Ideals of QSym, shuffle-compatibility and exterior peaks

Darij Grinberg (UMN)

28 February 2018
University of Washington

slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/seattle18.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf
project: https://github.com/darijgr/gzshuf

Section 1

Shuffle-compatibility

Reference:

- Ira M. Gessel, Yan Zhuang, *Shuffle-compatible permutation statistics*, arXiv:1706.00750.

- This project spins off from a paper by Ira Gessel and Yan Zhuang (`arXiv:1706.00750`), which Yan presented here last week.
  We prove a conjecture (shuffle-compatibility of Epk) and study a stronger version of shuffle-compatibility.

Permutations & permutation statistics: Definitions 1

- This project spins off from a paper by Ira Gessel and Yan Zhuang (arXiv:1706.00750), which Yan presented here last week.
  We prove a conjecture (shuffle-compatibility of Epk) and study a stronger version of shuffle-compatibility.
- Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- For $n \in \mathbb{N}$, an *n-permutation* means a tuple of n distinct positive integers.
  Example: $(3, 1, 7)$ is a 3-permutation, but $(2, 1, 2)$ is not.

Permutations & permutation statistics: Definitions 1

- This project spins off from a paper by Ira Gessel and Yan Zhuang (arXiv:1706.00750), which Yan presented here last week.
  We prove a conjecture (shuffle-compatibility of Epk) and study a stronger version of shuffle-compatibility.
- Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- For $n \in \mathbb{N}$, an *n-permutation* means a tuple of n distinct positive integers.
  Example: $(3, 1, 7)$ is a 3-permutation, but $(2, 1, 2)$ is not.
- A *permutation* means an n-permutation for some n.

Permutations & permutation statistics: Definitions 1

- This project spins off from a paper by Ira Gessel and Yan Zhuang (arXiv:1706.00750), which Yan presented here last week.
  We prove a conjecture (shuffle-compatibility of Epk) and study a stronger version of shuffle-compatibility.
- Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- For $n \in \mathbb{N}$, an *n-permutation* means a tuple of n distinct positive integers.
  Example: $(3, 1, 7)$ is a 3-permutation, but $(2, 1, 2)$ is not.
- A *permutation* means an n-permutation for some n.
  If π is an n-permutation, then $|\pi| := n$.

Permutations & permutation statistics: Definitions 1

- This project spins off from a paper by Ira Gessel and Yan Zhuang (arXiv:1706.00750), which Yan presented here last week.
  We prove a conjecture (shuffle-compatibility of Epk) and study a stronger version of shuffle-compatibility.
- Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- For $n \in \mathbb{N}$, an *n-permutation* means a tuple of n distinct positive integers.
  Example: $(3, 1, 7)$ is a 3-permutation, but $(2, 1, 2)$ is not.
- A *permutation* means an n-permutation for some n.
  If π is an n-permutation, then $|\pi| := n$.
  We say that π is *nonempty* if $n > 0$.

Permutations & permutation statistics: Definitions 1

- This project spins off from a paper by Ira Gessel and Yan Zhuang (arXiv:1706.00750), which Yan presented here last week.
  We prove a conjecture (shuffle-compatibility of Epk) and study a stronger version of shuffle-compatibility.
- Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- For $n \in \mathbb{N}$, an *n-permutation* means a tuple of n distinct positive integers.
  Example: $(3, 1, 7)$ is a 3-permutation, but $(2, 1, 2)$ is not.
- A *permutation* means an n-permutation for some n.
  If π is an n-permutation, then $|\pi| := n$.
  We say that π is *nonempty* if $n > 0$.
- If π is an n-permutation and $i \in \{1, 2, \ldots, n\}$, then π_i denotes the i-th entry of π.

Permutations & permutation statistics: Definitions 2

- Two n-permutations α and β (with the same n) are *order-equivalent* if all $i, j \in \{1, 2, \ldots, n\}$ satisfy $(\alpha_i < \alpha_j) \iff (\beta_i < \beta_j)$.
- Order-equivalence is an equivalence relation on permutations. Its equivalence classes are called *order-equivalence classes*.

- Two n-permutations α and β (with the same n) are *order-equivalent* if all $i, j \in \{1, 2, \ldots, n\}$ satisfy $(\alpha_i < \alpha_j) \iff (\beta_i < \beta_j)$.
- Order-equivalence is an equivalence relation on permutations. Its equivalence classes are called *order-equivalence classes*.
- A *permutation statistic* (henceforth just *statistic*) is a map st from the set of all permutations (to anywhere) that is constant on each order-equivalence class.
  Intuition: A statistic computes some "fingerprint" of a permutation that only depends on the relative order of its letters.

Permutations & permutation statistics: Definitions 2

- Two n-permutations α and β (with the same n) are *order-equivalent* if all $i, j \in \{1, 2, \ldots, n\}$ satisfy $(\alpha_i < \alpha_j) \iff (\beta_i < \beta_j)$.
- Order-equivalence is an equivalence relation on permutations. Its equivalence classes are called *order-equivalence classes*.
- A *permutation statistic* (henceforth just *statistic*) is a map st from the set of all permutations (to anywhere) that is constant on each order-equivalence class.
  Intuition: A statistic computes some "fingerprint" of a permutation that only depends on the relative order of its letters.
  Note: A statistic need not be integer-valued! It can be set-valued, or list-valued for example.

Examples of permutation statistics, 1: descents et al

- If π is an n-permutation, then a *descent* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$.
- The *descent set* $\operatorname{Des} \pi$ of a permutation π is the set of all descents of π.
  Thus, Des is a statistic.
  Example: $\operatorname{Des}(3, 1, 5, 2, 4) = \{1, 3\}$.

Examples of permutation statistics, 1: descents et al

- If π is an n-permutation, then a *descent* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$.
- The *descent set* $\operatorname{Des} \pi$ of a permutation π is the set of all descents of π.
  Thus, Des is a statistic.
  Example: $\operatorname{Des}(3, 1, 5, 2, 4) = \{1, 3\}$.
- The *descent number* $\operatorname{des} \pi$ of a permutation π is the number of all descents of π: that is, $\operatorname{des} \pi = |\operatorname{Des} \pi|$.
  Thus, des is a statistic.
  Example: $\operatorname{des}(3, 1, 5, 2, 4) = 2$.

Examples of permutation statistics, 1: descents et al

- If π is an n-permutation, then a *descent* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$.
- The *descent set* $\operatorname{Des} \pi$ of a permutation π is the set of all descents of π.
  Thus, Des is a statistic.
  Example: $\operatorname{Des}(3, 1, 5, 2, 4) = \{1, 3\}$.
- The *descent number* $\operatorname{des} \pi$ of a permutation π is the number of all descents of π: that is, $\operatorname{des} \pi = |\operatorname{Des} \pi|$.
  Thus, des is a statistic.
  Example: $\operatorname{des}(3, 1, 5, 2, 4) = 2$.
- The *major index* $\operatorname{maj} \pi$ of a permutation π is the **sum** of all descents of π.
  Thus, maj is a statistic.
  Example: $\operatorname{maj}(3, 1, 5, 2, 4) = 4$.

Examples of permutation statistics, 1: descents et al

- If π is an n-permutation, then a *descent* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$.
- The *descent set* $\operatorname{Des} \pi$ of a permutation π is the set of all descents of π.
  Thus, Des is a statistic.
  Example: $\operatorname{Des}(3, 1, 5, 2, 4) = \{1, 3\}$.
- The *descent number* $\operatorname{des} \pi$ of a permutation π is the number of all descents of π: that is, $\operatorname{des} \pi = |\operatorname{Des} \pi|$.
  Thus, des is a statistic.
  Example: $\operatorname{des}(3, 1, 5, 2, 4) = 2$.
- The *major index* $\operatorname{maj} \pi$ of a permutation π is the **sum** of all descents of π.
  Thus, maj is a statistic.
  Example: $\operatorname{maj}(3, 1, 5, 2, 4) = 4$.
- The *Coxeter length* inv (i.e., *number of inversions*) and the *set of inversions* are statistics, too.

Examples of permutation statistics, 2: peaks

- If π is an n-permutation, then a *peak* of π means an $i \in \{2, 3, \ldots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$.
  (Thus, peaks can only exist if $n \geq 3$.
  The name refers to the plot of π, where peaks are local maxima.)
- The *peak set* $\operatorname{Pk} \pi$ of a permutation π is the set of all peaks of π.
  Thus, Pk is a statistic.
  Examples:
  - $\operatorname{Pk}(3, 1, 5, 2, 4) = \{3\}$.
  - $\operatorname{Pk}(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
  - $\operatorname{Pk}(3, 2) = \{\}$.

Examples of permutation statistics, 2: peaks

- If π is an n-permutation, then a *peak* of π means an $i \in \{2, 3, \ldots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$.
  (Thus, peaks can only exist if $n \geq 3$.
  The name refers to the plot of π, where peaks are local maxima.)
- The *peak set* $\operatorname{Pk} \pi$ of a permutation π is the set of all peaks of π.
  Thus, Pk is a statistic.
  Examples:
  - $\operatorname{Pk}(3, 1, 5, 2, 4) = \{3\}$.
  - $\operatorname{Pk}(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
  - $\operatorname{Pk}(3, 2) = \{\}$.
- The *peak number* $\operatorname{pk} \pi$ of a permutation π is the number of all peaks of π: that is, $\operatorname{pk} \pi = |\operatorname{Pk} \pi|$.
  Thus, pk is a statistic.
  Example: $\operatorname{pk}(3, 1, 5, 2, 4) = 1$.

Examples of permutation statistics, 3: left peaks

- If π is an n-permutation, then a *left peak* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$.
  (Thus, left peaks are the same as peaks, except that 1 counts as a left peak if $\pi_1 > \pi_2$.)
- The *left peak set* $\operatorname{Lpk} \pi$ of a permutation π is the set of all left peaks of π.
  Thus, Lpk is a statistic.
  Examples:
  - $\operatorname{Lpk}(3, 1, 5, 2, 4) = \{1, 3\}$.
  - $\operatorname{Lpk}(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
  - $\operatorname{Lpk}(3, 2) = \{1\}$.
- The *left peak number* $\operatorname{lpk} \pi$ of a permutation π is the number of all left peaks of π: that is, $\operatorname{lpk} \pi = |\operatorname{Lpk} \pi|$.
  Thus, lpk is a statistic.
  Example: $\operatorname{lpk}(3, 1, 5, 2, 4) = 2$.

Examples of permutation statistics, 4: right peaks

- If π is an n-permutation, then a *right peak* of π means an $i \in \{2, 3, \ldots, n\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_{n+1} = 0$.
  (Thus, right peaks are the same as peaks, except that n counts as a right peak if $\pi_{n-1} < \pi_n$.)
- The *right peak set* $\operatorname{Rpk} \pi$ of a permutation π is the set of all right peaks of π.
  Thus, Rpk is a statistic.
  Examples:
  - $\operatorname{Rpk}(3, 1, 5, 2, 4) = \{3, 5\}$.
  - $\operatorname{Rpk}(1, 3, 2, 5, 4, 6) = \{2, 4, 6\}$.
  - $\operatorname{Rpk}(3, 2) = \{\}$.
- The *right peak number* $\operatorname{rpk} \pi$ of a permutation π is the number of all right peaks of π: that is, $\operatorname{rpk} \pi = |\operatorname{Rpk} \pi|$.
  Thus, rpk is a statistic.
  Example: $\operatorname{rpk}(3, 1, 5, 2, 4) = 2$.

Examples of permutation statistics, 5: exterior peaks

- If π is an n-permutation, then an *exterior peak* of π means an $i \in \{1, 2, \ldots, n\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$ *and* $\pi_{n+1} = 0$.
  (Thus, exterior peaks are the same as peaks, except that 1 counts if $\pi_1 > \pi_2$, and n counts if $\pi_{n-1} < \pi_n$.)
- The *exterior peak set* $\operatorname{Epk} \pi$ of a permutation π is the set of all exterior peaks of π.
  Thus, Epk is a statistic.
  Examples:
  - $\operatorname{Epk}(3, 1, 5, 2, 4) = \{1, 3, 5\}$.
  - $\operatorname{Epk}(1, 3, 2, 5, 4, 6) = \{2, 4, 6\}$.
  - $\operatorname{Epk}(3, 2) = \{1\}$.
- Thus, $\operatorname{Epk} \pi = \operatorname{Lpk} \pi \cup \operatorname{Rpk} \pi$ if $n \geq 2$.
- The *exterior peak number* $\operatorname{epk} \pi$ of a permutation π is the number of all exterior peaks of π: that is, $\operatorname{epk} \pi = |\operatorname{Epk} \pi|$.
  Thus, epk is a statistic.
  Example: $\operatorname{epk}(3, 1, 5, 2, 4) = 3$.

Shuffles of permutations

- Let π and σ be two permutations.
- We say that π and σ are *disjoint* if they have no letter in common.

- Let π and σ be two permutations.
- We say that π and σ are *disjoint* if they have no letter in common.
- Assume that π and σ are disjoint. Set $m = |\pi|$ and $n = |\sigma|$. An $(m+n)$-permutation τ is called a *shuffle* of π and σ if both π and σ appear as subsequences of τ. (And thus, no other letters can appear in τ.)
- We let $S(\pi, \sigma)$ be the set of all shuffles of π and σ.
- **Example:**

$$S((4,1),(2,5)) = \{(4,1,2,5), (4,2,1,5), (4,2,5,1), (2,4,1,5), (2,4,5,1), (2,5,4,1)\}.$$

Shuffles of permutations

- Let π and σ be two permutations.
- We say that π and σ are *disjoint* if they have no letter in common.
- Assume that π and σ are disjoint. Set $m = |\pi|$ and $n = |\sigma|$. An $(m+n)$-permutation τ is called a *shuffle* of π and σ if both π and σ appear as subsequences of τ.
  (And thus, no other letters can appear in τ.)
- We let $S(\pi, \sigma)$ be the set of all shuffles of π and σ.
- **Example:**

$$S((4,1),(2,5)) = \{(4,1,2,5),(4,2,1,5),(4,2,5,1),$$
$$(2,4,1,5),(2,4,5,1),(2,5,4,1)\}.$$

- Observe that π and σ have $\binom{m+n}{m}$ shuffles, in bijection with m-element subsets of $\{1, 2, \ldots, m+n\}$.

Shuffle-compatible statistics: definition

- A statistic st is said to be *shuffle-compatible* if for any two disjoint permutations π and σ, the multiset

$$\{\operatorname{st}\tau \mid \tau \in S(\pi, \sigma)\}_{\text{multiset}}$$

depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, $|\pi|$ and $|\sigma|$.

Shuffle-compatible statistics: definition

- A statistic st is said to be *shuffle-compatible* if for any two disjoint permutations π and σ, the multiset

$$\{\mathrm{st}\,\tau \mid \tau \in S(\pi, \sigma)\}_{\text{multiset}}$$

depends only on $\mathrm{st}\,\pi$, $\mathrm{st}\,\sigma$, $|\pi|$ and $|\sigma|$.

- In other words, st is shuffle-compatible if and only the distribution of st on the set $S(\pi, \sigma)$ stays unchaged if π and σ are replaced by two other disjoint permutations of the same size and same st-values.

Shuffle-compatible statistics: definition

- A statistic st is said to be *shuffle-compatible* if for any two disjoint permutations π and σ, the multiset

$$\{\operatorname{st}\tau \mid \tau \in S(\pi, \sigma)\}_{\text{multiset}}$$

depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, $|\pi|$ and $|\sigma|$.

- In other words, st is shuffle-compatible if and only the distribution of st on the set $S(\pi, \sigma)$ stays unchaged if π and σ are replaced by two other disjoint permutations of the same size and same st-values.
In particular, it has to stay unchanged if π and σ are replaced by two permutations order-equivalent to them: e.g., st must have the same distribution on the three sets

$$S((4,1),(2,5)), \qquad S((2,1),(3,5)), \qquad S((9,8),(2,3)).$$

Shuffle-compatible statistics: results of Gessel and Zhuang

- Gessel and Zhuang, in arXiv:1706.00750, prove that various important statistics are shuffle-compatible (but some are not).

Shuffle-compatible statistics: results of Gessel and Zhuang

- Gessel and Zhuang, in `arXiv:1706.00750`, prove that various important statistics are shuffle-compatible (but some are not).
- Statistics they show to be **shuffle-compatible**: Des, des, maj, Pk, Lpk, Rpk, lpk, rpk, epk, and various others.

Shuffle-compatible statistics: results of Gessel and Zhuang

- Gessel and Zhuang, in arXiv:1706.00750, prove that various important statistics are shuffle-compatible (but some are not).
- Statistics they show to be **shuffle-compatible**: Des, des, maj, Pk, Lpk, Rpk, lpk, rpk, epk, and various others.
- Statistics that are **not shuffle-compatible**: inv, $\operatorname{des} + \operatorname{maj}$, maj_2 (sending π to the sum of the squares of its descents), $(\operatorname{Pk}, \operatorname{des})$ (sending π to $(\operatorname{Pk} \pi, \operatorname{des} \pi)$), and others.

Shuffle-compatible statistics: results of Gessel and Zhuang

- Gessel and Zhuang, in arXiv:1706.00750, prove that various important statistics are shuffle-compatible (but some are not).
- Statistics they show to be **shuffle-compatible**: Des, des, maj, Pk, Lpk, Rpk, lpk, rpk, epk, and various others.
- Statistics that are **not shuffle-compatible**: inv, $\operatorname{des} + \operatorname{maj}$, maj_2 (sending π to the sum of the squares of its descents), $(\operatorname{Pk}, \operatorname{des})$ (sending π to $(\operatorname{Pk} \pi, \operatorname{des} \pi)$), and others.
- Their proofs use a mixture of enumerative combinatorics (including some known formulas of MacMahon, Stanley, ...), quasisymmetric functions, Hopf algebra theory, P-partitions (and variants by Stembridge and Petersen), Eulerian polynomials (based on earlier work by Zhuang, and even earlier work by Foata and Strehl).

Shuffle-compatible statistics: results of Gessel and Zhuang

- Gessel and Zhuang, in arXiv:1706.00750, prove that various important statistics are shuffle-compatible (but some are not).
- Statistics they show to be **shuffle-compatible**: Des, des, maj, Pk, Lpk, Rpk, lpk, rpk, epk, and various others.
- Statistics that are **not shuffle-compatible**: inv, $\operatorname{des} + \operatorname{maj}$, maj_2 (sending π to the sum of the squares of its descents), $(\operatorname{Pk}, \operatorname{des})$ (sending π to $(\operatorname{Pk} \pi, \operatorname{des} \pi)$), and others.
- Their proofs use a mixture of enumerative combinatorics (including some known formulas of MacMahon, Stanley, ...), quasisymmetric functions, Hopf algebra theory, P-partitions (and variants by Stembridge and Petersen), Eulerian polynomials (based on earlier work by Zhuang, and even earlier work by Foata and Strehl).
- The shuffle-compatibility of Epk is left unproven in Gessel/Zhuang. Proving this is our first goal.

Left- and right-shuffle-compatibility

- We further begin the study of a finer version of shuffle-compatibility: "left- and right-shuffle-compatibility".
- Given two disjoint nonempty permutations π and σ,
    - a *left shuffle* of π and σ is a shuffle of π and σ that starts with a letter of π;
    - a *right shuffle* of π and σ is a shuffle of π and σ that starts with a letter of σ.
- We let $S_{\prec}(\pi, \sigma)$ be the set of all left shuffles of π and σ. We let $S_{\succ}(\pi, \sigma)$ be the set of all right shuffles of π and σ.

Left- and right-shuffle-compatibility

- We further begin the study of a finer version of shuffle-compatibility: "left- and right-shuffle-compatibility".
- Given two disjoint nonempty permutations π and σ,
  - a *left shuffle* of π and σ is a shuffle of π and σ that starts with a letter of π;
  - a *right shuffle* of π and σ is a shuffle of π and σ that starts with a letter of σ.
- We let $S_{\prec}(\pi, \sigma)$ be the set of all left shuffles of π and σ. We let $S_{\succ}(\pi, \sigma)$ be the set of all right shuffles of π and σ.
- A statistic st is said to be *left-shuffle-compatible* if for any two disjoint nonempty permutations π and σ such that

  the first entry of π is greater than the first entry of σ,

  the multiset

  $$\{\operatorname{st} \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multiset}}$$

  depends only on $\operatorname{st} \pi$, $\operatorname{st} \sigma$, $|\pi|$ and $|\sigma|$.

Left- and right-shuffle-compatibility

- We further begin the study of a finer version of shuffle-compatibility: "left- and right-shuffle-compatibility".
- Given two disjoint nonempty permutations π and σ,
  - a *left shuffle* of π and σ is a shuffle of π and σ that starts with a letter of π;
  - a *right shuffle* of π and σ is a shuffle of π and σ that starts with a letter of σ.
- We let $S_{\prec}(\pi, \sigma)$ be the set of all left shuffles of π and σ. We let $S_{\succ}(\pi, \sigma)$ be the set of all right shuffles of π and σ.
- A statistic st is said to be *right-shuffle-compatible* if for any two disjoint nonempty permutations π and σ such that

  the first entry of π is greater than the first entry of σ,

  the multiset
  $$\{\operatorname{st}\tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multiset}}$$
  depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, $|\pi|$ and $|\sigma|$.

Left- and right-shuffle-compatibility

- We further begin the study of a finer version of shuffle-compatibility: "left- and right-shuffle-compatibility".
- Given two disjoint nonempty permutations π and σ,
  - a *left shuffle* of π and σ is a shuffle of π and σ that starts with a letter of π;
  - a *right shuffle* of π and σ is a shuffle of π and σ that starts with a letter of σ.
- We let $S_{\prec}(\pi, \sigma)$ be the set of all left shuffles of π and σ. We let $S_{\succ}(\pi, \sigma)$ be the set of all right shuffles of π and σ.
- A statistic st is said to be *right-shuffle-compatible* if for any two disjoint nonempty permutations π and σ such that

  the first entry of π is greater than the first entry of σ,

  the multiset
  $$\{\operatorname{st}\tau \mid \tau \in S_{\succ}(\pi, \sigma)\}_{\text{multiset}}$$
  depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, $|\pi|$ and $|\sigma|$.
- We'll show that Des, des, Lpk and Epk are left- and right-shuffle-compatible.

Section 2

The algebraic approach: QSym **and kernels**

Reference:

- Ira M. Gessel, Yan Zhuang, *Shuffle-compatible permutation statistics*, arXiv:1706.00750.
- Darij Grinberg, Victor Reiner, *Hopf Algebras in Combinatorics*, arXiv:1409.8356, and various other texts on combinatorial Hopf algebras.

Descent statistics

- Gessel and Zhuang prove **most** of their shuffle-compatibilities algebraically. Their methods involve combinatorial Hopf algebras (QSym and NSym).
- These methods work for **descent statistics** only. What is a descent statistic?

- Gessel and Zhuang prove **most** of their shuffle-compatibilities algebraically. Their methods involve combinatorial Hopf algebras (QSym and NSym).
- These methods work for **descent statistics** only. What is a descent statistic?
- A *descent statistic* is a statistic st such that $\operatorname{st}\pi$ depends only on $|\pi|$ and $\operatorname{Des}\pi$ (in other words: if π and σ are two n-permutations with $\operatorname{Des}\pi = \operatorname{Des}\sigma$, then $\operatorname{st}\pi = \operatorname{st}\sigma$).
  Intuition: A descent statistic is a statistic which "factors through Des in each size".

Compositions & descent compositions: definitions

- A *composition* is a finite list of positive integers.
  A *composition of* $n \in \mathbb{N}$ is a composition whose entries sum to n.

Compositions & descent compositions: definitions

- A *composition* is a finite list of positive integers.
  A *composition of $n \in \mathbb{N}$* is a composition whose entries sum to n.
- For example, $(1, 3, 2)$ is a composition of 6.

- A *composition* is a finite list of positive integers.
  A *composition of $n \in \mathbb{N}$* is a composition whose entries sum to n.
- For example, $(1, 3, 2)$ is a composition of 6.
- Let $n \in \mathbb{N}$, and let $[n-1] = \{1, 2, \ldots, n-1\}$.
  Then, there are mutually inverse bijections

$$\mathrm{Des} : \{\text{compositions of } n\} \to \{\text{subsets of } [n-1]\},$$

$$(i_1, i_2, \ldots, i_k) \mapsto \{i_1 + i_2 + \cdots + i_j \mid 1 \leq j \leq k-1\}$$

and

$$\mathrm{Comp} : \{\text{subsets of } [n-1]\} \to \{\text{compositions of } n\},$$

$$\{s_1 < s_2 < \cdots < s_k\} \mapsto (s_1 - s_0, s_2 - s_1, \ldots, s_{k+1} - s_k)$$

(using the notations $s_0 = 0$ and $s_{k+1} = n$).

Compositions & descent compositions: definitions

- A *composition* is a finite list of positive integers.
  A *composition of* $n \in \mathbb{N}$ is a composition whose entries sum to n.
- For example, $(1, 3, 2)$ is a composition of 6.
- Let $n \in \mathbb{N}$, and let $[n-1] = \{1, 2, \ldots, n-1\}$.
  Then, there are mutually inverse bijections Des and Comp between $\{\text{subsets of } [n-1]\}$ and $\{\text{compositions of } n\}$.
  If π is an n-permutation, then $\operatorname{Comp}(\operatorname{Des} \pi)$ is called the *descent composition* of π, and is written $\operatorname{Comp} \pi$.

Compositions & descent compositions: definitions

- A *composition* is a finite list of positive integers.
  A *composition of* $n \in \mathbb{N}$ is a composition whose entries sum to n.
- For example, $(1, 3, 2)$ is a composition of 6.
- Let $n \in \mathbb{N}$, and let $[n-1] = \{1, 2, \ldots, n-1\}$.
  Then, there are mutually inverse bijections Des and Comp between $\{\text{subsets of } [n-1]\}$ and $\{\text{compositions of } n\}$.
  If π is an n-permutation, then $\operatorname{Comp}(\operatorname{Des}\pi)$ is called the *descent composition* of π, and is written $\operatorname{Comp}\pi$.
- Thus, a descent statistic is a statistic st that factors through Comp (that is, $\operatorname{st}\pi$ depends only on $\operatorname{Comp}\pi$).

Compositions & descent compositions: definitions

- A *composition* is a finite list of positive integers.
  A *composition of $n \in \mathbb{N}$* is a composition whose entries sum to n.
- For example, $(1, 3, 2)$ is a composition of 6.
- Let $n \in \mathbb{N}$, and let $[n-1] = \{1, 2, \ldots, n-1\}$.
  Then, there are mutually inverse bijections Des and Comp between $\{\text{subsets of } [n-1]\}$ and $\{\text{compositions of } n\}$.
  If π is an n-permutation, then $\operatorname{Comp}(\operatorname{Des} \pi)$ is called the *descent composition* of π, and is written $\operatorname{Comp} \pi$.
- Thus, a descent statistic is a statistic st that factors through Comp (that is, $\operatorname{st} \pi$ depends only on $\operatorname{Comp} \pi$).
- If st is a descent statistic, then we use the notation $\operatorname{st} \alpha$ (where α is a composition) for $\operatorname{st} \pi$, where π is any permutation with $\operatorname{Comp} \pi = \alpha$.

Compositions & descent compositions: definitions

- A *composition* is a finite list of positive integers.
  A *composition of* $n \in \mathbb{N}$ is a composition whose entries sum to n.
- For example, $(1, 3, 2)$ is a composition of 6.
- Let $n \in \mathbb{N}$, and let $[n-1] = \{1, 2, \ldots, n-1\}$.
  Then, there are mutually inverse bijections Des and Comp between $\{\text{subsets of } [n-1]\}$ and $\{\text{compositions of } n\}$.
  If π is an n-permutation, then $\operatorname{Comp}(\operatorname{Des}\pi)$ is called the *descent composition* of π, and is written $\operatorname{Comp}\pi$.
- If st is a descent statistic, then we use the notation $\operatorname{st}\alpha$ (where α is a composition) for $\operatorname{st}\pi$, where π is any permutation with $\operatorname{Comp}\pi = \alpha$.
- **Warning:**

$$\operatorname{Des}((1, 5, 2) \text{ the composition}) = \{1, 6\};$$
$$\operatorname{Des}((1, 5, 2) \text{ the permutation}) = \{2\}.$$

Same for other statistics! Context must disambiguate.

Descent statistics: examples

- Almost all of our statistics so far are descent statistics. Examples:

Descent statistics: examples

- Almost all of our statistics so far are descent statistics. Examples:
- Des, des and maj are descent statistics.

Descent statistics: examples

- Almost all of our statistics so far are descent statistics. Examples:
- Des, des and maj are descent statistics.
- Pk is a descent statistic: If π is an n-permutation, then

  $$\operatorname{Pk}\pi = (\operatorname{Des}\pi) \setminus ((\operatorname{Des}\pi \cup \{0\}) + 1),$$

  where for any set K of integers and any integer a we set $K + a = \{k + a \mid k \in K\}$.
- Similarly, Lpk, Rpk and Epk are descent statistics.

Descent statistics: examples

- Almost all of our statistics so far are descent statistics. Examples:
- Des, des and maj are descent statistics.
- Pk is a descent statistic: If π is an n-permutation, then

$$\operatorname{Pk}\pi = (\operatorname{Des}\pi) \setminus ((\operatorname{Des}\pi \cup \{0\}) + 1),$$

  where for any set K of integers and any integer a we set $K + a = \{k + a \mid k \in K\}$.
- Similarly, Lpk, Rpk and Epk are descent statistics.
- inv is not a descent statistic: The permutations $(2, 1, 3)$ and $(3, 1, 2)$ have the same descents, but different numbers of inversions.

Descent statistics: examples

- Almost all of our statistics so far are descent statistics. Examples:
- Des, des and maj are descent statistics.
- Pk is a descent statistic: If π is an n-permutation, then

$$\operatorname{Pk}\pi = (\operatorname{Des}\pi) \setminus ((\operatorname{Des}\pi \cup \{0\}) + 1),$$

  where for any set K of integers and any integer a we set $K + a = \{k + a \mid k \in K\}$.
- Similarly, Lpk, Rpk and Epk are descent statistics.
- inv is not a descent statistic: The permutations $(2, 1, 3)$ and $(3, 1, 2)$ have the same descents, but different numbers of inversions.
- **Question (Gessel & Zhuang).** Is every shuffle-compatible statistic a descent statistic?

- Let's now talk about power series, which are crucial to the algebraic approach to shuffle-compatibility.
- Consider the ring $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates.

Power series & symmetric functions

- Let's now talk about power series, which are crucial to the algebraic approach to shuffle-compatibility.
- Consider the ring $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates.
- A formal power series f is said to be *bounded-degree* if the monomials it contains are bounded (from above) in degree.

Power series & symmetric functions

- Let's now talk about power series, which are crucial to the algebraic approach to shuffle-compatibility.
- Consider the ring $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates.
- A formal power series f is said to be *bounded-degree* if the monomials it contains are bounded (from above) in degree.
- A formal power series f is said to be *symmetric* if it is invariant under permutations of the indeterminates. Equivalently, if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1, i_2, \ldots, i_k$ are distinct and $j_1, j_2, \ldots, j_k$ are distinct.
- For example:
  - $1 + x_1 + x_2^3$ is bounded-degree but not symmetric.
  - $(1 + x_1)(1 + x_2)(1 + x_3) \cdots$ is symmetric but not bounded-degree.

Power series & symmetric functions

- Let's now talk about power series, which are crucial to the algebraic approach to shuffle-compatibility.
- Consider the ring $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates.
- A formal power series f is said to be *bounded-degree* if the monomials it contains are bounded (from above) in degree.
- A formal power series f is said to be *symmetric* if it is invariant under permutations of the indeterminates. Equivalently, if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1, i_2, \ldots, i_k$ are distinct and $j_1, j_2, \ldots, j_k$ are distinct.
- The symmetric bounded-degree power series form a subring Λ of $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$, called the *ring of symmetric functions* over $\mathbb{Q}$. This talk is not about it.

- We shall now define the quasisymmetric functions – a bigger algebra than Λ, but still with many of its nice properties.
- A formal power series f (still in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$) is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
  - Every symmetric power series is quasisymmetric.
  - $\sum\limits_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.

- We shall now define the quasisymmetric functions – a bigger algebra than Λ, but still with many of its nice properties.
- A formal power series f (still in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$) is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
  - Every symmetric power series is quasisymmetric.
  - $\sum\limits_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.
- Let QSym be the set of all quasisymmetric bounded-degree power series in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. This is a $\mathbb{Q}$-subalgebra, called the *ring of quasisymmetric functions* over $\mathbb{Q}$. (Gessel, 1980s.)

Quasisymmetric functions, part 1: definition

- We shall now define the quasisymmetric functions – a bigger algebra than Λ, but still with many of its nice properties.
- A formal power series f (still in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$) is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
  - Every symmetric power series is quasisymmetric.
  - $\sum\limits_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.
- Let QSym be the set of all quasisymmetric bounded-degree power series in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. This is a $\mathbb{Q}$-subalgebra, called the *ring of quasisymmetric functions* over $\mathbb{Q}$. (Gessel, 1980s.)
- We have $\Lambda \subseteq \mathrm{QSym} \subseteq \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$.

Quasisymmetric functions, part 1: definition

- We shall now define the quasisymmetric functions – a bigger algebra than Λ, but still with many of its nice properties.
- A formal power series f (still in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$) is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
    - Every symmetric power series is quasisymmetric.
    - $\sum\limits_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.
- Let QSym be the set of all quasisymmetric bounded-degree power series in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. This is a $\mathbb{Q}$-subalgebra, called the *ring of quasisymmetric functions* over $\mathbb{Q}$. (Gessel, 1980s.)
- The $\mathbb{Q}$-vector space QSym has several combinatorial bases. We will use two of them: the monomial basis and the fundamental basis.

- For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, define

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

$=$ sum of all monomials whose nonzero exponents are $\alpha_1, \alpha_2, \ldots, \alpha_k$ in **this** order.

  This is a homogeneous power series of degree $|\alpha|$ (the *size* of α, defined by $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_k$).
- Examples:
  - $M_{()} = 1$.
  - $M_{(1,1)} = \sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + \cdots$.
  - $M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots$.
  - $M_{(3)} = \sum_{i} x_i^3 = x_1^3 + x_2^3 + x_3^3 + \cdots$.

- For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, define

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$
$$= \text{sum of all monomials whose nonzero exponents are } \alpha_1, \alpha_2, \ldots, \alpha_k \text{ in } \textbf{this} \text{ order.}$$

  This is a homogeneous power series of degree $|\alpha|$ (the *size* of α, defined by $|\alpha| := \alpha_1 + \alpha_2 + \cdots + \alpha_k$).
- The family $(M_\alpha)_{\alpha \text{ is a composition}}$ is a basis of the $\mathbb{Q}$-vector space QSym, called the *monomial basis* (or M-basis).

Quasisymmetric functions, part 3: the fundamental basis

- For every composition $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, define

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ for all } j \in \operatorname{Des} \alpha}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$= \sum_{\substack{\beta \text{ is a composition of } n; \\ \operatorname{Des} \beta \supseteq \operatorname{Des} \alpha}} M_\beta, \qquad \text{where } n = |\alpha| .$$

  This is a homogeneous power series of degree $|\alpha|$ again.

- Examples:
  - $F_{()} = 1$.
  - $F_{(1,1)} = \sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + \cdots$.
  - $F_{(2,1)} = \sum_{i \leq j < k} x_i x_j x_k$.
  - $F_{(3)} = \sum_{i \leq j \leq k} x_i x_j x_k$.

- For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, define

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ for all } j \in \operatorname{Des} \alpha}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$= \sum_{\substack{\beta \text{ is a composition of } n; \\ \operatorname{Des} \beta \supseteq \operatorname{Des} \alpha}} M_\beta, \qquad \text{where } n = |\alpha| .$$

  This is a homogeneous power series of degree $|\alpha|$ again.
- The family $(F_\alpha)_{\alpha \text{ is a composition}}$ is a basis of the $\mathbb{Q}$-vector space QSym, called the *fundamental basis* (or F-basis). Sometimes, F_α is also denoted L_α.

The product formula for the F_α

- What connects QSym with shuffles of permutations is the following fact:

  Theorem. If π and σ are two disjoint permutations, then

$$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$

The product formula for the F_α

- What connects QSym with shuffles of permutations is the following fact:
  Theorem. If π and σ are two disjoint permutations, then
  $$F_{\operatorname{Comp} \pi} \cdot F_{\operatorname{Comp} \sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp} \tau}.$$
- This theorem yields that Des is shuffle-compatible. Why?

The product formula for the F_α

- What connects QSym with shuffles of permutations is the following fact:
  Theorem. If π and σ are two disjoint permutations, then
  $$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$
- This theorem yields that Des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $|\pi| = |\pi'|$ and $|\sigma| = |\sigma'|$ and $\operatorname{Des}\pi = \operatorname{Des}\pi'$ and $\operatorname{Des}\sigma = \operatorname{Des}\sigma'$. We must prove that
    $$\{\operatorname{Des}\tau \mid \tau \in S(\pi,\sigma)\}_{\text{multiset}} = \left\{\operatorname{Des}\tau \mid \tau \in S\left(\pi', \sigma'\right)\right\}_{\text{multiset}}.$$

- What connects QSym with shuffles of permutations is the following fact:
  Theorem. If π and σ are two disjoint permutations, then
  $$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$
- This theorem yields that Des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $\operatorname{Comp}\pi = \operatorname{Comp}\pi'$ and $\operatorname{Comp}\sigma = \operatorname{Comp}\sigma'$.
    We must prove that
    $$\{\operatorname{Comp}\tau \mid \tau \in S(\pi,\sigma)\}_{\text{multiset}} = \left\{\operatorname{Comp}\tau \mid \tau \in S\left(\pi',\sigma'\right)\right\}_{\text{multiset}}$$
    (this is equivalent to what we just said, since $\operatorname{Comp}\pi$ encodes the same data as $\operatorname{Des}\pi$ and $|\pi|$ together).

The product formula for the F_α

- What connects QSym with shuffles of permutations is the following fact:
  Theorem. If π and σ are two disjoint permutations, then
  $$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$
- This theorem yields that Des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $\operatorname{Comp}\pi = \operatorname{Comp}\pi'$ and $\operatorname{Comp}\sigma = \operatorname{Comp}\sigma'$.
    We must prove that
    $$\sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp}\tau} = \sum_{\tau \in S(\pi',\sigma')} F_{\operatorname{Comp}\tau}$$
    (this is equivalent to what we just said, since the F_α for α ranging over all compositions are linearly independent).

The product formula for the F_α

- What connects QSym with shuffles of permutations is the following fact:
  Theorem. If π and σ are two disjoint permutations, then

$$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$

- This theorem yields that Des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $\operatorname{Comp}\pi = \operatorname{Comp}\pi'$ and $\operatorname{Comp}\sigma = \operatorname{Comp}\sigma'$.
    We must prove that

$$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = F_{\operatorname{Comp}\pi'} \cdot F_{\operatorname{Comp}\sigma'}$$

    (this is equivalent to what we just said, by the Theorem above).

The product formula for the F_α

- What connects QSym with shuffles of permutations is the following fact:

  Theorem. If π and σ are two disjoint permutations, then

  $$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$

- This theorem yields that Des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $\operatorname{Comp}\pi = \operatorname{Comp}\pi'$ and $\operatorname{Comp}\sigma = \operatorname{Comp}\sigma'$.

    We must prove that

    $$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = F_{\operatorname{Comp}\pi'} \cdot F_{\operatorname{Comp}\sigma'}$$

    (this is equivalent to what we just said, by the Theorem above).

    But this follows from assumptions.

Shuffle-compatibility of des

- The same technique works for some other statistics. For example, we can show that des is shuffle-compatible.

Shuffle-compatibility of des

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, define the polynomial

$$f_{n,k} = x^n \binom{p - k + n}{n} \in \mathbb{Q}[p, x].$$

Shuffle-compatibility of des

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, define the polynomial
$$f_{n,k} = x^n \binom{p-k+n}{n} \in \mathbb{Q}[p, x].$$
- **Corollary (of preceding Theorem).** If π and σ are two disjoint permutations, with $n = |\pi|$ and $m = |\sigma|$, then
$$f_{n, \operatorname{des} \pi} \cdot f_{m, \operatorname{des} \sigma} = \sum_{\tau \in S(\pi,\sigma)} f_{n+m, \operatorname{des} \tau}.$$

Shuffle-compatibility of des

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, define the polynomial

$$f_{n,k} = x^n \binom{p-k+n}{n} \in \mathbb{Q}[p,x].$$

- **Corollary (of preceding Theorem).** If π and σ are two disjoint permutations, with $n = |\pi|$ and $m = |\sigma|$, then

$$f_{n,\operatorname{des}\pi} \cdot f_{m,\operatorname{des}\sigma} = \sum_{\tau \in S(\pi,\sigma)} f_{n+m,\operatorname{des}\tau}.$$

- **Proof idea (from Gessel/Zhuang).** There is a $\mathbb{Q}$-algebra homomorphism $\operatorname{QSym} \to \mathbb{Q}[p,x]$ sending each $g \in \operatorname{QSym}$ to

$$g\left(\underbrace{x, x, \ldots, x}_{p \text{ times}}, 0, 0, 0, \ldots\right)$$ (yes, this can be made sense of).

This is a variant of the *(generic) principal specialization*.

Shuffle-compatibility of des

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, define the polynomial

$$f_{n,k} = x^n \binom{p-k+n}{n} \in \mathbb{Q}[p, x].$$

- **Corollary (of preceding Theorem).** If π and σ are two disjoint permutations, with $n = |\pi|$ and $m = |\sigma|$, then

$$f_{n,\operatorname{des}\pi} \cdot f_{m,\operatorname{des}\sigma} = \sum_{\tau \in S(\pi,\sigma)} f_{n+m,\operatorname{des}\tau}.$$

- This corollary yields that des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $|\pi| = |\pi'|$ and $|\sigma| = |\sigma'|$ and $\operatorname{des}\pi = \operatorname{des}\pi'$ and $\operatorname{des}\sigma = \operatorname{des}\sigma'$. We must prove that

$$\{\operatorname{des}\tau \mid \tau \in S(\pi,\sigma)\}_{\text{multiset}} = \left\{\operatorname{des}\tau \mid \tau \in S(\pi',\sigma')\right\}_{\text{multiset}}.$$

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, define the polynomial

$$f_{n,k} = x^n \binom{p-k+n}{n} \in \mathbb{Q}[p, x].$$

- **Corollary (of preceding Theorem).** If π and σ are two disjoint permutations, with $n = |\pi|$ and $m = |\sigma|$, then

$$f_{n, \operatorname{des} \pi} \cdot f_{m, \operatorname{des} \sigma} = \sum_{\tau \in S(\pi,\sigma)} f_{n+m, \operatorname{des} \tau}.$$

- This corollary yields that des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $|\pi| = |\pi'|$ and $|\sigma| = |\sigma'|$ and $\operatorname{des} \pi = \operatorname{des} \pi'$ and $\operatorname{des} \sigma = \operatorname{des} \sigma'$. We must prove that

$$\sum_{\tau \in S(\pi,\sigma)} f_{n+m, \operatorname{des} \tau} = \sum_{\tau \in S(\pi',\sigma')} f_{n+m, \operatorname{des} \tau},$$

where $n = |\pi| = |\pi'|$ and $m = |\sigma| = |\sigma'|$ (this is equivalent to what we just said, since the $f_{n,k}$ for $n, k \in \mathbb{N}$ are linearly independent).

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, define the polynomial
$$f_{n,k} = x^n \dbinom{p-k+n}{n} \in \mathbb{Q}[p, x].$$
- **Corollary (of preceding Theorem).** If π and σ are two disjoint permutations, with $n = |\pi|$ and $m = |\sigma|$, then
$$f_{n,\operatorname{des}\pi} \cdot f_{m,\operatorname{des}\sigma} = \sum_{\tau \in S(\pi,\sigma)} f_{n+m,\operatorname{des}\tau}.$$
- This corollary yields that des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $|\pi| = |\pi'|$ and $|\sigma| = |\sigma'|$ and $\operatorname{des}\pi = \operatorname{des}\pi'$ and $\operatorname{des}\sigma = \operatorname{des}\sigma'$. We must prove that
$$f_{n,\operatorname{des}\pi} \cdot f_{m,\operatorname{des}\sigma} = f_{n,\operatorname{des}\pi'} \cdot f_{m,\operatorname{des}\sigma'}$$
(this is equivalent to what we just said, by the Corollary above).

Shuffle-compatibility of des

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, define the polynomial

$$f_{n,k} = x^n \binom{p-k+n}{n} \in \mathbb{Q}[p, x].$$

- **Corollary (of preceding Theorem).** If π and σ are two disjoint permutations, with $n = |\pi|$ and $m = |\sigma|$, then

$$f_{n,\operatorname{des}\pi} \cdot f_{m,\operatorname{des}\sigma} = \sum_{\tau \in S(\pi,\sigma)} f_{n+m,\operatorname{des}\tau}.$$

- This corollary yields that des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $|\pi| = |\pi'|$ and $|\sigma| = |\sigma'|$ and $\operatorname{des}\pi = \operatorname{des}\pi'$ and $\operatorname{des}\sigma = \operatorname{des}\sigma'$.
    We must prove that

$$f_{n,\operatorname{des}\pi} \cdot f_{m,\operatorname{des}\sigma} = f_{n,\operatorname{des}\pi'} \cdot f_{m,\operatorname{des}\sigma'}$$

    (this is equivalent to what we just said, by the Corollary above).
    But this follows from assumptions.

- The above arguments can be abstracted into a general criterion for shuffle-compatibility of a descent statistic (Gessel and Zhuang, in arXiv:1706.00750v2, Section 4.1). QSym and $\mathbb{Q}[p, x]$ get replaced by a "shuffle algebra" with an algebra homomorphism from QSym.
- We shall give our own variant of the criterion.

The kernel criterion for shuffle-compatibility, 2

- If st is a descent statistic, then two compositions α and β are said to be st-*equivalent* if $|\alpha| = |\beta|$ and $\operatorname{st}\alpha = \operatorname{st}\beta$.
  (Remember: $\operatorname{st}\alpha$ means $\operatorname{st}\pi$ for any permutation π satisfying $\operatorname{Comp}\pi = \alpha$.)

The kernel criterion for shuffle-compatibility, 2

- If st is a descent statistic, then two compositions α and β are said to be *st-equivalent* if $|\alpha| = |\beta|$ and $\operatorname{st} \alpha = \operatorname{st} \beta$. (Remember: $\operatorname{st} \alpha$ means $\operatorname{st} \pi$ for any permutation π satisfying $\operatorname{Comp} \pi = \alpha$.)
- The *kernel* $\mathcal{K}_{\text{st}}$ of a descent statistic st is the $\mathbb{Q}$-vector subspace of QSym spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st-equivalent compositions:

$$\mathcal{K}_{\text{st}} = \langle F_\alpha - F_\beta \mid |\alpha| = |\beta| \text{ and } \operatorname{st} \alpha = \operatorname{st} \beta \rangle_{\mathbb{Q}} .$$

- If st is a descent statistic, then two compositions α and β are said to be st-*equivalent* if $|\alpha| = |\beta|$ and $\operatorname{st}\alpha = \operatorname{st}\beta$. (Remember: $\operatorname{st}\alpha$ means $\operatorname{st}\pi$ for any permutation π satisfying $\operatorname{Comp}\pi = \alpha$.)
- The *kernel* $\mathcal{K}_{\mathrm{st}}$ of a descent statistic st is the $\mathbb{Q}$-vector subspace of QSym spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st-equivalent compositions:

$$\mathcal{K}_{\mathrm{st}} = \langle F_\alpha - F_\beta \ \mid \ |\alpha| = |\beta| \text{ and } \operatorname{st}\alpha = \operatorname{st}\beta \rangle_{\mathbb{Q}} .$$

- **Theorem.** The descent statistic st is shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is an ideal of QSym.

Section 3

The exterior peak set

References:

- Darij Grinberg, *Shuffle-compatible permutation statistics II: the exterior peak set*, draft.
- John R. Stembridge, *Enriched P-partitions*, Trans. Amer. Math. Soc. 349 (1997), no. 2, pp. 763–788.
- T. Kyle Petersen, *Enriched P-partitions and peak algebras*, Adv. in Math. 209 (2007), pp. 561–610.

Roadmap to Epk

- We will now outline our proof that Epk is shuffle-compatible.
- The main idea is to imitate the above proof for Des, but instead of $F_{\operatorname{Comp}\pi}$ we'll now have some different power series (not in QSym).

Roadmap to Epk

- We will now outline our proof that Epk is shuffle-compatible.
- The main idea is to imitate the above proof for Des, but instead of $F_{\operatorname{Comp}\pi}$ we'll now have some different power series (not in QSym).
- The idea is not new. This is how Pk, Lpk and Rpk were proven shuffle-compatible.

Roadmap to Epk

- We will now outline our proof that Epk is shuffle-compatible.
- The main idea is to imitate the above proof for Des, but instead of $F_{\operatorname{Comp}\pi}$ we'll now have some different power series (not in QSym).
- The idea is not new. This is how Pk, Lpk and Rpk were proven shuffle-compatible.
- The main tool is the concept of $\mathcal{Z}$**-enriched** P**-partitions**: a generalization of
  - P-partitions (Stanley 1972);
  - enriched P-partitions (Stembridge 1997);
  - left enriched P-partitions (Petersen 2007),

  which are used in the proofs for Des, Pk and Lpk, respectively.

Roadmap to Epk

- We will now outline our proof that Epk is shuffle-compatible.
- The main idea is to imitate the above proof for Des, but instead of $F_{\text{Comp}\,\pi}$ we'll now have some different power series (not in QSym).
- The idea is not new. This is how Pk, Lpk and Rpk were proven shuffle-compatible.
- The main tool is the concept of $\mathcal{Z}$-**enriched** P-**partitions**: a generalization of
  - P-partitions (Stanley 1972);
  - enriched P-partitions (Stembridge 1997);
  - left enriched P-partitions (Petersen 2007),

  which are used in the proofs for Des, Pk and Lpk, respectively. (Yes, the $F_{\text{Comp}\,\pi} \cdot F_{\text{Comp}\,\sigma}$ theorem we used in proving Des follows from the theory of P-partitions.)

Roadmap to Epk

- We will now outline our proof that Epk is shuffle-compatible.
- The main idea is to imitate the above proof for Des, but instead of $F_{\text{Comp}\,\pi}$ we'll now have some different power series (not in QSym).
- The idea is not new. This is how Pk, Lpk and Rpk were proven shuffle-compatible.
- The main tool is the concept of $\mathcal{Z}$**-enriched** P**-partitions**: a generalization of
    - P-partitions (Stanley 1972);
    - enriched P-partitions (Stembridge 1997);
    - left enriched P-partitions (Petersen 2007),

  which are used in the proofs for Des, Pk and Lpk, respectively. (Yes, the $F_{\text{Comp}\,\pi} \cdot F_{\text{Comp}\,\sigma}$ theorem we used in proving Des follows from the theory of P-partitions.)
- The idea is simple, but the proof has technical parts I am not showing.

Labeled posets

- A *labeled poset* means a pair (P, γ) consisting of a finite poset $P = (X, \leq)$ and an injective map $\gamma : X \to A$ into some totally ordered set A. The injective map γ is called the *labeling* of the labeled poset (P, γ).

$\mathcal{N}$ and $\mathcal{Z}$: definitions

- Fix a totally ordered set $\mathcal{N}$, and denote its strict order relation by $\prec$.
- Let $+$ and $-$ be two distinct symbols.
  Let $\mathcal{Z}$ be a subset of the set $\mathcal{N} \times \{+, -\}$.
- **Intuition:** $\mathcal{N}$ is a set of letters that will index our indeterminates.
  $\mathcal{Z}$ is a set of "signed letters", which are pairs of a letter in $\mathcal{N}$ and a sign in $\{+, -\}$. (Not all such pairs must lie in $\mathcal{Z}$.)

$\mathcal{N}$ and $\mathcal{Z}$: definitions

- Fix a totally ordered set $\mathcal{N}$, and denote its strict order relation by $\prec$.
- Let $+$ and $-$ be two distinct symbols.
  Let $\mathcal{Z}$ be a subset of the set $\mathcal{N} \times \{+, -\}$.
- **Intuition:** $\mathcal{N}$ is a set of letters that will index our indeterminates.
  $\mathcal{Z}$ is a set of "signed letters", which are pairs of a letter in $\mathcal{N}$ and a sign in $\{+, -\}$. (Not all such pairs must lie in $\mathcal{Z}$.)
- If $n \in \mathcal{N}$, then we will denote the two elements $(n, +)$ and $(n, -)$ of $\mathcal{N} \times \{+, -\}$ by $+n$ and $-n$, respectively.

$\mathcal{N}$ and $\mathcal{Z}$: definitions

- Fix a totally ordered set $\mathcal{N}$, and denote its strict order relation by $\prec$.
- Let $+$ and $-$ be two distinct symbols.
  Let $\mathcal{Z}$ be a subset of the set $\mathcal{N} \times \{+, -\}$.
- **Intuition:** $\mathcal{N}$ is a set of letters that will index our indeterminates.
  $\mathcal{Z}$ is a set of "signed letters", which are pairs of a letter in $\mathcal{N}$ and a sign in $\{+, -\}$. (Not all such pairs must lie in $\mathcal{Z}$.)
- If $n \in \mathcal{N}$, then we will denote the two elements $(n, +)$ and $(n, -)$ of $\mathcal{N} \times \{+, -\}$ by $+n$ and $-n$, respectively.
- Let us totally order the set $\mathcal{Z}$ in such a way that the (strict) order relation $\prec$ satisfies

$$(n, s) \prec (n', s') \text{ if and only if either } n \prec n' \text{ or } \left(n = n' \text{ and } s = - \text{ and } s' = +\right).$$

$\mathcal{N}$ and $\mathcal{Z}$: definitions

- Fix a totally ordered set $\mathcal{N}$, and denote its strict order relation by $\prec$.
- Let $+$ and $-$ be two distinct symbols.
  Let $\mathcal{Z}$ be a subset of the set $\mathcal{N} \times \{+, -\}$.
- **Intuition:** $\mathcal{N}$ is a set of letters that will index our indeterminates.
  $\mathcal{Z}$ is a set of "signed letters", which are pairs of a letter in $\mathcal{N}$ and a sign in $\{+, -\}$. (Not all such pairs must lie in $\mathcal{Z}$.)
- If $n \in \mathcal{N}$, then we will denote the two elements $(n, +)$ and $(n, -)$ of $\mathcal{N} \times \{+, -\}$ by $+n$ and $-n$, respectively.
- Let us totally order the set $\mathcal{Z}$ in such a way that the (strict) order relation $\prec$ satisfies

$$(n, s) \prec (n', s') \text{ if and only if either } n \prec n' \\ \text{or } \left(n = n' \text{ and } s = - \text{ and } s' = +\right).$$

- Let $\operatorname{Pow} \mathcal{N}$ be the ring of all power series over $\mathbb{Q}$ in the indeterminates x_n for $n \in \mathcal{N}$.

$\mathcal{N}$ and $\mathcal{Z}$: example

- For an example of the setting just introduced, take $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Then,

  $$\mathcal{Z} \subseteq \mathbb{N} \times \{+, -\} = \{-0, +0, -1, +1, -2, +2, \dots\}.$$

  Note: $-0 \neq +0$, since these are shorthands for pairs, not numbers.

$\mathcal{N}$ and $\mathcal{Z}$: example

- For an example of the setting just introduced, take $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Then,

$$\mathcal{Z} \subseteq \mathbb{N} \times \{+, -\} = \{-0, +0, -1, +1, -2, +2, \ldots\}\,.$$

  Note: $-0 \neq +0$, since these are shorthands for pairs, not numbers.

- The total order $\prec$ on $\mathcal{Z}$ is the restriction of

$$-0 \prec +0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots\,.$$

$\mathcal{N}$ and $\mathcal{Z}$: example

- For an example of the setting just introduced, take $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Then,

$$\mathcal{Z} \subseteq \mathbb{N} \times \{+, -\} = \{-0, +0, -1, +1, -2, +2, \ldots\}.$$

  Note: $-0 \neq +0$, since these are shorthands for pairs, not numbers.

- The total order $\prec$ on $\mathcal{Z}$ is the restriction of

$$-0 \prec +0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots.$$

- $\operatorname{Pow} \mathcal{N} = \mathbb{Q}[[x_0, x_1, x_2, \ldots]]$.

$\mathcal{Z}$-enriched (P, γ)-partitions: definition

- Now, let (P, γ) be a labeled poset. A *$\mathcal{Z}$-enriched (P, γ)-partition* means a map $f : P \to \mathcal{Z}$ such that for all $x < y$ in P, the following conditions hold:

  (i) We have $f(x) \preccurlyeq f(y)$.

  (ii) If $f(x) = f(y) = +n$ for some $n \in \mathcal{N}$, then $\gamma(x) < \gamma(y)$.

  (iii) If $f(x) = f(y) = -n$ for some $n \in \mathcal{N}$, then $\gamma(x) > \gamma(y)$.

  (Keep in mind: $\mathcal{N}$ and $\mathcal{Z}$ are fixed.)

$\mathcal{Z}$-enriched (P, γ)-partitions: definition

- Now, let (P, γ) be a labeled poset. A *$\mathcal{Z}$-enriched (P, γ)-partition* means a map $f : P \to \mathcal{Z}$ such that for all $x < y$ in P, the following conditions hold:

  (i) We have $f(x) \preccurlyeq f(y)$.
  (ii) If $f(x) = f(y) = +n$ for some $n \in \mathcal{N}$, then $\gamma(x) < \gamma(y)$.
  (iii) If $f(x) = f(y) = -n$ for some $n \in \mathcal{N}$, then $\gamma(x) > \gamma(y)$.

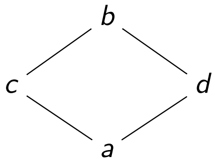
  (Keep in mind: $\mathcal{N}$ and $\mathcal{Z}$ are fixed.)

- **(Attempt at) intuition:** A $\mathcal{Z}$-enriched (P, γ)-partition is a map $f : P \to \mathcal{Z}$ (that is, assigning a signed letter to each poset element) which

  (i) is weakly increasing on P;
  (ii) + (iii) is occasionally strictly increasing, when γ and the sign of the f-value "are out of alignment".

$\mathcal{Z}$-enriched (P, γ)-partitions: example

- Let P be the poset with the following Hasse diagram:



and let $\gamma : P \to \mathbb{Z}$ be a labeling that satisfies $\gamma(a) < \gamma(b) < \gamma(c) < \gamma(d)$ (for example, γ could be the map that sends a, b, c, d to $2, 3, 5, 7$, respectively). Then, a $\mathcal{Z}$-enriched (P, γ)-partition is a map $f : P \to \mathcal{Z}$ satisfying the following conditions:

(i) We have $f(a) \preccurlyeq f(c) \preccurlyeq f(b)$ and $f(a) \preccurlyeq f(d) \preccurlyeq f(b)$.

(ii) We cannot have $f(c) = f(b) = +n$ with $n \in \mathcal{N}$. Also, we cannot have $f(d) = f(b) = +n$ with $n \in \mathcal{N}$.

(iii) We cannot have $f(a) = f(c) = -n$ with $n \in \mathcal{N}$. Also, we cannot have $f(a) = f(d) = -n$ with $n \in \mathcal{N}$.

$\mathcal{Z}$-enriched (P, γ)-partitions: revisiting the literature

- Consider again the case when $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Let us see what $\mathcal{Z}$-enriched (P, γ)-partitions are, depending on $\mathcal{Z}$.

$\mathcal{Z}$-enriched (P, γ)-partitions: revisiting the literature

- Consider again the case when $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Let us see what $\mathcal{Z}$-enriched (P, γ)-partitions are, depending on $\mathcal{Z}$.
- If $\mathcal{Z} = \mathbb{N} \times \{+\} = \{+0 \prec +1 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are just the (usual) (P, γ)-partitions into $\mathbb{N}$ (up to renaming n as $+n$).

$\mathcal{Z}$-enriched (P, γ)-partitions: revisiting the literature

- Consider again the case when $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Let us see what $\mathcal{Z}$-enriched (P, γ)-partitions are, depending on $\mathcal{Z}$.
- If $\mathcal{Z} = \mathbb{N} \times \{+\} = \{+0 \prec +1 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are just the (usual) (P, γ)-partitions into $\mathbb{N}$ (up to renaming n as $+n$).
- If $\mathcal{Z} = \mathbb{N} \times \{+, -\} =$ $\{-0 \prec +0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are Stembridge's enriched (P, γ)-partitions (up to renaming n as $n - 1$).

$\mathcal{Z}$-enriched (P, γ)-partitions: revisiting the literature

- Consider again the case when $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Let us see what $\mathcal{Z}$-enriched (P, γ)-partitions are, depending on $\mathcal{Z}$.
- If $\mathcal{Z} = \mathbb{N} \times \{+\} = \{+0 \prec +1 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are just the (usual) (P, γ)-partitions into $\mathbb{N}$ (up to renaming n as $+n$).
- If $\mathcal{Z} = \mathbb{N} \times \{+, -\} = \{-0 \prec +0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are Stembridge's enriched (P, γ)-partitions (up to renaming n as $n - 1$).
- If $\mathcal{Z} = (\mathbb{N} \times \{+, -\}) \setminus \{-0\} = \{+0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are Petersen's left enriched (P, γ)-partitions.

$\mathcal{Z}$-enriched (P, γ)-partitions: revisiting the literature

- Consider again the case when $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Let us see what $\mathcal{Z}$-enriched (P, γ)-partitions are, depending on $\mathcal{Z}$.
- If $\mathcal{Z} = \mathbb{N} \times \{+\} = \{+0 \prec +1 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are just the (usual) (P, γ)-partitions into $\mathbb{N}$ (up to renaming n as $+n$).
- If $\mathcal{Z} = \mathbb{N} \times \{+, -\} = \{-0 \prec +0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are Stembridge's enriched (P, γ)-partitions (up to renaming n as $n-1$).
- If $\mathcal{Z} = (\mathbb{N} \times \{+, -\}) \setminus \{-0\} = \{+0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are Petersen's left enriched (P, γ)-partitions.
- We shall later focus on the case when $\mathcal{N} = \mathbb{N} \cup \{\infty\}$ and $\mathcal{Z} = (\mathcal{N} \times \{+, -\}) \setminus \{-0, +\infty\}$.

$\mathcal{E}(P, \gamma)$ and $\mathcal{L}(P)$

- A few more notations are needed.
- If (P, γ) is a labeled poset, then $\mathcal{E}(P, \gamma)$ shall denote the set of all $\mathcal{Z}$-enriched (P, γ)-partitions.

$\mathcal{E}(P,\gamma)$ and $\mathcal{L}(P)$

- A few more notations are needed.
- If (P,γ) is a labeled poset, then $\mathcal{E}(P,\gamma)$ shall denote the set of all $\mathcal{Z}$-enriched (P,γ)-partitions.
- If P is any poset, then $\mathcal{L}(P)$ shall denote the set of all linear extensions of P.
  A linear extension of P shall be understood simultaneously as a totally ordered set extending P and as a list $(w_1, w_2, \ldots, w_n)$ of all elements of P such that no two integers $i < j$ satisfy $w_i \geq w_j$ in P.

Any $\mathcal{E}(P,\gamma)$-partition has its favorite linear extension

- **Proposition.** For any labeled poset (P,γ), we have

$$\mathcal{E}(P,\gamma) = \bigsqcup_{w \in \mathcal{L}(P)} \mathcal{E}(w,\gamma).$$

- This is a generalization of a standard result on P-partitions ("Stanley's main lemma"), and is proven by the same reasoning.

The power series $\Gamma_{\mathcal{Z}}(P, \gamma)$

- Let (P, γ) be a labeled poset. We define a power series $\Gamma_{\mathcal{Z}}(P, \gamma) \in \operatorname{Pow} \mathcal{N}$ by

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{f \in \mathcal{E}(P,\gamma)} \prod_{p \in P} x_{|f(p)|}.$$

Here, $|f(p)| \in \mathcal{N}$ is defined to be the first entry of $f(p)$ (recall: $f(p)$ is a pair of an element of $\mathcal{N}$ and a sign in $\{+, -\}$).

The power series $\Gamma_{\mathcal{Z}}(P, \gamma)$

- Let (P, γ) be a labeled poset. We define a power series $\Gamma_{\mathcal{Z}}(P, \gamma) \in \operatorname{Pow} \mathcal{N}$ by

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{f \in \mathcal{E}(P, \gamma)} \prod_{p \in P} x_{|f(p)|}.$$

  Here, $|f(p)| \in \mathcal{N}$ is defined to be the first entry of $f(p)$ (recall: $f(p)$ is a pair of an element of $\mathcal{N}$ and a sign in $\{+, -\}$).
- This generalizes the classical quasisymmetric P-partition enumerators (which give the fundamental basis F_α when P is totally ordered).

The power series $\Gamma_{\mathcal{Z}}(P, \gamma)$

- Let (P, γ) be a labeled poset. We define a power series $\Gamma_{\mathcal{Z}}(P, \gamma) \in \operatorname{Pow} \mathcal{N}$ by
$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{f \in \mathcal{E}(P,\gamma)} \prod_{p \in P} x_{|f(p)|}.$$
Here, $|f(p)| \in \mathcal{N}$ is defined to be the first entry of $f(p)$ (recall: $f(p)$ is a pair of an element of $\mathcal{N}$ and a sign in $\{+, -\}$).
- This generalizes the classical quasisymmetric P-partition enumerators (which give the fundamental basis F_α when P is totally ordered).
- **Corollary.** For any labeled poset (P, γ), we have
$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{w \in \mathcal{L}(P)} \Gamma_{\mathcal{Z}}(w, \gamma).$$

The power series $\Gamma_{\mathcal{Z}}(P,\gamma)$

- Let (P,γ) be a labeled poset. We define a power series $\Gamma_{\mathcal{Z}}(P,\gamma) \in \operatorname{Pow}\mathcal{N}$ by

$$\Gamma_{\mathcal{Z}}(P,\gamma) = \sum_{f \in \mathcal{E}(P,\gamma)} \prod_{p \in P} x_{|f(p)|}.$$

  Here, $|f(p)| \in \mathcal{N}$ is defined to be the first entry of $f(p)$ (recall: $f(p)$ is a pair of an element of $\mathcal{N}$ and a sign in $\{+,-\}$).
- This generalizes the classical quasisymmetric P-partition enumerators (which give the fundamental basis F_α when P is totally ordered).
- **Question.** Where do these $\Gamma_{\mathcal{Z}}(P,\gamma)$ live (other than in $\operatorname{Pow}\mathcal{N}$) ?
  I don't know a good answer; it should be a generalization of QSym.
  Jia Huang's work (arXiv:1506.02962v2) looks relevant.

Disjoint unions give product of Γ's

- Let P be any set. Let A be a totally ordered set. Let $\gamma : P \to A$ and $\delta : P \to A$ be two maps. We say that γ and δ are *order-equivalent* if the following holds: For every pair $(p, q) \in P \times P$, we have $\gamma(p) \leq \gamma(q)$ if and only if $\delta(p) \leq \delta(q)$.

Disjoint unions give product of Γ's

- Let P be any set. Let A be a totally ordered set. Let $\gamma : P \to A$ and $\delta : P \to A$ be two maps. We say that γ and δ are *order-equivalent* if the following holds: For every pair $(p, q) \in P \times P$, we have $\gamma(p) \leq \gamma(q)$ if and only if $\delta(p) \leq \delta(q)$.
- **Proposition.** Let (P, γ) and (Q, δ) be two labeled posets. Let $(P \sqcup Q, \varepsilon)$ be the labeled poset
  - for which $P \sqcup Q$ is the disjoint union of P and Q, and
  - whose labeling ε is such that the restriction of ε to P is order-equivalent to γ and such that the restriction of ε to Q is order-equivalent to δ.

  Then,
  $$\Gamma_{\mathcal{Z}}(P, \gamma) \cdot \Gamma_{\mathcal{Z}}(Q, \delta) = \Gamma_{\mathcal{Z}}(P \sqcup Q, \varepsilon).$$
- Again, the proof is simple.

From a permutation to a labeled poset

- Let $n \in \mathbb{N}$. Write $[n]$ for $\{1, 2, \ldots, n\}$.
  Let π be any n-permutation. Consider π as an injective map $[n] \to \{1, 2, 3, \ldots\}$ (sending i to π_i). Thus, $([n], \pi)$ is a labeled poset. We define $\Gamma_{\mathcal{Z}}(\pi)$ to be the power series $\Gamma_{\mathcal{Z}}([n], \pi)$.

From a permutation to a labeled poset

- Let $n \in \mathbb{N}$. Write $[n]$ for $\{1, 2, \ldots, n\}$.
  Let π be any n-permutation. Consider π as an injective map $[n] \to \{1, 2, 3, \ldots\}$ (sending i to π_i). Thus, $([n], \pi)$ is a labeled poset. We define $\Gamma_{\mathcal{Z}}(\pi)$ to be the power series $\Gamma_{\mathcal{Z}}([n], \pi)$.
- Explicitly:
  $$\Gamma_{\mathcal{Z}}(\pi) = \sum x_{|j_1|} x_{|j_2|} \cdots x_{|j_n|},$$
  where the sum is over all n-tuples $(j_1, j_2, \ldots, j_n) \in \mathcal{Z}^n$ having the properties that:
  (i) $j_1 \preccurlyeq j_2 \preccurlyeq \cdots \preccurlyeq j_n$;
  (ii) if $j_k = j_{k+1} = +s$ for some $s \in \mathcal{N}$, then $\pi_k < \pi_{k+1}$;
  (iii) if $j_k = j_{k+1} = -s$ for some $s \in \mathcal{N}$, then $\pi_k > \pi_{k+1}$.
- This $\Gamma_{\mathcal{Z}}(\pi)$ will serve as an analogue of $F_{\operatorname{Comp} \pi}$.

From a permutation to a labeled poset

- Let $n \in \mathbb{N}$. Write $[n]$ for $\{1, 2, \ldots, n\}$.
  Let π be any n-permutation. Consider π as an injective map $[n] \to \{1, 2, 3, \ldots\}$ (sending i to π_i). Thus, $([n], \pi)$ is a labeled poset. We define $\Gamma_{\mathcal{Z}}(\pi)$ to be the power series $\Gamma_{\mathcal{Z}}([n], \pi)$.
- **Proposition.** Let w be a finite totally ordered set with ground set W. Let $n = |W|$. Let $\overline{w}$ be the unique poset isomorphism $w \to [n]$. Let $\gamma : W \to \{1, 2, 3, \ldots\}$ be any injective map. Then, $\Gamma_{\mathcal{Z}}(w, \gamma) = \Gamma_{\mathcal{Z}}\left(\gamma \circ \overline{w}^{-1}\right)$.
- Again, this follows the roadmap of classical P-partition theory.

- Let $n \in \mathbb{N}$. Write $[n]$ for $\{1, 2, \ldots, n\}$.
  Let π be any n-permutation. Consider π as an injective map $[n] \to \{1, 2, 3, \ldots\}$ (sending i to π_i). Thus, $([n], \pi)$ is a labeled poset. We define $\Gamma_{\mathcal{Z}}(\pi)$ to be the power series $\Gamma_{\mathcal{Z}}([n], \pi)$.
- **Proposition.** Let w be a finite totally ordered set with ground set W. Let $n = |W|$. Let $\overline{w}$ be the unique poset isomorphism $w \to [n]$. Let $\gamma : W \to \{1, 2, 3, \ldots\}$ be any injective map. Then, $\Gamma_{\mathcal{Z}}(w, \gamma) = \Gamma_{\mathcal{Z}}\left(\gamma \circ \overline{w}^{-1}\right)$.
- Again, this follows the roadmap of classical P-partition theory.
- **Corollary.** Let (P, γ) be a labeled poset. Let $n = |P|$. Then,

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{\substack{x : P \to [n] \\ \text{bijective poset} \\ \text{homomorphism}}} \Gamma_{\mathcal{Z}}\left(\gamma \circ x^{-1}\right).$$

From a permutation to a labeled poset

- Let $n \in \mathbb{N}$. Write $[n]$ for $\{1, 2, \ldots, n\}$.
  Let π be any n-permutation. Consider π as an injective map $[n] \to \{1, 2, 3, \ldots\}$ (sending i to π_i). Thus, $([n], \pi)$ is a labeled poset. We define $\Gamma_{\mathcal{Z}}(\pi)$ to be the power series $\Gamma_{\mathcal{Z}}([n], \pi)$.
- **Proposition.** Let w be a finite totally ordered set with ground set W. Let $n = |W|$. Let $\overline{w}$ be the unique poset isomorphism $w \to [n]$. Let $\gamma : W \to \{1, 2, 3, \ldots\}$ be any injective map. Then, $\Gamma_{\mathcal{Z}}(w, \gamma) = \Gamma_{\mathcal{Z}}\left(\gamma \circ \overline{w}^{-1}\right)$.
- Again, this follows the roadmap of classical P-partition theory.
- **Corollary.** Let (P, γ) be a labeled poset. Let $n = |P|$. Then,

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{\substack{x:P \to [n] \\ \text{bijective poset} \\ \text{homomorphism}}} \Gamma_{\mathcal{Z}}\left(\gamma \circ x^{-1}\right).$$

- Thus, the $\Gamma_{\mathcal{Z}}$ of any labeled poset can be described in terms of the $\Gamma_{\mathcal{Z}}(\pi)$.

The product formula for the $\Gamma_{\mathcal{Z}}(P, \gamma)$

- Combining the above results, we see:
  Theorem. Let π and σ be two disjoint permutations. Then,

$$\Gamma_{\mathcal{Z}}(\pi) \cdot \Gamma_{\mathcal{Z}}(\sigma) = \sum_{\tau \in S(\pi, \sigma)} \Gamma_{\mathcal{Z}}(\tau).$$

- Combining the above results, we see:
  Theorem. Let π and σ be two disjoint permutations. Then,

$$\Gamma_{\mathcal{Z}}(\pi) \cdot \Gamma_{\mathcal{Z}}(\sigma) = \sum_{\tau \in S(\pi,\sigma)} \Gamma_{\mathcal{Z}}(\tau).$$

- This generalizes the

$$F_{\operatorname{Comp} \pi} \cdot F_{\operatorname{Comp} \sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp} \tau}$$

  formula in QSym (which you can recover by setting $\mathcal{N} = \mathbb{N}$ and $\mathcal{Z} = \mathbb{N} \times \{+\} = \{+0 \prec +1 \prec +2 \prec \cdots\}$).
- Likewise, you can recover similar results by Stembridge and Petersen from this.

Customizing the setting for Epk

- Remember: we want to show Epk is shuffle-compatible.
- Specialize the above setting as follows:
  - Set $\mathcal{N} = \{0, 1, 2, \ldots\} \cup \{\infty\}$, with total order given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.
  - Set

$$\begin{aligned} \mathcal{Z} &= (\mathcal{N} \times \{+, -\}) \setminus \{-0, +\infty\} \\ &= \{+0\} \cup \{+n \mid n \in \{1, 2, 3, \ldots\}\} \\ &\qquad \cup \{-n \mid n \in \{1, 2, 3, \ldots\}\} \cup \{-\infty\}. \end{aligned}$$

    Recall that the total order on $\mathcal{Z}$ has

$$+0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots \prec -\infty.$$

Fiber-ends

- Let $n \in \mathbb{N}$. Let $g : [n] \to \mathcal{N}$ be any map. We define a subset $\operatorname{FE}(g)$ of $[n]$ by

$$\operatorname{FE}(g) = \left\{\min\left(g^{-1}(h)\right) \mid h \in \{1, 2, 3, \ldots, \infty\}\right\} \cup \left\{\max\left(g^{-1}(h)\right) \mid h \in \{0, 1, 2, 3, \ldots\}\right\}$$

  (ignore the maxima/minima of empty fibers).
  In other words, $\operatorname{FE}(g)$ is the set comprising
  - the smallest elements of all nonempty fibers of g except for $g^{-1}(0)$ as well as
  - the largest elements of all nonempty fibers of g except for $g^{-1}(\infty)$.

K-series

- Let $n \in \mathbb{N}$. If Λ (no connection to symmetric functions) is any subset of $[n]$, then we define a power series $K_{n,\Lambda}^{\mathcal{Z}} \in \operatorname{Pow} \mathcal{N}$ by

$$K_{n,\Lambda}^{\mathcal{Z}} = \sum_{\substack{g:[n]\to\mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda \subseteq \operatorname{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\ldots\}|} x_{g(1)} x_{g(2)} \cdots x_{g(n)}.$$

- **Proposition.** Let $n \in \mathbb{N}$. Let π be an n-permutation. Then,

$$\Gamma_{\mathcal{Z}}(\pi) = K_{n,\operatorname{Epk}\pi}^{\mathcal{Z}}.$$

  This is proven by a counting argument (if a map g comes from an $([n], \pi)$-partition, then the fibers of g subdivide $[n]$ into intervals on which π is "V-shaped"; a peak can only occur at a border between two such intervals).

K-series

- Let $n \in \mathbb{N}$. If Λ (no connection to symmetric functions) is any subset of $[n]$, then we define a power series $K_{n,\Lambda}^{\mathcal{Z}} \in \operatorname{Pow} \mathcal{N}$ by

$$K_{n,\Lambda}^{\mathcal{Z}} = \sum_{\substack{g:[n]\to\mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda \subseteq \operatorname{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\ldots\}|} x_{g(1)} x_{g(2)} \cdots x_{g(n)}.$$

- **Proposition.** Let $n \in \mathbb{N}$. Let π be an n-permutation. Then,

$$\Gamma_{\mathcal{Z}}(\pi) = K_{n,\operatorname{Epk}\pi}^{\mathcal{Z}}.$$

- Thus, the product formula above specializes to

$$K_{n,\operatorname{Epk}\pi}^{\mathcal{Z}} \cdot K_{m,\operatorname{Epk}\sigma}^{\mathcal{Z}} = \sum_{\tau \in S(\pi,\sigma)} K_{n+m,\operatorname{Epk}\tau}^{\mathcal{Z}}.$$

- This formula is used to show that Epk is shuffle-compatible, but we need a bit more: we need to show that the "relevant" $K_{n,\Lambda}^{\mathcal{Z}}$ are linearly independent.

- Let $n \in \mathbb{N}$. If Λ (no connection to symmetric functions) is any subset of $[n]$, then we define a power series $K_{n,\Lambda}^{\mathcal{Z}} \in \operatorname{Pow} \mathcal{N}$ by

$$K_{n,\Lambda}^{\mathcal{Z}} = \sum_{\substack{g:[n]\to\mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda \subseteq \operatorname{FE}(g)}} 2^{|g([n]) \cap \{1,2,3,\ldots\}|} x_{g(1)} x_{g(2)} \cdots x_{g(n)}.$$

- **Proposition.** Let $n \in \mathbb{N}$. Let π be an n-permutation. Then,

$$\Gamma_{\mathcal{Z}}(\pi) = K_{n,\operatorname{Epk}\pi}^{\mathcal{Z}}.$$

- Thus, the product formula above specializes to

$$K_{n,\operatorname{Epk}\pi}^{\mathcal{Z}} \cdot K_{m,\operatorname{Epk}\sigma}^{\mathcal{Z}} = \sum_{\tau \in S(\pi,\sigma)} K_{n+m,\operatorname{Epk}\tau}^{\mathcal{Z}}.$$

- This formula is used to show that Epk is shuffle-compatible, but we need a bit more: we need to show that the "relevant" $K_{n,\Lambda}^{\mathcal{Z}}$ are linearly independent.
- Not all $K_{n,\Lambda}^{\mathcal{Z}}$ are linearly independent. Rather, we need to pick the right subset.

Lacunar subsets and linear independence

- A set S of integers is called *lacunar* if it contains no two consecutive integers.
- **Well-known fact:** The number of lacunar subsets of $[n]$ is the Fibonacci number f_{n+1}.

Lacunar subsets and linear independence

- A set S of integers is called *lacunar* if it contains no two consecutive integers.
- **Well-known fact:** The number of lacunar subsets of $[n]$ is the Fibonacci number f_{n+1}.
- **Lemma.** For each permutation π, the set $\operatorname{Epk} \pi$ is a nonempty lacunar subset of $[n]$.
  (And conversely – although we won't need it –, any such subset has the form $\operatorname{Epk} \pi$ for some π.)

Lacunar subsets and linear independence

- A set S of integers is called *lacunar* if it contains no two consecutive integers.
- **Well-known fact:** The number of lacunar subsets of $[n]$ is the Fibonacci number f_{n+1}.
- **Lemma.** For each permutation π, the set $\operatorname{Epk} \pi$ is a nonempty lacunar subset of $[n]$.
  (And conversely – although we won't need it –, any such subset has the form $\operatorname{Epk} \pi$ for some π.)
- **Lemma.** The family

$$\left(K_{n,\Lambda}^{\mathcal{Z}}\right)_{n \in \mathbb{N};\ \Lambda \subseteq [n] \text{ is lacunar and nonempty}}$$

  is $\mathbb{Q}$-linearly independent.
- This actually takes work to prove. But once proven, it completes the argument for the shuffle-compatibility of Epk.

- Recall: The *kernel* $\mathcal{K}_{\mathsf{st}}$ of a descent statistic st is the $\mathbb{Q}$-vector subspace of QSym spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st-equivalent compositions:

$$\mathcal{K}_{\mathsf{st}} = \langle F_\alpha - F_\beta \mid |\alpha| = |\beta| \text{ and } \mathsf{st}\,\alpha = \mathsf{st}\,\beta \rangle_{\mathbb{Q}} .$$

- Recall: The *kernel* $\mathcal{K}_{\text{st}}$ of a descent statistic st is the $\mathbb{Q}$-vector subspace of QSym spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st-equivalent compositions:

$$\mathcal{K}_{\text{st}} = \langle F_\alpha - F_\beta \mid |\alpha| = |\beta| \text{ and } \operatorname{st}\alpha = \operatorname{st}\beta \rangle_{\mathbb{Q}} .$$

- Since Epk is shuffle-compatible, its kernel $\mathcal{K}_{\text{Epk}}$ is an ideal of QSym. How can we describe it?
- Two ways: using the F-basis and using the M-basis.

The kernel $\mathcal{K}_{\text{Epk}}$ in terms of the F-basis

- If $J = (j_1, j_2, \ldots, j_m)$ and K are two compositions, then we write $J \to K$ if there exists an $\ell \in \{2, 3, \ldots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \ldots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m)$. (In other words, we write $J \to K$ if K can be obtained from J by "splitting" some non-initial entry $j_\ell > 2$ into two consecutive entries 1 and $j_\ell - 1$.)
- **Example.** Here are all instances of the $\to$ relation on compositions of size ≤ 5:

$$(1,3) \to (1,1,2), \qquad (1,4) \to (1,1,3),$$
$$(1,3,1) \to (1,1,2,1), \qquad (1,1,3) \to (1,1,1,2),$$
$$(2,3) \to (2,1,2).$$

- **Proposition.** The ideal $\mathcal{K}_{\text{Epk}}$ of QSym is spanned (as a $\mathbb{Q}$-vector space) by all differences of the form $F_J - F_K$, where J and K are two compositions satisfying $J \to K$.

The kernel $\mathcal{K}_{\mathrm{Epk}}$ in terms of the M-basis

- If $J = (j_1, j_2, \ldots, j_m)$ and K are two compositions, then we write $J \underset{M}{\to} K$ if there exists an $\ell \in \{2, 3, \ldots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \ldots, j_{\ell-1}, 2, j_\ell - 2, j_{\ell+1}, j_{\ell+2}, \ldots, j_m)$. (In other words, we write $J \underset{M}{\to} K$ if K can be obtained from J by "splitting" some non-initial entry $j_\ell > 2$ into two consecutive entries 2 and $j_\ell - 2$.)
- **Example.** Here are all instances of the $\underset{M}{\to}$ relation on compositions of size ≤ 5:

$$(1,3) \underset{M}{\to} (1,2,1), \qquad (1,4) \underset{M}{\to} (1,2,2),$$
$$(1,3,1) \underset{M}{\to} (1,2,1,1), \qquad (1,1,3) \underset{M}{\to} (1,1,2,1),$$
$$(2,3) \underset{M}{\to} (2,2,1).$$

- **Proposition.** The ideal $\mathcal{K}_{\mathrm{Epk}}$ of QSym is spanned (as a $\mathbb{Q}$-vector space) by all sums of the form $M_J + M_K$, where J and K are two compositions satisfying $J \underset{M}{\to} K$.

What about other statistics?

- **Question.** Do other descent statistics allow for similar descriptions of $\mathcal{K}_{\mathrm{st}}$?

Section 4

Left-/right-shuffle-compatibility

References:

- Darij Grinberg, *Shuffle-compatible permutation statistics II: the exterior peak set*, draft.
- Darij Grinberg, *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions*, Canad. J. Math. 69 (2017), pp. 21–53.

Left/right-shuffle-compatibility (repeated)

- We further begin the study of a finer version of shuffle-compatibility: "left/right-shuffle-compatibility".
- Given two disjoint nonempty permutations π and σ,
  - a *left shuffle* of π and σ is a shuffle of π and σ that starts with a letter of π;
  - a *right shuffle* of π and σ is a shuffle of π and σ that starts with a letter of σ.
- We let $S_{\prec}(\pi, \sigma)$ be the set of all left shuffles of π and σ. We let $S_{\succ}(\pi, \sigma)$ be the set of all right shuffles of π and σ.
- A statistic st is said to be *left-shuffle-compatible* if for any two disjoint nonempty permutations π and σ such that

  the first entry of π is greater than the first entry of σ,

  the multiset

  $$\{\operatorname{st} \tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multiset}}$$

  depends only on $\operatorname{st} \pi$, $\operatorname{st} \sigma$, $|\pi|$ and $|\sigma|$.
- We show that Des, des, Lpk and Epk are left- and right-shuffle-compatible. (But not maj or Rpk.)

- This proof will use a **dendriform algebra** structure on QSym, as well as two other operations and a bit of the Hopf algebra structure.
  I don't know of a combinatorial proof.
- This structure first appeared in:
  Darij Grinberg, *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions*, Canad. J. Math. 69 (2017), pp. 21–53.
  But the ideas go back to:
  - Glânffrwd P. Thomas, *Frames, Young tableaux, and Baxter sequences*, Advances in Mathematics, Volume 26, Issue 3, December 1977, Pages 275–289.
  - Jean-Christophe Novelli, Jean-Yves Thibon, *Construction of dendriform trialgebras*, arXiv:math/0510218.

  Something similar also appeared in: Aristophanes Dimakis, Folkert Müller-Hoissen, *Quasi-symmetric functions and the KP hierarchy*, Journal of Pure and Applied Algebra, Volume 214, Issue 4, April 2010, Pages 449–460.

Dendriform structure on QSym, part 1

- For any monomial $\mathfrak{m}$, let $\operatorname{Supp} \mathfrak{m}$ denote the set $\{i \mid x_i \text{ appears in } \mathfrak{m}\}$.
- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.

Dendriform structure on QSym, part 1

- For any monomial $\mathfrak{m}$, let $\operatorname{Supp}\mathfrak{m}$ denote the set $\{i \mid x_i \text{ appears in } \mathfrak{m}\}$.
- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.
- We define a binary operation $\prec$ on the $\mathbb{Q}$-vector space $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ as follows:
  - On monomials, it should be given by

$$\mathfrak{m} \prec \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) < \min(\operatorname{Supp}\mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) \geq \min(\operatorname{Supp}\mathfrak{n}) \end{cases}$$

    for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbb{Q}$-bilinear.
  - It should be continuous (i.e., its $\mathbb{Q}$-bilinearity also applies to infinite $\mathbb{Q}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $\left(x_2^2 x_4\right) \prec \left(x_3^2 x_5\right) = x_2^2 x_3^2 x_4 x_5$, but $\left(x_2^2 x_4\right) \prec \left(x_2^2 x_5\right) = 0$.

Dendriform structure on QSym, part 1

- For any monomial $\mathfrak{m}$, let $\operatorname{Supp}\mathfrak{m}$ denote the set $\{i \mid x_i \text{ appears in } \mathfrak{m}\}$.
- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.
- We define a binary operation $\succeq$ on the $\mathbb{Q}$-vector space $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ as follows:
  - On monomials, it should be given by
$$\mathfrak{m} \succeq \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) \geq \min(\operatorname{Supp}\mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) < \min(\operatorname{Supp}\mathfrak{n}) \end{cases}$$
    for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbb{Q}$-bilinear.
  - It should be continuous (i.e., its $\mathbb{Q}$-bilinearity also applies to infinite $\mathbb{Q}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $\left(x_2^2 x_4\right) \succeq \left(x_3^2 x_5\right) = 0$, but $\left(x_2^2 x_4\right) \succeq \left(x_2^2 x_5\right) = x_2^4 x_4 x_5$.

Dendriform structure on QSym, part 2

- We now have defined two binary operations $\prec$ and $\succeq$ on $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. They satisfy:

$$a \prec b + a \succeq b = ab;$$
$$(a \prec b) \prec c = a \prec (bc);$$
$$(a \succeq b) \prec c = a \succeq (b \prec c);$$
$$a \succeq (b \succeq c) = (ab) \succeq c.$$

Dendriform structure on QSym, part 2

- We now have defined two binary operations $\prec$ and $\succeq$ on $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. They satisfy:

$$a \prec b + a \succeq b = ab;$$
$$(a \prec b) \prec c = a \prec (bc);$$
$$(a \succeq b) \prec c = a \succeq (b \prec c);$$
$$a \succeq (b \succeq c) = (ab) \succeq c.$$

- This says that $(\mathbb{Q}[[x_1, x_2, x_3, \ldots]], \prec, \succeq)$ is a *dendriform algebra* in the sense of Loday (see, e.g., Zinbiel, *Encyclopedia of types of algebras 2010*, arXiv:1101.0267).

Dendriform structure on QSym, part 2

- We now have defined two binary operations $\prec$ and $\succeq$ on $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. They satisfy:

$$\begin{aligned} a \prec b + a \succeq b &= ab; \\ (a \prec b) \prec c &= a \prec (bc); \\ (a \succeq b) \prec c &= a \succeq (b \prec c); \\ a \succeq (b \succeq c) &= (ab) \succeq c. \end{aligned}$$

- This says that $(\mathbb{Q}[[x_1, x_2, x_3, \ldots]], \prec, \succeq)$ is a *dendriform algebra* in the sense of Loday (see, e.g., Zinbiel, *Encyclopedia of types of algebras 2010*, arXiv:1101.0267).
- QSym is closed under both operations $\prec$ and $\succeq$. Thus, QSym becomes a dendriform subalgebra of $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$.

The kernel criterion for left/right-shuffle-compatibility

- Recall the **Theorem:** The descent statistic st is shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is an ideal of QSym.

The kernel criterion for left/right-shuffle-compatibility

- Similarly, we have:
  - **Theorem.** The descent statistic st is left-shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is a $\prec$-ideal of QSym (that is: $\mathrm{QSym} \prec \mathcal{K}_{\mathrm{st}} \subseteq \mathcal{K}_{\mathrm{st}}$ and $\mathcal{K}_{\mathrm{st}} \prec \mathrm{QSym} \subseteq \mathcal{K}_{\mathrm{st}}$).
  - **Theorem.** The descent statistic st is right-shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is a $\succeq$-ideal of QSym (that is: $\mathrm{QSym} \succeq \mathcal{K}_{\mathrm{st}} \subseteq \mathcal{K}_{\mathrm{st}}$ and $\mathcal{K}_{\mathrm{st}} \succeq \mathrm{QSym} \subseteq \mathcal{K}_{\mathrm{st}}$).

The kernel criterion for left/right-shuffle-compatibility

- Similarly, we have:
  - **Theorem.** The descent statistic st is left-shuffle-compatible if and only if $\mathcal{K}_{\text{st}}$ is a $\prec$-ideal of QSym (that is: $\text{QSym} \prec \mathcal{K}_{\text{st}} \subseteq \mathcal{K}_{\text{st}}$ and $\mathcal{K}_{\text{st}} \prec \text{QSym} \subseteq \mathcal{K}_{\text{st}}$).
  - **Theorem.** The descent statistic st is right-shuffle-compatible if and only if $\mathcal{K}_{\text{st}}$ is a $\succeq$-ideal of QSym (that is: $\text{QSym} \succeq \mathcal{K}_{\text{st}} \subseteq \mathcal{K}_{\text{st}}$ and $\mathcal{K}_{\text{st}} \succeq \text{QSym} \subseteq \mathcal{K}_{\text{st}}$).
- **Corollary.** Let st be a descent statistic. If st has 2 of the 3 properties "shuffle-compatible", "left-shuffle-compatible" and "right-shuffle-compatible", then it has all 3.
  (To prove this, recall $ab = a \prec b + a \succeq b$.)

The kernel criterion for left/right-shuffle-compatibility

- Similarly, we have:
  - **Theorem.** The descent statistic st is left-shuffle-compatible if and only if $\mathcal{K}_{\text{st}}$ is a $\prec$-ideal of QSym (that is: $\text{QSym} \prec \mathcal{K}_{\text{st}} \subseteq \mathcal{K}_{\text{st}}$ and $\mathcal{K}_{\text{st}} \prec \text{QSym} \subseteq \mathcal{K}_{\text{st}}$).
  - **Theorem.** The descent statistic st is right-shuffle-compatible if and only if $\mathcal{K}_{\text{st}}$ is a $\succeq$-ideal of QSym (that is: $\text{QSym} \succeq \mathcal{K}_{\text{st}} \subseteq \mathcal{K}_{\text{st}}$ and $\mathcal{K}_{\text{st}} \succeq \text{QSym} \subseteq \mathcal{K}_{\text{st}}$).
- **Corollary.** Let st be a descent statistic. If st has 2 of the 3 properties "shuffle-compatible", "left-shuffle-compatible" and "right-shuffle-compatible", then it has all 3.
  (To prove this, recall $ab = a \prec b + a \succeq b$.)
- **Question.** Are there non-shuffle-compatible but left-shuffle-compatible descent statistics?
  (I don't know of any, but haven't looked far.)

The kernel criterion for left/right-shuffle-compatibility

- Similarly, we have:
    - **Theorem.** The descent statistic st is left-shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is a $\prec$-ideal of QSym (that is: $\mathrm{QSym} \prec \mathcal{K}_{\mathrm{st}} \subseteq \mathcal{K}_{\mathrm{st}}$ and $\mathcal{K}_{\mathrm{st}} \prec \mathrm{QSym} \subseteq \mathcal{K}_{\mathrm{st}}$).
    - **Theorem.** The descent statistic st is right-shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is a $\succeq$-ideal of QSym (that is: $\mathrm{QSym} \succeq \mathcal{K}_{\mathrm{st}} \subseteq \mathcal{K}_{\mathrm{st}}$ and $\mathcal{K}_{\mathrm{st}} \succeq \mathrm{QSym} \subseteq \mathcal{K}_{\mathrm{st}}$).
- **Corollary.** Let st be a descent statistic. If st has 2 of the 3 properties "shuffle-compatible", "left-shuffle-compatible" and "right-shuffle-compatible", then it has all 3.
  (To prove this, recall $ab = a \prec b + a \succeq b$.)
- Okay, but how do we actually prove that $\mathcal{K}_{\mathrm{st}}$ is a $\prec$-ideal of QSym ?

The dendriform product formula for the F_α

- An analogue of the product formula for $F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma}$:
  Theorem. Let π and σ be two disjoint nonempty permutations. Assume that

  the first entry of π is greater than the first entry of σ.

  Then,
  $$F_{\operatorname{Comp}\pi} \prec F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S_{\prec}(\pi,\sigma)} F_{\operatorname{Comp}\tau}$$
  and
  $$F_{\operatorname{Comp}\pi} \succeq F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S_{\succ}(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$

The dendriform product formula for the F_α

- An analogue of the product formula for $F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma}$:
  Theorem. Let π and σ be two disjoint nonempty permutations. Assume that

  the first entry of π is greater than the first entry of σ.

  Then,
  $$F_{\operatorname{Comp}\pi} \prec F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S_{\prec}(\pi,\sigma)} F_{\operatorname{Comp}\tau}$$
  and
  $$F_{\operatorname{Comp}\pi} \succeq F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S_{\succ}(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$
- This theorem yields that Des is left-shuffle-compatible and right-shuffle-compatible, just as the product formula showed that Des is shuffle-compatible.

The dendriform product formula for the F_α

- An analogue of the product formula for $F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma}$:
  Theorem. Let π and σ be two disjoint nonempty permutations. Assume that

  the first entry of π is greater than the first entry of σ.

  Then,

  $$F_{\operatorname{Comp}\pi} \prec F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S_{\prec}(\pi,\sigma)} F_{\operatorname{Comp}\tau}$$

  and

  $$F_{\operatorname{Comp}\pi} \succeq F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S_{\succ}(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$

- This theorem yields that Des is left-shuffle-compatible and right-shuffle-compatible, just as the product formula showed that Des is shuffle-compatible.
- Can we play the same game with Epk, using our $K^{\mathcal{Z}}_{n,\Lambda}$ series instead of F_α ?

The dendriform product formula for the F_α

- An analogue of the product formula for $F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma}$:
  Theorem. Let π and σ be two disjoint nonempty permutations. Assume that

  the first entry of π is greater than the first entry of σ.

  Then,
  $$F_{\operatorname{Comp}\pi} \prec F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S_{\prec}(\pi,\sigma)} F_{\operatorname{Comp}\tau}$$
  and
  $$F_{\operatorname{Comp}\pi} \succeq F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S_{\succ}(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$
- This theorem yields that Des is left-shuffle-compatible and right-shuffle-compatible, just as the product formula showed that Des is shuffle-compatible.
- Can we play the same game with Epk, using our $K^{\mathcal{Z}}_{n,\Lambda}$ series instead of F_α ?
  Not to my knowledge: I don't know of an analogue of the above theorem. Instead, I use a different approach.

- I need two other operations on quasisymmetric functions.
- We define a binary operation ϕ on the $\mathbb{Q}$-vector space $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ as follows:
  - On monomials, it should be given by
    $$\mathfrak{m} \phi \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \leq \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) > \min(\operatorname{Supp} \mathfrak{n}) \end{cases}$$
    for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbb{Q}$-bilinear.
  - It should be continuous (i.e., its $\mathbb{Q}$-bilinearity also applies to infinite $\mathbb{Q}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $\left(x_2^2 x_4\right) \phi \left(x_4^2 x_5\right) = x_2^2 x_4^3 x_5$ and $\left(x_2^2 x_4\right) \phi \left(x_3^2 x_5\right) = 0$.

The Φ and $\ast$ operations

- I need two other operations on quasisymmetric functions.
- We define a binary operation $\ast$ on the $\mathbb{Q}$-vector space $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ as follows:
  - On monomials, it should be given by
  $$\mathfrak{m} \ast \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \geq \min(\operatorname{Supp} \mathfrak{n}) \end{cases}$$
  for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbb{Q}$-bilinear.
  - It should be continuous (i.e., its $\mathbb{Q}$-bilinearity also applies to infinite $\mathbb{Q}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $(x_2^2 x_4) \ast (x_4^2 x_5) = 0$, but $(x_2^2 x_4) \ast (x_5^2 x_6) = x_2^2 x_4 x_5^2 x_6$.

The ᛄ and ᛯ operations

- QSym is closed under both operations ᛄ and ᛯ.
- Belgthor (ᛄ) and Tvimadur (ᛯ) are two calendar runes signifying two of the 19 years of the Metonic cycle. I sought two (unused) symbols that (roughly) look like "stacking one thing (monomial) atop another", allowing overlap (ᛄ) and disallowing overlap (ᛯ).

A crucial identity

- **Proposition.** For any $a \in \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ and $b \in \operatorname{QSym}$, we have

$$\sum_{(b)} \left(S\left(b_{(1)}\right) \mathbin{\Phi} a\right) b_{(2)} = a \prec b,$$

  where we use the Hopf algebra structure on QSym and the following notations:
  - S for the antipode of QSym;
  - Sweedler's notation $\sum_{(b)} b_{(1)} \otimes b_{(2)}$ for $\Delta(b)$.

A crucial identity

- **Proposition.** For any $a \in \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ and $b \in \operatorname{QSym}$, we have
$$\sum_{(b)} \left(S\left(b_{(1)}\right) \mathbin{\Phi} a \right) b_{(2)} = a \prec b,$$
where we use the Hopf algebra structure on QSym .
- This proposition was important in my study of "dual immaculate creation operators"; it is equally helpful here.
Corollary. Let M be an ideal of QSym. If $\operatorname{QSym} \mathbin{\Phi} M \subseteq M$, then $M \prec \operatorname{QSym} \subseteq M$.

A crucial identity

- **Proposition.** For any $a \in \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ and $b \in \mathrm{QSym}$, we have
$$\sum_{(b)} \left(S\left(b_{(1)}\right) \Phi\ a\right) b_{(2)} = a \prec b,$$
where we use the Hopf algebra structure on QSym .
- This proposition was important in my study of "dual immaculate creation operators"; it is equally helpful here.
Corollary. Let M be an ideal of QSym. If $\mathrm{QSym} \Phi M \subseteq M$, then $M \prec \mathrm{QSym} \subseteq M$.
- A similar identity for $\between$ yields:
Corollary. Let M be an ideal of QSym. If $\mathrm{QSym} \between M \subseteq M$, then $\mathrm{QSym} \succeq M \subseteq M$.

A crucial identity

- **Proposition.** For any $a \in \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ and $b \in \operatorname{QSym}$, we have

$$\sum_{(b)} \left(S\left(b_{(1)}\right) \Phi\ a \right) b_{(2)} = a \prec b,$$

where we use the Hopf algebra structure on QSym .

- This proposition was important in my study of "dual immaculate creation operators"; it is equally helpful here. **Corollary.** Let M be an ideal of QSym. If $\operatorname{QSym} \Phi\ M \subseteq M$, then $M \prec \operatorname{QSym} \subseteq M$.
- A similar identity for $\between$ yields: **Corollary.** Let M be an ideal of QSym. If $\operatorname{QSym} \between M \subseteq M$, then $\operatorname{QSym} \succeq M \subseteq M$.
- **Corollary.** Let M be an ideal of QSym that is a left Φ-ideal (that is, $\operatorname{QSym} \Phi\ M \subseteq M$) and a left $\between$-ideal (that is, $\operatorname{QSym} \between M \subseteq M$). Then, M is a $\prec$-ideal and a $\succeq$-ideal of QSym.

“Runic calculus”

- The operations ᛰ and ᛯ are associative and unital (with unity 1).

- The operations Φ and ᛯ are associative and unital (with unity 1).
- For any two nonempty (i.e., $\neq ()$) compositions α and β, we have

$$M_\alpha \mathbin{\Phi} M_\beta = M_{[\alpha,\beta]} + M_{\alpha\odot\beta};$$
$$M_\alpha \mathbin{\text{ᛯ}} M_\beta = M_{[\alpha,\beta]};$$
$$F_\alpha \mathbin{\Phi} F_\beta = F_{\alpha\odot\beta};$$
$$F_\alpha \mathbin{\text{ᛯ}} F_\beta = F_{[\alpha,\beta]},$$

where $[\alpha,\beta]$ and $\alpha \odot \beta$ are two compositions defined by

$$[(\alpha_1,\alpha_2,\ldots,\alpha_\ell),(\beta_1,\beta_2,\ldots,\beta_m)] = (\alpha_1,\alpha_2,\ldots,\alpha_\ell,\beta_1,\beta_2,\ldots,\beta_m)$$

and

$$(\alpha_1,\alpha_2,\ldots,\alpha_\ell)\odot(\beta_1,\beta_2,\ldots,\beta_m) = (\alpha_1,\alpha_2,\ldots,\alpha_{\ell-1},\alpha_\ell+\beta_1,\beta_2,\beta_3,\ldots,\beta_m).$$

"Runic calculus"

- The operations Φ and ᛤ are associative and unital (with unity 1).
- They satisfy

$$(a \Phi b) \text{ ᛤ } c - a \Phi (b \text{ ᛤ } c) = \varepsilon(b)(a \text{ ᛤ } c - a \Phi c);$$
$$(a \text{ ᛤ } b) \Phi c - a \text{ ᛤ } (b \Phi c) = \varepsilon(b)(a \Phi c - a \text{ ᛤ } c),$$

  where $\varepsilon : \mathbb{Q}[[x_1, x_2, x_3, \ldots]] \to \mathbb{Q}$ sends f to $f(0, 0, 0, \ldots)$.
- As a consequence,

$$(a \Phi b) \text{ ᛤ } c + (a \text{ ᛤ } b) \Phi c = a \Phi (b \text{ ᛤ } c) + a \text{ ᛤ } (b \Phi c).$$

  This says that (QSym, Φ, ᛤ) is a $As^{\langle 2 \rangle}$*-algebra* (in the sense of Loday).
- **Question.** What other identities do Φ, ᛤ, $\prec$ and $\succeq$ satisfy?

How to check left-/right-shuffle-compatibility

- Recall the **Corollary:** Let M be an ideal of QSym that is a left Φ-ideal (that is, $\operatorname{QSym} \mathbin{\Phi} M \subseteq M$) and a left Ж-ideal (that is, $\operatorname{QSym} \mathbin{\text{Ж}} M \subseteq M$). Then, M is a $\prec$-ideal and a $\succeq$-ideal of QSym.

How to check left-/right-shuffle-compatibility

- Recall the **Corollary:** Let M be an ideal of QSym that is a left Φ-ideal (that is, $\operatorname{QSym} \Phi M \subseteq M$) and a left $\between$-ideal (that is, $\operatorname{QSym} \between M \subseteq M$). Then, M is a $\prec$-ideal and a $\succeq$-ideal of QSym.
- Given a shuffle-compatible descent statistic st, we thus conclude that if $\mathcal{K}_{\mathrm{st}}$ is a left Φ-ideal and a left $\between$-ideal, then st is left-shuffle-compatible and right-shuffle-compatible.

How to check left-/right-shuffle-compatibility

- Recall the **Corollary:** Let M be an ideal of QSym that is a left Φ-ideal (that is, $\operatorname{QSym} \Phi M \subseteq M$) and a left $\between$-ideal (that is, $\operatorname{QSym} \between M \subseteq M$). Then, M is a $\prec$-ideal and a $\succeq$-ideal of QSym.
- Given a shuffle-compatible descent statistic st, we thus conclude that if $\mathcal{K}_{\mathrm{st}}$ is a left Φ-ideal and a left $\between$-ideal, then st is left-shuffle-compatible and right-shuffle-compatible.
- Fortunately, this is easy to apply:
  Proposition. Let st be a descent statistic.
  - $\mathcal{K}_{\mathrm{st}}$ is a left Φ-ideal of QSym if and only if st has the following property: If J and K are two st-equivalent nonempty compositions, and if G is any nonempty composition, then $G \odot J$ and $G \odot K$ are st-equivalent.
  - $\mathcal{K}_{\mathrm{st}}$ is a left $\between$-ideal of QSym if and only if st has the following property: If J and K are two st-equivalent nonempty compositions, and if G is any nonempty composition, then $[G, J]$ and $[G, K]$ are st-equivalent.

How to check left-/right-shuffle-compatibility

- Recall the **Corollary:** Let M be an ideal of QSym that is a left Φ-ideal (that is, $\text{QSym} \Phi M \subseteq M$) and a left $\between$-ideal (that is, $\text{QSym} \between M \subseteq M$). Then, M is a $\prec$-ideal and a $\succeq$-ideal of QSym.
- Given a shuffle-compatible descent statistic st, we thus conclude that if $\mathcal{K}_{\text{st}}$ is a left Φ-ideal and a left $\between$-ideal, then st is left-shuffle-compatible and right-shuffle-compatible.
- Fortunately, this is easy to apply:

  Proposition. Let st be a descent statistic.
  - $\mathcal{K}_{\text{st}}$ is a left Φ-ideal of QSym if and only if for each fixed nonempty composition A, the value $\text{st}(A \odot B)$ (for a nonempty composition B) is uniquely determined by $|B|$ and $\text{st}\, B$.
  - $\mathcal{K}_{\text{st}}$ is a left $\between$-ideal of QSym if and only if for each fixed nonempty composition A, the value $\text{st}([A, B])$ (for a nonempty composition B) is uniquely determined by $|B|$ and $\text{st}\, B$.

- Thus, proving that Epk is left- and right-shuffle-compatible requires showing that $\operatorname{Epk}(A \odot B)$ and $\operatorname{Epk}([A, B])$ (for nonempty compositions A and B) are uniquely determined by $|B|$ and $\operatorname{Epk} B$ when A is fixed.

Why Epk is left- and right-shuffle-compatible

- Thus, proving that Epk is left- and right-shuffle-compatible requires showing that $\operatorname{Epk}(A \odot B)$ and $\operatorname{Epk}([A, B])$ (for nonempty compositions A and B) are uniquely determined by $|B|$ and $\operatorname{Epk} B$ when A is fixed.
- This is not hard:

$$\operatorname{Epk}(A \odot B) = ((\operatorname{Epk} A) \setminus \{n\}) \cup (\operatorname{Epk} B + n);$$
$$\operatorname{Epk}([A, B]) = (\operatorname{Epk} A) \cup ((\operatorname{Epk} B + n) \setminus \{n+1\}),$$

where $n = |A|$.

Why Epk is left- and right-shuffle-compatible

- Thus, proving that Epk is left- and right-shuffle-compatible requires showing that $\operatorname{Epk}(A \odot B)$ and $\operatorname{Epk}([A, B])$ (for nonempty compositions A and B) are uniquely determined by $|B|$ and $\operatorname{Epk} B$ when A is fixed.
- This is not hard:

$$\operatorname{Epk}(A \odot B) = ((\operatorname{Epk} A) \setminus \{n\}) \cup (\operatorname{Epk} B + n);$$
$$\operatorname{Epk}([A, B]) = (\operatorname{Epk} A) \cup ((\operatorname{Epk} B + n) \setminus \{n+1\}),$$

  where $n = |A|$.
- Similarly,
  - Des is left- and right-shuffle-compatible (again);
  - des is left- and right-shuffle-compatible;
  - maj is **not** left- or right-shuffle-compatible ($\operatorname{maj}(A \odot B)$ and $\operatorname{maj}([A, B])$ depend not just on $|A|$, $|B|$, $\operatorname{maj} A$ and $\operatorname{maj} B$, but also on $\operatorname{des} B$).

Why Epk is left- and right-shuffle-compatible

- Thus, proving that Epk is left- and right-shuffle-compatible requires showing that $\operatorname{Epk}(A \odot B)$ and $\operatorname{Epk}([A, B])$ (for nonempty compositions A and B) are uniquely determined by $|B|$ and $\operatorname{Epk} B$ when A is fixed.
- This is not hard:

$$\operatorname{Epk}(A \odot B) = ((\operatorname{Epk} A) \setminus \{n\}) \cup (\operatorname{Epk} B + n);$$
$$\operatorname{Epk}([A, B]) = (\operatorname{Epk} A) \cup ((\operatorname{Epk} B + n) \setminus \{n+1\}),$$

  where $n = |A|$.
- Similarly,
  - $(\operatorname{des}, \operatorname{maj})$ is left- and right-shuffle-compatible;
  - Lpk is left- and right-shuffle-compatible;
  - Rpk is **not** left- or right-shuffle-compatible;
  - Pk is **not** left- or right-shuffle-compatible.
- More statistics remain to be analyzed.

- **Question (repeated).** Can a statistic be shuffle-compatible without being a descent statistic?
  (Would FQSym help in studying such statistics?)
- **Question (repeated).** Can a descent statistic be left-shuffle-compatible without being shuffle-compatible?
- **Question.** What mileage do we get out of $\mathcal{Z}$-enriched (P, γ)-partitions for other choices of $\mathcal{N}$ and $\mathcal{Z}$?
- **Question (repeated).** Where do the $\Gamma_{\mathcal{Z}}(P, \gamma)$ live?
- **Question.** Hsiao and Petersen have generalized enriched (P, γ)-partitions to "colored (P, γ)-partitions" (with $\{+, -\}$ replaced by an m-element set). Does this generalize our results?

Thanks to Ira Gessel and Yan Zhuang for initiating this direction (and for helpful discussions), and to Sara Billey for an invitation to Seattle.
And thanks to you for attending!

slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/seattle18.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf
project: https://github.com/darijgr/gzshuf