

t -unique reductions for Mészáros's subdivision algebra

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Abstract. Fix a commutative ring \mathbf{k} , two elements $\beta \in \mathbf{k}$ and $\alpha \in \mathbf{k}$ and a positive integer n . Let \mathcal{X} be the polynomial ring over \mathbf{k} in the $n(n-1)/2$ indeterminates $x_{i,j}$ for all $1 \leq i < j \leq n$. Consider the ideal \mathcal{J} of \mathcal{X} generated by all polynomials of the form $x_{i,j}x_{j,k} - x_{i,k}(x_{i,j} + x_{j,k} + \beta) - \alpha$ for $1 \leq i < j < k \leq n$. The quotient algebra \mathcal{X}/\mathcal{J} (at least for a certain choice of \mathbf{k} , β and α) has been introduced by Karola Mészáros as a commutative analogue of Anatol Kirillov's quasi-classical Yang-Baxter algebra. A natural question is to find a combinatorial basis of this quotient algebra. One can define the *pathless monomials*, i.e., the monomials in \mathcal{X} that have no divisors of the form $x_{i,j}x_{j,k}$ with $1 \leq i < j < k \leq n$. The residue classes of these pathless monomials indeed span the \mathbf{k} -module \mathcal{X}/\mathcal{J} ; however, they turn out (in general) to be \mathbf{k} -linearly dependent. More combinatorially: Reducing a given monomial in \mathcal{X} modulo the ideal \mathcal{J} by applying replacements of the form $x_{i,j}x_{j,k} \mapsto x_{i,k}(x_{i,j} + x_{j,k} + \beta) - \alpha$ always eventually leads to a \mathbf{k} -linear combination of pathless monomials, but the result may depend on the choices made in the process.

More recently, the study of Grothendieck polynomials has led Laura Escobar and Karola Mészáros to defining a \mathbf{k} -algebra homomorphism D from \mathcal{X} into the polynomial ring $\mathbf{k}[t_1, t_2, \dots, t_{n-1}]$ that sends each $x_{i,j}$ to t_i . For a certain class of monomials m (those corresponding to "noncrossing trees"), they have shown that whatever result one gets by reducing m modulo \mathcal{J} , the image of this result under D is independent on the choices made in the reduction process. Mészáros has conjectured that this property holds not only for this class of monomials, but for any polynomial $p \in \mathcal{X}$. We prove this result, in the following slightly stronger form: If $p \in \mathcal{X}$, and if $q \in \mathcal{X}$ is a \mathbf{k} -linear combination of pathless monomials satisfying $p \equiv q \pmod{\mathcal{J}}$, then $D(q)$ does not depend on q (as long as β , α and p are fixed).

We also find an actual basis of the \mathbf{k} -module \mathcal{X}/\mathcal{J} , using what we call *forkless monomials*.

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Introduction

We begin with an example that illustrates the main result of this work.²

Example 1. *Let us play a solitaire game. Fix a positive integer n and two numbers $\beta \in \mathbb{Q}$ and $\alpha \in \mathbb{Q}$, and let \mathcal{X} be the ring $\mathbb{Q}[x_{i,j} \mid 1 \leq i < j \leq n]$ of polynomials with rational coefficients in the $n(n-1)/2$ indeterminates $x_{i,j}$ with $1 \leq i < j \leq n$. (For example, if $n = 4$, then $\mathcal{X} = \mathbb{Q}[x_{1,2}, x_{1,3}, x_{1,4}, x_{2,3}, x_{2,4}, x_{3,4}]$.)*

Start with any polynomial $p \in \mathcal{X}$. The allowed move is the following: Pick a monomial \mathfrak{m} that appears in p and that is divisible by $x_{i,j}x_{j,k}$ for some $1 \leq i < j < k \leq n$. For example, $x_{1,2}x_{1,3}x_{2,4}$ is such a monomial (if it appears in p and if $n \geq 4$), because it is divisible by $x_{i,j}x_{j,k}$ for $(i, j, k) = (1, 2, 4)$. Choose one triple (i, j, k) with $1 \leq i < j < k \leq n$ and $x_{i,j}x_{j,k} \mid \mathfrak{m}$ (sometimes, there are several choices). Now, replace this monomial \mathfrak{m} by $(x_{i,k}(x_{i,j} + x_{j,k} + \beta) + \alpha) \mathfrak{m} / (x_{i,j}x_{j,k})$ in p . The game ends when no more moves are possible (i.e., no monomial \mathfrak{m} divisible by any $x_{i,j}x_{j,k}$ appears in the polynomial anymore).

It is easy to see that this game (a thinly veiled reduction procedure modulo an ideal of \mathcal{X}) always ends after finitely many moves. For instance, if we start with the polynomial $x_{1,2}x_{2,3}x_{1,3}$, then after just a single move we obtain the polynomial $(x_{1,3}(x_{1,2} + x_{2,3} + \beta) + \alpha) x_{1,3} = x_{1,2}x_{1,3}^2 + x_{1,3}^2x_{2,3} + \alpha x_{1,3} + \beta x_{1,3}^2$, from which no further moves are possible, so the game ends here.

*Unlike for many simpler games of this kind, the polynomial obtained at the end of the game **does** depend on the choices made during the game. For example, if we start with the polynomial $x_{1,2}x_{2,3}x_{3,4}$ for $\beta = 0$ and $\alpha = 0$, then we can end up with (at least) two different polynomials³ depending on which moves we make. However, as we will show in Theorem 1, if we apply the substitution $x_{i,j} \mapsto t_i$ to the polynomial obtained at the end of the game (where t_1, t_2, \dots, t_{n-1} are new indeterminates), then the result of the substitution will **not** depend on any choices made in the game.*

Why would one play a game like this? The interest in the reduction rule $(x_{i,k}(x_{i,j} + x_{j,k} + \beta)) \mathfrak{m} / (x_{i,j}x_{j,k})$ (this is a particular case of our above rule, when $\alpha = 0$) originates in Mészáros's study [8] of the abelianization of Kirillov's quasi-classical Yang-Baxter algebra (see [6] for a recent survey of the latter and its many variants). To define this abelianization, we let β be an indeterminate (unlike in Example 1, where it was an element of \mathbb{Q}). Furthermore, fix a positive integer n . The abelianization of the (n -th) quasi-classical Yang-Baxter algebra is the commutative $\mathbb{Q}[\beta]$ -algebra $\mathcal{S}(A_n)$ with

$$\begin{array}{llll} \text{generators} & x_{i,j} & \text{for all } 1 \leq i < j \leq n & \text{and} \\ \text{relations} & x_{i,j}x_{j,k} = x_{i,k}(x_{i,j} + x_{j,k} + \beta) & \text{for all } 1 \leq i < j < k \leq n. \end{array}$$

²The notations used in this Introduction are meant to be provisional. In the rest of this paper, we shall work with different notations (and in a more general setting), which will be introduced in Section 1.

³namely, $x_{1,2}x_{1,3}x_{1,4} + x_{1,2}x_{1,4}x_{3,4} + x_{1,3}x_{1,4}x_{2,4} + x_{1,3}x_{2,3}x_{2,4} + x_{1,4}x_{2,4}x_{3,4}$ and $x_{1,2}x_{1,3}x_{1,4} + x_{1,2}x_{1,4}x_{3,4} + x_{1,3}x_{1,4}x_{2,3} + x_{1,4}x_{2,3}x_{2,4} + x_{1,4}x_{2,4}x_{3,4}$

A natural question is to find an explicit basis of $\mathcal{S}(A_n)$ (as a \mathbb{Q} -vector space, or, if possible, as a $\mathbb{Q}[\beta]$ -module). One might try constructing such a basis using a reduction algorithm (or “straightening law”) that takes any element of $\mathcal{S}(A_n)$ (written as any polynomial in the generators $x_{i,j}$) and rewrites it in a “normal form”. The most obvious way one could try to construct such a reduction algorithm is by repeatedly rewriting products of the form $x_{i,j}x_{j,k}$ (with $1 \leq i < j < k \leq n$) as $x_{i,k}(x_{i,j} + x_{j,k} + \beta)$, until this is no longer possible. This is precisely the game that we played in Example 1 (with the only difference that β is now an indeterminate, not a number). Unfortunately, the result of the game turns out to depend on the choices made while playing it; consequently, the “normal form” it constructs is not literally a normal form, and instead of a basis of $\mathcal{S}(A_n)$ we only obtain a spanning set.⁴

Nevertheless, the result of the game is not meaningless. The idea to substitute t_i for $x_{i,j}$ (in the result, not in the original polynomial!) seems to have appeared in work of Postnikov, Stanley and Mészáros; some concrete formulas (for specific values of the initial polynomial and specific values of β) appear in [10, Exercise A22] (resulting in Catalan and Narayana numbers). Recent work on Grothendieck polynomials by Escobar and Mészáros [4, §5] has again brought up the notion of substituting t_i for $x_{i,j}$ in the polynomial obtained at the end of the game. This has led Mészáros to the conjecture that, after this substitution, the resulting polynomial no longer depends on the choices made during the game. She has proven this conjecture for a certain class of polynomials (those corresponding to “noncrossing trees”).

The main purpose of this paper is to outline a proof of Mészáros's conjecture in the general case. We shall work in a more general situation. First, instead of the relation $x_{i,j}x_{j,k} = x_{i,k}(x_{i,j} + x_{j,k} + \beta)$, we shall consider the “deformed” relation $x_{i,j}x_{j,k} = x_{i,k}(x_{i,j} + x_{j,k} + \beta) + \alpha$; this idea again goes back to the work of Kirillov (see [6, Definition 5.1 (1)] for a noncommutative variant of the quotient ring \mathcal{X}/\mathcal{J} , which he calls the “associative quasi-classical Yang–Baxter algebra of weight (α, β) ”). Second, we shall work over an arbitrary commutative ring \mathbf{k} instead of \mathbb{Q} (and our parameters α and β will be arbitrary elements of \mathbf{k}). Rather than working in an algebra like $\mathcal{S}(A_n)$, we shall work in the polynomial ring $\mathcal{X} = \mathbf{k}[x_{i,j} \mid 1 \leq i < j \leq n]$, and study the ideal \mathcal{J} generated by all elements of the form $x_{i,j}x_{j,k} - x_{i,k}(x_{i,j} + x_{j,k} + \beta) - \alpha$ for $1 \leq i < j < k \leq n$. Instead of focussing on the reduction algorithm itself, we shall generally study polynomials in \mathcal{X} that are congruent to each other modulo the ideal \mathcal{J} . A monomial in \mathcal{X} will be called “pathless” if it is not divisible by any monomial of the form $x_{i,j}x_{j,k}$ with $i < j < k$. A polynomial in \mathcal{X} will be called “pathless” if all monomials appearing in it are pathless. Thus, “pathless” polynomials are precisely the polynomials $p \in \mathcal{X}$ for which the game in Example 1 would end immediately if started at p . Our main result (Theorem 1) will show that if $p \in \mathcal{X}$ is a polynomial, and if $q \in \mathcal{X}$ is a pathless polynomial congruent to

⁴Surprisingly, a similar reduction algorithm **does** work for the (non-abelianized) quasi-classical Yang–Baxter algebra itself. This is one of Mészáros's results [8, Theorem 30].

p modulo \mathcal{J} , then the image of q under the substitution $x_{i,j} \mapsto t_i$ does not depend on q (but only on α , β and p). This, in particular, yields Mészáros's conjecture; but it is a stronger result, because it does not require that q is obtained from p by playing the game from Example 1 (all we ask for is that q be pathless and congruent to p modulo \mathcal{J}), and of course because of the more general setting.

After the proof of Theorem 1, we shall also outline an answer (Proposition 9) to the (easier) question of finding a basis for the quotient ring \mathcal{X}/\mathcal{J} . This basis will be obtained using an explicitly given Gröbner basis of the ideal \mathcal{J} .

A particular case of the conjecture stated in Example 1 (where $\alpha = 0$ and where the game starts with a monomial β) has recently been obtained independently by Mészáros and St. Dizier [7] using combinatorial and geometric methods.

This extended abstract is an abbreviated version of [5].

1 Definitions and results

Let us now start from scratch, and set the stage for the main result.

Definition 1. Let $\mathbb{N} = \{0, 1, 2, \dots\}$. Let $[m]$ be the set $\{1, 2, \dots, m\}$ for each $m \in \mathbb{N}$. Let \mathbf{k} be a commutative ring. (We fix \mathbf{k} throughout this paper.) Fix two elements β and α of \mathbf{k} . The word "monomial" shall always mean an element of a free abelian monoid (written multiplicatively). For example, the monomials in two indeterminates x and y are the $x^i y^j$ with $(i, j) \in \mathbb{N}^2$. Thus, monomials do not include coefficients.

Fix a positive integer n . Let $\mathcal{X} = \mathbf{k} [x_{i,j} \mid (i, j) \in [n]^2 \text{ satisfying } i < j]$ be the polynomial ring in the $n(n-1)/2$ indeterminates $x_{i,j}$ over \mathbf{k} . Let \mathfrak{M} be the set of all monomials in these indeterminates $x_{i,j}$.

Definition 2. A monomial $\mathfrak{m} \in \mathfrak{M}$ is said to be **pathless** if there exists no triple $(i, j, k) \in [n]^3$ satisfying $i < j < k$ and $x_{i,j} x_{j,k} \mid \mathfrak{m}$ (as monomials). A polynomial $p \in \mathcal{X}$ is said to be **pathless** if it is a \mathbf{k} -linear combination of pathless monomials.

Definition 3. Let \mathcal{J} be the ideal of \mathcal{X} generated by all elements of the form $x_{i,j} x_{j,k} - x_{i,k} (x_{i,j} + x_{j,k} + \beta) - \alpha$ for $(i, j, k) \in [n]^3$ satisfying $i < j < k$.

The following fact follows by a straightforward induction:

Proposition 1. Let $p \in \mathcal{X}$. There exists a pathless polynomial $q \in \mathcal{X}$ such that $p \equiv q \pmod{\mathcal{J}}$.

Proposition 1 corresponds to the fact that the game in Example 1 always ends. Since the polynomial left behind by the game is path-dependent, the q in Proposition 1 is not unique (in general).

The ideal \mathcal{J} is relevant to the so-called *subdivision algebra of root polytopes* (denoted by $\mathcal{S}(\beta)$ in [4, §5] and $\mathcal{S}(A_n)$ in [8, §1]). Namely, this latter algebra is defined as the

quotient \mathcal{X}/\mathcal{J} for a certain choice of \mathbf{k} , β and α . This algebra was first introduced by Mészáros in [8] as the abelianization of Kirillov's quasi-classical Yang-Baxter algebra.

In [4, §5 and §7], Escobar and Mészáros study the result of substituting t_i for each variable $x_{i,j}$ in a polynomial $f \in \mathcal{X}$. In our language, this leads to the following:

Definition 4. Let \mathcal{T}' be the polynomial ring $\mathbf{k}[t_1, t_2, \dots, t_{n-1}]$. We define a \mathbf{k} -algebra homomorphism $D : \mathcal{X} \rightarrow \mathcal{T}'$ by

$$D(x_{i,j}) = t_i \quad \text{for every } (i,j) \in [n]^2 \text{ satisfying } i < j.$$

The goal of this work is to prove the following fact, which (in a less general setting) was conjectured by Mészáros in a 2015 talk at MIT:

Theorem 1. Let $p \in \mathcal{X}$. Consider any pathless polynomial $q \in \mathcal{X}$ such that $p \equiv q \pmod{\mathcal{J}}$. Then, $D(q)$ does not depend on the choice of q (but merely on the choice of α , β and p).

It is not generally true that $D(q) = D(p)$.

2 The proof

The proof of Theorem 1 proceeds in several steps. First, we shall define four \mathbf{k} -algebras \mathcal{Q} , $\mathcal{T}'[[w]]$, \mathcal{T} and $\mathcal{T}[[w]]$ (with \mathcal{T}' being a subalgebra of \mathcal{T}) and three \mathbf{k} -linear maps A , B and E (with A and E being \mathbf{k} -algebra homomorphisms) forming a diagram

$$\begin{array}{ccccc} \mathcal{X} & \xrightarrow{A} & \mathcal{Q} & \xrightarrow{B} & \mathcal{T}[[w]] \\ & \searrow D & & & \uparrow \\ & & \mathcal{T}' & \xrightarrow{E} & \mathcal{T}'[[w]] \end{array}$$

(where the vertical arrow is a canonical injection) that is **not** commutative. We shall eventually show that: the homomorphism A annihilates the ideal \mathcal{J} (Proposition 2); the homomorphism E is injective (Proposition 3); and each pathless polynomial q satisfies $(E \circ D)(q) = (B \circ A)(q)$ (Corollary 1). These three facts will allow us to prove Theorem 1. Indeed, the first and the third will imply that each pathless polynomial in \mathcal{J} is annihilated by $E \circ D$; because of the second, this will show that it is also annihilated by D ; and from here, Theorem 1 will easily follow.

2.1 Laurent series, and the maps A , B and E

We begin by recalling the definition of (formal) Laurent series in n indeterminates r_1, r_2, \dots, r_n . We refer to [5] for the full details of the definition.

Definition 5. Consider n distinct symbols r_1, r_2, \dots, r_n .

(a) A **bi-infinite power series** means a formal expression of the form

$$\sum_{(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n} \lambda_{a_1, a_2, \dots, a_n} r_1^{a_1} r_2^{a_2} \cdots r_n^{a_n},$$

where $\lambda_{a_1, a_2, \dots, a_n}$ is an element of \mathbf{k} for each $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$.

(b) A bi-infinite power series $f = \sum_{(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n} \lambda_{a_1, a_2, \dots, a_n} r_1^{a_1} r_2^{a_2} \cdots r_n^{a_n}$ is said to be a **Laurent series** if there exists a $d \in \mathbb{Z}$ such that all $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n \setminus \{d, d+1, d+2, \dots\}^n$ satisfy $\lambda_{a_1, a_2, \dots, a_n} = 0$ (that is, no indeterminate appears with exponent $< d$ in the series). If this d can be taken to be 0, then f is said to be a **formal power series**.

(c) The Laurent series form a \mathbf{k} -algebra (where multiplication is defined in the same way as, e.g., for usual power series: by expanding and combining terms). Denote this \mathbf{k} -algebra by \mathcal{Q} . The formal power series form a \mathbf{k} -subalgebra $\mathbf{k}[[r_1, r_2, \dots, r_n]]$ of \mathcal{Q} .

The bi-infinite power series do not form a \mathbf{k} -algebra, since multiplying two bi-infinite power series would (in general) result in an infinite sum in front of each monomial.

Definition 6. A **Laurent monomial** means a formal expression of the form $r_1^{a_1} r_2^{a_2} \cdots r_n^{a_n}$ with $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$.

For each $i \in [n]$, we define a Laurent monomial q_i by $q_i = r_i r_{i+1} \cdots r_n$.

Notice that $q_i/q_j = r_i r_{i+1} \cdots r_{j-1}$ whenever $1 \leq i \leq j \leq n$. Thus, for any $i \in [n]$ and $j \in [n]$ satisfying $i < j$, the difference $1 - q_i/q_j = 1 - r_i r_{i+1} \cdots r_{j-1}$ is invertible in \mathcal{Q} (since it is a formal power series in $\mathbf{k}[[r_1, r_2, \dots, r_n]]$ having constant term 1).

Definition 7. Define a \mathbf{k} -algebra homomorphism $A : \mathcal{X} \rightarrow \mathcal{Q}$ by

$$A(x_{i,j}) = -\frac{q_i + \beta + \alpha/q_j}{1 - q_i/q_j} \quad \text{for all } (i, j) \in [n]^2 \text{ satisfying } i < j.$$

This is well-defined, since all denominators appearing here are invertible in \mathcal{Q} .

Proposition 2. We have $A(\mathcal{J}) = 0$.

This can be proven by straightforward computations.

Definition 8. (a) Let \mathcal{T} be the topological \mathbf{k} -algebra $\mathbf{k}[[t_1, t_2, \dots, t_n]]$. This is the ring of formal power series in the n indeterminates t_1, t_2, \dots, t_n over \mathbf{k} (equipped with the usual topology).

(b) We shall regard the canonical injections

$$\mathcal{T}' = \mathbf{k}[t_1, t_2, \dots, t_{n-1}] \hookrightarrow \mathbf{k}[t_1, t_2, \dots, t_n] \hookrightarrow \mathbf{k}[[t_1, t_2, \dots, t_n]] = \mathcal{T}$$

as inclusions. Thus, \mathcal{T}' becomes a \mathbf{k} -subalgebra of \mathcal{T} . Hence, $D : \mathcal{X} \rightarrow \mathcal{T}'$ becomes a \mathbf{k} -algebra homomorphism $\mathcal{X} \rightarrow \mathcal{T}$.

(c) We consider the \mathbf{k} -algebras $\mathcal{T}[[w]]$ and $\mathcal{T}'[[w]]$ (consisting of formal power series in a new indeterminate w over \mathcal{T} and over \mathcal{T}' , respectively). Thus, $\mathcal{T}[[w]]$ can be regarded as the ring of formal power series in the $n+1$ indeterminates t_1, t_2, \dots, t_n, w over \mathbf{k} . We endow $\mathcal{T}[[w]]$ with the topology transported from the latter ring.

Definition 9. We define a continuous \mathbf{k} -linear map $B : \mathcal{Q} \rightarrow \mathcal{T}[[w]]$ by setting

$$B(q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}) = \left(\prod_{\substack{i \in [n]; \\ a_i > 0}} t_i^{a_i} \right) \left(\prod_{\substack{i \in [n]; \\ a_i < 0}} w^{-a_i} \right) \quad \text{for each } (a_1, a_2, \dots, a_n) \in \mathbb{Z}^n. \quad (2.1)$$

This map B is well-defined, because each Laurent monomial (in \mathcal{Q}) can be written uniquely in the form $q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}$ with $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$, and because any given monomial in t_1, t_2, \dots, t_n, w appears on the right hand side of (2.1) for finitely many $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ only. Of course, B is (in general) not a \mathbf{k} -algebra homomorphism.

Definition 10. We define a \mathbf{k} -algebra homomorphism $E : \mathcal{T}' \rightarrow \mathcal{T}'[[w]]$ by

$$E(t_i) = -\frac{t_i + \beta + \alpha w}{1 - t_i w} \quad \text{for each } i \in [n-1].$$

This is well-defined, because for each $i \in [n-1]$, the power series $1 - t_i w$ is invertible in $\mathcal{T}'[[w]]$.

Proposition 3. The homomorphism E is injective.

Proof. Let $F : \mathcal{T}'[[w]] \rightarrow \mathcal{T}'$ be the \mathcal{T}' -algebra homomorphism that sends each formal power series $f \in \mathcal{T}'[[w]]$ (regarded as a formal power series in the single indeterminate w over \mathcal{T}') to its constant term $f(0) \in \mathcal{T}'$. Let $G : \mathcal{T}' \rightarrow \mathcal{T}'$ be the \mathbf{k} -algebra homomorphism that sends t_i to $-t_i - \beta$ for each $i \in [n-1]$. It is easy to check that $G \circ F \circ E = \text{id}$. Hence, the map E has a left inverse, and thus is injective. \square

2.2 Pathless monomials and subsets S of $[n-1]$

Next, we want to study the action of the compositions $B \circ A$ and $E \circ D$ on pathless monomials. We need more notations:

Definition 11. Let S be a subset of $[n-1]$.

(a) Let \mathfrak{P}_S be the set of all pairs $(i, j) \in S \times ([n] \setminus S)$ satisfying $i < j$.

(b) A monomial $\mathfrak{m} \in \mathfrak{M}$ is said to be **S-friendly** if it is a product of some of the indeterminates $x_{i,j}$ with $(i, j) \in \mathfrak{P}_S$. In other words, a monomial $\mathfrak{m} \in \mathfrak{M}$ is S -friendly if and only if every indeterminate $x_{i,j}$ that appears in \mathfrak{m} satisfies $i \in S$ and $j \notin S$.

We let \mathfrak{M}_S denote the set of all S -friendly monomials.

(c) We let \mathcal{X}_S denote the polynomial ring $\mathbf{k}[x_{i,j} \mid (i, j) \in \mathfrak{P}_S]$. This is clearly a subring of \mathcal{X} . The \mathbf{k} -module \mathcal{X}_S has a basis consisting of all S -friendly monomials $\mathfrak{m} \in \mathfrak{M}$.

(d) An n -tuple $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ is said to be **S-adequate** if and only if it satisfies $(a_i \geq 0$ for all $i \in S)$ and $(a_i \leq 0$ for all $i \in [n] \setminus S)$. We let \mathcal{Q}_S denote the subset of \mathcal{Q} consisting of all infinite \mathbf{k} -linear combinations of the Laurent monomials $q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}$ for S -adequate n -tuples $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$ (as long as these combinations belong to \mathcal{Q}). It is easy to see that \mathcal{Q}_S is a topological \mathbf{k} -subalgebra of \mathcal{Q} (since the entrywise sum of two S -adequate n -tuples is S -adequate again).

(e) We let \mathcal{T}_S denote the topological \mathbf{k} -algebra $\mathbf{k}[[t_i \mid i \in S]]$. This is a topological subalgebra of \mathcal{T} . Hence, the ring $\mathcal{T}_S[[w]]$ (that is, the ring of formal power series in the (single) variable w over \mathcal{T}_S) is a topological \mathbf{k} -subalgebra of the similarly-defined ring $\mathcal{T}[[w]]$.

(f) We define a \mathbf{k} -algebra homomorphism $A_S : \mathcal{X}_S \rightarrow \mathcal{Q}_S$ by

$$A_S(x_{i,j}) = -\frac{q_i + \beta + \alpha/q_j}{1 - q_i/q_j} \quad \text{for all } (i,j) \in \mathfrak{P}_S.$$

This is easily seen to be well-defined.

(g) We define a continuous \mathbf{k} -linear map $B_S : \mathcal{Q}_S \rightarrow \mathcal{T}_S[[w]]$ by setting

$$B_S(q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}) = \left(\prod_{i \in S} t_i^{a_i} \right) \left(\prod_{i \in [n] \setminus S} w^{-a_i} \right)$$

for each S -adequate $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$.

This is well-defined according to Proposition 4 (b) below.

(h) We let \mathcal{T}'_S denote the \mathbf{k} -algebra $\mathbf{k}[t_i \mid i \in S]$. This is a \mathbf{k} -subalgebra of \mathcal{T}' . Hence, the ring $\mathcal{T}'_S[[w]]$ (that is, the ring of formal power series in the (single) variable w over \mathcal{T}'_S) is a \mathbf{k} -subalgebra of the similarly-defined ring $\mathcal{T}'[[w]]$.

(i) We define a \mathbf{k} -algebra homomorphism $D_S : \mathcal{X}_S \rightarrow \mathcal{T}'_S$ by

$$D_S(x_{i,j}) = t_i \quad \text{for all } (i,j) \in \mathfrak{P}_S.$$

This is well-defined, since each $(i,j) \in \mathfrak{P}_S$ satisfies $i \in S$.

(j) We define a \mathbf{k} -algebra homomorphism $E_S : \mathcal{T}'_S \rightarrow \mathcal{T}'_S[[w]]$ by

$$E_S(t_i) = -\frac{t_i + \beta + \alpha w}{1 - t_i w} \quad \text{for each } i \in S.$$

This is well-defined (by the universal property of the polynomial ring \mathcal{T}'_S), because for each $i \in S$, the power series $1 - t_i w$ is invertible in $\mathcal{T}'_S[[w]]$ (indeed, its constant term is 1).

The following propositions are easy to verify:

Proposition 4. Let S be a subset of $[n - 1]$.

(a) We have

$$B(q_1^{a_1} q_2^{a_2} \cdots q_n^{a_n}) = \left(\prod_{i \in S} t_i^{a_i} \right) \left(\prod_{i \in [n] \setminus S} w^{-a_i} \right) \quad (2.2)$$

for each S -adequate n -tuple $(a_1, a_2, \dots, a_n) \in \mathbb{Z}^n$.

(b) The map B_S (defined in Definition 11 (g)) is well-defined.

Proposition 5. Let S be a subset of $[n - 1]$. Then, the diagrams

$$\begin{array}{ccccc} \mathcal{X}_S & \xrightarrow{A_S} & \mathcal{Q}_S & \xrightarrow{B_S} & \mathcal{T}_S [[w]] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{A} & \mathcal{Q} & \xrightarrow{B} & \mathcal{T} [[w]] \end{array} \quad (2.3)$$

and

$$\begin{array}{ccccc} \mathcal{X}_S & \xrightarrow{D_S} & \mathcal{T}'_S & \xrightarrow{E_S} & \mathcal{T}'_S [[w]] \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{X} & \xrightarrow{D} & \mathcal{T}' & \xrightarrow{E} & \mathcal{T}' [[w]] \end{array} \quad (2.4)$$

(where the vertical arrows are the obvious inclusion maps) are commutative.

Proposition 6. Let S be a subset of $[n - 1]$. Then, $B_S : \mathcal{Q}_S \rightarrow \mathcal{T}_S [[w]]$ is a continuous \mathbf{k} -algebra homomorphism.

Proposition 7. Let S be a subset of $[n - 1]$. Let $(i, j) \in \mathfrak{P}_S$. Then,

$$(E_S \circ D_S)(x_{i,j}) = (B_S \circ A_S)(x_{i,j}).$$

Proposition 8. Let $\mathfrak{m} \in \mathfrak{M}$ be a pathless monomial.

(a) There exists a subset S of $[n - 1]$ such that \mathfrak{m} is S -friendly.

(b) Let S be such a subset. Then, $\mathfrak{m} \in \mathcal{X}_S$ and $(E \circ D)(\mathfrak{m}) = (B_S \circ A_S)(\mathfrak{m})$.

Proof of Proposition 8. (a) Write \mathfrak{m} in the form $\mathfrak{m} = \prod_{\substack{(i,j) \in [n]^2 \\ i < j}} x_{i,j}^{a_{i,j}}$. For each $i \in [n - 1]$, define a $b_i \in \mathbb{N}$ by $b_i = \sum_{j=i+1}^n a_{i,j}$. Define a subset S of $[n - 1]$ by $S = \{i \in [n - 1] \mid b_i > 0\}$. An easy argument (using the pathlessness of \mathfrak{m}) then confirms that \mathfrak{m} is S -friendly. This proves Proposition 8 (a).

(b) We know that $\mathfrak{m} \in \mathcal{X}_S$ (since \mathfrak{m} is S -friendly). Now, we shall show that $(E \circ D)|_{\mathcal{X}_S} = B_S \circ A_S$ (if we regard $B_S \circ A_S$ as a map to $\mathcal{T} [[w]]$ and regard $E \circ D$ as a map to $\mathcal{T} [[w]]$). It is clearly enough to prove this on the generating family $(x_{i,j})_{(i,j) \in \mathfrak{P}_S}$ of the \mathbf{k} -algebra \mathcal{X}_S (since all of A_S, B_S, D, E are \mathbf{k} -algebra homomorphisms). In other words, it is enough to prove that $((E \circ D)|_{\mathcal{X}_S})(x_{i,j}) = (B_S \circ A_S)(x_{i,j})$ for each $(i, j) \in \mathfrak{P}_S$. So let us fix some $(i, j) \in \mathfrak{P}_S$. Proposition 5 shows that the diagram (2.4) is commutative. Thus, $(E \circ D)|_{\mathcal{X}_S} = E_S \circ D_S$ (provided that we regard $E_S \circ D_S$ as a map to $\mathcal{T}' [[w]]$), and thus

$$((E \circ D)|_{\mathcal{X}_S})(x_{i,j}) = (E_S \circ D_S)(x_{i,j}) = (B_S \circ A_S)(x_{i,j})$$

(by Proposition 7). This completes our proof of $(E \circ D)|_{\mathcal{X}_S} = B_S \circ A_S$. Hence, $(E \circ D)(\mathfrak{m}) = (B_S \circ A_S)(\mathfrak{m})$ (since $\mathfrak{m} \in \mathcal{X}_S$). This proves Proposition 8 (b). \square

Corollary 1. *Let $q \in \mathcal{X}$ be pathless. Then, $(E \circ D)(q) = (B \circ A)(q)$.*

Proof. WLOG assume that q is a pathless monomial \mathfrak{m} (by linearity). Proposition 8 (a) shows that there exists a subset S of $[n - 1]$ such that this \mathfrak{m} is S -friendly. Consider this S . Proposition 8 (b) yields $\mathfrak{m} \in \mathcal{X}_S$ and $(E \circ D)(\mathfrak{m}) = (B_S \circ A_S)(\mathfrak{m})$. But the commutativity of the diagram (2.3) shows that $(B_S \circ A_S)(\mathfrak{m}) = (B \circ A)(\mathfrak{m})$. Thus, $(E \circ D)(\mathfrak{m}) = (B_S \circ A_S)(\mathfrak{m}) = (B \circ A)(\mathfrak{m})$. Since $q = \mathfrak{m}$, this rewrites as $(E \circ D)(q) = (B \circ A)(q)$. \square

2.3 Proof of Theorem 1

Lemma 1. *Let $p \in \mathcal{X}$ be a pathless polynomial such that $p \in \mathcal{J}$. Then, $D(p) = 0$.*

Proof. We have $A(\mathcal{J}) = 0$ (by Proposition 2); thus, $A(p) = 0$ (since $p \in \mathcal{J}$). Hence, $B(A(p)) = B(0) = 0$. But Corollary 1 (applied to $q = p$) yields $(E \circ D)(p) = (B \circ A)(p) = B(A(p)) = 0$. Thus, $E(D(p)) = (E \circ D)(p) = 0$. Since the \mathbf{k} -linear map E is injective (by Proposition 3), we thus conclude that $D(p) = 0$. \square

Proof of Theorem 1. We need to prove that if f and g are two pathless polynomials $q \in \mathcal{X}$ such that $p \equiv q \pmod{\mathcal{J}}$, then $D(f) = D(g)$. So let f and g be two pathless polynomials $q \in \mathcal{X}$ such that $p \equiv q \pmod{\mathcal{J}}$. Thus, $p \equiv f \pmod{\mathcal{J}}$ and $p \equiv g \pmod{\mathcal{J}}$. Hence, $f \equiv p \equiv g \pmod{\mathcal{J}}$, so that $f - g \in \mathcal{J}$. Also, the polynomial $f - g \in \mathcal{X}$ is pathless (since f and g are pathless). Thus, Lemma 1 (applied to $f - g$ instead of p) shows that $D(f - g) = 0$. Thus, $0 = D(f - g) = D(f) - D(g)$ (since D is a \mathbf{k} -algebra homomorphism). In other words, $D(f) = D(g)$. This proves Theorem 1. \square

3 Forkless polynomials and a basis of \mathcal{X}/\mathcal{J}

We have thus answered one of the major questions about the ideal \mathcal{J} ; but we have begged perhaps the most obvious one: Can we find a basis of the \mathbf{k} -module \mathcal{X}/\mathcal{J} ? This turns out to be much simpler than the above; the key is to use a different strategy. Instead of reducing polynomials to pathless polynomials, we shall reduce them to *forkless* polynomials, defined as follows:

Definition 12. *A monomial $\mathfrak{m} \in \mathfrak{M}$ is said to be **forkless** if there exists no triple $(i, j, k) \in [n]^3$ satisfying $i < j < k$ and $x_{i,j}x_{i,k} \mid \mathfrak{m}$ (as monomials).*

*A polynomial $p \in \mathcal{X}$ is said to be **forkless** if it is a \mathbf{k} -linear combination of forkless monomials.*

Theorem 2. *Let $p \in \mathcal{X}$. There exists a **unique** forkless $q \in \mathcal{X}$ such that $p \equiv q \pmod{\mathcal{J}}$.*

Proposition 9. *The projections of the forkless monomials $\mathfrak{m} \in \mathfrak{M}$ onto the quotient ring \mathcal{X}/\mathcal{J} form a basis of the \mathbf{k} -module \mathcal{X}/\mathcal{J} .*

Theorem 2 and Proposition 9 are clearly equivalent. We prove them using the theory of Gröbner bases (see, e.g., [1] for an introduction). While we give some details on the proof in [5], in this abstract let us merely state the main step:

Proposition 10. *Equip the set \mathfrak{M} of monomials with any term order (i.e., well-order that respects the monoid structure of \mathfrak{M}) satisfying the following condition: For every $(i, j, k) \in [n]^3$ satisfying $i < j < k$, we have $x_{i,k} > x_{j,k}$ and $x_{i,j} > x_{j,k}$.*

Then, the set $\left\{ x_{i,k}x_{i,j} - x_{i,j}x_{j,k} + x_{i,k}x_{j,k} + \beta x_{i,k} + \alpha \mid (i, j, k) \in [n]^3 \text{ satisfying } i < j < k \right\}$ is a Gröbner basis of the ideal \mathcal{J} of \mathcal{X} (with respect to this order).

4 Further questions

Question 1. (a) *Is \mathcal{J} the kernel of the map $A : \mathcal{X} \rightarrow \mathcal{Q}$ from Definition 7?*

(b) *Consider the polynomial ring $\mathbf{k}[\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n]$ in n indeterminates $\tilde{q}_1, \tilde{q}_2, \dots, \tilde{q}_n$ over \mathbf{k} . Let \mathcal{Q}_{rat} denote the localization of this polynomial ring at the multiplicative subset generated by all differences of the form $\tilde{q}_i - \tilde{q}_j$ (for $1 \leq i < j \leq n$). Then, the morphism $A : \mathcal{X} \rightarrow \mathcal{Q}$ factors through a \mathbf{k} -algebra homomorphism $\tilde{A} : \mathcal{X} \rightarrow \mathcal{Q}_{\text{rat}}$ which sends each $x_{i,j}$ to $-\frac{\tilde{q}_i + \beta + \alpha / \tilde{q}_j}{1 - \tilde{q}_i / \tilde{q}_j} = -\frac{\tilde{q}_i \tilde{q}_j + \beta \tilde{q}_j + \alpha}{\tilde{q}_j - \tilde{q}_i} \in \mathcal{Q}_{\text{rat}}$. Is \mathcal{J} the kernel of this latter homomorphism \tilde{A} ?*

Of course, if the answer to Question 1 (a) is positive, then so is the answer to Question 1 (b). This question is interesting partly because a positive answer to part (b) would provide a realization of $\mathcal{X} / \mathcal{J}$ as a subalgebra of a localized polynomial ring in (only) n indeterminates. This subalgebra would probably not be the whole \mathcal{Q}_{rat} .

As a step towards Question 1 (b), we have found a basis of the \mathbf{k} -module \mathcal{Q}_{rat} :

Proposition 11. *In \mathcal{Q}_{rat} , consider the family of all elements of the form $\prod_{i=1}^n g_i$, where each g_i has either the form $1 / (\tilde{q}_i - \tilde{q}_j)^m$ for some $j \in \{i+1, i+2, \dots, n\}$ and $m > 0$ or the form \tilde{q}_i^k for some $k \in \mathbb{N}$. This family is a basis of the \mathbf{k} -module \mathcal{Q}_{rat} .*

This family is similar to the forkless monomials in Proposition 9, but it is “larger”.

If $\beta = 0$ and $\alpha = 0$, then the subdivision algebra $\mathcal{X} / \mathcal{J}$ is a known construct: it is the Orlik-Terao algebra [9] of the braid arrangement. We may thus regard $\mathcal{X} / \mathcal{J}$ as a deformation of a specific Orlik-Terao algebra, and ask for a generalization:

Question 2. (a) *Can an arbitrary Orlik-Terao algebra be deformed by two parameters β and α , generalizing our $\mathcal{X} / \mathcal{J}$?*

(b) *Our basis of forkless monomials for $\mathcal{X} / \mathcal{J}$ can be regarded as an “**nb**c basis” in the sense of [2] (except that our monomials are not required to be squarefree). Indeed, if we totally order the monomials $x_{i,j}$ in such a way that $x_{i,j} > x_{u,v}$ whenever $i < u$, then the broken circuits of the graphical matroid of K_n are precisely the sets of the form $\{\{i, j\}, \{i, k\}\}$ for $i < j < k$; but these*

correspond to the precise monomials $x_{i,j}x_{i,k}$ that a forkless monomial cannot be divisible by. Does this extend to the arbitrary Orlik-Terao algebras?

(c) Does Theorem 1 extend to Orlik-Terao algebras?

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