if $|A| > |B|$.
- **(c)** We have $\widetilde{\nabla}_{B,A} = \sum\limits_{\substack{V \subseteq B; \\ |V|=|A|}} \nabla_{V,A}$.
- **(d)** We have $\nabla_{B,A} = \nabla_{[n]\setminus B,\ [n]\setminus A}$.
- **(e)** If $|A| = |B|$, then $\nabla_{B,A} = \widetilde{\nabla}_{B,A}$.
- **(f)** The antipode S satisfies $S(\nabla_{B,A}) = \nabla_{A,B}$.
- **(g)** The antipode S satisfies $S\left(\widetilde{\nabla}_{B,A}\right) = \widetilde{\nabla}_{[n]\setminus A,\ [n]\setminus B}$.

Proposition 1.1.1 **(c)** shows that the elements $\nabla_{B,A}$ and $\widetilde{\nabla}_{B,A}$ have the same span as B and A range over the subsets of $[n]$, or even as B ranges over all subsets of $[n]$ while A is fixed.

You might wonder: What is this span? What is its dimension? This will be answered later (Corollary 2.6.1).

1.2. The product rule

What is more interesting is that the span of the $\nabla_{B,A}$ is a nonunital $\mathbf{k}$-subalgebra of $\mathcal{A}$. It has an explicit multiplication rule, which we shall state after a quick definition:

Definition 1.2.1. For any two subsets B and C of $[n]$, we define the positive integer

$$\omega_{B,C} := |B \cap C|! \cdot |B \setminus C|! \cdot |C \setminus B|! \cdot |[n] \setminus (B \cup C)|! \in \mathbb{Z}.$$

Theorem 1.2.2. Let A, B, C, D be four subsets of $[n]$ such that $|A| = |B|$ and $|C| = |D|$. Then,

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|}} (-1)^{|U|-|B\cap C|} \binom{|U|}{|B \cap C|} \nabla_{U,V}.$$

Proof idea. We shall first show that

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{\substack{w \in S_n; \\ |w(A) \cap D| = |B \cap C|}} w. \qquad (1)$$

[*Proof of (1):* Each permutation w appearing in the product $\nabla_{D,C} \nabla_{B,A}$ has the property that $|w(A) \cap D| = |B \cap C|$ (because it can be written as $w = uv$ with $u(C) = D$ and $v(A) = B$, and therefore we have $\underbrace{w(A)}_{=u(v(A))} \cap \underbrace{D}_{=u(C)} = u\left(\underbrace{v(A)}_{=B}\right) \cap u(C) = u(B) \cap u(C) = u(B \cap C)$, so that $|w(A) \cap D| = |B \cap C|$). It remains to show that each w with this property appears exactly $\omega_{B,C}$ times in this product. In other words, given a permutation $w \in S_n$ satisfy $|w(A) \cap D| = |B \cap C|$, we must show that there are exactly $\omega_{B,C}$ ways to decompose w as $w = uv$ with $u(C) = D$ and $v(A) = B$. But this is an exercise in counting: We want to count the permutations $v \in S_n$ satisfying $v(A) = B$ and $\left(wv^{-1}\right)(C) = D$. Such a permutation v must send A to B and send $w^{-1}(D)$ to C. In other words, it must send the four subsets $A \cap w^{-1}(D)$, $A \setminus w^{-1}(D)$, $w^{-1}(D) \setminus A$ and $[n] \setminus \left(A \cup w^{-1}(D)\right)$ to the respectively equinumerous subsets $B \cap C$, $B \setminus C$, $C \setminus B$ and $[n] \setminus (B \cup C)$, respectively. The number of ways to do this is

$$|B \cap C|! \cdot |B \setminus C|! \cdot |C \setminus B|! \cdot |[n] \setminus (B \cup C)|!,$$

which is exactly $\omega_{B,C}$. Thus, the proof of (1) is complete.]

In view of (1), it only remains to show that

$$\sum_{\substack{w \in S_n; \\ |w(A) \cap D| = |B \cap C|}} w = \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|}} (-1)^{|U|-|B\cap C|} \binom{|U|}{|B \cap C|} \nabla_{U,V}.$$

After expanding the right hand side as a sum of permutations, we can compare coefficients of a given coefficient $w \in S_n$ on both sides of this equality. That is, we must now prove that

$$\begin{cases} 1, & \text{if } |w(A) \cap D| = |B \cap C|; \\ 0, & \text{else} \end{cases} = \sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|; \\ w(V)=U}} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|}$$

for each permutation $w \in S_n$. This is fairly easy: We have

$$\sum_{\substack{U \subseteq D, \\ V \subseteq A; \\ |U|=|V|; \\ w(V)=U}} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|} = \sum_{U \subseteq w(A) \cap D} (-1)^{|U|-|B \cap C|} \binom{|U|}{|B \cap C|}$$

(since the set V is uniquely determined by U via $w(V) = U$, and is a subset of A if and only if we have $U \subseteq w(A) \cap D$), and it remains to recall the easy combinatorial identity

$$\sum_{U \subseteq Z} (-1)^{|U|-k} \binom{|U|}{k} = \begin{cases} 1, & \text{if } |Z| = k; \\ 0, & \text{else} \end{cases}$$

that holds for any finite set Z and any $k \in \mathbb{N}$. □

We can restate Theorem 1.2.2 as follows:

Theorem 1.2.3. Let A, B, C, D be four subsets of $[n]$ such that $|A| = |B|$ and $|C| = |D|$. Then,

$$\nabla_{D,C} \nabla_{B,A} = \omega_{B,C} \sum_{V \subseteq A} (-1)^{|V|-|B \cap C|} \binom{|V|}{|B \cap C|} \widetilde{\nabla}_{D,V}.$$

Proof. Theorem 1.2.2 yields

$$\begin{aligned}
\nabla_{D,C}\nabla_{B,A} &= \omega_{B,C} \sum_{\substack{U\subseteq D,\\ V\subseteq A;\\ |U|=|V|}} (-1)^{|U|-|B\cap C|} \binom{|U|}{|B\cap C|} \nabla_{U,V} \\
&= \omega_{B,C} \sum_{\substack{U\subseteq D,\\ V\subseteq A;\\ |U|=|V|}} (-1)^{|V|-|B\cap C|} \binom{|V|}{|B\cap C|} \nabla_{U,V} \qquad \left(\begin{array}{c}\text{due to the } |U|=|V| \\ \text{condition}\end{array}\right) \\
&= \omega_{B,C} \sum_{V\subseteq A} (-1)^{|V|-|B\cap C|} \binom{|V|}{|B\cap C|} \underbrace{\sum_{\substack{U\subseteq D;\\ |U|=|V|}} \nabla_{U,V}}_{\substack{=\widetilde{\nabla}_{D,V} \\ \text{(by Proposition 1.1.1 (c))}}} \\
&= \omega_{B,C} \sum_{V\subseteq A} (-1)^{|V|-|B\cap C|} \binom{|V|}{|B\cap C|} \widetilde{\nabla}_{D,V}.
\end{aligned}$$

□

1.3. The D-filtration

We shall next derive some nilpotency-type consequence from the multiplication rule.

For the rest of this section, we fix a subset D of $[n]$. We define

$$\mathcal{F}_k := \operatorname{span}\left\{ \widetilde{\nabla}_{D,C} \ \mid\ C \subseteq [n] \text{ with } |C| \leq k \right\}$$

for each $k \in \mathbb{Z}$. Of course, $\mathcal{F}_n \supseteq \mathcal{F}_{n-1} \supseteq \cdots \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_{-1} = 0$. It is easy to see that $\mathcal{F}_0$ is spanned by $\widetilde{\nabla}_{D,\varnothing} = \nabla_{\varnothing,\varnothing} = \sum\limits_{w\in S_n} w$.

Definition 1.3.1. For any subset $C \subseteq [n]$ and any $k \in \mathbb{N}$, we define the integer

$$\delta_{D,C,k} := \sum_{\substack{B\subseteq D;\\ |B|=k}} \omega_{B,C} (-1)^{k-|B\cap C|} \binom{k}{|B\cap C|} \in \mathbb{Z}.$$

Now, we note the following:

Proposition 1.3.2. Let $C \subseteq [n]$ satisfy $|C| = |D|$. Let $k \in \mathbb{N}$. Then,

$$\left(\nabla_{D,C} - \delta_{D,C,k}\right) \mathcal{F}_k \subseteq \mathcal{F}_{k-1}.$$

Proof. By the definition of $\mathcal{F}_k$, it suffices to show that

$$\left(\nabla_{D,C} - \delta_{D,C,k}\right) \widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1} \tag{2}$$

for each $A \subseteq [n]$ with $|A| \leq k$.

To prove this, we fix $A \subseteq [n]$ with $|A| \leq k$. Then, Proposition 1.1.1 **(c)** yields

$$\widetilde{\nabla}_{D,A} = \sum_{\substack{V \subseteq D; \\ |V| = |A|}} \nabla_{V,A} = \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \nabla_{B,A}. \tag{3}$$

Multiplying this equality by $\nabla_{D,C}$ from the left, we obtain

$$\nabla_{D,C} \widetilde{\nabla}_{D,A} = \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \nabla_{D,C} \nabla_{B,A}. \tag{4}$$

However, for each subset $B \subseteq D$ satisfying $|B| = |A|$, we can use Theorem 1.2.3 to obtain

$$\begin{aligned} \nabla_{D,C} \nabla_{B,A} &= \omega_{B,C} \sum_{V \subseteq A} (-1)^{|V| - |B \cap C|} \binom{|V|}{|B \cap C|} \underbrace{\widetilde{\nabla}_{D,V}}_{\substack{\in \mathcal{F}_{k-1} \text{ unless } V = A \\ \text{(since } |V| < |A| \leq k \text{ unless } V = A)}} \\ &\equiv \omega_{B,C} (-1)^{|A| - |B \cap C|} \binom{|A|}{|B \cap C|} \widetilde{\nabla}_{D,A} \mod \mathcal{F}_{k-1}. \end{aligned} \tag{5}$$

Recall that $|A| \leq k$. Hence, we are in one of the following two cases:

Case 1: We have $|A| = k$.

Case 2: We have $|A| < k$.

Let us first consider Case 1. In this case, we have $|A| = k$. Hence, (4) becomes

$$\begin{aligned} \nabla_{D,C} \widetilde{\nabla}_{D,A} &= \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \nabla_{D,C} \nabla_{B,A} \\ &\equiv \sum_{\substack{B \subseteq D; \\ |B| = |A|}} \omega_{B,C} (-1)^{|A| - |B \cap C|} \binom{|A|}{|B \cap C|} \widetilde{\nabla}_{D,A} \qquad \text{(by (5))} \\ &= \underbrace{\sum_{\substack{B \subseteq D; \\ |B| = k}} \omega_{B,C} (-1)^{k - |B \cap C|} \binom{k}{|B \cap C|}}_{= \delta_{D,C,k}} \widetilde{\nabla}_{D,A} \qquad \text{(since } |A| = k) \\ &= \delta_{D,C,k} \widetilde{\nabla}_{D,A} \mod \mathcal{F}_{k-1}. \end{aligned}$$

In other words, $\nabla_{D,C} \widetilde{\nabla}_{D,A} - \delta_{D,C,k} \widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$. In other words, $\left(\nabla_{D,C} - \delta_{D,C,k}\right) \widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$. Hence, (2) is proved in Case 1.

Let us now consider Case 2. In this case, we have $|A| < k$. Hence, $|A| \leq k-1$, so that $\widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$. Now, (4) becomes

$$\begin{aligned}\nabla_{D,C}\widetilde{\nabla}_{D,A} &= \sum_{\substack{B\subseteq D;\\ |B|=|A|}} \nabla_{D,C}\nabla_{B,A} \\ &\equiv \sum_{\substack{B\subseteq D;\\ |B|=|A|}} \omega_{B,C}\,(-1)^{|A|-|B\cap C|}\binom{|A|}{|B\cap C|}\widetilde{\nabla}_{D,A} \qquad \text{(by (5))} \\ &\in \mathcal{F}_{k-1} \qquad \left(\text{since } \widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}\right).\end{aligned}$$

Combining this with $\delta_{D,C,k}\widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$ (since $\widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$), we obtain $\nabla_{D,C}\widetilde{\nabla}_{D,A} - \delta_{D,C,k}\widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$. In other words, $(\nabla_{D,C} - \delta_{D,C,k})\,\widetilde{\nabla}_{D,A} \in \mathcal{F}_{k-1}$. Hence, (2) is proved in Case 2.

We have now proved (2) in both Cases 1 and 2. Thus, (2) always holds, and Proposition 1.3.2 is proved. $\square$

Definition 1.3.3. Let $\alpha = (\alpha_C)_{C\subseteq[n];\ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Then, we set

$$\nabla_{D,\alpha} := \sum_{\substack{C\subseteq[n];\\ |C|=|D|}} \alpha_C \nabla_{D,C} \in \mathcal{A}.$$

Furthermore, for each $k \in \mathbb{N}$, we set

$$\delta_{D,\alpha,k} := \sum_{\substack{C\subseteq[n];\\ |C|=|D|}} \alpha_C \delta_{D,C,k} \in \mathbf{k}.$$

Proposition 1.3.4. Let $\alpha = (\alpha_C)_{C\subseteq[n];\ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Let $k \in \mathbb{N}$. Then,

$$(\nabla_{D,\alpha} - \delta_{D,\alpha,k})\,\mathcal{F}_k \subseteq \mathcal{F}_{k-1}.$$

Proof. Proposition 1.3.2 yields $(\nabla_{D,C} - \delta_{D,C,k})\,\mathcal{F}_k \subseteq \mathcal{F}_{k-1}$ for each $C \subseteq [n]$ satisfying $|C| = |D|$. Multiply this relation by α_C and sum up over all $C \subseteq [n]$ satisfying $|C| = |D|$. The result is Proposition 1.3.4. $\square$

Proposition 1.3.5. Let $\alpha = (\alpha_C)_{C\subseteq[n];\ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Then, for each integer $m \geq -1$, we have

$$\left(\prod_{k=0}^{m}(\nabla_{D,\alpha} - \delta_{D,\alpha,k})\right)\mathcal{F}_m = 0.$$

Proof. Induction on m. The *base case* is obvious, since $\mathcal{F}_{-1} = 0$. The *induction step* uses Proposition 1.3.4. $\square$

1.4. The triangularity theorem

We can now state our main theorem (still using Definition 1.3.3):

Theorem 1.4.1. Let $\alpha = (\alpha_C)_{C \subseteq [n];\ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Then,

$$\left(\prod_{k=0}^{|D|} \left(\nabla_{D,\alpha} - \delta_{D,\alpha,k} \right) \right) \nabla_{D,\alpha} = 0.$$

Proof. For each subset C of $[n]$ satisfying $|C| = |D|$, we have $\nabla_{D,C} = \widetilde{\nabla}_{D,C}$ (by Proposition 1.1.1 **(e)**) and thus $\nabla_{D,C} = \widetilde{\nabla}_{D,C} \in \mathcal{F}_{|C|} = \mathcal{F}_{|D|}$ (since $|C| = |D|$). Thus, $\nabla_{D,\alpha} \in \mathcal{F}_{|D|}$ as well (since $\nabla_{D,\alpha}$ is a $\mathbf{k}$-linear combination of such $\nabla_{D,C}$'s). However, Proposition 1.3.5 (applied to $m = |D|$) yields

$$\left(\prod_{k=0}^{|D|} \left(\nabla_{D,\alpha} - \delta_{D,\alpha,k} \right) \right) \mathcal{F}_{|D|} = 0.$$

Combine these two facts, and conclude. $\square$

Using the antipode S of $\mathcal{A}$, we can obtain a reflected version of Theorem 1.4.1:

Theorem 1.4.2. Let $\alpha = (\alpha_C)_{C \subseteq [n];\ |C|=|D|}$ be a family of scalars in $\mathbf{k}$ indexed by the $|D|$-element subsets of $[n]$. Set

$$\nabla_{\alpha,D} := \sum_{\substack{C \subseteq [n]; \\ |C|=|D|}} \alpha_C \nabla_{C,D} \in \mathcal{A}.$$

Then,

$$\left(\prod_{k=0}^{|D|} \left(\nabla_{\alpha,D} - \delta_{D,\alpha,k} \right) \right) \nabla_{\alpha,D} = 0.$$

Proof. The antipode S is a $\mathbf{k}$-algebra anti-homomorphism, and sends $\nabla_{D,\alpha}$ to $\nabla_{\alpha,D}$ (since it sends $\nabla_{D,C}$ to $\nabla_{C,D}$ for each C). Thus, Theorem 1.4.2 follows easily by applying the antipode to Theorem 1.4.1. (Note that we don't have to reverse the order of factors in the product, since all these factors commute with each other.) $\square$

Corollary 1.4.3. Let B and D be two subsets of $[n]$. For each $k \in \mathbb{N}$, we set

$$\widetilde{\delta}_{D,B,k} := \sum_{\substack{C \subseteq B; \\ |C|=|D|}} \delta_{D,C,k} \in \mathbb{Z}.$$

Then,

$$\left(\prod_{k=0}^{|D|} \left(\widetilde{\nabla}_{B,D} - \delta_{D,B,k} \right) \right) \widetilde{\nabla}_{B,D} = 0.$$

Proof. Define a family $\alpha = (\alpha_C)_{C \subseteq [n];\ |C|=|D|}$ of scalars in $\mathbf{k}$ by setting

$$\alpha_C = \begin{cases} 1, & \text{if } C \subseteq B; \\ 0, & \text{if } C \not\subseteq B \end{cases} \qquad \text{for each } C \subseteq [n].$$

Then, Proposition 1.1.1 **(c)** yields

$$\widetilde{\nabla}_{B,D} = \sum_{\substack{V \subseteq B; \\ |V|=|D|}} \nabla_{V,D} = \sum_{\substack{V \subseteq [n]; \\ |V|=|D|}} \alpha_V \nabla_{V,D} = \nabla_{\alpha,D},$$

where $\nabla_{\alpha,D}$ is defined as in Theorem 1.4.2. Hence, Corollary 1.4.3 yields Theorem 1.4.2, once we realize that the $\delta_{D,\alpha,k}$ from Theorem 1.4.2 is precisely the $\widetilde{\delta}_{D,B,k}$. $\square$

Corollary 1.4.3 shows that the element $\widetilde{\nabla}_{B,D}$ has a minimal polynomial that factors entirely into linear factors. Moreover, there are at most $|D| + 2$ factors, and one of them is X or else there are at most $|D| + 1$ of them.

Question 1.4.4. Can we simplify the formula for $\widetilde{\delta}_{D,B,k}$?

1.5. A table of minimal polynomials

For any subsets A and B of $[n]$, we let $\kappa_{A,B}$ be the sum (in $\mathbb{Z}[S_n]$) of all permutations $w \in S_n$ that satisfy $w(A) \cap B = \varnothing$ (that is, $w(a) \notin B$ for all $a \in A$). Then, $\kappa_{A,B}$ is simply $\widetilde{\nabla}_{[n]\setminus B,\ A}$. Thus, Corollary 1.4.3 shows that the element $\kappa_{A,B}$ has a minimal polynomial that factors into at most $|A| + 2$ factors. Note that these factors will sometimes have multiplicities (e.g., the case of $n = 6$ and $a = 3$ and $b = 2$ and $c = 1$).

Let us collect a table of these minimal polynomials. We observe that the minimal polynomial of $\kappa_{B,A}$ depends only on the three numbers $a := |A|$, $b := |B|$ and $c := |A \cap B|$ (since any two pairs (A, B) that agree in these three numbers can be obtained from each other by the action of some permutation $\sigma \in S_n$, and therefore the corresponding elements $\kappa_{B,A}$ are conjugate to each other in $\mathcal{A}$). Hence, we can rename $\kappa_{B,A}$ as $\kappa_{a,b,c}$.

We also note that $\kappa_{a,b,c} = 0$ if $c > a$ or $b > a$ or $a + b > n$. Hence, we only need to consider the cases $a, b \in [0, n]$ and $a + b \leq n$ and $c \in [0, \min\{a, b\}]$.

Moreover, $\kappa_{B,A}$ is the antipode of $\kappa_{A,B}$ (by Proposition 1.1.1 **(g)**), and the antipode preserves minimal polynomials. Thus, we only need to consider the case $a \leq b$.

This being said, here is a table of minpols (= minimal polynomials) of $\kappa_{a,b,c}$'s produced by SageMath:

Let $n = 1$.
For $b = 0$, the minpol is $x - 1$.

Let $n = 2$.
For $b = 0$, the minpol is $(x - 2)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $x - 1$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x - 1)(x + 1)$.

Let $n = 3$.
For $b = 0$, the minpol is $(x - 6)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $(x - 4)(x - 1)x$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x - 4)x(x + 2)$.
For $a = 2$ and $b = 1$ and $c = 0$, the minpol is $(x - 2)x$.
For $a = 2$ and $b = 1$ and $c = 1$, the minpol is $(x - 2)x(x + 1)$.

Let $n = 4$.
For $b = 0$, the minpol is $(x - 24)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $(x - 18)(x - 2)x$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x - 18)x(x + 6)$.
For $a = 2$ and $b = 1$ and $c = 0$, the minpol is $(x - 12)(x - 4)x$.
For $a = 2$ and $b = 1$ and $c = 1$, the minpol is $(x - 12)x(x + 4)$.
For $a = 3$ and $b = 1$ and $c = 0$, the minpol is $(x - 6)x$.
For $a = 3$ and $b = 1$ and $c = 1$, the minpol is $(x - 6)x(x + 2)$.
For $a = 2$ and $b = 2$ and $c = 0$, the minpol is $(x - 4)x$.
For $a = 2$ and $b = 2$ and $c = 1$, the minpol is $(x - 4)(x + 2)x^2$.
For $a = 2$ and $b = 2$ and $c = 2$, the minpol is $(x - 4)x(x + 4)$.

Let $n = 5$.
For $b = 0$, the minpol is $(x - 120)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $(x - 96)(x - 6)x$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x - 96)x(x + 24)$.
For $a = 2$ and $b = 1$ and $c = 0$, the minpol is $(x - 72)(x - 12)x$.
For $a = 2$ and $b = 1$ and $c = 1$, the minpol is $(x - 72)x(x + 18)$.
For $a = 3$ and $b = 1$ and $c = 0$, the minpol is $(x - 48)(x - 18)x$.

For $a = 3$ and $b = 1$ and $c = 1$, the minpol is $(x-48)x(x+12)$.
For $a = 4$ and $b = 1$ and $c = 0$, the minpol is $(x-24)x$.
For $a = 4$ and $b = 1$ and $c = 1$, the minpol is $(x-24)x(x+6)$.
For $a = 2$ and $b = 2$ and $c = 0$, the minpol is $(x-36)(x-16)(x-4)x$.
For $a = 2$ and $b = 2$ and $c = 1$, the minpol is $(x-36)x(x+4)$.
For $a = 2$ and $b = 2$ and $c = 2$, the minpol is $(x-36)(x-12)x(x+24)$.
For $a = 3$ and $b = 2$ and $c = 0$, the minpol is $(x-12)x$.
For $a = 3$ and $b = 2$ and $c = 1$, the minpol is $(x-12)(x-2)x(x+4)$.
For $a = 3$ and $b = 2$ and $c = 2$, the minpol is $(x-12)(x-4)x(x+8)$.

Let $n = 6$.
For $b = 0$, the minpol is $(x-720)x$.
For $a = 1$ and $b = 1$ and $c = 0$, the minpol is $(x-600)(x-24)x$.
For $a = 1$ and $b = 1$ and $c = 1$, the minpol is $(x-600)x(x+120)$.
For $a = 2$ and $b = 1$ and $c = 0$, the minpol is $(x-480)(x-48)x$.
For $a = 2$ and $b = 1$ and $c = 1$, the minpol is $(x-480)x(x+96)$.
For $a = 3$ and $b = 1$ and $c = 0$, the minpol is $(x-360)(x-72)x$.
For $a = 3$ and $b = 1$ and $c = 1$, the minpol is $(x-360)x(x+72)$.
For $a = 4$ and $b = 1$ and $c = 0$, the minpol is $(x-240)(x-96)x$.
For $a = 4$ and $b = 1$ and $c = 1$, the minpol is $(x-240)x(x+48)$.
For $a = 5$ and $b = 1$ and $c = 0$, the minpol is $(x-120)x$.
For $a = 5$ and $b = 1$ and $c = 1$, the minpol is $(x-120)x(x+24)$.
For $a = 2$ and $b = 2$ and $c = 0$, the minpol is $(x-288)(x-72)(x-8)x$.
For $a = 2$ and $b = 2$ and $c = 1$, the minpol is $(x-288)x(x+12)(x+36)$.
For $a = 2$ and $b = 2$ and $c = 2$, the minpol is $(x-288)(x-48)x(x+144)$.
For $a = 3$ and $b = 2$ and $c = 0$, the minpol is $(x-144)(x-72)(x-24)x$.
For $a = 3$ and $b = 2$ and $c = 1$, the minpol is $(x-144)(x+16)x^2$.
For $a = 3$ and $b = 2$ and $c = 2$, the minpol is $(x-144)(x-24)x(x+72)$.
For $a = 4$ and $b = 2$ and $c = 0$, the minpol is $(x-48)x$.
For $a = 4$ and $b = 2$ and $c = 1$, the minpol is $(x-48)(x-12)x(x+12)$.
For $a = 4$ and $b = 2$ and $c = 2$, the minpol is $(x-48)(x-8)x(x+24)$.
For $a = 3$ and $b = 3$ and $c = 0$, the minpol is $(x-36)x$.
For $a = 3$ and $b = 3$ and $c = 1$, the minpol is $(x-36)(x-12)x(x+4)(x+12)$.
For $a = 3$ and $b = 3$ and $c = 2$, the minpol is $(x-36)(x-12)x(x+4)(x+12)$.
For $a = 3$ and $b = 3$ and $c = 3$, the minpol is $(x-36)x(x+36)$.

1.6. Aside: The formal Nabla-algebra

We take a tangent and address a question that is suggested by Theorem 1.2.2 but takes us out of the symmetric group algebra $\mathcal{A}$. Namely, let us see what happens if we take the multiplication rule in Theorem 1.2.2 literally while forgetting what the $\nabla_{B,A}$ are.

Theorem 1.6.1. For any two subsets A and B of $[n]$ satisfying $|A| = |B|$, introduce a formal symbol $\Delta_{B,A}$. Thus, we have introduced altogether $\sum\limits_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ symbols $\Delta_{B,A}$. Let $\mathcal{D}$ be the free $\mathbf{k}$-module with basis $(\Delta_{B,A})_{A,B\subseteq[n] \text{ with } |A|=|B|}$. Define a multiplication on $\mathcal{D}$ by

$$\Delta_{D,C}\Delta_{B,A} := \omega_{B,C} \sum_{\substack{U\subseteq D,\\ V\subseteq A;\\ |U|=|V|}} (-1)^{|U|-|B\cap C|} \binom{|U|}{|B\cap C|} \Delta_{U,V}.$$

(Recall Definition 1.2.1, which defines the $\omega_{B,C}$ here.) Then, $\mathcal{D}$ becomes a nonunital $\mathbf{k}$-algebra.

Proof omitted due to excessive ugliness.

Question 1.6.2. The above proof idea is clearly in bad taste. There should be a more conceptual proof that identifies $\mathcal{D}$ as some existing (nonunital) $\mathbf{k}$-algebra (what nonunital $\mathbf{k}$-algebra has dimension $\binom{2n}{n}$ over $\mathbf{k}$?) or at least with a subquotient of a such.

Example 1.6.3. Let $n = 1$. Then, the $\mathbf{k}$-module $\mathcal{D}$ in Theorem 1.6.1 has basis (u, v) with $u = \Delta_{\varnothing,\varnothing}$ and $v = \Delta_{\{1\},\{1\}}$. The multiplication on $\mathcal{D}$ defined ibidem is given by

$$uu = uv = vu = u, \qquad vv = v.$$

Thus, the nonunital $\mathbf{k}$-algebra $\mathcal{D}$ is isomorphic to the $\mathbf{k}$-algebra $\mathbf{k}[x] / (x^2 - x)$, and therefore has a unity (namely, v).

Example 1.6.4. Let $n = 2$. Then, the $\mathbf{k}$-module $\mathcal{D}$ in Theorem 1.6.1 has basis $(u, v_{11}, v_{12}, v_{21}, v_{22}, w)$ with $u = \Delta_{\varnothing,\varnothing}$ and $v_{ij} = \Delta_{\{i\},\{j\}}$ and $w = \Delta_{[2],[2]}$. The multiplication on $\mathcal{D}$ defined ibidem is given by

$$\begin{aligned}
uu = uw = wu = 2u, &\qquad uv_{ij} = v_{ij}u = u,\\
v_{dc}v_{ba} = u - v_{da} &\qquad \text{if } b \neq c;\\
v_{dc}v_{ba} = v_{da} &\qquad \text{if } b = c,\\
v_{ij}w = v_{i1} + v_{i2}, &\qquad wv_{ij} = v_{1j} + v_{2j},\\
ww = 2w. &
\end{aligned}$$

This nonunital $\mathbf{k}$-algebra $\mathcal{D}$ has a unity if and only if 2 is invertible in $\mathbf{k}$. This unity is $\frac{1}{4}(v_{11} + v_{22} - v_{12} - v_{21} + 2w)$.

Question 1.6.5. Does the $\mathbf{k}$-algebra $\mathcal{D}$ in Theorem 1.6.1 have a unity if $n!$ is invertible in $\mathbf{k}$? (I suspect that the answer is "yes".)

1.7. Other rook sums?

Encouraged by the above results, we can define an element

$$\nabla_T := \sum_{\substack{w \in S_n; \\ (i,w(i)) \in T \text{ for each } i \in [n]}} w$$

for any subset T of $[n] \times [n]$. This is the sum of the n-rook placements on the (arbitrary) board T. It is tempting to conjecture that the minimal polynomial of ∇_T will always factor nicely, but this is not true: If $n = 5$ and $T = \{(i,j) \mid j \neq i+1\}$, then the minimal polynomial of ∇_T has irreducible factors of degrees 1, 4, 5 and 6 (over $\mathbb{Q}$).

Of course, some boards do behave nicely: If $T = \{(i,j) \in [n] \times [n] \mid i \neq j\}$, then ∇_T is the sum of all derangements in S_n, thus a central element of $\mathbb{Z}[S_n]$, and hence the minimal polynomial of ∇_T factors (since the center of $\mathbb{Q}[S_n]$ is split semisimple).

2. Row-to-row sums in the symmetric group algebra

2.1. Definitions

As we recall, n is a nonnegative integer and $\mathbf{k}$ a commutative ring. We work in the group algebra $\mathcal{A} = \mathbf{k}[S_n]$ of the symmetric group S_n.

A *set decomposition* of a set U shall mean a tuple $(U_1, U_2, \ldots, U_k)$ of disjoint subsets of U such that $U_1 \cup U_2 \cup \cdots \cup U_k = U$. The subsets $U_1, U_2, \ldots, U_k$ are called the *blocks* of this set decomposition $(U_1, U_2, \ldots, U_k)$. The number k of these blocks is called the *length* of this set decomposition. The length of a set decomposition $\mathbf{U}$ is called $\ell(\mathbf{U})$.

A *set composition* of a set U shall mean a set decomposition of U whose blocks are all nonempty. Clearly, any set decomposition of U can be transformed into a set composition of U by removing all empty blocks.

Let $\mathrm{SC}(n)$ denote the set of all set compositions of $[n]$.

If $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ are two set decompositions of $[n]$ having the same length, then we define the element

$$\nabla_{\mathbf{B},\mathbf{A}} := \sum_{\substack{w \in S_n; \\ w(A_i) = B_i \text{ for all } i}} w \qquad \text{of } \mathcal{A}.$$

This will be called a *row-to-row sum*. We observe some easy properties:

Proposition 2.1.1. Let $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ and $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ be two set decompositions of $[n]$ having the same length. Then:

(a) We have $\nabla_{\mathbf{B},\mathbf{A}} = 0$ unless each $i \in [k]$ satisfies $|A_i| = |B_i|$.

(b) The element $\nabla_{\mathbf{B},\mathbf{A}}$ does not change if we permute the blocks of $\mathbf{A}$ and the blocks of $\mathbf{B}$ using the same permutation. In other words, for any permutation $\sigma \in S_k$, we have $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{\mathbf{B}\sigma,\mathbf{A}\sigma}$, where $\mathbf{A}\sigma := \left(A_{\sigma(1)}, A_{\sigma(2)}, \ldots, A_{\sigma(k)}\right)$ and $\mathbf{B}\sigma := \left(B_{\sigma(1)}, B_{\sigma(2)}, \ldots, B_{\sigma(k)}\right)$.

(c) The element $\nabla_{\mathbf{B},\mathbf{A}}$ does not change if we remove empty blocks from $\mathbf{A}$ and from $\mathbf{B}$, provided that these blocks are in the same positions in both $\mathbf{A}$ and $\mathbf{B}$.

(d) The antipode S of $\mathcal{A}$ satisfies $S\left(\nabla_{\mathbf{B},\mathbf{A}}\right) = \nabla_{\mathbf{A},\mathbf{B}}$.

Moreover, these row-to-row sums $\nabla_{\mathbf{B},\mathbf{A}}$ generalize the rectangular rook sums $\nabla_{B,A}$ from Section 1:

Proposition 2.1.2. Let A and B be two subsets of $[n]$. Define the two set decompositions $\mathbf{A} := (A,\ [n] \setminus A)$ and $\mathbf{B} := (B,\ [n] \setminus B)$ of $[n]$. Then, $\nabla_{\mathbf{B},\mathbf{A}} = \nabla_{B,A}$.

Remark 2.1.3. It might be more convenient to rewrite the row-to-row elements using colorings instead of set (de)compositions. Namely, a *coloring* of $[n]$ means a map $f : [n] \to C$ to some set C. If $C = [k]$ for some $k \in \mathbb{N}$, then such a coloring f can be regarded as a set decomposition of $[n]$ of length k, where the i-th block is $f^{-1}(i)$ for each $i \in [k]$. The image $f(j)$ of an element $j \in [n]$ under a coloring $f : [n] \to C$ is called the *color* of j (under f). Now, the row-to-row sum $\nabla_{g,f}$ corresponding to two colorings f and g of $[n]$ is the sum of all permutations $w \in S_n$ that satisfy $g \circ w = f$. (This is a "preservation of colors" condition.)

Remark 2.1.4. Let $u \in S_n$ be any permutation. Let $\mathbf{A}$ be the set composition $(\{1\}, \{2\}, \ldots, \{n\})$ of $[n]$, and let $\mathbf{B}$ be the set composition $(\{u(1)\}, \{u(2)\}, \ldots, \{u(n)\})$ of $[n]$. Then, $\nabla_{\mathbf{B},\mathbf{A}} = u$. Thus, the row-to-row sums $\nabla_{\mathbf{B},\mathbf{A}}$ in general are not as special as their particular cases the rectangular rook sums $\nabla_{B,A}$. In particular, the minimal polynomials of general row-to-row sums $\nabla_{\mathbf{B},\mathbf{A}}$ cannot be factored into linear factors over $\mathbb{Z}$.

This all was easy. Let us now come to deeper facts.

2.2. The two ideals

For each subset U of $[n]$, we define the element

$$\nabla_U^- := \sum_{\substack{w \in S_n; \\ w(i)=i \text{ for all } i \in [n] \setminus U}} (-1)^w w \in \mathcal{A}.$$

This is called the *antisymmetrizer* of U. Note that it equals 1 if $|U| \leq 1$. Another way to rephrase the definition of ∇_U^- is

$$\nabla_U^- := \sum_{w \in S_U} (-1)^w w \in \mathcal{A},$$

where S_U denotes the symmetric group on the set U (embedded into S_n in the standard way).

Definition 2.2.1. Let $k \in \mathbb{N}$. We define two $\mathbf{k}$-submodules $\mathcal{I}_k$ and $\mathcal{J}_k$ of $\mathcal{A}$ by

$$\mathcal{I}_k := \operatorname{span}\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \operatorname{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k\}$$

and

$$\mathcal{J}_k := \mathcal{A} \cdot \operatorname{span}\left\{\nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1\right\} \cdot \mathcal{A}.$$

Proposition 2.2.2. Let $k \in \mathbb{N}$. Then:

(a) Both $\mathcal{I}_k$ and $\mathcal{J}_k$ are ideals of $\mathcal{A}$.

(b) We have

$$\begin{aligned}\mathcal{J}_k &= \mathcal{A} \cdot \operatorname{span}\left\{\nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1\right\} \\ &= \operatorname{span}\left\{\nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1\right\} \cdot \mathcal{A}.\end{aligned}$$

(c) The antipode S satisfies $S(\mathcal{I}_k) = \mathcal{I}_k$ and $S(\mathcal{J}_k) = \mathcal{J}_k$.

(d) We have

$$\mathcal{I}_k = \operatorname{span}\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A} \text{ and } \mathbf{B} \text{ are set decompositions of } [n] \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k\}.$$

(e) We have

$$\mathcal{J}_k = \mathcal{A} \cdot \operatorname{span}\left\{\nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } > k\right\} \cdot \mathcal{A}.$$

Proof. **(a)** Clearly, $\mathcal{J}_k$ is an ideal of $\mathcal{A}$. It remains to show that so is $\mathcal{I}_k$. To this purpose, it suffices to show that

$$u\nabla_{\mathbf{B},\mathbf{A}}v = \nabla_{u\mathbf{B},v^{-1}\mathbf{A}} \tag{6}$$

for any permutations $u, v \in S_n$ and any set compositions $\mathbf{A}, \mathbf{B} \in \mathrm{SC}(n)$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B})$, where the symmetric group S_n acts on the set $\mathrm{SC}(n)$ by the formula

$$w(B_1, B_2, \ldots, B_i) = (w(B_1), \ w(B_2), \ \ldots, \ w(B_i)).$$

But this is easy.

(b) It is easy to see that any permutation $w \in S_n$ and any subset U of $[n]$ satisfy

$$w\nabla_U^- = \nabla_{w(U)}^- w. \tag{7}$$

This equality yields

$$\begin{aligned}
&\mathcal{A} \cdot \operatorname{span}\left\{\nabla_U^- \ \mid \ U \text{ is a subset of } [n] \text{ having size } k+1\right\} \\
&= \operatorname{span}\left\{\nabla_U^- \ \mid \ U \text{ is a subset of } [n] \text{ having size } k+1\right\} \cdot \mathcal{A}.
\end{aligned}$$

From this, part **(b)** easily follows.

(c) The equality $S(\mathcal{I}_k) = \mathcal{I}_k$ follows from Proposition 2.1.1. The equality $S(\mathcal{J}_k) = \mathcal{J}_k$ follows from the equality $S(\nabla_U^-) = \nabla_U^-$, which holds for each $U \subseteq [n]$.

(d) Proposition 2.1.1 **(c)** yields that if $\mathbf{A}$ and $\mathbf{B}$ are set decompositions of $[n]$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$, then the row-to-row sum $\nabla_{\mathbf{B},\mathbf{A}}$ can be rewritten as $\nabla_{\mathbf{D},\mathbf{C}}$ for two set compositions $\mathbf{C}, \mathbf{D}$ of $[n]$ satisfying $\ell(\mathbf{C}) = \ell(\mathbf{D}) \leq \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$ (namely, $\mathbf{C}$ and $\mathbf{D}$ are obtained from $\mathbf{A}$ and $\mathbf{B}$ by removing all empty blocks). Thus, $\mathcal{I}_k$ does not change if we replace the set compositions in the definition of $\mathcal{I}_k$ by set decompositions. Hence, part **(d)** is proved.

(e) Clearly,

$$\mathcal{J}_k \subseteq \mathcal{A} \cdot \operatorname{span}\left\{\nabla_U^- \ \mid \ U \text{ is a subset of } [n] \text{ having size } > k\right\} \cdot \mathcal{A}$$

(since any subset of size $k+1$ has size $> k$). It remains to prove the reverse inclusion. Since $\mathcal{J}_k$ is an ideal of $\mathcal{A}$, it suffices to show that $\nabla_U^- \in \mathcal{J}_k$ whenever U is a subset of $[n]$ having size $> k$. So let U be a subset of $[n]$ having size $> k$. Then, U has a subset V of size $k+1$. Consider this V. Now, ∇_U^- can be written as $\nabla_U^- = \nabla_V^- f$ for some $f \in \mathcal{A}$, since the symmetric group S_V is a subgroup of S_U. Consider this f. We have $\nabla_V^- \in \mathcal{J}_k$ (by the definition of $\mathcal{J}_k$, since V is a subset of $[n]$ having size $k+1$) and thus $\nabla_V^- f \in \mathcal{J}_k$ (since $\mathcal{J}_k$ is an ideal of $\mathcal{A}$). In other words, $\nabla_U^- \in \mathcal{J}_k$ (since $\nabla_U^- = \nabla_V^- f$). This proves part **(e)**. $\square$

2.3. Annihilators and the bilinear form

If $\mathcal{B}$ is any subset of $\mathcal{A}$, then we define the two subsets

$$\operatorname{LAnn}\mathcal{B} := \{a \in \mathcal{A} \mid ab = 0 \text{ for all } b \in \mathcal{B}\} \qquad \text{and}$$
$$\operatorname{RAnn}\mathcal{B} := \{a \in \mathcal{A} \mid ba = 0 \text{ for all } b \in \mathcal{B}\}$$

of $\mathcal{A}$. We call them the *left annihilator* and the *right annihilator* of $\mathcal{B}$, respectively.

Moreover, we define the $\mathbf{k}$-bilinear form

$$\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \to \mathbf{k},$$

which sends the pair (u, v) to $\begin{cases} 1, & \text{if } u = v; \\ 0, & \text{if } u \neq v \end{cases}$ for any two permutations $u, v \in S_n$. This is the standard nondegenerate symmetric bilinear form on $\mathcal{A} = \mathbf{k}[S_n]$ known from representation theory.

If $\mathcal{B}$ is any subset of $\mathcal{A}$, then we define the subset

$$\mathcal{B}^{\perp} := \{a \in \mathcal{A} \mid \langle a, b \rangle = 0 \text{ for all } b \in \mathcal{B}\}$$

of $\mathcal{A}$. This is called the *orthogonal complement* of $\mathcal{B}$ in $\mathcal{A}$. Note that it does not change if we replace $\langle a, b \rangle$ by $\langle b, a \rangle$ in its definition, since the form $\langle \cdot, \cdot \rangle$ is symmetric.

Definition 2.3.1. Let $k \in \mathbb{N}$. Let $w \in S_n$ be a permutation. We say that w *avoids* $12 \cdots (k+1)$ if there exists no $(k+1)$-element subset U of $[n]$ such that the restriction $w\mid_U$ is increasing (i.e., if there exist no $k+1$ elements $i_1 < i_2 < \cdots < i_{k+1}$ such that $w(i_1) < w(i_2) < \cdots < w(i_{k+1})$). We let $\operatorname{Av}_n(k+1)$ denote the set of all permutations $w \in S_n$ that avoid $12 \cdots (k+1)$.

Note that this notion of "avoiding $12 \cdots (k+1)$" is taken from the theory of pattern avoidance.

Question 2.3.2. Explore the relation between the following and [BaiRai01, §8].

2.4. The main theorem

Theorem 2.4.1. Let $k \in \mathbb{N}$. Then:

- **(a)** We have $\mathcal{I}_k = \mathcal{J}_k^{\perp} = \operatorname{LAnn}\mathcal{J}_k = \operatorname{RAnn}\mathcal{J}_k$.
- **(b)** We have $\mathcal{J}_k = \mathcal{I}_k^{\perp} = \operatorname{LAnn}\mathcal{I}_k = \operatorname{RAnn}\mathcal{I}_k$.
- **(c)** The $\mathbf{k}$-module $\mathcal{I}_k$ is free of rank $|\operatorname{Av}_n(k+1)|$.
- **(d)** The $\mathbf{k}$-module $\mathcal{J}_k$ is free of rank $|S_n \setminus \operatorname{Av}_n(k+1)|$.

(e) The $\mathbf{k}$-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w \in S_n \setminus \mathrm{Av}_n(k+1)}$. (Here, $\overline{w}$ denotes the projection of $w \in \mathcal{A}$ onto the quotient $\mathcal{A}/\mathcal{I}_k$.)

(f) The $\mathbf{k}$-module $\mathcal{A}/\mathcal{J}_k$ is free with basis $(\overline{w})_{w \in \mathrm{Av}_n(k+1)}$.

(g) Assume that $n!$ is invertible in $\mathbf{k}$. Then, $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum) as $\mathbf{k}$-module. Moreover, $\mathcal{I}_k$ and $\mathcal{J}_k$ are nonunital subalgebras of $\mathcal{A}$ and satisfy $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as $\mathbf{k}$-algebras.

We note that the ideals $\mathcal{I}_k$ and $\mathcal{J}_k$ of $\mathcal{A}$ have a simple representation-theoretical interpretation when $\mathbf{k}$ is a field of characteristic 0; this is discussed in Proposition 2.7.1 further below.

2.5. Lemmas for the proof of the main theorem

The proof of this will be conveyed via a series of lemmas.

Lemma 2.5.1. Let $\mathcal{B}$ be a left ideal of $\mathcal{A}$. Then:

(a) We have $\mathcal{B}^{\perp} = \mathrm{LAnn}\left(S\left(\mathcal{B}\right)\right)$.

(b) If $S\left(\mathcal{B}\right) = \mathcal{B}$, then $\mathcal{B}^{\perp} = \mathrm{LAnn}\,\mathcal{B} = \mathrm{RAnn}\,\mathcal{B}$.

Proof. Let $\mathrm{coeff}_1 : \mathcal{A} \to \mathbf{k}$ be the map that sends each element of $\mathcal{A} = \mathbf{k}\left[S_n\right]$ to the coefficient of the identity permutation id in this element. In other words, $\mathrm{coeff}_1 : \mathcal{A} \to \mathbf{k}$ is the $\mathbf{k}$-linear map that sends the permutation $\mathrm{id} \in S_n$ to 1 while sending any non-identity permutation $w \in S_n$ to 0.

It is easy to see that the bilinear form $\langle \cdot, \cdot \rangle$ can be expressed as follows: For any $a, b \in \mathcal{A}$, we have

$$\begin{aligned}
\langle a, b \rangle &= \mathrm{coeff}_1\left(S\left(a\right) b\right) && (8)\\
&= \mathrm{coeff}_1\left(b S\left(a\right)\right) && (9)\\
&= \mathrm{coeff}_1\left(S\left(b\right) a\right) && (10)\\
&= \mathrm{coeff}_1\left(a S\left(b\right)\right). && (11)
\end{aligned}$$

(a) Let $a \in \mathrm{LAnn}\left(S\left(\mathcal{B}\right)\right)$. Then, $aS\left(b\right) = 0$ for all $b \in \mathcal{B}$. Hence, (11) yields $\langle a, b \rangle = \mathrm{coeff}_1\left(\underbrace{aS\left(b\right)}_{=0}\right) = \mathrm{coeff}_1 0 = 0$ for all $b \in \mathcal{B}$. In other words, $a \in \mathcal{B}^{\perp}$. Thus, we have shown that $\mathrm{LAnn}\left(S\left(\mathcal{B}\right)\right) \subseteq \mathcal{B}^{\perp}$.

Conversely, let $c \in \mathcal{B}^{\perp}$. Then, $\langle c, b \rangle = 0$ for all $b \in \mathcal{B}$. Now, let $b \in \mathcal{B}$ be arbitrary. Then, for every $w \in S_n$, we have $wb \in \mathcal{B}$ (since $\mathcal{B}$ is a left ideal of $\mathcal{A}$) and therefore

$\langle c, wb \rangle = 0$ (since $c \in \mathcal{B}^{\perp}$), so that

$$\begin{aligned} 0 &= \langle c, wb \rangle = \operatorname{coeff}_1 (cS(wb)) && \text{(by (11))} \\ &= \operatorname{coeff}_1 \left(cS(b) w^{-1} \right) && \left(\text{since } S(wb) = S(b) \underbrace{S(w)}_{=w^{-1}} = S(b) w^{-1} \right) \\ &= \left(\text{the coefficient of id in } cS(b) w^{-1} \right) \\ &= (\text{the coefficient of } w \text{ in } cS(b)) . \end{aligned}$$

Since this holds for each $w \in S_n$, we thus obtain $cS(b) = 0$. Since this holds for each $b \in \mathcal{B}$, we conclude that $c \in \operatorname{LAnn}(S(\mathcal{B}))$. Thus, we have shown that $\mathcal{B}^{\perp} \subseteq \operatorname{LAnn}(S(\mathcal{B}))$. Combining this with $\operatorname{LAnn}(S(\mathcal{B})) \subseteq \mathcal{B}^{\perp}$, we obtain $\mathcal{B}^{\perp} = \operatorname{LAnn}(S(\mathcal{B}))$. Thus, Lemma 2.5.1 **(a)** is proved.

(b) Assume that $S(\mathcal{B}) = \mathcal{B}$. Then, Lemma 2.5.1 **(a)** yields $\mathcal{B}^{\perp} = \operatorname{LAnn}(S(\mathcal{B})) = \operatorname{LAnn} \mathcal{B}$ (since $S(\mathcal{B}) = \mathcal{B}$). Furthermore, since the bilinear form $\langle \cdot, \cdot \rangle$ is S-invariant (i.e., satisfies $\langle S(a), S(b) \rangle = \langle a, b \rangle$ for all $a, b \in \mathcal{A}$), we can easily see that $(S(\mathcal{B}))^{\perp} = S(\mathcal{B}^{\perp})$. In view of $S(\mathcal{B}) = \mathcal{B}$, we can rewrite this as $\mathcal{B}^{\perp} = S(\mathcal{B}^{\perp})$. However, S is a $\mathbf{k}$-algebra anti-automorphism. Thus, $\operatorname{RAnn}(S(\mathcal{B})) = S(\operatorname{LAnn} \mathcal{B})$. In view of $S(\mathcal{B}) = \mathcal{B}$, we can rewrite this as $\operatorname{RAnn} \mathcal{B} = S\left(\underbrace{\operatorname{LAnn} \mathcal{B}}_{=\mathcal{B}^{\perp}} \right) = S(\mathcal{B}^{\perp}) = \mathcal{B}^{\perp}$. Thus, $\mathcal{B}^{\perp} = \operatorname{RAnn} \mathcal{B}$. Combined with $\mathcal{B}^{\perp} = \operatorname{LAnn} \mathcal{B}$, this completes the proof of Lemma 2.5.1 **(b)**. ☐

Lemma 2.5.2. Let $\mathcal{M}$ be a free $\mathbf{k}$-module with a basis $(m_i)_{i \in I}$. Let J and K be two disjoint subsets of I such that $J \cup K = I$. Let $\mathcal{N}$ be a $\mathbf{k}$-submodule of $\mathcal{M}$ such that the quotient module $\mathcal{M}/\mathcal{N}$ has a basis $(\overline{m_i})_{i \in J}$. (Here, as usual, $\overline{m}$ denotes the projection of any vector $m \in \mathcal{M}$ onto the quotient $\mathcal{M}/\mathcal{N}$.)

Then:

(a) The $\mathbf{k}$-module $\mathcal{N}$ is free of rank $|K|$.

(b) There exists a $\mathbf{k}$-linear projection $\pi : \mathcal{M} \to \mathcal{N}$ (that is, a $\mathbf{k}$-linear map $\pi : \mathcal{M} \to \mathcal{N}$ such that $\pi |_{\mathcal{N}} = \operatorname{id}$).

Proof. For each $k \in K$, the vector $\overline{m_k} \in \mathcal{M}/\mathcal{N}$ can be written as a $\mathbf{k}$-linear combination of the family $(\overline{m_i})_{i \in J}$ (since this family is a basis of $\mathcal{M}/\mathcal{N}$). In other words, there exist coefficients $c_{k,j}$ for all $k \in K$ and $j \in J$ such that each $k \in K$ satisfies

$$\overline{m_k} = \sum_{j \in J} c_{k,j} \overline{m_j}. \tag{12}$$

Now, let us set

$$v_k := m_k - \sum_{j \in J} c_{k,j} m_j \tag{13}$$

for each $k \in K$. This element v_k belongs to $\mathcal{N}$ (since $\overline{v_k} = \overline{m_k - \sum_{j \in J} c_{k,j} m_j} = \overline{m_k} - \sum_{j \in J} c_{k,j} \overline{m_j} = \overline{0}$ by (12)). Thus, $(v_k)_{k \in K}$ is a family of vectors in $\mathcal{N}$. This family is easily seen to be $\mathbf{k}$-linearly independent (since the only vector in this family that contains a given m_k is v_k). Moreover, it spans the $\mathbf{k}$-module $\mathcal{N}$, because any vector in $\mathcal{N}$ can be reduced (modulo this family) to a $\mathbf{k}$-linear combination of the m_j with $j \in J$, and the latter combination must have zeroes for its coefficients for our vector to belong to $\mathcal{N}$ (since $(\overline{m_i})_{i \in J}$ is linearly independent in $\mathcal{M}/\mathcal{N}$). Thus, the family $(v_k)_{k \in K}$ is a basis of $\mathcal{N}$. Therefore, $\mathcal{N}$ is free of rank $|K|$. This proves Lemma 2.5.2 **(a)**.

(b) Let $\pi : \mathcal{M} \to \mathcal{N}$ be the $\mathbf{k}$-linear map that sends each basis element m_i of $\mathcal{M}$ to

$$\begin{cases} v_i, & \text{if } i \in K; \\ 0, & \text{if } i \in J. \end{cases}$$

This is well-defined (since $(m_i)_{i \in I}$ is a basis of $\mathcal{M}$, and since each $i \in I$ belongs to exactly one of the sets K and J). It is easy to see that this map π sends v_k to v_k for each $k \in K$ (because applying π to the right hand side of (13) kills all m_j addends while sending m_k to v_k). Thus, $\pi \mid_{\mathcal{N}} = \operatorname{id}$ (since $(v_k)_{k \in K}$ is a basis of $\mathcal{N}$). Thus, π is a projection. This proves Lemma 2.5.2 **(b)**. □

Lemma 2.5.3. Let $k \in \mathbb{N}$. Then, $\mathcal{I}_k \mathcal{J}_k = \mathcal{J}_k \mathcal{I}_k = 0$.

Proof. Let us first show that $\mathcal{I}_k \mathcal{J}_k = 0$. Indeed, we have

$$\mathcal{I}_k = \operatorname{span} \{ \nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \operatorname{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k \}$$

and

$$\mathcal{J}_k = \operatorname{span} \left\{ \nabla_U^- \mid U \text{ is a subset of } [n] \text{ having size } k+1 \right\} \cdot \mathcal{A}$$

(by Proposition 2.2.2 **(b)**). Thus, in order to prove that $\mathcal{I}_k \mathcal{J}_k = 0$, it suffices to show that $\nabla_{\mathbf{B},\mathbf{A}} \nabla_U^- = 0$ for all set compositions $\mathbf{A}, \mathbf{B} \in \operatorname{SC}(n)$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$ and all subsets U of $[n]$ having size $k+1$. So let us show this. We fix two set compositions $\mathbf{A}, \mathbf{B} \in \operatorname{SC}(n)$ satisfying $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$ and a subset U of $[n]$ having size $k+1$.

We have $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k < k+1 = |U|$. In other words, $\mathbf{A}$ has fewer blocks than U has elements. Hence, by the pigeonhole principle, there exist two distinct elements u and v of U that belong to the same block of $\mathbf{A}$. Pick such u and v. Let $\tau \in S_U$ be the transposition that swaps u and v. Then, $\nabla_U^- = (1 - \tau) q$ for some $q \in \mathbf{k}[S_U]$ (since $\langle \tau \rangle = \{1, \tau\}$ is a subgroup of the group S_U). Consider this q. But (6) yields

$$\nabla_{\mathbf{B},\mathbf{A}} \tau = \nabla_{\mathbf{B},\tau^{-1}\mathbf{A}} = \nabla_{\mathbf{B},\mathbf{A}}$$

(since $\tau^{-1}\mathbf{A} = \mathbf{A}$ (because u and v belong to the same block of $\mathbf{A}$)). Thus,

$$\nabla_{\mathbf{B},\mathbf{A}} \underbrace{\nabla_U^-}_{=(1-\tau)q} = \underbrace{\nabla_{\mathbf{B},\mathbf{A}} (1-\tau)}_{\substack{=\nabla_{\mathbf{B},\mathbf{A}} - \nabla_{\mathbf{B},\mathbf{A}}\tau \\ =0 \\ (\text{since } \nabla_{\mathbf{B},\mathbf{A}}\tau = \nabla_{\mathbf{B},\mathbf{A}})}} q = 0.$$

This completes our proof of $\mathcal{I}_k \mathcal{J}_k = 0$.

It remains to prove that $\mathcal{J}_k \mathcal{I}_k = 0$. This can be done similarly, but can also be derived from $\mathcal{I}_k \mathcal{J}_k = 0$ easily: Since S is an algebra anti-automorphism, we have

$$S(\mathcal{I}_k \mathcal{J}_k) = \underbrace{S(\mathcal{J}_k)}_{\substack{=\mathcal{J}_k \\ (\text{by Proposition 2.2.2 } \textbf{(c)})}} \underbrace{S(\mathcal{I}_k)}_{\substack{=\mathcal{I}_k \\ (\text{by Proposition 2.2.2 } \textbf{(c)})}} = \mathcal{J}_k \mathcal{I}_k.$$

Thus, $\mathcal{J}_k \mathcal{I}_k = S(\mathcal{I}_k \mathcal{J}_k) = 0$ (since $\mathcal{I}_k \mathcal{J}_k = 0$). The proof of Lemma 2.5.3 is thus complete. □

Lemma 2.5.4. Let $k \in \mathbb{N}$. Then, the quotient module $\mathcal{A}/\mathcal{I}_k$ is spanned by the family $(\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(k+1)}$.

Proof. It suffices to prove that

$$\overline{u} \in \operatorname{span}\left((\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \right) \qquad \text{for each } u \in S_n. \tag{14}$$

To prove this, we proceed by induction on u in lexicographic order. Thus, we fix a permutation $v \in S_n$, and we assume (as the induction hypothesis) that (14) holds for every $u < v$ in lexicographic order. We must now prove (14) for $u = v$.

If $v \in S_n \setminus \operatorname{Av}_n(k+1)$, then this is trivial. Thus, we WLOG assume that $v \notin S_n \setminus \operatorname{Av}_n(k+1)$. Hence, $v \in \operatorname{Av}_n(k+1)$. Therefore, by a variant of the Erdös–Szekeres theorem, there exists a set decomposition $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ of $[n]$ such that all restrictions $v\mid_{A_1}, v\mid_{A_2}, \ldots, v\mid_{A_k}$ are decreasing[1]. Consider this set decomposition $\mathbf{A} = (A_1, A_2, \ldots, A_k)$. Define a further set decomposition $\mathbf{B} = (B_1, B_2, \ldots, B_k)$ of $[n]$ by

$$B_i := v(A_i) \qquad \text{for each } i \in [k].$$

[1]Let us recall how this can be shown: Let $\mathbf{v}$ be the sequence $(v(1), v(2), \ldots, v(n))$. Then, $\mathbf{v}$ has no increasing subsequence of length $> k$ (because $v \in \operatorname{Av}_n(k+1)$). For each $i \in \{1, 2, \ldots, k\}$, we let

$$A_i := \{ j \in [n] \mid \text{the longest increasing subsequence of } \mathbf{v} \text{ ending with } v(j) \text{ has length } i \}.$$

These k sets $A_1, A_2, \ldots, A_k$ are clearly disjoint, and their union is $[n]$ (since the sequence $\mathbf{v}$ has no increasing subsequence of length $> k$). In other words, $\mathbf{A} = (A_1, A_2, \ldots, A_k)$ is a set decomposition of $[n]$. By its definition, it is easy to see that all restrictions $v\mid_{A_1}, v\mid_{A_2}, \ldots, v\mid_{A_k}$ are decreasing (because if two elements $p < q$ of A_i satisfied $v(p) < v(q)$, then the longest increasing subsequence of $\mathbf{v}$ ending with $v(q)$ would be longer than the one ending with $v(p)$).

Thus, $v \in S_n$ is a permutation satisfying $v(A_i) = B_i$ for all i. Hence, the row-to-row sum

$$\nabla_{\mathbf{B},\mathbf{A}} = \sum_{\substack{w \in S_n; \\ w(A_i)=B_i \text{ for all } i}} w$$

contains the permutation v as one of its addends. All its remaining addends can be obtained from v by permuting the values of v on A_1, permuting the values of v on A_2, and so on. Any such permutation decreases v in lexicographic order (because the restrictions $v\mid_{A_1}, v\mid_{A_2}, \ldots, v\mid_{A_k}$ are decreasing). Thus, all the addends of $\nabla_{\mathbf{B},\mathbf{A}}$ except for v are lexicographically smaller than v. Hence,

$$\nabla_{\mathbf{B},\mathbf{A}} = v + (\text{some permutations } w < v).$$

Therefore,

$$v = \nabla_{\mathbf{B},\mathbf{A}} - (\text{some permutations } w < v). \tag{15}$$

But we have $\ell(\mathbf{A}) = \ell(\mathbf{B}) = k$. Hence, Proposition 2.2.2 **(d)** yields $\nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{I}_k$. Thus, projecting the equality (15) onto the quotient $\mathcal{A}/\mathcal{I}_k$, we obtain

$$\begin{aligned}
\overline{v} &= \underbrace{\overline{\nabla_{\mathbf{B},\mathbf{A}}}}_{\substack{=0 \\ (\text{since } \nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{I}_k)}} - \overline{(\text{some permutations } w < v)} \\
&= -\underbrace{\overline{(\text{some permutations } w < v)}}_{\substack{\in \operatorname{span}\left((\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(k+1)}\right) \\ (\text{by our induction hypothesis})}} \in \operatorname{span}\left((\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(k+1)}\right).
\end{aligned}$$

In other words, (14) holds for $u = v$. This completes the induction. Thus, Lemma 2.5.4 is proved. □

Lemma 2.5.5. Let $k \in \mathbb{N}$. Then, the quotient module $\mathcal{A}/\mathcal{J}_k$ is spanned by the family $(\overline{w})_{w \in \operatorname{Av}_n(k+1)}$.

Proof. It suffices to prove that

$$\overline{u} \in \operatorname{span}\left((\overline{w})_{w \in \operatorname{Av}_n(k+1)}\right) \qquad \text{for each } u \in S_n. \tag{16}$$

To prove this, we proceed by induction on u in reverse lexicographic order. Thus, we fix a permutation $v \in S_n$, and we assume (as the induction hypothesis) that (16) holds for every $u > v$ in lexicographic order. We must now prove (16) for $u = v$.

If $v \in \operatorname{Av}_n(k+1)$, then this is trivial. Thus, we WLOG assume that $v \notin \operatorname{Av}_n(k+1)$. Hence, $v \in S_n \setminus \operatorname{Av}_n(k+1)$. Therefore, there exists a $(k+1)$-element subset U of $[n]$ such that the restriction $v\mid_U$ is increasing. Consider this U. Thus, the sum

$$v\nabla_U^- = \sum_{\substack{w \in S_n \text{ agrees with } v \text{ on} \\ \text{all elements outside of } U}} \pm w$$

contains the permutation v as one of its addends. All its remaining addends can be obtained from v by permuting the values of v on U. Any such permutation increases v in lexicographic order (because the restriction $v\mid_U$ is increasing). Thus, all the addends of $v\nabla_U^-$ except for v are lexicographically larger than v. Hence,

$$v\nabla_U^- = v \pm (\text{some permutations } w > v).$$

Therefore,

$$v = v\nabla_U^- \pm (\text{some permutations } w > v). \tag{17}$$

But the definition of $\mathcal{J}_k$ yields $v\nabla_U^- \in \mathcal{J}_k$. Thus, projecting the equality (17) onto the quotient $\mathcal{A}/\mathcal{J}_k$, we obtain

$$\begin{aligned}\overline{v} &= \underbrace{\overline{v\nabla_U^-}}_{\substack{=0 \\ (\text{since } v\nabla_U^- \in \mathcal{J}_k)}} \pm \overline{(\text{some permutations } w > v)} \\ &= -\underbrace{\overline{(\text{some permutations } w > v)}}_{\substack{\in \operatorname{span}\left((\overline{w})_{w\in \operatorname{Av}_n(k+1)}\right) \\ (\text{by our induction hypothesis})}} \in \operatorname{span}\left((\overline{w})_{w\in \operatorname{Av}_n(k+1)}\right).\end{aligned}$$

In other words, (16) holds for $u = v$. This completes the induction. Thus, Lemma 2.5.5 is proved. □

Lemma 2.5.6. Let $k \in \mathbb{N}$. Let $(\alpha_w)_{w\in \operatorname{Av}_n(k+1)} \in \mathbf{k}^{\operatorname{Av}_n(k+1)}$ be a family of scalars satisfying

$$\sum_{w\in \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k^\perp.$$

Then, $\alpha_w = 0$ for all $w \in \operatorname{Av}_n(k+1)$.

Proof. Assume the contrary. Thus, there exist some $w \in \operatorname{Av}_n(k+1)$ such that $\alpha_w \neq 0$. Let v be the lexicographically smallest such w. Thus, $\alpha_v \neq 0$, but

$$\alpha_w = 0 \qquad \text{for every } w \in S_n \text{ satisfying } w < v. \tag{18}$$

As in the proof of Lemma 2.5.4, we can construct set decompositions $\mathbf{A}$ and $\mathbf{B}$ of $[n]$ such that $\ell(\mathbf{A}) = \ell(\mathbf{B}) = k$ and $\nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{I}_k$ and

$$\nabla_{\mathbf{B},\mathbf{A}} = v + (\text{some permutations } w < v) \tag{19}$$

hold. Consider these $\mathbf{A}$ and $\mathbf{B}$. From $\sum\limits_{w\in \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k^\perp$ and $\nabla_{\mathbf{B},\mathbf{A}} \in \mathcal{I}_k$, we conclude that $\left\langle \sum\limits_{w\in \operatorname{Av}_n(k+1)} \alpha_w w,\ \nabla_{\mathbf{B},\mathbf{A}} \right\rangle = 0$. Thus,

$$\begin{aligned} 0 &= \left\langle \sum_{w\in \operatorname{Av}_n(k+1)} \alpha_w w,\ \nabla_{\mathbf{B},\mathbf{A}} \right\rangle \\ &= \sum_{w\in \operatorname{Av}_n(k+1)} \underbrace{\alpha_w}_{\substack{=0 \text{ if } w<v \\ \text{(by (18))}}} \underbrace{\left\langle w,\ \nabla_{\mathbf{B},\mathbf{A}} \right\rangle}_{\substack{=0 \text{ if } w>v \\ \text{(by (19))}}} \\ &= \alpha_v \underbrace{\left\langle v,\ \nabla_{\mathbf{B},\mathbf{A}} \right\rangle}_{\substack{=1 \\ \text{(by (19))}}} = \alpha_v \neq 0, \end{aligned}$$

which is absurd. This completes the proof of Lemma 2.5.6. $\square$

Lemma 2.5.7. Let $k \in \mathbb{N}$. Let $(\alpha_w)_{w\in S_n \setminus \operatorname{Av}_n(k+1)} \in \mathbf{k}^{S_n\setminus \operatorname{Av}_n(k+1)}$ be a family of scalars satisfying

$$\sum_{w\in S_n\setminus \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{J}_k^{\perp}.$$

Then, $\alpha_w = 0$ for all $w \in S_n \setminus \operatorname{Av}_n(k+1)$.

Proof. Assume the contrary. Thus, there exist some $w \in S_n \setminus \operatorname{Av}_n(k+1)$ such that $\alpha_w \neq 0$. Let v be the lexicographically largest such w. Thus, $\alpha_v \neq 0$, but

$$\alpha_w = 0 \qquad \text{for every } w \in S_n \text{ satisfying } w > v. \tag{20}$$

As in the proof of Lemma 2.5.5, we can construct a $(k+1)$-element subset U of $[n]$ such that $v\nabla_U^- \in \mathcal{J}_k$ and

$$v\nabla_U^- = v \pm (\text{some permutations } w > v). \tag{21}$$

Consider this U. From $\sum\limits_{w\in S_n\setminus \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{J}_k^{\perp}$ and $v\nabla_U^- \in \mathcal{J}_k$, we conclude that $\left\langle \sum\limits_{w\in S_n\setminus \operatorname{Av}_n(k+1)} \alpha_w w,\ v\nabla_U^- \right\rangle = 0$. Thus,

$$\begin{aligned} 0 &= \left\langle \sum_{w\in S_n\setminus \operatorname{Av}_n(k+1)} \alpha_w w,\ v\nabla_U^- \right\rangle \\ &= \sum_{w\in S_n\setminus \operatorname{Av}_n(k+1)} \underbrace{\alpha_w}_{\substack{=0 \text{ if } w>v \\ \text{(by (20))}}} \underbrace{\left\langle w,\ v\nabla_U^- \right\rangle}_{\substack{=0 \text{ if } w<v \\ \text{(by (21))}}} \\ &= \alpha_v \underbrace{\left\langle v,\ v\nabla_U^- \right\rangle}_{\substack{=1 \\ \text{(by (21))}}} = \alpha_v \neq 0, \end{aligned}$$

which is absurd. This completes the proof of Lemma 2.5.7. $\square$

Lemma 2.5.8. Let $k \in \mathbb{N}$. Then, the $\mathbf{k}$-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(k+1)}$.

Proof. The family $(\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(k+1)}$ spans this $\mathbf{k}$-module $\mathcal{A}/\mathcal{I}_k$, as we know from Lemma 2.5.4. It remains to prove that it is $\mathbf{k}$-linearly independent.

Let $(\alpha_w)_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \in \mathbf{k}^{S_n \setminus \operatorname{Av}_n(k+1)}$ be a family of scalars satisfying

$$\sum_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \alpha_w \overline{w} = 0. \tag{22}$$

We thus need to show that $\alpha_w = 0$ for all $w \in S_n \setminus \operatorname{Av}_n(k+1)$.

However, (22) means that $\sum\limits_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k$. But Lemma 2.5.3 yields $\mathcal{I}_k \mathcal{J}_k = 0$. Thus, $\mathcal{I}_k \subseteq \operatorname{LAnn} \mathcal{J}_k$. However, Proposition 2.2.2 **(c)** yields $S(\mathcal{J}_k) = \mathcal{J}_k$. Furthermore, $\mathcal{J}_k$ is an ideal of $\mathcal{A}$ (by Proposition 2.2.2 **(a)**). Thus, Lemma 2.5.1 **(b)** (applied to $\mathcal{B} = \mathcal{J}_k$) yields $\mathcal{J}_k^\perp = \operatorname{LAnn}(\mathcal{J}_k) = \operatorname{RAnn}(\mathcal{J}_k)$. Thus,

$$\sum_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k \subseteq \operatorname{LAnn} \mathcal{J}_k = \mathcal{J}_k^\perp.$$

Lemma 2.5.7 thus yields that $\alpha_w = 0$ for all $w \in S_n \setminus \operatorname{Av}_n(k+1)$. This completes the proof of Lemma 2.5.8. □

Lemma 2.5.9. Let $k \in \mathbb{N}$. Then, the $\mathbf{k}$-module $\mathcal{A}/\mathcal{J}_k$ is free with basis $(\overline{w})_{w \in \operatorname{Av}_n(k+1)}$.

Proof. Analogous to the proof of Lemma 2.5.8. (Of course, use Lemma 2.5.5 and Lemma 2.5.6 instead of Lemma 2.5.4 and Lemma 2.5.7 now.) □

Lemma 2.5.10. Let $k \in \mathbb{N}$. Then,

$$\mathcal{I}_k = \mathcal{J}_k^\perp = \operatorname{LAnn} \mathcal{J}_k = \operatorname{RAnn} \mathcal{J}_k.$$

Proof. Proposition 2.2.2 **(c)** yields $S(\mathcal{J}_k) = \mathcal{J}_k$. Furthermore, $\mathcal{J}_k$ is an ideal of $\mathcal{A}$ (by Proposition 2.2.2 **(a)**). Thus, Lemma 2.5.1 **(b)** (applied to $\mathcal{B} = \mathcal{J}_k$) yields $\mathcal{J}_k^\perp = \operatorname{LAnn} \mathcal{J}_k = \operatorname{RAnn} \mathcal{J}_k$. Thus, it remains to prove that $\mathcal{I}_k = \mathcal{J}_k^\perp$.

Lemma 2.5.3 yields $\mathcal{I}_k \mathcal{J}_k = 0$. Thus, $\mathcal{I}_k \subseteq \operatorname{LAnn} \mathcal{J}_k = \mathcal{J}_k^\perp$. Thus, we only need to show that $\mathcal{J}_k^\perp \subseteq \mathcal{I}_k$.

Let $a \in \mathcal{J}_k^\perp$. We must prove that $a \in \mathcal{I}_k$.

Lemma 2.5.4 shows that the quotient module $\mathcal{A}/\mathcal{I}_k$ is spanned by the family $(\overline{w})_{w \in S_n \setminus \operatorname{Av}_n(k+1)}$. Hence, the projection $\overline{a} \in \mathcal{A}/\mathcal{I}_k$ can be written as a $\mathbf{k}$-linear combination of this family. In other words, we can write $\overline{a}$ as

$$\overline{a} = \sum_{w \in S_n \setminus \operatorname{Av}_n(k+1)} \alpha_w \overline{w} \tag{23}$$

for some family $(\alpha_w)_{w\in S_n\setminus \operatorname{Av}_n(k+1)} \in \mathbf{k}^{S_n\setminus \operatorname{Av}_n(k+1)}$ of scalars. Consider this family. We can rewrite (23) as

$$a - \sum_{w\in S_n\setminus \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{I}_k \subseteq \mathcal{J}_k^{\perp}.$$

Since $a \in \mathcal{J}_k^{\perp}$, this yields $\sum\limits_{w\in S_n\setminus \operatorname{Av}_n(k+1)} \alpha_w w \in \mathcal{J}_k^{\perp}$. By Lemma 2.5.7, we thus conclude that $\alpha_w = 0$ for all $w \in S_n \setminus \operatorname{Av}_n(k+1)$. Thus, (23) rewrites as $\overline{a} = \sum\limits_{w\in S_n\setminus \operatorname{Av}_n(k+1)} 0\overline{w} = 0$, so that $a \in \mathcal{I}_k$. This completes our proof of Lemma 2.5.10. ◻

Lemma 2.5.11. Let $k \in \mathbb{N}$. Then,

$$\mathcal{J}_k = \mathcal{I}_k^{\perp} = \operatorname{LAnn}\mathcal{I}_k = \operatorname{RAnn}\mathcal{I}_k.$$

Proof. Analogous to the proof of Lemma 2.5.10. (Of course, use Lemma 2.5.5 and Lemma 2.5.6 instead of Lemma 2.5.4 and Lemma 2.5.7 now.) ◻

Lemma 2.5.12. Assume that $n!$ is invertible in $\mathbf{k}$. Let A and B be two disjoint subsets of S_n such that $A \cup B = S_n$. Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals of $\mathcal{A}$ such that $\mathcal{I} = \operatorname{LAnn}\mathcal{J}$ and $\mathcal{J} = \operatorname{LAnn}\mathcal{I}$. Assume that the family $(\overline{w})_{w\in A}$ is a basis of the $\mathbf{k}$-module $\mathcal{A}/\mathcal{I}$, and that the family $(\overline{w})_{w\in B}$ is a basis of the $\mathbf{k}$-module $\mathcal{A}/\mathcal{J}$. Then, $\mathcal{A} = \mathcal{I} \oplus \mathcal{J}$ (internal direct sum) as $\mathbf{k}$-module. Moreover, $\mathcal{I}$ and $\mathcal{J}$ are nonunital subalgebras of $\mathcal{A}$ and satisfy $\mathcal{A} \cong \mathcal{I} \times \mathcal{J}$ as $\mathbf{k}$-algebras.

Proof. From $\mathcal{I} = \operatorname{LAnn}\mathcal{J}$, we obtain $\mathcal{I}\mathcal{J} = 0$. Similarly, $\mathcal{J}\mathcal{I} = 0$.

Lemma 2.5.2 **(b)** (applied to $\mathcal{M} = \mathcal{A}$ and $\mathcal{N} = \mathcal{I}$ and $I = S_n$ and $J = A$ and $K = B$) yields that there exists a $\mathbf{k}$-linear projection $\pi : \mathcal{A} \to \mathcal{I}$ (that is, a $\mathbf{k}$-linear map $\pi : \mathcal{A} \to \mathcal{I}$ such that $\pi\mid_{\mathcal{I}} = \operatorname{id}$). Consider this π.

Note that $\mathcal{I}$ is an ideal of $\mathcal{A}$, thus a left ideal of $\mathcal{A}$, hence a left $\mathcal{A}$-submodule of $\mathcal{A}$. Moreover, $|S_n| = n!$ is invertible in $\mathbf{k}$. Hence, the standard proof of the Maschke theorem (via averaging the projection π over S_n) yields that there exists a $\mathbf{k}$-linear projection $\pi' : \mathcal{A} \to \mathcal{I}$ that is a left $\mathcal{A}$-module homomorphism[2]. Consider this π'.

Let $e := \pi'(1) \in \mathcal{I}$. Then, we claim that

$$ue = u \qquad \text{for each } u \in \mathcal{I}. \tag{24}$$

[*Proof of (24):* Let $u \in \mathcal{I}$. Then, $\pi'(u) = u$ (since π' is a projection). However, π' is a left $\mathcal{A}$-module homomorphism. Thus, $\pi'(u1) = u\underbrace{\pi'(1)}_{=e} = ue$, so that $ue = \pi'\left(\underbrace{u1}_{=u}\right) = \pi'(u) = u$. This proves (24).]

[2]Explicitly, π' can be constructed as follows:

$$\pi'(a) = \frac{1}{|S_n|}\sum_{\sigma\in S_n} \sigma\pi\left(\sigma^{-1}a\right) \qquad \text{for each } a \in \mathcal{A}.$$

Clearly, $\mathcal{I}$ is a nonunital subalgebra of $\mathcal{A}$ (since $\mathcal{I}$ is an ideal of $\mathcal{A}$). From (24), we see that this algebra $\mathcal{I}$ has a right unity (namely, e). A similar argument (using right instead of left $\mathcal{A}$-modules) yields that $\mathcal{I}$ has a left unity. Thus, a standard argument shows that $\mathcal{I}$ has a unity (since any associative operation that has a left neutral element and a right neutral element has a neutral element). In other words, $\mathcal{I}$ is a unital algebra (although its unity is not that of $\mathcal{A}$). Let $1_{\mathcal{I}}$ denote its unity.

Set $g := 1 - 1_{\mathcal{I}}$. Then, each $u \in \mathcal{I}$ satisfies

$$gu = (1 - 1_{\mathcal{I}})\,u = u - \underbrace{1_{\mathcal{I}}u}_{\substack{=u \\ \text{(since } 1_{\mathcal{I}} \text{ is the} \\ \text{unity of } \mathcal{I})}} = u - u = 0$$

In other words, $g \in \operatorname{LAnn}\mathcal{I} = \mathcal{J}$. Moreover, each $v \in \mathcal{J}$ satisfies $1_{\mathcal{I}}v = 0$ (since $\underbrace{1_{\mathcal{I}}}_{\in \mathcal{I}} \underbrace{v}_{\in \mathcal{J}} \in \mathcal{I}\mathcal{J} = 0$) and thus

$$\underbrace{g}_{=1-1_{\mathcal{I}}} v = (1 - 1_{\mathcal{I}})\,v = v - \underbrace{1_{\mathcal{I}}v}_{=0} = v.$$

Hence, g is a left unity of the algebra $\mathcal{J}$ (since $g \in \mathcal{J}$). A similar computation shows that g is a right unity of $\mathcal{J}$. Hence, g is a unity of $\mathcal{J}$. We shall thus rename g as $1_{\mathcal{J}}$ now.

Moreover, each $u \in \mathcal{I} \cap \mathcal{J}$ satisfies

$$\begin{aligned} u &= \underbrace{u}_{\in \mathcal{I} \cap \mathcal{J} \subseteq \mathcal{I}} \underbrace{1_{\mathcal{J}}}_{\in \mathcal{J}} \qquad (\text{since } u \in \mathcal{I} \cap \mathcal{J} \subseteq \mathcal{J}) \\ &\in \mathcal{I}\mathcal{J} = 0 \end{aligned}$$

and thus $u = 0$. In other words, $\mathcal{I} \cap \mathcal{J} = 0$. Furthermore, each $a \in \mathcal{A}$ satisfies

$$a = 1_{\mathcal{I}}a + \underbrace{(1 - 1_{\mathcal{I}})}_{=g=1_{\mathcal{J}}} a = \underbrace{1_{\mathcal{I}}a}_{\substack{\in \mathcal{I} \\ \text{(since } \mathcal{I} \text{ is an ideal} \\ \text{and } 1_{\mathcal{I}} \in \mathcal{I})}} + \underbrace{1_{\mathcal{J}}a}_{\substack{\in \mathcal{J} \\ \text{(since } \mathcal{J} \text{ is an ideal} \\ \text{and } 1_{\mathcal{J}} \in \mathcal{J})}} \in \mathcal{I} + \mathcal{J}.$$

This shows that $\mathcal{I} + \mathcal{J} = \mathcal{A}$. Combining this with $\mathcal{I} \cap \mathcal{J} = 0$, we conclude that $\mathcal{A} = \mathcal{I} \oplus \mathcal{J}$ (internal direct sum) as $\mathbf{k}$-module.

Recalling further that $\mathcal{I}\mathcal{J} = \mathcal{J}\mathcal{I} = 0$, we conclude that $\mathcal{A} \cong \mathcal{I} \times \mathcal{J}$ as $\mathbf{k}$-algebras (via the isomorphism that sends $(i, j) \in \mathcal{I} \times \mathcal{J}$ to $i + j \in \mathcal{A}$). Thus, the proof of Lemma 2.5.12 is complete. $\square$

2.6. Proof of the main theorem

Proof of Theorem 2.4.1. **(a)** This is just Lemma 2.5.10.

(b) This is just Lemma 2.5.11.

(c) Lemma 2.5.8 yields that the $\mathbf{k}$-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w\in S_n \setminus \mathrm{Av}_n(k+1)}$. Hence, Lemma 2.5.2 **(a)** (applied to $\mathcal{M} = \mathcal{A}$ and $\mathcal{N} = \mathcal{I}_k$ and $I = S_n$ and $J = S_n \setminus \mathrm{Av}_n(k+1)$ and $K = \mathrm{Av}_n(k+1)$) yields that the $\mathbf{k}$-module $\mathcal{I}_k$ is free of rank $|\mathrm{Av}_n(k+1)|$. This proves Theorem 2.4.1 **(c)**.

(d) This is proved similarly to part **(c)**, but using Lemma 2.5.9 instead of Lemma 2.5.8.

(e) This is just Lemma 2.5.8.

(f) This is just Lemma 2.5.9.

(g) Proposition 2.2.2 **(a)** yields that both $\mathcal{I}_k$ and $\mathcal{J}_k$ are ideals of $\mathcal{A}$. Theorem 2.4.1 **(a)** yields $\mathcal{I}_k = \mathrm{LAnn}\,\mathcal{J}_k$. Theorem 2.4.1 **(b)** yields $\mathcal{J}_k = \mathrm{LAnn}\,\mathcal{I}_k$. Clearly, $S_n \setminus \mathrm{Av}_n(k+1)$ and $\mathrm{Av}_n(k+1)$ are two disjoint subsets of S_n such that $(S_n \setminus \mathrm{Av}_n(k+1)) \cup \mathrm{Av}_n(k+1) = S_n$. Theorem 2.4.1 **(e)** says that the $\mathbf{k}$-module $\mathcal{A}/\mathcal{I}_k$ is free with basis $(\overline{w})_{w\in S_n \setminus \mathrm{Av}_n(k+1)}$. Theorem 2.4.1 **(f)** says that the $\mathbf{k}$-module $\mathcal{A}/\mathcal{J}_k$ is free with basis $(\overline{w})_{w\in \mathrm{Av}_n(k+1)}$. Hence, Lemma 2.5.12 (applied to $A = S_n \setminus \mathrm{Av}_n(k+1)$ and $B = \mathrm{Av}_n(k+1)$ and $\mathcal{I} = \mathcal{I}_k$ and $\mathcal{J} = \mathcal{J}_k$) yields that $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum) as $\mathbf{k}$-module, and moreover, $\mathcal{I}_k$ and $\mathcal{J}_k$ are nonunital subalgebras of $\mathcal{A}$ and satisfy $\mathcal{A} \cong \mathcal{I}_k \times \mathcal{J}_k$ as $\mathbf{k}$-algebras. This proves Theorem 2.4.1 **(g)**. $\square$

Corollary 2.6.1. We have

$$\mathcal{I}_2 = \mathrm{span}\left\{\nabla_{B,A} \ \mid\ A, B \subseteq [n]\right\}. \tag{25}$$

Moreover, the $\mathbf{k}$-module $\mathcal{I}_2$ is free of rank $|\mathrm{Av}_n(3)|$, which is the Catalan number C_n.

Proof. The equality (25) follows from the definition of $\mathcal{I}_2$ using Proposition 2.1.2. The "Moreover" claim follows from Theorem 2.4.1 **(c)**. $\square$

2.7. The Specht module connection

The following proposition discusses the representation-theoretical significance of the ideals $\mathcal{I}_k$ and $\mathcal{J}_k$. We use some basic representation theory, including the concept of a Specht module (see, e.g., [EGHLSVY11, §5.12]).

Proposition 2.7.1. Assume that $\mathbf{k}$ is a field of characteristic 0. For each partition λ of n, let S^λ denote the corresponding Specht module (a left $\mathcal{A}$-module). For each $a \in \mathcal{A}$ and each partition λ of n, we let $a_\lambda \in \mathrm{End}\left(S^\lambda\right)$ denote the action of a on the Specht module S^λ.

Consider the map

$$\begin{aligned} \mathrm{AW} : \mathcal{A} &\to \prod_{\lambda \vdash n} \mathrm{End}\left(S^\lambda\right), \\ a &\mapsto (a_\lambda)_{\lambda \vdash n}. \end{aligned}$$

This map AW is known to be a $\mathbf{k}$-algebra isomorphism. (This follows from the Artin–Wedderburn decomposition of $\mathcal{A}$, since the S^λ are the absolutely irreducible $\mathcal{A}$-modules; alternatively, this can be derived from [Ruther48, §17, Theorem 12].)

For each subset U of $\{\lambda \mid \lambda \vdash n\}$, we consider the subproduct $\prod\limits_{\lambda \in U} \operatorname{End}\left(S^\lambda\right)$ of $\prod\limits_{\lambda \vdash n} \operatorname{End}\left(S^\lambda\right)$. The preimage of this subproduct under AW is thus an ideal of $\mathcal{A}$, and will be denoted by $\mathcal{A}_U$.

Now, let $k \in \mathbb{N}$. Then,

$$\mathcal{I}_k = \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) \leq k\}} \qquad \text{and} \qquad \mathcal{J}_k = \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) > k\}}.$$

The proof of this proposition will rely on the following general fact:

Lemma 2.7.2. Let $\mathcal{M}$ be a $\mathbf{k}$-module. Let $\mathcal{I}$ and $\mathcal{J}$ be two $\mathbf{k}$-submodules of $\mathcal{M}$ such that $\mathcal{M} = \mathcal{I} \oplus \mathcal{J}$ (internal direct sum). Let $\mathcal{U}$ and $\mathcal{V}$ be two $\mathbf{k}$-submodules of $\mathcal{M}$ such that $\mathcal{I} \subseteq \mathcal{U}$ and $\mathcal{J} \subseteq \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = 0$. Then, $\mathcal{I} = \mathcal{U}$ and $\mathcal{J} = \mathcal{V}$.

Proof. Let $u \in \mathcal{U}$. We shall show that $u \in \mathcal{I}$.

We have $u \in \mathcal{U} \subseteq \mathcal{M} = \mathcal{I} \oplus \mathcal{J}$. Thus, $u = i + j$ for some $i \in \mathcal{I}$ and $j \in \mathcal{J}$. Consider these i and j. From $u = i + j$, we obtain $j = \underbrace{u}_{\in \mathcal{U}} - \underbrace{i}_{\in \mathcal{I} \subseteq \mathcal{U}} \subseteq \mathcal{U} - \mathcal{U} \subseteq \mathcal{U}$. Combining this with $j \in \mathcal{J} \subseteq \mathcal{V}$, we obtain $j \in \mathcal{U} \cap \mathcal{V} = 0$. In other words, $j = 0$. Hence, $u = i + \underbrace{j}_{=0} = i \in \mathcal{I}$.

Forget that we fixed u. We thus have shown that $u \in \mathcal{I}$ for each $u \in \mathcal{U}$. In other words, $\mathcal{U} \subseteq \mathcal{I}$. Combined with $\mathcal{I} \subseteq \mathcal{U}$, this yields $\mathcal{I} = \mathcal{U}$. Similarly, we can show $\mathcal{J} = \mathcal{V}$. This proves Lemma 2.7.2. □

Proof of Proposition 2.7.1. For each $U \subseteq \{\lambda \mid \lambda \vdash n\}$, we have

$$\begin{aligned}
\mathcal{A}_U &= \{a \in \mathcal{A} \mid \text{the } \lambda\text{-th entry of } \operatorname{AW}(a) \text{ is 0 for all } \lambda \notin U\} \\
&= \{a \in \mathcal{A} \mid a_\lambda = 0 \text{ for all } \lambda \notin U\} \\
&= \left\{a \in \mathcal{A} \mid aS^\lambda = 0 \text{ for all } \lambda \notin U\right\}.
\end{aligned}$$

Hence, in order to prove that $\mathcal{I}_k \subseteq \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) \leq k\}}$, it suffices to show that $aS^\lambda = 0$ for all $a \in \mathcal{I}_k$ and all partitions $\lambda \vdash n$ that don't satisfy $\ell(\lambda) \leq k$. Let us prove this.

Let $\lambda \vdash n$ be a partition that doesn't satisfy $\ell(\lambda) \leq k$. Thus, $\ell(\lambda) > k$. We must prove that $aS^\lambda = 0$ for each $a \in \mathcal{I}_k$.

Since $\mathcal{I}_k = \operatorname{span}\{\nabla_{\mathbf{B},\mathbf{A}} \mid \mathbf{A}, \mathbf{B} \in \operatorname{SC}(n) \text{ with } \ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k\}$, it suffices to prove that $\nabla_{\mathbf{B},\mathbf{A}} S^\lambda = 0$ for any two compositions $\mathbf{A}, \mathbf{B} \in \operatorname{SC}(n)$ with $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$. So let us consider two compositions $\mathbf{A}, \mathbf{B} \in \operatorname{SC}(n)$ with $\ell(\mathbf{A}) = \ell(\mathbf{B}) \leq k$. We must prove that $\nabla_{\mathbf{B},\mathbf{A}} S^\lambda = 0$.

Recall that $S^\lambda = \mathcal{A} a_\lambda b_\lambda$, where a_λ is the Young symmetrizer of λ, and where b_λ is the Young antisymmetrizer of λ. Hence, $S^\lambda = \underbrace{\mathcal{A} a_\lambda}_{\subseteq \mathcal{A}} b_\lambda \subseteq \mathcal{A} b_\lambda$. It thus suffices to show that $\nabla_{\mathbf{B},\mathbf{A}} \mathcal{A} b_\lambda = 0$ (since $\nabla_{\mathbf{B},\mathbf{A}} S^\lambda = 0$ will then follow). In other words, it suffices to show that $\nabla_{\mathbf{B},\mathbf{A}} w b_\lambda = 0$ for any $w \in S_n$. But this is not hard: We have $w b_\lambda = b_T w$, where T is a certain Young tableau of shape λ (filled with the entries $1, 2, \ldots, n$ in some order, not necessarily standard). The first column of this tableau T contains $\ell(\lambda)$ entries, and thus contains more than k entries (since $\ell(\lambda) > k$). Hence, at least two entries of this column belong to the same block of $\mathbf{A}$ (by the pigeonhole principle, since $\mathbf{A}$ has only $\ell(\mathbf{A}) \leq k$ blocks). Pick two such entries. Let $\tau \in S_n$ be the transposition that swaps these two entries. This transposition τ thus preserves the blocks of $\mathbf{A}$, and therefore preserves $\nabla_{\mathbf{B},\mathbf{A}}$ from the right (i.e., satisfies $\nabla_{\mathbf{B},\mathbf{A}} \tau = \nabla_{\mathbf{B},\mathbf{A}}$). On the other hand, this transposition τ swaps two entries in the first column of T, and thus belongs to the column group of T. Hence, $b_T = (\tau - 1)\eta$ for some $\eta \in \mathcal{A}$. Thus,

$$\nabla_{\mathbf{B},\mathbf{A}} \underbrace{w b_\lambda}_{= b_T w} = \nabla_{\mathbf{B},\mathbf{A}} \underbrace{b_T}_{= (\tau - 1)\eta} w = \underbrace{\nabla_{\mathbf{B},\mathbf{A}} (\tau - 1)}_{\substack{= \nabla_{\mathbf{B},\mathbf{A}} \tau - \nabla_{\mathbf{B},\mathbf{A}} \\ = 0 \\ (\text{since } \nabla_{\mathbf{B},\mathbf{A}} \tau = \nabla_{\mathbf{B},\mathbf{A}})}} \eta w = 0.$$

This completes our proof of $\nabla_{\mathbf{B},\mathbf{A}} S^\lambda = 0$.

Thus, as explained above, we have shown that

$$\mathcal{I}_k \subseteq \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) \leq k\}}. \tag{26}$$

A similar argument shows that

$$\mathcal{J}_k \subseteq \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) > k\}}. \tag{27}$$

(Here, however, we need to argue that $\nabla_U^- \mathcal{A} a_\lambda = 0$ whenever U is a subset of $[n]$ having size $k+1$ and whenever $\lambda \vdash n$ is a partition satisfying $\ell(\lambda) \leq k$. This again relies on the pigeonhole principle, now to argue that two elements of U belong to the same row of our tableau T.)

However, if X and Y are two disjoint subsets of $\{\lambda \ \mid \ \lambda \vdash n\}$, then $\mathcal{A}_X \cap \mathcal{A}_Y = 0$ (since the subproducts $\prod\limits_{\lambda \in X} \operatorname{End}(S^\lambda)$ and $\prod\limits_{\lambda \in Y} \operatorname{End}(S^\lambda)$ of $\prod\limits_{\lambda \vdash n} \operatorname{End}(S^\lambda)$ have intersection 0). Thus,

$$\mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) \leq k\}} \cap \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) > k\}} = 0.$$

However, Theorem 2.4.1 **(g)** yields $\mathcal{A} = \mathcal{I}_k \oplus \mathcal{J}_k$ (internal direct sum). Thus, Lemma 2.7.2 (applied to $\mathcal{M} = \mathcal{A}$ and $\mathcal{I} = \mathcal{I}_k$ and $\mathcal{J} = \mathcal{J}_k$ and $\mathcal{U} = \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) \leq k\}}$ and $\mathcal{V} = \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) > k\}}$) yields

$$\mathcal{I}_k = \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) \leq k\}} \qquad \text{and} \qquad \mathcal{J}_k = \mathcal{A}_{\{\lambda \vdash n \ \mid \ \ell(\lambda) > k\}}.$$

This proves Proposition 2.7.1. □

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