Commutator nilpotency for somewhere-to-below shuffles

Darij Grinberg

version 2.0, November 3, 2023

Abstract. Given a positive integer n, we consider the group algebra of the symmetric group S_n. In this algebra, we define n elements $t_1, t_2, \ldots, t_n$ by the formula

$$t_\ell := \operatorname{cyc}_\ell + \operatorname{cyc}_{\ell,\ell+1} + \operatorname{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \operatorname{cyc}_{\ell,\ell+1,\ldots,n},$$

where $\operatorname{cyc}_{\ell,\ell+1,\ldots,k}$ denotes the cycle that sends $\ell \mapsto \ell+1 \mapsto \ell+2 \mapsto \cdots \mapsto k \mapsto \ell$. These n elements are called the *somewhere-to-below shuffles* due to an interpretation as card-shuffling operators.

In this paper, we show that their commutators $\left[t_i, t_j\right] = t_i t_j - t_j t_i$ are nilpotent, and specifically that

$$\left[t_i, t_j\right]^{\lceil (n-j)/2 \rceil + 1} = 0 \qquad \text{for any } i, j \in \{1, 2, \ldots, n\}$$

and

$$\left[t_i, t_j\right]^{j-i+1} = 0 \qquad \text{for any } 1 \leq i \leq j \leq n.$$

We discuss some further identities and open questions.

Mathematics Subject Classifications: 05E99, 20C30, 60J10.

Keywords: symmetric group, permutations, card shuffling, top-to-random shuffle, group algebra, filtration, nilpotency, substitutional analysis.

Contents

1. Introduction

The *somewhere-to-below shuffles* $t_1, t_2, \ldots, t_n$ (and their linear combinations, called the *one-sided cycle shuffles*) are certain elements in the group algebra of a symmetric group S_n. They have been introduced in [GriLaf22] by Lafrenière and the present author, and are a novel generalization of the top-to-random shuffle (also known as the *Tsetlin library*). They are defined by the formula

$$t_\ell := \operatorname{cyc}_\ell + \operatorname{cyc}_{\ell,\ell+1} + \operatorname{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \operatorname{cyc}_{\ell,\ell+1,\ldots,n} \in \mathbf{k}\left[S_n\right],$$

where $\operatorname{cyc}_{\ell,\ell+1,\ldots,k}$ denotes the cycle that sends $\ell \mapsto \ell+1 \mapsto \ell+2 \mapsto \cdots \mapsto k \mapsto \ell$ (and leaves all remaining elements of $\{1,2,\ldots,n\}$ unchanged).

One of the main results of [GriLaf22] was the construction of a basis $(a_w)_{w \in S_n}$ of the group algebra in which multiplication by these shuffles acts as an upper-triangular matrix (i.e., for which $a_w t_\ell$ equals a linear combination of a_u's with $u \leq w$ for a certain total order on S_n). Consequences of this fact (or, more precisely, of a certain filtration that entails this fact) include an explicit description of the eigenvalues of each one-sided cycle shuffle, as well as analogous properties of some related shuffles.

Another consequence of the joint triangularizability of $t_1, t_2, \ldots, t_n$ is the fact that the commutators $\left[t_i, t_j\right] := t_i t_j - t_j t_i$ are nilpotent (since the commutator of two upper-triangular matrices is strictly upper-triangular and thus nilpotent). Explicitly, this means that $\left[t_i, t_j\right]^{n!} = 0$, since the $t_1, t_2, \ldots, t_n$ act on a free module of rank $n!$. However, experiments have suggested that the minimal $m \in \mathbb{N}$ satisfying $\left[t_i, t_j\right]^m = 0$ is far smaller than $n!$, and in fact is bounded from above by n.

In the present paper, we shall prove this. Concretely, we will prove the following results (the notation $[m]$ means the set $\{1,2,\ldots,m\}$):

- **Corollary 8.18.** We have $\left[t_i, t_j\right]^{\lceil (n-j)/2 \rceil + 1} = 0$ for any $i, j \in [n]$.
- **Theorem 8.15.** Let $j \in [n]$ and $m \in \mathbb{N}$ be such that $2m \geq n - j + 2$. Let $i_1, i_2, \ldots, i_m$ be m elements of $[j]$ (not necessarily distinct). Then,
$$\left[t_{i_1}, t_j\right] \left[t_{i_2}, t_j\right] \cdots \left[t_{i_m}, t_j\right] = 0.$$
- **Corollary 9.11.** We have $\left[t_i, t_j\right]^{j-i+1} = 0$ for any $1 \leq i \leq j \leq n$.
- **Theorem 9.10.** Let $j \in [n]$, and let m be a positive integer. Let $k_1, k_2, \ldots, k_m$ be m elements of $[j]$ (not necessarily distinct) satisfying $m \geq j - k_m + 1$. Then,
$$\left[t_{k_1}, t_j\right] \left[t_{k_2}, t_j\right] \cdots \left[t_{k_m}, t_j\right] = 0.$$

Along the way, we will also prove the following helpful facts:

- **Theorem 7.5.** We have $\left(1 + s_j\right) \left[t_i, t_j\right] = 0$ for any $1 \leq i \leq j < n$, where s_j denotes the transposition swapping j with $j+1$.

- **Theorem 5.1.** For any $i \in [n-1]$, we have $t_{i+1}t_i = (t_i - 1)\, t_i = t_i\, (t_i - 1)$.
- **Theorem 6.1.** For any $i \in [n-2]$, we have $t_{i+2}\, (t_i - 1) = (t_i - 1)\, (t_{i+1} - 1)$.
- **Corollary 5.2.** For any $i \in [n-1]$, we have $[t_i, t_{i+1}] = t_i\, (t_{i+1} - (t_i - 1))$ and $[t_i, t_{i+1}]\, t_i = [t_i, t_{i+1}]^2 = 0$.

These results can be regarded as first steps towards understanding the $\mathbf{k}$-subalgebra $\mathbf{k}\, [t_1, t_2, \ldots, t_n]$ of $\mathbf{k}\, [S_n]$ that is generated by the somewhere-to-below shuffles. So far, very little is known about this $\mathbf{k}$-subalgebra, except for its simultaneous triangularizability (a consequence of [GriLaf22, Theorem 4.1]). One might ask for its dimension as a $\mathbf{k}$-module (when $\mathbf{k}$ is a field). Here is some numerical data for $\mathbf{k} = \mathbb{Q}$ and $n \leq 8$:

n	1	2	3	4	5	6	7	8
$\dim\left(\mathbb{Q}\, [t_1, t_2, \ldots, t_n]\right)$	1	2	4	9	23	66	212	761

(1)

As of September 14th, 2023, this sequence of dimensions is not in the OEIS. We note that each single somewhere-to-below shuffle by itself is easily understood using the well-known theory of the top-to-random shuffle[1], but this approach says nothing about the interactions between two or more of the n somewhere-to-below shuffles.

Acknowledgements The author would like to thank Sarah Brauner and Nadia Lafrenière for inspiring discussions. The SageMath CAS [SageMath] was indispensable at every stage of the research presented here.

2. Notations and notions

2.1. Basic notations

Let $\mathbf{k}$ be any commutative ring. (The reader can safely take $\mathbf{k} = \mathbb{Z}$.)

Let $\mathbb{N} := \{0, 1, 2, \ldots\}$ be the set of all nonnegative integers.

For any integers a and b, we set

$$[a, b] := \{k \in \mathbb{Z} \mid a \leq k \leq b\} = \{a, a+1, \ldots, b\}.$$

This is an empty set if $a > b$. In general, $[a, b]$ is called an *integer interval*.

[1]Indeed, the first somewhere-to-below shuffle t_1 is known as the *top-to-random shuffle*, and has been discussed, e.g., in [Hendri72, Donnel91, Phatar91, Fill96, BiHaRo99, DiFiPi92, Palmes10, Grinbe18]. More generally, for each $\ell \in [n]$, the somewhere-to-below shuffle t_ℓ is exactly the top-to-random shuffle of the symmetric group algebra $\mathbf{k}\, [S_{n-\ell+1}]$, transported into $\mathbf{k}\, [S_n]$ using the embedding $S_{n-\ell+1} \hookrightarrow S_n$ that renames the numbers $1, 2, \ldots, n-\ell+1$ as $\ell, \ell+1, \ldots, n$. Thus, we know (e.g.) that the minimal polynomial of t_ℓ over a characteristic-0 field $\mathbf{k}$ is $\prod_{i=0}^{n-\ell-1} (x - i) \cdot (x - (n - \ell + 1))$ (by [DiFiPi92, Theorem 4.1]).

For each $n \in \mathbb{Z}$, let $[n] := [1, n] = \{1, 2, \ldots, n\}$.

Fix an integer $n \in \mathbb{N}$. Let S_n be the n-th symmetric group, i.e., the group of all permutations of $[n]$. We multiply permutations in the "continental" way: that is, $(\pi\sigma)(i) = \pi(\sigma(i))$ for all $\pi, \sigma \in S_n$ and $i \in [n]$.

For any k distinct elements $i_1, i_2, \ldots, i_k$ of $[n]$, we let $\operatorname{cyc}_{i_1, i_2, \ldots, i_k}$ be the permutation in S_n that sends $i_1, i_2, \ldots, i_{k-1}, i_k$ to $i_2, i_3, \ldots, i_k, i_1$, respectively while leaving all remaining elements of $[n]$ unchanged. This permutation is known as a *cycle*. Note that $\operatorname{cyc}_i = \operatorname{id}$ for any single $i \in [n]$.

For any $i \in [n-1]$, we let $s_i := \operatorname{cyc}_{i,i+1} \in S_n$. This permutation s_i is called a *simple transposition*, as it swaps i with $i+1$ while leaving all other elements of $[n]$ unchanged. It clearly satisfies

$$s_i^2 = \operatorname{id}. \tag{2}$$

Furthermore, two simple transpositions s_i and s_j commute whenever $|i - j| > 1$. This latter fact is known as *reflection locality*.

2.2. Some elements of $\mathbf{k}[S_n]$

Consider the group algebra $\mathbf{k}[S_n]$. In this algebra, define n elements $t_1, t_2, \ldots, t_n$ by setting[2]

$$t_\ell := \operatorname{cyc}_\ell + \operatorname{cyc}_{\ell,\ell+1} + \operatorname{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \operatorname{cyc}_{\ell,\ell+1,\ldots,n} \in \mathbf{k}[S_n] \tag{3}$$

for each $\ell \in [n]$. Thus, in particular, $t_n = \operatorname{cyc}_n = \operatorname{id} = 1$ (where 1 means the unity of $\mathbf{k}[S_n]$). We shall refer to the n elements $t_1, t_2, \ldots, t_n$ as the *somewhere-to-below shuffles*. These shuffles were studied in [GriLaf22] (where, in particular, their probabilistic meaning was discussed, which explains the origin of their name).

2.3. Commutators

If a and b are two elements of some ring, then $[a, b]$ shall denote their commutator $ab - ba$. This notation clashes with our above-defined notation $[a, b]$ for the interval $\{k \in \mathbb{Z} \mid a \leq k \leq b\}$ (when a and b are two integers), but we don't expect any confusion to arise in practice, since we will only use the notation $[a, b]$ for $ab - ba$ when a and b are visibly elements of the ring $\mathbf{k}[S_n]$ (as opposed to integers).

3. Elementary computations in S_n

In this section, we will perform some simple computations in the symmetric group S_n. The results of these computations will later become ingredients in some of our proofs.

[2]We view S_n as a subset of $\mathbf{k}[S_n]$ in the obvious way.

3.1. The cycles $(v \Longrightarrow w)$

Definition 3.1. Let $v, w \in [n]$ satisfy $v \leq w$. Then, $(v \Longrightarrow w)$ shall denote the permutation $\operatorname{cyc}_{v,v+1,\ldots,w}$.

The symbol "$\Longrightarrow$" in this notation $(v \Longrightarrow w)$ has nothing to do with logical implication; instead, it is meant to summon an image of a "current" flowing from v to w. The symbol "$\Longrightarrow$" is understood to bind less strongly than addition or subtraction; thus, for example, the expression "$(v+1 \Longrightarrow w)$" means $((v+1) \Longrightarrow w)$.

Every $v \in [n]$ satisfies

$$(v \Longrightarrow v) = \operatorname{cyc}_v = \operatorname{id} = 1. \tag{4}$$

The following is just a little bit less obvious:

Proposition 3.2. Let $v, w \in [n]$ satisfy $v \leq w$. Then, $(v \Longrightarrow w) = s_v s_{v+1} \cdots s_{w-1}$.

Proof. Easy verification. □

Proposition 3.3. Let $v, w \in [n]$ satisfy $v < w$. Then:

(a) We have $(v \Longrightarrow w) = s_v (v+1 \Longrightarrow w)$.

(b) We have $(v \Longrightarrow w) = (v \Longrightarrow w-1) s_{w-1}$.

Proof. Easy verification (easiest using Proposition 3.2). □

3.2. Rewriting rules for products of cycles

Next we recall how conjugation in S_n acts on cycles:

Proposition 3.4. Let $\sigma \in S_n$. Let $i_1, i_2, \ldots, i_k$ be k distinct elements of $[n]$. Then,

$$\sigma \operatorname{cyc}_{i_1,i_2,\ldots,i_k} \sigma^{-1} = \operatorname{cyc}_{\sigma(i_1),\sigma(i_2),\ldots,\sigma(i_k)}. \tag{5}$$

Proof. Well-known. □

Proposition 3.4 allows us to prove several relations between the cycles $(v \Longrightarrow w)$. We shall collect a catalogue of such relations now in order to have them at arm's reach in later proofs.

Lemma 3.5. Let $i, j, v, w \in [n]$ be such that $w \geq v > j \geq i$. Then,

$$(j+1 \Longrightarrow v)(i \Longrightarrow w) = (i \Longrightarrow w)(j \Longrightarrow v-1).$$

Proof. Let $\sigma := (i \Longrightarrow w)$. We have $i \leq j$ (since $j \geq i$) and $v - 1 \leq w - 1$ (since $w \geq v$). Thus, the numbers $j, j+1, \ldots, v-1$ all belong to the interval $[i, w-1]$. Hence, the permutation $\sigma = (i \Longrightarrow w) = \operatorname{cyc}_{i,i+1,\ldots,w}$ sends these numbers to $j+1, j+2, \ldots, v$, respectively. In other words,

$$(\sigma(j), \sigma(j+1), \ldots, \sigma(v-1)) = (j+1, j+2, \ldots, v).$$

However, from $(i \Longrightarrow w) = \sigma$ and $(j \Longrightarrow v-1) = \operatorname{cyc}_{j,j+1,\ldots,v-1}$, we obtain

$$\begin{aligned}
&(i \Longrightarrow w)(j \Longrightarrow v-1)(i \Longrightarrow w)^{-1} \\
&= \sigma \operatorname{cyc}_{j,j+1,\ldots,v-1} \sigma^{-1} = \operatorname{cyc}_{\sigma(j),\sigma(j+1),\ldots,\sigma(v-1)} \qquad \text{(by (5))} \\
&= \operatorname{cyc}_{j+1,j+2,\ldots,v} \qquad \text{(since } (\sigma(j), \sigma(j+1), \ldots, \sigma(v-1)) = (j+1, j+2, \ldots, v)\text{)} \\
&= (j+1 \Longrightarrow v).
\end{aligned}$$

In other words, $(i \Longrightarrow w)(j \Longrightarrow v-1) = (j+1 \Longrightarrow v)(i \Longrightarrow w)$. Thus, Lemma 3.5 is proved. □

Lemma 3.6. Let $i, v, w \in [n]$ be such that $v > w \geq i$. Then,

$$(i+1 \Longrightarrow v)(i \Longrightarrow w) = (i \Longrightarrow w+1)(i \Longrightarrow v).$$

Proof. We have $i < v$ (since $v > i$). Thus, Proposition 3.3 **(a)** yields

$$(i \Longrightarrow v) = s_i (i+1 \Longrightarrow v). \tag{6}$$

On the other hand, from $v > w$, we obtain $v \geq w+1$, so that $w+1 \leq v \leq n$ and therefore $w+1 \in [n]$. Furthermore, $v \geq w+1 > w \geq i \geq i$. Thus, Lemma 3.5 (applied to i, $w+1$ and v instead of j, v and w) yields

$$\begin{aligned}
(i+1 \Longrightarrow w+1)(i \Longrightarrow v) &= (i \Longrightarrow v)\left(i \Longrightarrow \underbrace{(w+1)-1}_{=w}\right) \\
&= (i \Longrightarrow v)(i \Longrightarrow w).
\end{aligned} \tag{7}$$

However, Proposition 3.3 **(a)** yields $(i \Longrightarrow w+1) = s_i (i+1 \Longrightarrow w+1)$ (since $i \leq w < w+1$). Hence,

$$\begin{aligned}
\underbrace{(i \Longrightarrow w+1)}_{=s_i(i+1 \Longrightarrow w+1)}(i \Longrightarrow v) &= s_i \underbrace{(i+1 \Longrightarrow w+1)(i \Longrightarrow v)}_{\substack{=(i \Longrightarrow v)(i \Longrightarrow w) \\ \text{(by (7))}}} = s_i \underbrace{(i \Longrightarrow v)}_{\substack{=s_i(i+1 \Longrightarrow v) \\ \text{(by (6))}}}(i \Longrightarrow w) \\
&= \underbrace{s_i s_i}_{\substack{=s_i^2=\text{id} \\ \text{(by (2))}}}(i+1 \Longrightarrow v)(i \Longrightarrow w) = (i+1 \Longrightarrow v)(i \Longrightarrow w).
\end{aligned}$$

This proves Lemma 3.6. □

Lemma 3.7. Let $i, u, v \in [n]$ be such that $i < u < v$. Then,

$$s_u (i \Longrightarrow v) = (i \Longrightarrow v) s_{u-1}.$$

Proof. Let $\sigma := (i \Longrightarrow v)$. Then, $i \leq u - 1$ (since $i < u$) and $u \leq v - 1$ (since $u < v$). Therefore, the numbers $u - 1$ and u both belong to the interval $[i, v-1]$. Hence, the permutation $\sigma = (i \Longrightarrow v) = \operatorname{cyc}_{i,i+1,\ldots,w}$ sends these numbers to u and $u+1$, respectively. In other words,

$$\sigma (u-1) = u \qquad \text{and} \qquad \sigma (u) = u + 1.$$

However, from $(i \Longrightarrow v) = \sigma$ and $s_{u-1} = \operatorname{cyc}_{u-1,u}$, we obtain

$$\begin{aligned} (i \Longrightarrow v) s_{u-1} (i \Longrightarrow v)^{-1} &= \sigma \operatorname{cyc}_{u-1,u} \sigma^{-1} = \operatorname{cyc}_{\sigma(u-1),\sigma(u)} && \text{(by (5))} \\ &= \operatorname{cyc}_{u,u+1} && (\text{since } \sigma(u-1) = u \text{ and } \sigma(u) = u+1) \\ &= s_u. \end{aligned}$$

In other words, $(i \Longrightarrow v) s_{u-1} = s_u (i \Longrightarrow v)$. Thus, Lemma 3.7 is proved. □

Finally, the following fact is easy to check:

Lemma 3.8. Let $i, j, k \in [n]$ be such that $i \leq j \leq k$. Then,

$$(i \Longrightarrow k) = (i \Longrightarrow j) (j \Longrightarrow k).$$

Proof. Proposition 3.2 yields $(i \Longrightarrow j) = s_i s_{i+1} \cdots s_{j-1}$ and $(j \Longrightarrow k) = s_j s_{j+1} \cdots s_{k-1}$. Multiplying these equalities by each other, we find

$$(i \Longrightarrow j) (j \Longrightarrow k) = \left(s_i s_{i+1} \cdots s_{j-1}\right) \left(s_j s_{j+1} \cdots s_{k-1}\right) = s_i s_{i+1} \cdots s_{k-1}.$$

Comparing this with

$$(i \Longrightarrow k) = s_i s_{i+1} \cdots s_{k-1} \qquad \text{(by Proposition 3.2)},$$

we obtain $(i \Longrightarrow k) = (i \Longrightarrow j) (j \Longrightarrow k)$. This proves Lemma 3.8. □

4. Basic properties of somewhere-to-below shuffles

We now return to the group algebra $\mathbf{k}[S_n]$. We begin by rewriting the definition of the somewhere-to-below shuffle t_ℓ:

Proposition 4.1. Let $\ell \in [n]$. Then,

$$t_\ell = \sum_{w=\ell}^{n} (\ell \Longrightarrow w) .$$

Proof. From (3), we have

$$\begin{aligned} t_\ell &= \operatorname{cyc}_\ell + \operatorname{cyc}_{\ell,\ell+1} + \operatorname{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \operatorname{cyc}_{\ell,\ell+1,\ldots,n} \\ &= \sum_{w=\ell}^{n} \underbrace{\operatorname{cyc}_{\ell,\ell+1,\ldots,w}}_{\substack{=(\ell\Longrightarrow w) \\ \text{(by the definition of } (\ell\Longrightarrow w))}} = \sum_{w=\ell}^{n} (\ell \Longrightarrow w) . \end{aligned}$$

This proves Proposition 4.1. $\square$

Corollary 4.2. Let $\ell \in [n-1]$. Then, $t_\ell = 1 + s_\ell t_{\ell+1}$.

Proof. Proposition 4.1 yields

$$\begin{aligned} t_\ell &= \sum_{w=\ell}^{n} (\ell \Longrightarrow w) = \underbrace{(\ell \Longrightarrow \ell)}_{=1} + \sum_{w=\ell+1}^{n} \underbrace{(\ell \Longrightarrow w)}_{\substack{=s_\ell(\ell+1\Longrightarrow w) \\ \text{(by Proposition 3.3 \textbf{(a)})}}} \\ &= 1 + \sum_{w=\ell+1}^{n} s_\ell (\ell+1 \Longrightarrow w) = 1 + s_\ell \sum_{w=\ell+1}^{n} (\ell+1 \Longrightarrow w) . \end{aligned}$$

Comparing this with

$$1 + s_\ell \underbrace{t_{\ell+1}}_{\substack{=\sum_{w=\ell+1}^{n}(\ell+1\Longrightarrow w) \\ \text{(by Proposition 4.1)}}} = 1 + s_\ell \sum_{w=\ell+1}^{n} (\ell+1 \Longrightarrow w) ,$$

we obtain $t_\ell = 1 + s_\ell t_{\ell+1}$, qed. $\square$

We state another simple property of the t_ℓ's:

Lemma 4.3. Let $\ell \in [n]$. Let $\sigma \in S_n$. Assume that σ leaves all the elements $\ell, \ell+1, \ldots, n$ unchanged. Then, σ commutes with t_ℓ in $\mathbf{k}[S_n]$.

Proof. The permutation σ leaves all the elements $\ell, \ell+1, \ldots, n$ unchanged, and thus commutes with each cycle $\operatorname{cyc}_{\ell,\ell+1,\ldots,w}$ with $w \geq \ell$ (because the latter cycle permutes only elements of $\{\ell, \ell+1, \ldots, n\}$). Hence, the permutation σ also commutes with the sum $\sum_{w=\ell}^{n} \operatorname{cyc}_{\ell,\ell+1,\ldots,w}$ of these cycles. Since the definition of t_ℓ yields

$$t_\ell = \operatorname{cyc}_\ell + \operatorname{cyc}_{\ell,\ell+1} + \operatorname{cyc}_{\ell,\ell+1,\ell+2} + \cdots + \operatorname{cyc}_{\ell,\ell+1,\ldots,n} = \sum_{w=\ell}^{n} \operatorname{cyc}_{\ell,\ell+1,\ldots,w},$$

we can rewrite this as follows: The permutation σ commutes with t_ℓ. This proves Lemma 4.3. $\square$

Specifically, we will need only the following particular case of Lemma 4.3:

Lemma 4.4. Let $i,k,j \in [n]$ be such that $i \leq k < j$. Then,

$$(i \Longrightarrow k)\, t_j = t_j\, (i \Longrightarrow k) \tag{8}$$

and

$$\left[(i \Longrightarrow k)\,,\ t_j\right] = 0. \tag{9}$$

Proof. The permutation $(i \Longrightarrow k) = \operatorname{cyc}_{i,i+1,\ldots,k}$ leaves all the elements $k+1, k+2, \ldots, n$ unchanged, and thus leaves all the elements $j, j+1, \ldots, n$ unchanged (since the latter elements are a subset of the former elements (because $k < j$)). Hence, Lemma 4.3 (applied to $\ell = j$ and $\sigma = (i \Longrightarrow k)$) shows that $(i \Longrightarrow k)$ commutes with t_j in $\mathbf{k}[S_n]$. In other words, $(i \Longrightarrow k)\, t_j = t_j\, (i \Longrightarrow k)$. This proves (8).

Now, the definition of a commutator yields

$$\left[(i \Longrightarrow k)\,,\ t_j\right] = (i \Longrightarrow k)\, t_j - t_j\, (i \Longrightarrow k) = 0$$

(since $(i \Longrightarrow k)\, t_j = t_j\, (i \Longrightarrow k)$). This proves (9). Thus, Lemma 4.4 is completely proved. $\square$

5. The identities $t_{i+1}t_i = (t_i - 1)\, t_i = t_i\, (t_i - 1)$ and $\left[t_i, t_{i+1}\right]^2 = 0$

5.1. The identity $t_{i+1}t_i = (t_i - 1)\, t_i = t_i\, (t_i - 1)$

We are now ready to prove the first really surprising result:

Theorem 5.1. Let $i \in [n-1]$. Then,

$$\begin{aligned} t_{i+1}t_i &= (t_i - 1)\, t_i \qquad (10)\\ &= t_i\, (t_i - 1)\,. \qquad (11) \end{aligned}$$

Proof. From Proposition 4.1, we obtain

$$\begin{aligned} t_i &= \sum_{w=i}^{n} (i \Longrightarrow w) \qquad (12)\\ &= \underbrace{(i \Longrightarrow i)}_{=\mathrm{id}=1} + \sum_{w=i+1}^{n} (i \Longrightarrow w) \qquad \left(\begin{array}{c}\text{here, we have split off the}\\ \text{addend for } w = i \text{ from the sum}\end{array}\right)\\ &= 1 + \sum_{w=i+1}^{n} (i \Longrightarrow w)\,. \end{aligned}$$

In other words,

$$t_i - 1 = \sum_{w=i+1}^{n} (i \Longrightarrow w) . \tag{13}$$

Moreover, (12) becomes

$$t_i = \sum_{w=i}^{n} (i \Longrightarrow w) = \sum_{v=i}^{n} (i \Longrightarrow v) . \tag{14}$$

Also, Proposition 4.1 (applied to $\ell = i+1$) yields

$$t_{i+1} = \sum_{w=i+1}^{n} (i+1 \Longrightarrow w) = \sum_{v=i+1}^{n} (i+1 \Longrightarrow v) .$$

Multiplying this equality by (12), we obtain

$$\begin{aligned}
t_{i+1} t_i &= \sum_{v=i+1}^{n} (i+1 \Longrightarrow v) \cdot \sum_{w=i}^{n} (i \Longrightarrow w) \\
&= \sum_{v=i+1}^{n} \underbrace{\sum_{w=i}^{n} (i+1 \Longrightarrow v)(i \Longrightarrow w)}_{\substack{= \sum_{w=i}^{v-1} (i+1 \Longrightarrow v)(i \Longrightarrow w) + \sum_{w=v}^{n} (i+1 \Longrightarrow v)(i \Longrightarrow w) \\ \text{(since } i < v \leq n\text{)}}} \\
&= \sum_{v=i+1}^{n} \left(\sum_{w=i}^{v-1} (i+1 \Longrightarrow v)(i \Longrightarrow w) + \sum_{w=v}^{n} (i+1 \Longrightarrow v)(i \Longrightarrow w) \right) \\
&= \sum_{v=i+1}^{n} \sum_{w=i}^{v-1} \underbrace{(i+1 \Longrightarrow v)(i \Longrightarrow w)}_{\substack{= (i \Longrightarrow w+1)(i \Longrightarrow v) \\ \text{(by Lemma 3.6} \\ \text{(since } w \leq v-1 < v \text{ and thus } v > w \geq i\text{))}}} + \sum_{v=i+1}^{n} \sum_{w=v}^{n} \underbrace{(i+1 \Longrightarrow v)(i \Longrightarrow w)}_{\substack{= (i \Longrightarrow w)(i \Longrightarrow v-1) \\ \text{(by Lemma 3.5, applied to } j=i \\ \text{(since } w \geq v \geq i+1 > i \geq i\text{))}}} \\
&= \sum_{v=i+1}^{n} \underbrace{\sum_{w=i}^{v-1} (i \Longrightarrow w+1)(i \Longrightarrow v)}_{\substack{= \sum_{w=i+1}^{v} (i \Longrightarrow w)(i \Longrightarrow v) \\ \text{(here, we have substituted } w \text{ for } w+1\text{)}}} + \underbrace{\sum_{v=i+1}^{n} \sum_{w=v}^{n} (i \Longrightarrow w)(i \Longrightarrow v-1)}_{\substack{= \sum_{v=i}^{n-1} \sum_{w=v+1}^{n} (i \Longrightarrow w)(i \Longrightarrow v) \\ \text{(here, we have substituted } v \text{ for } v-1\text{)}}} \\
&= \sum_{v=i+1}^{n} \sum_{w=i+1}^{v} (i \Longrightarrow w)(i \Longrightarrow v) + \sum_{v=i}^{n-1} \sum_{w=v+1}^{n} (i \Longrightarrow w)(i \Longrightarrow v) .
\end{aligned}$$

Comparing this with

$$
\begin{aligned}
(t_i - 1)\, t_i &= \sum_{w=i+1}^{n} (i \Longrightarrow w) \cdot \sum_{v=i}^{n} (i \Longrightarrow v) \\
&\qquad \left(\text{by multiplying the two equalities (13) and (14)}\right) \\
&= \sum_{v=i}^{n} \underbrace{\sum_{w=i+1}^{n} (i \Longrightarrow w)\,(i \Longrightarrow v)}_{\substack{= \sum_{w=i+1}^{v} (i\Longrightarrow w)(i\Longrightarrow v) + \sum_{w=v+1}^{n} (i\Longrightarrow w)(i\Longrightarrow v) \\ \text{(since } i\leq v\leq n\text{)}}} \\
&= \sum_{v=i}^{n} \left(\sum_{w=i+1}^{v} (i \Longrightarrow w)\,(i \Longrightarrow v) + \sum_{w=v+1}^{n} (i \Longrightarrow w)\,(i \Longrightarrow v) \right) \\
&= \underbrace{\sum_{v=i}^{n} \sum_{w=i+1}^{v} (i \Longrightarrow w)\,(i \Longrightarrow v)}_{\substack{= \sum_{w=i+1}^{i} (i\Longrightarrow w)(i\Longrightarrow i) + \sum_{v=i+1}^{n} \sum_{w=i+1}^{v} (i\Longrightarrow w)(i\Longrightarrow v) \\ \text{(here, we have split off the addend for } v=i \text{ from the sum)}}} \\
&\qquad + \underbrace{\sum_{v=i}^{n} \sum_{w=v+1}^{n} (i \Longrightarrow w)\,(i \Longrightarrow v)}_{\substack{= \sum_{w=n+1}^{n} (i\Longrightarrow w)(i\Longrightarrow n) + \sum_{v=i}^{n-1} \sum_{w=v+1}^{n} (i\Longrightarrow w)(i\Longrightarrow v) \\ \text{(here, we have split off the addend for } v=n \text{ from the sum)}}} \\
&= \underbrace{\sum_{w=i+1}^{i} (i \Longrightarrow w)\,(i \Longrightarrow i)}_{=(\text{empty sum})=0} + \sum_{v=i+1}^{n} \sum_{w=i+1}^{v} (i \Longrightarrow w)\,(i \Longrightarrow v) \\
&\qquad + \underbrace{\sum_{w=n+1}^{n} (i \Longrightarrow w)\,(i \Longrightarrow n)}_{=(\text{empty sum})=0} + \sum_{v=i}^{n-1} \sum_{w=v+1}^{n} (i \Longrightarrow w)\,(i \Longrightarrow v) \\
&= \sum_{v=i+1}^{n} \sum_{w=i+1}^{v} (i \Longrightarrow w)\,(i \Longrightarrow v) + \sum_{v=i}^{n-1} \sum_{w=v+1}^{n} (i \Longrightarrow w)\,(i \Longrightarrow v),
\end{aligned}
$$

we obtain $t_{i+1}t_i = (t_i - 1)\, t_i$. This proves (10). From this, (11) follows, since $(t_i - 1)\, t_i = t_i^2 - t_i = t_i\,(t_i - 1)$. Thus, Theorem 5.1 is proved. □

5.2. The identity $[t_i, t_{i+1}]^2 = 0$

Corollary 5.2. Let $i \in [n-1]$. Then,

$$[t_i, t_{i+1}] = t_i (t_{i+1} - (t_i - 1)) \tag{15}$$

and

$$[t_i, t_{i+1}] t_i = 0 \tag{16}$$

and

$$[t_i, t_{i+1}]^2 = 0. \tag{17}$$

Proof. The definition of a commutator yields

$$[t_i, t_{i+1}] = t_i t_{i+1} - \underbrace{t_{i+1} t_i}_{\substack{= t_i (t_i - 1) \\ \text{(by (11))}}} = t_i t_{i+1} - t_i (t_i - 1) = t_i (t_{i+1} - (t_i - 1)).$$

This proves the equality (15). Multiplying both sides of this equality by t_i on the right, we obtain

$$[t_i, t_{i+1}] t_i = t_i \underbrace{(t_{i+1} - (t_i - 1)) t_i}_{= t_{i+1} t_i - (t_i - 1) t_i} = t_i \underbrace{(t_{i+1} t_i - (t_i - 1) t_i)}_{\substack{=0 \\ \text{(by (10))}}} = 0.$$

This proves (16). Now,

$$[t_i, t_{i+1}]^2 = [t_i, t_{i+1}] \underbrace{[t_i, t_{i+1}]}_{\substack{= t_i (t_{i+1} - (t_i - 1)) \\ \text{(by (15))}}} = \underbrace{[t_i, t_{i+1}] t_i}_{\substack{=0 \\ \text{(by (16))}}} (t_{i+1} - (t_i - 1)) = 0.$$

This proves (17). Thus, Corollary 5.2 is proved. □

6. The identities $t_{i+2} (t_i - 1) = (t_i - 1) (t_{i+1} - 1)$ and $[t_i, t_{i+2}] (t_i - 1) = t_{i+1} [t_i, t_{i+1}]$

6.1. The identity $t_{i+2} (t_i - 1) = (t_i - 1) (t_{i+1} - 1)$

The next theorem is a "next-level" analogue of Theorem 5.1:

Theorem 6.1. Let $i \in [n-2]$. Then,

$$t_{i+2} (t_i - 1) = (t_i - 1) (t_{i+1} - 1).$$

Proof of Theorem 6.1. From $i \in [n-2]$, we obtain $i + 1 \in [2, n-1] \subseteq [n-1]$. Hence, (11) (applied to $i + 1$ instead of i) yields $t_{(i+1)+1} t_{i+1} = t_{i+1} (t_{i+1} - 1)$. In view of $(i + 1) + 1 = i + 2$, we can rewrite this as

$$t_{i+2} t_{i+1} = t_{i+1} (t_{i+1} - 1). \tag{18}$$

Furthermore, Corollary 4.2 (applied to $\ell = i$) yields $t_i = 1 + s_i t_{i+1}$ (since $i \in [n-2] \subseteq [n-1]$). Hence,

$$t_i - 1 = s_i t_{i+1}. \tag{19}$$

The definition of $(i \Longrightarrow i+1)$ yields $(i \Longrightarrow i+1) = \operatorname{cyc}_{i,i+1,\ldots,i+1} = \operatorname{cyc}_{i,i+1} = s_i$. However, (8) (applied to $k = i+1$ and $j = i+2$) yields $(i \Longrightarrow i+1)\, t_{i+2} = t_{i+2}\,(i \Longrightarrow i+1)$. In view of $(i \Longrightarrow i+1) = s_i$, we can rewrite this as $s_i t_{i+2} = t_{i+2} s_i$. In other words, $t_{i+2} s_i = s_i t_{i+2}$.

Now,

$$t_{i+2} \underbrace{(t_i - 1)}_{\substack{= s_i t_{i+1} \\ \text{(by (19))}}} = \underbrace{t_{i+2} s_i}_{= s_i t_{i+2}} t_{i+1} = s_i \underbrace{t_{i+2} t_{i+1}}_{\substack{= t_{i+1}(t_{i+1}-1) \\ \text{(by (18))}}} = \underbrace{s_i t_{i+1}}_{\substack{= t_i - 1 \\ \text{(by (19))}}} (t_{i+1} - 1) = (t_i - 1)(t_{i+1} - 1).$$

Thus, Theorem 6.1 is proved. □

Remark 6.2. The similarity between Theorem 5.1 and Theorem 6.1 might suggest that the two theorems are the first two instances of a general identity of the form $t_{i+k}(t_i - \ell) = (t_i - m)$ (something) for certain integers ℓ and m. Unfortunately, such an identity most likely does not exist for $k = 3$. Indeed, using SageMath, we have verified that for $n = 6$ and $\mathbf{k} = \mathbb{Q}$, the product $t_4(t_1 - \ell)$ is not a right multiple of $t_1 - m$ for any $m \in \{0,1,2,3,4,6\}$ and $\ell \in [0,12]$. (The restriction $m \in \{0,1,2,3,4,6\}$ ensures that $t_1 - m$ is not invertible; otherwise, the claim would be trivial and uninteresting.)

We also observe that $t_4 t_1$ does not belong to the $\mathbb{Q}$-linear span of the elements 1, t_i and $t_i t_j$ for $i \leq j$ when $n = 5$. This is another piece of evidence suggesting that the pattern of Theorems 5.1 and 6.1 does not continue.

Generally, for $\mathbf{k} = \mathbb{Q}$, the span of all products of the form $t_i t_j$ with $i, j \in [n]$ inside $\mathbb{Q}[S_n]$ seems to have dimension $n^2 - 3n + 4$ (verified using SageMath for all $n \leq 14$). The discrepancy between this dimension and the naive maximum guess n^2 is fully explained by the $n-1$ identities $t_{i+1} t_i = t_i^2 - t_i$ from Theorem 5.1, the $n-2$ identities $t_{i+2} t_i - t_{i+2} = t_i t_{i+1} - t_i - t_{i+1} + 1$ from Theorem 6.1, and the $n-1$ obvious identities $t_i t_n = t_n t_i$ that come from $t_n = 1$ (assuming that these $3n-4$ identities are linearly independent). This suggests that Theorem 5.1 and Theorem 6.1 exhaust the interesting quadratic identities between the t_i.

6.2. The identity $[t_i, t_{i+2}](t_i - 1) = t_{i+1}[t_i, t_{i+1}]$

Corollary 6.3. Let $i \in [n-2]$. Then,

$$[t_i, t_{i+2}](t_i - 1) = t_{i+1}[t_i, t_{i+1}].$$

Proof. From $i \in [n-2]$, we obtain $i + 1 \in [2, n-1] \subseteq [n-1]$. Thus, (11) (applied to $i+1$ instead of i) yields $t_{(i+1)+1} t_{i+1} = t_{i+1}(t_{i+1} - 1)$. In view of $(i+1) + 1 = i + 2$, we can rewrite this as

$$t_{i+2} t_{i+1} = t_{i+1}(t_{i+1} - 1). \tag{20}$$

However, we have

$$t_i \underbrace{t_{i+2}(t_i-1)}_{\substack{=(t_i-1)(t_{i+1}-1) \\ \text{(by Theorem 6.1)}}} = \underbrace{t_i(t_i-1)}_{\substack{=t_{i+1}t_i \\ \text{(by (11))}}}(t_{i+1}-1) = t_{i+1}t_i(t_{i+1}-1)$$
$$= t_{i+1}t_it_{i+1} - t_{i+1}t_i \tag{21}$$

and

$$t_{i+2}\underbrace{t_i(t_i-1)}_{\substack{=t_{i+1}t_i \\ \text{(by (11))}}} = \underbrace{t_{i+2}t_{i+1}}_{\substack{=t_{i+1}(t_{i+1}-1) \\ \text{(by (20))}}} t_i = t_{i+1}(t_{i+1}-1)t_i$$
$$= t_{i+1}t_{i+1}t_i - t_{i+1}t_i. \tag{22}$$

Now, the definition of a commutator yields $[t_i, t_{i+1}] = t_it_{i+1} - t_{i+1}t_i$ and $[t_i, t_{i+2}] = t_it_{i+2} - t_{i+2}t_i$. Hence,

$$\begin{aligned}
\underbrace{[t_i, t_{i+2}]}_{=t_it_{i+2}-t_{i+2}t_i}(t_i-1) &= (t_it_{i+2} - t_{i+2}t_i)(t_i-1) \\
&= t_it_{i+2}(t_i-1) - t_{i+2}t_i(t_i-1) \\
&= (t_{i+1}t_it_{i+1} - t_{i+1}t_i) - (t_{i+1}t_{i+1}t_i - t_{i+1}t_i) \\
&\qquad \text{(here, we subtracted the equality (22) from (21))} \\
&= t_{i+1}t_it_{i+1} - t_{i+1}t_{i+1}t_i = t_{i+1}\underbrace{(t_it_{i+1} - t_{i+1}t_i)}_{=[t_i,t_{i+1}]} = t_{i+1}[t_i, t_{i+1}].
\end{aligned}$$

This proves Corollary 6.3. □

7. The identity $(1+s_j)[t_i, t_j] = 0$ for all $i \le j$

7.1. The identity $(1+s_j)[t_{j-1}, t_j] = 0$

We shall next prove the following:

Lemma 7.1. Let $i \in [n-2]$. Then,

$$(1+s_{i+1})[t_i, t_{i+1}] = 0.$$

Proof. Set $a := t_i$ and $b := t_{i+1}$.

From (2), we obtain $s_{i+1}^2 = \mathrm{id} = 1$. Hence, $s_{i+1}^2 - 1 = 0$.

The definition of a commutator yields

$$\begin{aligned}
[a-1,\ b-1] &= (a-1)(b-1) - (b-1)(a-1) \\
&= (ab - a - b + 1) - (ba - b - a + 1) \\
&= ab - ba = [a, b].
\end{aligned} \tag{23}$$

From $i \in [n-2]$, we obtain $i+1 \in [2, n-1] \subseteq [n-1]$. Hence, Corollary 4.2 (applied to $\ell = i+1$) yields $t_{i+1} = 1 + s_{i+1} t_{(i+1)+1} = 1 + s_{i+1} t_{i+2}$ (since $(i+1)+1 = i+2$). Hence, $b = t_{i+1} = 1 + s_{i+1} t_{i+2}$. Therefore, $b - 1 = s_{i+1} t_{i+2}$. Thus,

$$s_{i+1} \underbrace{(b-1)}_{= s_{i+1} t_{i+2}} = \underbrace{s_{i+1} s_{i+1}}_{= s_{i+1}^2 = 1} t_{i+2} = t_{i+2}. \tag{24}$$

However, Theorem 6.1 yields

$$t_{i+2} (t_i - 1) = (t_i - 1)(t_{i+1} - 1).$$

In view of $a = t_i$ and $b = t_{i+1}$, we can rewrite this as

$$t_{i+2} (a-1) = (a-1)(b-1).$$

Hence,

$$(a-1)(b-1) = \underbrace{t_{i+2}}_{\substack{= s_{i+1}(b-1) \\ \text{(by (24))}}} (a-1) = s_{i+1} (b-1)(a-1). \tag{25}$$

Now, (23) becomes

$$\begin{aligned} [a, b] &= [a-1, \ b-1] = \underbrace{(a-1)(b-1)}_{\substack{= s_{i+1}(b-1)(a-1) \\ \text{(by (25))}}} - (b-1)(a-1) \\ &= s_{i+1}(b-1)(a-1) - (b-1)(a-1) \\ &= (s_{i+1} - 1)(b-1)(a-1). \end{aligned}$$

Multiplying both sides of this equality by $1 + s_{i+1}$ from the left, we obtain

$$(1 + s_{i+1}) [a, b] = \underbrace{(1 + s_{i+1})(s_{i+1} - 1)}_{\substack{= (s_{i+1}+1)(s_{i+1}-1) \\ = s_{i+1}^2 - 1 = 0}} (b-1)(a-1) = 0.$$

In view of $a = t_i$ and $b = t_{i+1}$, we can rewrite this as $(1 + s_{i+1}) [t_i, t_{i+1}] = 0$. This proves Lemma 7.1. $\square$

The following is just a restatement of Lemma 7.1:

Lemma 7.2. Let $j \in [2, n-1]$. Then,

$$(1 + s_j) [t_{j-1}, t_j] = 0.$$

Proof. We have $j - 1 \in [n-2]$ (since $j \in [2, n-1]$). Hence, Lemma 7.1 (applied to $i = j-1$) yields

$$\left(1 + s_{(j-1)+1}\right) \left[t_{j-1}, t_{(j-1)+1}\right] = 0.$$

In view of $(j-1)+1 = j$, we can rewrite this as $(1 + s_j) [t_{j-1}, t_j] = 0$. This proves Lemma 7.2. $\square$

7.2. Expressing $\left[t_i, t_j\right]$ via $\left[t_{j-1}, t_j\right]$

The following lemma is useful for reducing questions about $\left[t_i, t_j\right]$ to questions about $\left[t_{j-1}, t_j\right]$:

Lemma 7.3. Let $i, j \in [n]$ satisfy $i < j$. Then:

(a) We have
$$\left[t_i, t_j\right] = \left[s_i s_{i+1} \cdots s_{j-1}, t_j\right] t_j.$$

(b) We have
$$\left[t_i, t_j\right] = \left(s_i s_{i+1} \cdots s_{j-2}\right) \left[t_{j-1}, t_j\right].$$

Proof. A well-known identity for commutators says that if R is a ring, then any three elements $a, b, c \in R$ satisfy
$$[ab, c] = [a, c]\, b + a\, [b, c]. \tag{26}$$

Hence, if R is a ring, then any two elements $a, b \in R$ satisfy
$$\begin{aligned} [ab, b] &= [a, b]\, b + a \underbrace{[b, b]}_{\substack{= bb - bb \\ = 0}} \qquad \text{(by (26), applied to } c = b\text{)} \\ &= [a, b]\, b + a0 = [a, b]\, b. \end{aligned} \tag{27}$$

(a) Proposition 4.1 yields

$$\begin{aligned}
t_i &= \sum_{w=i}^{n} (i \Longrightarrow w) = \sum_{k=i}^{n} (i \Longrightarrow k) \\
&= \sum_{k=i}^{j-1} (i \Longrightarrow k) + \sum_{k=j}^{n} \underbrace{(i \Longrightarrow k)}_{\substack{=(i\Longrightarrow j)(j\Longrightarrow k) \\ \text{(by Lemma 3.8,} \\ \text{since } i\leq j\leq k)}} \qquad (\text{since } i < j \leq n) \\
&= \sum_{k=i}^{j-1} (i \Longrightarrow k) + \underbrace{\sum_{k=j}^{n} (i \Longrightarrow j)(j \Longrightarrow k)}_{=(i\Longrightarrow j) \sum_{k=j}^{n} (j\Longrightarrow k)} \\
&= \sum_{k=i}^{j-1} (i \Longrightarrow k) + (i \Longrightarrow j) \underbrace{\sum_{k=j}^{n} (j \Longrightarrow k)}_{\substack{= \sum_{w=j}^{n} (j\Longrightarrow w) = t_j \\ \text{(since Proposition 4.1} \\ \text{yields } t_j = \sum_{w=j}^{n} (j\Longrightarrow w))}} \\
&= \sum_{k=i}^{j-1} (i \Longrightarrow k) + (i \Longrightarrow j)\, t_j .
\end{aligned}$$

Thus,

$$\begin{aligned}
\left[t_i, t_j\right] &= \left[\sum_{k=i}^{j-1} (i \Longrightarrow k) + (i \Longrightarrow j)\, t_j,\ t_j\right] \\
&= \sum_{k=i}^{j-1} \underbrace{\left[(i \Longrightarrow k),\ t_j\right]}_{\substack{=0 \\ \text{(by (9) (since } i\leq k<j))}} + \left[(i \Longrightarrow j)\, t_j,\ t_j\right] \\
&\qquad (\text{since the commutator } [a,b] \text{ is bilinear in } a \text{ and } b) \\
&= \underbrace{\sum_{k=i}^{j-1} 0}_{=0} + \left[(i \Longrightarrow j)\, t_j,\ t_j\right] = \left[(i \Longrightarrow j)\, t_j,\ t_j\right] \\
&= \left[(i \Longrightarrow j), t_j\right] t_j \qquad \left(\text{by (27), applied to } a = (i \Longrightarrow j) \text{ and } b = t_j\right) \\
&= \left[s_i s_{i+1} \cdots s_{j-1}, t_j\right] t_j
\end{aligned}$$

(since Proposition 3.2 yields $(i \Longrightarrow j) = s_i s_{i+1} \cdots s_{j-1}$). This proves Lemma 7.3 **(a)**.

(b) Set $a := s_i s_{i+1} \cdots s_{j-2}$ and $b := s_{j-1}$ and $c := t_j$. Thus,

$$ab = \left(s_i s_{i+1} \cdots s_{j-2}\right) s_{j-1} = s_i s_{i+1} \cdots s_{j-1}. \tag{28}$$

However, $i \leq j-1$ (since $i < j$). Hence, Proposition 3.2 (applied to $v = i$ and $w = j-1$) yields $(i \Longrightarrow j-1) = s_i s_{i+1} \cdots s_{(j-1)-1} = s_i s_{i+1} \cdots s_{j-2} = a$ (since $a = s_i s_{i+1} \cdots s_{j-2}$).

Now, $i \leq j-1 < j$. Hence, (9) (applied to $k = j-1$) yields $\left[(i \Longrightarrow j-1),\ t_j\right] = 0$. In view of $(i \Longrightarrow j-1) = a$ and $t_j = c$, we can rewrite this as $[a,c] = 0$. Hence, (26) becomes

$$[ab,c] = \underbrace{[a,c]}_{=0} b + a\,[b,c] = a\,[b,c]\,. \tag{29}$$

On the other hand, applying Lemma 7.3 **(a)** to $j-1$ instead of j, we obtain

$$\left[t_{j-1}, t_j\right] = \left[s_{j-1}, t_j\right] t_j. \tag{30}$$

However, Lemma 7.3 **(a)** yields

$$\begin{aligned}
\left[t_i, t_j\right] &= \left[\underbrace{s_i s_{i+1} \cdots s_{j-1}}_{\substack{=ab \\ \text{(by (28))}}}, \underbrace{t_j}_{=c}\right] t_j = \underbrace{[ab,c]}_{\substack{=a[b,c] \\ \text{(by (29))}}} t_j = \underbrace{a}_{=s_i s_{i+1} \cdots s_{j-2}} \left[\underbrace{b}_{=s_{j-1}}, \underbrace{c}_{=t_j}\right] t_j \\
&= \left(s_i s_{i+1} \cdots s_{j-2}\right) \underbrace{\left[s_{j-1}, t_j\right] t_j}_{\substack{=\left[t_{j-1}, t_j\right] \\ \text{(by (30))}}} = \left(s_i s_{i+1} \cdots s_{j-2}\right) \left[t_{j-1}, t_j\right].
\end{aligned}$$

This proves Lemma 7.3 **(b)**. ☐

Corollary 7.4. Let $i \in [n]$ and $j \in [2,n]$ be such that $i \leq j$. Then,

$$\left[t_i, t_j\right] t_{j-1} = 0.$$

Proof. If $i = j$, then this is obvious (because $i = j$ entails $\left[t_i, t_j\right] = \left[t_j, t_j\right] = t_j t_j - t_j t_j = 0$ and therefore $\underbrace{\left[t_i, t_j\right]}_{=0} t_{j-1} = 0$). Hence, for the rest of this proof, we WLOG assume that $i \neq j$.

Combining $i \leq j$ with $i \neq j$, we obtain $i < j$. Hence, Lemma 7.3 **(b)** yields

$$\left[t_i, t_j\right] = \left(s_i s_{i+1} \cdots s_{j-2}\right) \left[t_{j-1}, t_j\right].$$

On the other hand, $j-1 \in [n-1]$ (since $j \in [2,n]$). Thus, (16) (applied to $j-1$ instead of i) yields $\left[t_{j-1}, t_{j-1+1}\right] t_{j-1} = 0$. In other words, $\left[t_{j-1}, t_j\right] t_{j-1} = 0$ (since $j-1+1 = j$). Thus,

$$\underbrace{\left[t_i, t_j\right]}_{=\left(s_i s_{i+1} \cdots s_{j-2}\right)\left[t_{j-1}, t_j\right]} t_{j-1} = \left(s_i s_{i+1} \cdots s_{j-2}\right) \underbrace{\left[t_{j-1}, t_j\right] t_{j-1}}_{=0} = 0.$$

This proves Corollary 7.4. ☐

7.3. The identity $\left(1+s_j\right)\left[t_i,t_j\right]=0$ for all $i\leq j$

We are now ready to prove the following surprising result:

Theorem 7.5. Let $i,j\in[n-1]$ satisfy $i\leq j$. Then,

$$\left(1+s_j\right)\left[t_i,t_j\right]=0.$$

Proof. If $i=j$, then this is obvious (since $i=j$ entails $\left[t_i,t_j\right]=\left[t_j,t_j\right]=0$). Hence, we WLOG assume that $i\neq j$. Thus, $i<j$ (since $i\leq j$).

The transpositions $s_i,s_{i+1},\ldots,s_{j-2}$ all commute with s_j (by reflection locality, since the numbers $i,i+1,\ldots,j-2$ differ by more than 1 from j). Thus, their product $s_is_{i+1}\cdots s_{j-2}$ commutes with s_j as well. In other words,

$$s_j\left(s_is_{i+1}\cdots s_{j-2}\right)=\left(s_is_{i+1}\cdots s_{j-2}\right)s_j.$$

Thus, in $\mathbf{k}\left[S_n\right]$, we have

$$\begin{aligned}\left(1+s_j\right)\left(s_is_{i+1}\cdots s_{j-2}\right)&=s_is_{i+1}\cdots s_{j-2}+\underbrace{s_j\left(s_is_{i+1}\cdots s_{j-2}\right)}_{=\left(s_is_{i+1}\cdots s_{j-2}\right)s_j}\\&=s_is_{i+1}\cdots s_{j-2}+\left(s_is_{i+1}\cdots s_{j-2}\right)s_j\\&=\left(s_is_{i+1}\cdots s_{j-2}\right)\left(1+s_j\right).\end{aligned}\tag{31}$$

However, Lemma 7.3 **(b)** yields $\left[t_i,t_j\right]=\left(s_is_{i+1}\cdots s_{j-2}\right)\left[t_{j-1},t_j\right]$ (since $i<j$). Hence,

$$\begin{aligned}\left(1+s_j\right)\underbrace{\left[t_i,t_j\right]}_{=\left(s_is_{i+1}\cdots s_{j-2}\right)\left[t_{j-1},t_j\right]}&=\underbrace{\left(1+s_j\right)\left(s_is_{i+1}\cdots s_{j-2}\right)}_{\substack{=\left(s_is_{i+1}\cdots s_{j-2}\right)\left(1+s_j\right)\\ \text{(by (31))}}}\left[t_{j-1},t_j\right]\\&=\left(s_is_{i+1}\cdots s_{j-2}\right)\underbrace{\left(1+s_j\right)\left[t_{j-1},t_j\right]}_{\substack{=0\\ \text{(by Lemma 7.2)}}}=0.\end{aligned}$$

This proves Theorem 7.5. ◻

Corollary 7.6. Let $n\geq 2$ and $i\in[n]$. Then, $t_{n-1}\left[t_i,t_{n-1}\right]=0$.

Proof. This is true for $i=n$ (because $t_n=1$ and thus $\left[t_n,t_{n-1}\right]=\left[1,t_{n-1}\right]=1t_{n-1}-t_{n-1}1=t_{n-1}-t_{n-1}=0$ and therefore $t_{n-1}\underbrace{\left[t_n,t_{n-1}\right]}_{=0}=0$). Hence, we WLOG assume that $i\neq n$. Therefore, $i\in[n]\setminus\{n\}=[n-1]$. Also, $n-1\in[n-1]$ (since $n\geq 2$).

The definition of t_{n-1} yields $t_{n-1}=\underbrace{\mathrm{cyc}_{n-1}}_{=1}+\underbrace{\mathrm{cyc}_{n-1,n}}_{=s_{n-1}}=1+s_{n-1}$.

However, Theorem 7.5 (applied to $j=n-1$) yields $\left(1+s_{n-1}\right)\left[t_i,t_{n-1}\right]=0$. In view of $t_{n-1}=1+s_{n-1}$, we can rewrite this as $t_{n-1}\left[t_i,t_{n-1}\right]=0$. This proves Corollary 7.6. ◻

8. The identity $\left[t_i, t_j\right]^{\lceil (n-j)/2 \rceil + 1} = 0$ for all $i, j \in [n]$

8.1. The elements s_k^+ and the left ideals $H_{k,j}$

We now introduce two crucial notions for the proof of our first main theorem:

Definition 8.1. We set $\mathbf{A} := \mathbf{k}\left[S_n\right]$. Furthermore, for any $i \in [n-1]$, we set

$$s_i^+ := s_i + 1 \in \mathbf{A}.$$

We also set $s_i^+ := 1 \in \mathbf{A}$ for all integers $i \notin [n-1]$. Thus, s_i^+ is defined for all integers i.

Definition 8.2. Let k and j be two integers. Then, we define

$$H_{k,j} := \sum_{\substack{u \in [j,k]; \\ u \equiv k \operatorname{mod} 2}} \mathbf{A} s_u^+.$$

This is a left ideal of $\mathbf{A}$. Note that

$$H_{k,j} = 0 \qquad \text{whenever } k < j. \tag{32}$$

Example 8.3. We have

$$H_{7,3} = \sum_{\substack{u \in [3,7]; \\ u \equiv 7 \operatorname{mod} 2}} \mathbf{A} s_u^+ = \mathbf{A} s_3^+ + \mathbf{A} s_5^+ + \mathbf{A} s_7^+ \qquad \text{and}$$

$$H_{7,2} = \sum_{\substack{u \in [2,7]; \\ u \equiv 7 \operatorname{mod} 2}} \mathbf{A} s_u^+ = \mathbf{A} s_3^+ + \mathbf{A} s_5^+ + \mathbf{A} s_7^+,$$

so that $H_{7,2} = H_{7,3}$. Similarly, $H_{7,4} = H_{7,5} = \mathbf{A} s_5^+ + \mathbf{A} s_7^+$ and $H_{7,6} = H_{7,7} = \mathbf{A} s_7^+$.

Let us prove some basic properties of the left ideals $H_{k,j}$:

Remark 8.4. Let k be an integer such that $k \notin [n-1]$. Let $j \in [n]$ satisfy $j \leq k$. Then, $H_{k,j} = \mathbf{A}$.

Proof. Since $k \notin [n-1]$, we have $s_k^+ = 1$ (by the definition of s_k^+). Also, $k \in [j,k]$ (since $j \leq k$).

Recall that $H_{k,j}$ is defined as the sum $\sum_{\substack{u \in [j,k]; \\ u \equiv k \operatorname{mod} 2}} \mathbf{A} s_u^+$. But this sum contains the addend $\mathbf{A} s_k^+$ (since $k \in [j,k]$ and $k \equiv k \operatorname{mod} 2$). Hence, $\sum_{\substack{u \in [j,k]; \\ u \equiv k \operatorname{mod} 2}} \mathbf{A} s_u^+ \supseteq \mathbf{A} \underbrace{s_k^+}_{=1} =$

$\mathbf{A}1 = \mathbf{A}$. Now,

$$H_{k,j} = \sum_{\substack{u \in [j,k]; \\ u \equiv k \operatorname{mod} 2}} \mathbf{A}s_u^+ \supseteq \mathbf{A},$$

so that $H_{k,j} = \mathbf{A}$. This proves Remark 8.4. □

Lemma 8.5. Let k and j be two integers. Then, $H_{k,j} \subseteq H_{k,j-1}$.

Proof. Definition 8.2 yields $H_{k,j} = \sum_{\substack{u \in [j,k]; \\ u \equiv k \operatorname{mod} 2}} \mathbf{A}s_u^+$ and $H_{k,j-1} = \sum_{\substack{u \in [j-1,k]; \\ u \equiv k \operatorname{mod} 2}} \mathbf{A}s_u^+$. But clearly, any addend of the former sum is an addend of the latter sum as well (since each $u \in [j,k]$ satisfies $u \in [j,k] \subseteq [j-1,k]$). Thus, the former sum is a subset of the latter. In other words, $H_{k,j} \subseteq H_{k,j-1}$. This proves Lemma 8.5. □

Lemma 8.6. Let v, w and j be three integers such that $v \leq w$ and $v \equiv w \operatorname{mod} 2$. Then, $H_{v,j} \subseteq H_{w,j}$.

Proof. Definition 8.2 yields

$$H_{v,j} = \sum_{\substack{u \in [j,v]; \\ u \equiv v \operatorname{mod} 2}} \mathbf{A}s_u^+ \qquad \text{and} \tag{33}$$

$$H_{w,j} = \sum_{\substack{u \in [j,w]; \\ u \equiv w \operatorname{mod} 2}} \mathbf{A}s_u^+. \tag{34}$$

However, each $u \in [j,v]$ satisfying $u \equiv v \operatorname{mod} 2$ is also an element of $[j,w]$ (since $u \leq v \leq w$) and satisfies $u \equiv w \operatorname{mod} 2$ (since $u \equiv v \equiv w \operatorname{mod} 2$). Thus, any addend of the sum $\sum_{\substack{u \in [j,v]; \\ u \equiv v \operatorname{mod} 2}} \mathbf{A}s_u^+$ is also an addend of the sum $\sum_{\substack{u \in [j,w]; \\ u \equiv w \operatorname{mod} 2}} \mathbf{A}s_u^+$. Therefore, the former sum is a subset of the latter sum. In other words, $\sum_{\substack{u \in [j,v]; \\ u \equiv v \operatorname{mod} 2}} \mathbf{A}s_u^+ \subseteq \sum_{\substack{u \in [j,w]; \\ u \equiv w \operatorname{mod} 2}} \mathbf{A}s_u^+$. In view of (33) and (34), we can rewrite this as $H_{v,j} \subseteq H_{w,j}$. This proves Lemma 8.6. □

Lemma 8.7. Let k and j be two integers such that $k \equiv j \operatorname{mod} 2$. Then, $H_{k,j-1} = H_{k,j}$.

Proof. Definition 8.2 yields

$$H_{k,j} = \sum_{\substack{u \in [j,k]; \\ u \equiv k \operatorname{mod} 2}} \mathbf{A}s_u^+ \qquad \text{and} \tag{35}$$

$$H_{k,j-1} = \sum_{\substack{u \in [j-1,k]; \\ u \equiv k \operatorname{mod} 2}} \mathbf{A}s_u^+. \tag{36}$$

However, each element $u \in [j,k]$ satisfying $u \equiv k \mod 2$ is also an element of $[j-1,k]$ (since $u \in [j,k] \subseteq [j-1,k]$). Conversely, each element $u \in [j-1,k]$ satisfying $u \equiv k \mod 2$ is also an element of $[j,k]$ (since otherwise, it would equal $j-1$, so that we would have $j-1 = u \equiv k \equiv j \mod 2$, but this would contradict $j-1 \not\equiv j \mod 2$). Therefore, the elements $u \in [j-1,k]$ satisfying $u \equiv k \mod 2$ are precisely the elements $u \in [j,k]$ satisfying $u \equiv k \mod 2$. In other words, the sum on the right hand side of (36) ranges over the same set as the sum on the right hand side of (35). Therefore, the right hand sides of the equalities (36) and (35) are equal. Hence, their left hand sides must also be equal. In other words, $H_{k,j-1} = H_{k,j}$. This proves Lemma 8.7. □

The following easy property follows from Lemma 3.7:

Lemma 8.8. Let $i,u,v \in [n]$ be such that $i < u < v$. Then,

$$s_u^+ (i \Longrightarrow v) = (i \Longrightarrow v) s_{u-1}^+.$$

Proof. We have $u \in [n-1]$ (since $u < v \leq n$ and $u > i \geq 1$) and thus $s_u^+ = s_u + 1$.

Also, $u > i \geq 1$, so that $u - 1 > 0$. Hence, $u - 1 \in [n-1]$ (since $u - 1 < u < v \leq n$ and $u - 1 > 0$) and thus $s_{u-1}^+ = s_{u-1} + 1$. Hence,

$$\begin{aligned}
&\underbrace{s_u^+}_{=s_u+1} (i \Longrightarrow v) - (i \Longrightarrow v) \underbrace{s_{u-1}^+}_{=s_{u-1}+1} \\
&= (s_u + 1)(i \Longrightarrow v) - (i \Longrightarrow v)(s_{u-1} + 1) \\
&= s_u (i \Longrightarrow v) + (i \Longrightarrow v) - (i \Longrightarrow v) s_{u-1} - (i \Longrightarrow v) \\
&= \underbrace{s_u (i \Longrightarrow v)}_{\substack{=(i \Longrightarrow v) s_{u-1} \\ \text{(by Lemma 3.7)}}} - (i \Longrightarrow v) s_{u-1} = (i \Longrightarrow v) s_{u-1} - (i \Longrightarrow v) s_{u-1} = 0.
\end{aligned}$$

Thus, $s_u^+ (i \Longrightarrow v) = (i \Longrightarrow v) s_{u-1}^+$. This proves Lemma 8.8. □

Let us also derive a simple consequence from Corollary 4.2:

Lemma 8.9. Let $j \in [2,n]$. Then,

$$s_{j-1}^+ \left(t_j - \left(t_{j-1} - 1\right)\right) = 0.$$

Proof. From $j \in [2,n]$, we obtain $j - 1 \in [n-1]$. Hence, the definition of s_{j-1}^+ yields $s_{j-1}^+ = s_{j-1} + 1 = 1 + s_{j-1}$.

Corollary 4.2 (applied to $\ell = j-1$) yields $t_{j-1} = 1 + s_{j-1} t_{(j-1)+1} = 1 + s_{j-1} t_j$ (since $(j-1)+1 = j$). Thus, $t_{j-1} - 1 = s_{j-1} t_j$, so that

$$t_j - \underbrace{\left(t_{j-1} - 1\right)}_{=s_{j-1} t_j} = t_j - s_{j-1} t_j = \left(1 - s_{j-1}\right) t_j.$$

Thus,

$$\underbrace{s_{j-1}^{+}}_{=1+s_{j-1}} \underbrace{\left(t_j - \left(t_{j-1} - 1\right)\right)}_{=\left(1-s_{j-1}\right)t_j} = \underbrace{\left(1 + s_{j-1}\right)\left(1 - s_{j-1}\right)}_{\substack{=1-s_{j-1}^2=0 \\ \text{(since (2) yields } s_{j-1}^2=\text{id}=1)}} t_j = 0.$$

This proves Lemma 8.9. □

8.2. The fuse

The next lemma will help us analyze the behavior of the ideals $H_{k,j}$ under repeated multiplication by t_j's:

Lemma 8.10. Let $j \in [n]$ and $k \in [n+1]$ be such that $j < k$. Then:

(a) If $k \not\equiv j \bmod 2$, then $s_k^+ t_j \in H_{k-1,j}$.

(b) If $k \equiv j \bmod 2$, then $s_k^+ \left(t_j - 1\right) \in H_{k-1,j}$.

Proof. If $k = n+1$, then both parts of Lemma 8.10 hold for fairly obvious reasons[3]. Hence, for the rest of this proof, we WLOG assume that $k \neq n+1$. Therefore, $k \in [n+1] \setminus \{n+1\} = [n]$, so that $k \leq n$.

Recall that $H_{k-1,j}$ is a left ideal of $\mathbf{A}$, therefore an additive subgroup of $\mathbf{A}$.

We have $j < k$, so that $j \leq k-1$. Hence, $k-1 \in [j, k-1]$.

The definition of $H_{k-1,j}$ yields

$$H_{k-1,j} = \sum_{\substack{u \in [j,k-1]; \\ u \equiv k-1 \bmod 2}} \mathbf{A} s_u^+ . \tag{37}$$

Since $\mathbf{A}s_{k-1}^+$ is an addend of the sum on the right hand side here (because $k-1 \in [j, k-1]$ and $k-1 \equiv k-1 \bmod 2$), we thus conclude that $\mathbf{A}s_{k-1}^+ \subseteq H_{k-1,j}$.

Proposition 4.1 yields

$$t_j = \sum_{w=j}^{n} (j \Longrightarrow w) .$$

[3] *Proof.* Assume that $k = n+1$. Then, $k-1 = n \notin [n-1]$ and $j \leq k-1$ (since $j < k$), so that $H_{k-1,j} = \mathbf{A}$ (by Remark 8.4, applied to $k-1$ instead of k). Thus, both elements $s_k^+ t_j$ and $s_k^+ \left(t_j - 1\right)$ belong to $H_{k-1,j}$ (since they both belong to $\mathbf{A}$). Therefore, both parts of Lemma 8.10 hold. Qed.

Hence,

$$\begin{aligned}
s_k^+ t_j &= s_k^+ \sum_{w=j}^{n} (j \Longrightarrow w) = \sum_{w=j}^{n} s_k^+ (j \Longrightarrow w) \\
&= \underbrace{\sum_{w=j}^{k} s_k^+ (j \Longrightarrow w)}_{= s_k^+ \sum\limits_{w=j}^{k} (j \Longrightarrow w)} + \sum_{w=k+1}^{n} \underbrace{s_k^+ (j \Longrightarrow w)}_{\substack{= (j \Longrightarrow w) s_{k-1}^+ \\ \text{(by Lemma 8.8,} \\ \text{since } j<k<w)}} \qquad (\text{since } j \leq k \leq n) \\
&= s_k^+ \sum_{w=j}^{k} (j \Longrightarrow w) + \sum_{w=k+1}^{n} (j \Longrightarrow w) \, s_{k-1}^+ . \qquad (38)
\end{aligned}$$

We now need a better understanding of the sums on the right hand side. For this purpose, we observe that every $w \in [j, n-1]$ satisfies

$$\begin{aligned}
(j \Longrightarrow w) + \underbrace{(j \Longrightarrow w+1)}_{\substack{= (j \Longrightarrow w) s_w \\ \text{(by Proposition 3.3 \textbf{(b)})}}} &= (j \Longrightarrow w) + (j \Longrightarrow w) \, s_w \\
&= (j \Longrightarrow w) \underbrace{(1 + s_w)}_{\substack{= s_w + 1 = s_w^+ \\ \text{(by the definition of } s_w^+)}} = (j \Longrightarrow w) \, s_w^+ \\
&\in \mathbf{A} s_w^+ . \qquad (39)
\end{aligned}$$

(a) Assume that $k \not\equiv j \bmod 2$. Thus, the integer $k - j$ is odd, so that $k - j + 1$ is even.

Now,

$$\begin{aligned}
\sum_{w=j}^{k}(j\Longrightarrow w) &= (j\Longrightarrow j)+(j\Longrightarrow j+1)+(j\Longrightarrow j+2)+\cdots+(j\Longrightarrow k)\\
&= \underbrace{((j\Longrightarrow j)+(j\Longrightarrow j+1))}_{\substack{\in \mathbf{A}s_j^+ \\ \text{(by (39))}}}\\
&\qquad + \underbrace{((j\Longrightarrow j+2)+(j\Longrightarrow j+3))}_{\substack{\in \mathbf{A}s_{j+2}^+ \\ \text{(by (39))}}}\\
&\qquad + \underbrace{((j\Longrightarrow j+4)+(j\Longrightarrow j+5))}_{\substack{\in \mathbf{A}s_{j+4}^+ \\ \text{(by (39))}}}\\
&\qquad + \cdots\\
&\qquad + \underbrace{((j\Longrightarrow k-1)+(j\Longrightarrow k))}_{\substack{\in \mathbf{A}s_{k-1}^+ \\ \text{(by (39))}}}\\
&\qquad\qquad \left(\begin{array}{c}\text{here, we have split our sum into pairs of}\\ \text{consecutive addends, since } k-j+1 \text{ is even}\end{array}\right)\\
&\in \mathbf{A}s_j^+ + \mathbf{A}s_{j+2}^+ + \mathbf{A}s_{j+4}^+ + \cdots + \mathbf{A}s_{k-1}^+ = \sum_{\substack{u\in[j,k-1];\\ u\equiv k-1 \bmod 2}} \mathbf{A}s_u^+\\
&= H_{k-1,j} \qquad \text{(by (37))}.
\end{aligned}$$

Now, (38) becomes

$$\begin{aligned}
s_k^+ t_j &= s_k^+ \underbrace{\sum_{w=j}^{k}(j\Longrightarrow w)}_{\in H_{k-1,j}} + \sum_{w=k+1}^{n} \underbrace{(j\Longrightarrow w)\, s_{k-1}^+}_{\in \mathbf{A}s_{k-1}^+ \subseteq H_{k-1,j}}\\
&\in s_k^+ H_{k-1,j} + \sum_{w=k+1}^{n} H_{k-1,j} \subseteq H_{k-1,j} \qquad \left(\text{since } H_{k-1,j} \text{ is a left ideal of } \mathbf{A}\right).
\end{aligned}$$

This proves Lemma 8.10 **(a)**.

(b) Assume that $k\equiv j \bmod 2$. Thus, the integer $k-j$ is even.

Now,

$$\begin{aligned}
\sum_{w=j+1}^{k} (j \Longrightarrow w) &= (j \Longrightarrow j+1) + (j \Longrightarrow j+2) + (j \Longrightarrow j+3) + \cdots + (j \Longrightarrow k) \\
&= \underbrace{((j \Longrightarrow j+1) + (j \Longrightarrow j+2))}_{\substack{\in \mathbf{A}s^+_{j+1} \\ \text{(by (39))}}} \\
&\qquad + \underbrace{((j \Longrightarrow j+3) + (j \Longrightarrow j+4))}_{\substack{\in \mathbf{A}s^+_{j+3} \\ \text{(by (39))}}} \\
&\qquad + \underbrace{((j \Longrightarrow j+5) + (j \Longrightarrow j+6))}_{\substack{\in \mathbf{A}s^+_{j+5} \\ \text{(by (39))}}} \\
&\qquad + \cdots \\
&\qquad + \underbrace{((j \Longrightarrow k-1) + (j \Longrightarrow k))}_{\substack{\in \mathbf{A}s^+_{k-1} \\ \text{(by (39))}}} \\
&\qquad \qquad \left(\begin{array}{c}\text{here, we have split our sum into pairs of} \\ \text{consecutive addends, since } k-j \text{ is even}\end{array}\right) \\
&\in \mathbf{A}s^+_{j+1} + \mathbf{A}s^+_{j+3} + \mathbf{A}s^+_{j+5} + \cdots + \mathbf{A}s^+_{k-1} = \sum_{\substack{u \in [j,k-1]; \\ u \equiv k-1 \bmod 2}} \mathbf{A}s^+_u \\
&= H_{k-1,j} \qquad \text{(by (37))} .
\end{aligned} \tag{40}$$

But

$$\begin{aligned}
\sum_{w=j}^{k} (j \Longrightarrow w) &= \underbrace{(j \Longrightarrow j)}_{=\mathrm{id}=1} + \sum_{w=j+1}^{k} (j \Longrightarrow w) \\
&\qquad \left(\begin{array}{c}\text{here, we have split off the} \\ \text{addend for } w=j \text{ from the sum}\end{array}\right) \\
&= 1 + \sum_{w=j+1}^{k} (j \Longrightarrow w) .
\end{aligned}$$

Now, (38) becomes

$$\begin{aligned} s_k^+ t_j &= s_k^+ \underbrace{\sum_{w=j}^{k} (j \Longrightarrow w)}_{=1+\sum\limits_{w=j+1}^{k}(j\Longrightarrow w)} + \sum_{w=k+1}^{n} (j \Longrightarrow w)\, s_{k-1}^+ \\ &= s_k^+ \left(1 + \sum_{w=j+1}^{k} (j \Longrightarrow w)\right) + \sum_{w=k+1}^{n} (j \Longrightarrow w)\, s_{k-1}^+ \\ &= s_k^+ + s_k^+ \sum_{w=j+1}^{k} (j \Longrightarrow w) + \sum_{w=k+1}^{n} (j \Longrightarrow w)\, s_{k-1}^+. \end{aligned}$$

Subtracting s_k^+ from both sides of this equality, we find

$$\begin{aligned} s_k^+ t_j - s_k^+ &= s_k^+ \underbrace{\sum_{w=j+1}^{k} (j \Longrightarrow w)}_{\substack{\in H_{k-1,j} \\ \text{(by (40))}}} + \sum_{w=k+1}^{n} \underbrace{(j \Longrightarrow w)\, s_{k-1}^+}_{\in \mathbf{A} s_{k-1}^+ \subseteq H_{k-1,j}} \\ &\in s_k^+ H_{k-1,j} + \sum_{w=k+1}^{n} H_{k-1,j} \\ &\subseteq H_{k-1,j} \qquad \left(\text{since } H_{k-1,j} \text{ is a left ideal of } \mathbf{A}\right). \end{aligned}$$

Hence, $s_k^+ \left(t_j - 1\right) = s_k^+ t_j - s_k^+ \in H_{k-1,j}$. This proves Lemma 8.10 **(b)**. □

From Lemma 8.10 and Lemma 8.9, we can easily obtain the following:

Lemma 8.11. Let $j \in [2, n]$ and $u \in [n]$ be such that $u \geq j - 1$ and $u \equiv j - 1 \bmod 2$. Then,

$$s_u^+ \left(t_j - \left(t_{j-1} - 1\right)\right) \in H_{u-1,j}.$$

Proof. If $u = j - 1$, then this follows from

$$\begin{aligned} s_{j-1}^+ \left(t_j - \left(t_{j-1} - 1\right)\right) &= 0 \qquad \text{(by Lemma 8.9)} \\ &\in H_{(j-1)-1,j} \qquad \left(\text{since } H_{(j-1)-1,j} \text{ is a left ideal}\right). \end{aligned}$$

Thus, for the rest of this proof, we WLOG assume that $u \neq j - 1$. Combining this with $u \geq j - 1$, we obtain $u > j - 1$. Therefore, $u \geq (j - 1) + 1 = j$. Moreover, $u \neq j$ (since $u \equiv j - 1 \not\equiv j \bmod 2$). Combining this with $u \geq j$, we obtain $u > j$. Thus, $u \geq j + 1$.

Also, from $j \in [2, n]$, we obtain $j - 1 \in [n - 1] \subseteq [n]$. From $u > j$, we obtain $j < u$. Moreover, $u \equiv j - 1 \not\equiv j \bmod 2$. Hence, Lemma 8.10 **(a)** (applied to $k = u$)

yields $s_u^+ t_j \in H_{u-1,j}$ (since $j < u$). Furthermore, Lemma 8.10 **(b)** (applied to u and $j-1$ instead of k and j) yields $s_u^+ \left(t_{j-1} - 1\right) \in H_{u-1,j-1}$ (since $u \equiv j - 1 \operatorname{mod} 2$ and $j - 1 < j < u$). Moreover, $H_{u-1,j} \subseteq H_{u-1,j-1}$ (by Lemma 8.5, applied to $k = u - 1$).

Finally, from $u \equiv j - 1 \operatorname{mod} 2$, we obtain $u - 1 \equiv (j-1) - 1 = j - 2 \equiv j \operatorname{mod} 2$. Hence, Lemma 8.7 (applied to $k = u - 1$) yields $H_{u-1,j-1} = H_{u-1,j}$.

Altogether, we now have

$$s_u^+ \left(t_j - \left(t_{j-1} - 1\right)\right) = \underbrace{s_u^+ t_j}_{\in H_{u-1,j}} - \underbrace{s_u^+ \left(t_{j-1} - 1\right)}_{\in H_{u-1,j-1} = H_{u-1,j}} \in H_{u-1,j} - H_{u-1,j} \subseteq H_{u-1,j}$$

(since $H_{u-1,j}$ is a left ideal of $\mathbf{A}$). This proves Lemma 8.11. □

Using Lemma 8.11 with Lemma 8.10 **(a)**, we can obtain the following:

Lemma 8.12. Let $j \in [n]$ and $k \in [n+1]$ be such that $1 < j \leq k$ and $k \equiv j \operatorname{mod} 2$. Then,

$$s_k^+ \left[t_{j-1}, t_j\right] \in H_{k-2,j}.$$

Proof. From $j \in [n]$ and $1 < j$, we obtain $j \in [2, n]$. Thus, $j - 1 \in [n-1]$.

Hence, (15) (applied to $i = j - 1$) yields

$$\left[t_{j-1}, t_j\right] = t_{j-1} \left(t_j - \left(t_{j-1} - 1\right)\right). \tag{41}$$

Multiplying this equality by s_k^+ from the left, we obtain

$$s_k^+ \left[t_{j-1}, t_j\right] = s_k^+ t_{j-1} \left(t_j - \left(t_{j-1} - 1\right)\right). \tag{42}$$

However, $k - 1 \equiv j - 1 \operatorname{mod} 2$ (since $k \equiv j \operatorname{mod} 2$). Furthermore, we have $j - 1 \in [n-1] \subseteq [n]$ and $j - 1 < j \leq k$ and $k \equiv j \not\equiv j - 1 \operatorname{mod} 2$. Thus, Lemma 8.10 **(a)** (applied to $j - 1$ instead of j) yields

$$s_k^+ t_{j-1} \in H_{k-1,j-1} = \sum_{\substack{u \in [j-1, k-1]; \\ u \equiv k-1 \operatorname{mod} 2}} \mathbf{A} s_u^+ \qquad \left(\text{by the definition of } H_{k-1,j-1}\right).$$

In other words, we can write $s_k^+ t_{j-1}$ in the form

$$s_k^+ t_{j-1} = \sum_{\substack{u \in [j-1, k-1]; \\ u \equiv k-1 \operatorname{mod} 2}} a_u s_u^+, \tag{43}$$

where $a_u \in \mathbf{A}$ is an element for each $u \in [j-1, k-1]$ satisfying $u \equiv k - 1 \operatorname{mod} 2$.

Consider these elements a_u. Now, (42) becomes

$$\begin{aligned} s_k^+ \left[t_{j-1}, t_j\right] &= s_k^+ t_{j-1} \left(t_j - \left(t_{j-1} - 1\right)\right) \\ &= \left(\sum_{\substack{u \in [j-1, k-1]; \\ u \equiv k-1 \operatorname{mod} 2}} a_u s_u^+ \right) \left(t_j - \left(t_{j-1} - 1\right)\right) \qquad \text{(by (43))} \\ &= \sum_{\substack{u \in [j-1, k-1]; \\ u \equiv k-1 \operatorname{mod} 2}} a_u s_u^+ \left(t_j - \left(t_{j-1} - 1\right)\right). \end{aligned} \tag{44}$$

However, every $u \in [j-1, k-1]$ satisfying $u \equiv k-1 \mod 2$ satisfies

$$s_u^+ \left(t_j - \left(t_{j-1} - 1\right)\right) \in H_{k-2,j}. \tag{45}$$

[*Proof of (45):* Let $u \in [j-1, k-1]$ be such that $u \equiv k-1 \mod 2$.

We have $j \in [2, n]$. Moreover, $u \in [j-1, k-1]$ shows that $u \geq j-1$ and $u \leq k-1 \leq n$ (since $k \leq n+1$). Thus, $u \in [n]$ (since $u \leq n$). Furthermore, $u \equiv k-1 \equiv j-1 \mod 2$. Thus, Lemma 8.11 yields $s_u^+ \left(t_j - \left(t_{j-1} - 1\right)\right) \in H_{u-1,j}$.

However, from $u \leq k-1$, we obtain $u-1 \leq (k-1)-1 = k-2$. Moreover, from $u \equiv k-1 \mod 2$, we obtain $u-1 \equiv (k-1)-1 = k-2 \mod 2$. These two facts entail $H_{u-1,j} \subseteq H_{k-2,j}$ (by Lemma 8.6, applied to $u-1$ and $k-2$ instead of v and w).

Hence, $s_u^+ \left(t_j - \left(t_{j-1} - 1\right)\right) \in H_{u-1,j} \subseteq H_{k-2,j}$. This proves (45).]

Now, (44) becomes

$$s_k^+ \left[t_{j-1}, t_j\right] = \sum_{\substack{u \in [j-1, k-1]; \\ u \equiv k-1 \mod 2}} a_u \underbrace{s_u^+ \left(t_j - \left(t_{j-1} - 1\right)\right)}_{\substack{\in H_{k-2,j} \\ \text{(by (45))}}} \in \sum_{\substack{u \in [j-1, k-1]; \\ u \equiv k-1 \mod 2}} a_u H_{k-2,j} \subseteq H_{k-2,j}$$

(since $H_{k-2,j}$ is a left ideal of $\mathbf{A}$). This proves Lemma 8.12. □

Lemma 8.13. Let $i, j \in [n]$ and $k \in [n+1]$ be such that $i \leq j$ and $k \equiv j \mod 2$. Then,

$$H_{k,j} \left[t_i, t_j\right] \subseteq H_{k-2,j}.$$

Proof. This is obvious for $i = j$ (since we have $\left[t_i, t_j\right] = \left[t_j, t_j\right] = 0$ in this case). Thus, we WLOG assume that $i \neq j$. Hence, $i < j$ (since $i \leq j$). Therefore, $1 \leq i < j$.

Set $a := s_i s_{i+1} \cdots s_{j-2}$. We have $i < j$. Thus, Lemma 7.3 **(b)** yields

$$\left[t_i, t_j\right] = \underbrace{\left(s_i s_{i+1} \cdots s_{j-2}\right)}_{=a} \left[t_{j-1}, t_j\right] = a \left[t_{j-1}, t_j\right].$$

However, it is easy to see that

$$s_u^+ a = a s_u^+ \qquad \text{for each } u \in [j, k]. \tag{46}$$

[*Proof of (46):* Let $u \in [j, k]$. We must prove that $s_u^+ a = a s_u^+$.

If $u \notin [n-1]$, then $s_u^+ = 1$ (by the definition of s_u^+), and thus this claim boils down to $1a = a1$, which is obvious. Thus, we WLOG assume that $u \in [n-1]$. Hence, $s_u^+ = s_u + 1$.

However, from $u \in [j, k]$, we obtain $u \geq j$. Hence, $j \leq u$, so that $j-2 \leq u-2$. Thus, each of the integers $i, i+1, \ldots, j-2$ has a distance larger than 1 from u. Hence, each of the transpositions $s_i, s_{i+1}, \ldots, s_{j-2}$ commutes with s_u (by reflection locality). Therefore, the product $s_i s_{i+1} \cdots s_{j-2}$ of these transpositions also commutes with s_u. In other words, a commutes with s_u (since $a = s_i s_{i+1} \cdots s_{j-2}$). In other words, $s_u a = a s_u$. Now,

$$\underbrace{s_u^+}_{=s_u+1} a = (s_u + 1)\, a = \underbrace{s_u a}_{=a s_u} + a = a s_u + a = a \underbrace{(s_u + 1)}_{=s_u^+} = a s_u^+ .$$

This proves (46).]

Using (46), we can easily see the following: For each $u \in [j,k]$ satisfying $u \equiv k \mod 2$, we have

$$s_u^+ \left[t_i, t_j\right] \in H_{k-2,j}. \tag{47}$$

[*Proof of (47):* Let $u \in [j,k]$ be such that $u \equiv k \mod 2$. From (46), we obtain $s_u^+ a = a s_u^+$. Hence,

$$s_u^+ \underbrace{\left[t_i, t_j\right]}_{=a\left[t_{j-1}, t_j\right]} = \underbrace{s_u^+ a}_{=a s_u^+} \left[t_{j-1}, t_j\right] = a s_u^+ \left[t_{j-1}, t_j\right]. \tag{48}$$

However, $u \in [j,k] \subseteq [k] \subseteq [n+1]$ and $1 < j \leq u$ and $u \equiv k \equiv j \mod 2$. Thus, Lemma 8.12 (applied to u instead of k) yields $s_u^+ \left[t_{j-1}, t_j\right] \in H_{u-2,j}$.

Furthermore, $u - 2 \leq k - 2$ (since $u \leq k$) and $u - 2 \equiv k - 2 \mod 2$ (since $u \equiv k \mod 2$). Hence, Lemma 8.6 (applied to $v = u - 2$ and $w = k - 2$) yields $H_{u-2,j} \subseteq H_{k-2,j}$.

Now, (48) becomes

$$\begin{aligned} s_u^+ \left[t_i, t_j\right] &= a \underbrace{s_u^+ \left[t_{j-1}, t_j\right]}_{\in H_{u-2,j}} \in a H_{u-2,j} \subseteq H_{u-2,j} \qquad \left(\text{since } H_{u-2,j} \text{ is a left ideal}\right) \\ &\subseteq H_{k-2,j}. \end{aligned}$$

This proves (47).]

Now,

$$\begin{aligned} \underbrace{H_{k,j}}_{\substack{= \sum\limits_{\substack{u \in [j,k];\\ u \equiv k \bmod 2}} \mathbf{A} s_u^+ \\ \text{(by the definition of } H_{k,j})}} \left[t_i, t_j\right] &= \left(\sum_{\substack{u \in [j,k];\\ u \equiv k \bmod 2}} \mathbf{A} s_u^+ \right) \left[t_i, t_j\right] = \sum_{\substack{u \in [j,k];\\ u \equiv k \bmod 2}} \mathbf{A} \underbrace{s_u^+ \left[t_i, t_j\right]}_{\substack{\in H_{k-2,j} \\ \text{(by (47))}}} \\ &\subseteq \sum_{\substack{u \in [j,k];\\ u \equiv k \bmod 2}} \mathbf{A} H_{k-2,j} \subseteq H_{k-2,j} \end{aligned}$$

(since $H_{k-2,j}$ is a left ideal). This proves Lemma 8.13. $\square$

If $i, j \in [n]$ and $k \in [n+1]$ are such that $i \leq j$ and $k \equiv j \bmod 2$, then we can apply Lemma 8.13 recursively, yielding

$$\begin{aligned} H_{k,j} \left[t_i, t_j\right] &\subseteq H_{k-2,j}, \\ H_{k,j} \left[t_i, t_j\right]^2 &\subseteq H_{k-4,j}, \\ H_{k,j} \left[t_i, t_j\right]^3 &\subseteq H_{k-6,j}, \\ &\ldots . \end{aligned}$$

Eventually, the right hand side will be 0, and thus we obtain $H_{k,j} \left[t_i, t_j\right]^s = 0$ for some $s \in \mathbb{N}$. By picking k appropriately (specifically, setting $k = n$ or $k = n+1$ depending on the parity of $n - j$), we can ensure that $H_{k,j} = \mathbf{A}$, and thus this equality $H_{k,j} \left[t_i, t_j\right]^s = 0$ yields $\left[t_i, t_j\right]^s = 0$. Thus, Lemma 8.13 "lays a fuse" for proving the nilpotency of $\left[t_i, t_j\right]$. We shall now elaborate on this.

8.3. Products of $\left[t_i, t_j\right]$'s for a fixed j

Lemma 8.14. Let $j \in [n]$ and $m \in \mathbb{N}$. Let r be the unique element of $\{n, n+1\}$ that is congruent to j modulo 2. (That is, $r = \begin{cases} n, & \text{if } n \equiv j \bmod 2; \\ n+1, & \text{otherwise.} \end{cases}$)

Let $i_1, i_2, \ldots, i_m$ be m elements of $[j]$ (not necessarily distinct). Then,

$$\left[t_{i_1}, t_j\right] \left[t_{i_2}, t_j\right] \cdots \left[t_{i_m}, t_j\right] \in H_{r-2m,j}.$$

Proof. We induct on m:

Base case: We have $r \geq n$ (by the definition of r), so that $r \notin [n-1]$ and $j \leq r$ (since $j \leq n \leq r$). Hence, Remark 8.4 (applied to $k = r$) yields $H_{r,j} = \mathbf{A}$. Now

$$\left[t_{i_1}, t_j\right] \left[t_{i_2}, t_j\right] \cdots \left[t_{i_0}, t_j\right] = (\text{empty product}) = 1 \in \mathbf{A} = H_{r,j} = H_{r-2\cdot 0,j}$$

(since $r = r - 2 \cdot 0$). In other words, Lemma 8.14 is proved for $m = 0$.

Induction step: Let $m \in \mathbb{N}$. Assume (as the induction hypothesis) that

$$\left[t_{i_1}, t_j\right] \left[t_{i_2}, t_j\right] \cdots \left[t_{i_m}, t_j\right] \in H_{r-2m,j} \tag{49}$$

whenever $i_1, i_2, \ldots, i_m$ are m elements of $[j]$. We must prove that

$$\left[t_{i_1}, t_j\right] \left[t_{i_2}, t_j\right] \cdots \left[t_{i_{m+1}}, t_j\right] \in H_{r-2(m+1),j} \tag{50}$$

whenever $i_1, i_2, \ldots, i_{m+1}$ are $m+1$ elements of $[j]$.

So let $i_1, i_2, \ldots, i_{m+1}$ be $m+1$ elements of $[j]$. We have $r - 2m \equiv r \equiv j \bmod 2$ (by the definition of r) and $i_{m+1} \in [j] \subseteq [n]$ and $i_{m+1} \leq j$ (since $i_{m+1} \in [j]$). Hence, Lemma 8.13 (applied to $k = r - 2m$ and $i = i_{m+1}$) yields[4]

$$H_{r-2m,j}\left[t_{i_{m+1}}, t_j\right] \subseteq H_{r-2m-2,j}.$$

Now,

$$\begin{aligned}\left[t_{i_1}, t_j\right]\left[t_{i_2}, t_j\right]\cdots\left[t_{i_{m+1}}, t_j\right] &= \underbrace{\left(\left[t_{i_1}, t_j\right]\left[t_{i_2}, t_j\right]\cdots\left[t_{i_m}, t_j\right]\right)}_{\substack{\in H_{r-2m,j} \\ \text{(by (49))}}}\cdot\left[t_{i_{m+1}}, t_j\right] \\ &\in H_{r-2m,j}\left[t_{i_{m+1}}, t_j\right] \subseteq H_{r-2m-2,j} = H_{r-2(m+1),j}\end{aligned}$$

(since $r - 2m - 2 = r - 2(m+1)$). In other words, (50) holds. This completes the induction step. Thus, Lemma 8.14 is proved. □

We can now prove our first main result:

Theorem 8.15. Let $j \in [n]$ and $m \in \mathbb{N}$ be such that $2m \geq n - j + 2$. Let $i_1, i_2, \ldots, i_m$ be m elements of $[j]$ (not necessarily distinct). Then,

$$\left[t_{i_1}, t_j\right]\left[t_{i_2}, t_j\right]\cdots\left[t_{i_m}, t_j\right] = 0.$$

Proof. Let r be the element of $\{n, n+1\}$ defined in Lemma 8.14. Then, $r \leq n+1$, so that

$$\underbrace{r}_{\leq n+1} - \underbrace{2m}_{\geq n-j+2} \leq (n+1) - (n-j+2) = j - 1 < j.$$

Thus, $H_{r-2m,j} = 0$ (by (32)). But Lemma 8.14 yields

$$\left[t_{i_1}, t_j\right]\left[t_{i_2}, t_j\right]\cdots\left[t_{i_m}, t_j\right] \in H_{r-2m,j} = 0.$$

In other words, $\left[t_{i_1}, t_j\right]\left[t_{i_2}, t_j\right]\cdots\left[t_{i_m}, t_j\right] = 0$. This proves Theorem 8.15. □

8.4. The identity $\left[t_i, t_j\right]^{\lceil (n-j)/2\rceil + 1} = 0$ for any $i, j \in [n]$

[4] Strictly speaking, this argument works only if $r - 2m \in [n+1]$ (since Lemma 8.13 requires $k \in [n+1]$). However, in all remaining cases, we can get to the same result in an even simpler way: Namely, assume that $r - 2m \notin [n+1]$. Thus, $r - 2m$ is either ≤ 0 or $> n+1$. Since $r - 2m$ cannot be $> n+1$ (because $r - 2\underbrace{m}_{\geq 0} \leq r \leq n+1$), we thus conclude that $r - 2m \leq 0$. Hence, $r - 2m \leq 0 < j$ and therefore $H_{r-2m,j} = 0$ (by (32)). Hence,

$$\underbrace{H_{r-2m,j}}_{=0}\left[t_{i_{m+1}}, t_j\right] = 0 \subseteq H_{r-2m-2,j}.$$

Lemma 8.16. Let $i, j \in [n]$ and $m \in \mathbb{N}$ be such that $2m \geq n - j + 2$ and $i \leq j$. Then, $[t_i, t_j]^m = 0$.

Proof. We have $i \in [j]$ (since $i \leq j$). Hence, Theorem 8.15 (applied to $i_k = i$) yields $\underbrace{[t_i, t_j] [t_i, t_j] \cdots [t_i, t_j]}_{m \text{ times}} = 0$. In other words, $[t_i, t_j]^m = 0$. This proves Lemma 8.16. $\square$

Corollary 8.17. Let $i, j \in [n]$ and $m \in \mathbb{N}$ be such that $2m \geq n - j + 2$. Then, $[t_i, t_j]^m = 0$.

Proof. If $i \leq j$, then Corollary 8.17 follows directly from Lemma 8.16. Thus, we WLOG assume that we don't have $i \leq j$. Hence, $i > j$.

Therefore, $j < i$, so that $j \leq i$. Moreover, $2m \geq n - \underbrace{j}_{<i} + 2 > n - i + 2$. Hence, we can apply Lemma 8.16 to j and i instead of i and j. We thus obtain $[t_j, t_i]^m = 0$. However, $[t_i, t_j] = -[t_j, t_i]$ (since any two elements a and b of a ring satisfy $[a, b] = -[b, a]$). Hence, $[t_i, t_j]^m = (-[t_j, t_i])^m = (-1)^m \underbrace{[t_j, t_i]^m}_{=0} = 0$. This proves Corollary 8.17. $\square$

Corollary 8.18. For any $x \in \mathbb{R}$, let $\lceil x \rceil$ denote the smallest integer that is $\geq x$. Let $i, j \in [n]$. Then, $[t_i, t_j]^{\lceil (n-j)/2 \rceil + 1} = 0$.

Proof. We have $2 \left(\underbrace{\lceil (n-j)/2 \rceil}_{\geq (n-j)/2} + 1 \right) \geq 2((n-j)/2 + 1) = n - j + 2$. Thus, Corollary 8.17 (applied to $m = \lceil (n-j)/2 \rceil + 1$) yields $[t_i, t_j]^{\lceil (n-j)/2 \rceil + 1} = 0$. This proves Corollary 8.18. $\square$

8.5. Can we lift the $i_1, i_2, \ldots, i_m \in [j]$ restriction?

Remark 8.19. Theorem 8.15 does not hold if we drop the $i_1, i_2, \ldots, i_m \in [j]$ restriction. For instance, for $n = 6$ and $j = 3$, we have

$$[t_1, t_3] [t_5, t_3] [t_4, t_3] [t_1, t_3] \neq 0 \qquad \text{despite } 2 \cdot 4 \geq n - j + 2.$$

Another counterexample is obtained for $n = 4$ and $j = 2$, since $[t_3, t_2] [t_1, t_2] \neq 0$.

Despite these counterexamples, the restriction can be lifted in some particular cases. Here is a particularly simple instance:

Corollary 8.20. Assume that $n \geq 2$. Let $u, v \in [n]$. Then, $[t_u, t_{n-1}] [t_v, t_{n-1}] = 0$.

Proof. We are in one of the following three cases:

Case 1: We have $u = n$.

Case 2: We have $v = n$.

Case 3: Neither u nor v equals n.

Let us first consider Case 1. In this case, we have $u = n$. Hence, $t_u = t_n = 1$ and thus $[t_u, t_{n-1}] = [1, t_{n-1}] = 0$ (since $[1, x] = 0$ for each x). Hence, $\underbrace{[t_u, t_{n-1}]}_{=0} [t_v, t_{n-1}] = 0$. Thus, Corollary 8.20 is proved in Case 1.

A similar argument proves Corollary 8.20 in Case 2.

Let us now consider Case 3. In this case, neither u nor v equals n. In other words, u and v are both $\neq n$. Thus, u and v are elements of $[n] \setminus \{n\} = [n-1]$. Hence, Theorem 8.15 (applied to $j = n-1$ and $m = 2$ and $(i_1, i_2, \ldots, i_m) = (u, v)$) yields $[t_u, t_{n-1}] [t_v, t_{n-1}] = 0$ (since $2 \cdot 2 = 4 \geq 3 = n - (n-1) + 2$). Thus, Corollary 8.20 is proved in Case 3.

We have now proved Corollary 8.20 in all three Cases 1, 2 and 3. $\square$

Proposition 8.21. Assume that $n \geq 3$. Then:

(a) We have $[t_i, t_{n-2}] [s_{n-1}, s_{n-2}] = 0$ for all $i \in [n-2]$.

(b) We have $[t_i, t_{n-2}] [t_{n-1}, t_{n-2}] = 0$ for all $i \in [n]$.

(c) We have $[t_u, t_{n-2}] [t_v, t_{n-2}] [t_w, t_{n-2}] = 0$ for all $u, v, w \in [n]$.

Proof sketch. **(a)** This is easily checked for $i = n-3$ and for $i = n-2$. [5] In all other cases, Lemma 7.3 **(b)** lets us rewrite $[t_i, t_{n-2}]$ as $(s_i s_{i+1} \cdots s_{n-4}) [t_{n-3}, t_{n-2}]$, and thus it remains to prove that $[t_{n-3}, t_{n-2}] [s_{n-1}, s_{n-2}] = 0$, which is exactly the $i = n-3$ case. Thus, Proposition 8.21 **(a)** is proved.

(b) This is easily checked for $i = n-1$ and for $i = n$. In all other cases, we have $i \in [n-2]$, and an easy computation shows that $[t_{n-1}, t_{n-2}] = [s_{n-1}, s_{n-2}] (1 + s_{n-1})$, so that the claim follows from Proposition 8.21 **(a)**. Thus, Proposition 8.21 **(b)** is proved.

(c) Let $u, v, w \in [n]$. We must prove that $[t_u, t_{n-2}] [t_v, t_{n-2}] [t_w, t_{n-2}] = 0$. If any of u, v, w equals n, then this is clear (since $t_n = 1$ and thus $[t_n, t_{n-2}] = [1, t_{n-2}] = 0$). Thus, WLOG assume that $u, v, w \in [n-1]$.

If $v = n-1$, then $[t_u, t_{n-2}] [t_v, t_{n-2}] = [t_u, t_{n-2}] [t_{n-1}, t_{n-2}] = 0$ (by Proposition 8.21 **(b)**), so that our claim holds. Likewise, our claim can be shown if $w = n-1$. Thus, WLOG assume that neither v nor w equals $n-1$. Hence, $v, w \in [n-2]$.

[5] Indeed, the case of $i = n-2$ is obvious (since $[t_{n-2}, t_{n-2}] = 0$). The case of $i = n-3$ requires some calculations, which can be made simpler by checking that $[t_{n-3}, t_{n-2}]$ is an element $a \in \mathbf{k}[S_n]$ satisfying $a = as_{n-2} = as_{n-1}$. (Explicitly, $[t_{n-3}, t_{n-2}] = (1 - s_{n-2}) s_{n-3} b$, where b is the sum of all six permutations in S_n that fix each of $1, 2, \ldots, n-3$.)

Therefore, Theorem 8.15 shows that $[t_u, t_{n-2}] [t_v, t_{n-2}] = 0$, which yields our claim again. This proves Proposition 8.21 **(c)**. □

9. The identity $[t_i, t_j]^{j-i+1} = 0$ for all $i \leq j$

We now approach the proof of another remarkable theorem: the identity $[t_i, t_j]^{j-i+1} = 0$, which holds for all $i, j \in [n]$ satisfying $i \leq j$. Some more work must be done before we can prove this.

9.1. The elements $\mu_{i,j}$ for $i \in [j-1]$

We first introduce a family of elements of the group algebra $\mathbf{k}[S_n]$.

Definition 9.1. Set $\mathbf{A} = \mathbf{k}[S_n]$.

Definition 9.2. Let $j \in [n]$, and let $i \in [j-1]$. Then, $j - 1 \geq 1$ (since $i \in [j-1]$ entails $1 \leq i \leq j-1$), so that $j - 1 \in [n]$. Hence, the elements $(i \Longrightarrow j-1) \in S_n$ and $t_{j-1} \in \mathbf{k}[S_n]$ are well-defined.

Now, we define an element

$$\mu_{i,j} := (i \Longrightarrow j-1)\, t_{j-1} \in \mathbf{A}.$$

Lemma 9.3. Let $j \in [n]$, and let $i \in [j-1]$. Then,

$$\begin{aligned} [t_i, t_j] &= (i \Longrightarrow j-1)\, [t_{j-1}, t_j] && (51) \\ &= \mu_{i,j} \left(t_j - t_{j-1} + 1\right). && (52) \end{aligned}$$

Proof. From $i \in [j-1]$, we obtain $1 \leq i \leq j-1$, so that $j - 1 \geq 1$. Thus, $j - 1 \in [n-1]$ (since $j - 1 < j \leq n$). Hence, (15) (applied to $j-1$ instead of i) yields

$$[t_{j-1}, t_{j-1+1}] = t_{j-1} \left(t_{j-1+1} - \left(t_{j-1} - 1\right)\right).$$

Since $j - 1 + 1 = j$, we can rewrite this as

$$[t_{j-1}, t_j] = t_{j-1} \left(t_j - \left(t_{j-1} - 1\right)\right). \tag{53}$$

We have $i \leq j-1$. Hence, Proposition 3.2 (applied to $v = i$ and $w = j-1$) yields

$$(i \Longrightarrow j-1) = s_i s_{i+1} \cdots s_{(j-1)-1} = s_i s_{i+1} \cdots s_{j-2}. \tag{54}$$

However, $i \leq j-1 < j$. Thus, Lemma 7.3 **(b)** yields

$$[t_i, t_j] = \underbrace{\left(s_i s_{i+1} \cdots s_{j-2}\right)}_{\substack{= (i \Longrightarrow j-1) \\ \text{(by (54))}}} [t_{j-1}, t_j] = (i \Longrightarrow j-1)\, [t_{j-1}, t_j].$$

This proves (51). Furthermore,

$$\left[t_i, t_j\right] = (i \Longrightarrow j-1) \underbrace{\left[t_{j-1}, t_j\right]}_{\substack{=t_{j-1}\left(t_j - \left(t_{j-1}-1\right)\right) \\ \text{(by (53))}}} = \underbrace{(i \Longrightarrow j-1)\, t_{j-1}}_{\substack{=\mu_{i,j} \\ \text{(by the definition of } \mu_{i,j}\text{)}}} \underbrace{\left(t_j - \left(t_{j-1}-1\right)\right)}_{=t_j - t_{j-1}+1}$$
$$= \mu_{i,j}\left(t_j - t_{j-1} + 1\right).$$

This proves (52). Thus, Lemma 9.3 is proved. □

Lemma 9.4. Let R be a ring. Let $a, b, c \in R$ be three elements satisfying $ca = ac$ and $cb = bc$. Then,

$$c\,[a,b] = [a,b]\,c.$$

Proof. The definition of a commutator yields $[a,b] = ab - ba$. Thus,

$$c \underbrace{[a,b]}_{=ab-ba} = c\,(ab - ba) = \underbrace{ca}_{=ac} b - \underbrace{cb}_{=bc} a = a \underbrace{cb}_{=bc} - b \underbrace{ca}_{=ac}$$
$$= abc - bac = \underbrace{(ab - ba)}_{=[a,b]} c = [a,b]\,c.$$

This proves Lemma 9.4. □

Lemma 9.5. Let $i, j, k \in [n]$ be such that $i \leq k < j-1$. Then,

$$\left[t_i, t_j\right] \mu_{k,j} = \mu_{k+1,j} \left[t_i, t_{j-1}\right].$$

Proof. We have $j-1 \geq j-1 > k \geq i$. Thus, Lemma 3.5 (applied to k, $j-1$ and $j-1$ instead of j, v and w) yields

$$(k+1 \Longrightarrow j-1)\,(i \Longrightarrow j-1) = (i \Longrightarrow j-1)\left(k \Longrightarrow \underbrace{(j-1)-1}_{=j-2}\right)$$
$$= (i \Longrightarrow j-1)\,(k \Longrightarrow j-2)\,. \tag{55}$$

From $i < j-1$, we obtain $i \leq j-2$ and thus $i \in [j-2] \subseteq [j-1]$. Likewise, $k \in [j-1]$ (since $k < j-1$).

Furthermore, Proposition 3.3 **(b)** (applied to $v = k$ and $w = j-1$) yields

$$(k \Longrightarrow j-1) = (k \Longrightarrow (j-1)-1)\, s_{(j-1)-1} \qquad \text{(since } k < j-1\text{)}$$
$$= (k \Longrightarrow j-2)\, s_{j-2} \tag{56}$$

(since $(j-1)-1 = j-2$). The same argument (applied to i instead of k) yields

$$(i \Longrightarrow j-1) = (i \Longrightarrow j-2)\, s_{j-2} \tag{57}$$

(since $i < j-1$).

We have $k \leq j-2$ (since $k < j-1$) and $j-2 < j$. Thus, (8) (applied to k and $j-2$ instead of i and k) yields

$$(k \Longrightarrow j-2)\, t_j = t_j\, (k \Longrightarrow j-2)\,. \tag{58}$$

Furthermore, we have $k \leq j-2$ and $j-2 < j-1$. Thus, (8) (applied to k, $j-2$ and $j-1$ instead of i, k and j) yields

$$(k \Longrightarrow j-2)\, t_{j-1} = t_{j-1}\, (k \Longrightarrow j-2)\,. \tag{59}$$

The same argument (but using i instead of k) shows that

$$(i \Longrightarrow j-2)\, t_{j-1} = t_{j-1}\, (i \Longrightarrow j-2) \tag{60}$$

(since $i \leq j-2$).

From (58) and (59), we obtain

$$(k \Longrightarrow j-2)\, [t_{j-1}, t_j] = [t_{j-1}, t_j]\, (k \Longrightarrow j-2) \tag{61}$$

(by Lemma 9.4, applied to $R = \mathbf{A}$, $a = t_{j-1}$, $b = t_j$ and $c = (k \Longrightarrow j-2)$).

Now, the definition of $\mu_{k,j}$ yields $\mu_{k,j} = (k \Longrightarrow j-1)\, t_{j-1}$. Hence,

$$\begin{aligned}
&\underbrace{[t_i, t_j]}_{\substack{=(i \Longrightarrow j-1)[t_{j-1},t_j] \\ \text{(by (51))}}} \underbrace{\mu_{k,j}}_{=(k \Longrightarrow j-1)t_{j-1}} \\
&= (i \Longrightarrow j-1)\, [t_{j-1}, t_j]\, \underbrace{(k \Longrightarrow j-1)}_{\substack{=(k \Longrightarrow j-2)s_{j-2} \\ \text{(by (56))}}} t_{j-1} \\
&= (i \Longrightarrow j-1)\, \underbrace{[t_{j-1}, t_j]\, (k \Longrightarrow j-2)}_{\substack{=(k \Longrightarrow j-2)[t_{j-1},t_j] \\ \text{(by (61))}}} s_{j-2} t_{j-1} \\
&= \underbrace{(i \Longrightarrow j-1)\, (k \Longrightarrow j-2)}_{\substack{=(k+1 \Longrightarrow j-1)(i \Longrightarrow j-1) \\ \text{(by (55))}}} [t_{j-1}, t_j]\, s_{j-2} t_{j-1} \\
&= (k+1 \Longrightarrow j-1)\, \underbrace{(i \Longrightarrow j-1)}_{\substack{=(i \Longrightarrow j-2)s_{j-2} \\ \text{(by (57))}}} [t_{j-1}, t_j]\, s_{j-2} t_{j-1} \\
&= (k+1 \Longrightarrow j-1)\, (i \Longrightarrow j-2)\, s_{j-2}\, [t_{j-1}, t_j]\, s_{j-2} t_{j-1}.
\end{aligned} \tag{62}$$

Next, we shall simplify the product $s_{j-2}\, [t_{j-1}, t_j]\, s_{j-2} t_{j-1}$ on the right hand side. Lemma 7.3 **(b)** (applied to $j-2$ instead of i) yields that

$$\begin{aligned}
[t_{j-2}, t_j] &= \underbrace{\left(s_{j-2} s_{(j-2)+1} \cdots s_{j-2}\right)}_{=s_{j-2}} [t_{j-1}, t_j] \qquad \text{(since } j-2 < j\text{)} \\
&= s_{j-2}\, [t_{j-1}, t_j]\,.
\end{aligned} \tag{63}$$

Furthermore, $i \leq j-2$, so that $j-2 \geq i \geq 1$. Combining this with $j-2 \leq n-2$ (since $j \leq n$), we obtain $j-2 \in [n-2] \subseteq [n-1]$. Hence, Corollary 4.2 (applied to $\ell = j-2$) yields $t_{j-2} = 1 + s_{j-2} \underbrace{t_{(j-2)+1}}_{=t_{j-1}} = 1 + s_{j-2} t_{j-1}$. Hence,

$$t_{j-2} - 1 = s_{j-2} t_{j-1}. \tag{64}$$

Multiplying the equalities (63) and (64) together, we obtain

$$\left[t_{j-2}, t_j\right] \left(t_{j-2} - 1\right) = s_{j-2} \left[t_{j-1}, t_j\right] s_{j-2} t_{j-1}. \tag{65}$$

On the other hand, Corollary 6.3 (applied to $i = j-2$) yields

$$\left[t_{j-2}, t_{j-2+2}\right] \left(t_{j-2} - 1\right) = t_{j-2+1} \left[t_{j-2}, t_{j-2+1}\right] \qquad (\text{since } j-2 \in [n-2]) .$$

In view of $j-2+2 = j$ and $j-2+1 = j-1$, we can rewrite this as

$$\left[t_{j-2}, t_j\right] \left(t_{j-2} - 1\right) = t_{j-1} \left[t_{j-2}, t_{j-1}\right] .$$

Comparing this with (65), we obtain

$$s_{j-2} \left[t_{j-1}, t_j\right] s_{j-2} t_{j-1} = t_{j-1} \left[t_{j-2}, t_{j-1}\right] .$$

Hence, (62) becomes

$$\begin{aligned} \left[t_i, t_j\right] \mu_{k,j} &= (k+1 \Longrightarrow j-1)\,(i \Longrightarrow j-2) \underbrace{s_{j-2} \left[t_{j-1}, t_j\right] s_{j-2} t_{j-1}}_{=t_{j-1}\left[t_{j-2}, t_{j-1}\right]} \\ &= (k+1 \Longrightarrow j-1) \underbrace{(i \Longrightarrow j-2)\, t_{j-1}}_{\substack{=t_{j-1}(i \Longrightarrow j-2) \\ (\text{by } (60))}} \left[t_{j-2}, t_{j-1}\right] \\ &= (k+1 \Longrightarrow j-1)\, t_{j-1}\, (i \Longrightarrow j-2) \left[t_{j-2}, t_{j-1}\right] . \end{aligned} \tag{66}$$

But $k \leq j-2$, so that $k+1 \leq j-1$. Thus, $k+1 \in [j-1]$. Hence, the definition of $\mu_{k+1,j}$ yields

$$\mu_{k+1,j} = (k+1 \Longrightarrow j-1)\, t_{j-1}. \tag{67}$$

Furthermore, $i \leq j-2 = (j-1)-1$, so that $i \in [(j-1)-1]$. Hence, (51) (applied to $j-1$ instead of j) yields

$$\begin{aligned} \left[t_i, t_{j-1}\right] &= (i \Longrightarrow (j-1)-1) \left[t_{(j-1)-1}, t_{j-1}\right] \\ &= (i \Longrightarrow j-2) \left[t_{j-2}, t_{j-1}\right] \end{aligned} \tag{68}$$

(since $(j-1)-1 = j-2$). Multiplying the equalities (67) and (68), we obtain

$$\mu_{k+1,j} \left[t_i, t_{j-1}\right] = (k+1 \Longrightarrow j-1)\, t_{j-1}\, (i \Longrightarrow j-2) \left[t_{j-2}, t_{j-1}\right] .$$

Comparing this with (66), we obtain $\left[t_i, t_j\right] \mu_{k,j} = \mu_{k+1,j} \left[t_i, t_{j-1}\right]$. This proves Lemma 9.5. □

We can combine Lemma 9.3 and Lemma 9.5 into a single result:

Lemma 9.6. Let $j \in [n]$, and let $i \in [j]$ and $k \in [j-1]$. Then, we have

$$\left([t_i, t_j]\, \mu_{k,j} = 0\right) \text{ or } \left([t_i, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some } \ell \in [k+1, j-1]\right).$$

Proof. If $i = j$, then this holds for obvious reasons[6]. Hence, for the rest of this proof, we WLOG assume that $i \neq j$.

We have $i \leq j$ (since $i \in [j]$). Combining this with $i \neq j$, we obtain $i < j$, so that $i \leq j-1$. In other words, $i \in [j-1]$.

From $k \in [j-1]$, we obtain $1 \leq k \leq j-1$, so that $j-1 \geq 1$ and therefore $j \geq 2$. Hence, $j \in [2, n]$.

We are in one of the following three cases:

Case 1: We have $k \geq j-1$.

Case 2: We have $i > k$.

Case 3: We have neither $k \geq j-1$ nor $i > k$.

Let us first consider Case 1. In this case, we have $k \geq j-1$. Combining this with $k \leq j-1$, we obtain $k = j-1$.

The definition of $\mu_{k,j}$ yields

$$\mu_{k,j} = \left(\underbrace{k}_{=j-1} \Longrightarrow j-1\right) t_{j-1} = \underbrace{(j-1 \Longrightarrow j-1)}_{\substack{=1 \\ \text{(by (4))}}} t_{j-1} = t_{j-1}.$$

Hence,

$$[t_i, t_j]\, \underbrace{\mu_{k,j}}_{=t_{j-1}} = [t_i, t_j]\, t_{j-1} = 0 \qquad \text{(by Corollary 7.4)}.$$

Thus, we have $\left([t_i, t_j]\, \mu_{k,j} = 0\right)$ or $\left([t_i, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some } \ell \in [k+1, j-1]\right)$. This proves Lemma 9.6 in Case 1.

Let us next consider Case 2. In this case, we have $i > k$. Hence, $i \geq k+1$. Combined with $i \leq j-1$, this entails $i \in [k+1, j-1]$. Furthermore, (52) shows that

$$[t_i, t_j] = \mu_{i,j} \underbrace{(t_j - t_{j-1} + 1)}_{\in \mathbf{A}} \in \mu_{i,j}\mathbf{A}.$$

We now know that $i \in [k+1, j-1]$ and $[t_i, t_j] \in \mu_{i,j}\mathbf{A}$. Therefore, $[t_i, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A}$ for some $\ell \in [k+1, j-1]$ (namely, for $\ell = i$). Thus, we have $\left([t_i, t_j]\, \mu_{k,j} = 0\right)$ or $\left([t_i, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some } \ell \in [k+1, j-1]\right)$. This proves Lemma 9.6 in Case 2.

[6] *Proof.* Assume that $i = j$. Then, $[t_i, t_j] = [t_j, t_j] = 0$ (since $[a, a] = 0$ for any element a of any ring). Hence, $\underbrace{[t_i, t_j]}_{=0} \mu_{k,j} = 0$. Therefore, we clearly have $\left([t_i, t_j]\, \mu_{k,j} = 0\right)$ or $\left([t_i, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some } \ell \in [k+1, j-1]\right)$. Thus, Lemma 9.6 is proved under the assumption that $i = j$.

Finally, let us consider Case 3. In this case, we have neither $k \geq j-1$ nor $i > k$. In other words, we have $k < j-1$ and $i \leq k$. Thus, $i \leq k < j-1$. Hence, Lemma 9.5 yields

$$[t_i, t_j] \, \mu_{k,j} = \mu_{k+1,j} \underbrace{[t_i, t_{j-1}]}_{\in \mathbf{A}} \in \mu_{k+1,j} \mathbf{A}.$$

Furthermore, $k < j-1$, so that $k \leq (j-1)-1$. In other words, $k+1 \leq j-1$. Hence, $k+1 \in [k+1, j-1]$.

We now know that $k+1 \in [k+1, j-1]$ and $[t_i, t_j] \, \mu_{k,j} \in \mu_{k+1,j} \mathbf{A}$. Hence, $[t_i, t_j] \, \mu_{k,j} \in \mu_{\ell,j} \mathbf{A}$ for some $\ell \in [k+1, j-1]$ (namely, for $\ell = k+1$). Thus, we have $\left([t_i, t_j] \, \mu_{k,j} = 0\right)$ or $\left([t_i, t_j] \, \mu_{k,j} \in \mu_{\ell,j} \mathbf{A} \text{ for some } \ell \in [k+1, j-1]\right)$. This proves Lemma 9.6 in Case 3.

We have now proved Lemma 9.6 in each of the three Cases 1, 2 and 3. Hence, this lemma is proved in all situations. □

9.2. Products of $[t_i, t_j]$'s for a fixed j redux

For the sake of convenience, we shall restate Lemma 9.6 in a simpler form. To this purpose, we extend Definition 9.2 somewhat:

Definition 9.7. Let $j \in [n]$, and let i be a positive integer. In Definition 9.7, we have defined $\mu_{i,j}$ whenever $i \in [j-1]$. We now set

$$\mu_{i,j} := 0 \in \mathbf{A} \qquad \text{whenever } i \notin [j-1] \, .$$

Thus, $\mu_{i,j}$ is defined for all positive integers i (not just for $i \in [j-1]$). For example, $\mu_{j,j} = 0$ (since $j \notin [j-1]$).

Using this extended meaning of $\mu_{i,j}$, we can rewrite Lemma 9.6 as follows:

Lemma 9.8. Let $j \in [n]$, and let $i \in [j]$. Let k be a positive integer. Then,

$$[t_i, t_j] \, \mu_{k,j} \in \mu_{\ell,j} \mathbf{A} \text{ for some integer } \ell \geq k+1.$$

Proof. If $k \geq j$, then this holds for obvious reasons[7]. Hence, for the rest of this proof, we WLOG assume that $k < j$. Thus, $k \in [j-1]$ (since k is a positive integer). Therefore, Lemma 9.6 yields that we have

$$\left([t_i, t_j] \, \mu_{k,j} = 0\right) \text{ or } \left([t_i, t_j] \, \mu_{k,j} \in \mu_{\ell,j} \mathbf{A} \text{ for some } \ell \in [k+1, j-1]\right).$$

[7] *Proof.* Assume that $k \geq j$. Thus, $k \geq j > j-1$, so that $k \notin [j-1]$ and therefore $\mu_{k,j} = 0$ (by Definition 9.7). Hence,

$$[t_i, t_j] \underbrace{\mu_{k,j}}_{=0} = 0 = \mu_{k+1,j} \cdot \underbrace{0}_{\in \mathbf{A}} \in \mu_{k+1,j} \mathbf{A}.$$

Hence, $[t_i, t_j] \, \mu_{k,j} \in \mu_{\ell,j} \mathbf{A}$ for some integer $\ell \geq k+1$ (namely, for $\ell = k+1$). Thus, Lemma 9.8 is proved under the assumption that $k \geq j$.

In other words, we are in one of the following cases:

Case 1: We have $[t_i, t_j]\, \mu_{k,j} = 0$.

Case 2: We have $[t_i, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A}$ for some $\ell \in [k+1, j-1]$.

Let us first consider Case 1. In this case, we have $[t_i, t_j]\, \mu_{k,j} = 0$. Hence, $[t_i, t_j]\, \mu_{k,j} = 0 = \mu_{k+1,j} \cdot \underbrace{0}_{\in \mathbf{A}} \in \mu_{k+1,j}\mathbf{A}$. Hence, $[t_i, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A}$ for some integer $\ell \geq k+1$ (namely, for $\ell = k+1$). Thus, Lemma 9.8 is proved in Case 1.

Let us now consider Case 2. In this case, we have $[t_i, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A}$ for some $\ell \in [k+1, j-1]$. Hence, we have $[t_i, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A}$ for some integer $\ell \geq k+1$ (because any $\ell \in [k+1, j-1]$ is an integer $\geq k+1$). Thus, Lemma 9.8 is proved in Case 2.

We have now proved Lemma 9.8 in both Cases 1 and 2. Hence, Lemma 9.8 is proved in all situations. □

The next lemma is similar to Lemma 8.14, and will play a similar role:

Lemma 9.9. Let $j \in [n]$. Let k be a positive integer, and let $m \in \mathbb{N}$. Let $i_1, i_2, \ldots, i_m$ be m elements of $[j]$ (not necessarily distinct). Then,

$$[t_{i_m}, t_j]\, [t_{i_{m-1}}, t_j] \cdots [t_{i_1}, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some integer } \ell \geq k+m.$$

Proof. We shall show that for each $v \in \{0, 1, \ldots, m\}$, we have

$$[t_{i_v}, t_j]\, [t_{i_{v-1}}, t_j] \cdots [t_{i_1}, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some integer } \ell \geq k+v. \tag{69}$$

In fact, we shall prove (69) by induction on v:

Base case: Let us check that (69) holds for $v = 0$. Indeed,

$$\underbrace{[t_{i_0}, t_j]\, [t_{i_{0-1}}, t_j] \cdots [t_{i_1}, t_j]}_{=(\text{empty product})=1}\, \mu_{k,j} = \mu_{k,j} = \mu_{k,j} \underbrace{1}_{\in \mathbf{A}} \in \mu_{k,j}\mathbf{A}.$$

Thus, $[t_{i_0}, t_j]\, [t_{i_{0-1}}, t_j] \cdots [t_{i_1}, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A}$ for some integer $\ell \geq k+0$ (namely, for $\ell = k$). In other words, (69) holds for $v = 0$. This completes the base case.

Induction step: Let $v \in \{0, 1, \ldots, m-1\}$. Assume (as the induction hypothesis) that (69) holds for v. We must prove that (69) holds for $v+1$ instead of v. In other words, we must prove that

$$[t_{i_{v+1}}, t_j]\, [t_{i_v}, t_j] \cdots [t_{i_1}, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some integer } \ell \geq k+(v+1).$$

Our induction hypothesis says that (69) holds for v. In other words, it says that

$$[t_{i_v}, t_j]\, [t_{i_{v-1}}, t_j] \cdots [t_{i_1}, t_j]\, \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some integer } \ell \geq k+v.$$

Let us denote this integer ℓ by w. Thus, $w \geq k+v$ is an integer and satisfies

$$[t_{i_v}, t_j]\, [t_{i_{v-1}}, t_j] \cdots [t_{i_1}, t_j]\, \mu_{k,j} \in \mu_{w,j}\mathbf{A}. \tag{70}$$

However, $w \geq k + v \geq k$, so that w is a positive integer. Also, $i_{v+1} \in [j]$. Thus, Lemma 9.8 (applied to i_{v+1} and w instead of i and k) yields that

$$\left[t_{i_{v+1}}, t_j\right] \mu_{w,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some integer } \ell \geq w + 1. \tag{71}$$

Consider this ℓ. Thus, $\ell \geq \underbrace{w}_{\geq k+v} + 1 \geq k + v + 1 = k + (v+1)$. Furthermore,

$$\begin{aligned}
&\underbrace{\left[t_{i_{v+1}}, t_j\right] \left[t_{i_v}, t_j\right] \cdots \left[t_{i_1}, t_j\right]}_{=\left[t_{i_{v+1}}, t_j\right] \cdot \left(\left[t_{i_v}, t_j\right]\left[t_{i_{v-1}}, t_j\right] \cdots \left[t_{i_1}, t_j\right]\right)} \mu_{k,j} = \left[t_{i_{v+1}}, t_j\right] \cdot \underbrace{\left(\left[t_{i_v}, t_j\right] \left[t_{i_{v-1}}, t_j\right] \cdots \left[t_{i_1}, t_j\right]\right) \mu_{k,j}}_{\substack{\in \mu_{w,j}\mathbf{A} \\ \text{(by (70))}}} \\
&\qquad \in \underbrace{\left[t_{i_{v+1}}, t_j\right] \mu_{w,j}}_{\substack{\in \mu_{\ell,j}\mathbf{A} \\ \text{(by (71))}}} \mathbf{A} \subseteq \mu_{\ell,j} \underbrace{\mathbf{A}\mathbf{A}}_{\subseteq \mathbf{A}} \subseteq \mu_{\ell,j}\mathbf{A}.
\end{aligned}$$

Thus, we have found an integer $\ell \geq k + (v+1)$ that satisfies $\left[t_{i_{v+1}}, t_j\right] \left[t_{i_v}, t_j\right] \cdots \left[t_{i_1}, t_j\right] \mu_{k,j} \in \mu_{\ell,j}\mathbf{A}$. Hence, we have shown that

$$\left[t_{i_{v+1}}, t_j\right] \left[t_{i_v}, t_j\right] \cdots \left[t_{i_1}, t_j\right] \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some integer } \ell \geq k + (v+1).$$

In other words, (69) holds for $v + 1$ instead of v. This completes the induction step. Thus, (69) is proved by induction on v.

Therefore, we can apply (69) to $v = m$. We obtain

$$\left[t_{i_m}, t_j\right] \left[t_{i_{m-1}}, t_j\right] \cdots \left[t_{i_1}, t_j\right] \mu_{k,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some integer } \ell \geq k + m.$$

This proves Lemma 9.9. □

Now, we can show our second main result:

Theorem 9.10. Let $j \in [n]$, and let m be a positive integer. Let $k_1, k_2, \ldots, k_m$ be any m elements of $[j]$ (not necessarily distinct) satisfying $m \geq j - k_m + 1$. Then,

$$\left[t_{k_1}, t_j\right] \left[t_{k_2}, t_j\right] \cdots \left[t_{k_m}, t_j\right] = 0.$$

Proof. If $k_m = j$, then this claim is obvious[8]. Hence, for the rest of this proof, we WLOG assume that $k_m \neq j$. Combining this with $k_m \leq j$ (since $k_m \in [j]$), we obtain $k_m < j$. Hence, $k_m \in [j-1]$. Therefore, (52) (applied to $i = k_m$) yields

$$\left[t_{k_m}, t_j\right] = \mu_{k_m,j} \left(t_j - t_{j-1} + 1\right). \tag{72}$$

[8]*Proof.* Assume that $k_m = j$. Thus, $\left[t_{k_m}, t_j\right] = \left[t_j, t_j\right] = 0$ (since $[a, a] = 0$ for any element a of any ring). In other words, the last factor of the product $\left[t_{k_1}, t_j\right] \left[t_{k_2}, t_j\right] \cdots \left[t_{k_m}, t_j\right]$ is 0. Thus, this whole product must equal 0. In other words, $\left[t_{k_1}, t_j\right] \left[t_{k_2}, t_j\right] \cdots \left[t_{k_m}, t_j\right] = 0$. This proves Theorem 9.10 under the assumption that $k_m = j$.

Now, we have $m-1 \in \mathbb{N}$ (since m is a positive integer). Let us define an $(m-1)$-tuple $(i_1, i_2, \ldots, i_{m-1})$ of elements of $[j]$ by

$$(i_1, i_2, \ldots, i_{m-1}) := (k_{m-1}, k_{m-2}, \ldots, k_1)$$

(that is, $i_v := k_{m-v}$ for each $v \in [m-1]$). Then, $i_1, i_2, \ldots, i_{m-1}$ are $m-1$ elements of $[j]$. Hence, Lemma 9.9 (applied to $m-1$ and k_m instead of m and k) yields

$$\left[t_{i_{m-1}}, t_j\right] \left[t_{i_{(m-1)-1}}, t_j\right] \cdots \left[t_{i_1}, t_j\right] \mu_{k_m,j} \in \mu_{\ell,j}\mathbf{A} \text{ for some integer } \ell \geq k_m + (m-1).$$

Consider this ℓ. We have

$$\ell \geq k_m + (m-1) = k_m + \underbrace{m}_{\geq j-k_m+1} - 1 \geq k_m + j - k_m + 1 - 1 = j > j-1,$$

so that $\ell \notin [j-1]$. Therefore, $\mu_{\ell,j} = 0$ (by Definition 9.7). Hence,

$$\left[t_{i_{m-1}}, t_j\right] \left[t_{i_{(m-1)-1}}, t_j\right] \cdots \left[t_{i_1}, t_j\right] \mu_{k_m,j} \in \underbrace{\mu_{\ell,j}}_{=0} \mathbf{A} = 0\mathbf{A} = 0.$$

In other words,

$$\left[t_{i_{m-1}}, t_j\right] \left[t_{i_{(m-1)-1}}, t_j\right] \cdots \left[t_{i_1}, t_j\right] \mu_{k_m,j} = 0. \tag{73}$$

However, from the equality $(i_1, i_2, \ldots, i_{m-1}) = (k_{m-1}, k_{m-2}, \ldots, k_1)$, we immediately obtain $\left(i_{m-1}, i_{(m-1)-1}, \ldots, i_1\right) = (k_1, k_2, \ldots, k_{m-1})$. Therefore,

$$\left[t_{i_{m-1}}, t_j\right] \left[t_{i_{(m-1)-1}}, t_j\right] \cdots \left[t_{i_1}, t_j\right] = \left[t_{k_1}, t_j\right] \left[t_{k_2}, t_j\right] \cdots \left[t_{k_{m-1}}, t_j\right].$$

Thus, we can rewrite (73) as

$$\left[t_{k_1}, t_j\right] \left[t_{k_2}, t_j\right] \cdots \left[t_{k_{m-1}}, t_j\right] \mu_{k_m,j} = 0. \tag{74}$$

Now,

$$\begin{aligned}\left[t_{k_1}, t_j\right] \left[t_{k_2}, t_j\right] \cdots \left[t_{k_m}, t_j\right] &= \left[t_{k_1}, t_j\right] \left[t_{k_2}, t_j\right] \cdots \left[t_{k_{m-1}}, t_j\right] \underbrace{\left[t_{k_m}, t_j\right]}_{\substack{=\mu_{k_m,j}\left(t_j - t_{j-1}+1\right) \\ \text{(by (72))}}} \\ &= \underbrace{\left[t_{k_1}, t_j\right] \left[t_{k_2}, t_j\right] \cdots \left[t_{k_{m-1}}, t_j\right] \mu_{k_m,j}}_{\substack{=0 \\ \text{(by (74))}}} \left(t_j - t_{j-1} + 1\right) = 0.\end{aligned}$$

This proves Theorem 9.10. □

9.3. The identity $\left[t_i, t_j\right]^{j-i+1} = 0$ for all $i \leq j$

As a particular case of Theorem 9.10, we obtain the following:

Corollary 9.11. Let $i, j \in [n]$ be such that $i \leq j$. Then, $\left[t_i, t_j\right]^{j-i+1} = 0$.

Proof. We have $j - i \geq 0$ (since $i \leq j$) and thus $j - i + 1 \geq 1$. Hence, $j - i + 1$ is a positive integer. Moreover, i is an element of $[j]$ (since $i \leq j$) and we have $j - i + 1 \geq j - i + 1$. Hence, Theorem 9.10 (applied to $m = j - i + 1$ and $k_r = i$) yields $\underbrace{\left[t_i, t_j\right] \left[t_i, t_j\right] \cdots \left[t_i, t_j\right]}_{j-i+1 \text{ times}} = 0$. Thus, $\left[t_i, t_j\right]^{j-i+1} = \underbrace{\left[t_i, t_j\right] \left[t_i, t_j\right] \cdots \left[t_i, t_j\right]}_{j-i+1 \text{ times}} = 0$. This proves Corollary 9.11. $\square$

10. Further directions

10.1. More identities?

A few other properties of somewhere-to-below shuffles can be shown. For example, the proofs of the following two propositions are left to the reader:

Proposition 10.1. We have $t_i = \sum\limits_{k=i}^{j-1} s_i s_{i+1} \cdots s_{k-1} + s_i s_{i+1} \cdots s_{j-1} t_j$ for any $1 \leq i < j \leq n$.

Proposition 10.2. Let $i, j \in [n-1]$ be such that $i \leq j$. Then, $\left[t_i, t_j\right] = \left[s_i s_{i+1} \cdots s_{j-1},\ s_j\right] t_{j+1} t_j$.

Proposition 10.3. Set $B_i := \prod\limits_{k=0}^{i-1} (t_1 - k)$ for each $i \in [0, n]$. Then, $B_i = t_i B_{i-1}$ for each $i \in [n]$.

We wonder to what extent the identities that hold for $t_1, t_2, \ldots, t_n$ can be described. For instance, we can ask:

Question 10.4.

- **(a)** What are generators and relations for the $\mathbb{Q}$-algebra $\mathbb{Q}\left[t_1, t_2, \ldots, t_n\right]$ for a given $n \in \mathbb{N}$?
- **(b)** Fix $k \in \mathbb{N}$. What identities hold for $t_1, t_2, \ldots, t_k$ for **all** n ? Is there a single algebra that "governs" the relations between $t_1, t_2, t_3, \ldots$ that hold independently of n ?
- **(c)** If a relation between $t_1, t_2, \ldots, t_k$ holds for all sufficiently high $n \geq k$, must it then hold for all $n \geq k$?

We suspect that these questions are hard to answer, as we saw in Remark 6.2 that even the quadratic relations between $t_1, t_2, \ldots, t_n$ exhibit some rather finicky behavior. The dimension of $\mathbb{Q}[t_1, t_2, \ldots, t_n]$ as a $\mathbb{Q}$-vector space does not seem to follow a simple rule either (see (1) for the first few values), although there appear to be some patterns in how this dimension is generated[9].

Another question, which we have already touched upon in Subsection 8.5, is the following:

Question 10.5. Fix $j \in [n]$. What is the smallest $h \in \mathbb{N}$ such that we have $[t_{i_1}, t_j][t_{i_2}, t_j] \cdots [t_{i_h}, t_j] = 0$ for all $i_1, i_2, \ldots, i_h \in [n]$ (as opposed to holding only for $i_1, i_2, \ldots, i_h \in [j]$)?

10.2. Optimal exponents?

Corollary 8.18 and Corollary 9.11 give two different answers to the question "what powers of $[t_i, t_j]$ are 0?". One might dare to ask for the **smallest** such power (more precisely, the smallest such exponent). In other words:

Question 10.6. Given $i, j \in [n]$, what is the smallest $m \in \mathbb{N}$ such that $[t_i, t_j]^m = 0$? (We assume $\mathbf{k} = \mathbb{Z}$ here to avoid small-characteristic cancellations.)

We conjecture that this smallest m is $\min\{j - i + 1,\ \lceil (n-j)/2 \rceil + 1\}$ whenever $i < j$ (so that whichever of Corollary 8.18 and Corollary 9.11 gives the better bound actually gives the optimal bound). Using SageMath, this conjecture has been verified for all $n \leq 12$.

10.3. Generalizing to the Hecke algebra

The *type-A Hecke algebra* (also known as the *type-A Iwahori-Hecke algebra*) is a deformation of the group algebra $\mathbf{k}[S_n]$ involving a new parameter $q \in \mathbf{k}$. It is commonly denoted by $\mathcal{H} = \mathcal{H}_q(S_n)$; it has a basis $(T_w)_{w \in S_n}$ indexed by the permutations $w \in S_n$, but its multiplication is more complicated than composing the indexing permutations. We refer to [Mathas99] for the definition and a deep study of this algebra. We can define the q-deformed somewhere-to-below shuffles $t_1^{\mathcal{H}}, t_2^{\mathcal{H}}, \ldots, t_n^{\mathcal{H}}$ by

$$t_\ell^{\mathcal{H}} := T_{\mathrm{cyc}_\ell} + T_{\mathrm{cyc}_{\ell,\ell+1}} + T_{\mathrm{cyc}_{\ell,\ell+1,\ell+2}} + \cdots + T_{\mathrm{cyc}_{\ell,\ell+1,\ldots,n}} \in \mathcal{H}.$$

Surprisingly, it seems that many of the properties of the original somewhere-to-below $t_1, t_2, \ldots, t_n$ still hold for these deformations. In particular:

[9] Namely, for all $n \leq 8$, we have verified that the algebra $\mathbb{Q}[t_1, t_2, \ldots, t_n]$ is generated by products of m somewhere-to-below shuffles with $m \in \{0, 1, \ldots, n-1\}$, and moreover, only one such product for $m = n - 1$ is needed.

Conjecture 10.7. Corollary 9.11 and Corollary 8.18 both seem to hold in $\mathcal{H}$ when the t_ℓ are replaced by the $t_\ell^{\mathcal{H}}$.

This generalization is not automatic. Our above proofs do not directly apply to $\mathcal{H}$, as (for example) Lemma 3.6 does not generalize to $\mathcal{H}$. The $\mathcal{H}$-generalization of Theorem 5.1 appears to be

$$q t_{i+1}^{\mathcal{H}} t_i^{\mathcal{H}} = \left(t_i^{\mathcal{H}} - 1\right) t_i^{\mathcal{H}} = t_i^{\mathcal{H}} \left(t_i^{\mathcal{H}} - 1\right) \tag{75}$$

(verified using SageMath for all $n \leq 11$). (The q on the left hand side is necessary; the product $t_{i+1}^{\mathcal{H}} t_i^{\mathcal{H}}$ is not a $\mathbb{Z}$-linear combination of 1, $t_i^{\mathcal{H}}$ and $\left(t_i^{\mathcal{H}}\right)^2$ when $q = 0$.) Our proof of Theorem 5.1 does not seem to adapt to (75), and while we suspect that proving (75) won't be too difficult, it is merely the first step.

10.4. One-sided cycle shuffles

We return to $\mathbf{k}[S_n]$.

The $\mathbf{k}$-linear combinations $\lambda_1 t_1 + \lambda_2 t_2 + \cdots + \lambda_n t_n$ (with $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbf{k}$) of the somewhere-to-below shuffles are called the *one-sided cycle shuffles*. They have been studied in [GriLaf22]. Again, the main result of [GriLaf22] entails that their commutators are nilpotent, but we can ask "how nilpotent?".

This question remains wide open, not least due to its computational complexity (even the $n = 6$ case brings SageMath to its limits). All that I can say with surety is that the commutators of one-sided cycle shuffles don't vanish as quickly (under taking powers) as the $\left[t_i, t_j\right]$'s.

Example 10.8. For instance, let us set $n = 6$ and choose arbitrary $a, b, c, d, e, a', b', c', d', e' \in \mathbf{k}$, and then introduce the elements

$$u := at_1 + bt_2 + ct_3 + dt_4 + et_5 \qquad \text{and}$$
$$u' := a't_1 + b't_2 + c't_3 + d't_4 + e't_5$$

(two completely generic one-sided shuffles, except that omit t_6 terms since $t_6 = 1$ does not influence the commutator). Then, 10 minutes of torturing SageMath reveals that $[u, u']^6 = 0$, but $[u, u']^5$ is generally nonzero.

Even this example is misleadingly well-behaved. For $n = 7$, it is not hard to find two one-sided cycle shuffles u, u' such that $[u, u']^n \neq 0$.

Question 10.9. For each given n, what is the smallest (or at least a reasonably small) $m \in \mathbb{N}$ such that every two one-sided cycle shuffles u, u' satisfy $[u, u']^m = 0$?

References

[BiHaRo99] P. Bidigare, P. Hanlon, and D. Rockmore. A combinatorial description of the spectrum for the Tsetlin library and its generalization to hyperplane arrangements. *Duke Mathematical Journal*, 99(1):135–174, 1999. doi:10.1215/S0012-7094-99-09906-4.

[DiFiPi92] P. Diaconis, J. A. Fill, and J. Pitman. Analysis of top to random shuffles. *Combinatorics, Probability and Computing*, 1(2):135–155, 1992. URL `https://statweb.stanford.edu/~cgates/PERSI/papers/randomshuff92.pdf`.

[Donnel91] P. Donnelly. The heaps process, libraries, and size-biased permutations. *Journal of Applied Probability*, 28(2):321–335, 1991. doi:10.2307/3214869.

[Fill96] J. A. Fill. An exact formula for the move-to-front rule for self-organizing lists. *Journal of Theoretical Probability*, 9(1):113–160, 1996. doi:10.1007/BF02213737.

[GriLaf22] D. Grinberg and N. Lafrenière. The one-sided cycle shuffles in the symmetric group algebra. arXiv:2212.06274v3, 2023.

[Grinbe18] D. Grinberg. Answers to "is this sum of cycles invertible in $\mathbb{Q}S_n$?". MathOverflow thread #308536. URL `https://mathoverflow.net/questions/308536/is-this-sum-of-cycles-invertible-in-mathbb-qs-n`.

[Hendri72] W. J. Hendricks. The stationary distribution of an interesting Markov chain. *J. Appl. Probability*, 9:231–233, 1972. URL `https://doi.org/10.2307/3212655`.

[Mathas99] A. Mathas. *Iwahori-Hecke Algebras and Schur Algebras of the Symmetric Group*, volume 15 of *University Lecture Series*. American Mathematical Society, 1999. URL `https://bookstore.ams.org/ulect-15`.

[Palmes10] C. Palmes. Top-to-random-shuffles. diploma thesis at Westfälische Wilhelms-Universität Münster, `https://www.uni-muenster.de/Stochastik/alsmeyer/Diplomarbeiten/Palmes.pdf`, 2010.

[Phatar91] R. M. Phatarfod. On the matrix occurring in a linear search problem. *Journal of Applied Probability*, 28(2):336–346, 1991. doi:10.1017/s0021900200039723.

[SageMath] The SageMath developers. *SageMath, (Version 10.1)*, 2023. `https://www.sagemath.org`.