A few classical results on tensor, symmetric and exterior powers

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Version 0.3 (June 13, 2017) (not proofread!)

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0.1. Version

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0.2. Introduction

In this note, I am going to give proofs to a few results about tensor products as well as tensor, pseudoexterior, symmetric and exterior powers of k-modules (where k is a commutative ring with 1). None of the results is new, as I have seen them used all around literature as if they were well-known and/or completely trivial. I have not yet found a place where they are actually proved (though I have not looked far), so I am doing it here.

This note is not completely new: The first four Subsections (0.4, 0.5, 0.6 and 0.7) as well as the proof of Proposition 38 are lifted from my diploma thesis [3], while Subsections 0.8 and 0.9 are translated from an additional section of [4] which was written by me.

0.3. Basic conventions

Before we come to the actual body of this note, let us fix some conventions to prevent misunderstandings from happening:

Convention 1. In this note, $\mathbb{N}$ will mean the set $\{0, 1, 2, 3, \ldots\}$ (rather than the set $\{1, 2, 3, \ldots\}$, which is denoted by $\mathbb{N}$ by various other authors).

For each $n \in \mathbb{N}$, we let S_n denote the n-th symmetric group (defined as the group of all permutations of the set $\{1, 2, \ldots, n\}$).

Convention 2. In this note, a *ring* will always mean an associative ring with 1. If k is a commutative ring, then a *k-algebra* will mean a (not necessarily commutative, but necessarily associative) k-algebra with 1. Sometimes we will use the word "algebra" as an abbreviation for "k-algebra". If L is a k-algebra, then a *left L-module* is always supposed to be a left L-module on which the unity of L acts as the identity. Whenever we use the tensor product sign $\otimes$ without an index, we mean $\otimes_k$.

0.4. Tensor products

The goal of this note is *not* to define tensor products; we assume that the reader already knows what they are. But let us recall one possible way to define the tensor product of several k-modules (assuming that the tensor product of **two** k-modules is already defined):

Definition 3. Let k be a commutative ring. Let $n \in \mathbb{N}$.
Now, by induction over n, we are going to define a k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ for any n arbitrary k-modules V_1, V_2, ..., V_n:
Induction base: For $n = 0$, we define $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ as the k-module k.
Induction step: Let $p \in \mathbb{N}$. Assuming that we have defined a k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_p$ for any p arbitrary k-modules V_1, V_2, ..., V_p, we now define a k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_{p+1}$ for any $p+1$ arbitrary k-modules V_1, V_2, ..., V_{p+1} by the equation

$$V_1 \otimes V_2 \otimes \cdots \otimes V_{p+1} = V_1 \otimes (V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1}) . \qquad (1)$$

Here, $V_1 \otimes (V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1})$ is to be understood as the tensor product of the k-module V_1 with the k-module $V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1}$ (note that the k-module $V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1}$ is already defined because we assumed that we have defined a k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_p$ for any p arbitrary k-modules V_1, V_2, ..., V_p). This completes the inductive definition.
Thus we have defined a k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ for any n arbitrary k-modules V_1, V_2, ..., V_n for any $n \in \mathbb{N}$. This k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is called the *tensor product* of the k-modules V_1, V_2, ..., V_n.

Remark 4. **(a)** Definition 3 is not the only possible definition of the tensor product of several k-modules. One could obtain a different definition by replacing the equation (1) by

$$V_1 \otimes V_2 \otimes \cdots \otimes V_{p+1} = (V_1 \otimes V_2 \otimes \cdots \otimes V_p) \otimes V_{p+1}.$$

This definition would have given us a *different* k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ for any n arbitrary k-modules V_1, V_2, ..., V_n for any $n \in \mathbb{N}$ than the one defined in Definition 3. However, this k-module would still be *canonically isomorphic* to the one defined in Definition 3, and thus it is commonly considered to be "more or less the same k-module".
There is yet another definition of $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, which proceeds by taking the free k-module on the set $V_1 \times V_2 \times \cdots \times V_n$ and factoring it modulo a certain submodule. This definition gives yet *another* k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_n$, but this module is also canonically isomorphic to the k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ defined in Definition 3, and thus can be considered to be "more or less the same k-module".
(b) Definition 3, applied to $n = 1$, defines the tensor product of *one* k-module V_1 as $V_1 \otimes k$. This takes some getting used to, since it seems more natural to define the tensor product of one k-module V_1 simply as V_1. But this isn't really different because there is a canonical isomorphism of k-modules $V_1 \cong V_1 \otimes k$, so most people consider V_1 to be "more or less the same k-module" as $V_1 \otimes k$.

Convention 5. A remark about notation is appropriate at this point:
There are two different conflicting notions of a "pure tensor" in a tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ of n arbitrary k-modules V_1, V_2, ..., V_n, where $n \geq 1$. The one notion defines a "pure tensor" as an element of the form $v \otimes T$ for some $v \in V_1$ and some $T \in V_2 \otimes V_3 \otimes \cdots \otimes V_n$ [1]. The other notion defines a "pure tensor" as an element of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ for some $(v_1, v_2, \ldots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$. These two notions are not equivalent. In this note, we are going to yield right of way to the second of these notions, i. e. we are going to define a pure tensor in $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ as an element of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ for some $(v_1, v_2, \ldots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$. The first notion, however, will also be used - but we will not call it a "pure tensor" but rather a "left-induced tensor". Thus we define a *left-induced tensor* in $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ as an element of the form $v \otimes T$ for some $v \in V_1$ and some $T \in V_2 \otimes V_3 \otimes \cdots \otimes V_n$.
We note that the k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ is generated by its left-induced tensors, but also generated by its pure tensors.

We also recall the definition of the tensor product of several k-module homomorphisms (assuming that the notion of the tensor product of **two** k-module homomorphisms is already defined):

[1] In fact, if we look at Definition 3, we see that the k-module $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ was defined as $V_1 \otimes (V_2 \otimes V_3 \otimes \cdots \otimes V_n)$, so it is the k-module $A \otimes B$ where $A = V_1$ and $B = V_2 \otimes V_3 \otimes \cdots \otimes V_n$. Since the usual definition of a pure tensor in $A \otimes B$ defines it as an element of the form $v \otimes T$ for some $v \in A$ and $T \in B$, it thus is logical to say that a pure tensor in $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ means an element of the form $v \otimes T$ for $v \in V_1$ and $T \in V_2 \otimes V_3 \otimes \cdots \otimes V_n$.

Definition 6. Let k be a commutative ring. Let $n \in \mathbb{N}$.
Now, by induction over n, we are going to define a k-module homomorphism $f_1 \otimes f_2 \otimes \cdots \otimes f_n : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to W_1 \otimes W_2 \otimes \cdots \otimes W_n$ whenever $V_1, V_2, \ldots, V_n$ are n arbitrary k-modules, $W_1, W_2, \ldots, W_n$ are n arbitrary k-modules, and $f_1 : V_1 \to W_1$, $f_2 : V_2 \to W_2, \ldots, f_n : V_n \to W_n$ are n arbitrary k-module homomorphisms:
Induction base: For $n = 0$, we define $f_1 \otimes f_2 \otimes \cdots \otimes f_n$ as the identity map $\mathrm{id} : k \to k$.
Induction step: Let $p \in \mathbb{N}$. Assume that we have defined a k-module homomorphism $f_1 \otimes f_2 \otimes \cdots \otimes f_p : V_1 \otimes V_2 \otimes \cdots \otimes V_p \to W_1 \otimes W_2 \otimes \cdots \otimes W_p$ whenever $V_1, V_2, \ldots, V_p$ are p arbitrary k-modules, $W_1, W_2, \ldots, W_p$ are p arbitrary k-modules, and $f_1 : V_1 \to W_1$, $f_2 : V_2 \to W_2, \ldots, f_p : V_p \to W_p$ are p arbitrary k-module homomorphisms. Now let us define a k-module homomorphism $f_1 \otimes f_2 \otimes \cdots \otimes f_{p+1} : V_1 \otimes V_2 \otimes \cdots \otimes V_{p+1} \to W_1 \otimes W_2 \otimes \cdots \otimes W_{p+1}$ whenever $V_1, V_2, \ldots, V_{p+1}$ are $p+1$ arbitrary k-modules, $W_1, W_2, \ldots, W_{p+1}$ are $p+1$ arbitrary k-modules, and $f_1 : V_1 \to W_1$, $f_2 : V_2 \to W_2$, $\ldots, f_{p+1} : V_{p+1} \to W_{p+1}$ are $p+1$ arbitrary k-module homomorphisms. Namely, we define this homomorphism $f_1 \otimes f_2 \otimes \cdots \otimes f_{p+1}$ to be $f_1 \otimes (f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1})$.
Here, $f_1 \otimes (f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1})$ is to be understood as the tensor product of the k-module homomorphism $f_1 : V_1 \to W_1$ with the k-module homomorphism $f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1} : V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1} \to W_2 \otimes W_3 \otimes \cdots \otimes W_{p+1}$ (note that the k-module homomorphism $f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1} : V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1} \to W_2 \otimes W_3 \otimes \cdots \otimes W_{p+1}$ is already defined (because we assumed that we have defined a k-module homomorphism $f_1 \otimes f_2 \otimes \cdots \otimes f_p : V_1 \otimes V_2 \otimes \cdots \otimes V_p \to W_1 \otimes W_2 \otimes \cdots \otimes W_p$ whenever $V_1, V_2, \ldots, V_p$ are p arbitrary k-modules, $W_1, W_2, \ldots, W_p$ are p arbitrary k-modules, and $f_1 : V_1 \to W_1$, $f_2 : V_2 \to W_2, \ldots, f_p : V_p \to W_p$ are p arbitrary k-module homomorphisms)). This completes the inductive definition.
Thus we have defined a k-module homomorphism $f_1 \otimes f_2 \otimes \cdots \otimes f_n : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to W_1 \otimes W_2 \otimes \cdots \otimes W_n$ whenever $V_1, V_2, \ldots, V_n$ are n arbitrary k-modules, W_1, W_2, $\ldots, W_n$ are n arbitrary k-modules, and $f_1 : V_1 \to W_1$, $f_2 : V_2 \to W_2, \ldots, f_n : V_n \to W_n$ are n arbitrary k-module homomorphisms. This k-module homomorphism $f_1 \otimes f_2 \otimes \cdots \otimes f_n$ is called the *tensor product* of the k-module homomorphisms f_1, $f_2, \ldots, f_n$.

Finally let us agree on a rather harmless abuse of notation:

Convention 7. Let k be a commutative ring. Let V be a k-module.
We are going to identify the three k-modules $V \otimes k$, $k \otimes V$ and V with each other (due to the canonical isomorphisms $V \to V \otimes k$ and $V \to k \otimes V$).

0.5. Tensor powers of k-modules

Next we define a particular case of tensor products of k-modules, namely the tensor powers. Here is the classical definition of this notion:

Definition 8. Let k be a commutative ring. Let $n \in \mathbb{N}$. For any k-module V, we define a k-module $V^{\otimes n}$ by $V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}}$. This k-module $V^{\otimes n}$ is called the *n-th tensor power* of the k-module V.

Remark 9. Let k be a commutative ring, and let V be a k-module. Then, $V^{\otimes 0} = k$ (because $V^{\otimes n} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{0 \text{ times}} = (\text{tensor product of zero } k\text{-modules}) = k$ according to the induction base of Definition 3) and $V^{\otimes 1} = V \otimes k$ (because $V^{\otimes 1} = \underbrace{V \otimes V \otimes \cdots \otimes V}_{1 \text{ times}} = V \otimes k$ according to the induction step of Definition 3). Since we identify $V \otimes k$ with V, we thus have $V^{\otimes 1} = V$.

Convention 10. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V and V' be k-modules, and let $f : V \to V'$ be a k-module homomorphism. Then, $f^{\otimes n}$ denotes the k-module homomorphism $\underbrace{f \otimes f \otimes \cdots \otimes f}_{n \text{ times}} : \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} \to \underbrace{V' \otimes V' \otimes \cdots \otimes V'}_{n \text{ times}}$. Since $\underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} = V^{\otimes n}$ and $\underbrace{V' \otimes V' \otimes \cdots \otimes V'}_{n \text{ times}} = V'^{\otimes n}$, this $f^{\otimes n}$ is thus a k-module homomorphism from $V^{\otimes n}$ to $V'^{\otimes n}$.

0.6. The tensor algebra

First let us agree on a convention which simplifies working with direct sums:

Convention 11. Let k be a commutative ring. Let S be a set. For every $s \in S$, let V_s be a k-module. For every $t \in S$, we are going to identify the k-module V_t with the image of V_t under the canonical injection $V_t \to \bigoplus_{s \in S} V_s$. This is an abuse of notation, but a relatively harmless one. It allows us to consider V_t as a k-submodule of the direct sum $\bigoplus_{s \in S} V_s$.

Secondly, we make a convention that simplifies working with the tensor powers of a k-module:

Convention 12. Let k be a commutative ring. For every k-module V, every $n \in \mathbb{N}$ and every $i \in \{0, 1, \ldots, n\}$, we are going to identify the k-module $V^{\otimes i} \otimes V^{\otimes (n-i)}$ with the k-module $V^{\otimes n}$ (using the canonical isomorphism $V^{\otimes i} \otimes V^{\otimes (n-i)} \cong V^{\otimes n}$). In other words, for every k-module V, every $a \in \mathbb{N}$ and every $b \in \mathbb{N}$, we are going to identify the k-module $V^{\otimes a} \otimes V^{\otimes b}$ with the k-module $V^{\otimes (a+b)}$.

The tensor powers $V^{\otimes n}$ of a k-module V can be combined to a k-module $\otimes V$ which turns out to have an algebra structure: that of the so-called tensor algebra. Let us recall its definition (which can easily shown to be well-defined):

Definition 13. Let k be a commutative ring.
(a) Let V be a k-module. The *tensor algebra* $\otimes V$ of V over k is defined to be the k-algebra formed by the k-module $\bigoplus_{i \in \mathbb{N}} V^{\otimes i} = V^{\otimes 0} \oplus V^{\otimes 1} \oplus V^{\otimes 2} \oplus \cdots$ equipped with a multiplication which is defined by

$$\left(\begin{array}{c} (a_i)_{i \in \mathbb{N}} \cdot (b_i)_{i \in \mathbb{N}} = \left(\sum_{i=0}^{n} a_i \otimes b_{n-i} \right)_{n \in \mathbb{N}} \\ \text{for every } (a_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} V^{\otimes i} \text{ and } (b_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} V^{\otimes i} \end{array} \right) \quad (2)$$

(where for every $n \in \mathbb{N}$ and every $i \in \{0, 1, \ldots, n\}$, the tensor $a_i \otimes b_{n-i} \in V^{\otimes i} \otimes V^{\otimes (n-i)}$ is considered as an element of $V^{\otimes n}$ due to the canonical identification $V^{\otimes i} \otimes V^{\otimes (n-i)} \cong V^{\otimes n}$ which was defined in Convention 12).
The k-module $\otimes V$ itself (without the k-algebra structure) is called the *tensor module* of V.
(b) Let V and W be two k-modules, and let $f : V \to W$ be a k-module homomorphism. The k-module homomorphisms $f^{\otimes i} : V^{\otimes i} \to W^{\otimes i}$ for all $i \in \mathbb{N}$ can be combined together to a k-module homomorphism from $V^{\otimes 0} \oplus V^{\otimes 1} \oplus V^{\otimes 2} \oplus \cdots$ to $W^{\otimes 0} \oplus W^{\otimes 1} \oplus W^{\otimes 2} \oplus \cdots$. This homomorphism is called $\otimes f$. Since $V^{\otimes 0} \oplus V^{\otimes 1} \oplus V^{\otimes 2} \oplus \cdots = \otimes V$ and $W^{\otimes 0} \oplus W^{\otimes 1} \oplus W^{\otimes 2} \oplus \cdots = \otimes W$, we see that this homomorphism $\otimes f$ is a k-module homomorphism from $\otimes V$ to $\otimes W$. Moreover, it follows easily from (2) that this $\otimes f$ is actually a k-algebra homomorphism from $\otimes V$ to $\otimes W$.
(c) Let V be a k-module. Then, according to Convention 11, we consider $V^{\otimes n}$ as a k-submodule of the direct sum $\bigoplus_{i \in \mathbb{N}} V^{\otimes i} = \otimes V$ for every $n \in \mathbb{N}$. In particular, every element of k is considered to be an element of $\otimes V$ by means of the canonical embedding $k = V^{\otimes 0} \subseteq \otimes V$, and every element of V is considered to be an element of $\otimes V$ by means of the canonical embedding $V = V^{\otimes 1} \subseteq \otimes V$. The element $1 \in k \subseteq \otimes V$ is easily seen to be the unity of the tensor algebra $\otimes V$.

Remark 14. The formula (2) (which defines the multiplication on the tensor algebra $\otimes V$) is often put in words by saying that "the multiplication in the tensor algebra $\otimes V$ is given by the tensor product". This informal statement tempts many authors (including myself in [2]) to use the sign $\otimes$ for multiplication in the algebra $\otimes V$, that is, to write $u \otimes v$ for the product of any two elements u and v of the tensor algebra $\otimes V$. This notation, however, can collide with the notation $u \otimes v$ for the tensor product of two vectors u and v in a k-module.[2] Due to this possibility of collision, we are *not* going to use the sign $\otimes$ for multiplication in the algebra $\otimes V$ in this paper. Instead we will use the sign $\cdot$ for this multiplication. However, due to (2), we still have

$$\left(a \cdot b = a \otimes b \qquad \text{for any } n \in \mathbb{N}, \text{ any } m \in \mathbb{N}, \text{ any } a \in V^{\otimes n} \text{ and any } b \in V^{\otimes m}\right), \tag{3}$$

where $a \otimes b$ is considered to be an element of $V^{\otimes (n+m)}$ by means of the identification of $V^{\otimes n} \otimes V^{\otimes m}$ with $V^{\otimes (n+m)}$.

The k-algebra $\otimes V$ is also denoted by $T(V)$ by many authors.

[2]For example, if z is a vector in the k-module V, then we can define two elements u and v of $\otimes V$ by $u = 1 + z$ and $v = 1 - z$ (where 1 and z are considered to be elements of $\otimes V$ according to Definition 13 **(c)**), and while the product of these elements u and v in $\otimes V$ is the element $(1 + z) \cdot (1 - z) = 1 \cdot 1 - 1 \cdot z + 1 \cdot z - z \otimes z = 1 - z \otimes z \in \otimes V$, the tensor product of these elements u and v is the element $(1 + z) \otimes (1 - z)$ of $(k \oplus V) \otimes (k \oplus V) \cong k \oplus V \oplus V \oplus (V \otimes V)$, which is a different element of a totally different k-module. So if we would use one and the same notation $u \otimes v$ for both the product of u and v in $\otimes V$ and the tensor product of u and v in $(k \oplus V) \otimes (k \oplus V)$, we would have ambiguous notations.

0.7. A variation on the nine lemma

The following fact is one of several algebraic statements related to the nine lemma, but having both weaker assertions and weaker conditions. We record it here to use it later:

Proposition 15. Let k be a commutative ring. Let A, B, C and D be k-modules, and let $x : A \to B$, $y : A \to C$, $z : B \to D$ and $w : C \to D$ be k-linear maps such that the diagram

$$\begin{CD} A @>x>> B \\ @VyVV @VVzV \\ C @>>w> D \end{CD} \qquad (4)$$

commutes. Assume that $\operatorname{Ker} z \subseteq x(\operatorname{Ker} y)$. Further assume that y is surjective. Then, $\operatorname{Ker} w = y(\operatorname{Ker} x)$.

Proof of Proposition 15. We know that the diagram

$$\begin{CD} A @>x>> B \\ @VyVV @VVzV \\ C @>>w> D \end{CD}$$

commutes. In other words, $w \circ y = z \circ x$.

We have

$$w(y(\operatorname{Ker} x)) = \underbrace{(w \circ y)}_{=z \circ x}(\operatorname{Ker} x) = (z \circ x)(\operatorname{Ker} x) = z\left(\underbrace{x(\operatorname{Ker} x)}_{=0}\right) = z(0) = 0$$
$$(\text{since } z \text{ is } k\text{-linear}),$$

and thus $y(\operatorname{Ker} x) \subseteq \operatorname{Ker} w$. We will now prove that $\operatorname{Ker} w \subseteq y(\operatorname{Ker} x)$:

Let $c \in \operatorname{Ker} w$ be arbitrary. Then, $w(c) = 0$. Now, since y is surjective, there exists some $a \in A$ such that $c = y(a)$. Consider this a. Then,

$$0 = w\left(\underbrace{c}_{=y(a)}\right) = w(y(a)) = \underbrace{(w \circ y)}_{=z \circ x}(a) = (z \circ x)(a) = z(x(a)),$$

so that $x(a) \in \operatorname{Ker} z \subseteq x(\operatorname{Ker} y)$. Thus, there exists some $a' \in \operatorname{Ker} y$ such that $x(a) = x(a')$. Consider this a'. Since x is k-linear, we have $x(a - a') = \underbrace{x(a)}_{=x(a')} - x(a') = x(a') - x(a') = 0$, so that $a - a' \in \operatorname{Ker} x$. Thus, $y(a - a') \in y(\operatorname{Ker} x)$. But since

$$y(a - a') = \underbrace{y(a)}_{=c} - \underbrace{y(a')}_{=0 \text{ (since } a' \in \operatorname{Ker} y)} \qquad (\text{since } y \text{ is } k\text{-linear})$$
$$= c - 0 = c,$$

this rewrites as $c \in y(\operatorname{Ker} x)$.

We have thus shown that every $c \in \operatorname{Ker} w$ satisfies $c \in y(\operatorname{Ker} x)$. Thus, $\operatorname{Ker} w \subseteq y(\operatorname{Ker} x)$. Combined with $y(\operatorname{Ker} x) \subseteq \operatorname{Ker} w$, this yields $\operatorname{Ker} w = y(\operatorname{Ker} x)$. This proves Proposition 15. □

Note that we would not lose any generality if we would replace k by $\mathbb{Z}$ in the statement of Proposition 15, because every k-module is an abelian group, i. e., a $\mathbb{Z}$-module (with additional structure). We could actually generalize Proposition 15 by replacing "k-modules" by "groups" (not necessarily abelian), but we will not have any use for Proposition 15 in this generality here.

0.8. Another diagram theorem about the nine lemma configuration

The next fact we will use is, again, about the nine lemma configuration:

Proposition 16. Let k be a ring. Let

$$\begin{CD} A_1 @>{a_1}>> A_2 @>{a_2}>> A_3 @>>> 0 \\ @V{u_1}VV @V{u_2}VV @V{u_3}VV \\ B_1 @>{b_1}>> B_2 @>{b_2}>> B_3 @>>> 0 \\ @V{v_1}VV @V{v_2}VV @V{v_3}VV \\ C_1 @>{c_1}>> C_2 @>{c_2}>> C_3 @>>> 0 \end{CD} \tag{5}$$

be a commutative diagram of k-left modules. Assume that every row of the diagram (6) is an exact sequence, and that every column of the diagram (6) is an exact sequence. Then,

$$\operatorname{Ker}(c_2 \circ v_2) = \operatorname{Ker}(v_3 \circ b_2) = b_1(B_1) + u_2(A_2).$$

Actually we will show something a bit stronger:

Proposition 17. Let k be a ring. Let

$$\begin{CD} A_1 @>{a_1}>> A_2 @>{a_2}>> A_3 \\ @V{u_1}VV @V{u_2}VV @V{u_3}VV \\ B_1 @>{b_1}>> B_2 @>{b_2}>> B_3 \\ @V{v_1}VV @V{v_2}VV @V{v_3}VV \\ C_1 @>{c_1}>> C_2 @>{c_2}>> C_3 \end{CD} \tag{6}$$

be a commutative diagram of k-left modules. Assume that every row of the diagram (6) is an exact sequence, and that every column of the diagram (6) is an exact sequence. Also assume that a_2 is surjective. Then,

$$\operatorname{Ker}(c_2 \circ v_2) = \operatorname{Ker}(v_3 \circ b_2) = b_1(B_1) + u_2(A_2).$$

Proof of Proposition 17. Since (6) is a commutative diagram, we have $c_2 \circ v_2 = v_3 \circ b_2$ and $b_2 \circ u_2 = u_3 \circ a_2$.

Since every row of the diagram (6) is an exact sequence, we have $b_2 \circ b_1 = 0$.

Since every column of the diagram (6) is an exact sequence, we have $v_3 \circ u_3 = 0$.

From $c_2 \circ v_2 = v_3 \circ b_2$, we conclude $\operatorname{Ker}(c_2 \circ v_2) = \operatorname{Ker}(v_3 \circ b_2)$. Thus, it only remains to prove that $\operatorname{Ker}(v_3 \circ b_2) = b_1(B_1) + u_2(A_2)$. Since $b_1(B_1) + u_2(A_2) \subseteq \operatorname{Ker}(v_3 \circ b_2)$ is obvious (because

$$\begin{aligned}
&(v_3 \circ b_2)(b_1(B_1) + u_2(A_2)) \\
&= v_3(b_2(b_1(B_1) + u_2(A_2))) = \underbrace{v_3(b_2(b_1(B_1)))}_{=v_3((b_2 \circ b_1)(B_1))} + \underbrace{v_3(b_2(u_2(A_2)))}_{=(v_3 \circ b_2 \circ u_2)(A_2)} \\
&= v_3\left(\underbrace{(b_2 \circ b_1)}_{=0}(B_1)\right) + \left(v_3 \circ \underbrace{b_2 \circ u_2}_{=u_3 \circ a_2}\right)(A_2) = \underbrace{v_3(0(B_1))}_{=0} + \left(\underbrace{v_3 \circ u_3}_{=0} \circ a_2\right)(A_2) \\
&= 0 + \underbrace{(0 \circ a_2)(A_2)}_{=0} = 0
\end{aligned}$$

), we must now only show that $\operatorname{Ker}(v_3 \circ b_2) \subseteq b_1(B_1) + u_2(A_2)$.

Let $t \in \operatorname{Ker}(v_3 \circ b_2)$ be arbitrary. Then, $(v_3 \circ b_2)(t) = 0$, so that $v_3(b_2(t)) = (v_3 \circ b_2)(t) = 0$ and thus $b_2(t) \in \operatorname{Ker} v_3 = u_3(A_3)$ (because every column of the diagram (6) is an exact sequence). Thus, there exists some $x \in A_3$ such that $b_2(t) = u_3(x)$. Consider this x. Since $a_2 : A_2 \to A_3$ is surjective, we have $x = a_2(x')$ for some $x' \in A_2$. Consider this x'. Now,

$$b_2(t - u_2(x')) = \underbrace{b_2(t)}_{=u_3(x)} - \underbrace{b_2(u_2(x'))}_{=(b_2 \circ u_2)(x')} = u_3(x) - \underbrace{(b_2 \circ u_2)}_{=u_3 \circ a_2}(x') = u_3(x) - u_3\left(\underbrace{a_2(x')}_{=x}\right) = 0,$$

so that $t - u_2(x') \in \operatorname{Ker} b_2 = b_1(B_1)$ (because every row of the diagram (6) is an exact sequence). Thus, $t = \underbrace{t - u_2(x')}_{\in b_1(B_1)} + \underbrace{u_2(x')}_{\in u_2(A_2)} \in b_1(B_1) + u_2(A_2)$.

We thus have shown that every $t \in \operatorname{Ker}(v_3 \circ b_2)$ satisfies $t \in b_1(B_1) + u_2(A_2)$. Consequently, $\operatorname{Ker}(v_3 \circ b_2) \subseteq b_1(B_1) + u_2(A_2)$. Combined with $b_1(B_1) + u_2(A_2) \subseteq \operatorname{Ker}(v_3 \circ b_2)$, this yields $\operatorname{Ker}(v_3 \circ b_2) = b_1(B_1) + u_2(A_2)$. Combined with $\operatorname{Ker}(c_2 \circ v_2) = \operatorname{Ker}(v_3 \circ b_2)$, this now completes the proof of Proposition 17. ☐

Proof of Proposition 16. Since the diagram (5) is commutative, the diagram (6) must also be commutative (because the diagram (6) is a subdiagram of the diagram (5)). Also, the map a_2 is surjective (since every row of the diagram (5) is an exact sequence). Therefore, we can apply Proposition 17, and conclude that $\operatorname{Ker}(c_2 \circ v_2) = \operatorname{Ker}(v_3 \circ b_2) = b_1(B_1) + u_2(A_2)$. This proves Proposition 16. ☐

0.9. $\operatorname{Ker}(f \otimes g)$ when f and g are surjective

Theorem 18. Let k be a commutative ring. Let V, W, V' and W' be four k-modules. Let $f : V \to V'$ and $g : W \to W'$ be two surjective k-linear maps. Let i_V be the canonical inclusion $\operatorname{Ker} f \to V$. Let i_W be the canonical inclusion $\operatorname{Ker} g \to W$. Then,

$$\operatorname{Ker}(f \otimes g) = (i_V \otimes \operatorname{id})((\operatorname{Ker} f) \otimes W) + (\operatorname{id} \otimes i_W)(V \otimes (\operatorname{Ker} g)).$$

Remark 19. **(a)** In Theorem 18, the condition that f and g be surjective cannot be removed (otherwise, $V = \mathbb{Z}$, $W = \mathbb{Z}$, $V' = \mathbb{Z}/4\mathbb{Z}$, $W' = \mathbb{Z}/4\mathbb{Z}$, $f = \left(x \mapsto \overline{2x}\right)$, $g = \left(x \mapsto \overline{2x}\right)$ would be a counterexample), but it can be replaced by some other conditions (see Lemma 21 and Corollary 20). (Here is a more complicated counterexample to show that having only g surjective is not yet enough: $V = \mathbb{Z}$, $W = \mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})$, $V' = \mathbb{Z}$, $W' = \mathbb{Z}/4\mathbb{Z}$, $f = (x \mapsto 2x)$, $g = \left((x, \alpha) \mapsto \overline{2x} + \alpha\right)$.)

(b) If the k-module V is flat in Theorem 18, then the map $\operatorname{id} \otimes i_W$ is injective (as can be easily seen), and therefore many people prefer to identify the image $(\operatorname{id} \otimes i_W)(V \otimes (\operatorname{Ker} g))$ with $V \otimes (\operatorname{Ker} g)$. Similarly, the image $(i_V \otimes \operatorname{id})((\operatorname{Ker} f) \otimes W)$ can be identified with $(\operatorname{Ker} f) \otimes W$ when the k-module W is flat. It is common among algebraists to perform these identifications when k is a field (because when k is a field, both k-modules V and W are flat), and sometimes even when k is not, but we will not perform these identifications here.

Proof of Theorem 18. The sequence

$$0 \longrightarrow \operatorname{Ker} f \xrightarrow{i_V} V \xrightarrow{f} V' \longrightarrow 0$$

is exact (since i_V is the inclusion map $\operatorname{Ker} f \to V$, while f is surjective). Since the tensor product is right exact, this yields that

$$\left(\begin{array}{c} \text{the sequences} \\ (\operatorname{Ker} f) \otimes (\operatorname{Ker} g) \xrightarrow{i_V \otimes \operatorname{id}} V \otimes (\operatorname{Ker} g) \xrightarrow{f \otimes \operatorname{id}} V' \otimes (\operatorname{Ker} g) \longrightarrow 0, \\ (\operatorname{Ker} f) \otimes W \xrightarrow{i_V \otimes \operatorname{id}} V \otimes W \xrightarrow{f \otimes \operatorname{id}} V' \otimes W \longrightarrow 0 \text{ and} \\ (\operatorname{Ker} f) \otimes W' \xrightarrow{i_V \otimes \operatorname{id}} V \otimes W' \xrightarrow{f \otimes \operatorname{id}} V' \otimes W' \longrightarrow 0 \text{ are exact} \end{array}\right). \tag{7}$$

On the other hand, the sequence

$$0 \longrightarrow \operatorname{Ker} g \xrightarrow{i_W} W \xrightarrow{g} W' \longrightarrow 0$$

is exact (since i_W is the inclusion map $\operatorname{Ker} g \to W$, while g is surjective). Since the tensor product is right exact, this yields that

$$\left(\begin{array}{c} \text{the sequences} \\ (\operatorname{Ker} f) \otimes (\operatorname{Ker} g) \xrightarrow{\operatorname{id} \otimes i_W} (\operatorname{Ker} f) \otimes W \xrightarrow{\operatorname{id} \otimes g} (\operatorname{Ker} f) \otimes W' \longrightarrow 0, \\ V \otimes (\operatorname{Ker} g) \xrightarrow{\operatorname{id} \otimes i_W} V \otimes W \xrightarrow{\operatorname{id} \otimes g} V \otimes W' \longrightarrow 0 \text{ and} \\ V' \otimes (\operatorname{Ker} g) \xrightarrow{\operatorname{id} \otimes i_W} V' \otimes W \xrightarrow{\operatorname{id} \otimes g} V' \otimes W' \longrightarrow 0 \text{ are exact} \end{array}\right). \tag{8}$$

Now, the diagram

$$\begin{array}{ccccccc}
(\operatorname{Ker} f) \otimes (\operatorname{Ker} g) & \xrightarrow{\operatorname{id} \otimes i_W} & (\operatorname{Ker} f) \otimes W & \xrightarrow{\operatorname{id} \otimes g} & (\operatorname{Ker} f) \otimes W' & \longrightarrow & 0 \\
\downarrow{\scriptstyle i_V \otimes \operatorname{id}} & & \downarrow{\scriptstyle i_V \otimes \operatorname{id}} & & \downarrow{\scriptstyle i_V \otimes \operatorname{id}} & & \\
V \otimes (\operatorname{Ker} g) & \xrightarrow{\operatorname{id} \otimes i_W} & V \otimes W & \xrightarrow{\operatorname{id} \otimes g} & V \otimes W' & \longrightarrow & 0 \\
\downarrow{\scriptstyle f \otimes \operatorname{id}} & & \downarrow{\scriptstyle f \otimes \operatorname{id}} & & \downarrow{\scriptstyle f \otimes \operatorname{id}} & & \\
V' \otimes (\operatorname{Ker} g) & \xrightarrow{\operatorname{id} \otimes i_W} & V' \otimes W & \xrightarrow{\operatorname{id} \otimes g} & V' \otimes W' & \longrightarrow & 0
\end{array} \tag{9}$$

is commutative (because

$$(i_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes i_W) = i_V \otimes i_W = (\mathrm{id} \otimes i_W) \circ (i_V \otimes \mathrm{id});$$
$$(i_V \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g) = i_V \otimes g = (\mathrm{id} \otimes g) \circ (i_V \otimes \mathrm{id});$$
$$(f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes i_W) = f \otimes i_W = (\mathrm{id} \otimes i_W) \circ (f \otimes \mathrm{id});$$
$$(f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g) = f \otimes g = (\mathrm{id} \otimes g) \circ (f \otimes \mathrm{id})$$

). Every row of this diagram is an exact sequence (due to (8)), and every column of this diagram is an exact sequence (due to (7)). Thus, Proposition 16 (applied to the diagram (9) instead of the diagram (5)) yields that

$$\begin{aligned}\mathrm{Ker}\left((\mathrm{id} \otimes g) \circ (f \otimes \mathrm{id})\right) &= \mathrm{Ker}\left((f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g)\right) \\ &= (\mathrm{id} \otimes i_W)\left(V \otimes (\mathrm{Ker}\, g)\right) + (i_V \otimes \mathrm{id})\left((\mathrm{Ker}\, f) \otimes W\right).\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{Ker} \underbrace{(f \otimes g)}_{=(f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g)} &= \mathrm{Ker}\left((f \otimes \mathrm{id}) \circ (\mathrm{id} \otimes g)\right) \\ &= (\mathrm{id} \otimes i_W)\left(V \otimes (\mathrm{Ker}\, g)\right) + (i_V \otimes \mathrm{id})\left((\mathrm{Ker}\, f) \otimes W\right) \\ &= (i_V \otimes \mathrm{id})\left((\mathrm{Ker}\, f) \otimes W\right) + (\mathrm{id} \otimes i_W)\left(V \otimes (\mathrm{Ker}\, g)\right).\end{aligned}$$

This proves Theorem 18. □

Let us notice a corollary of this theorem:

Corollary 20. Let k be a commutative ring. Let V, W, V' and W' be four k-modules. Let $f : V \to V'$ and $g : W \to W'$ be two k-linear maps. Assume that $f(V)$ and W' are flat k-modules. Let i_V be the canonical inclusion $\mathrm{Ker}\, f \to V$. Let i_W be the canonical inclusion $\mathrm{Ker}\, g \to W$. Then,

$$\mathrm{Ker}\,(f \otimes g) = (i_V \otimes \mathrm{id})\left((\mathrm{Ker}\, f) \otimes W\right) + (\mathrm{id} \otimes i_W)\left(V \otimes (\mathrm{Ker}\, g)\right).$$

Note that this Corollary 20 does not require f or g to be surjective, but instead it requires $f(V)$ and W' to be flat (which is always satisfied if k is a field, for example).

To show this, we first prove:

Lemma 21. Let k be a commutative ring. Let V, W, V' and W' be four k-modules. Let $f : V \to V'$ and $g : W \to W'$ be two k-linear maps. Assume that V' is a flat k-module, and that f is surjective. Let i_V be the canonical inclusion $\mathrm{Ker}\, f \to V$. Let i_W be the canonical inclusion $\mathrm{Ker}\, g \to W$. Then,

$$\mathrm{Ker}\,(f \otimes g) = (i_V \otimes \mathrm{id})\left((\mathrm{Ker}\, f) \otimes W\right) + (\mathrm{id} \otimes i_W)\left(V \otimes (\mathrm{Ker}\, g)\right).$$

Lemma 22. Let k be a commutative ring. Let V, W and A be three k-modules such that the k-module A is flat. Let $i : V \to W$ be an injective k-module homomorphism.
(a) The k-module homomorphism $\mathrm{id} \otimes i : A \otimes V \to A \otimes W$ is injective.
(b) The k-module homomorphism $i \otimes \mathrm{id} : V \otimes A \to W \otimes A$ is injective.

Proof of Lemma 22. Let p be the canonical projection $p : W \to W \diagup (i(V))$. Then, p is a surjective k-module homomorphism, and $\operatorname{Ker} p = i(V)$. Thus,

$$0 \longrightarrow V \xrightarrow{\ i\ } W \xrightarrow{\ p\ } W \diagup (i(V)) \longrightarrow 0 \tag{10}$$

is a short exact sequence (since p is surjective, since i is injective, and since $\operatorname{Ker} p = i(V)$). Since tensoring with A is an exact functor (because A is a flat k-module), this yields that

$$0 \longrightarrow A \otimes V \xrightarrow{\operatorname{id} \otimes i} A \otimes W \xrightarrow{\operatorname{id} \otimes p} A \otimes (W \diagup (i(V))) \longrightarrow 0$$

is a short exact sequence. Therefore, $\operatorname{id} \otimes i : A \otimes V \to A \otimes W$ is injective. This proves Lemma 22 **(a)**.

Since (10) is a short exact sequence, and since tensoring with A is an exact functor (because A is a flat k-module), we find that

$$0 \longrightarrow V \otimes A \xrightarrow{i \otimes \operatorname{id}} W \otimes A \xrightarrow{p \otimes \operatorname{id}} (W \diagup (i(V))) \otimes A \longrightarrow 0$$

is a short exact sequence. Therefore, $i \otimes \operatorname{id} : V \otimes A \to W \otimes A$ is injective. This proves Lemma 22 **(b)**. $\square$

Proof of Lemma 21. Define a k-linear map $g_1 : W \to g(W)$ by

$$(g_1(w) = g(w) \qquad \text{for every } w \in W)$$

(this is well-defined since $g(w) \in g(W)$ for every $w \in W$). Let m_W be the canonical inclusion $g(W) \to W'$. Clearly, every $w \in W$ satisfies

$$\begin{aligned}(m_W \circ g_1)(w) &= m_W(g_1(w)) = g_1(w) \qquad \text{(since } m_W \text{ is the canonical inclusion)} \\ &= g(w).\end{aligned}$$

Thus, $m_W \circ g_1 = g$.

Also, g_1 is surjective, because every $x \in g(W)$ satisfies $x \in g_1(W)$ [3]. Also,

$$\operatorname{Ker} g = \left\{ w \in W \mid \underbrace{g(w)}_{=g_1(w)} = 0 \right\} = \{ w \in W \mid g_1(w) = 0 \} = \operatorname{Ker} g_1.$$

Thus, i_W is the canonical inclusion $\operatorname{Ker} g_1 \to W$ (since i_W is the canonical inclusion $\operatorname{Ker} g \to W$). Thus, Theorem 18 (applied to $g(W)$ and g_1 instead of W' and g) shows that

$$\begin{aligned}\operatorname{Ker}(f \otimes g_1) &= (i_V \otimes \operatorname{id})((\operatorname{Ker} f) \otimes W) + (\operatorname{id} \otimes i_W)\left(V \otimes \underbrace{(\operatorname{Ker} g_1)}_{=\operatorname{Ker} g} \right) \\ &= (i_V \otimes \operatorname{id})((\operatorname{Ker} f) \otimes W) + (\operatorname{id} \otimes i_W)(V \otimes (\operatorname{Ker} g)).\end{aligned} \tag{11}$$

[3] *Proof.* Let $x \in g(W)$ be arbitrary. Then, there exists some $w \in W$ such that $x = g(w)$. Consider this w. Then, $x = g(w) = g_1(w) \in g_1(W)$, qed.

Now, m_W is injective (since m_W is a canonical inclusion). Thus, applying Lemma 22 **(a)** to m_W, $g(W)$, W' and V' instead of i, V, W and A, we obtain that the map $\operatorname{id} \otimes m_W : V' \otimes g(W) \to V' \otimes W'$ is injective. In other words, $\operatorname{Ker}(\operatorname{id} \otimes m_W) = 0$.

Now, it is known that whenever A, B, C, A', B', C' are six k-modules, and $\alpha : A \to B$, $\beta : B \to C$, $\gamma : A' \to B'$ and $\delta : B' \to C'$ are four k-linear maps, then $(\beta \otimes \delta) \circ (\alpha \otimes \gamma) = (\beta \circ \alpha) \otimes (\delta \circ \gamma)$. Applying this fact to $A = V$, $B = V'$, $C = V'$, $A' = W$, $B' = g(W)$, $C' = W'$, $\alpha = f$, $\beta = \operatorname{id}$, $\gamma = g_1$ and $\delta = m_W$, we obtain

$$(\operatorname{id} \otimes m_W) \circ (f \otimes g_1) = \underbrace{(\operatorname{id} \circ f)}_{=f} \otimes \left(\underbrace{m_W \circ g_1}_{=g} \right) = f \otimes g.$$

Now, let $x \in \operatorname{Ker}(f \otimes g_1)$ be arbitrary. Then, $(f \otimes g_1)(x) = 0$. Now,

$$\underbrace{(f \otimes g)}_{=(\operatorname{id} \otimes m_W) \circ (f \otimes g_1)} (x) = ((\operatorname{id} \otimes m_W) \circ (f \otimes g_1))(x) = (\operatorname{id} \otimes m_W) \left(\underbrace{(f \otimes g_1)(x)}_{=0} \right)$$
$$= (\operatorname{id} \otimes m_W)(0) = 0,$$

so that $x \in \operatorname{Ker}(f \otimes g)$. Thus we have seen that every $x \in \operatorname{Ker}(f \otimes g_1)$ satisfies $x \in \operatorname{Ker}(f \otimes g)$. In other words, $\operatorname{Ker}(f \otimes g_1) \subseteq \operatorname{Ker}(f \otimes g)$.

On the other hand, let $y \in \operatorname{Ker}(f \otimes g)$ be arbitrary. Then,

$$(\operatorname{id} \otimes m_W)((f \otimes g_1)(y)) = \left(\underbrace{(\operatorname{id} \otimes m_W) \circ (f \otimes g_1)}_{=f \otimes g} \right)(y) = (f \otimes g)(y) = 0$$

(since $y \in \operatorname{Ker}(f \otimes g)$), so that $(f \otimes g_1)(y) \in \operatorname{Ker}(\operatorname{id} \otimes m_W) = 0$. Thus, $(f \otimes g_1)(y) = 0$, so that $y \in \operatorname{Ker}(f \otimes g_1)$. Thus we have shown that every $y \in \operatorname{Ker}(f \otimes g)$ satisfies $y \in \operatorname{Ker}(f \otimes g_1)$. In other words, $\operatorname{Ker}(f \otimes g) \subseteq \operatorname{Ker}(f \otimes g_1)$.

Combined with $\operatorname{Ker}(f \otimes g_1) \subseteq \operatorname{Ker}(f \otimes g)$, this yields $\operatorname{Ker}(f \otimes g) = \operatorname{Ker}(f \otimes g_1)$. Thus, (11) becomes

$$\operatorname{Ker}(f \otimes g) = (i_V \otimes \operatorname{id})((\operatorname{Ker} f) \otimes W) + (\operatorname{id} \otimes i_W)(V \otimes (\operatorname{Ker} g)).$$

This proves Lemma 21. □

Proof of Corollary 20. Define a k-linear map $f_1 : V \to f(V)$ by

$$(f_1(v) = f(v) \qquad \text{for every } v \in V)$$

(this is well-defined since $f(v) \in f(V)$ for every $v \in V$). Let m_V be the canonical inclusion $f(V) \to V'$. Clearly, every $v \in V$ satisfies

$$(m_V \circ f_1)(v) = m_V(f_1(v)) = f_1(v) \qquad \text{(since } m_V \text{ is the canonical inclusion)}$$
$$= f(v).$$

Thus, $m_V \circ f_1 = f$.

Also, f_1 is surjective, because every $x \in f(V)$ satisfies $x \in f_1(V)$ [4]. Also,

$$\operatorname{Ker} f = \left\{ v \in V \mid \underbrace{f(v)}_{=f_1(v)} = 0 \right\} = \{ v \in V \mid f_1(v) = 0 \} = \operatorname{Ker} f_1.$$

Thus, i_V is the canonical inclusion $\operatorname{Ker} f_1 \to V$ (since i_V is the canonical inclusion $\operatorname{Ker} f \to V$). Thus, Lemma 21 (applied to $f(V)$ and f_1 instead of V' and f) shows that

$$\begin{aligned} \operatorname{Ker}(f_1 \otimes g) &= (i_V \otimes \operatorname{id}) \left(\underbrace{(\operatorname{Ker} f_1)}_{=\operatorname{Ker} f} \otimes W \right) + (\operatorname{id} \otimes i_W)(V \otimes (\operatorname{Ker} g)) \\ &= (i_V \otimes \operatorname{id})((\operatorname{Ker} f) \otimes W) + (\operatorname{id} \otimes i_W)(V \otimes (\operatorname{Ker} g)). \end{aligned} \tag{12}$$

Now, m_V is injective (since m_V is a canonical inclusion). Thus, applying Lemma 22 **(b)** to m_V, $f(V)$, V' and W' instead of i, V, W and A, we obtain that the map $m_V \otimes \operatorname{id} : f(V) \otimes W' \to V' \otimes W'$ is injective. In other words, $\operatorname{Ker}(m_V \otimes \operatorname{id}) = 0$.

Now, it is known that whenever A, B, C, A', B', C' are six k-modules, and $\alpha : A \to B$, $\beta : B \to C$, $\gamma : A' \to B'$ and $\delta : B' \to C'$ are four k-linear maps, then $(\beta \otimes \delta) \circ (\alpha \otimes \gamma) = (\beta \circ \alpha) \otimes (\delta \circ \gamma)$. Applying this fact to $A = V$, $B = f(V)$, $C = V'$, $A' = W$, $B' = W'$, $C' = W'$, $\alpha = f_1$, $\beta = m_V$, $\gamma = g$ and $\delta = \operatorname{id}$, we obtain

$$(m_V \otimes \operatorname{id}) \circ (f_1 \otimes g) = \underbrace{(m_V \circ f_1)}_{=f} \otimes \underbrace{(\operatorname{id} \circ g)}_{=g} = f \otimes g.$$

Now, let $x \in \operatorname{Ker}(f_1 \otimes g)$ be arbitrary. Then, $(f_1 \otimes g)(x) = 0$. Now,

$$\begin{aligned} \underbrace{(f \otimes g)}_{=(m_V \otimes \operatorname{id}) \circ (f_1 \otimes g)} (x) &= ((m_V \otimes \operatorname{id}) \circ (f_1 \otimes g))(x) = (m_V \otimes \operatorname{id}) \left(\underbrace{(f_1 \otimes g)(x)}_{=0} \right) \\ &= (m_V \otimes \operatorname{id})(0) = 0, \end{aligned}$$

so that $x \in \operatorname{Ker}(f \otimes g)$. Thus we have seen that every $x \in \operatorname{Ker}(f_1 \otimes g)$ satisfies $x \in \operatorname{Ker}(f \otimes g)$. In other words, $\operatorname{Ker}(f_1 \otimes g) \subseteq \operatorname{Ker}(f \otimes g)$.

On the other hand, let $y \in \operatorname{Ker}(f \otimes g)$ be arbitrary. Then,

$$(m_V \otimes \operatorname{id})((f_1 \otimes g)(y)) = \left(\underbrace{(m_V \otimes \operatorname{id}) \circ (f_1 \otimes g)}_{=f \otimes g} \right)(y) = (f \otimes g)(y) = 0$$

(since $y \in \operatorname{Ker}(f \otimes g)$), so that $(f_1 \otimes g)(y) \in \operatorname{Ker}(m_V \otimes \operatorname{id}) = 0$. Thus, $(f_1 \otimes g)(y) = 0$, so that $y \in \operatorname{Ker}(f_1 \otimes g)$. Thus we have shown that every $y \in \operatorname{Ker}(f \otimes g)$ satisfies $y \in \operatorname{Ker}(f_1 \otimes g)$. In other words, $\operatorname{Ker}(f \otimes g) \subseteq \operatorname{Ker}(f_1 \otimes g)$.

Combined with $\operatorname{Ker}(f_1 \otimes g) \subseteq \operatorname{Ker}(f \otimes g)$, this yields $\operatorname{Ker}(f \otimes g) = \operatorname{Ker}(f_1 \otimes g)$. Thus, (12) becomes

$$\operatorname{Ker}(f \otimes g) = (i_V \otimes \operatorname{id})((\operatorname{Ker} f) \otimes W) + (\operatorname{id} \otimes i_W)(V \otimes (\operatorname{Ker} g)).$$

This proves Corollary 20. □

[4] *Proof.* Let $x \in f(V)$ be arbitrary. Then, there exists some $v \in V$ such that $x = f(v)$. Consider this v. Then, $x = f(v) = f_1(v) \in f_1(V)$, qed.

We notice a triviality on tensor products of surjective maps:

Lemma 23. Let k be a commutative ring. Let V, W, V' and W' be four k-modules. Let $f : V \to V'$ and $g : W \to W'$ be two surjective k-linear maps. Then, the map $f \otimes g : V \otimes W \to V' \otimes W'$ is surjective.

Proof of Lemma 23. Let $T \in V' \otimes W'$ be arbitrary. Then, we can write the tensor T in the form $T = \sum_{i=1}^{n} \alpha_i \otimes \beta_i$ for some $n \in \mathbb{N}$, some elements α_1, α_2, ..., α_n of V' and some elements β_1, β_2, ..., β_n of W'. Consider this n, these α_1, α_2, ..., α_n and these β_1, β_2, ..., β_n.

For every $i \in \{1, 2, \ldots, n\}$, there exists some $v_i \in V$ such that $\alpha_i = f(v_i)$ (since f is surjective). Consider this v_i.

For every $i \in \{1, 2, \ldots, n\}$, there exists some $w_i \in W$ such that $\beta_i = g(w_i)$ (since g is surjective). Consider this w_i.

Now,

$$\begin{aligned} T &= \sum_{i=1}^{n} \underbrace{\alpha_i}_{=f(v_i)} \otimes \underbrace{\beta_i}_{=g(w_i)} = \sum_{i=1}^{n} \underbrace{f(v_i) \otimes g(w_i)}_{=(f\otimes g)(v_i \otimes w_i)} = \sum_{i=1}^{n} (f \otimes g)(v_i \otimes w_i) \\ &= (f \otimes g)\left(\sum_{i=1}^{n} v_i \otimes w_i \right) \qquad \text{(since } f \otimes g \text{ is } k\text{-linear)} \\ &\in (f \otimes g)(V \otimes W). \end{aligned}$$

So we have proven that every $T \in V' \otimes W'$ satisfies $T \in (f \otimes g)(V \otimes W)$. Thus, $f \otimes g$ is surjective, so that Lemma 23 is proven. $\square$

0.10. Extension to n modules

We can trivially generalize Lemma 23 to several k-modules:

Lemma 24. Let k be a commutative ring. Let $n \in \mathbb{N}$. For any $i \in \{1, 2, \ldots, n\}$, let V_i and V_i' be two k-modules, and let $f_i : V_i \to V_i'$ be a surjective k-module homomorphism. Then, the map $f_1 \otimes f_2 \otimes \cdots \otimes f_n : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to V_1' \otimes V_2' \otimes \cdots \otimes V_n'$ is surjective.

Proof of Lemma 24. We are going to prove Lemma 24 by induction over n:

Induction base: For $n = 0$, Lemma 24 holds (because for $n = 0$, the map $f_1 \otimes f_2 \otimes \cdots \otimes f_n : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to V_1' \otimes V_2' \otimes \cdots \otimes V_n'$ is the identity map $\mathrm{id} : k \to k$ and therefore surjective). Thus, the induction base is complete.

Induction step: Let $p \in \mathbb{N}$ be arbitrary. Assume that Lemma 24 holds for $n = p$.

Now let us prove that Lemma 24 holds for $n = p + 1$. So let V_i and V_i' be two k-modules for every $i \in \{1, 2, \ldots, p + 1\}$, and let $f_i : V_i \to V_i'$ be a surjective k-module homomorphism for every $i \in \{1, 2, \ldots, p + 1\}$.

According to Definition 6, we have $f_1 \otimes f_2 \otimes \cdots \otimes f_{p+1} = f_1 \otimes (f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1})$.

We know that V_i and V_i' are two k-modules for every $i \in \{1, 2, \ldots, p + 1\}$. Thus, V_i and V_i' are two k-modules for every $i \in \{2, 3, \ldots, p + 1\}$. Substituting $i + 1$ for i in this fact, we obtain that V_{i+1} and V_{i+1}' are two k-modules for every $i \in \{1, 2, \ldots, p\}$.

We know that $f_i : V_i \to V_i'$ is a surjective k-module homomorphism for every $i \in \{1, 2, \dots, p+1\}$. Thus, $f_i : V_i \to V_i'$ is a surjective k-module homomorphism for every $i \in \{2, 3, \dots, p+1\}$. Substituting $i+1$ for i in this fact, we obtain that f_{i+1} is a surjective k-module homomorphism for every $i \in \{1, 2, \dots, p\}$.

Applying Lemma 24 to p, V_{i+1}, V_{i+1}' and f_{i+1} instead of n, V_i, V_i' and f_i (this is allowed, because we have assumed that Lemma 24 holds for $n = p$), we see that the map $f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1}$ is surjective.

We know that $f_i : V_i \to V_i'$ is a surjective k-module homomorphism for every $i \in \{1, 2, \dots, p+1\}$. Applying this to $i = 1$, we conclude that $f_1 : V_1 \to V_1'$ is a surjective k-module homomorphism.

Applying Lemma 23 to $V = V_1$, $V' = V_1'$, $W = V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1}$, $W' = V_2' \otimes V_3' \otimes \cdots \otimes V_{p+1}'$, $f = f_1$ and $g = f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1}$, we now conclude that the map $f_1 \otimes (f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1})$ is surjective. Since $f_1 \otimes f_2 \otimes \cdots \otimes f_{p+1} = f_1 \otimes (f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1})$, this yields that the map $f_1 \otimes f_2 \otimes \cdots \otimes f_{p+1}$ is surjective.

We have thus proven that if V_i and V_i' are two k-modules for every $i \in \{1, 2, \dots, p+1\}$, and $f_i : V_i \to V_i'$ is a surjective k-module homomorphism for every $i \in \{1, 2, \dots, p+1\}$, then the map $f_1 \otimes f_2 \otimes \cdots \otimes f_{p+1} : V_1 \otimes V_2 \otimes \cdots \otimes V_{p+1} \to V_1' \otimes V_2' \otimes \cdots \otimes V_{p+1}'$ is surjective. In other words, we have proven that Lemma 24 holds for $n = p+1$. This completes the induction step, and thus Lemma 24 is proven. $\square$

Now let us extend Theorem 18 to n modules:

Theorem 25. Let k be a commutative ring. Let $n \in \mathbb{N}$. For any $i \in \{1, 2, \dots, n\}$, let V_i and V_i' be two k-modules, and let $f_i : V_i \to V_i'$ be a surjective k-module homomorphism. For any $i \in \{1, 2, \dots, n\}$, let $\mathfrak{i}_i$ be the canonical inclusion $\operatorname{Ker} f_i \to V_i$. Then,

$$\begin{aligned}
&\operatorname{Ker}(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \\
&= \sum_{i=1}^{n} \left(\underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{i-1 \text{ times}} \otimes \mathfrak{i}_i \otimes \underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n-i \text{ times}} \right) \\
&\qquad (V_1 \otimes V_2 \otimes \cdots \otimes V_{i-1} \otimes (\operatorname{Ker} f_i) \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_n) . \qquad (13)
\end{aligned}$$

Before we show this, we need an (almost trivial) lemma:

Lemma 26. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let A and B be two k-modules. For any $i \in \{1, 2, \dots, n\}$, let B_i be a k-submodule of B. Let B' be the k-submodule $\sum_{i=1}^{n} B_i$ of B.

For any k-module C and any k-submodule D of C, we let $\mathrm{inc}_{D,C}$ denote the canonical inclusion map $D \to C$.

Then,

$$(\mathrm{id} \otimes \mathrm{inc}_{B',B})(A \otimes B') = \sum_{i=1}^{n} (\mathrm{id} \otimes \mathrm{inc}_{B_i,B})(A \otimes B_i)$$

(as k-submodules of $A \otimes B$).

Proof of Lemma 26. Since $\sum_{i=1}^{n} B_i = B'$, we have $B_i \subseteq B'$ for every $i \in \{1, 2, \ldots, n\}$.

The maps $\operatorname{inc}_{B_i,B} : B_i \to B$ for all $i \in \{1, 2, \ldots, n\}$ give rise to a map $\sum_{i=1}^{n} \operatorname{inc}_{B_i,B} : \bigoplus_{i=1}^{n} B_i \to B$. Similarly, the maps $\operatorname{id} \otimes \operatorname{inc}_{B_i,B} : A \otimes B_i \to A \otimes B$ for all $i \in \{1, 2, \ldots, n\}$ give rise to a map $\sum_{i=1}^{n} \operatorname{id} \otimes \operatorname{inc}_{B_i,B} : \bigoplus_{i=1}^{n} (A \otimes B_i) \to A \otimes B$.

Since the tensor product is known to commute with direct sums, there is a canonical k-module isomorphism $A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \to \bigoplus_{i=1}^{n} (A \otimes B_i)$. Denote this isomorphism by I. By the universal property of this I, the diagram

$$\begin{array}{ccc} A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) & \xrightarrow{I} & \bigoplus_{i=1}^{n} (A \otimes B_i) \\ & \searrow^{\operatorname{id} \otimes \left(\sum_{i=1}^{n} \operatorname{inc}_{B_i,B} \right)} & \downarrow{\scriptstyle \sum_{i=1}^{n} \left(\operatorname{id} \otimes \operatorname{inc}_{B_i,B} \right)} \\ & & A \otimes B \end{array}$$

commutes. In other words,

$$\operatorname{id} \otimes \left(\sum_{i=1}^{n} \operatorname{inc}_{B_i,B} \right) = \left(\sum_{i=1}^{n} (\operatorname{id} \otimes \operatorname{inc}_{B_i,B}) \right) \circ I,$$

so that

$$\begin{aligned} \left(\operatorname{id} \otimes \left(\sum_{i=1}^{n} \operatorname{inc}_{B_i,B} \right) \right) \left(A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \right) &= \left(\left(\sum_{i=1}^{n} (\operatorname{id} \otimes \operatorname{inc}_{B_i,B}) \right) \circ I \right) \left(A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \right) \\ &= \left(\sum_{i=1}^{n} (\operatorname{id} \otimes \operatorname{inc}_{B_i,B}) \right) \underbrace{\left(I \left(A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \right) \right)}_{\substack{= \bigoplus_{i=1}^{n} (A \otimes B_i) \\ \text{(since } I \text{ is an isomorphism)}}} \\ &= \left(\sum_{i=1}^{n} (\operatorname{id} \otimes \operatorname{inc}_{B_i,B}) \right) \left(\bigoplus_{i=1}^{n} (A \otimes B_i) \right) \\ &= \sum_{i=1}^{n} (\operatorname{id} \otimes \operatorname{inc}_{B_i,B}) (A \otimes B_i). \qquad (14) \end{aligned}$$

On the other hand, the maps $\operatorname{inc}_{B_i,B'} : B_i \to B'$ for all $i \in \{1, 2, \ldots, n\}$ (these maps are well-defined since $B_i \subseteq B'$ for every $i \in \{1, 2, \ldots, n\}$) give rise to a map $\sum_{i=1}^{n} \operatorname{inc}_{B_i,B'} : \bigoplus_{i=1}^{n} B_i \to B'$. Every $i \in \{1, 2, \ldots, n\}$ satisfies $\operatorname{inc}_{B_i,B} = \operatorname{inc}_{B',B} \circ \operatorname{inc}_{B_i,B'}$, so that

$$\sum_{i=1}^{n} \operatorname{inc}_{B_i,B'} = \sum_{i=1}^{n} (\operatorname{inc}_{B',B} \circ \operatorname{inc}_{B_i,B'}) = \operatorname{inc}_{B',B} \circ \left(\sum_{i=1}^{n} \operatorname{inc}_{B_i,B'} \right).$$

Hence,

$$\mathrm{id} \otimes \sum_{i=1}^{n} \mathrm{inc}_{B_i,B'} = \mathrm{id} \otimes \left(\mathrm{inc}_{B',B} \circ \left(\sum_{i=1}^{n} \mathrm{inc}_{B_i,B'} \right) \right)$$
$$= (\mathrm{id} \otimes \mathrm{inc}_{B',B}) \circ \left(\mathrm{id} \otimes \left(\sum_{i=1}^{n} \mathrm{inc}_{B_i,B'} \right) \right)$$

as maps from $A \otimes \left(\bigoplus_{i=1}^{n} B_i \right)$ to $A \otimes B$ (where id always denotes id_A). Hence,

$$\left(\mathrm{id} \otimes \left(\sum_{i=1}^{n} \mathrm{inc}_{B_i,B} \right) \right) \left(A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \right)$$
$$= \left((\mathrm{id} \otimes \mathrm{inc}_{B',B}) \circ \left(\mathrm{id} \otimes \left(\sum_{i=1}^{n} \mathrm{inc}_{B_i,B'} \right) \right) \right) \left(A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \right)$$
$$= (\mathrm{id} \otimes \mathrm{inc}_{B',B}) \left(\left(\mathrm{id} \otimes \left(\sum_{i=1}^{n} \mathrm{inc}_{B_i,B'} \right) \right) \left(A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \right) \right). \qquad (15)$$

But since $\mathrm{inc}_{B_i,B'}$ is the inclusion map $B_i \to B'$ for every $i \in \{1, 2, \ldots, n\}$, it is clear that the image of the map $\sum_{i=1}^{n} \mathrm{inc}_{B_i,B'} : \bigoplus_{i=1}^{n} B_i \to B'$ is $\sum_{i=1}^{n} B_i = B'$. In other words, the map $\sum_{i=1}^{n} \mathrm{inc}_{B_i,B'} : \bigoplus_{i=1}^{n} B_i \to B'$ is surjective. Hence, the map $\mathrm{id} \otimes \left(\sum_{i=1}^{n} \mathrm{inc}_{B_i,B'} \right) : A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \to A \otimes B'$ is surjective as well (by Lemma 23, applied to $V = A$, $V' = A$, $W = \bigoplus_{i=1}^{n} B_i$, $W' = B'$, $f = \mathrm{id}$ and $g = \sum_{i=1}^{n} \mathrm{inc}_{B_i,B'}$), so that $\left(\mathrm{id} \otimes \left(\sum_{i=1}^{n} \mathrm{inc}_{B_i,B'} \right) \right) \left(A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \right) = A \otimes B'$. Thus, (15) simplifies to

$$\left(\mathrm{id} \otimes \left(\sum_{i=1}^{n} \mathrm{inc}_{B_i,B} \right) \right) \left(A \otimes \left(\bigoplus_{i=1}^{n} B_i \right) \right) = (\mathrm{id} \otimes \mathrm{inc}_{B',B}) (A \otimes B').$$

Compared with (14), we obtain

$$\sum_{i=1}^{n} (\mathrm{id} \otimes \mathrm{inc}_{B_i,B}) (A \otimes B_i) = (\mathrm{id} \otimes \mathrm{inc}_{B',B}) (A \otimes B').$$

This proves Lemma 26. □

Another lemma:

Lemma 27. Let A, B and C be three k-modules. Let $f : B \to C$ be a k-module map. Then,

$$(\mathrm{id} \otimes f) (A \otimes B) = \left(\mathrm{id} \otimes \mathrm{inc}_{f(B),C} \right) (A \otimes (f (B)))$$

as k-submodules of $A \otimes C$.

Proof of Lemma 27. We define a map $f' : B \to f(B)$ by

$$\left(f'(x) = f(x) \qquad \text{for every } x \in B\right).$$

(This map is well-defined, since $f(x) \in f(B)$ for every $x \in B$.) Then, every $x \in B$ satisfies

$$f(x) = f'(x) = \operatorname{inc}_{f(B),C}(f'(x)) = \left(\operatorname{inc}_{f(B),C} \circ f'\right)(x).$$

Thus, $f = \operatorname{inc}_{f(B),C} \circ f'$. Hence,

$$\operatorname{id} \otimes f = \operatorname{id} \otimes \left(\operatorname{inc}_{f(B),C} \circ f'\right) = \left(\operatorname{id} \otimes \operatorname{inc}_{f(B),C}\right) \circ \left(\operatorname{id} \otimes f'\right),$$

where id means id_A. Thus,

$$\begin{aligned}(\operatorname{id} \otimes f)(A \otimes B) &= \left(\left(\operatorname{id} \otimes \operatorname{inc}_{f(B),C}\right) \circ (\operatorname{id} \otimes f')\right)(A \otimes B) \\ &= \left(\operatorname{id} \otimes \operatorname{inc}_{f(B),C}\right)\left((\operatorname{id} \otimes f')(A \otimes B)\right). \qquad (16)\end{aligned}$$

Now, the map $f' : B \to f(B)$ is surjective[5], so that the map $\operatorname{id} \otimes f' : A \otimes B \to A \otimes (f(B))$ is surjective as well (by Lemma 23, applied to A, A, B, $f(B)$, id, f' instead of V, V', W, W', f, g, respectively). Thus, $(\operatorname{id} \otimes f')(A \otimes B) = A \otimes (f(B))$. Hence, (16) becomes

$$(\operatorname{id} \otimes f)(A \otimes B) = \left(\operatorname{id} \otimes \operatorname{inc}_{f(B),C}\right)(A \otimes (f(B))).$$

This proves Lemma 27. □

Proof of Theorem 25. We are going to prove Theorem 25 by induction over n:

Induction base: For $n = 0$, Theorem 25 holds (because for $n = 0$, the map $f_1 \otimes f_2 \otimes \cdots \otimes f_n : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to V_1' \otimes V_2' \otimes \cdots \otimes V_n'$ is the identity map $\operatorname{id} : k \to k$ and therefore its kernel $\operatorname{Ker}(f_1 \otimes f_2 \otimes \cdots \otimes f_n)$ is 0, while the right hand side of (13) is also 0 when $n = 0$). Thus, the induction base is complete.

Induction step: Let $p \in \mathbb{N}$ be arbitrary. Assume that Theorem 25 holds for $n = p$.

Now let us prove that Theorem 25 holds for $n = p + 1$. So let V_i and V_i' be two k-modules for every $i \in \{1, 2, \ldots, p+1\}$, and let $f_i : V_i \to V_i'$ be a surjective k-module homomorphism for every $i \in \{1, 2, \ldots, p+1\}$.

According to Definition 6, we have $f_1 \otimes f_2 \otimes \cdots \otimes f_{p+1} = f_1 \otimes (f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1})$.

We know that $f_i : V_i \to V_i'$ is a surjective k-module homomorphism for every $i \in \{1, 2, \ldots, p+1\}$. Thus, $f_i : V_i \to V_i'$ is a surjective k-module homomorphism for every $i \in \{2, 3, \ldots, p+1\}$. Substituting $i+1$ for i in this fact, we obtain that f_{i+1} is a surjective k-module homomorphism for every $i \in \{1, 2, \ldots, p\}$. Thus, Lemma 24 (applied to p, V_{i+1}, V_{i+1}' and f_{i+1} instead of n, V_i, V_i' and f_i) yields that the map $f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1}$ is surjective. On the other hand, the map f_1 is surjective (since f_i is surjective for every $i \in \{1, 2, \ldots, p+1\}$).

Now let $V = V_1$, $V' = W_1$, $f = f_1$ and $i_V = \mathfrak{i}_1$. Then, f is a surjective k-linear map (since f_1 is a surjective k-linear map), and i_V is the canonical inclusion $\operatorname{Ker} f \to V$ (since $i_V = \mathfrak{i}_1$ is the canonical inclusion $\operatorname{Ker} f_1 \to V_1$).

[5] *Proof.* Let $y \in f(B)$. Then, there exists some $x \in B$ such that $y = f(x)$ (by the definition of $f(B)$). Consider this x. Then, $f'(x) = f(x) = y$.

Hence, we have shown that for every $y \in f(B)$, there exists some $x \in B$ such that $y = f'(x)$. In other words, the map $f' : B \to f(B)$ is surjective, qed.

Further let $W = V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1}$, $W' = W_2 \otimes W_3 \otimes \cdots \otimes W_{p+1}$ and $g = f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1}$. Then, g is a surjective k-linear map (since $f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1}$ is a surjective k-linear map). Let i_W be the canonical inclusion $\operatorname{Ker} g \to W$. Then, Theorem 18 yields

$$\operatorname{Ker}(f \otimes g) = (i_V \otimes \operatorname{id})((\operatorname{Ker} f) \otimes W) + (\operatorname{id} \otimes i_W)(V \otimes (\operatorname{Ker} g)). \qquad (17)$$

Note that $V = V_1$ and $W = V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1}$ yield $V \otimes W = V_1 \otimes (V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1}) = V_1 \otimes V_2 \otimes \cdots \otimes V_{p+1}$.

We now define some more abbreviations. For every $i \in \{1, 2, \ldots, p+1\}$, let K_i denote the k-module

$$V_1 \otimes V_2 \otimes \cdots \otimes V_{i-1} \otimes (\operatorname{Ker} f_i) \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_{p+1}.$$

For every $i \in \{1, 2, \ldots, p+1\}$, let κ_i denote the k-linear map

$$\underbrace{\operatorname{id} \otimes \operatorname{id} \otimes \cdots \otimes \operatorname{id}}_{i-1 \text{ times}} \otimes \mathfrak{i}_i \otimes \underbrace{\operatorname{id} \otimes \operatorname{id} \otimes \cdots \otimes \operatorname{id}}_{p+1-i \text{ times}} : K_i \to V \otimes W$$

(this is well-defined since $K_i = V_1 \otimes V_2 \otimes \cdots \otimes V_{i-1} \otimes (\operatorname{Ker} f_i) \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_{p+1}$ and $V \otimes W = V_1 \otimes V_2 \otimes \cdots \otimes V_{p+1}$).

For every $i \in \{2, 3, \ldots, p+1\}$, let M_i denote the k-module

$$V_2 \otimes V_3 \otimes \cdots \otimes V_{i-1} \otimes (\operatorname{Ker} f_i) \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_{p+1}.$$

For every $i \in \{2, 3, \ldots, p+1\}$, let μ_i denote the k-linear map

$$\underbrace{\operatorname{id} \otimes \operatorname{id} \otimes \cdots \otimes \operatorname{id}}_{i-2 \text{ times}} \otimes \mathfrak{i}_i \otimes \underbrace{\operatorname{id} \otimes \operatorname{id} \otimes \cdots \otimes \operatorname{id}}_{p+1-i \text{ times}} : M_i \to W$$

(this is well-defined since $M_i = V_2 \otimes V_3 \otimes \cdots \otimes V_{i-1} \otimes (\operatorname{Ker} f_i) \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_{p+1}$ and $W = V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1}$).

For any k-module C and any k-submodule D of C, we let $\operatorname{inc}_{D,C}$ denote the canonical inclusion map $D \to C$. Then,

$$\operatorname{inc}_{\operatorname{Ker} f_i, V_i} = (\text{the canonical inclusion map } \operatorname{Ker} f_i \to V_i) = \mathfrak{i}_i$$

for every $i \in \{1, 2, \ldots, p+1\}$. On the other hand,

$$\operatorname{inc}_{\operatorname{Ker} g, W} = (\text{the canonical inclusion map } \operatorname{Ker} g \to W) = i_W$$

(by definition of i_W).

Applying Theorem 25 to p, V_{i+1}, V'_{i+1} and f_{i+1} instead of n, V_i, V'_i and f_i (this is allowed, because we have assumed that Theorem 25 holds for $n = p$), we see that

$$\begin{aligned}
&\operatorname{Ker}\left(f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1}\right) \\
&= \sum_{i=1}^{p} \left(\underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{i-1 \text{ times}} \otimes \mathfrak{i}_{i+1} \otimes \underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{p-i \text{ times}} \right) \\
&\qquad\qquad \left(V_2 \otimes V_3 \otimes \cdots \otimes V_i \otimes \left(\operatorname{Ker} f_{i+1}\right) \otimes V_{i+2} \otimes V_{i+3} \otimes \cdots \otimes V_{p+1}\right) \\
&= \sum_{i=2}^{p+1} \underbrace{\left(\underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{i-2 \text{ times}} \otimes \mathfrak{i}_{i} \otimes \underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{p-i+1 \text{ times}} \right)}_{=\mu_i} \\
&\qquad\qquad \underbrace{\left(V_2 \otimes V_3 \otimes \cdots \otimes V_{i-1} \otimes \left(\operatorname{Ker} f_{i}\right) \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_{p+1}\right)}_{=M_i} \\
&\qquad \left(\text{here, we substituted } i \text{ for } i+1 \text{ in the sum}\right) \\
&= \sum_{i=2}^{p+1} \mu_i \left(M_i\right).
\end{aligned}$$

Since $f_2 \otimes f_3 \otimes \cdots \otimes f_{p+1} = g$, this rewrites as

$$\operatorname{Ker} g = \sum_{i=2}^{p+1} \mu_i \left(M_i\right) = \sum_{i=1}^{p} \mu_{i+1} \left(M_{i+1}\right) \tag{18}$$

(here, we substituted i for $i-1$). But now it is easy to see that

$$\left(\mathrm{id} \otimes i_W\right) \left(V \otimes \left(\operatorname{Ker} g\right)\right) = \sum_{i=2}^{p+1} \kappa_i \left(K_i\right). \tag{19}$$

Proof of (19). Let $i \in \{2, 3, \ldots, p+1\}$. Then,

$$\begin{aligned}
\kappa_i &= \underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{i-1 \text{ times}} \otimes \mathfrak{i}_{i} \otimes \underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{p+1-i \text{ times}} \\
&= \mathrm{id} \otimes \underbrace{\left(\underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{i-2 \text{ times}} \otimes \mathfrak{i}_{i} \otimes \underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{p+1-i \text{ times}} \right)}_{=\mu_i} = \mathrm{id} \otimes \mu_i
\end{aligned}$$

and

$$\begin{aligned}
K_i &= V_1 \otimes V_2 \otimes \cdots \otimes V_{i-1} \otimes \left(\operatorname{Ker} f_i\right) \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_{p+1} \\
&= \underbrace{V_1}_{=V} \otimes \underbrace{\left(V_2 \otimes V_3 \otimes \cdots \otimes V_{i-1} \otimes \left(\operatorname{Ker} f_i\right) \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_{p+1}\right)}_{=M_i} \\
&= V \otimes M_i.
\end{aligned}$$

Thus,

$$\kappa_i \left(K_i\right) = \left(\mathrm{id} \otimes \mu_i\right) \left(V \otimes M_i\right) = \left(\mathrm{id} \otimes \operatorname{inc}_{\mu_i(M_i),W}\right) \left(V \otimes \left(\mu_i \left(M_i\right)\right)\right)$$

(by Lemma 27, applied to $A = V$, $B = M_i$, $C = W$ and $f = \mu_i$).

But since $\mu_i(M_i) \subseteq \operatorname{Ker} g$ (by (18)), we have

$$\operatorname{inc}_{\mu_i(M_i),W} = \underbrace{\operatorname{inc}_{\operatorname{Ker} g,W}}_{=i_W} \circ \operatorname{inc}_{\mu_i(M_i),\operatorname{Ker} g}$$
$$\left(\begin{array}{c}\text{since any three } k\text{-modules } A,\ B,\ C \text{ such that } A \subseteq B \subseteq C \text{ satisfy} \\ \operatorname{inc}_{A,C} = \operatorname{inc}_{B,C} \circ \operatorname{inc}_{A,B}\end{array}\right)$$
$$= i_W \circ \operatorname{inc}_{\mu_i(M_i),\operatorname{Ker} g}.$$

Now forget that we fixed $i \in \{2, 3, \ldots, p+1\}$. Due to (18), we can apply Lemma 26 to $n = p$, $A = V$, $B = W$, $B' = \operatorname{Ker} g$ and $B_i = \mu_{i+1}(M_{i+1})$. Applying it yields that

$$\begin{aligned}(\operatorname{id} \otimes \operatorname{inc}_{\operatorname{Ker} g,W})(V \otimes (\operatorname{Ker} g)) &= \sum_{i=1}^{p} \left(\operatorname{id} \otimes \operatorname{inc}_{\mu_{i+1}(M_{i+1}),W}\right)(V \otimes (\mu_{i+1}(M_{i+1}))) \\ &= \sum_{i=2}^{p+1} \underbrace{\left(\operatorname{id} \otimes \operatorname{inc}_{\mu_i(M_i),W}\right)(V \otimes (\mu_i(M_i)))}_{=\kappa_i(K_i)} \\ &\qquad \text{(here, we substituted } i \text{ for } i+1 \text{ in the sum)} \\ &= \sum_{i=2}^{p+1} \kappa_i(K_i),\end{aligned}$$

and thus (19) is proven.

On the other hand, we defined the k-module K_i as

$$V_1 \otimes V_2 \otimes \cdots \otimes V_{i-1} \otimes (\operatorname{Ker} f_i) \otimes V_{i+1} \otimes V_{i+2} \otimes \cdots \otimes V_{p+1}$$

for every $i \in \{1, 2, \ldots, p+1\}$. Applied to $i = 1$, this yields

$$K_1 = (\operatorname{Ker} f_1) \otimes V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1} = \left(\operatorname{Ker} \underbrace{f_1}_{=f}\right) \otimes \underbrace{(V_2 \otimes V_3 \otimes \cdots \otimes V_{p+1})}_{=W} = (\operatorname{Ker} f) \otimes W. \tag{20}$$

We further defined the map κ_i as

$$\underbrace{\operatorname{id} \otimes \operatorname{id} \otimes \cdots \otimes \operatorname{id}}_{i-1 \text{ times}} \otimes \mathfrak{i}_i \otimes \underbrace{\operatorname{id} \otimes \operatorname{id} \otimes \cdots \otimes \operatorname{id}}_{p+1-i \text{ times}} : K_i \to V \otimes W$$

for every $i \in \{1, 2, \ldots, p+1\}$. Applied to $i = 1$, this yields

$$\kappa_1 = \mathfrak{i}_1 \otimes \underbrace{\operatorname{id} \otimes \operatorname{id} \otimes \cdots \otimes \operatorname{id}}_{p+1-1 \text{ times}} = \underbrace{\mathfrak{i}_1}_{=i_V} \otimes \underbrace{\left(\underbrace{\operatorname{id} \otimes \operatorname{id} \otimes \cdots \otimes \operatorname{id}}_{p+1-1 \text{ times}}\right)}_{=\operatorname{id}} = i_V \otimes \operatorname{id}. \tag{21}$$

Now, (17) becomes

$$\operatorname{Ker}(f\otimes g) = \underbrace{(i_V\otimes\operatorname{id})}_{=\kappa_1 \text{ (by (21))}} \left(\underbrace{(\operatorname{Ker} f)\otimes W}_{=K_1 \text{ (by (20))}}\right) + \underbrace{(\operatorname{id}\otimes i_W)(V\otimes(\operatorname{Ker} g))}_{=\sum_{i=2}^{p+1}\kappa_i(K_i) \text{ (by (19))}}$$
$$= \kappa_1(K_1) + \sum_{i=2}^{p+1}\kappa_i(K_i) = \sum_{i=1}^{p+1}\kappa_i(K_i).$$

Since

$$\underbrace{f}_{=f_1}\otimes\underbrace{g}_{=f_2\otimes f_3\otimes\cdots\otimes f_{p+1}} = f_1\otimes(f_2\otimes f_3\otimes\cdots\otimes f_{p+1}) = f_1\otimes f_2\otimes\cdots\otimes f_{p+1},$$

this rewrites as

$$\operatorname{Ker}(f_1\otimes f_2\otimes\cdots\otimes f_{p+1})$$
$$= \sum_{i=1}^{p+1} \underbrace{\kappa_i}_{=\underbrace{\operatorname{id}\otimes\operatorname{id}\otimes\cdots\otimes\operatorname{id}}_{i-1\text{ times}}\otimes\mathfrak{i}_i\otimes\underbrace{\operatorname{id}\otimes\operatorname{id}\otimes\cdots\otimes\operatorname{id}}_{p+1-i\text{ times}}} \left(\underbrace{K_i}_{=V_1\otimes V_2\otimes\cdots\otimes V_{i-1}\otimes(\operatorname{Ker} f_i)\otimes V_{i+1}\otimes V_{i+2}\otimes\cdots\otimes V_{p+1}}\right)$$
$$= \sum_{i=1}^{p+1}\left(\underbrace{\operatorname{id}\otimes\operatorname{id}\otimes\cdots\otimes\operatorname{id}}_{i-1\text{ times}}\otimes\mathfrak{i}_i\otimes\underbrace{\operatorname{id}\otimes\operatorname{id}\otimes\cdots\otimes\operatorname{id}}_{p+1-i\text{ times}}\right)$$
$$(V_1\otimes V_2\otimes\cdots\otimes V_{i-1}\otimes(\operatorname{Ker} f_i)\otimes V_{i+1}\otimes V_{i+2}\otimes\cdots\otimes V_{p+1}).$$

We thus have proven that (13) holds for $n = p+1$. This completes the induction step. Thus, the induction proof of Theorem 25 is complete. □

0.11. The tensor algebra case

Before we actually come to the tensor algebra, let us bring Theorem 25 to a nicer form when all the f_i are equal:

Theorem 28. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V and V' be two k-modules, and let $f : V \to V'$ be a surjective k-module homomorphism. Let $\mathfrak{i}$ be the canonical inclusion $\operatorname{Ker} f \to V$. Then,

$$\operatorname{Ker}\left(f^{\otimes n}\right) = \sum_{i=1}^{n}\left(\operatorname{id}_{V^{\otimes(i-1)}}\otimes\mathfrak{i}\otimes\operatorname{id}_{V^{\otimes(n-i)}}\right)\left(V^{\otimes(i-1)}\otimes(\operatorname{Ker} f)\otimes V^{\otimes(n-i)}\right). \quad (22)$$

Notice that the left hand side of the equation (22) is a subset of $V^{\otimes n}$, while the i-th addend on the right hand side is a subset of $V^{\otimes(i-1)}\otimes V\otimes V^{\otimes(n-i)}$. To make sense of the equation (22), the set $V^{\otimes n}$ thus must be equal to $V^{\otimes(i-1)} \otimes V \otimes V^{\otimes(n-i)}$ for every $i \in \{1, 2, \ldots, n\}$. Fortunately, this is guaranteed by Convention 12 [6].

For the proof of Theorem 28, we will use one more convention:

Convention 29. Let k be a commutative ring, and let A, B and C be three k-modules. Then, we identify the k-module $(A \otimes B) \otimes C$ with the k-module $A \otimes (B \otimes C)$ by means of the k-module isomorphism

$$(A \otimes B) \otimes C \to A \otimes (B \otimes C), \\ (a \otimes b) \otimes c \mapsto a \otimes (b \otimes c).$$

Note that we will only use Convention 29 in the proof of Theorem 28, but nowhere else in this text.

Remark 30. As a consequence of Convention 29, it can be easily seen that the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_n$ of any k-modules V_1, V_2, ..., V_n can be computed by means of any bracketing. For instance, when $n = 4$, this means that

$$V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)) = V_1 \otimes ((V_2 \otimes V_3) \otimes V_4) = (V_1 \otimes V_2) \otimes (V_3 \otimes V_4) \\ = (V_1 \otimes (V_2 \otimes V_3)) \otimes V_4 = ((V_1 \otimes V_2) \otimes V_3) \otimes V_4$$

for any four k-modules V_1, V_2, V_3, V_4.

Remark 31. Convention 29 is compatible with Convention 12. In fact, Conventions 29 and 7 combined make Convention 12 redundant, in the following sense: If we identify $(A \otimes B) \otimes C$ with $A \otimes (B \otimes C)$ for all k-modules A, B and C (as in Convention 29), and identify $V \otimes k$, $k \otimes V$ and V for all k-modules V (as in Convention 7), then automatically $V^{\otimes a} \otimes V^{\otimes b}$ becomes identical with $V^{\otimes(a+b)}$ for all k-modules V and $a \in \mathbb{N}$ and $b \in \mathbb{N}$ (and this identification is the same as the one given in Convention 12).

Proof of Theorem 28. During this proof, we are going to use Convention 29.

Now, we apply Theorem 25 to $V_i = V$, $V_i' = V'$, $f_i = f$ and $\mathfrak{i}_i = \mathfrak{i}$. As a result, we

[6] In fact, using Convention 12, we have

$$V^{\otimes(i-1)} \otimes \underbrace{V}_{= V^{\otimes 1} \text{ (by Remark 9)}} \otimes V^{\otimes(n-i)} \\ = \underbrace{V^{\otimes(i-1)} \otimes V^{\otimes 1}}_{= V^{\otimes(i-1+1)} \text{ (by Convention 12)}} \otimes V^{\otimes(n-i)} \\ = \underbrace{V^{\otimes(i-1+1)}}_{= V^{\otimes i}} \otimes V^{\otimes(n-i)} = V^{\otimes i} \otimes V^{\otimes(n-i)} = V^{\otimes(i+n-i)} \qquad \text{(by Convention 12)} \\ = V^{\otimes n}.$$

obtain

$$\begin{aligned}
&\operatorname{Ker}\left(\underbrace{f\otimes f\otimes\cdots\otimes f}_{n\text{ times}}\right)\\
&=\sum_{i=1}^{n}\left(\underbrace{\mathrm{id}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{i-1\text{ times}}\otimes\mathfrak{i}\otimes\underbrace{\mathrm{id}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{n-i\text{ times}}\right)\\
&\qquad\qquad\left(\underbrace{V\otimes V\otimes\cdots\otimes V}_{i-1\text{ times}}\otimes(\operatorname{Ker}f)\otimes\underbrace{V\otimes V\otimes\cdots\otimes V}_{n-i\text{ times}}\right).
\end{aligned}\tag{23}$$

Now, since we are using Convention 29, we can write

$$\begin{aligned}
&\underbrace{V\otimes V\otimes\cdots\otimes V}_{i-1\text{ times}}\otimes(\operatorname{Ker}f)\otimes\underbrace{V\otimes V\otimes\cdots\otimes V}_{n-i\text{ times}}\\
&=\underbrace{\left(\underbrace{V\otimes V\otimes\cdots\otimes V}_{i-1\text{ times}}\right)}_{=V^{\otimes(i-1)}}\otimes(\operatorname{Ker}f)\otimes\underbrace{\left(\underbrace{V\otimes V\otimes\cdots\otimes V}_{n-i\text{ times}}\right)}_{=V^{\otimes(n-i)}}\\
&=V^{\otimes(i-1)}\otimes(\operatorname{Ker}f)\otimes V^{\otimes(n-i)}
\end{aligned}$$

and correspondingly

$$\begin{aligned}
&\underbrace{\mathrm{id}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{i-1\text{ times}}\otimes\mathfrak{i}\otimes\underbrace{\mathrm{id}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{n-i\text{ times}}\\
&=\underbrace{\left(\underbrace{\mathrm{id}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{i-1\text{ times}}\right)}_{=\mathrm{id}_{V^{\otimes(i-1)}}}\otimes\mathfrak{i}\otimes\underbrace{\left(\underbrace{\mathrm{id}\otimes\mathrm{id}\otimes\cdots\otimes\mathrm{id}}_{n-i\text{ times}}\right)}_{=\mathrm{id}_{V^{\otimes(n-i)}}}\\
&=\mathrm{id}_{V^{\otimes(i-1)}}\otimes\mathfrak{i}\otimes\mathrm{id}_{V^{\otimes(n-i)}}
\end{aligned}$$

for every $i \in \{1, 2, \ldots, n\}$. Now, since $f^{\otimes n} = \underbrace{f \otimes f \otimes \cdots \otimes f}_{n \text{ times}}$, we have

$$\begin{aligned}
\operatorname{Ker}\left(f^{\otimes n}\right) &= \operatorname{Ker}\left(\underbrace{f \otimes f \otimes \cdots \otimes f}_{n \text{ times}}\right) \\
&= \sum_{i=1}^{n} \underbrace{\left(\underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{i-1 \text{ times}} \otimes \mathfrak{i} \otimes \underbrace{\mathrm{id} \otimes \mathrm{id} \otimes \cdots \otimes \mathrm{id}}_{n-i \text{ times}}\right)}_{= \mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \mathrm{id}_{V^{\otimes(n-i)}}} \\
&\qquad \underbrace{\left(\underbrace{V \otimes V \otimes \cdots \otimes V}_{i-1 \text{ times}} \otimes (\operatorname{Ker} f) \otimes \underbrace{V \otimes V \otimes \cdots \otimes V}_{n-i \text{ times}}\right)}_{= V^{\otimes(i-1)} \otimes (\operatorname{Ker} f) \otimes V^{\otimes(n-i)}} \\
&\qquad \text{(by (23))} \\
&= \sum_{i=1}^{n} \left(\mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \mathrm{id}_{V^{\otimes(n-i)}}\right) \left(V^{\otimes(i-1)} \otimes (\operatorname{Ker} f) \otimes V^{\otimes(n-i)}\right).
\end{aligned}$$

This proves Theorem 28. □

Now, our claim about the tensor algebra:

Theorem 32. Let k be a commutative ring. Let V and V' be two k-modules, and let $f : V \to V'$ be a surjective k-module homomorphism. Then, the kernel of the map $\otimes f : \otimes V \to \otimes V'$ is

$$\operatorname{Ker}(\otimes f) = (\otimes V) \cdot (\operatorname{Ker} f) \cdot (\otimes V).$$

Here, $\operatorname{Ker} f$ is considered a k-submodule of $\otimes V$ by means of the inclusion $\operatorname{Ker} f \subseteq V = V^{\otimes 1} \subseteq \otimes V$.

We are going to derive this theorem from Theorem 28. For this we need the following lemma:

Lemma 33. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let $i \in \{0, 1, \ldots, n\}$. Let V be a k-module, and let W be a k-submodule of V. Let $\mathfrak{i}$ be the canonical inclusion $W \to V$. Then,

$$\left(\mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \mathrm{id}_{V^{\otimes(n-i)}}\right) \left(V^{\otimes(i-1)} \otimes W \otimes V^{\otimes(n-i)}\right) = V^{\otimes(i-1)} \cdot W \cdot V^{\otimes(n-i)},$$

where we identify $V^{\otimes n}$ with a k-submodule of $\otimes V$ as in Definition 13 **(c)**.

To prove this lemma, we make a convention:

Convention 34. (a) Whenever k is a commutative ring, M is a k-module, and S is a subset of M, we denote by $\langle S \rangle$ the k-submodule of M generated by the elements of S. This k-submodule $\langle S \rangle$ is called the *k-linear span* (or simply the *k-span*) of S.

(b) Whenever k is a commutative ring, M is a k-module, Φ is a set, and $P : \Phi \to M$ is a map (not necessarily a linear map), we denote by $\langle P(v) \mid v \in \Phi \rangle$ the k-submodule $\langle \{P(v) \mid v \in \Phi\} \rangle$ of M. (In other words, $\langle P(v) \mid v \in \Phi \rangle$ is the k-submodule of M generated by the elements $P(v)$ for all $v \in \Phi$.)

Note that some authors use the notation $\langle S \rangle$ for various other things (e. g., the two-sided ideal generated by S, or the Lie subalgebra generated by S), but *we will only use it for the k-submodule generated by S (as defined in Convention 34 **(a)**).*

The following fact was proven in [3], §1.7 (but is basically trivial):

Proposition 35. Let k be a commutative ring. Let M be a k-module. Let S be a subset of M.
(a) Let Q be a k-submodule of M such that $S \subseteq Q$. Then, $\langle S \rangle \subseteq Q$.
(b) Let R be a k-module, and $f : M \to R$ be a k-module homomorphism. Then, $f(\langle S \rangle) = \langle f(S) \rangle$.

Now let us come to the proof of Lemma 33:

Proof of Lemma 33. The tensor product $V^{\otimes(i-1)} \otimes W \otimes V^{\otimes(n-i)}$ is generated (as a k-module) by its pure tensors. In other words,

$$\begin{aligned} V^{\otimes(i-1)} \otimes W \otimes V^{\otimes(n-i)} &= \left\langle u \otimes v \otimes w \mid (u,v,w) \in V^{\otimes(i-1)} \times W \times V^{\otimes(n-i)} \right\rangle \\ &= \left\langle \left\{ u \otimes v \otimes w \mid (u,v,w) \in V^{\otimes(i-1)} \times W \times V^{\otimes(n-i)} \right\} \right\rangle \end{aligned}$$

Thus,

$$\begin{aligned} &\left(\mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \mathrm{id}_{V^{\otimes(n-i)}}\right)\left(V^{\otimes(i-1)} \otimes W \otimes V^{\otimes(n-i)}\right) \\ &= \left(\mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \mathrm{id}_{V^{\otimes(n-i)}}\right)\left(\left\langle \left\{ u \otimes v \otimes w \mid (u,v,w) \in V^{\otimes(i-1)} \times W \times V^{\otimes(n-i)} \right\} \right\rangle\right) \\ &= \left\langle \left(\mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \mathrm{id}_{V^{\otimes(n-i)}}\right)\left(\left\{ u \otimes v \otimes w \mid (u,v,w) \in V^{\otimes(i-1)} \times W \times V^{\otimes(n-i)} \right\}\right) \right\rangle \\ &\qquad \left(\begin{array}{c} \text{by Proposition 35 } \textbf{(b)}\text{, applied to } M = V^{\otimes(i-1)} \otimes W \otimes V^{\otimes(n-i)}, \\ R = V^{\otimes n},\ f = \mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \mathrm{id}_{V^{\otimes(n-i)}} \text{ and} \\ S = \left\{ u \otimes v \otimes w \mid (u,v,w) \in V^{\otimes(i-1)} \times W \times V^{\otimes(n-i)} \right\} \end{array}\right). \end{aligned}$$

Since

$$\begin{aligned} &\left(\mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \mathrm{id}_{V^{\otimes(n-i)}}\right)\left(\left\{ u \otimes v \otimes w \mid (u,v,w) \in V^{\otimes(i-1)} \times W \times V^{\otimes(n-i)} \right\}\right) \\ &= \left\{ \underbrace{\left(\mathrm{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \mathrm{id}_{V^{\otimes(n-i)}}\right)(u \otimes v \otimes w)}_{= \mathrm{id}_{V^{\otimes(i-1)}}(u) \otimes \mathfrak{i}(v) \otimes \mathrm{id}_{V^{\otimes(n-i)}}(w)} \mid (u,v,w) \in V^{\otimes(i-1)} \times W \times V^{\otimes(n-i)} \right\} \\ &= \left\{ \underbrace{\mathrm{id}_{V^{\otimes(i-1)}}(u)}_{=u} \otimes \underbrace{\mathfrak{i}(v)}_{\substack{=v \text{ (since } \mathfrak{i} \text{ is the} \\ \text{inclusion map)}}} \otimes \underbrace{\mathrm{id}_{V^{\otimes(n-i)}}(w)}_{=w} \mid (u,v,w) \in V^{\otimes(i-1)} \times W \times V^{\otimes(n-i)} \right\} \\ &= \left\{ u \otimes v \otimes w \mid (u,v,w) \in V^{\otimes(i-1)} \times W \times V^{\otimes(n-i)} \right\}, \end{aligned}$$

this rewrites as

$$\begin{aligned}
&\left(\mathrm{id}_{V^{\otimes(i-1)}}\otimes\mathfrak{i}\otimes\mathrm{id}_{V^{\otimes(n-i)}}\right)\left(V^{\otimes(i-1)}\otimes W\otimes V^{\otimes(n-i)}\right)\\
&=\left\langle\left\{u\otimes v\otimes w\ \mid\ (u,v,w)\in V^{\otimes(i-1)}\times W\times V^{\otimes(n-i)}\right\}\right\rangle\\
&=\left\langle u\otimes v\otimes w\ \mid\ (u,v,w)\in V^{\otimes(i-1)}\times W\times V^{\otimes(n-i)}\right\rangle. \qquad (24)
\end{aligned}$$

On the other hand,

$$V^{\otimes(i-1)}\cdot W\cdot V^{\otimes(n-i)}=\left\langle u\cdot v\cdot w\ \mid\ (u,v,w)\in V^{\otimes(i-1)}\times W\times V^{\otimes(n-i)}\right\rangle, \qquad (25)$$

where the $\cdot$ sign stands for multiplication inside the k-algebra $\otimes V$. But for every $(u,v,w)\in V^{\otimes(i-1)}\times W\times V^{\otimes(n-i)}$, we have $u\cdot v\cdot w=u\otimes v\otimes w$ [7], so that (25) becomes

$$\begin{aligned}
V^{\otimes(i-1)}\cdot W\cdot V^{\otimes(n-i)}&=\left\langle u\otimes v\otimes w\ \mid\ (u,v,w)\in V^{\otimes(i-1)}\times W\times V^{\otimes(n-i)}\right\rangle\\
&=\left(\mathrm{id}_{V^{\otimes(i-1)}}\otimes\mathfrak{i}\otimes\mathrm{id}_{V^{\otimes(n-i)}}\right)\left(V^{\otimes(i-1)}\otimes W\otimes V^{\otimes(n-i)}\right)
\end{aligned}$$

(by (24)). This proves Lemma 33. □

Proof of Theorem 32. Let $\mathfrak{i}$ be the canonical inclusion $\operatorname{Ker} f\to V$.

We have

$$\begin{aligned}
\otimes V&=\bigoplus_{i\in\mathbb{N}}V^{\otimes i}=\sum_{i\in\mathbb{N}}V^{\otimes i}\qquad\text{(since direct sums are sums)} \qquad (26)\\
&=\sum_{j\in\mathbb{N}}V^{\otimes j}\qquad\text{(here, we renamed the index } i \text{ as } j\text{)}. \qquad (27)
\end{aligned}$$

But the map $\otimes f$ is defined as the direct sum of the k-module homomorphisms

[7] *Proof.* We have $u\in V^{\otimes(i-1)}$ and $v\in W\subseteq V=V^{\otimes 1}$. Hence, $u\cdot v=u\otimes v$ (by (3), applied to u, v, $i-1$ and 1 instead of a, b, n and m) and thus $u\cdot v=\underbrace{u}_{\in V^{\otimes(i-1)}}\otimes\underbrace{v}_{\in V^{\otimes 1}}\in V^{\otimes(i-1)}\otimes V^{\otimes 1}=V^{\otimes i}$.

Combined with $w\in V^{\otimes(n-i)}$, this leads to $(u\otimes v)\cdot w=(u\otimes v)\otimes w$ (by (3), applied to $u\otimes v$, w, i and $n-i$ instead of a, b, n and m). Thus, $\underbrace{u\cdot v}_{=u\otimes v}\cdot w=(u\otimes v)\cdot w=(u\otimes v)\otimes w=u\otimes v\otimes w$, qed.

$f^{\otimes i} : V^{\otimes i} \to W^{\otimes i}$ for all $i \in \mathbb{N}$. Hence,

$$\begin{aligned}
\operatorname{Ker}(\otimes f) &= \bigoplus_{i\in\mathbb{N}} \operatorname{Ker}\left(f^{\otimes i}\right) = \sum_{i\in\mathbb{N}} \operatorname{Ker}\left(f^{\otimes i}\right) \qquad \text{(since direct sums are sums)} \\
&= \sum_{n\in\mathbb{N}} \operatorname{Ker}\left(f^{\otimes n}\right) \qquad \text{(here, we renamed the index } i \text{ as } n \text{ in the sum)} \\
&= \sum_{n\in\mathbb{N}} \sum_{i=1}^{n} \underbrace{\left(\operatorname{id}_{V^{\otimes(i-1)}} \otimes \mathfrak{i} \otimes \operatorname{id}_{V^{\otimes(n-i)}}\right)\left(V^{\otimes(i-1)} \otimes (\operatorname{Ker} f) \otimes V^{\otimes(n-i)}\right)}_{\substack{=V^{\otimes(i-1)}\cdot(\operatorname{Ker} f)\cdot V^{\otimes(n-i)} \\ \text{(by Lemma 33, applied to } W=\operatorname{Ker} f)}} \qquad \text{(by (22))} \\
&= \sum_{n\in\mathbb{N}} \sum_{i=1}^{n} V^{\otimes(i-1)} \cdot (\operatorname{Ker} f) \cdot V^{\otimes(n-i)} = \sum_{n\in\mathbb{N}} \sum_{i=0}^{n-1} V^{\otimes i} \cdot (\operatorname{Ker} f) \cdot \underbrace{V^{\otimes(n-(i+1))}}_{=V^{\otimes(n-1-i)}} \\
&\qquad \text{(here, we substituted } i+1 \text{ for } i \text{ in the second sum)} \\
&= \sum_{n\in\mathbb{N}} \sum_{i=0}^{n-1} V^{\otimes i} \cdot (\operatorname{Ker} f) \cdot V^{\otimes(n-1-i)} \\
&= \sum_{\substack{n\in\mathbb{N}; \\ n\geq 1}} \sum_{i=0}^{n-1} V^{\otimes i} \cdot (\operatorname{Ker} f) \cdot V^{\otimes(n-1-i)} + \underbrace{\sum_{i=0}^{0-1} V^{\otimes i} \cdot (\operatorname{Ker} f) \cdot V^{\otimes(0-1-i)}}_{=(\text{empty sum})=0} \\
&= \sum_{\substack{n\in\mathbb{N}; \\ n\geq 1}} \sum_{i=0}^{n-1} V^{\otimes i} \cdot (\operatorname{Ker} f) \cdot V^{\otimes(n-1-i)} = \underbrace{\sum_{n\in\mathbb{N}} \sum_{i=0}^{n}}_{=\sum_{i\in\mathbb{N}} \sum_{\substack{n\in\mathbb{N}; \\ n\geq i}}} V^{\otimes i} \cdot (\operatorname{Ker} f) \cdot V^{\otimes(n-i)} \\
&\qquad \text{(here, we substituted } n \text{ for } n-1 \text{ in the first sum)} \\
&= \sum_{i\in\mathbb{N}} \sum_{\substack{n\in\mathbb{N}; \\ n\geq i}} V^{\otimes i} \cdot (\operatorname{Ker} f) \cdot V^{\otimes(n-i)} = \sum_{i\in\mathbb{N}} \sum_{j\in\mathbb{N}} V^{\otimes i} \cdot (\operatorname{Ker} f) \cdot V^{\otimes j} \\
&\qquad \text{(here, we substituted } j \text{ for } n-i \text{ in the second sum)} \\
&= \underbrace{\left(\sum_{i\in\mathbb{N}} V^{\otimes i}\right)}_{=\otimes V} \cdot (\operatorname{Ker} f) \cdot \underbrace{\left(\sum_{j\in\mathbb{N}} V^{\otimes j}\right)}_{=\otimes V} = (\otimes V) \cdot (\operatorname{Ker} f) \cdot (\otimes V).
\end{aligned}$$

This proves Theorem 32. □

0.12. The pseudoexterior algebra

We are now going to introduce the pseudoexterior algebra $\operatorname{Exter} V$ of a k-module V. There are two ways to do this: one is by constructing $\operatorname{Exter} V$ as a direct sum of pseudoexterior powers $\operatorname{Exter}^n V$ (so these pseudoexterior powers must be defined first); the other is by directly constructing $\operatorname{Exter} V$ as a quotient of the tensor algebra $\otimes V$ modulo a certain two-sided ideal (and then we can construct the pseudoexterior powers $\operatorname{Exter}^n V$ as homogeneous components of this $\operatorname{Exter} V$). It is not immediately clear (although not difficult) to prove that these two ways yield one and the same (up to canonical isomorphism) k-algebra $\operatorname{Exter} V$. We are going to reconcile these two ways by first proving some properties of the two-sided ideal that we want to factor the tensor algebra $\otimes V$ by; once these are shown, it will be easy to see that both definitions of $\operatorname{Exter} V$ are the same. We delay the definition of $\operatorname{Exter} V$ until that moment. So let us first define the pseudoexterior powers $\operatorname{Exter}^n V$:

Definition 36. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Let $Q_n(V)$ be the k-submodule

$$\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n \right\rangle$$

of the k-module $V^{\otimes n}$ (where we are using Convention 34, and are denoting the n-th symmetric group by S_n).
The factor k-module $V^{\otimes n} / Q_n(V)$ is called the *n-th pseudoexterior power* of the k-module V and will be denoted by $\operatorname{Exter}^n V$. We denote by $\operatorname{exter}_{V,n}$ the canonical projection $V^{\otimes n} \to V^{\otimes n} / Q_n(V) = \operatorname{Exter}^n V$. Clearly, this map $\operatorname{exter}_{V,n}$ is a surjective k-module homomorphism.

Warning 37. This n-th pseudoexterior power $\operatorname{Exter}^n V$ is called **pseudo**exterior for a reason: it is not exactly the same as the n-th exterior power $\wedge^n V$ (which we will introduce in Definition 65). While the difference between $\operatorname{Exter}^n V$ and $\wedge^n V$ is not that large (in particular, they are identic when 2 is invertible in k, as Theorem 82 will show), this difference exists and should not be forgotten.
Most literature only works with the n-th exterior power $\wedge^n V$, because the n-th pseudoexterior power $\operatorname{Exter}^n V$ is much less interesting in the general case. **However**, a number of texts which are only concerned with the case when 2 is invertible in k define the n-th exterior power $\wedge^n V$ by our Definition 36; i. e., what they call the n-th exterior power $\wedge^n V$ is what we call the n-th pseudoexterior power $\operatorname{Exter}^n V$. Fortunately this does not conflict with our notation as long as 2 is invertible in k (because when 2 is invertible in k, Theorem 82 **(c)** yields $\wedge^n V = \operatorname{Exter}^n V$).

This is not the pseudoexterior algebra $\operatorname{Exter} V$ yet, but only the n-th pseudoexterior power $\operatorname{Exter}^n V$; we will compose the pseudoexterior algebra from these later. First, here is an alternative description of the module $Q_n(V)$ from this definition:

Proposition 38. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Then,

$$Q_n(V) = \sum_{i=1}^{n-1} \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle,$$

where τ_i denotes the transposition $(i, i+1) \in S_n$.

This proposition is classical and can be concluded from the definition of $Q_n(V)$ and the fact that the transpositions τ_1, τ_2, ..., τ_{n-1} generate the symmetric group S_n. Here are the details of this proof:

Proof of Proposition 38. Let T denote the subset

$$\left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n \right\}$$

of $V^{\otimes n}$. Then,

$$\begin{aligned}\langle T\rangle &= \left\langle \left\{ v_1\otimes v_2\otimes\cdots\otimes v_n - (-1)^\sigma v_{\sigma(1)}\otimes v_{\sigma(2)}\otimes\cdots\otimes v_{\sigma(n)} \mid ((v_1,v_2,\ldots,v_n),\sigma)\in V^n\times S_n\right\}\right\rangle \\ &= \left\langle v_1\otimes v_2\otimes\cdots\otimes v_n - (-1)^\sigma v_{\sigma(1)}\otimes v_{\sigma(2)}\otimes\cdots\otimes v_{\sigma(n)} \mid ((v_1,v_2,\ldots,v_n),\sigma)\in V^n\times S_n\right\rangle \\ &= Q_n(V) \qquad \text{(by Definition 36)}.\end{aligned}$$

On the other hand,

$$v_1\otimes v_2\otimes\cdots\otimes v_n - (-1)^\sigma v_{\sigma(1)}\otimes v_{\sigma(2)}\otimes\cdots\otimes v_{\sigma(n)} \in T \qquad \text{for every } ((v_1,v_2,\ldots,v_n),\sigma)\in V^n\times S_n \tag{28}$$

(since

$T = \left\{ v_1\otimes v_2\otimes\cdots\otimes v_n - (-1)^\sigma v_{\sigma(1)}\otimes v_{\sigma(2)}\otimes\cdots\otimes v_{\sigma(n)} \mid ((v_1,v_2,\ldots,v_n),\sigma)\in V^n\times S_n\right\}$

).

On the other hand, let Z denote the k-submodule

$$\sum_{i=1}^{n-1}\left\langle v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\rangle$$

of $V^{\otimes n}$. Then,

$$\left\langle v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_\mathbf{I}(1)}\otimes v_{\tau_\mathbf{I}(2)}\otimes\cdots\otimes v_{\tau_\mathbf{I}(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\rangle \subseteq Z \tag{29}$$

for every $\mathbf{I}\in\{1,2,\ldots,n-1\}$. Now,

$$\begin{aligned}&\left\{ w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_\mathbf{I}(1)}\otimes w_{\tau_\mathbf{I}(2)}\otimes\cdots\otimes w_{\tau_\mathbf{I}(n)} \mid (w_1,w_2,\ldots,w_n)\in V^n\right\} \\ &= \left\{ v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_\mathbf{I}(1)}\otimes v_{\tau_\mathbf{I}(2)}\otimes\cdots\otimes v_{\tau_\mathbf{I}(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\} \\ &\qquad \text{(here, we renamed } (w_1,w_2,\ldots,w_n) \text{ as } (v_1,v_2,\ldots,v_n)) \\ &\subseteq \left\langle \left\{ v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_\mathbf{I}(1)}\otimes v_{\tau_\mathbf{I}(2)}\otimes\cdots\otimes v_{\tau_\mathbf{I}(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\}\right\rangle \\ &= \left\langle v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_\mathbf{I}(1)}\otimes v_{\tau_\mathbf{I}(2)}\otimes\cdots\otimes v_{\tau_\mathbf{I}(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\rangle \subseteq Z\end{aligned}$$

for every $\mathbf{I}\in\{1,2,\ldots,n-1\}$. Thus,

$$w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_\mathbf{I}(1)}\otimes w_{\tau_\mathbf{I}(2)}\otimes\cdots\otimes w_{\tau_\mathbf{I}(n)} \in Z \qquad \text{for every } (w_1,w_2,\ldots,w_n)\in V^n \text{ and every } \mathbf{I}\in\{1,2,\ldots,n-1\}. \tag{30}$$

We are now going to show that $Z = \langle T\rangle$.

First, let us prove that $Z\subseteq\langle T\rangle$. In fact, every $i\in\{1,2,\ldots,n-1\}$ satisfies

$$\left\{ v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\} \subseteq \langle T\rangle$$

(since every $(v_1,v_2,\ldots,v_n)\in V^n$ satisfies

$$\begin{aligned}&v_1\otimes v_2\otimes\cdots\otimes v_n + \underbrace{v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)}}_{=-(-1)v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)}} \\ &= v_1\otimes v_2\otimes\cdots\otimes v_n - \underbrace{(-1)}_{\substack{=(-1)^{\tau_i} \\ \text{(since } \tau_i \text{ is a transposition,} \\ \text{so that } (-1)^{\tau_i}=-1)}} v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \\ &= v_1\otimes v_2\otimes\cdots\otimes v_n - (-1)^{\tau_i} v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \\ &\in T \qquad \text{(due to (28), applied to } ((v_1,v_2,\ldots,v_n),\tau_i) \text{ instead of } ((v_1,v_2,\ldots,v_n),\sigma)) \\ &\subseteq \langle T\rangle\end{aligned}$$

). Now,

$$Z = \sum_{i=1}^{n-1} \underbrace{\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \ \mid \ (v_1, v_2, \ldots, v_n) \in V^n \right\rangle}_{\subseteq \langle T \rangle}$$

$$\subseteq \sum_{i=1}^{n-1} \langle T \rangle \subseteq \langle T \rangle \qquad \text{(since } \langle T \rangle \text{ is a } k\text{-module)}.$$

Now, let us show that $\langle T \rangle \subseteq Z$. To that aim, we will show that $T \subseteq Z$.

In fact, let $((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n$ be arbitrary. Then, $(v_1, v_2, \ldots, v_n) \in V^n$ and $\sigma \in S_n$. Now, it is known that every element of the symmetric group S_n can be written as a product of some transpositions from the set $\{\tau_1, \tau_2, \ldots, \tau_{n-1}\}$. Applying this to the element $\sigma \in S_n$, we conclude that σ can be written as a product of some transpositions from the set $\{\tau_1, \tau_2, \ldots, \tau_{n-1}\}$. In other words, there exists a natural number $m \in \mathbb{N}$ and a sequence $(i_1, i_2, \ldots, i_m) \in \{1, 2, \ldots, n-1\}^m$ such that $\sigma = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_m}$. Consider this m and this $(i_1, i_2, \ldots, i_m)$. For every $j \in \{0, 1, \ldots, m\}$, let σ_j denote the permutation $\tau_{i_1} \tau_{i_2} \cdots \tau_{i_j} \in S_n$. Thus, $\sigma_0 = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_0} = (\text{empty product}) = \text{id}$ and $\sigma_m = \tau_{i_1} \tau_{i_2} \cdots \tau_{i_m} = \sigma$. Moreover, every $j \in \{1, 2, \ldots, m\}$ satisfies $(-1)^{\sigma_{j-1}} v_{\sigma_{j-1}(1)} \otimes$

$v_{\sigma_{j-1}(2)} \otimes \cdots \otimes v_{\sigma_{j-1}(n)} - (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)} \in Z$. [8] Thus,

$$\sum_{j=1}^{m} \underbrace{\left((-1)^{\sigma_{j-1}} v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \cdots \otimes v_{\sigma_{j-1}(n)} - (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)} \right)}_{\in Z} \in \sum_{j=1}^{m} Z \subseteq Z$$

(since Z is a k-module). Since

$$\sum_{j=1}^{m} \left((-1)^{\sigma_{j-1}} v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \cdots \otimes v_{\sigma_{j-1}(n)} - (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)} \right)$$
$$= v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \qquad (31)$$

[9], this rewrites as

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \in Z.$$

We have thus shown that every $((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n$ satisfies $v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \in Z$. Thus,

$$\left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n \right\} \subseteq Z.$$

[8] *Proof.* Let $j \in \{1, 2, \ldots, m\}$ be arbitrary. Then,

$$\underbrace{\sigma_{j-1}}_{\substack{=\tau_{i_1}\tau_{i_2}\cdots\tau_{i_{j-1}} \\ \text{(by the formula } \sigma_j = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_j}, \\ \text{applied to } j-1 \text{ instead of } j)}} \tau_{i_j} = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_{j-1}}\tau_{i_j} = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_j} = \sigma_j,$$

so that $\sigma_j = \sigma_{j-1}\tau_{i_j}$. Denote i_j by $\mathbf{I}$. Then, $\sigma_j = \sigma_{j-1}\tau_{i_j}$ rewrites as $\sigma_j = \sigma_{j-1}\tau_{\mathbf{I}}$. Thus,

$$(-1)^{\sigma_j} = (-1)^{\sigma_{j-1}\tau_{\mathbf{I}}} = (-1)^{\sigma_{j-1}} \underbrace{(-1)^{\tau_{\mathbf{I}}}}_{=-1 \text{ (since } \tau_{\mathbf{I}} \text{ is a transposition)}} = -(-1)^{\sigma_{j-1}}.$$

Now, define an n-tuple $(w_1, w_2, \ldots, w_n) \in V^n$ by $\left(w_\rho = v_{\sigma_{j-1}(\rho)} \text{ for every } \rho \in \{1, 2, \ldots, n\} \right)$. Then, $(w_1, w_2, \ldots, w_n) = \left(v_{\sigma_{j-1}(1)}, v_{\sigma_{j-1}(2)}, \ldots, v_{\sigma_{j-1}(n)} \right)$, so that $w_1 \otimes w_2 \otimes \cdots \otimes w_n = v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \cdots \otimes v_{\sigma_{j-1}(n)}$. On the other hand, every $\xi \in \{1, 2, \ldots, n\}$ satisfies

$$w_{\tau_{\mathbf{I}}(\xi)} = v_{\sigma_{j-1}(\tau_{\mathbf{I}}(\xi))} \qquad \left(\text{by the formula } w_\rho = v_{\sigma_{j-1}(\rho)}, \text{ applied to } \rho = \tau_{\mathbf{I}}(\xi) \right)$$
$$= v_{\sigma_j(\xi)} \qquad \left(\text{since } \sigma_{j-1}(\tau_{\mathbf{I}}(\xi)) = \underbrace{(\sigma_{j-1}\tau_{\mathbf{I}})}_{=\sigma_j}(\xi) = \sigma_j(\xi) \right).$$

Thus, $\left(w_{\tau_{\mathbf{I}}(1)}, w_{\tau_{\mathbf{I}}(2)}, \ldots, w_{\tau_{\mathbf{I}}(n)} \right) = \left(v_{\sigma_j(1)}, v_{\sigma_j(2)}, \ldots, v_{\sigma_j(n)} \right)$, so that $w_{\tau_{\mathbf{I}}(1)} \otimes w_{\tau_{\mathbf{I}}(2)} \otimes \cdots \otimes w_{\tau_{\mathbf{I}}(n)} = v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)}$.

Now,

$$(-1)^{\sigma_{j-1}} \underbrace{v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \cdots \otimes v_{\sigma_{j-1}(n)}}_{=w_1 \otimes w_2 \otimes \cdots \otimes w_n} - \underbrace{(-1)^{\sigma_j}}_{=-(-1)^{\sigma_{j-1}}} \underbrace{v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)}}_{=w_{\tau_{\mathbf{I}}(1)} \otimes w_{\tau_{\mathbf{I}}(2)} \otimes \cdots \otimes w_{\tau_{\mathbf{I}}(n)}}$$
$$= (-1)^{\sigma_{j-1}} w_1 \otimes w_2 \otimes \cdots \otimes w_n - \left(-(-1)^{\sigma_{j-1}} \right) w_{\tau_{\mathbf{I}}(1)} \otimes w_{\tau_{\mathbf{I}}(2)} \otimes \cdots \otimes w_{\tau_{\mathbf{I}}(n)}$$
$$= (-1)^{\sigma_{j-1}} \underbrace{\left(w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_{\mathbf{I}}(1)} \otimes w_{\tau_{\mathbf{I}}(2)} \otimes \cdots \otimes w_{\tau_{\mathbf{I}}(n)} \right)}_{\in Z \text{ (by (30))}} \in Z \qquad (\text{since } Z \text{ is a } k\text{-module}),$$

qed.

[9] *Proof of (31).* We distinguish between two cases: the case when $m > 0$, and the case when $m = 0$.

Since $\left\{v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n\right\} = T$, this rewrites as $T \subseteq Z$. Proposition 35 **(a)** (applied to $V^{\otimes n}$, T and Z instead of M, S and Q) thus yields that $\langle T \rangle \subseteq Z$. Combined with $Z \subseteq \langle T \rangle$, this yields $Z = \langle T \rangle$.

In the case when $m > 0$, we have

$$\begin{aligned}
&\sum_{j=1}^{m} \left((-1)^{\sigma_{j-1}} v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \cdots \otimes v_{\sigma_{j-1}(n)} - (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)}\right) \\
&= \sum_{j=1}^{m} (-1)^{\sigma_{j-1}} v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \cdots \otimes v_{\sigma_{j-1}(n)} - \sum_{j=1}^{m} (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)} \\
&= \underbrace{\sum_{j=0}^{m-1} (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)}}_{=(-1)^{\sigma_0} v_{\sigma_0(1)} \otimes v_{\sigma_0(2)} \otimes \cdots \otimes v_{\sigma_0(n)} + \sum_{j=1}^{m-1} (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)}} \\
&\qquad - \underbrace{\sum_{j=1}^{m} (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)}}_{=\sum_{j=1}^{m-1} (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)} + (-1)^{\sigma_m} v_{\sigma_m(1)} \otimes v_{\sigma_m(2)} \otimes \cdots \otimes v_{\sigma_m(n)}} \\
&\qquad \left(\text{here, we substituted } j \text{ for } j-1 \text{ in the first sum}\right) \\
&= \left((-1)^{\sigma_0} v_{\sigma_0(1)} \otimes v_{\sigma_0(2)} \otimes \cdots \otimes v_{\sigma_0(n)} + \sum_{j=1}^{m-1} (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)}\right) \\
&\qquad - \left(\sum_{j=1}^{m-1} (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)} + (-1)^{\sigma_m} v_{\sigma_m(1)} \otimes v_{\sigma_m(2)} \otimes \cdots \otimes v_{\sigma_m(n)}\right) \\
&= \underbrace{(-1)^{\sigma_0}}_{=1 \text{ (since } \sigma_0 = \mathrm{id})} \underbrace{v_{\sigma_0(1)} \otimes v_{\sigma_0(2)} \otimes \cdots \otimes v_{\sigma_0(n)}}_{\substack{= v_1 \otimes v_2 \otimes \cdots \otimes v_n \\ \text{(since } \sigma_0 = \mathrm{id} \text{ yields} \\ v_{\sigma_0(1)} \otimes v_{\sigma_0(2)} \otimes \cdots \otimes v_{\sigma_0(n)} = v_{\mathrm{id}(1)} \otimes v_{\mathrm{id}(2)} \otimes \cdots \otimes v_{\mathrm{id}(n)} \\ = v_1 \otimes v_2 \otimes \cdots \otimes v_n)}} - \underbrace{(-1)^{\sigma_m}}_{\substack{=(-1)^{\sigma} \\ \text{(since } \sigma_m = \sigma)}} \underbrace{v_{\sigma_m(1)} \otimes v_{\sigma_m(2)} \otimes \cdots \otimes v_{\sigma_m(n)}}_{\substack{= v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \\ \text{(since } \sigma_m = \sigma)}} \\
&= v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)},
\end{aligned}$$

so that (31) is proven in the case when $m > 0$.

In the case when $m = 0$, we have

$$\begin{aligned}
&\sum_{j=1}^{m} \left((-1)^{\sigma_{j-1}} v_{\sigma_{j-1}(1)} \otimes v_{\sigma_{j-1}(2)} \otimes \cdots \otimes v_{\sigma_{j-1}(n)} - (-1)^{\sigma_j} v_{\sigma_j(1)} \otimes v_{\sigma_j(2)} \otimes \cdots \otimes v_{\sigma_j(n)}\right) \\
&= (\text{empty sum}) = 0 \\
&= \underbrace{(-1)^{\mathrm{id}}}_{=1} \underbrace{v_{\mathrm{id}(1)} \otimes v_{\mathrm{id}(2)} \otimes \cdots \otimes v_{\mathrm{id}(n)}}_{= v_1 \otimes v_2 \otimes \cdots \otimes v_n} - \underbrace{(-1)^{\mathrm{id}} v_{\mathrm{id}(1)} \otimes v_{\mathrm{id}(2)} \otimes \cdots \otimes v_{\mathrm{id}(n)}}_{\substack{=(-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \\ \text{(since } m=0 \text{ leads to } \sigma_m = \sigma_0 \text{, so that} \\ \mathrm{id} = \sigma_0 = \sigma_m = \sigma)}} \\
&= v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^{\sigma} v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)},
\end{aligned}$$

so that (31) is proven in the case when $m = 0$.

Thus, (31) is proven in both possible cases. This completes the proof of (31).

We now have

$$\sum_{i=1}^{n-1} \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle$$
$$= Z = \langle T \rangle = Q_n(V).$$

This proves Proposition 38. □

A trivial corollary from Proposition 38:

Corollary 39. Let k be a commutative ring. Let V be a k-module. Then,

$$Q_2(V) = \left\langle v_1 \otimes v_2 + v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\rangle.$$

Proof of Corollary 39. Applying Proposition 38 to $n = 2$, we obtain

$$Q_2(V) = \sum_{i=1}^{2-1} \left\langle v_1 \otimes v_2 + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \mid (v_1, v_2) \in V^2 \right\rangle$$
$$= \left\langle v_1 \otimes v_2 + v_{\tau_1(1)} \otimes v_{\tau_1(2)} \mid (v_1, v_2) \in V^2 \right\rangle = \left\langle v_1 \otimes v_2 + v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\rangle$$

(since $\tau_1(1) = 2$ and $\tau_1(2) = 1$). This proves Corollary 39. □

Next something very basic:

Lemma 40. Let k be a commutative. Let P be a k-algebra.
(a) Let X and Y be two sets, and $a : X \to P$ and $b : Y \to P$ be two maps. Then,

$$\langle a(x) \mid x \in X \rangle \cdot \langle b(y) \mid y \in Y \rangle = \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle.$$

(b) Let X, Y and Z be three sets, and $a : X \to P$, $b : Y \to P$ and $c : Z \to P$ be three maps. Then,

$$\langle a(x) \mid x \in X \rangle \cdot \langle b(y) \mid y \in Y \rangle \cdot \langle c(z) \mid z \in Z \rangle$$
$$= \langle a(x) b(y) c(z) \mid (x, y, z) \in X \times Y \times Z \rangle.$$

Proof of Lemma 40. **(a)** Let $X' = \langle a(x) \mid x \in X \rangle$ and $Y' = \langle b(y) \mid y \in Y \rangle$.

We will now prove that $X'Y' \subseteq \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle$ and $\langle a(x) b(y) \mid (x, y) \in X \times Y \rangle \subseteq X'Y'$.

Proof of $X'Y' \subseteq \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle$*:*

By the definition of the product of two k-submodules, we have

$$X'Y' = \langle pq \mid (p, q) \in X' \times Y' \rangle = \langle \{pq \mid (p, q) \in X' \times Y'\} \rangle. \tag{32}$$

Now, let $(p, q) \in X' \times Y'$ be arbitrary. Then, $p \in X' = \langle a(x) \mid x \in X \rangle$, so that we can find some $n \in \mathbb{N}$, some elements $x_1, x_2, \ldots, x_n$ of X and some elements $\lambda_1, \lambda_2, \ldots,$

λ_n of k such that $p = \sum_{i=1}^{n} \lambda_i a(x_i)$. Consider this n, these $x_1, x_2, \ldots, x_n$ and these $\lambda_1, \lambda_2, \ldots, \lambda_n$.

Since $(p, q) \in X' \times Y'$, we have $q \in Y' = \langle b(y) \mid y \in Y \rangle$, so that we can find some $m \in \mathbb{N}$, some elements $y_1, y_2, \ldots, y_m$ of Y and some elements $\mu_1, \mu_2, \ldots, \mu_m$ of k such that $q = \sum_{j=1}^{m} \mu_j b(y_j)$. Consider this m, these $y_1, y_2, \ldots, y_m$ and these $\mu_1, \mu_2, \ldots, \mu_m$.

Since $p = \sum_{i=1}^{n} \lambda_i a(x_i)$ and $q = \sum_{j=1}^{m} \mu_j b(y_j)$, we have

$$pq = \sum_{i=1}^{n} \lambda_i a(x_i) \cdot \sum_{j=1}^{m} \mu_j b(y_j) = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \mu_j \underbrace{a(x_i) b(y_j)}_{\substack{\in \{a(x)b(y) \mid (x,y) \in X \times Y\} \\ \subseteq \langle \{a(x)b(y) \mid (x,y) \in X \times Y\} \rangle \\ = \langle a(x)b(y) \mid (x,y) \in X \times Y \rangle}}$$
$$\subseteq \sum_{i=1}^{n} \sum_{j=1}^{m} \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle$$
$$\subseteq \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle \qquad \left(\text{since } \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle \text{ is a } k\text{-module}\right).$$

Since this holds for all $(p, q) \in X' \times Y'$, we have thus proven that

$$\{pq \mid (p, q) \in X' \times Y'\} \subseteq \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle .$$

Therefore, Proposition 35 **(a)** (applied to P, $\{pq \mid (p, q) \in X' \times Y'\}$ and $\langle a(x) b(y) \mid (x, y) \in X \times Y \rangle$ instead of M, S and Q) yields that

$$\langle \{pq \mid (p, q) \in X' \times Y'\} \rangle \subseteq \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle .$$

Combined with (32), this yields

$$X'Y' \subseteq \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle .$$

We have thus proven that $X'Y' \subseteq \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle$.

Proof of $\langle a(x) b(y) \mid (x, y) \in X \times Y \rangle \subseteq X'Y'$*:* We have

$$X' = \langle a(x) \mid x \in X \rangle = \langle \{a(x) \mid x \in X\} \rangle \supseteq \{a(x) \mid x \in X\} .$$

Thus, $a(x) \in X'$ for every $x \in X$. Similarly, $b(y) \in Y'$ for every $y \in Y$.

Now, let $(x, y) \in X \times Y$ be arbitrary. Then, $x \in X$ and $y \in Y'$, so that $a(x) \in X'$ and $b(y) \in Y'$ (as we just have seen). Hence, $a(x) b(y) \in X'Y'$.

We have thus shown that $a(x) b(y) \in X'Y'$ for every $(x, y) \in X \times Y$. In other words, $\{a(x) b(y) \mid (x, y) \in X \times Y\} \subseteq X'Y'$. Therefore, Proposition 35 **(a)** (applied to P, $\{a(x) b(y) \mid (x, y) \in X \times Y\}$ and $X'Y'$ instead of M, S and Q) yields that

$$\langle \{a(x) b(y) \mid (x, y) \in X \times Y\} \rangle \subseteq X'Y'.$$

Since $\langle \{a(x) b(y) \mid (x, y) \in X \times Y\} \rangle = \langle a(x) b(y) \mid (x, y) \in X \times Y \rangle$, we have thus proven $\langle a(x) b(y) \mid (x, y) \in X \times Y \rangle \subseteq X'Y'$.

Combined with $X'Y' \subseteq \langle a(x) b(y) \mid (x,y) \in X \times Y \rangle$, this yields that $X'Y' = \langle a(x) b(y) \mid (x,y) \in X \times Y \rangle$. Since $X' = \langle a(x) \mid x \in X \rangle$ and $Y' = \langle b(y) \mid y \in Y \rangle$, this rewrites as follows:

$$\langle a(x) \mid x \in X \rangle \cdot \langle b(y) \mid y \in Y \rangle = \langle a(x) b(y) \mid (x,y) \in X \times Y \rangle .$$

Thus, Lemma 40 **(a)** is proven.

(b) Define a map $d : X \times Y \to P$ by

$$(d(x,y) = a(x) b(y) \text{ for all } (x,y) \in X \times Y) .$$

Now, Lemma 40 **(a)** yields

$$\begin{aligned} \langle a(x) \mid x \in X \rangle \cdot \langle b(y) \mid y \in Y \rangle &= \left\langle \underbrace{a(x) b(y)}_{=d(x,y)} \mid (x,y) \in X \times Y \right\rangle \\ &= \langle d(x,y) \mid (x,y) \in X \times Y \rangle = \langle d(x) \mid x \in X \times Y \rangle \end{aligned}$$

(here, we renamed the index (x,y) as x). Also, $\langle c(z) \mid z \in Z \rangle = \langle c(y) \mid y \in Z \rangle$ (here, we renamed the index z as y). But Lemma 40 **(a)** (applied to $X \times Y$, Z, d and c instead of X, Y, a and b) yields

$$\langle d(x) \mid x \in X \times Y \rangle \cdot \langle c(y) \mid y \in Z \rangle = \langle d(x) \cdot c(y) \mid (x,y) \in (X \times Y) \times Z \rangle .$$

Thus,

$$\begin{aligned} & \underbrace{\langle a(x) \mid x \in X \rangle \cdot \langle b(y) \mid y \in Y \rangle}_{=\langle d(x) \mid x \in X \times Y \rangle} \cdot \underbrace{\langle c(z) \mid z \in Z \rangle}_{=\langle c(y) \mid y \in Z \rangle} \\ &= \langle d(x) \mid x \in X \times Y \rangle \cdot \langle c(y) \mid y \in Z \rangle = \langle d(x) \cdot c(y) \mid (x,y) \in (X \times Y) \times Z \rangle \\ &= \langle d(t) \cdot c(z) \mid (t,z) \in (X \times Y) \times Z \rangle \\ &\qquad \text{(here, we renamed the index } (x,y) \text{ as } (t,z)) \\ &= \langle d(x,y) \cdot c(z) \mid ((x,y),z) \in (X \times Y) \times Z \rangle \\ &\qquad \text{(here, we renamed the index } (t,z) \text{ as } ((x,y),z)) \\ &= \left\langle \underbrace{d(x,y)}_{=a(x)b(y)} \cdot c(z) \mid (x,y,z) \in X \times Y \times Z \right\rangle \\ &\qquad \text{(here, we substituted the triple } (x,y,z) \text{ for the pair } ((x,y),z)) \\ &= \langle a(x) b(y) c(z) \mid (x,y,z) \in X \times Y \times Z \rangle . \end{aligned}$$

This proves Lemma 40 **(b)**. □

The following lemma will help us in making use of Proposition 38:

Lemma 41. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Let $i \in \{1, 2, \ldots, n-1\}$.
Then,

$$\begin{aligned} & \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \rangle \\ & = V^{\otimes(i-1)} \cdot (Q_2(V)) \cdot V^{\otimes(n-1-i)}, \end{aligned}$$

where τ_i denotes the transposition $(i, i+1) \in S_n$. Here, we consider $V^{\otimes n}$ as a k-submodule of $\otimes V$.

Proof of Lemma 41. Define a map $a : V^{i-1} \to V^{\otimes(i-1)}$ by

$$\left(a\left(v_1, v_2, \ldots, v_{i-1}\right) = v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \qquad \text{for every } \left(v_1, v_2, \ldots, v_{i-1}\right) \in V^{i-1}\right).$$

Define a map $b : V^2 \to V^{\otimes 2}$ by

$$\left(b\left(v_i, v_{i+1}\right) = v_i \otimes v_{i+1} + v_{i+1} \otimes v_i \qquad \text{for every } \left(v_i, v_{i+1}\right) \in V^2\right).$$

Define a map $c : V^{n-1-i} \to V^{\otimes(n-1-i)}$ by

$$\left(c\left(v_{i+2}, v_{i+3}, \ldots, v_n\right) = v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n \qquad \text{for every } \left(v_{i+2}, v_{i+3}, \ldots, v_n\right) \in V^{n-1-i}\right).$$

Since $V^{\otimes(i-1)}$, $V^{\otimes 2}$ and $V^{\otimes(n-1-i)}$ are k-submodules of $\otimes V$, we can consider all three maps a, b and c as maps to the set $\otimes V$.

It is now easy to see that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} = a\left(v_1, v_2, \ldots, v_{i-1}\right) \cdot b\left(v_i, v_{i+1}\right) \cdot c\left(v_{i+2}, v_{i+3}, \ldots, v_n\right),$$

where the multiplication on the right hand side is the multiplication in the tensor algebra $\otimes V$. [10] Thus,

$$\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle$$
$$= \left\langle a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) \cdot c(v_{i+2}, v_{i+3}, \ldots, v_n) \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle$$
$$= \left\langle a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) \cdot c(v_{i+2}, v_{i+3}, \ldots, v_n) \mid ((v_1, v_2, \ldots, v_{i-1}), (v_i, v_{i+1}), (v_{i+2}, v_{i+3}, \ldots, v_n)) \in V^{i-1} \times V^2 \times V^{n-1-i} \right\rangle$$
$$\left(\begin{array}{c}\text{here, we substituted the triple } ((v_1, v_2, \ldots, v_{i-1}), (v_i, v_{i+1}), (v_{i+2}, v_{i+3}, \ldots, v_n)) \\ \text{for the } n\text{-tuple } (v_1, v_2, \ldots, v_n)\end{array}\right)$$
$$= \left\langle a(x)\, b(y)\, c(z) \mid (x, y, z) \in V^{i-1} \times V^2 \times V^{n-1-i} \right\rangle \tag{34}$$
$$\left(\text{here, we renamed } ((v_1, v_2, \ldots, v_{i-1}), (v_i, v_{i+1}), (v_{i+2}, v_{i+3}, \ldots, v_n)) \text{ as } (x, y, z)\right).$$

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[10] *Proof.* Let $(v_1, v_2, \ldots, v_n) \in V^n$. Then, recalling Convention 12, we have

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n = \underbrace{(v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1})}_{=a(v_1, v_2, \ldots, v_{i-1})} \otimes (v_i \otimes v_{i+1}) \otimes \underbrace{(v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n)}_{=c(v_{i+2}, v_{i+3}, \ldots, v_n)}$$
$$= a(v_1, v_2, \ldots, v_{i-1}) \otimes (v_i \otimes v_{i+1}) \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n). \tag{33}$$

On the other hand, every $j \in \{1, 2, \ldots, i-1\}$ satisfies $\tau_i(j) = j$ (since τ_i is the transposition $(i, i+1)$) and thus $v_{\tau_i(j)} = v_j$. In other words, we have the equalities $v_{\tau_i(1)} = v_1$, $v_{\tau_i(2)} = v_2$, ..., $v_{\tau_i(i-1)} = v_{i-1}$. Taking the tensor product of these equalities yields

$$v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(i-1)} = v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} = a(v_1, v_2, \ldots, v_{i-1}).$$

Every $j \in \{i+2, i+3, \ldots, n\}$ satisfies $\tau_i(j) = j$ (since τ_i is the transposition $(i, i+1)$) and thus $v_{\tau_i(j)} = v_j$. In other words, we have the equalities $v_{\tau_i(i+2)} = v_{i+2}$, $v_{\tau_i(i+3)} = v_{i+3}$, ..., $v_{\tau_i(n)} = v_n$. Taking the tensor product of these equalities yields

$$v_{\tau_i(i+2)} \otimes v_{\tau_i(i+3)} \otimes \cdots \otimes v_{\tau_i(n)} = v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n = c(v_{i+2}, v_{i+3}, \ldots, v_n).$$

Since τ_i is the transposition $(i, i+1)$, we have $\tau_i(i) = i+1$ and $\tau_i(i+1) = i$. These equalities yield $v_{\tau_i(i)} = v_{i+1}$ and $v_{\tau_i(i+1)} = v_i$, respectively.

Now,

$$v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)}$$
$$= \underbrace{(v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(i-1)})}_{=a(v_1, v_2, \ldots, v_{i-1})} \otimes \left(\underbrace{v_{\tau_i(i)}}_{=v_{i+1}} \otimes \underbrace{v_{\tau_i(i+1)}}_{=v_i}\right) \otimes \underbrace{(v_{\tau_i(i+2)} \otimes v_{\tau_i(i+3)} \otimes \cdots \otimes v_{\tau_i(n)})}_{=c(v_{i+2}, v_{i+3}, \ldots, v_n)}$$
$$= a(v_1, v_2, \ldots, v_{i-1}) \otimes (v_{i+1} \otimes v_i) \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n).$$

Adding this to (33), we get

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)}$$
$$= a(v_1, v_2, \ldots, v_{i-1}) \otimes (v_i \otimes v_{i+1}) \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n) + a(v_1, v_2, \ldots, v_{i-1}) \otimes (v_{i+1} \otimes v_i) \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n)$$
$$= a(v_1, v_2, \ldots, v_{i-1}) \otimes \underbrace{(v_i \otimes v_{i+1} + v_{i+1} \otimes v_i)}_{=b(v_i, v_{i+1})} \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n)$$
$$= a(v_1, v_2, \ldots, v_{i-1}) \otimes b(v_i, v_{i+1}) \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n).$$

On the other hand, (3) (applied to $a(v_1, v_2, \ldots, v_{i-1})$, $b(v_i, v_{i+1})$, $i-1$ and 2 instead of a, b, n and m) yields

$$a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) = a(v_1, v_2, \ldots, v_{i-1}) \otimes b(v_i, v_{i+1}).$$

Also, (3) (applied to $a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1})$, $c(v_{i+2}, v_{i+3}, \ldots, v_n)$, $i+1$ and $n-1-i$ instead of a, b, n and m) yields

$$a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) \cdot c(v_{i+2}, v_{i+3}, \ldots, v_n)$$
$$= \underbrace{(a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}))}_{=a(v_1, v_2, \ldots, v_{i-1}) \otimes b(v_i, v_{i+1})} \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n)$$
$$= a(v_1, v_2, \ldots, v_{i-1}) \otimes b(v_i, v_{i+1}) \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n)$$
$$= v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)},$$

qed.

But Lemma 40 **(b)** (applied to $X = V^{i-1}$, $Y = V^2$, $Z = V^{n-1-i}$ and $P = \otimes V$) yields

$$\left\langle a(x) \mid x \in V^{i-1} \right\rangle \cdot \left\langle b(y) \mid y \in V^2 \right\rangle \cdot \left\langle c(z) \mid z \in V^{n-1-i} \right\rangle$$
$$= \left\langle a(x) b(y) c(z) \mid (x,y,z) \in V^{i-1} \times V^2 \times V^{n-1-i} \right\rangle.$$

Compared to (34), this yields

$$\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle$$
$$= \left\langle a(x) \mid x \in V^{i-1} \right\rangle \cdot \left\langle b(y) \mid y \in V^2 \right\rangle \cdot \left\langle c(z) \mid z \in V^{n-1-i} \right\rangle. \tag{35}$$

But

$$\left\langle a(x) \mid x \in V^{i-1} \right\rangle = \left\langle \underbrace{a(v_1, v_2, \ldots, v_{i-1})}_{= v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1}} \mid (v_1, v_2, \ldots, v_{i-1}) \in V^{i-1} \right\rangle$$
$$\text{(here, we renamed } x \text{ as } (v_1, v_2, \ldots, v_{i-1}))$$
$$= \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \mid (v_1, v_2, \ldots, v_{i-1}) \in V^{i-1} \right\rangle = V^{\otimes(i-1)}$$

(since the k-module $V^{\otimes(i-1)}$ is generated by its pure tensors, i. e., by tensors of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1}$ with $(v_1, v_2, \ldots, v_{i-1}) \in V^{i-1}$). Also,

$$\left\langle c(z) \mid z \in V^{n-1-i} \right\rangle = \left\langle \underbrace{c(v_{i+2}, v_{i+3}, \ldots, v_n)}_{= v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n} \mid (v_{i+2}, v_{i+3}, \ldots, v_n) \in V^{n-1-i} \right\rangle$$
$$\text{(here, we renamed } z \text{ as } (v_{i+2}, v_{i+3}, \ldots, v_n))$$
$$= \left\langle v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n \mid (v_{i+2}, v_{i+3}, \ldots, v_n) \in V^{n-1-i} \right\rangle = V^{\otimes(n-1-i)}$$

(since the k-module $V^{\otimes(n-1-i)}$ is generated by its pure tensors, i. e., by tensors of the form $v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n$ with $(v_{i+2}, v_{i+3}, \ldots, v_n) \in V^{n-1-i}$). Also,

$$\left\langle b(y) \mid y \in V^2 \right\rangle = \left\langle \underbrace{b(v_i, v_{i+1})}_{= v_i \otimes v_{i+1} + v_{i+1} \otimes v_i} \mid (v_i, v_{i+1}) \in V^2 \right\rangle \qquad \text{(here, we renamed } y \text{ as } (v_i, v_{i+1}))$$
$$= \left\langle v_i \otimes v_{i+1} + v_{i+1} \otimes v_i \mid (v_i, v_{i+1}) \in V^2 \right\rangle$$
$$= \left\langle v_1 \otimes v_2 + v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\rangle$$
$$\text{(here, we renamed } (v_i, v_{i+1}) \text{ as } (v_1, v_2))$$
$$= Q_2(V) \qquad \text{(by Corollary 39)}.$$

Thus, (35) becomes

$$\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle$$
$$= \underbrace{\left\langle a(x) \mid x \in V^{i-1} \right\rangle}_{= V^{\otimes(i-1)}} \cdot \underbrace{\left\langle b(y) \mid y \in V^2 \right\rangle}_{= Q_2(V)} \cdot \underbrace{\left\langle c(z) \mid z \in V^{n-1-i} \right\rangle}_{= V^{\otimes(n-1-i)}}$$
$$= V^{\otimes(i-1)} \cdot (Q_2(V)) \cdot V^{\otimes(n-1-i)},$$

so that Lemma 41 is proven. □

This lemma yields:

Corollary 42. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Then,

$$Q_n(V) = \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (Q_2(V)) \cdot V^{\otimes(n-1-i)}$$

(this is an equality between k-submodules of $\otimes V$, where $Q_n(V)$ becomes such a k-submodule by means of the inclusion $Q_n(V) \subseteq V^{\otimes n} \subseteq \otimes V$). Here, the multiplication on the right hand side is multiplication inside the k-algebra $\otimes V$.

Proof of Corollary 42. For every $i \in \{1, 2, \ldots, i-1\}$, let τ_i denote the transposition $(i, i+1) \in S_n$. Then, by Proposition 38, we have

$$Q_n(V) = \sum_{i=1}^{n-1} \underbrace{\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle}_{= V^{\otimes(i-1)} \cdot (Q_2(V)) \cdot V^{\otimes(n-1-i)} \text{ (by Lemma 41)}}$$
$$= \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (Q_2(V)) \cdot V^{\otimes(n-1-i)}.$$

Thus, Corollary 42 is proven. □

We now claim that:

Theorem 43. Let k be a commutative ring. Let V be a k-module. We know that $Q_n(V)$ is a k-submodule of $V^{\otimes n}$ for every $n \in \mathbb{N}$. Thus, $\bigoplus_{n \in \mathbb{N}} Q_n(V)$ is a k-submodule of $\bigoplus_{n \in \mathbb{N}} V^{\otimes n} = \otimes V$. This k-submodule satisfies

$$\bigoplus_{n \in \mathbb{N}} Q_n(V) = (\otimes V) \cdot (Q_2(V)) \cdot (\otimes V).$$

Proof of Theorem 43. Working inside $\otimes V$, we have

$$\begin{aligned}
\bigoplus_{n\in\mathbb{N}} Q_n(V) &= \sum_{n\in\mathbb{N}} \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (Q_2(V)) \cdot V^{\otimes(n-1-i)} \qquad \text{(by Corollary 42)} \\
&= \sum_{n\in\mathbb{N}} \underbrace{\sum_{i=0}^{n-2}}_{=\sum_{\substack{i\in\mathbb{N};\\ n-2\geq i}} = \sum_{\substack{i\in\mathbb{N};\\ n\geq i+2}}} V^{\otimes i} \cdot (Q_2(V)) \cdot \underbrace{V^{\otimes(n-1-(i+1))}}_{=V^{\otimes(n-1-i-1)}=V^{\otimes(n-(i+2))}} \\
&= \underbrace{\sum_{n\in\mathbb{N}} \sum_{\substack{i\in\mathbb{N};\\ n\geq i+2}}}_{=\sum_{i\in\mathbb{N}} \sum_{\substack{n\in\mathbb{N};\\ n\geq i+2}}} V^{\otimes i} \cdot (Q_2(V)) \cdot V^{\otimes(n-(i+2))} \\
&\qquad \text{(here, we substituted } i \text{ for } i-1 \text{ in the second sum)} \\
&= \sum_{i\in\mathbb{N}} \sum_{\substack{n\in\mathbb{N};\\ n\geq i+2}} V^{\otimes i} \cdot (Q_2(V)) \cdot V^{\otimes(n-(i+2))} = \sum_{i\in\mathbb{N}} V^{\otimes i} \cdot (Q_2(V)) \cdot \sum_{\substack{n\in\mathbb{N};\\ n\geq i+2}} V^{\otimes(n-(i+2))} \\
&= \sum_{i\in\mathbb{N}} V^{\otimes i} \cdot (Q_2(V)) \cdot \sum_{j\in\mathbb{N}} V^{\otimes j} \\
&\qquad \text{(here, we substituted } j \text{ for } n-(i+2) \text{ in the second sum)} \\
&= \underbrace{\left(\sum_{i\in\mathbb{N}} V^{\otimes i}\right)}_{=\otimes V \text{ (by (26))}} \cdot (Q_2(V)) \cdot \underbrace{\left(\sum_{j\in\mathbb{N}} V^{\otimes j}\right)}_{=\otimes V \text{ (by (27))}} \\
&= (\otimes V) \cdot (Q_2(V)) \cdot (\otimes V).
\end{aligned}$$

This proves Theorem 43. □

Now we can finally define the pseudoexterior algebra:

Definition 44. Let k be a commutative ring. Let V be a k-module.
By Theorem 43, the two k-submodules $\bigoplus_{n\in\mathbb{N}} Q_n(V)$ and $(\otimes V) \cdot (Q_2(V)) \cdot (\otimes V)$ of $\otimes V$ are identic (where $\bigoplus_{n\in\mathbb{N}} Q_n(V)$ becomes a k-submodule of $\otimes V$ in the same way as explained in Theorem 43). We denote these two identic k-submodules by $Q(V)$.
In other words, we define $Q(V)$ by

$$Q(V) = \bigoplus_{n\in\mathbb{N}} Q_n(V) = (\otimes V) \cdot (Q_2(V)) \cdot (\otimes V).$$

Since $Q(V) = (\otimes V) \cdot (Q_2(V)) \cdot (\otimes V)$, it is clear that $Q(V)$ is a two-sided ideal of the k-algebra $\otimes V$.

Now we define a k-module $\operatorname{Exter} V$ as the direct sum $\bigoplus_{n\in\mathbb{N}} \operatorname{Exter}^n V$. Then,

$$\operatorname{Exter} V = \bigoplus_{n\in\mathbb{N}} \underbrace{\overset{n}{\operatorname{Exter}} V}_{=V^{\otimes n}/Q_n(V)} = \bigoplus_{n\in\mathbb{N}} \left(V^{\otimes n} \diagup Q_n(V)\right) \cong \underbrace{\left(\bigoplus_{n\in\mathbb{N}} V^{\otimes n}\right)}_{=\otimes V} \diagup \underbrace{\left(\bigoplus_{n\in\mathbb{N}} Q_n(V)\right)}_{=Q(V)}$$
$$= (\otimes V) \diagup Q(V).$$

This is a canonical isomorphism, so we will use it to identify $\operatorname{Exter} V$ with $(\otimes V) \diagup Q(V)$. Since $Q(V)$ is a two-sided ideal of the k-algebra $\otimes V$, the quotient k-module $(\otimes V) \diagup Q(V)$ canonically becomes a k-algebra. Since $\operatorname{Exter} V = (\otimes V) \diagup Q(V)$, this means that $\operatorname{Exter} V$ becomes a k-algebra. We refer to this k-algebra as the *pseudoexterior algebra* of the k-module V.
We denote by exter_V the canonical projection $\otimes V \to (\otimes V) \diagup Q(V) = \operatorname{Exter} V$. Clearly, this map exter_V is a surjective k-algebra homomorphism. Besides, due to $\otimes V = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$ and $Q(V) = \bigoplus_{n\in\mathbb{N}} Q_n(V)$, it is clear that the canonical projection $\otimes V \to (\otimes V) \diagup Q(V)$ is the direct sum of the canonical projections $V^{\otimes n} \to V^{\otimes n} \diagup Q_n(V)$ over all $n \in \mathbb{N}$. Since the canonical projection $\otimes V \to (\otimes V) \diagup Q(V)$ is the map exter_V, whereas the canonical projection $V^{\otimes n} \to V^{\otimes n} \diagup Q_n(V)$ is the map $\operatorname{exter}_{V,n}$, this rewrites as follows: The map exter_V is the direct sum of the maps $\operatorname{exter}_{V,n}$ over all $n \in \mathbb{N}$.

We now prove a first, almost trivial result about the $Q(V)$:

Lemma 45. Let k be a commutative ring. Let V and W be two k-modules. Let $f : V \to W$ be a k-module homomorphism.
(a) Then, the k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ satisfies $(\otimes f)(Q(V)) \subseteq Q(W)$. Also, for every $n \in \mathbb{N}$, the k-module homomorphism $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ satisfies $f^{\otimes n}(Q_n(V)) \subseteq Q_n(W)$.
(b) Assume that f is surjective. Then, the k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ satisfies $(\otimes f)(Q(V)) = Q(W)$. Also, for every $n \in \mathbb{N}$, the k-module homomorphism $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ satisfies $f^{\otimes n}(Q_n(V)) = Q_n(W)$.

Proof of Lemma 45. **(a)** Fix some $n \in \mathbb{N}$. For every $i \in \{1, 2, \ldots, n-1\}$, let τ_i denote the transposition $(i, i+1) \in S_n$. Then, the definition of $Q_n(V)$ yields

$$Q_n(V) = \sum_{i=1}^{n-1} \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle, \tag{36}$$

whereas the definition of $Q_n(W)$ yields

$$Q_n(W) = \sum_{i=1}^{n-1} \left\langle w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} \mid (w_1, w_2, \ldots, w_n) \in W^n \right\rangle. \tag{37}$$

Now it is easy to see that every $i \in \{1, 2, \ldots, n-1\}$ satisfies

$$f^{\otimes n}\left(\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \ \mid\ (v_1, v_2, \ldots, v_n) \in V^n \right\rangle\right)$$
$$\subseteq \left\langle w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} \ \mid\ (w_1, w_2, \ldots, w_n) \in W^n \right\rangle . \tag{38}$$

[11] Now, (36) yields

$$f^{\otimes n}\left(Q_n\left(V\right)\right)$$
$$= f^{\otimes n}\left(\sum_{i=1}^{n-1}\left\langle v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\rangle\right)$$
$$= \sum_{i=1}^{n-1}\underbrace{f^{\otimes n}\left(\left\langle v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\rangle\right)}_{\subseteq\left\langle w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\rangle \text{ (by (38))}}$$
$$\left(\text{since } f^{\otimes n} \text{ is linear}\right)$$
$$\subseteq \sum_{i=1}^{n-1}\left\langle w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\rangle$$
$$= Q_n\left(W\right).$$

---

[11] *Proof.* Fix some $i\in\{1,2,\ldots,n-1\}$. Let S be the set

$$\left\{v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\}.$$

It is clear that every $(v_1,v_2,\ldots,v_n)\in V^n$ satisfies

$$f^{\otimes n}\left(v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)}\right)$$
$$= f(v_1)\otimes f(v_2)\otimes\cdots\otimes f(v_n) + f\left(v_{\tau_i(1)}\right)\otimes f\left(v_{\tau_i(2)}\right)\otimes\cdots\otimes f\left(v_{\tau_i(n)}\right) \qquad \left(\text{by the definition of } f^{\otimes n}\right)$$
$$\in \left\{w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\}$$
$$\subseteq \left\langle\left\{w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\}\right\rangle$$
$$= \left\langle w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\rangle.$$

In other words,

$$\left\{f^{\otimes n}\left(v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)}\right) \mid (v_1,v_2,\ldots,v_n)\in V^n\right\}$$
$$\subseteq \left\langle w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\rangle.$$

Since

$$\left\{f^{\otimes n}\left(v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)}\right) \mid (v_1,v_2,\ldots,v_n)\in V^n\right\}$$
$$= f^{\otimes n}\left(\underbrace{\left\{v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\}}_{=S}\right) = f^{\otimes n}(S),$$

this rewrites as

$$f^{\otimes n}(S)\subseteq \left\langle w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\rangle.$$

By Proposition 35 **(a)** (applied to $f^{\otimes n}(S)$, $W^{\otimes n}$ and $\left\langle w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\rangle$ instead of S, M and Q), this yields

$$\left\langle f^{\otimes n}(S)\right\rangle\subseteq \left\langle w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\rangle.$$

But by Proposition 35 **(b)** (applied to $f^{\otimes n}$, $V^{\otimes n}$ and $W^{\otimes n}$ instead of f, M and R), we have $f^{\otimes n}(\langle S\rangle) = \langle f^{\otimes n}(S)\rangle$. Thus,

$$f^{\otimes n}(\langle S\rangle) = \left\langle f^{\otimes n}(S)\right\rangle\subseteq \left\langle w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\rangle.$$

Since

$$\langle S\rangle = \left\langle\left\{v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\}\right\rangle$$
$$\left(\text{because } S = \left\{v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\}\right)$$
$$= \left\langle v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\rangle,$$

this becomes

$$f^{\otimes n}\left(\left\langle v_1\otimes v_2\otimes\cdots\otimes v_n + v_{\tau_i(1)}\otimes v_{\tau_i(2)}\otimes\cdots\otimes v_{\tau_i(n)} \mid (v_1,v_2,\ldots,v_n)\in V^n\right\rangle\right)$$
$$\subseteq \left\langle w_1\otimes w_2\otimes\cdots\otimes w_n + w_{\tau_i(1)}\otimes w_{\tau_i(2)}\otimes\cdots\otimes w_{\tau_i(n)} \mid (w_1,w_2,\ldots,w_n)\in W^n\right\rangle,$$

qed.

Thus, we have shown that

$$\text{for every } n \in \mathbb{N}, \text{ we have } f^{\otimes n}\left(Q_n\left(V\right)\right) \subseteq Q_n\left(W\right). \tag{39}$$

Now forget that we fixed n. Since the map $\otimes f$ is the direct sum of the maps $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ for all $n \in \mathbb{N}$, we have $\left(\otimes f\right)\left(x\right) = f^{\otimes n}\left(x\right)$ for every $n \in \mathbb{N}$ and every $x \in V^{\otimes n}$. Thus, for every $n \in \mathbb{N}$, we have $\left(\otimes f\right)\left(Q_n\left(V\right)\right) = \left\{ \underbrace{\left(\otimes f\right)\left(x\right)}_{\substack{=f^{\otimes n}(x) \\ (\text{since } x \in Q_n(V) \subseteq V^{\otimes n})}} \mid x \in Q_n\left(V\right) \right\} = \left\{ f^{\otimes n}\left(x\right) \mid x \in Q_n\left(V\right) \right\} = f^{\otimes n}\left(Q_n\left(V\right)\right)$.

The definition of $Q\left(W\right)$ yields $Q\left(W\right) = \bigoplus_{n \in \mathbb{N}} Q_n\left(W\right)$. Since direct sums are sums, this rewrites as $Q\left(W\right) = \sum_{n \in \mathbb{N}} Q_n\left(W\right)$.

Now, $Q\left(V\right) = \bigoplus_{n \in \mathbb{N}} Q_n\left(V\right) = \sum_{n \in \mathbb{N}} Q_n\left(V\right)$ (since direct sums are sums) and thus

$$\left(\otimes f\right)\left(Q\left(V\right)\right) = \left(\otimes f\right)\left(\sum_{n \in \mathbb{N}} Q_n\left(V\right)\right) = \sum_{n \in \mathbb{N}} \underbrace{\left(\otimes f\right)\left(Q_n\left(V\right)\right)}_{\substack{=f^{\otimes n}(Q_n(V)) \subseteq Q_n(W) \\ (\text{by (39)})}} \subseteq \sum_{n \in \mathbb{N}} Q_n\left(W\right) = Q\left(W\right).$$

This completes the proof of Lemma 45 **(a).**

(b) Fix some $n \in \mathbb{N}$. For every $i \in \{1, 2, \ldots, n-1\}$, it is easy to prove (using the surjectivity of f) that

$$\begin{aligned} & f^{\otimes n}\left(\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid \left(v_1, v_2, \ldots, v_n\right) \in V^n \right\rangle\right) \\ & = \left\langle w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} \mid \left(w_1, w_2, \ldots, w_n\right) \in W^n \right\rangle. \end{aligned} \tag{40}$$

Proof of (40). Fix some $i \in \{1, 2, \ldots, n-1\}$. Then, every $\left(w_1, w_2, \ldots, w_n\right) \in W^n$ satisfies

$$\begin{aligned} & w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} \\ & \in f^{\otimes n}\left(\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid \left(v_1, v_2, \ldots, v_n\right) \in V^n \right\rangle\right). \end{aligned}$$

[12] In other words,

$$\begin{aligned} & \left\{ w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} \mid \left(w_1, w_2, \ldots, w_n\right) \in W^n \right\} \\ & \subseteq f^{\otimes n}\left(\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid \left(v_1, v_2, \ldots, v_n\right) \in V^n \right\rangle\right). \end{aligned}$$

Applying Proposition 35 **(a)** to $\left\{ w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} \mid \left(w_1, w_2, \ldots, w_n\right) \in W^n \right\}$, $W^{\otimes n}$ and

[12] *Proof.* Let $\left(w_1, w_2, \ldots, w_n\right) \in W^n$. For every $i \in \{1, 2, \ldots, n\}$, there exists some $z_i \in V$ such that $w_i = f\left(z_i\right)$ (since f is surjective). Fix such a z_i for each $i \in \{1, 2, \ldots, n\}$. Then, $w_1 = f\left(z_1\right)$, $w_2 = f\left(z_2\right)$, ..., $w_n = f\left(z_n\right)$. Taking the tensor product of these equalities, we get $w_1 \otimes w_2 \otimes \cdots \otimes w_n = f\left(z_1\right) \otimes f\left(z_2\right) \otimes \cdots \otimes f\left(z_n\right)$. Also, since $w_i = f\left(z_i\right)$ for each $i \in \{1, 2, \ldots, n\}$, we have $w_{\tau_i(1)} = f\left(z_{\tau_i(1)}\right)$, $w_{\tau_i(2)} = f\left(z_{\tau_i(2)}\right)$, ..., $w_{\tau_i(n)} = f\left(z_{\tau_i(n)}\right)$. Taking the tensor product of these equalities, we get $w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} = f\left(z_{\tau_i(1)}\right) \otimes f\left(z_{\tau_i(2)}\right) \otimes \cdots \otimes f\left(z_{\tau_i(n)}\right)$.

$f^{\otimes n}\left(\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle\right)$ instead of S, M and Q), we conclude from this that

$$\left\langle \left\{ w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} \mid (w_1, w_2, \ldots, w_n) \in W^n \right\} \right\rangle$$
$$\subseteq f^{\otimes n}\left(\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle\right).$$

Thus,

$$\left\langle w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} \mid (w_1, w_2, \ldots, w_n) \in W^n \right\rangle$$
$$= \left\langle \left\{ w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)} \mid (w_1, w_2, \ldots, w_n) \in W^n \right\} \right\rangle$$
$$\subseteq f^{\otimes n}\left(\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle\right).$$

Combined with (38), this yields (40). Thus, we have proven (40).

In the proof of Lemma 45 **(a)**, we have used (38) to show that $f^{\otimes n}(Q_n(V)) \subseteq Q_n(W)$. In the same way, we can use (40) to prove that $f^{\otimes n}(Q_n(V)) = Q_n(W)$ (in the situation of Lemma 45 **(b)**).

So we have shown that

$$\text{for every } n \in \mathbb{N}, \text{ we have } f^{\otimes n}(Q_n(V)) = Q_n(W). \qquad (41)$$

In the proof of Lemma 45 **(a)**, we have used (39) to conclude that $(\otimes f)(Q(V)) \subseteq Q(W)$. In the same way, we can use (41) to conclude that $(\otimes f)(Q(V)) = Q(W)$ (in the situation of Lemma 45 **(b)**). This completes the proof of Lemma 45 **(b)**. □

[*Remark.* The above proof of Lemma 45 uses the definition of $Q(V)$ as $\bigoplus_{n \in \mathbb{N}} Q_n(V)$. We could just as well have proven Lemma 45 using the definition of $Q(V)$ as $(\otimes V) \cdot (Q_2(V)) \cdot (\otimes V)$.]

The pseudoexterior algebra is (just as most other constructions we did above) functorial in V. This means that:

Now, by the definition of $f^{\otimes n}$, we have

$$f^{\otimes n}\left(z_1 \otimes z_2 \otimes \cdots \otimes z_n + z_{\tau_i(1)} \otimes z_{\tau_i(2)} \otimes \cdots \otimes z_{\tau_i(n)}\right)$$
$$= \underbrace{f(z_1) \otimes f(z_2) \otimes \cdots \otimes f(z_n)}_{= w_1 \otimes w_2 \otimes \cdots \otimes w_n} + \underbrace{f\left(z_{\tau_i(1)}\right) \otimes f\left(z_{\tau_i(2)}\right) \otimes \cdots \otimes f\left(z_{\tau_i(n)}\right)}_{= w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)}}$$
$$= w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)},$$

so that

$$w_1 \otimes w_2 \otimes \cdots \otimes w_n + w_{\tau_i(1)} \otimes w_{\tau_i(2)} \otimes \cdots \otimes w_{\tau_i(n)}$$
$$= f^{\otimes n}\left(\underbrace{z_1 \otimes z_2 \otimes \cdots \otimes z_n + z_{\tau_i(1)} \otimes z_{\tau_i(2)} \otimes \cdots \otimes z_{\tau_i(n)}}_{\substack{\in \left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\} \\ \subseteq \left\langle \left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\} \right\rangle \\ = \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle}}\right)$$
$$\in f^{\otimes n}\left(\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle\right),$$

qed.

Definition 46. Let k be a commutative ring. Let V and W be two k-modules. Let $f : V \to W$ be a k-module homomorphism. Then, the k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ satisfies $(\otimes f)(Q(V)) \subseteq Q(W)$ (by Lemma 45 **(a)**), and thus gives rise to a k-algebra homomorphism $(\otimes V) \diagup Q(V) \to (\otimes W) \diagup Q(W)$. This latter k-algebra homomorphism will be denoted by $\operatorname{Exter} f$. Since $(\otimes V) \diagup Q(V) = \operatorname{Exter} V$ and $(\otimes W) \diagup Q(W) = \operatorname{Exter} W$, this homomorphism $\operatorname{Exter} f : (\otimes V) \diagup Q(V) \to (\otimes W) \diagup Q(W)$ is actually a homomorphism from $\operatorname{Exter} V$ to $\operatorname{Exter} W$.
By the construction of $\operatorname{Exter} f$, the diagram

$$\begin{array}{ccc} \otimes V & \xrightarrow{\otimes f} & \otimes W \\ {\scriptstyle \operatorname{exter}_V}\downarrow & & \downarrow{\scriptstyle \operatorname{exter}_W} \\ \operatorname{Exter} V & \xrightarrow[\operatorname{Exter} f]{} & \operatorname{Exter} W \end{array} \qquad (42)$$

commutes (since exter_V is the canonical projection $\otimes V \to \operatorname{Exter} V$ and since exter_W is the canonical projection $\otimes W \to \operatorname{Exter} W$).

As a consequence of Lemma 45 **(b)**, we have:

Proposition 47. Let k be a commutative ring. Let V and W be two k-modules. Let $f : V \to W$ be a surjective k-module homomorphism. Then:
(a) The k-module homomorphism $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ is surjective for every $n \in \mathbb{N}$.
(b) The k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ is surjective.
(c) The k-algebra homomorphism $\operatorname{Exter} f : \operatorname{Exter} V \to \operatorname{Exter} W$ is surjective.

Proof of Proposition 47. **(a)** Let $n \in \mathbb{N}$. Lemma 24 (applied to $f_i = f$) yields that the map $\underbrace{f \otimes f \otimes \cdots \otimes f}_{n \text{ times}}$ is surjective. Since $f^{\otimes n} = \underbrace{f \otimes f \otimes \cdots \otimes f}_{n \text{ times}}$, this yields that the map $f^{\otimes n}$ is surjective. This proves Proposition 47 **(a)**.

(b) We defined the map $\otimes f$ as the direct sum of the maps $f^{\otimes n}$ for all $n \in \mathbb{N}$. Since the maps $f^{\otimes n}$ are surjective (by Proposition 47 **(a)**), this yields that the map $\otimes f$ is surjective (since the direct sum of surjective maps is always surjective). This proves Proposition 47 **(b)**.

(c) Since the diagram (42) commutes, we have $\operatorname{exter}_W \circ (\otimes f) = (\operatorname{Exter} f) \circ \operatorname{exter}_V$. Now, the map exter_W is surjective (since it is the canonical projection $\otimes W \to \operatorname{Exter} W$), and the map $\otimes f$ is surjective (by Proposition 47 **(b)**). Hence, the map $\operatorname{exter}_W \circ (\otimes f)$ is surjective (since the composition of surjective maps is always surjective). Since $\operatorname{exter}_W \circ (\otimes f) = (\operatorname{Exter} f) \circ \operatorname{exter}_V$, this yields that the map $(\operatorname{Exter} f) \circ \operatorname{exter}_V$ is surjective. Hence, the map $\operatorname{Exter} f$ is surjective (because if α and β are two maps such that the composition $\alpha \circ \beta$ is surjective, then α must itself be surjective). This proves Proposition 47 **(c)**. □

0.13. The kernel of $\operatorname{Exter} f$

We will now formulate a result about the kernel of $\operatorname{Exter} f$ for a k-module map f (similar to Theorem 32, but with a twist):

Theorem 48. Let k be a commutative ring. Let V and V' be two k-modules, and let $f : V \to V'$ be a surjective k-module homomorphism. Then, the kernel of the map $\operatorname{Exter} f : \operatorname{Exter} V \to \operatorname{Exter} V'$ is

$$\begin{aligned}\operatorname{Ker}(\operatorname{Exter} f) &= (\operatorname{Exter} V) \cdot \operatorname{exter}_V(\operatorname{Ker} f) \cdot (\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V(\operatorname{Ker} f) \\ &= \operatorname{exter}_V(\operatorname{Ker} f) \cdot (\operatorname{Exter} V).\end{aligned}$$

Here, $\operatorname{Ker} f$ is considered a k-submodule of $\otimes V$ by means of the inclusion $\operatorname{Ker} f \subseteq V = V^{\otimes 1} \subseteq \otimes V$.

Note that the $\operatorname{Ker}(\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V(\operatorname{Ker} f) \cdot (\operatorname{Exter} V)$ part of this theorem will be a rather quick application of Proposition 15 to the results of Theorem 32 and Proposition 47 **(c)**. It is slightly less clear how to show the $(\operatorname{Exter} V) \cdot \operatorname{exter}_V(\operatorname{Ker} f) \cdot (\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V(\operatorname{Ker} f) = \operatorname{exter}_V(\operatorname{Ker} f) \cdot (\operatorname{Exter} V)$ part. We will do this using the following lemma:

Lemma 49. Let k be a commutative ring. Let V be a k-module. Let A be a k-algebra, and let $\pi : \otimes V \to A$ be a surjective k-algebra homomorphism. Let M be a k-submodule of A such that $M \cdot \pi(V) \subseteq M$. Then, M is a right ideal of A.

Proof of Lemma 49. We claim that for every $n \in \mathbb{N}$, we have

$$M \cdot \pi\left(V^{\otimes n}\right) \subseteq M. \tag{43}$$

Proof of (43). We are going to prove (43) by induction over n:

Induction base: For $n = 0$, we have

$$M \cdot \pi\Bigg(\underbrace{V^{\otimes n}}_{=V^{\otimes 0}=k=k\cdot 1}\Bigg) = M \cdot \underbrace{\pi(k \cdot 1)}_{\substack{=k\cdot\pi(1) \\ \text{(since } \pi \text{ is } k\text{-linear)}}} = M \cdot k \cdot \underbrace{\pi(1)}_{\substack{=1 \text{ (since } \pi \text{ is a} \\ k\text{-algebra homomorphism)}}} = M \cdot k \cdot 1 = M$$

(since M is a k-module). Thus, (43) is true for $n = 0$. This completes the induction base.

Induction step: Let $m \in \mathbb{N}$. Assume that (43) holds for $n = m$. We now must prove that (43) holds for $n = m + 1$.

Since (43) holds for $n = m$, we have $M \cdot \pi(V^{\otimes m}) \subseteq M$. Since $V^{\otimes(m+1)} = V \cdot V^{\otimes m}$ [13], we have

$$\pi\left(V^{\otimes(m+1)}\right) = \pi\left(V \cdot V^{\otimes m}\right) = \pi(V) \cdot \pi\left(V^{\otimes m}\right) \qquad \text{(since } \pi \text{ is a } k\text{-algebra homomorphism)},$$

[13] *Proof.* Every $v \in V$ and $w \in V^{\otimes m}$ satisfy $v \cdot w = v \otimes w$ (by (3), applied to $n = 1$, $a = v$ and $b = w$). In other words, every $(v, w) \in V \times V^{\otimes m}$ satisfies $v \cdot w = v \otimes w$. Thus,

$$V \cdot V^{\otimes m} = \Bigg\langle \underbrace{v \cdot w}_{=v\otimes w} \mid (v, w) \in V \times V^{\otimes m} \Bigg\rangle = \left\langle v \otimes w \mid (v, w) \in V \times V^{\otimes m} \right\rangle.$$

Compared with $V^{\otimes(m+1)} = V \otimes V^{\otimes m} = \langle v \otimes w \mid (v, w) \in V \times V^{\otimes m} \rangle$ (since a tensor product is generated by its pure tensors), this yields $V^{\otimes(m+1)} = V \cdot V^{\otimes m}$, qed.

so that

$$M \cdot \pi\left(V^{\otimes(m+1)}\right) = \underbrace{M \cdot \pi(V)}_{\subseteq M} \cdot \pi\left(V^{\otimes m}\right) \subseteq M \cdot \pi\left(V^{\otimes m}\right) \subseteq M.$$

In other words, (43) holds for $n = m + 1$. This completes the induction step.

Thus, the induction proof of (43) is complete.

Now that (43) is proven, we notice that

$$\otimes V = \bigoplus_{n\in\mathbb{N}} V^{\otimes n} = \sum_{n\in\mathbb{N}} V^{\otimes n} \qquad \text{(since direct sums are sums)},$$

so that

$$\pi(\otimes V) = \pi\left(\sum_{n\in\mathbb{N}} V^{\otimes n}\right) = \sum_{n\in\mathbb{N}} \pi\left(V^{\otimes n}\right) \qquad (\text{since } \pi \text{ is linear}).$$

Since $\pi(\otimes V) = A$ (because π is surjective), this becomes $A = \sum_{n\in\mathbb{N}} \pi(V^{\otimes n})$. Thus,

$$M \cdot A = M \cdot \sum_{n\in\mathbb{N}} \pi\left(V^{\otimes n}\right) = \sum_{n\in\mathbb{N}} \underbrace{M \cdot \pi\left(V^{\otimes n}\right)}_{\subseteq M \text{ (by (43))}} \subseteq \sum_{n\in\mathbb{N}} M \subseteq M$$

(since M is a k-module). In other words, M is a right ideal of A. This proves Lemma 49. □

Corollary 50. Let k be a commutative ring. Let V be a k-module, and let W be a k-submodule of V. Then,

$$(\operatorname{Exter} V) \cdot \operatorname{exter}_V(W) \cdot (\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V(W) = \operatorname{exter}_V(W) \cdot (\operatorname{Exter} V).$$

Here, W is considered a k-submodule of $\otimes V$ by means of the inclusion $W \subseteq V = V^{\otimes 1} \subseteq \otimes V$.

Proof of Corollary 50. **(i)** We have $V \cdot W + Q_2(V) = W \cdot V + Q_2(V)$ (as k-submodules of $\otimes V$).

Proof. Let $(v, w) \in V \times W$ be arbitrary. Then,

$$\underbrace{v \cdot w}_{\substack{= v \otimes w \\ \text{(by (27), applied to} \\ a=v,\ b=w,\ n=1,\ m=1)}} + \underbrace{w \cdot v}_{\substack{= w \otimes v \\ \text{(by (27), applied to} \\ a=w,\ b=v,\ n=1,\ m=1)}}$$
$$= v \otimes w + w \otimes v \in \left\{v_1 \otimes v_2 + v_2 \otimes v_1 \mid (v_1, v_2) \in V^2\right\}$$
$$\subseteq \left\langle \left\{v_1 \otimes v_2 + v_2 \otimes v_1 \mid (v_1, v_2) \in V^2\right\} \right\rangle = \left\langle v_1 \otimes v_2 + v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\rangle = Q_2(V)$$

(by Corollary 39), so that

$$v \cdot w \in Q_2(V) - \underbrace{w}_{\in W} \cdot \underbrace{v}_{\in V} = Q_2(V) - W \cdot V = Q_2(V) + W \cdot V \qquad (\text{since } W \cdot V \text{ is a } k\text{-module}).$$

Since this holds for all $(v, w) \in V \times W$, we thus have

$$\{v \cdot w \mid (v, w) \in V \times W\} \subseteq Q_2(V) + W \cdot V.$$

Applying Proposition 35 **(a)** to $\otimes V$, $\{v \cdot w \mid (v,w) \in V \times W\}$ and $Q_2(V) + W \cdot V$ instead of M, S and Q, we see that this yields

$$\langle \{v \cdot w \mid (v,w) \in V \times W\} \rangle \subseteq Q_2(V) + W \cdot V.$$

Since

$$\langle \{v \cdot w \mid (v,w) \in V \times W\} \rangle = \langle v \cdot w \mid (v,w) \in V \times W \rangle = V \cdot W,$$

this rewrites as $V \cdot W \subseteq Q_2(V) + W \cdot V$. Thus,

$$V \cdot W + Q_2(V) \subseteq (Q_2(V) + W \cdot V) + Q_2(V) = W \cdot V + \underbrace{Q_2(V) + Q_2(V)}_{\substack{\subseteq Q_2(V) \\ (\text{since } Q_2(V) \text{ is a } k\text{-module})}} \subseteq W \cdot V + Q_2(V).$$

Combining this with the fact that $W \cdot V + Q_2(V) \subseteq V \cdot W + Q_2(V)$ (which can be proven completely analogously), we obtain that $V \cdot W + Q_2(V) = W \cdot V + Q_2(V)$. This proves **(i)**.

(ii) We have $\operatorname{exter}_V(V) \cdot \operatorname{exter}_V(W) = \operatorname{exter}_V(W) \cdot \operatorname{exter}_V(V)$ (as k-submodules of $\operatorname{Exter} V$).

Proof. Since exter_V is a k-algebra homomorphism, we have

$$\operatorname{exter}_V(V \cdot W + Q_2(V)) = \operatorname{exter}_V(V) \cdot \operatorname{exter}_V(W) + \operatorname{exter}_V(Q_2(V)).$$

But since $\operatorname{exter}_V(Q_2(V)) = 0$ (because exter_V is the canonical projection $\otimes V \to (\otimes V) \diagup Q(V)$, and thus $Q(V) = \operatorname{Ker} \operatorname{exter}_V$, so that $Q_2(V) \subseteq \bigoplus_{n \in \mathbb{N}} Q_n(V) = Q(V) = \operatorname{Ker} \operatorname{exter}_V$), this rewrites as

$$\operatorname{exter}_V(V \cdot W + Q_2(V)) = \operatorname{exter}_V(V) \cdot \operatorname{exter}_V(W) + 0 = \operatorname{exter}_V(V) \cdot \operatorname{exter}_V(W).$$

Similarly,

$$\operatorname{exter}_V(W \cdot V + Q_2(V)) = \operatorname{exter}_V(W) \cdot \operatorname{exter}_V(V).$$

Now,

$$\begin{aligned} \operatorname{exter}_V(V) \cdot \operatorname{exter}_V(W) &= \operatorname{exter}_V(V \cdot W + Q_2(V)) = \operatorname{exter}_V(W \cdot V + Q_2(V)) \\ &\qquad (\text{since } V \cdot W + Q_2(V) = W \cdot V + Q_2(V) \text{ by } \textbf{(i)}) \\ &= \operatorname{exter}_V(W) \cdot \operatorname{exter}_V(V), \end{aligned}$$

and thus **(ii)** is proven.

(iii) We have $(\operatorname{Exter} V) \cdot \operatorname{exter}_V(W) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V(W) \cdot (\operatorname{Exter} V)$.

Proof. We have

$$(\operatorname{Exter} V) \cdot \underbrace{\operatorname{exter}_V(W) \cdot \operatorname{exter}_V(V)}_{= \operatorname{exter}_V(V) \cdot \operatorname{exter}_V(W)} = \underbrace{(\operatorname{Exter} V) \cdot \operatorname{exter}_V(V)}_{\subseteq \operatorname{Exter} V} \cdot \operatorname{exter}_V(W) \subseteq (\operatorname{Exter} V) \cdot \operatorname{exter}_V(W).$$

By Lemma 49 (applied to $A = \operatorname{Exter} V$, $\pi = \operatorname{exter}_V$ and $M = (\operatorname{Exter} V) \cdot \operatorname{exter}_V(W)$), this yields that $(\operatorname{Exter} V) \cdot \operatorname{exter}_V(W)$ is a right ideal of $\operatorname{Exter} V$. In other words, $(\operatorname{Exter} V) \cdot \operatorname{exter}_V(W) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V(W) \cdot (\operatorname{Exter} V)$. This proves **(iii)**.

(iv) We have $\operatorname{exter}_V(W) \cdot (\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V(W) \cdot (\operatorname{Exter} V)$.

Proof. The proof of **(iv)** is analogous to the proof of **(iii)** (but this time we need an analogue of Lemma 49 for left instead of right ideals).

(v) Corollary 50 clearly follows by combining **(iii)** and **(iv)**. The proof of Corollary 50 is thus complete. □

Proof of Theorem 48. Applying the commutative diagram (42) to $W = V'$, we obtain the commutative diagram

$$\begin{array}{ccc} \otimes V & \xrightarrow{\otimes f} & \otimes V' \\ {\scriptstyle \operatorname{exter}_V}\downarrow & & \downarrow{\scriptstyle \operatorname{exter}_{V'}} \\ \operatorname{Exter} V & \xrightarrow[\operatorname{Exter} f]{} & \operatorname{Exter} V' \end{array} . \qquad (44)$$

But it is easy to see that

$$\operatorname{Ker} \operatorname{exter}_{V'} \subseteq (\otimes f) (\operatorname{Ker} \operatorname{exter}_V)$$

[14] and that the map exter_V is surjective (since exter_V is the canonical projection $\otimes V \to \operatorname{Exter} V$). Hence, we can apply Proposition 15 to the commutative diagram (44), and conclude that $\operatorname{Ker} (\operatorname{Exter} f) = \operatorname{exter}_V (\operatorname{Ker} (\otimes f))$. Since $\operatorname{Ker} (\otimes f) = (\otimes V) \cdot (\operatorname{Ker} f) \cdot (\otimes V)$ by Theorem 32, this becomes

$$\operatorname{Ker} (\operatorname{Exter} f) = \operatorname{exter}_V ((\otimes V) \cdot (\operatorname{Ker} f) \cdot (\otimes V)) = \operatorname{exter}_V (\otimes V) \cdot \operatorname{exter}_V (\operatorname{Ker} f) \cdot \operatorname{exter}_V (\otimes V)$$
$$(\text{since } \operatorname{exter}_V \text{ is a } k\text{-algebra homomorphism}) .$$

Since $\operatorname{exter}_V (\otimes V) = \operatorname{Exter} V$ (because exter_V is surjective), this becomes

$$\operatorname{Ker} (\operatorname{Exter} f) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V (\operatorname{Ker} f) \cdot (\operatorname{Exter} V) .$$

Combined with the equality

$$(\operatorname{Exter} V) \cdot \operatorname{exter}_V (\operatorname{Ker} f) \cdot (\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V (\operatorname{Ker} f) = \operatorname{exter}_V (\operatorname{Ker} f) \cdot (\operatorname{Exter} V)$$

(which follows from Corollary 50, applied to $W = \operatorname{Ker} f$), this yields

$$\operatorname{Ker} (\operatorname{Exter} f) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V (\operatorname{Ker} f) \cdot (\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V (\operatorname{Ker} f)$$
$$= \operatorname{exter}_V (\operatorname{Ker} f) \cdot (\operatorname{Exter} V) .$$

This proves Theorem 48. □

Here is a way to rewrite Theorem 48:

Corollary 51. Let k be a commutative ring. Let V be a k-module. Let W be a k-submodule of V, and let $f : V \to V \diagup W$ be the canonical projection.
(a) Then, the kernel of the map $\operatorname{Exter} f : \operatorname{Exter} V \to \operatorname{Exter} (V \diagup W)$ is

$$\operatorname{Ker} (\operatorname{Exter} f) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V (W) \cdot (\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V (W)$$
$$= \operatorname{exter}_V (W) \cdot (\operatorname{Exter} V) .$$

[14] *Proof.* Since exter_V is the canonical projection $\otimes V \to (\otimes V) \diagup Q(V)$, we have $\operatorname{Ker} \operatorname{exter}_V = Q(V)$. Similarly, $\operatorname{Ker} \operatorname{exter}_{V'} = Q(V')$. But Lemma 45 **(b)** (applied to $W = V'$) yields that $(\otimes f)(Q(V)) = Q(V')$. Thus,

$$\operatorname{Ker} \operatorname{exter}_{V'} = Q(V') = (\otimes f) \left(\underbrace{Q(V)}_{= \operatorname{Ker} \operatorname{exter}_V} \right) = (\otimes f) (\operatorname{Ker} \operatorname{exter}_V) ,$$

qed.

Here, W is considered a k-submodule of $\otimes V$ by means of the inclusion $W \subseteq V = V^{\otimes 1} \subseteq \otimes V$.
(b) We have

$$(\operatorname{Exter} V) \diagup ((\operatorname{Exter} V) \cdot \operatorname{exter}_V (W)) \cong \operatorname{Exter} (V \diagup W) \qquad \text{as } k\text{-modules.}$$

Proof of Corollary 51. Since f is the canonical projection $V \to V \diagup W$, it is clear that f is surjective and that $\operatorname{Ker} f = W$. Now, we can apply Theorem 48 to $V' = V \diagup W$ and conclude that

$$\begin{aligned} \operatorname{Ker} (\operatorname{Exter} f) &= (\operatorname{Exter} V) \cdot \operatorname{exter}_V (\operatorname{Ker} f) \cdot (\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V (\operatorname{Ker} f) \\ &= \operatorname{exter}_V (\operatorname{Ker} f) \cdot (\operatorname{Exter} V) . \end{aligned}$$

Since $\operatorname{Ker} f = W$, this simplifies to

$$\begin{aligned} \operatorname{Ker} (\operatorname{Exter} f) &= (\operatorname{Exter} V) \cdot \operatorname{exter}_V (W) \cdot (\operatorname{Exter} V) = (\operatorname{Exter} V) \cdot \operatorname{exter}_V (W) \\ &= \operatorname{exter}_V (W) \cdot (\operatorname{Exter} V) . \end{aligned}$$

This proves Corollary 51 **(a)**.

Since the map $f : V \to V \diagup W$ is surjective, the map $\operatorname{Exter} f : \operatorname{Exter} V \to \operatorname{Exter} (V \diagup W)$ is also surjective (by Proposition 47 **(c)**, applied to $V \diagup W$ instead of W), and thus we have $(\operatorname{Exter} f) (\operatorname{Exter} V) = \operatorname{Exter} (V \diagup W)$. But by the isomorphism theorem, $(\operatorname{Exter} f) (\operatorname{Exter} V) \cong (\operatorname{Exter} V) \diagup \operatorname{Ker} (\operatorname{Exter} f)$ as k-modules. Thus,

$$\begin{aligned} \operatorname{Exter} (V \diagup W) &= (\operatorname{Exter} f) (\operatorname{Exter} V) \cong (\operatorname{Exter} V) \diagup \underbrace{\operatorname{Ker} (\operatorname{Exter} f)}_{= (\operatorname{Exter} V) \cdot \operatorname{exter}_V (W)} \\ &= (\operatorname{Exter} V) \diagup ((\operatorname{Exter} V) \cdot \operatorname{exter}_V (W)) \qquad \text{as } k\text{-modules.} \end{aligned}$$

This proves Corollary 51 **(b)**. □

0.14. The symmetric algebra

In the previous two subsections (Subsections 0.12 and 0.13), we have studied the pseudoexterior algebra $\operatorname{Exter} V$ of a k-module V. Many properties of the pseudoexterior algebra $\operatorname{Exter} V$ are shared by its more well-known analogue - the symmetric algebra $\operatorname{Sym} V$. Pretty much all of our above-proven properties of $\operatorname{Exter} V$ have analogues for $\operatorname{Sym} V$. We are now going to formulate these analogues, without proving them (because their proofs are completely analogous to the proofs of the properties of $\operatorname{Exter} V$ that we did above). First, before we define the symmetric algebra $\operatorname{Sym} V$, let us define the symmetric powers $\operatorname{Sym}^n V$:

Definition 52. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Let $K_n (V)$ be the k-submodule

$$\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n - v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \ \mid \ ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n \right\rangle$$

of the k-module $V^{\otimes n}$ (where we are using Convention 34, and are denoting the n-th symmetric group by S_n).

The factor k-module $V^{\otimes n} \diagup K_n(V)$ is called the *n-th symmetric power* of the k-module V and will be denoted by $\operatorname{Sym}^n V$. We denote by $\operatorname{sym}_{V,n}$ the canonical projection $V^{\otimes n} \to V^{\otimes n} \diagup K_n(V) = \operatorname{Sym}^n V$. Clearly, this map $\operatorname{sym}_{V,n}$ is a surjective k-module homomorphism.

We should understand these notions $K_n(V)$, $\operatorname{Sym}^n V$ and $\operatorname{sym}_{V,n}$ as analogues of the notions $Q_n(V)$, $\operatorname{Exter}^n V$ and $\operatorname{exter}_{V,n}$ from Definition 36, respectively. Here is an analogue of Proposition 38:

Proposition 53. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Then,

$$K_n(V) = \sum_{i=1}^{n-1} \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle,$$

where τ_i denotes the transposition $(i, i+1) \in S_n$.

Proof of Proposition 53. The proof of this Proposition 53 is completely analogous to the proof of Proposition 38 (up to some replacing of $+$ signs by $-$ signs and some removal of powers of -1) and can be found in §5.1 of the long (detailed) version of [3]. □

Here is the analogue of Corollary 39:

Corollary 54. Let k be a commutative ring. Let V be a k-module. Then,

$$K_2(V) = \left\langle v_1 \otimes v_2 - v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\rangle.$$

Proof of Corollary 54. Again, the proof of Corollary 54 is completely analogous to the proof of Corollary 39. □

Next, the analogue of Lemma 41:

Lemma 55. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Let $i \in \{1, 2, \ldots, n-1\}$.
Then,

$$\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n - v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle$$
$$= V^{\otimes(i-1)} \cdot (K_2(V)) \cdot V^{\otimes(n-1-i)},$$

where τ_i denotes the transposition $(i, i+1) \in S_n$. Here, we consider $V^{\otimes n}$ as a k-submodule of $\otimes V$.

Proof of Lemma 55. The proof of Lemma 55 is completely analogous to the proof of Lemma 41. □

Next, the analogue of Corollary 42:

Corollary 56. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Then,

$$K_n(V) = \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (K_2(V)) \cdot V^{\otimes(n-1-i)}$$

(this is an equality between k-submodules of $\otimes V$, where $K_n(V)$ becomes such a k-submodule by means of the inclusion $K_n(V) \subseteq V^{\otimes n} \subseteq \otimes V$). Here, the multiplication on the right hand side is multiplication inside the k-algebra $\otimes V$.

Proof of Corollary 56. The proof of Corollary 56 is completely analogous to the proof of Corollary 42. □

Now the analogue of Theorem 43:

Theorem 57. Let k be a commutative ring. Let V be a k-module. We know that $K_n(V)$ is a k-submodule of $V^{\otimes n}$ for every $n \in \mathbb{N}$. Thus, $\bigoplus_{n\in\mathbb{N}} K_n(V)$ is a k-submodule of $\bigoplus_{n\in\mathbb{N}} V^{\otimes n} = \otimes V$. This k-submodule satisfies

$$\bigoplus_{n\in\mathbb{N}} K_n(V) = (\otimes V) \cdot (K_2(V)) \cdot (\otimes V).$$

Proof of Theorem 57. The proof of Theorem 57 is completely analogous to the proof of Theorem 43. □

Now we can finally define the symmetric algebra, similarly to Definition 44:

Definition 58. Let k be a commutative ring. Let V be a k-module.
By Theorem 57, the two k-submodules $\bigoplus_{n\in\mathbb{N}} K_n(V)$ and $(\otimes V) \cdot (K_2(V)) \cdot (\otimes V)$ of $\otimes V$ are identic (where $\bigoplus_{n\in\mathbb{N}} K_n(V)$ becomes a k-submodule of $\otimes V$ in the same way as explained in Theorem 57). We denote these two identic k-submodules by $K(V)$. In other words, we define $K(V)$ by

$$K(V) = \bigoplus_{n\in\mathbb{N}} K_n(V) = (\otimes V) \cdot (K_2(V)) \cdot (\otimes V).$$

Since $K(V) = (\otimes V) \cdot (K_2(V)) \cdot (\otimes V)$, it is clear that $K(V)$ is a two-sided ideal of the k-algebra $\otimes V$.
Now we define a k-module $\operatorname{Sym} V$ as the direct sum $\bigoplus_{n\in\mathbb{N}} \operatorname{Sym}^n V$. Then,

$$\operatorname{Sym} V = \bigoplus_{n\in\mathbb{N}} \underbrace{\operatorname{Sym}^n V}_{= V^{\otimes n} / K_n(V)} = \bigoplus_{n\in\mathbb{N}} \left(V^{\otimes n} / K_n(V)\right) \cong \underbrace{\left(\bigoplus_{n\in\mathbb{N}} V^{\otimes n}\right)}_{= \otimes V} \Big/ \underbrace{\left(\bigoplus_{n\in\mathbb{N}} K_n(V)\right)}_{= K(V)}$$
$$= (\otimes V) / K(V).$$

This is a canonical isomorphism, so we will use it to identify $\operatorname{Sym} V$ with $(\otimes V) \diagup K(V)$. Since $K(V)$ is a two-sided ideal of the k-algebra $\otimes V$, the quotient k-module $(\otimes V) \diagup K(V)$ canonically becomes a k-algebra. Since $\operatorname{Sym} V = (\otimes V) \diagup K(V)$, this means that $\operatorname{Sym} V$ becomes a k-algebra. We refer to this k-algebra as the *symmetric algebra* of the k-module V.
We denote by sym_V the canonical projection $\otimes V \to (\otimes V) \diagup K(V) = \operatorname{Sym} V$. Clearly, this map sym_V is a surjective k-algebra homomorphism. Besides, due to $\otimes V = \bigoplus_{n \in \mathbb{N}} V^{\otimes n}$ and $K(V) = \bigoplus_{n \in \mathbb{N}} K_n(V)$, it is clear that the canonical projection $\otimes V \to (\otimes V) \diagup K(V)$ is the direct sum of the canonical projections $V^{\otimes n} \to V^{\otimes n} \diagup K_n(V)$ over all $n \in \mathbb{N}$. Since the canonical projection $\otimes V \to (\otimes V) \diagup K(V)$ is the map sym_V, whereas the canonical projection $V^{\otimes n} \to V^{\otimes n} \diagup K_n(V)$ is the map $\operatorname{sym}_{V,n}$, this rewrites as follows: The map sym_V is the direct sum of the maps $\operatorname{sym}_{V,n}$ over all $n \in \mathbb{N}$.
When $v_1, v_2, \ldots, v_n$ are some elements of V, one often abbreviates the element $\operatorname{sym}_V(v_1 \otimes v_2 \otimes \cdots \otimes v_n)$ of $\operatorname{Sym} V$ by $v_1 v_2 \cdots v_n$. (We will not use this abbreviation in this following.)

We should think of the notions $K(V)$, $\operatorname{Sym} V$ and sym_V as analogues of the notions $Q(V)$, $\operatorname{Exter} V$ and exter_V from Definition 44, respectively. The next result provides an analogue of Lemma 45:

Lemma 59. Let k be a commutative ring. Let V and W be two k-modules. Let $f : V \to W$ be a k-module homomorphism.
(a) Then, the k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ satisfies $(\otimes f)(K(V)) \subseteq K(W)$. Also, for every $n \in \mathbb{N}$, the k-module homomorphism $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ satisfies $f^{\otimes n}(K_n(V)) \subseteq K_n(W)$.
(b) Assume that f is surjective. Then, the k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ satisfies $(\otimes f)(K(V)) = K(W)$. Also, for every $n \in \mathbb{N}$, the k-module homomorphism $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ satisfies $f^{\otimes n}(K_n(V)) = K_n(W)$.

The following definition mirrors Definition 46:

Definition 60. Let k be a commutative ring. Let V and W be two k-modules. Let $f : V \to W$ be a k-module homomorphism. Then, the k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ satisfies $(\otimes f)(K(V)) \subseteq K(W)$ (by Lemma 59 **(a)**), and thus gives rise to a k-algebra homomorphism $(\otimes V) \diagup K(V) \to (\otimes W) \diagup K(W)$. This latter k-algebra homomorphism will be denoted by $\operatorname{Sym} f$. Since $(\otimes V) \diagup K(V) = \operatorname{Sym} V$ and $(\otimes W) \diagup K(W) = \operatorname{Sym} W$, this homomorphism $\operatorname{Sym} f : (\otimes V) \diagup K(V) \to (\otimes W) \diagup K(W)$ is actually a homomorphism from $\operatorname{Sym} V$ to $\operatorname{Sym} W$.
By the construction of $\operatorname{Sym} f$, the diagram

$$\begin{array}{ccc} \otimes V & \xrightarrow{\otimes f} & \otimes W \\ {\scriptstyle \operatorname{sym}_V} \downarrow & & \downarrow {\scriptstyle \operatorname{sym}_W} \\ \operatorname{Sym} V & \xrightarrow[\operatorname{Sym} f]{} & \operatorname{Sym} W \end{array} \qquad (45)$$

commutes (since sym_V is the canonical projection $\otimes V \to \operatorname{Sym} V$ and since sym_W is the canonical projection $\otimes W \to \operatorname{Sym} W$).

Needless to say, the notion $\operatorname{Sym} f$ introduced in this definition is an analogue of the notion $\operatorname{Exter} f$ introduced in Definition 46.

Here is the analogue of Proposition 47:

Proposition 61. Let k be a commutative ring. Let V and W be two k-modules. Let $f : V \to W$ be a surjective k-module homomorphism. Then:
(a) The k-module homomorphism $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ is surjective for every $n \in \mathbb{N}$.
(b) The k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ is surjective.
(c) The k-algebra homomorphism $\operatorname{Sym} f : \operatorname{Sym} V \to \operatorname{Sym} W$ is surjective.

Proof of Proposition 61. The proof of this Proposition 61 is completely analogous to the proof of Proposition 47 (and parts **(a)** and **(b)** are even the same). □

So much for analogues of the results of Subsection 0.12. Now let us formulate the analogues of the results of Subsection 0.13. First, the analogue of Theorem 48:

Theorem 62. Let k be a commutative ring. Let V and V' be two k-modules, and let $f : V \to V'$ be a surjective k-module homomorphism. Then, the kernel of the map $\operatorname{Sym} f : \operatorname{Sym} V \to \operatorname{Sym} V'$ is

$$\begin{aligned} \operatorname{Ker}(\operatorname{Sym} f) &= (\operatorname{Sym} V) \cdot \operatorname{sym}_V(\operatorname{Ker} f) \cdot (\operatorname{Sym} V) = (\operatorname{Sym} V) \cdot \operatorname{sym}_V(\operatorname{Ker} f) \\ &= \operatorname{sym}_V(\operatorname{Ker} f) \cdot (\operatorname{Sym} V). \end{aligned}$$

Here, $\operatorname{Ker} f$ is considered a k-submodule of $\otimes V$ by means of the inclusion $\operatorname{Ker} f \subseteq V = V^{\otimes 1} \subseteq \otimes V$.

Proof of Theorem 62. The proof of this Theorem 62 is completely analogous to that of Theorem 48. □

The analogue of Corollary 50 comes next:

Corollary 63. Let k be a commutative ring. Let V be a k-module, and let W be a k-submodule of V. Then,

$$(\operatorname{Sym} V) \cdot \operatorname{sym}_V(W) \cdot (\operatorname{Sym} V) = (\operatorname{Sym} V) \cdot \operatorname{sym}_V(W) = \operatorname{sym}_V(W) \cdot (\operatorname{Sym} V).$$

Here, W is considered a k-submodule of $\otimes V$ by means of the inclusion $W \subseteq V = V^{\otimes 1} \subseteq \otimes V$.

Proof of Corollary 63. Expectedly, the proof of Corollary 63 is analogous to the proof of Corollary 50. □

Finally, the analogue of Corollary 51:

Corollary 64. Let k be a commutative ring. Let V be a k-module. Let W be a k-submodule of V, and let $f : V \to V \diagup W$ be the canonical projection.
(a) Then, the kernel of the map $\operatorname{Sym} f : \operatorname{Sym} V \to \operatorname{Sym}(V \diagup W)$ is

$$\operatorname{Ker}(\operatorname{Sym} f) = (\operatorname{Sym} V) \cdot \operatorname{sym}_V(W) \cdot (\operatorname{Sym} V) = (\operatorname{Sym} V) \cdot \operatorname{sym}_V(W) = \operatorname{sym}_V(W) \cdot (\operatorname{Sym} V).$$

Here, W is considered a k-submodule of $\otimes V$ by means of the inclusion $W \subseteq V = V^{\otimes 1} \subseteq \otimes V$.
(b) We have

$$(\operatorname{Sym} V) \diagup ((\operatorname{Sym} V) \cdot \operatorname{sym}_V (W)) \cong \operatorname{Sym} (V \diagup W) \qquad \text{as } k\text{-modules.}$$

Proof of Corollary 64. The proof of Corollary 64 is analogous to the proof of Corollary 51. □

0.15. The exterior algebra

Now we are going to study the exterior algebra $\wedge V$ of a k-module V. This algebra is rather similar, but not completely analogous to $\operatorname{Exter} V$ and $\operatorname{Sym} V$. We are going to again formulate properties similar to corresponding properties of $\operatorname{Exter} V$ and $\operatorname{Sym} V$; but this time, some of these properties will require different proofs, so we will not always be able to skip their proofs by referring to analogy. Still some of the proofs will be very similar to the corresponding proofs for $\operatorname{Exter} V$ we gave in Subsections 0.12 and 0.13 (some others will be not). First, before we define the exterior algebra $\wedge V$, let us define the exterior powers $\wedge^n V$:

Definition 65. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Let $R_n (V)$ be the k-submodule

$$\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid ((v_1, v_2, \ldots, v_n), (i, j)) \in V^n \times \{1, 2, \ldots, n\}^2 ;\ i \neq j;\ v_i = v_j \right\rangle$$

of the k-module $V^{\otimes n}$ (where we are using Convention 34).
The factor k-module $V^{\otimes n} \diagup R_n (V)$ is called the *n-th exterior power* of the k-module V and will be denoted by $\wedge^n V$. We denote by $\operatorname{wedge}_{V,n}$ the canonical projection $V^{\otimes n} \to V^{\otimes n} \diagup R_n (V) = \wedge^n V$. Clearly, this map $\operatorname{wedge}_{V,n}$ is a surjective k-module homomorphism.

We should understand these notions $R_n (V)$, $\wedge^n V$ and $\operatorname{wedge}_{V,n}$ as analogues of the notions $Q_n (V)$, $\operatorname{Exter}^n V$ and $\operatorname{exter}_{V,n}$ from Definition 36, respectively. First something very basic - an analogue of Corollary 39:

Corollary 66. Let k be a commutative ring. Let V be a k-module. Then,

$$R_2 (V) = \langle v \otimes v \mid v \in V \rangle .$$

Proof of Corollary 66. We have the inclusions

$$\left\{ v_1 \otimes v_2 \mid ((v_1, v_2), (i, j)) \in V^2 \times \{1, 2\}^2 ;\ i \neq j;\ v_i = v_j \right\} \subseteq \{ v \otimes v \mid v \in V \}$$

[15] and

$$\{ v \otimes v \mid v \in V \} \subseteq \left\{ v_1 \otimes v_2 \mid ((v_1, v_2), (i, j)) \in V^2 \times \{1, 2\}^2 ;\ i \neq j;\ v_i = v_j \right\}$$

[15] *Proof.* Let $p \in \left\{ v_1 \otimes v_2 \mid ((v_1, v_2), (i, j)) \in V^2 \times \{1, 2\}^2 ;\ i \neq j;\ v_i = v_j \right\}$. Then, there exists some $((v_1, v_2), (i, j)) \in V^2 \times \{1, 2\}^2$ such that $i \neq j$ and $v_i = v_j$ and $p = v_1 \otimes v_2$. Consider this

[16]. Combining these two inclusions, we get

$$\left\{v_1 \otimes v_2 \mid \left((v_1, v_2), (i,j)\right) \in V^2 \times \{1,2\}^2;\ i \neq j;\ v_i = v_j\right\} = \{v \otimes v \mid v \in V\}.$$

But by the definition of $R_2(V)$, we have

$$\begin{aligned} R_2(V) &= \left\langle v_1 \otimes v_2 \mid \left((v_1, v_2), (i,j)\right) \in V^2 \times \{1,2\}^2;\ i \neq j;\ v_i = v_j\right\rangle \\ &= \left\langle \underbrace{\left\{v_1 \otimes v_2 \mid \left((v_1, v_2), (i,j)\right) \in V^2 \times \{1,2\}^2;\ i \neq j;\ v_i = v_j\right\}}_{=\{v \otimes v \mid v \in V\}} \right\rangle \\ &= \langle \{v \otimes v \mid v \in V\} \rangle = \langle v \otimes v \mid v \in V \rangle. \end{aligned}$$

This proves Corollary 66. □

Here is an analogue of Proposition 38:

Proposition 67. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Then,

$$R_n(V) = \sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle.$$

While the proof of this proposition is not too much harder than that of Proposition 38, it is better understood when split into lemmas. Here is the first one:

Lemma 68. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Let $\widetilde{R}_n(V)$ denote the k-submodule

$$\sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle$$

of $V^{\otimes n}$. Then, $Q_n(V) \subseteq \widetilde{R}_n(V)$.

$\left((v_1, v_2), (i,j)\right)$. Then, $(i,j) \in \{1,2\}^2$. Since $i \neq j$, this yields that either ($i = 1$ and $j = 2$) or ($j = 1$ and $i = 2$). In each of these two cases, we have $v_1 = v_2$ (in fact, in the case ($i = 1$ and $j = 2$), the equation $v_i = v_j$ rewrites as $v_1 = v_2$; and in the other case ($j = 1$ and $i = 2$), the equation $v_i = v_j$ rewrites as $v_2 = v_1$, so that $v_1 = v_2$). Hence, we have $v_1 = v_2$. Thus, $p = \underbrace{v_1}_{=v_2} \otimes v_2 = v_2 \otimes v_2 \in \{v \otimes v \mid v \in V\}$.

We have thus shown that $p \in \{v \otimes v \mid v \in V\}$ for every $p \in \left\{v_1 \otimes v_2 \mid \left((v_1, v_2), (i,j)\right) \in V^2 \times \{1,2\}^2;\ i \neq j;\ v_i = v_j\right\}$. Hence, $\left\{v_1 \otimes v_2 \mid \left((v_1, v_2), (i,j)\right) \in V^2 \times \{1,2\}^2;\ i \neq j;\ v_i = v_j\right\} \subseteq \{v \otimes v \mid v \in V\}$, qed.

[16] *Proof.* Let $p \in \{v \otimes v \mid v \in V\}$. Then, there exists $v \in V$ such that $p = v \otimes v$. Consider this v. Then, there exists some $\left((v_1, v_2), (i,j)\right) \in V^2 \times \{1,2\}^2$ with $i \neq j$ and $v_i = v_j$ such that $p = v_1 \otimes v_2$ (namely, $\left((v_1, v_2), (i,j)\right) = \left((v, v), (1,2)\right)$). Hence, $p \in \left\{v_1 \otimes v_2 \mid \left((v_1, v_2), (i,j)\right) \in V^2 \times \{1,2\}^2;\ i \neq j;\ v_i = v_j\right\}$. Since we have proven this for every $p \in \{v \otimes v \mid v \in V\}$, we have thus shown that $\{v \otimes v \mid v \in V\} \subseteq \left\{v_1 \otimes v_2 \mid \left((v_1, v_2), (i,j)\right) \in V^2 \times \{1,2\}^2;\ i \neq j;\ v_i = v_j\right\}$.

Proof of Lemma 68. **(i)** Every $i \in \{1, 2, \ldots, n-1\}$ satisfies

$$\begin{aligned} &\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle \\ &\subseteq \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n; \ v_i = v_{i+1} \rangle, \end{aligned} \tag{46}$$

where τ_i denotes the transposition $(i, i+1) \in S_n$.

Proof. Fix some $i \in \{1, 2, \ldots, n-1\}$. Now let $\mathfrak{W}$ be the set

$$\{w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid (w_1, w_2, \ldots, w_n) \in V^n; \ w_i = w_{i+1}\}.$$

Then,

$$w_1 \otimes w_2 \otimes \cdots \otimes w_n \in \mathfrak{W} \qquad \text{for every } (w_1, w_2, \ldots, w_n) \in V^n \text{ satisfying } w_i = w_{i+1}. \tag{47}$$

Fix some arbitrary $(v_1, v_2, \ldots, v_n) \in V^n$. Define a tensor $A \in V^{\otimes(i-1)}$ by $A = v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1}$. Define a tensor $C \in V^{\otimes(n-1-i)}$ by $C = v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n$. Then, recalling Convention 12, we have

$$\begin{aligned} v_1 \otimes v_2 \otimes \cdots \otimes v_n &= \underbrace{(v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1})}_{=A} \otimes (v_i \otimes v_{i+1}) \otimes \underbrace{(v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n)}_{=C} \\ &= A \otimes (v_i \otimes v_{i+1}) \otimes C. \end{aligned} \tag{48}$$

On the other hand, every $j \in \{1, 2, \ldots, i-1\}$ satisfies $\tau_i(j) = j$ (since τ_i is the transposition $(i, i+1)$) and thus $v_{\tau_i(j)} = v_j$. In other words, we have the equalities $v_{\tau_i(1)} = v_1$, $v_{\tau_i(2)} = v_2$, $\ldots$, $v_{\tau_i(i-1)} = v_{i-1}$. Taking the tensor product of these equalities yields

$$v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(i-1)} = v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} = A.$$

Every $j \in \{i+2, i+3, \ldots, n\}$ satisfies $\tau_i(j) = j$ (since τ_i is the transposition $(i, i+1)$) and thus $v_{\tau_i(j)} = v_j$. In other words, we have the equalities $v_{\tau_i(i+2)} = v_{i+2}$, $v_{\tau_i(i+3)} = v_{i+3}$, $\ldots$, $v_{\tau_i(n)} = v_n$. Taking the tensor product of these equalities yields

$$v_{\tau_i(i+2)} \otimes v_{\tau_i(i+3)} \otimes \cdots \otimes v_{\tau_i(n)} = v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n = C.$$

Since τ_i is the transposition $(i, i+1)$, we have $\tau_i(i) = i+1$ and $\tau_i(i+1) = i$. These equalities yield $v_{\tau_i(i)} = v_{i+1}$ and $v_{\tau_i(i+1)} = v_i$, respectively.

Now,

$$\begin{aligned} &v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \\ &= \underbrace{(v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(i-1)})}_{=A} \otimes \left(\underbrace{v_{\tau_i(i)}}_{=v_{i+1}} \otimes \underbrace{v_{\tau_i(i+1)}}_{=v_i} \right) \otimes \underbrace{(v_{\tau_i(i+2)} \otimes v_{\tau_i(i+3)} \otimes \cdots \otimes v_{\tau_i(n)})}_{=C} \\ &= A \otimes (v_{i+1} \otimes v_i) \otimes C. \end{aligned}$$

Adding this to (48), we get

$$\begin{aligned} &v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \\ &= A \otimes (v_i \otimes v_{i+1}) \otimes C + A \otimes (v_{i+1} \otimes v_i) \otimes C \\ &= A \otimes (v_i \otimes v_{i+1} + v_{i+1} \otimes v_i) \otimes C. \end{aligned} \tag{49}$$

But it is easy to see that

$$A \otimes p \otimes p \otimes C \in \mathfrak{W} \qquad \text{for every } p \in V. \tag{50}$$

[17] Since $\mathfrak{W} \subseteq \langle \mathfrak{W} \rangle$, this yields

$$A \otimes p \otimes p \otimes C \in \langle \mathfrak{W} \rangle \qquad \text{for every } p \in V. \tag{52}$$

Now,

$$\begin{aligned} v_i \otimes v_{i+1} + v_{i+1} \otimes v_i &= \underbrace{\left(v_i \otimes v_i + v_i \otimes v_{i+1} + v_{i+1} \otimes v_i + v_{i+1} \otimes v_{i+1}\right)}_{=(v_i+v_{i+1})\otimes(v_i+v_{i+1})} - v_i \otimes v_i - v_{i+1} \otimes v_{i+1} \\ &= \left(v_i + v_{i+1}\right) \otimes \left(v_i + v_{i+1}\right) - v_i \otimes v_i - v_{i+1} \otimes v_{i+1}. \end{aligned}$$

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[17] *Proof of (50).* Let $p \in V$. Define an n-tuple $(w_1, w_2, \ldots, w_n) \in V^n$ by

$$\left(w_\ell = \begin{cases} v_\ell, \text{ if } \ell < i; \\ p, \text{ if } \ell = i; \\ p, \text{ if } \ell = i+1; \\ v_\ell, \text{ if } \ell > i+1 \end{cases} \qquad \text{for every } \ell \in \{1, 2, \ldots, n\} \right). \tag{51}$$

Then, every $\ell \in \{1, 2, \ldots, i-1\}$ satisfies $w_\ell = \begin{cases} v_\ell, \text{ if } \ell < i; \\ p, \text{ if } \ell = i; \\ p, \text{ if } \ell = i+1; \\ v_\ell, \text{ if } \ell > i+1 \end{cases} = v_\ell$ (since $\ell < i$). In other words, we have the equalities $w_1 = v_1$, $w_2 = v_2$, $\ldots$, $w_{i-1} = v_{i-1}$. Taking the tensor product of these equalities, we get $w_1 \otimes w_2 \otimes \cdots \otimes w_{i-1} = v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} = A$.

Also, every $\ell \in \{i+2, i+3, \ldots, n\}$ satisfies $w_\ell = \begin{cases} v_\ell, \text{ if } \ell < i; \\ p, \text{ if } \ell = i; \\ p, \text{ if } \ell = i+1; \\ v_\ell, \text{ if } \ell > i+1 \end{cases} = v_\ell$ (since $\ell > i+1$). In other words, we have the equalities $w_{i+2} = v_{i+2}$, $w_{i+3} = v_{i+3}$, $\ldots$, $w_n = v_n$. Taking the tensor product of these equalities, we get $w_{i+2} \otimes w_{i+3} \otimes \cdots \otimes w_n = v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n = C$.

Applying (51) to $\ell = i$, we get $w_i = \begin{cases} v_i, \text{ if } i < i; \\ p, \text{ if } i = i; \\ p, \text{ if } i = i+1; \\ v_i, \text{ if } i > i+1 \end{cases} = p$ (since $i = i$). Applying (51) to $\ell = i+1$, we get $w_i = \begin{cases} v_{i+1}, \text{ if } i+1 < i; \\ p, \text{ if } i+1 = i; \\ p, \text{ if } i+1 = i+1; \\ v_{i+1}, \text{ if } i+1 > i+1 \end{cases} = p$ (since $i+1 = i+1$).

Now,

$$w_1 \otimes w_2 \otimes \cdots \otimes w_n = \underbrace{\left(w_1 \otimes w_2 \otimes \cdots \otimes w_{i-1}\right)}_{=A} \otimes \underbrace{w_i}_{=p} \otimes \underbrace{w_{i+1}}_{=p} \otimes \underbrace{\left(w_{i+2} \otimes w_{i+3} \otimes \cdots \otimes w_n\right)}_{=C} = A \otimes p \otimes p \otimes C.$$

Since $w_1 \otimes w_2 \otimes \cdots \otimes w_n \in \mathfrak{W}$ (by (47)), we thus have $A \otimes p \otimes p \otimes C \in \mathfrak{W}$. This proves (50).

Thus, (49) becomes

$$\begin{aligned}
&v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \\
&= A \otimes \underbrace{(v_i \otimes v_{i+1} + v_{i+1} \otimes v_i)}_{=(v_i+v_{i+1})\otimes(v_i+v_{i+1})-v_i\otimes v_i - v_{i+1}\otimes v_{i+1}} \otimes C \\
&= A \otimes ((v_i + v_{i+1}) \otimes (v_i + v_{i+1}) - v_i \otimes v_i - v_{i+1} \otimes v_{i+1}) \otimes C \\
&= \underbrace{A \otimes (v_i + v_{i+1}) \otimes (v_i + v_{i+1}) \otimes C}_{\in \langle \mathfrak{W} \rangle \text{ (by (52), applied to } p = v_i + v_{i+1})} - \underbrace{A \otimes v_i \otimes v_i \otimes C}_{\in \langle \mathfrak{W} \rangle \text{ (by (52), applied to } p = v_i)} - \underbrace{A \otimes v_{i+1} \otimes v_{i+1} \otimes C}_{\in \langle \mathfrak{W} \rangle \text{ (by (52), applied to } p = v_{i+1})} \\
&\in \langle \mathfrak{W} \rangle - \langle \mathfrak{W} \rangle - \langle \mathfrak{W} \rangle \subseteq \langle \mathfrak{W} \rangle \qquad (\text{since } \langle \mathfrak{W} \rangle \text{ is a } k\text{-module}).
\end{aligned}$$

We now forget that we fixed $(v_1, v_2, \ldots, v_n)$. What we have proven is that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \in \langle \mathfrak{W} \rangle .$$

In other words,

$$\left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\} \subseteq \langle \mathfrak{W} \rangle .$$

Hence, Proposition 35 **(a)** (applied to $V^{\otimes n}$, $\left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\}$ and $\langle \mathfrak{W} \rangle$ instead of M, S and Q) yields

$$\left\langle \left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\} \right\rangle \subseteq \langle \mathfrak{W} \rangle .$$

Thus,

$$\begin{aligned}
&\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle \\
&= \left\langle \left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\} \right\rangle \\
&\subseteq \langle \mathfrak{W} \rangle = \langle \{ w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid (w_1, w_2, \ldots, w_n) \in V^n ;\ w_i = w_{i+1} \} \rangle \\
&\qquad (\text{since } \mathfrak{W} = \{ w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid (w_1, w_2, \ldots, w_n) \in V^n ;\ w_i = w_{i+1} \}) \\
&= \langle w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid (w_1, w_2, \ldots, w_n) \in V^n ;\ w_i = w_{i+1} \rangle \\
&= \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n ;\ v_i = v_{i+1} \rangle \\
&\qquad (\text{here, we renamed } (w_1, w_2, \ldots, w_n) \text{ as } (v_1, v_2, \ldots, v_n)).
\end{aligned}$$

This proves **(i)**.

(ii) For every $i \in \{1, 2, \ldots, n-1\}$, let τ_i denote the transposition $(i, i+1) \in S_n$. By Proposition 38, we have

$$\begin{aligned}
Q_n(V) &= \sum_{i=1}^{n-1} \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n + v_{\tau_i(1)} \otimes v_{\tau_i(2)} \otimes \cdots \otimes v_{\tau_i(n)} \mid (v_1, v_2, \ldots, v_n) \in V^n \right\rangle \\
&\subseteq \sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n ;\ v_i = v_{i+1} \rangle \qquad (\text{by (46)}) \\
&= \widetilde{R}_n(V) .
\end{aligned}$$

This proves Lemma 68. □

Our next step is the following lemma:

Lemma 69. In the situation of Lemma 68, we have $\widetilde{R}_n(V) \subseteq R_n(V)$.

Proof of Lemma 69. First fix some $\mathbf{I} \in \{1, 2, \ldots, n-1\}$. Fix some $(w_1, w_2, \ldots, w_n) \in V^n$ satisfying $w_\mathbf{I} = w_{\mathbf{I}+1}$. Then, the pair $((w_1, w_2, \ldots, w_n), (\mathbf{I}, \mathbf{I}+1)) \in V^n \times \{1, 2, \ldots, n\}^2$ satisfies $\mathbf{I} \neq \mathbf{I}+1$ and $v_\mathbf{I} = v_{\mathbf{I}+1}$. Therefore,

$$\begin{aligned}
& w_1 \otimes w_2 \otimes \cdots \otimes w_n \\
& \in \left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid ((v_1, v_2, \ldots, v_n), (i,j)) \in V^n \times \{1, 2, \ldots, n\}^2;\ i \neq j;\ v_i = v_j \right\} \\
& \subseteq \left\langle \left\{ v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid ((v_1, v_2, \ldots, v_n), (i,j)) \in V^n \times \{1, 2, \ldots, n\}^2;\ i \neq j;\ v_i = v_j \right\} \right\rangle \\
& = \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid ((v_1, v_2, \ldots, v_n), (i,j)) \in V^n \times \{1, 2, \ldots, n\}^2;\ i \neq j;\ v_i = v_j \right\rangle \\
& = R_n(V).
\end{aligned}$$

Now forget that we fixed some $(w_1, w_2, \ldots, w_n) \in V^n$ satisfying $w_\mathbf{I} = w_{\mathbf{I}+1}$. We have thus proven that every $(w_1, w_2, \ldots, w_n) \in V^n$ satisfying $w_\mathbf{I} = w_{\mathbf{I}+1}$ satisfies $w_1 \otimes w_2 \otimes \cdots \otimes w_n \in R_n(V)$. In other words, we have proven that

$$\{ w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid (w_1, w_2, \ldots, w_n) \in V^n;\ w_\mathbf{I} = w_{\mathbf{I}+1} \} \subseteq R_n(V).$$

Hence, Proposition 35 **(a)** (applied to $V^{\otimes n}$, $\{ w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid (w_1, w_2, \ldots, w_n) \in V^n;\ w_\mathbf{I} = w_{\mathbf{I}+1} \}$ and $R_n(V)$ instead of M, S and Q) yields

$$\langle \{ w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid (w_1, w_2, \ldots, w_n) \in V^n;\ w_\mathbf{I} = w_{\mathbf{I}+1} \} \rangle \subseteq R_n(V).$$

So we have

$$\begin{aligned}
& \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_\mathbf{I} = v_{\mathbf{I}+1} \rangle \\
& = \langle w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid (w_1, w_2, \ldots, w_n) \in V^n;\ w_\mathbf{I} = w_{\mathbf{I}+1} \rangle \\
& \qquad \left(\text{here, we renamed } (v_1, v_2, \ldots, v_n) \text{ as } (w_1, w_2, \ldots, w_n) \right) \\
& = \langle \{ w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid (w_1, w_2, \ldots, w_n) \in V^n;\ w_\mathbf{I} = w_{\mathbf{I}+1} \} \rangle \subseteq R_n(V). \qquad (53)
\end{aligned}$$

Now forget that we fixed some $\mathbf{I} \in \{1, 2, \ldots, n-1\}$. We have now proven that every $\mathbf{I} \in \{1, 2, \ldots, n-1\}$ satisfies (53). Now,

$$\begin{aligned}
\widetilde{R}_n(V) &= \sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle \\
&= \sum_{\mathbf{I}=1}^{n-1} \underbrace{\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_\mathbf{I} = v_{\mathbf{I}+1} \rangle}_{\subseteq R_n(V) \text{ (by (53))}} \\
&\qquad \left(\text{here, we renamed the summation index } i \text{ as } \mathbf{I} \right) \\
&\subseteq \sum_{\mathbf{I}=1}^{n-1} R_n(V) \subseteq R_n(V) \qquad \left(\text{since } R_n(V) \text{ is a } k\text{-module} \right).
\end{aligned}$$

This proves Lemma 69. □

Our final lemma is:

Lemma 70. In the situation of Lemma 68, we have $R_n(V) \subseteq \widetilde{R}_n(V)$.

Proof of Lemma 70. **(i)** It is clear that every $\mathbf{I} \in \{1, 2, \ldots, n-1\}$ satisfies

$$\begin{aligned}
&\{v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_{\mathbf{I}} = v_{\mathbf{I}+1}\} \\
&\subseteq \langle \{v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_{\mathbf{I}} = v_{\mathbf{I}+1}\} \rangle \\
&= \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_{\mathbf{I}} = v_{\mathbf{I}+1} \rangle \\
&\subseteq \sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle = \widetilde{R}_n(V).
\end{aligned}$$

In other words, for every $\mathbf{I} \in \{1, 2, \ldots, n-1\}$,

$$\text{every } (v_1, v_2, \ldots, v_n) \in V^n \text{ such that } v_{\mathbf{I}} = v_{\mathbf{I}+1} \text{ satisfies } v_1 \otimes v_2 \otimes \cdots \otimes v_n \in \widetilde{R}_n(V). \tag{54}$$

(ii) Every $((w_1, w_2, \ldots, w_n), (\mathbf{I}, \mathbf{J})) \in V^n \times \{1, 2, \ldots, n\}^2$ satisfying $\mathbf{I} \neq \mathbf{J}$ and $w_{\mathbf{I}} = w_{\mathbf{J}}$ must satisfy $w_1 \otimes w_2 \otimes \cdots \otimes w_n \in \widetilde{R}_n(V)$.

Proof. Fix some $((w_1, w_2, \ldots, w_n), (\mathbf{I}, \mathbf{J})) \in V^n \times \{1, 2, \ldots, n\}^2$ satisfying $\mathbf{I} \neq \mathbf{J}$ and $w_{\mathbf{I}} = w_{\mathbf{J}}$.

Then, $(w_1, w_2, \ldots, w_n) \in V^n$ and $(\mathbf{I}, \mathbf{J}) \in \{1, 2, \ldots, n\}^2$.

We can WLOG assume that $\mathbf{I} \leq \mathbf{J}$ (since otherwise, we could just transpose $\mathbf{I}$ with $\mathbf{J}$, and nothing would change (because each of the conditions $\mathbf{I} \neq \mathbf{J}$ and $w_{\mathbf{I}} = w_{\mathbf{J}}$ is clearly symmetric with respect to $\mathbf{I}$ and $\mathbf{J}$)). So let us assume this. Then, $\mathbf{I} < \mathbf{J}$ (since $\mathbf{I} \leq \mathbf{J}$ and $\mathbf{I} \neq \mathbf{J}$). Thus, $\mathbf{I} < \mathbf{J} \leq n$, so that $\mathbf{I} \leq n-1$ (since $\mathbf{I}$ and n are integers), and thus $\mathbf{I} + 1 \leq n$. This allows us to speak of the vector $w_{\mathbf{I}+1}$.

Now, there clearly exists a permutation $\tau \in S_n$ such that $\tau(\mathbf{I}) = \mathbf{I}$ and $\tau(\mathbf{I}+1) = \mathbf{J}$.[18] Consider such a τ. From $\tau(\mathbf{I}) = \mathbf{I}$, we obtain $w_{\tau(\mathbf{I})} = w_{\mathbf{I}} = w_{\mathbf{J}} = w_{\tau(\mathbf{I}+1)}$ (since $\mathbf{J} = \tau(\mathbf{I}+1)$).

Now, since $(w_1, w_2, \ldots, w_n) \in V^n$ and $\tau \in S_n$, we have $((w_1, w_2, \ldots, w_n), \tau) \in V^n \times S_n$, so that

$$\begin{aligned}
&w_1 \otimes w_2 \otimes \cdots \otimes w_n - (-1)^\tau w_{\tau(1)} \otimes w_{\tau(2)} \otimes \cdots \otimes w_{\tau(n)} \\
&\in \left\{v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n\right\} \\
&\subseteq \left\langle \left\{v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n\right\} \right\rangle \\
&= \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n \right\rangle \\
&= Q_n(V) \subseteq \widetilde{R}_n(V) \qquad \text{(by Lemma 68)}.
\end{aligned}$$

[18] *Proof.* We distinguish between two cases:

Case 1: We have $\mathbf{J} = \mathbf{I} + 1$.

Case 2: We have $\mathbf{J} \neq \mathbf{I} + 1$.

First consider Case 1. In this case, the permutation $\mathrm{id} \in S_n$ satisfies $\mathrm{id}(\mathbf{I}) = \mathbf{I}$ and $\mathrm{id}(\mathbf{I}+1) = \mathbf{I}+1 = \mathbf{J}$. Hence, in Case 1, there exists a permutation $\tau \in S_n$ such that $\tau(\mathbf{I}) = \mathbf{I}$ and $\tau(\mathbf{I}+1) = \mathbf{J}$ (namely, $\tau = \mathrm{id}$).

Now let us consider Case 2. In this case, $\mathbf{J} \neq \mathbf{I} + 1$. Hence, the transposition $(\mathbf{J}, \mathbf{I}+1) \in S_n$ is well-defined, and it satisfies $(\mathbf{J}, \mathbf{I}+1)(\mathbf{I}) = \mathbf{I}$ (since $\mathbf{J} \neq \mathbf{I}$ and $\mathbf{I}+1 \neq \mathbf{I}$) and $(\mathbf{J}, \mathbf{I}+1)(\mathbf{I}+1) = \mathbf{J}$. Hence, in Case 2, there exists a permutation $\tau \in S_n$ such that $\tau(\mathbf{I}) = \mathbf{I}$ and $\tau(\mathbf{I}+1) = \mathbf{J}$ (namely, $\tau = (\mathbf{J}, \mathbf{I}+1)$).

We have thus proven in each of the two possible cases that there exists a permutation $\tau \in S_n$ such that $\tau(\mathbf{I}) = \mathbf{I}$ and $\tau(\mathbf{I}+1) = \mathbf{J}$.

This completes the proof that there always exists a permutation $\tau \in S_n$ such that $\tau(\mathbf{I}) = \mathbf{I}$ and $\tau(\mathbf{I}+1) = \mathbf{J}$.

On the other hand, the n-tuple $\left(w_{\tau(1)}, w_{\tau(2)}, \ldots, w_{\tau(n)}\right) \in V^n$ satisfies $w_{\tau(\mathbf{I})} = w_{\tau(\mathbf{I}+1)}$. Hence, (54) (applied to $(v_1, v_2, \ldots, v_n) = \left(w_{\tau(1)}, w_{\tau(2)}, \ldots, w_{\tau(n)}\right)$) yields $w_{\tau(1)} \otimes w_{\tau(2)} \otimes \cdots \otimes w_{\tau(n)} \in \widetilde{R}_n(V)$.

Now,

$$\begin{aligned}
& w_1 \otimes w_2 \otimes \cdots \otimes w_n \\
& = \underbrace{\left(w_1 \otimes w_2 \otimes \cdots \otimes w_n - (-1)^\tau w_{\tau(1)} \otimes w_{\tau(2)} \otimes \cdots \otimes w_{\tau(n)}\right)}_{\in \widetilde{R}_n(V)} + (-1)^\tau \underbrace{w_{\tau(1)} \otimes w_{\tau(2)} \otimes \cdots \otimes w_{\tau(n)}}_{\in \widetilde{R}_n(V)} \\
& \in \widetilde{R}_n(V) + (-1)^\tau \widetilde{R}_n(V) \subseteq \widetilde{R}_n(V) \qquad \left(\text{since } \widetilde{R}_n(V) \text{ is a } k\text{-module}\right).
\end{aligned}$$

This proves **(ii)**.

(iii) According to **(ii)**, every $((w_1, w_2, \ldots, w_n), (\mathbf{I}, \mathbf{J})) \in V^n \times \{1, 2, \ldots, n\}^2$ satisfying $\mathbf{I} \neq \mathbf{J}$ and $w_{\mathbf{I}} = w_{\mathbf{J}}$ must satisfy $w_1 \otimes w_2 \otimes \cdots \otimes w_n \in \widetilde{R}_n(V)$.

In other words,

$$\begin{aligned}
& \left\{w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid ((w_1, w_2, \ldots, w_n), (\mathbf{I}, \mathbf{J})) \in V^n \times \{1, 2, \ldots, n\}^2 ;\ \mathbf{I} \neq \mathbf{J};\ w_{\mathbf{I}} = w_{\mathbf{J}}\right\} \\
& \subseteq \widetilde{R}_n(V).
\end{aligned}$$

Thus, Proposition 35 **(a)** (applied to $V^{\otimes n}$, $\left\{w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid ((w_1, w_2, \ldots, w_n), (\mathbf{I}, \mathbf{J})) \in V^n \times \{1, 2, \ldots, n\}^2 ;\ \mathbf{I} \neq \mathbf{J};\ w_{\mathbf{I}} = w_{\mathbf{J}}\right\}$ and $\widetilde{R}_n(V)$ instead of M, S and Q) yields

$$\begin{aligned}
& \left\langle\left\{w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid ((w_1, w_2, \ldots, w_n), (\mathbf{I}, \mathbf{J})) \in V^n \times \{1, 2, \ldots, n\}^2 ;\ \mathbf{I} \neq \mathbf{J};\ w_{\mathbf{I}} = w_{\mathbf{J}}\right\}\right\rangle \\
& \subseteq \widetilde{R}_n(V).
\end{aligned}$$

Now,

$$\begin{aligned}
& R_n(V) \\
& = \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid ((v_1, v_2, \ldots, v_n), (i, j)) \in V^n \times \{1, 2, \ldots, n\}^2 ;\ i \neq j;\ v_i = v_j\right\rangle \\
& = \left\langle\left\{w_1 \otimes w_2 \otimes \cdots \otimes w_n \mid ((w_1, w_2, \ldots, w_n), (\mathbf{I}, \mathbf{J})) \in V^n \times \{1, 2, \ldots, n\}^2 ;\ \mathbf{I} \neq \mathbf{J};\ w_{\mathbf{I}} = w_{\mathbf{J}}\right\}\right\rangle \\
& \subseteq \widetilde{R}_n(V).
\end{aligned}$$

This proves Lemma 70. □

Proof of Proposition 67. Lemma 70 yields $R_n(V) \subseteq \widetilde{R}_n(V)$. Lemma 69 yields $\widetilde{R}_n(V) \subseteq R_n(V)$. Combining these two inclusions, we obtain $\widetilde{R}_n(V) = R_n(V)$. Thus,

$$R_n(V) = \widetilde{R}_n(V) = \sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1}\rangle .$$

This proves Proposition 67. □

The analogue of Lemma 41 looks as follows:

Lemma 71. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Let $i \in \{1, 2, \ldots, n-1\}$.
Then,

$$\begin{aligned}&\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \right\rangle \\ &= V^{\otimes(i-1)} \cdot (R_2(V)) \cdot V^{\otimes(n-1-i)}.\end{aligned}$$

Here, we consider $V^{\otimes n}$ as a k-submodule of $\otimes V$.

Proof of Lemma 71. Let V^{Δ} be the k-submodule $\{(v, v) \mid v \in V\}$ of V^2. Then, $V^{\Delta} = \{(v, w) \in V^2 \mid v = w\}$. Hence, for every $(v_i, v_{i+1}) \in V^2$, we have

$$(v_i, v_{i+1}) \in V^{\Delta} \text{ if and only if } v_i = v_{i+1}. \tag{55}$$

Define a map $a : V^{i-1} \to V^{\otimes(i-1)}$ by

$$\left(a(v_1, v_2, \ldots, v_{i-1}) = v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \qquad \text{for every } (v_1, v_2, \ldots, v_{i-1}) \in V^{i-1}\right).$$

Define a map $b : V^{\Delta} \to V^{\otimes 2}$ by

$$\left(b(v_i, v_{i+1}) = v_i \otimes v_{i+1} \qquad \text{for every } (v_i, v_{i+1}) \in V^{\Delta}\right).$$

(Of course, every $(v_i, v_{i+1}) \in V^{\Delta}$ in fact satisfies $v_i = v_{i+1}$ by the definition of V^{Δ}; but we still use different letters for v_i and v_{i+1} here to make this notation match another one.) Define a map $c : V^{n-1-i} \to V^{\otimes(n-1-i)}$ by

$$\left(c(v_{i+2}, v_{i+3}, \ldots, v_n) = v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n \qquad \text{for every } (v_{i+2}, v_{i+3}, \ldots, v_n) \in V^{n-1-i}\right).$$

Since $V^{\otimes(i-1)}$, $V^{\otimes 2}$ and $V^{\otimes(n-1-i)}$ are k-submodules of $\otimes V$, we can consider all three maps a, b and c as maps to the set $\otimes V$.

It is now easy to see that every $(v_1, v_2, \ldots, v_n) \in V^n$ such that $v_i = v_{i+1}$ satisfies $(v_i, v_{i+1}) \in V^{\Delta}$ and

$$v_1 \otimes v_2 \otimes \cdots \otimes v_n = a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) \cdot c(v_{i+2}, v_{i+3}, \ldots, v_n),$$

where the multiplication on the right hand side is the multiplication in the tensor algebra $\otimes V$. [19] Thus,

$$\begin{aligned}
&\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle \\
&= \langle a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) \cdot c(v_{i+2}, v_{i+3}, \ldots, v_n) \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle \\
&= \left\langle a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) \cdot c(v_{i+2}, v_{i+3}, \ldots, v_n) \right. \\
&\qquad \left. \mid ((v_1, v_2, \ldots, v_{i-1}), (v_i, v_{i+1}), (v_{i+2}, v_{i+3}, \ldots, v_n)) \in V^{i-1} \times V^2 \times V^{n-1-i};\ \underbrace{v_i = v_{i+1}}_{\substack{\text{by (55), this is} \\ \text{equivalent to} \\ (v_i, v_{i+1}) \in V^\Delta}} \right\rangle \\
&\qquad \left(\begin{array}{c} \text{here, we substituted the triple } ((v_1, v_2, \ldots, v_{i-1}), (v_i, v_{i+1}), (v_{i+2}, v_{i+3}, \ldots, v_n)) \\ \text{for the } n\text{-tuple } (v_1, v_2, \ldots, v_n) \end{array} \right) \\
&= \langle a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) \cdot c(v_{i+2}, v_{i+3}, \ldots, v_n) \\
&\quad \mid ((v_1, v_2, \ldots, v_{i-1}), (v_i, v_{i+1}), (v_{i+2}, v_{i+3}, \ldots, v_n)) \in V^{i-1} \times V^2 \times V^{n-1-i};\ (v_i, v_{i+1}) \in V^\Delta \rangle \\
&= \left\langle a(x) b(y) c(z) \mid \underbrace{(x, y, z) \in V^{i-1} \times V^2 \times V^{n-1-i};\ y \in V^\Delta}_{\text{this is equivalent to } (x,y,z) \in V^{i-1} \times V^\Delta \times V^{n-1-i}} \right\rangle \\
&\qquad (\text{here, we renamed } ((v_1, v_2, \ldots, v_{i-1}), (v_i, v_{i+1}), (v_{i+2}, v_{i+3}, \ldots, v_n)) \text{ as } (x, y, z)) \\
&= \left\langle a(x) b(y) c(z) \mid (x, y, z) \in V^{i-1} \times V^\Delta \times V^{n-1-i} \right\rangle. \qquad (56)
\end{aligned}$$

But Lemma 40 **(b)** (applied to $X = V^{i-1}$, $Y = V^\Delta$, $Z = V^{n-1-i}$ and $P = \otimes V$) yields

$$\begin{aligned}
&\left\langle a(x) \mid x \in V^{i-1} \right\rangle \cdot \left\langle b(y) \mid y \in V^\Delta \right\rangle \cdot \left\langle c(z) \mid z \in V^{n-1-i} \right\rangle \\
&= \left\langle a(x) b(y) c(z) \mid (x, y, z) \in V^{i-1} \times V^\Delta \times V^{n-1-i} \right\rangle.
\end{aligned}$$

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[19] *Proof.* Let $(v_1, v_2, \ldots, v_n) \in V^n$ be such that $v_i = v_{i+1}$. Then, $\left(\underbrace{v_i}_{= v_{i+1}}, v_{i+1} \right) = (v_{i+1}, v_{i+1}) \in \{(v, v) \mid v \in V\} = V^\Delta$. Recalling Convention 12, we have

$$\begin{aligned}
v_1 \otimes v_2 \otimes \cdots \otimes v_n &= \underbrace{(v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1})}_{= a(v_1, v_2, \ldots, v_{i-1})} \otimes \underbrace{(v_i \otimes v_{i+1})}_{= b(v_i, v_{i+1})} \otimes \underbrace{(v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n)}_{= c(v_{i+2}, v_{i+3}, \ldots, v_n)} \\
&= a(v_1, v_2, \ldots, v_{i-1}) \otimes b(v_i, v_{i+1}) \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n).
\end{aligned}$$

On the other hand, (3) (applied to $a(v_1, v_2, \ldots, v_{i-1})$, $b(v_i, v_{i+1})$, $i-1$ and 2 instead of a, b, n and m) yields

$$a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) = a(v_1, v_2, \ldots, v_{i-1}) \otimes b(v_i, v_{i+1}).$$

Also, (3) (applied to $a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1})$, $c(v_{i+2}, v_{i+3}, \ldots, v_n)$, $i+1$ and $n-1-i$ instead of a, b, n and m) yields

$$\begin{aligned}
&a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}) \cdot c(v_{i+2}, v_{i+3}, \ldots, v_n) \\
&= \underbrace{(a(v_1, v_2, \ldots, v_{i-1}) \cdot b(v_i, v_{i+1}))}_{= a(v_1, v_2, \ldots, v_{i-1}) \otimes b(v_i, v_{i+1})} \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n) \\
&= a(v_1, v_2, \ldots, v_{i-1}) \otimes b(v_i, v_{i+1}) \otimes c(v_{i+2}, v_{i+3}, \ldots, v_n) = v_1 \otimes v_2 \otimes \cdots \otimes v_n,
\end{aligned}$$

qed.

Compared to (56), this yields

$$\begin{aligned}
&\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \ \mid \ (v_1, v_2, \ldots, v_n) \in V^n; \ v_i = v_{i+1} \rangle \\
&= \left\langle a(x) \ \mid \ x \in V^{i-1} \right\rangle \cdot \left\langle b(y) \ \mid \ y \in V^{\Delta} \right\rangle \cdot \left\langle c(z) \ \mid \ z \in V^{n-1-i} \right\rangle . \qquad (57)
\end{aligned}$$

But

$$\begin{aligned}
\left\langle a(x) \ \mid \ x \in V^{i-1} \right\rangle &= \left\langle \underbrace{a(v_1, v_2, \ldots, v_{i-1})}_{= v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1}} \ \mid \ (v_1, v_2, \ldots, v_{i-1}) \in V^{i-1} \right\rangle \\
&\qquad \text{(here, we renamed } x \text{ as } (v_1, v_2, \ldots, v_{i-1})) \\
&= \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1} \ \mid \ (v_1, v_2, \ldots, v_{i-1}) \in V^{i-1} \right\rangle = V^{\otimes (i-1)}
\end{aligned}$$

(since the k-module $V^{\otimes(i-1)}$ is generated by its pure tensors, i. e., by tensors of the form $v_1 \otimes v_2 \otimes \cdots \otimes v_{i-1}$ with $(v_1, v_2, \ldots, v_{i-1}) \in V^{i-1}$). Also,

$$\begin{aligned}
\left\langle c(z) \ \mid \ z \in V^{n-1-i} \right\rangle &= \left\langle \underbrace{c(v_{i+2}, v_{i+3}, \ldots, v_n)}_{= v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n} \ \mid \ (v_{i+2}, v_{i+3}, \ldots, v_n) \in V^{n-1-i} \right\rangle \\
&\qquad \text{(here, we renamed } z \text{ as } (v_{i+2}, v_{i+3}, \ldots, v_n)) \\
&= \left\langle v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n \ \mid \ (v_{i+2}, v_{i+3}, \ldots, v_n) \in V^{n-1-i} \right\rangle = V^{\otimes (n-1-i)}
\end{aligned}$$

(since the k-module $V^{\otimes(n-1-i)}$ is generated by its pure tensors, i. e., by tensors of the form $v_{i+2} \otimes v_{i+3} \otimes \cdots \otimes v_n$ with $(v_{i+2}, v_{i+3}, \ldots, v_n) \in V^{n-1-i}$). Also, the map

$$V \to V^{\Delta}, \qquad v \mapsto (v, v) \qquad (58)$$

is a bijection (this follows easily by the definition of V^{Δ}), and thus we have

$$\begin{aligned}
\left\langle b(y) \ \mid \ y \in V^{\Delta} \right\rangle &= \left\langle \underbrace{b(v, v)}_{\substack{= v \otimes v \\ \text{(by the definition of } b)}} \ \mid \ v \in V \right\rangle \\
&\qquad \left(\begin{array}{c} \text{here, we substituted } (v, v) \text{ for } y \text{, because} \\ \text{the map (58) is a bijection} \end{array} \right) \\
&= \langle v \otimes v \ \mid \ v \in V \rangle = R_2(V) \qquad \text{(by Corollary 66)} .
\end{aligned}$$

Thus, (57) becomes

$$\begin{aligned}
&\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \ \mid \ (v_1, v_2, \ldots, v_n) \in V^n; \ v_i = v_{i+1} \rangle \\
&= \underbrace{\left\langle a(x) \ \mid \ x \in V^{i-1} \right\rangle}_{= V^{\otimes(i-1)}} \cdot \underbrace{\left\langle b(y) \ \mid \ y \in V^{\Delta} \right\rangle}_{= R_2(V)} \cdot \underbrace{\left\langle c(z) \ \mid \ z \in V^{n-1-i} \right\rangle}_{= V^{\otimes(n-1-i)}} \\
&= V^{\otimes(i-1)} \cdot (R_2(V)) \cdot V^{\otimes(n-1-i)},
\end{aligned}$$

so that Lemma 71 is proven. □

Next, the analogue of Corollary 42:

Corollary 72. Let k be a commutative ring. Let V be a k-module. Let $n \in \mathbb{N}$. Then,

$$R_n(V) = \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (R_2(V)) \cdot V^{\otimes(n-1-i)}$$

(this is an equality between k-submodules of $\otimes V$, where $R_n(V)$ becomes such a k-submodule by means of the inclusion $R_n(V) \subseteq V^{\otimes n} \subseteq \otimes V$). Here, the multiplication on the right hand side is multiplication inside the k-algebra $\otimes V$.

Proof of Corollary 72. By Proposition 67, we have

$$R_n(V) = \sum_{i=1}^{n-1} \underbrace{\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle}_{= V^{\otimes(i-1)} \cdot (R_2(V)) \cdot V^{\otimes(n-1-i)} \text{ (by Lemma 71)}}$$
$$= \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (R_2(V)) \cdot V^{\otimes(n-1-i)}.$$

Thus, Corollary 72 is proven. □

Now the analogue of Theorem 43:

Theorem 73. Let k be a commutative ring. Let V be a k-module. We know that $R_n(V)$ is a k-submodule of $V^{\otimes n}$ for every $n \in \mathbb{N}$. Thus, $\bigoplus_{n \in \mathbb{N}} R_n(V)$ is a k-submodule of $\bigoplus_{n \in \mathbb{N}} V^{\otimes n} = \otimes V$. This k-submodule satisfies

$$\bigoplus_{n \in \mathbb{N}} R_n(V) = (\otimes V) \cdot (R_2(V)) \cdot (\otimes V).$$

Proof of Theorem 73. The proof of Theorem 73 using Corollary 72 is completely analogous to the proof of Theorem 43 using Corollary 42. □

Now we can finally define the exterior algebra, similarly to Definition 44:

Definition 74. Let k be a commutative ring. Let V be a k-module.
By Theorem 73, the two k-submodules $\bigoplus_{n \in \mathbb{N}} R_n(V)$ and $(\otimes V) \cdot (R_2(V)) \cdot (\otimes V)$ of $\otimes V$ are identic (where $\bigoplus_{n \in \mathbb{N}} R_n(V)$ becomes a k-submodule of $\otimes V$ in the same way as explained in Theorem 73). We denote these two identic k-submodules by $R(V)$. In other words, we define $R(V)$ by

$$R(V) = \bigoplus_{n \in \mathbb{N}} R_n(V) = (\otimes V) \cdot (R_2(V)) \cdot (\otimes V).$$

Since $R(V) = (\otimes V) \cdot (R_2(V)) \cdot (\otimes V)$, it is clear that $R(V)$ is a two-sided ideal of the k-algebra $\otimes V$.

Now we define a k-module $\wedge V$ as the direct sum $\bigoplus_{n\in\mathbb{N}} \wedge^n V$. Then,

$$\wedge V = \bigoplus_{n\in\mathbb{N}} \underbrace{\wedge^n V}_{=V^{\otimes n}\diagup R_n(V)} = \bigoplus_{n\in\mathbb{N}} \left(V^{\otimes n}\diagup R_n(V)\right) \cong \underbrace{\left(\bigoplus_{n\in\mathbb{N}} V^{\otimes n}\right)}_{=\otimes V} \diagup \underbrace{\left(\bigoplus_{n\in\mathbb{N}} R_n(V)\right)}_{=R(V)}$$
$$= (\otimes V)\diagup R(V).$$

This is a canonical isomorphism, so we will use it to identify $\wedge V$ with $(\otimes V)\diagup R(V)$. Since $R(V)$ is a two-sided ideal of the k-algebra $\otimes V$, the quotient k-module $(\otimes V)\diagup R(V)$ canonically becomes a k-algebra. Since $\wedge V = (\otimes V)\diagup R(V)$, this means that $\wedge V$ becomes a k-algebra. We refer to this k-algebra as the *exterior algebra* of the k-module V.

We denote by wedge_V the canonical projection $\otimes V \to (\otimes V)\diagup R(V) = \wedge V$. Clearly, this map wedge_V is a surjective k-algebra homomorphism. Besides, due to $\otimes V = \bigoplus_{n\in\mathbb{N}} V^{\otimes n}$ and $R(V) = \bigoplus_{n\in\mathbb{N}} R_n(V)$, it is clear that the canonical projection $\otimes V \to (\otimes V)\diagup R(V)$ is the direct sum of the canonical projections $V^{\otimes n} \to V^{\otimes n}\diagup R_n(V)$ over all $n\in\mathbb{N}$. Since the canonical projection $\otimes V \to (\otimes V)\diagup R(V)$ is the map wedge_V, whereas the canonical projection $V^{\otimes n} \to V^{\otimes n}\diagup R_n(V)$ is the map $\operatorname{wedge}_{V,n}$, this rewrites as follows: The map wedge_V is the direct sum of the maps $\operatorname{wedge}_{V,n}$ over all $n\in\mathbb{N}$.

When $v_1, v_2, \ldots, v_n$ are some elements of V, one often abbreviates the element $\operatorname{wedge}_V(v_1\otimes v_2\otimes\cdots\otimes v_n)$ of $\wedge V$ by $v_1\wedge v_2\wedge\cdots\wedge v_n$. (We will not use this abbreviation in this following.)

We should think of the notions $R(V)$, $\wedge V$ and wedge_V as analogues of the notions $Q(V)$, $\operatorname{Exter} V$ and exter_V from Definition 44, respectively. The next result provides an analogue of Lemma 45:

Lemma 75. Let k be a commutative ring. Let V and W be two k-modules. Let $f : V \to W$ be a k-module homomorphism.

(a) Then, the k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ satisfies $(\otimes f)(R(V)) \subseteq R(W)$. Also, for every $n\in\mathbb{N}$, the k-module homomorphism $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ satisfies $f^{\otimes n}(R_n(V)) \subseteq R_n(W)$.

(b) Assume that f is surjective. Then, the k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ satisfies $(\otimes f)(R(V)) = R(W)$. Also, for every $n\in\mathbb{N}$, the k-module homomorphism $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ satisfies $f^{\otimes n}(R_n(V)) = R_n(W)$.

We can prove Lemma 75 by imitating the proof of Lemma 45 with some minor changes, but let us instead give a different proof for a change:

Proof of Lemma 75. First, let us prepare.

Corollary 66 yields $R_2(V) = \langle v\otimes v \mid v\in V\rangle$. Corollary 66 (applied to W instead of V) yields $R_2(W) = \langle v\otimes v \mid v\in W\rangle = \langle w\otimes w \mid w\in W\rangle$.

Now, every $v\in V$ satisfies

$$(\otimes f)(v\otimes v) = f(v)\otimes f(v) \qquad \text{(by the definition of } \otimes f\text{)}$$
$$\in \{w\otimes w \mid w\in W\}.$$

In other words,

$$\{(\otimes f)(v \otimes v) \mid v \in V\} \subseteq \{w \otimes w \mid w \in W\}. \tag{59}$$

Thus,

$$(\otimes f)(\{v \otimes v \mid v \in V\}) = \{(\otimes f)(v \otimes v) \mid v \in V\} \subseteq \{w \otimes w \mid w \in W\}.$$

But Proposition 35 **(b)** (applied to $\otimes V$, $\{v \otimes v \mid v \in V\}$, $\otimes W$ and $\otimes f$ instead of M, S, R and f) yields $(\otimes f)(\langle\{v \otimes v \mid v \in V\}\rangle) = \langle(\otimes f)(\{v \otimes v \mid v \in V\})\rangle$. Now,

$$R_2(V) = \langle v \otimes v \mid v \in V\rangle = \langle\{v \otimes v \mid v \in V\}\rangle,$$

so that

$$(\otimes f)(R_2(V)) = (\otimes f)(\langle\{v \otimes v \mid v \in V\}\rangle) = \left\langle \underbrace{(\otimes f)(\{v \otimes v \mid v \in V\})}_{\subseteq\{w \otimes w \mid w \in W\}} \right\rangle$$
$$\subseteq \langle\{w \otimes w \mid w \in W\}\rangle = \langle w \otimes w \mid w \in W\rangle = R_2(W). \tag{60}$$

By Corollary 72, we have

$$R_n(V) = \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (R_2(V)) \cdot V^{\otimes(n-1-i)} \tag{61}$$

for every $n \in \mathbb{N}$. Corollary 72 (applied to W instead of V) yields

$$R_n(W) = \sum_{i=1}^{n-1} W^{\otimes(i-1)} \cdot (R_2(W)) \cdot W^{\otimes(n-1-i)} \tag{62}$$

for every $n \in \mathbb{N}$.

The map $\otimes f$ is the direct sum of the maps $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ for $n \in \mathbb{N}$. Hence, for every $n \in \mathbb{N}$, the restriction $(\otimes f)\mid_{V^{\otimes n}}$ of the map $\otimes f$ to $V^{\otimes n}$ is the map $f^{\otimes n}$ (at least if we ignore the technicality that the targets of the maps $\otimes f$ and $f^{\otimes n}$ are different).

It is also clear that

$$(\otimes f)\left(V^{\otimes j}\right) \subseteq W^{\otimes j} \qquad \text{for every } j \in \mathbb{N} \tag{63}$$

(since $\otimes f$ is the direct sum of the maps $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ for $n \in \mathbb{N}$).

(a) For every $n \in \mathbb{N}$, we have

$$(\otimes f)(R_n(V)) = (\otimes f)\left(\sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (R_2(V)) \cdot V^{\otimes(n-1-i)}\right) \qquad (\text{by } (61))$$
$$= \sum_{i=1}^{n-1} \underbrace{(\otimes f)\left(V^{\otimes(i-1)}\right)}_{\subseteq W^{\otimes(i-1)} \text{ (by (63))}} \cdot \underbrace{(\otimes f)(R_2(V))}_{\subseteq R_2(W) \text{ (by (60))}} \cdot \underbrace{(\otimes f)\left(V^{\otimes(n-1-i)}\right)}_{\subseteq W^{\otimes(n-1-i)} \text{ (by (63))}}$$
$$(\text{since } \otimes f \text{ is a } k\text{-algebra homomorphism})$$
$$\subseteq \sum_{i=1}^{n-1} W^{\otimes(i-1)} \cdot (R_2(W)) \cdot W^{\otimes(n-1-i)} = R_n(W).$$

Since $(\otimes f)(R_n(V)) = \underbrace{((\otimes f)\mid_{V^{\otimes n}})}_{=f^{\otimes n}}(R_n(V)) = f^{\otimes n}(R_n(V))$, this rewrites as $f^{\otimes n}(R_n(V)) \subseteq R_n(W)$.

We have $R(V) = \bigoplus_{n\in\mathbb{N}} R_n(V) = \sum_{n\in\mathbb{N}} R_n(V)$ (because direct sums are sums) and $R(W) = \sum_{n\in\mathbb{N}} R_n(W)$ (similarly). Since $R(V) = \sum_{n\in\mathbb{N}} R_n(V)$, we have

$$(\otimes f)(R(V)) = (\otimes f)\left(\sum_{n\in\mathbb{N}} R_n(V)\right) = \sum_{n\in\mathbb{N}} \underbrace{(\otimes f)(R_n(V))}_{\subseteq R_n(W)} \qquad \text{(since } \otimes f \text{ is } k\text{-linear)}$$
$$\subseteq \sum_{n\in\mathbb{N}} R_n(W) = R(W).$$

This completes the proof of Lemma 75 **(a)**.

(b) Assume that the map f is surjective.

Every $w \in W$ satisfies $w\otimes w \in \{(\otimes f)(v\otimes v) \mid v \in V\}$. [20] In other words, $\{w\otimes w \mid w\in W\} \subseteq \{(\otimes f)(v\otimes v) \mid v\in V\}$. Combined with (59), this yields

$$\{(\otimes f)(v\otimes v) \mid v\in V\} = \{w\otimes w \mid w \in W\}. \tag{64}$$

Now, in the same way as we used (59) to prove (60), we can use (64) to prove that

$$(\otimes f)(R_n(V)) = R_n(W). \tag{65}$$

For every $n\in\mathbb{N}$, we have $(\otimes f)(V^{\otimes n}) = \underbrace{((\otimes f)\mid_{V^{\otimes n}})}_{=f^{\otimes n}}(V^{\otimes n}) = f^{\otimes n}(V^{\otimes n}) = W^{\otimes n}$ (since $f^{\otimes n}$ is surjective by Proposition 47 **(a)**). Renaming n as j in this statement, we see that

$$(\otimes f)\left(V^{\otimes j}\right) \subseteq W^{\otimes j} \qquad \text{for every } j\in\mathbb{N}. \tag{66}$$

Now, for every $n\in\mathbb{N}$, we have

$$(\otimes f)(R_n(V)) = (\otimes f)\left(\sum_{i=1}^{n-1} V^{\otimes(i-1)}\cdot(R_2(V))\cdot V^{\otimes(n-1-i)}\right) \qquad \text{(by (61))}$$
$$= \sum_{i=1}^{n-1} \underbrace{(\otimes f)\left(V^{\otimes(i-1)}\right)}_{=W^{\otimes(i-1)} \text{ (by (66))}} \cdot \underbrace{(\otimes f)(R_2(V))}_{=R_2(W) \text{ (by (65))}} \cdot \underbrace{(\otimes f)\left(V^{\otimes(n-1-i)}\right)}_{=W^{\otimes(n-1-i)} \text{ (by (66))}}$$
$$\text{(since } \otimes f \text{ is a } k\text{-algebra homomorphism)}$$
$$= \sum_{i=1}^{n-1} W^{\otimes(i-1)}\cdot(R_2(W))\cdot W^{\otimes(n-1-i)} = R_n(W).$$

Since $(\otimes f)(R_n(V)) = \underbrace{((\otimes f)\mid_{V^{\otimes n}})}_{=f^{\otimes n}}(R_n(V)) = f^{\otimes n}(R_n(V))$, this rewrites as $f^{\otimes n}(R_n(V)) = R_n(W)$.

[20] *Proof.* Let $w\in W$ be arbitrary. Then, there exists some $z\in V$ such that $w = f(z)$ (since f is surjective). Consider this z. Then, $w\otimes w = f(z)\otimes f(z) = (\otimes f)(z\otimes z)$ (by the definition of $\otimes f$), so that $w\otimes w \in \{(\otimes f)(v\otimes v) \mid v\in V\}$, qed.

We have $R(V) = \bigoplus_{n\in\mathbb{N}} R_n(V) = \sum_{n\in\mathbb{N}} R_n(V)$ (because direct sums are sums) and $R(W) = \sum_{n\in\mathbb{N}} R_n(W)$ (similarly). Since $R(V) = \sum_{n\in\mathbb{N}} R_n(V)$, we have

$$(\otimes f)(R(V)) = (\otimes f)\left(\sum_{n\in\mathbb{N}} R_n(V)\right) = \sum_{n\in\mathbb{N}} \underbrace{(\otimes f)(R_n(V))}_{=R_n(W)} \qquad \text{(since } \otimes f \text{ is } k\text{-linear)}$$
$$= \sum_{n\in\mathbb{N}} R_n(W) = R(W).$$

This completes the proof of Lemma 75 **(b)**. □

The following definition mirrors Definition 46:

Definition 76. Let k be a commutative ring. Let V and W be two k-modules. Let $f : V \to W$ be a k-module homomorphism. Then, the k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ satisfies $(\otimes f)(R(V)) \subseteq R(W)$ (by Lemma 75 **(a)**), and thus gives rise to a k-algebra homomorphism $(\otimes V) \diagup R(V) \to (\otimes W) \diagup R(W)$. This latter k-algebra homomorphism will be denoted by $\wedge f$. Since $(\otimes V) \diagup R(V) = \wedge V$ and $(\otimes W) \diagup R(W) = \wedge W$, this homomorphism $\wedge f : (\otimes V) \diagup R(V) \to (\otimes W) \diagup R(W)$ is actually a homomorphism from $\wedge V$ to $\wedge W$.
By the construction of $\wedge f$, the diagram

$$\begin{array}{ccc} \otimes V & \xrightarrow{\otimes f} & \otimes W \\ {\scriptstyle \mathrm{wedge}_V}\downarrow & & \downarrow{\scriptstyle \mathrm{wedge}_W} \\ \wedge V & \xrightarrow[\wedge f]{} & \wedge W \end{array} \qquad (67)$$

commutes (since wedge_V is the canonical projection $\otimes V \to \wedge V$ and since wedge_W is the canonical projection $\otimes W \to \wedge W$).

Needless to say, the notion $\wedge f$ introduced in this definition is an analogue of the notion $\mathrm{Exter}\, f$ introduced in Definition 46.

Here is the analogue of Proposition 47:

Proposition 77. Let k be a commutative ring. Let V and W be two k-modules. Let $f : V \to W$ be a surjective k-module homomorphism. Then:
(a) The k-module homomorphism $f^{\otimes n} : V^{\otimes n} \to W^{\otimes n}$ is surjective for every $n \in \mathbb{N}$.
(b) The k-algebra homomorphism $\otimes f : \otimes V \to \otimes W$ is surjective.
(c) The k-algebra homomorphism $\wedge f : \wedge V \to \wedge W$ is surjective.

Proof of Proposition 77. The proof of this Proposition 77 is completely analogous to the proof of Proposition 47 (and parts **(a)** and **(b)** are even the same). □

So much for analogues of the results of Subsection 0.12. Now let us formulate the analogues of the results of Subsection 0.13. First, the analogue of Theorem 48:

Theorem 78. Let k be a commutative ring. Let V and V' be two k-modules, and let $f : V \to V'$ be a surjective k-module homomorphism. Then, the kernel of the map $\wedge f : \wedge V \to \wedge V'$ is

$$\operatorname{Ker}(\wedge f) = (\wedge V) \cdot \operatorname{wedge}_V (\operatorname{Ker} f) \cdot (\wedge V) = (\wedge V) \cdot \operatorname{wedge}_V (\operatorname{Ker} f) = \operatorname{wedge}_V (\operatorname{Ker} f) \cdot (\wedge V) .$$

Here, $\operatorname{Ker} f$ is considered a k-submodule of $\otimes V$ by means of the inclusion $\operatorname{Ker} f \subseteq V = V^{\otimes 1} \subseteq \otimes V$.

Proof of Theorem 78. The proof of this Theorem 78 is completely analogous to that of Theorem 48. □

The analogue of Corollary 50 comes next:

Corollary 79. Let k be a commutative ring. Let V be a k-module, and let W be a k-submodule of V. Then,

$$(\wedge V) \cdot \operatorname{wedge}_V (W) \cdot (\wedge V) = (\wedge V) \cdot \operatorname{wedge}_V (W) = \operatorname{wedge}_V (W) \cdot (\wedge V) .$$

Here, W is considered a k-submodule of $\otimes V$ by means of the inclusion $W \subseteq V = V^{\otimes 1} \subseteq \otimes V$.

Proof of Corollary 79. Expectedly, the proof of Corollary 79 is analogous to the proof of Corollary 50. □

Finally, the analogue of Corollary 51:

Corollary 80. Let k be a commutative ring. Let V be a k-module. Let W be a k-submodule of V, and let $f : V \to V / W$ be the canonical projection.
(a) Then, the kernel of the map $\wedge f : \wedge V \to \wedge (V / W)$ is

$$\operatorname{Ker}(\wedge f) = (\wedge V) \cdot \operatorname{wedge}_V (W) \cdot (\wedge V) = (\wedge V) \cdot \operatorname{wedge}_V (W) = \operatorname{wedge}_V (W) \cdot (\wedge V) .$$

Here, W is considered a k-submodule of $\otimes V$ by means of the inclusion $W \subseteq V = V^{\otimes 1} \subseteq \otimes V$.
(b) We have

$$(\wedge V) / ((\wedge V) \cdot \operatorname{wedge}_V (W)) \cong \wedge (V / W) \qquad \text{as } k\text{-modules.}$$

Proof of Corollary 80. The proof of Corollary 80 is analogous to the proof of Corollary 51. □

0.16. The relation between the exterior and pseudoexterior algebras

The name "pseudoexterior" for the algebra $\operatorname{Exter} V$ introduced in Definition 44 already suggests a close relation to the exterior algebra $\wedge V$. Indeed such a relation is given by the following two theorems:

Theorem 81. Let k be a commutative ring. Let V be a k-module.
(a) We have $Q_n(V) \subseteq R_n(V)$ for all $n \in \mathbb{N}$.
(b) We have $Q(V) \subseteq R(V)$.
(c) For every $n \in \mathbb{N}$, the projection $\operatorname{wedge}_{V,n} : V^{\otimes n} \to \wedge^n V$ factors through the projection $\operatorname{exter}_{V,n} : V^{\otimes n} \to \operatorname{Exter}^n V$.
(d) The projection $\operatorname{wedge}_V : \otimes V \to \wedge V$ factors through the projection $\operatorname{exter}_V : \otimes V \to \operatorname{Exter} V$.

Theorem 82. Let k be a commutative ring in which 2 is invertible. Let V be a k-module.
(a) We have $Q_n(V) = R_n(V)$ for all $n \in \mathbb{N}$.
(b) We have $Q(V) = R(V)$.
(c) For every $n \in \mathbb{N}$, we have $\wedge^n V = \operatorname{Exter}^n V$ and $\operatorname{wedge}_{V,n} = \operatorname{exter}_{V,n}$.
(d) We have $\wedge V = \operatorname{Exter} V$ and $\operatorname{wedge}_V = \operatorname{exter}_V$.

Proof of Theorem 81. **(a)** Let us use the notations of Lemma 68. For every $n \in \mathbb{N}$, we have

$$\begin{aligned} Q_n(V) &\subseteq \widetilde{R}_n(V) && \text{(by Lemma 68)} \\ &\subseteq R_n(V) && \text{(by Lemma 69)}. \end{aligned}$$

This proves Theorem 81 **(a)**.

(b) We have

$$Q(V) = \bigoplus_{n \in \mathbb{N}} \underbrace{Q_n(V)}_{\substack{\subseteq R_n(V) \\ \text{(by Theorem 81 (a))}}} \subseteq \bigoplus_{n \in \mathbb{N}} R_n(V) = R(V).$$

This proves Theorem 81 **(b)**.

(c) Let $n \in \mathbb{N}$. The canonical projection $V^{\otimes n} \to V^{\otimes n} / R_n(V)$ factors through the canonical projection $V^{\otimes n} \to V^{\otimes n} / Q_n(V)$ (because $Q_n(V) \subseteq R_n(V)$ by Theorem 81 **(a)**). Since the canonical projection $V^{\otimes n} \to V^{\otimes n} / R_n(V)$ is the map $\operatorname{wedge}_{V,n} : V^{\otimes n} \to \wedge^n V$, and since the canonical projection $V^{\otimes n} \to V^{\otimes n} / Q_n(V)$ is the map $\operatorname{exter}_{V,n} : V^{\otimes n} \to \operatorname{Exter}^n V$, this rewrites as follows: The map $\operatorname{wedge}_{V,n} : V^{\otimes n} \to \wedge^n V$ factors through the map $\operatorname{exter}_{V,n} : V^{\otimes n} \to \operatorname{Exter}^n V$. This proves Theorem 81 **(c)**.

(d) The canonical projection $\otimes V \to (\otimes V) / R(V)$ factors through the canonical projection $\otimes V \to (\otimes V) / Q(V)$ (because $Q(V) \subseteq R(V)$ by Theorem 81 **(b)**). Since the canonical projection $\otimes V \to (\otimes V) / R(V)$ is the map $\operatorname{wedge}_V : \otimes V \to \wedge V$, and since the canonical projection $\otimes V \to (\otimes V) / Q(V)$ is the map $\operatorname{exter}_V : \otimes V \to \operatorname{Exter} V$, this rewrites as follows: The map $\operatorname{wedge}_V : \otimes V \to \wedge V$ factors through the map $\operatorname{exter}_V : \otimes V \to \operatorname{Exter} V$. This proves Theorem 81 **(d)**. □

Proof of Theorem 82. The main step is to prove that $Q_2(V) = R_2(V)$. Let us do this now:

Corollary 39 yields

$$Q_2(V) = \left\langle v_1 \otimes v_2 + v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\rangle.$$

Every $v \in V$ satisfies

$$\begin{aligned} v \otimes v &= \frac{1}{2}\left(v \otimes v + v \otimes v\right) \qquad \text{(since 2 is invertible in } k\text{)} \\ &= \frac{1}{2} v \otimes v + v \otimes \frac{1}{2} v \in \left\{v_1 \otimes v_2 + v_2 \otimes v_1 \ \mid \ (v_1, v_2) \in V^2\right\} \end{aligned}$$

(since the tensor $\frac{1}{2} v \otimes v + v \otimes \frac{1}{2} v$ has the form $v_1 \otimes v_2 + v_2 \otimes v_1$ for $(v_1, v_2) = \left(\frac{1}{2} v, v\right)$). In other words,

$$\{v \otimes v \ \mid \ v \in V\} \subseteq \left\{v_1 \otimes v_2 + v_2 \otimes v_1 \ \mid \ (v_1, v_2) \in V^2\right\}.$$

Now, Corollary 66 yields

$$\begin{aligned} R_2(V) &= \langle v \otimes v \ \mid \ v \in V\rangle = \left\langle \underbrace{\{v \otimes v \ \mid \ v \in V\}}_{\subseteq \{v_1 \otimes v_2 + v_2 \otimes v_1 \ \mid \ (v_1, v_2) \in V^2\}} \right\rangle \subseteq \left\langle \left\{v_1 \otimes v_2 + v_2 \otimes v_1 \ \mid \ (v_1, v_2) \in V^2\right\} \right\rangle \\ &= \left\langle v_1 \otimes v_2 + v_2 \otimes v_1 \ \mid \ (v_1, v_2) \in V^2 \right\rangle = Q_2(V). \end{aligned}$$

Combined with $Q_2(V) \subseteq R_2(V)$ (which follows from Theorem 81 **(a)**, applied to $n = 2$), this yields $Q_2(V) = R_2(V)$.

(a) Let $n \in \mathbb{N}$. Both $Q_n(V)$ and $R_n(V)$ are k-submodules of $V^{\otimes n}$ and thus k-submodules of $\otimes V$ (since $V^{\otimes n} \subseteq \otimes V$). Using the multiplication on the k-algebra $\otimes V$, we have

$$\begin{aligned} Q_n(V) &= \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot \underbrace{(Q_2(V))}_{=R_2(V)} \cdot V^{\otimes(n-1-i)} \qquad \text{(by Corollary 42)} \\ &= \sum_{i=1}^{n-1} V^{\otimes(i-1)} \cdot (R_2(V)) \cdot V^{\otimes(n-1-i)} = R_n(V) \qquad \text{(by Corollary 72)}. \end{aligned}$$

This proves Theorem 82 **(a)**.

(b) We have

$$Q(V) = \bigoplus_{n \in \mathbb{N}} \underbrace{Q_n(V)}_{\substack{=R_n(V) \\ \text{(by Theorem 82 (a))}}} = \bigoplus_{n \in \mathbb{N}} R_n(V) = R(V).$$

This proves Theorem 82 **(b)**.

(c) Let $n \in \mathbb{N}$. Then, $V^{\otimes n} / R_n(V) = V^{\otimes n} / Q_n(V)$ (because $R_n(V) = Q_n(V)$ by Theorem 82 **(a)**). Thus, $\wedge^n V = V^{\otimes n} / R_n(V) = V^{\otimes n} / Q_n(V) = \operatorname{Exten}^n V$.

Since the canonical projection $V^{\otimes n} \to V^{\otimes n} / R_n(V)$ is the map $\operatorname{wedge}_{V,n} : V^{\otimes n} \to \wedge^n V$, and since the canonical projection $V^{\otimes n} \to V^{\otimes n} / Q_n(V)$ is the map $\operatorname{exter}_{V,n} : V^{\otimes n} \to \operatorname{Exter}^n V$, we have $\operatorname{wedge}_{V,n} = \operatorname{exter}_{V,n}$ (because $R_n(V) = Q_n(V)$). This proves Theorem 82 **(c)**.

(d) We have $(\otimes V) / R(V) = (\otimes V) / Q(V)$ (because $R(V) = Q(V)$ by Theorem 82 **(b)**). Since the canonical projection $\otimes V \to (\otimes V) / R(V)$ is the map $\operatorname{wedge}_V : \otimes V \to \wedge V$, and since the canonical projection $\otimes V \to (\otimes V) / Q(V)$ is the map $\operatorname{exter}_V : \otimes V \to \operatorname{Exter} V$, we have $\operatorname{wedge}_V = \operatorname{exter}_V$ (because $R(V) = Q(V)$). This proves Theorem 82 **(d)**. $\square$

0.17. The symmetric algebra is commutative

In this section we are going to continue the study of the symmetric algebra that we started in Section 0.14 and prove some results which don't have direct analogues for $\operatorname{Exter} V$ and $\wedge V$ (although some analogues for $\wedge V$ can be found with a little more effort, which we are not going to make).

The main result of this section will be:

Theorem 83. Let k be a commutative ring. Let V be a k-module. Then, the k-algebra $\operatorname{Sym} V$ is commutative.

The standard proof of this theorem proceeds by double induction over the degrees of the tensors that must be shown to commute. We are going to show a slightly slicker version of this proof here, which replaces the double induction by a double application of Lemma 49 (which, in its proof, hides an induction). The intermediate step between these two applications will be the following lemma:

Lemma 84. Let k be a commutative ring. Let V be a k-module. Every $v \in V$ and every $p \in \operatorname{Sym} V$ satisfy $\operatorname{sym}_V(v) \cdot p = p \cdot \operatorname{sym}_V(v)$.

(Of course, the notations we are using here and everywhere throughout this section are the notations of Section 0.14.)

Proof of Lemma 84. Let $v \in V$. Let M be the subset

$$\{q \in \operatorname{Sym} V \mid \operatorname{sym}_V(v) \cdot q = q \cdot \operatorname{sym}_V(v)\}$$

of $\operatorname{Sym} V$. We are going to prove that M is the whole $\operatorname{Sym} V$.

First of all, we have $0 \in M$ [21]. Furthermore, every $\alpha \in k$, $\beta \in k$, $p \in M$ and $r \in M$ satisfy $\alpha p + \beta r \in M$. [22] In other words, M is a k-submodule of $\operatorname{Sym} V$.

Second, $1 \in M$ (with 1 denoting the unity of the k-algebra $\operatorname{Sym} V$) [23].

On the other hand, every $(p, s) \in M \times \operatorname{sym}_V(V)$ satisfy $p \cdot s \in M$ [24]. In other words, $\{p \cdot s \mid (p, s) \in M \times \operatorname{sym}_V(V)\} \subseteq M$. By Proposition 35 **(a)** (applied to $\operatorname{Sym} V$, $\{p \cdot s \mid (p, s) \in M \times \operatorname{sym}_V(V)\}$ and M instead of M, S and Q), this yields

$$\langle \{p \cdot s \mid (p, s) \in M \times \operatorname{sym}_V(V)\} \rangle \subseteq M.$$

[21] *Proof.* Clearly, $\operatorname{sym}_V(v) \cdot 0 = 0 = 0 \cdot \operatorname{sym}_V(v)$, so that $0 \in \{q \in \operatorname{Sym} V \mid \operatorname{sym}_V(v) \cdot q = q \cdot \operatorname{sym}_V(v)\} = M$.

[22] *Proof.* Let $\alpha \in k$, $\beta \in k$, $p \in M$ and $r \in M$.

Since $p \in M = \{q \in \operatorname{Sym} V \mid \operatorname{sym}_V(v) \cdot q = q \cdot \operatorname{sym}_V(v)\}$, we have $\operatorname{sym}_V(v) \cdot p = p \cdot \operatorname{sym}_V(v)$. Similarly, $\operatorname{sym}_V(v) \cdot r = r \cdot \operatorname{sym}_V(v)$. Now,

$$\begin{aligned}\operatorname{sym}_V(v) \cdot (\alpha p + \beta r) &= \alpha \underbrace{\operatorname{sym}_V(v) \cdot p}_{= p \cdot \operatorname{sym}_V(v)} + \beta \underbrace{\operatorname{sym}_V(v) \cdot r}_{= r \cdot \operatorname{sym}_V(v)} = \alpha p \cdot \operatorname{sym}_V(v) + \beta r \cdot \operatorname{sym}_V(v) \\ &= (\alpha p + \beta r) \cdot \operatorname{sym}_V(v).\end{aligned}$$

In other words, $\alpha p + \beta r \in \{q \in \operatorname{Sym} V \mid \operatorname{sym}_V(v) \cdot q = q \cdot \operatorname{sym}_V(v)\} = M$, qed.

[23] *Proof.* Clearly, $\operatorname{sym}_V(v) \cdot 1 = \operatorname{sym}_V(v) = 1 \cdot \operatorname{sym}_V(v)$, so that $1 \in \{q \in \operatorname{Sym} V \mid \operatorname{sym}_V(v) \cdot q = q \cdot \operatorname{sym}_V(v)\} = M$.

[24] *Proof.* Let $(p, s) \in M \times \operatorname{sym}_V(V)$. Then, $p \in M$ and $s \in \operatorname{sym}_V(V)$. Since $s \in \operatorname{sym}_V(V)$, there exists some $w \in V$ such that $s = \operatorname{sym}_V(w)$. Consider this w.

Since $p \in M = \{q \in \operatorname{Sym} V \mid \operatorname{sym}_V(v) \cdot q = q \cdot \operatorname{sym}_V(v)\}$, we have $\operatorname{sym}_V(v) \cdot p = p \cdot \operatorname{sym}_V(v)$. We have $v \in V = V^{\otimes 1}$ and $w \in V = V^{\otimes 1}$. Hence, $v \cdot w = v \otimes w$ (by (3), applied to v, w, 1 and

Now,

$$M \cdot \operatorname{sym}_V(V) = \langle p \cdot s \mid (p, s) \in M \times \operatorname{sym}_V(V) \rangle = \langle \{p \cdot s \mid (p, s) \in M \times \operatorname{sym}_V(V)\} \rangle \subseteq M.$$

By Lemma 49 (applied to $A = \operatorname{Sym} V$ and $\pi = \operatorname{sym}_V$), this yields that M is a right ideal of $\operatorname{Sym} V$. Thus, $M \cdot \operatorname{Sym} V \subseteq M$. But since $1 \in M$, we have $1 \cdot \operatorname{Sym} V \subseteq M \cdot \operatorname{Sym} V \subseteq M$, so that $\operatorname{Sym} V = 1 \cdot \operatorname{Sym} V \subseteq M$. Combined with $M \subseteq \operatorname{Sym} V$, this yields $M = \operatorname{Sym} V$. Hence, every $p \in \operatorname{Sym} V$ satisfies $p \in M = \{q \in \operatorname{Sym} V \mid \operatorname{sym}_V(v) \cdot q = q \cdot \operatorname{sym}_V(v)\}$, so that $\operatorname{sym}_V(v) \cdot p = p \cdot \operatorname{sym}_V(v)$. This proves Lemma 84. □

Using Lemma 84, we will now prove Theorem 83:

Proof of Theorem 83. **a)** Let $t \in \operatorname{Sym} V$. We are going to prove that $tp = pt$ for every $p \in \operatorname{Sym} V$.

Proof. Let M be the subset

$$\{q \in \operatorname{Sym} V \mid tq = qt\}$$

of $\operatorname{Sym} V$. We are going to prove that M is the whole $\operatorname{Sym} V$.

First of all, we have $0 \in M$ [25]. Also, every $\alpha \in k$, $\beta \in k$, $p \in M$ and $r \in M$ satisfy $\alpha p + \beta r \in M$. [26] In other words, M is a k-submodule of $\operatorname{Sym} V$.

Second, $1 \in M$ (with 1 denoting the unity of the k-algebra $\operatorname{Sym} V$) [27].

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1 instead of a, b, n and m) and similarly $w \cdot v = w \otimes v$. Thus,

$$\begin{aligned}
\underbrace{v \cdot w}_{= v \otimes w} - \underbrace{w \cdot v}_{= w \otimes v} &= v \otimes w - w \otimes v \\
&\in \left\{v_1 \otimes v_2 - v_2 \otimes v_1 \mid (v_1, v_2) \in V^2\right\} \\
&\subseteq \left\langle \left\{v_1 \otimes v_2 - v_2 \otimes v_1 \mid (v_1, v_2) \in V^2\right\} \right\rangle \\
&= \left\langle v_1 \otimes v_2 - v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\rangle = K_2(V) \qquad \text{(by Corollary 54)} \\
&\subseteq \bigoplus_{n \in \mathbb{N}} K_n(V) = K(V).
\end{aligned}$$

In other words, $v \cdot w \equiv w \cdot v \operatorname{mod} K(V)$. Since sym_V is the projection $\otimes V \to (\otimes V) \diagup K(V)$, this rewrites as $\operatorname{sym}_V(v \cdot w) = \operatorname{sym}_V(w \cdot v)$. Since sym_V is a k-algebra homomorphism, we have $\operatorname{sym}_V(v \cdot w) = \operatorname{sym}_V(v) \cdot \operatorname{sym}_V(w)$ and $\operatorname{sym}_V(w \cdot v) = \operatorname{sym}_V(w) \cdot \operatorname{sym}_V(v)$.

Now,

$$\begin{aligned}
\underbrace{\operatorname{sym}_V(v) \cdot p}_{= p \cdot \operatorname{sym}_V(v)} \cdot \underbrace{s}_{= \operatorname{sym}_V(w)} &= p \cdot \underbrace{\operatorname{sym}_V(v) \cdot \operatorname{sym}_V(w)}_{\substack{= \operatorname{sym}_V(v \cdot w) = \operatorname{sym}_V(w \cdot v) \\ = \operatorname{sym}_V(w) \cdot \operatorname{sym}_V(v)}} = p \cdot \underbrace{\operatorname{sym}_V(w)}_{= s} \cdot \operatorname{sym}_V(v) \\
&= p \cdot s \cdot \operatorname{sym}_V(v).
\end{aligned}$$

In other words, $p \cdot s \in \{q \in \operatorname{Sym} V \mid \operatorname{sym}_V(v) \cdot q = q \cdot \operatorname{sym}_V(v)\} = M$, qed.

[25] *Proof.* Clearly, $t0 = 0 = 0t$, so that $0 \in \{q \in \operatorname{Sym} V \mid tq = qt\} = M$.

[26] *Proof.* Let $\alpha \in k$, $\beta \in k$, $p \in M$ and $r \in M$.

Since $p \in M = \{q \in \operatorname{Sym} V \mid tq = qt\}$, we have $tp = pt$. Similarly, $tr = rt$. Now,

$$t(\alpha p + \beta r) = \alpha \underbrace{tp}_{= pt} + \beta \underbrace{tr}_{= rt} = \alpha pt + \beta rt = (\alpha p + \beta r) t.$$

In other words, $\alpha p + \beta r \in \{q \in \operatorname{Sym} V \mid tq = qt\} = M$, qed.

[27] *Proof.* Clearly, $t \cdot 1 = t = 1t$, so that $1 \in \{q \in \operatorname{Sym} V \mid tq = qt\} = M$.

On the other hand, every $(p,s) \in M \times \operatorname{sym}_V(V)$ satisfy $p \cdot s \in M$ [28]. In other words, $\{p \cdot s \mid (p,s) \in M \times \operatorname{sym}_V(V)\} \subseteq M$. By Proposition 35 **(a)** (applied to $\operatorname{Sym} V$, $\{p \cdot s \mid (p,s) \in M \times \operatorname{sym}_V(V)\}$ and M instead of M, S and Q), this yields

$$\langle \{p \cdot s \mid (p,s) \in M \times \operatorname{sym}_V(V)\} \rangle \subseteq M.$$

Now,

$$M \cdot \operatorname{sym}_V(V) = \langle p \cdot s \mid (p,s) \in M \times \operatorname{sym}_V(V) \rangle = \langle \{p \cdot s \mid (p,s) \in M \times \operatorname{sym}_V(V)\} \rangle \subseteq M.$$

By Lemma 49 (applied to $A = \operatorname{Sym} V$ and $\pi = \operatorname{sym}_V$), this yields that M is a right ideal of $\operatorname{Sym} V$. Thus, $M \cdot \operatorname{Sym} V \subseteq M$. But since $1 \in M$, we have $1 \cdot \operatorname{Sym} V \subseteq M \cdot \operatorname{Sym} V \subseteq M$, so that $\operatorname{Sym} V = 1 \cdot \operatorname{Sym} V \subseteq M$. Combined with $M \subseteq \operatorname{Sym} V$, this yields $M = \operatorname{Sym} V$. Hence, every $p \in \operatorname{Sym} V$ satisfies $p \in M = \{q \in \operatorname{Sym} V \mid tq = qt\}$, so that $tp = pt$. This proves **a)**.

b) Forget that we fixed t. We have proven that every $t \in \operatorname{Sym} V$ and every $p \in \operatorname{Sym} V$ satisfy $tp = pt$ (by part **a)**). In other words, the k-algebra $\operatorname{Sym} V$ is commutative. Theorem 83 is proven. □

Theorem 83 is a result on the nature of the factor algebra $\operatorname{Sym} V = (\otimes V) \diagup K(V)$, so unsurprisingly it gives us an insight about the ideal $K(V)$ itself - namely, a new characterization of this ideal:

Corollary 85. Let k be a commutative ring. Let V be a k-module. Then,

$$K(V) = (\otimes V) \cdot \left\langle pq - qp \mid (p,q) \in (\otimes V)^2 \right\rangle \cdot (\otimes V).$$

This Corollary is usually formulated as follows: The ideal $K(V)$ is the commutator ideal of the k-algebra $\otimes V$. This is actually often used as an alternative definition of $K(V)$.

Proof of Corollary 85. **a)** Let us first show that $K(V) \subseteq (\otimes V) \cdot \left\langle pq - qp \mid (p,q) \in (\otimes V)^2 \right\rangle \cdot (\otimes V)$.

Proof. Every $(v_1, v_2) \in V^2$ satisfies $v_1 \otimes v_2 - v_2 \otimes v_1 = v_1 v_2 - v_2 v_1$. [29] Hence,

$$\begin{aligned} &\left\{v_1 \otimes v_2 - v_2 \otimes v_1 \mid (v_1, v_2) \in V^2\right\} \\ &= \left\{v_1 v_2 - v_2 v_1 \mid (v_1, v_2) \in V^2\right\} = \left\{pq - qp \mid (p,q) \in V^2\right\} \\ &\qquad \left(\text{here, we renamed } (v_1, v_2) \text{ as } (p,q)\right) \\ &\subseteq \left\{pq - qp \mid (p,q) \in (\otimes V)^2\right\} \qquad \left(\text{since } V^2 \subseteq (\otimes V)^2\right). \end{aligned}$$

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[28] *Proof.* Let $(p,s) \in M \times \operatorname{sym}_V(V)$. Then, $p \in M$ and $s \in \operatorname{sym}_V(V)$. Since $s \in \operatorname{sym}_V(V)$, there exists some $v \in V$ such that $s = \operatorname{sym}_V(v)$. Consider this v. Lemma 84 (applied to t instead of p) yields $\operatorname{sym}_V(v) \cdot t = t \cdot \operatorname{sym}_V(v)$. Since $\operatorname{sym}_V(v) = s$, this becomes $s \cdot t = t \cdot s$.

Since $p \in M = \{q \in \operatorname{Sym} V \mid tq = qt\}$, we have $tp = pt$. Now, $\underbrace{tp}_{=pt} s = p \underbrace{t \cdot s}_{=s \cdot t = st} = pst$. In other words, $ps \in \{q \in \operatorname{Sym} V \mid tq = qt\} = M$, so that $p \cdot s = ps \in M$, qed.

[29] *Proof.* Let $(v_1, v_2) \in V^2$. Then, $v_1 \in V = V^{\otimes 1}$ and $v_2 \in V = V^{\otimes 1}$. Hence, $v_1 \cdot v_2 = v_1 \otimes v_2$ (by (3), applied to v_1, v_2, 1 and 1 instead of a, b, n and m) and similarly $v_2 \cdot v_1 = v_2 \otimes v_1$. Hence,

$$\underbrace{v_1 \otimes v_2}_{= v_1 \cdot v_2 = v_1 v_2} - \underbrace{v_2 \otimes v_1}_{= v_2 \cdot v_1 = v_2 v_1} = v_1 v_2 - v_2 v_1,$$

qed.

Thus,

$$\left\langle \left\{ v_1 \otimes v_2 - v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\} \right\rangle \subseteq \left\langle \left\{ pq - qp \mid (p, q) \in (\otimes V)^2 \right\} \right\rangle .$$

But by Corollary 54, we have

$$\begin{aligned} K_2(V) &= \left\langle v_1 \otimes v_2 - v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\rangle = \left\langle \left\{ v_1 \otimes v_2 - v_2 \otimes v_1 \mid (v_1, v_2) \in V^2 \right\} \right\rangle \\ &\subseteq \left\langle \left\{ pq - qp \mid (p, q) \in (\otimes V)^2 \right\} \right\rangle = \left\langle pq - qp \mid (p, q) \in (\otimes V)^2 \right\rangle . \end{aligned}$$

Hence,

$$(\otimes V) \cdot (K_2(V)) \cdot (\otimes V) \subseteq (\otimes V) \cdot \left\langle pq - qp \mid (p, q) \in (\otimes V)^2 \right\rangle \cdot (\otimes V) .$$

Since $(\otimes V) \cdot (K_2(V)) \cdot (\otimes V) = K(V)$, this rewrites as

$$K(V) \subseteq (\otimes V) \cdot \left\langle pq - qp \mid (p, q) \in (\otimes V)^2 \right\rangle \cdot (\otimes V) .$$

This proves part **a)**.

b) Now we will prove that $(\otimes V) \cdot \left\langle pq - qp \mid (p, q) \in (\otimes V)^2 \right\rangle \cdot (\otimes V) \subseteq K(V)$.

Proof. Every $(p, q) \in (\otimes V)^2$ satisfy $pq - qp \in K(V)$ [30]. In other words, $\left\{ pq - qp \mid (p, q) \in (\otimes V)^2 \right\} \subseteq K(V)$. Hence, Proposition 35 **(a)** (applied to $\otimes V$, $\left\{ pq - qp \mid (p, q) \in (\otimes V)^2 \right\}$ and $K(V)$ instead of M, S and Q) yields

$$\left\langle \left\{ pq - qp \mid (p, q) \in (\otimes V)^2 \right\} \right\rangle \subseteq K(V) .$$

Thus,

$$\left\langle pq - qp \mid (p, q) \in (\otimes V)^2 \right\rangle = \left\langle \left\{ pq - qp \mid (p, q) \in (\otimes V)^2 \right\} \right\rangle \subseteq K(V) .$$

Hence,

$$(\otimes V) \cdot \left\langle pq - qp \mid (p, q) \in (\otimes V)^2 \right\rangle \cdot (\otimes V) \subseteq (\otimes V) \cdot (K(V)) \cdot (\otimes V) \subseteq K(V)$$

(since $K(V)$ is a two-sided ideal of $\otimes V$). This proves part **b)**.

c) Combining $K(V) \subseteq (\otimes V) \cdot \left\langle pq - qp \mid (p, q) \in (\otimes V)^2 \right\rangle \cdot (\otimes V)$ (which we know from part **a)**) with $(\otimes V) \cdot \left\langle pq - qp \mid (p, q) \in (\otimes V)^2 \right\rangle \cdot (\otimes V) \subseteq K(V)$ (which we know from part **b)**), we obtain $K(V) = (\otimes V) \cdot \left\langle pq - qp \mid (p, q) \in (\otimes V)^2 \right\rangle \cdot (\otimes V)$. This proves Corollary 85. □

0.18. Some universal properties

We shall next discuss some universal properties for the pseudoexterior powers $\operatorname{Exter}^n V$, the symmetric powers $\operatorname{Sym}^n V$ and the exterior powers $\wedge^n V$.

Let us first recall the definition of a multilinear map:

[30] *Proof.* Let $(p, q) \in (\otimes V)^2$. Then,

$$\begin{aligned} \operatorname{sym}_V(pq) &= \operatorname{sym}_V(p) \cdot \operatorname{sym}_V(q) && (\text{since } \operatorname{sym}_V \text{ is a } k\text{-algebra homomorphism}) \\ &= \operatorname{sym}_V(q) \cdot \operatorname{sym}_V(p) && (\text{since } \operatorname{Sym} V \text{ is commutative by Theorem 83}) \\ &= \operatorname{sym}_V(qp) && (\text{since } \operatorname{sym}_V \text{ is a } k\text{-algebra homomorphism}) . \end{aligned}$$

In other words, $pq \equiv qp \mod K(V)$ (since sym_V is the projection $\otimes V \to (\otimes V) \diagup K(V)$). In other words, $pq - qp \in K(V)$, qed.

Definition 86. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let $V_1, V_2, \ldots, V_n$ be k-modules.

Let W be any k-module, and let $f : V_1 \times V_2 \times \cdots \times V_n \to W$ be a map. We say that the map f is *multilinear* if and only if for each $i \in \{1, 2, \ldots, n\}$ and each

$$(v_1, v_2, \ldots, v_{i-1}, v_{i+1}, v_{i+2}, \ldots, v_n) \in V_1 \times V_2 \times \cdots \times V_{i-1} \times V_{i+1} \times V_{i+2} \times \cdots \times V_n,$$

the map

$$V_i \to W,$$
$$v \mapsto f(v_1, v_2, \ldots, v_{i-1}, v, v_{i+1}, v_{i+2}, \ldots, v_n)$$

is k-linear.

Now, we can state the classical universal property of a tensor product:

Proposition 87. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let $V_1, V_2, \ldots, V_n$ be k-modules.

Let W be any k-module, and let $f : V_1 \times V_2 \times \cdots \times V_n \to W$ be a multilinear map. Then, there exists a unique k-linear map $f_\otimes : V_1 \otimes V_2 \otimes \cdots \otimes V_n \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V_1 \times V_2 \times \cdots \times V_n$ satisfies $f_\otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n)$.

Proposition 87 is the classical result that allows one to construct maps from a tensor product comfortably.

The particular case of Proposition 87 when all of $V_1, V_2, \ldots, V_n$ are identical will be the most useful to us:

Corollary 88. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V be a k-module.

Let W be any k-module, and let $f : V^n \to W$ be a multilinear map. Then, there exists a unique k-linear map $f_\otimes : V^{\otimes n} \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_\otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n)$.

Proof of Corollary 88. The map f is a multilinear map $V^n \to W$. In other words, the map f is a multilinear map $\underbrace{V \times V \times \cdots \times V}_{n \text{ times}} \to W$ (since $V^n = \underbrace{V \times V \times \cdots \times V}_{n \text{ times}}$). Thus, Proposition 87 (applied to $V_i = V$) shows that there exists a unique k-linear map $f_\otimes : \underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} \to W$ such that every $(v_1, v_2, \ldots, v_n) \in \underbrace{V \times V \times \cdots \times V}_{n \text{ times}}$ satisfies $f_\otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n)$. Since $\underbrace{V \otimes V \otimes \cdots \otimes V}_{n \text{ times}} = V^{\otimes n}$ and $\underbrace{V \times V \times \cdots \times V}_{n \text{ times}} = V^n$, this rewrites as follows: There exists a unique k-linear map $f_\otimes : V^{\otimes n} \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_\otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n)$. This proves Corollary 88. $\square$

We shall now use Corollary 88 to derive a universal property for the pseudoexterior powers $\operatorname{Exter}^n V$. We first state an almost obvious fact:

Lemma 89. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V be a k-module. Let W be any k-module, and let $f : V^n \to W$ be any map. Then, there exists **at most one** k-linear map $f_{\operatorname{Exter}} : \operatorname{Exter}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\operatorname{Exter}} (\operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n)) = f (v_1, v_2, \ldots, v_n)$.

Proof of Lemma 89. Let α and β be two k-linear maps $f_{\operatorname{Exter}} : \operatorname{Exter}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\operatorname{Exter}} (\operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n)) = f (v_1, v_2, \ldots, v_n)$. We shall show that $\alpha = \beta$.

We know that α is a k-linear map $f_{\operatorname{Exter}} : \operatorname{Exter}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\operatorname{Exter}} (\operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n)) = f (v_1, v_2, \ldots, v_n)$. In other words, α is a k-linear map $\operatorname{Exter}^n V \to W$ and has the property that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies

$$\alpha (\operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n)) = f (v_1, v_2, \ldots, v_n) . \tag{68}$$

The same argument (applied to β instead of α) shows that β is a k-linear map $\operatorname{Exter}^n V \to W$ and has the property that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies

$$\beta (\operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n)) = f (v_1, v_2, \ldots, v_n) . \tag{69}$$

Now, the map $\alpha - \beta$ is k-linear (since the mps α and β are k-linear). Hence, $\operatorname{Ker} (\alpha - \beta)$ is a k-submodule of $\operatorname{Exter}^n V$.

Define a subset S of $\operatorname{Exter}^n V$ by

$$S = \{\operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n) \mid (v_1, v_2, \ldots, v_n) \in V^n\} . \tag{70}$$

Then, $S \subseteq \operatorname{Ker} (\alpha - \beta)$ [31]. Hence, Proposition 35 **(a)** (applied to $M = \operatorname{Exter}^n V$ and $Q = \operatorname{Ker} (\alpha - \beta)$) shows that $\langle S \rangle \subseteq \operatorname{Ker} (\alpha - \beta)$.

On the other hand, define a subset S' of $V^{\otimes n}$ by

$$S' = \{v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n\} . \tag{71}$$

Then,

$$\begin{aligned} \operatorname{exter}_{V,n} (S') &= \operatorname{exter}_{V,n} (\{v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n\}) \\ &\qquad (\text{by (71)}) \\ &= \{\operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n) \mid (v_1, v_2, \ldots, v_n) \in V^n\} \\ &= S \qquad (\text{by (70)}) . \end{aligned} \tag{72}$$

[31] *Proof.* Let $s \in S$. Thus, $s \in S = \{\operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n) \mid (v_1, v_2, \ldots, v_n) \in V^n\}$. In other words, $s = \operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n)$ for some $(v_1, v_2, \ldots, v_n) \in V^n$. Consider this $(v_1, v_2, \ldots, v_n)$.

Applying the map α to the equality $s = \operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n)$, we obtain

$$\alpha (s) = \alpha (\operatorname{exter}_{V,n} (v_1 \otimes v_2 \otimes \cdots \otimes v_n)) = f (v_1, v_2, \ldots, v_n)$$

(by (68)). The same argument (applied to β instead of α) shows that $\beta (s) = f (v_1, v_2, \ldots, v_n)$. Thus, $\alpha (s) = f (v_1, v_2, \ldots, v_n) = \beta (s)$. Now, $(\alpha - \beta) (s) = \underbrace{\alpha (s)}_{=\beta(s)} - \beta (s) = \beta (s) - \beta (s) = 0$, so that $s \in \operatorname{Ker} (\alpha - \beta)$.

Now, let us forget that we fixed s. We thus have shown that $s \in \operatorname{Ker} (\alpha - \beta)$ for each $s \in S$. In other words, $S \subseteq \operatorname{Ker} (\alpha - \beta)$.

However, the tensor product $V^{\otimes n}$ is generated (as a k-module) by its pure tensors. In other words,

$$V^{\otimes n} = \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n \rangle = \left\langle \underbrace{\{v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n\}}_{=S'} \right\rangle = \langle S' \rangle .$$

Applying the map $\operatorname{exter}_{V,n}$ to both sides of this equality, we obtain

$$\operatorname{exter}_{V,n}\left(V^{\otimes n}\right) = \operatorname{exter}_{V,n}\left(\langle S' \rangle\right) = \left\langle \underbrace{\operatorname{exter}_{V,n}\left(S'\right)}_{\substack{=S \\ \text{(by (72))}}} \right\rangle$$
$$\left(\begin{array}{c}\text{by Proposition 35 } \textbf{(b)} \text{ (applied to} \\ V^{\otimes n},\ S',\ \operatorname{Exter}^n V \text{ and } \operatorname{exter}_{V,n} \text{ instead of } M,\ S,\ R \text{ and } f)\end{array}\right)$$
$$= \langle S \rangle .$$

But the map $\operatorname{exter}_{V,n}$ is surjective. Hence, $\operatorname{Exter}^n V = \operatorname{exter}_{V,n}\left(V^{\otimes n}\right) = \langle S \rangle \subseteq \operatorname{Ker}(\alpha - \beta)$. In other words, $\alpha - \beta = 0$. Hence, $\alpha = \beta$.

Now, forget that we fixed α and β. We thus have shown that if α and β are two k-linear maps $f_{\operatorname{Exter}} : \operatorname{Exter}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\operatorname{Exter}}\left(\operatorname{exter}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\right) = f(v_1, v_2, \ldots, v_n)$, then $\alpha = \beta$. In other words, there exists **at most one** k-linear map $f_{\operatorname{Exter}} : \operatorname{Exter}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\operatorname{Exter}}\left(\operatorname{exter}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\right) = f(v_1, v_2, \ldots, v_n)$. This proves Lemma 89. □

We shall furthermore need a definition:

Definition 90. Let $n \in \mathbb{N}$. Let V be a set.

Let W be a $\mathbb{Z}$-module. Let $f : V^n \to W$ be a map. We say that the map f is *antisymmetric* if and only if each $(v_1, v_2, \ldots, v_n) \in V^n$ and $\gamma \in S_n$ satisfy

$$f\left(v_{\gamma(1)}, v_{\gamma(2)}, \ldots, v_{\gamma(n)}\right) = (-1)^{\gamma} f(v_1, v_2, \ldots, v_n) .$$

Now, we can state a universal property for the pseudoexterior powers $\operatorname{Exter}^n V$:

Corollary 91. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V be a k-module.

Let W be any k-module, and let $f : V^n \to W$ be an antisymmetric multilinear map. (The notion of "antisymmetric" makes sense here because the k-module W is clearly a $\mathbb{Z}$-module.) Then, there exists a unique k-linear map $f_{\operatorname{Exter}} : \operatorname{Exter}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\operatorname{Exter}}\left(\operatorname{exter}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\right) = f(v_1, v_2, \ldots, v_n)$.

Before we prove this, let us recall a classical fact from abstract algebra – viz. the universal property of quotient modules (also known as the homomorphism theorem for k-modules):

Proposition 92. Let k be a commutative ring. Let V be a k-module. Let I be a k-submodule of V. Let π_I be the canonical projection $V \to V \diagup I$.

Let W be any k-module, and let $f : V \to W$ be a k-linear map satisfying $f(I) = 0$. Then, there exists a unique k-linear map $f' : V \diagup I \to W$ satisfying $f = f' \circ \pi_I$.

Proof of Corollary 91. Corollary 88 shows that there exists a unique k-linear map $f_\otimes : V^{\otimes n} \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_\otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n)$. Consider this $f_\otimes$. The map $f_\otimes$ is k-linear; thus, $\operatorname{Ker}(f_\otimes)$ is a k-submodule of $V^{\otimes n}$.

We know that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies

$$f_\otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n) . \tag{73}$$

The map f is antisymmetric. In other words, each $(v_1, v_2, \ldots, v_n) \in V^n$ and $\gamma \in S_n$ satisfy

$$f\left(v_{\gamma(1)}, v_{\gamma(2)}, \ldots, v_{\gamma(n)}\right) = (-1)^\gamma f(v_1, v_2, \ldots, v_n) \tag{74}$$

(by the definition of "antisymmetric").

Define a subset T of $V^{\otimes n}$ by

$$T = \left\{v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n\right\} .$$

Thus,

$$\begin{aligned}
&\langle T \rangle \\
&= \left\langle \left\{v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n\right\} \right\rangle \\
&= \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n \right\rangle \\
&= Q_n(V)
\end{aligned}$$

(since $Q_n(V)$ is defined to be
$\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n \right\rangle$).

Recall that $\operatorname{exter}_{V,n}$ is the canonical projection $V^{\otimes n} \to V^{\otimes n} \diagup Q_n(V)$.

Now, $T \subseteq \operatorname{Ker}(f_\otimes)$ [32]. Thus, $f_\otimes \left(\underbrace{T}_{\subseteq \operatorname{Ker}(f_\otimes)} \right) \subseteq f_\otimes (\operatorname{Ker}(f_\otimes)) = 0$, so that $f_\otimes(T) = 0$. But Proposition 35 **(b)** (applied to $V^{\otimes n}$, T, W and $f_\otimes$ instead of M, S, R and

[32] *Proof.* Let $t \in T$. Then,

$$t \in T = \left\{v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)} \mid ((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n\right\} .$$

In other words, t has the form $t = v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$ for some $((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n$. Consider this $((v_1, v_2, \ldots, v_n), \sigma)$.

It is known that $(-1)^\sigma \in \{1, -1\}$. But each $g \in \{1, -1\}$ satisfies $g^2 = 1$. Applying this to $g = (-1)^\sigma$, we obtain $((-1)^\sigma)^2 = 1$.

From $((v_1, v_2, \ldots, v_n), \sigma) \in V^n \times S_n$, we obtain $(v_1, v_2, \ldots, v_n) \in V^n$ and $\sigma \in S_n$. From (73), we obtain $f_\otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n)$. From (73) (applied to $\left(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}\right)$ instead of $(v_1, v_2, \ldots, v_n)$), we obtain

$$\begin{aligned}
f_\otimes \left(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}\right) &= f\left(v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(n)}\right) \\
&= (-1)^\sigma f(v_1, v_2, \ldots, v_n) \qquad \text{(by (74) (applied to } \gamma = \sigma)) .
\end{aligned}$$

f) yields $f_\otimes(\langle T\rangle) = \left\langle \underbrace{f_\otimes(T)}_{=0} \right\rangle = \langle 0\rangle = 0$. Since $\langle T\rangle = Q_n(V)$, this rewrites as $f_\otimes(Q_n(V)) = 0$.

Hence, Proposition 92 (applied to $V^{\otimes n}$, $Q_n(V)$, $\operatorname{exter}_{V,n}$, W and $f_\otimes$ instead of V, I, π_I, W and f) yields that there exists a unique k-linear map $f' : V^{\otimes n}/Q_n(V) \to W$ satisfying $f_\otimes = f' \circ \operatorname{exter}_{V,n}$. Consider this f'.

The map f' is a k-linear map $V^{\otimes n}/Q_n(V) \to W$. In other words, the map f' is a k-linear map $\operatorname{Exter}^n V \to W$ (since $V^{\otimes n}/Q_n(V) = \operatorname{Exter}^n V$). Every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies

$$\begin{aligned} & f'(\operatorname{exter}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)) \\ &= \underbrace{(f' \circ \operatorname{exter}_{V,n})}_{=f_\otimes}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f_\otimes(v_1 \otimes v_2 \otimes \cdots \otimes v_n) \\ &= f(v_1, v_2, \ldots, v_n) \qquad \text{(by (73))}. \end{aligned}$$

Thus, f' is a k-linear map $\operatorname{Exter}^n V \to W$ and has the property that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f'(\operatorname{exter}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)) = f(v_1, v_2, \ldots, v_n)$. Hence, there exists **at least one** k-linear map $f_{\text{Exter}} : \operatorname{Exter}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\text{Exter}}(\operatorname{exter}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)) = f(v_1, v_2, \ldots, v_n)$ (namely, $f_{\text{Exter}} = f'$). Since we also know that there exists **at most one** such map (in fact, this follows from Lemma 89), we can therefore conclude that there exists a **unique** k-linear map $f_{\text{Exter}} : \operatorname{Exter}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\text{Exter}}(\operatorname{exter}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)) = f(v_1, v_2, \ldots, v_n)$. This proves Corollary 91. □

We can similarly deal with symmetric powers. First, we state an analogue to Lemma 89:

Lemma 93. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V be a k-module. Let W be any k-module, and let $f : V^n \to W$ be any map. Then, there exists **at most one** k-linear map $f_{\text{Sym}} : \operatorname{Sym}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\text{Sym}}\left(\operatorname{sym}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\right) = f(v_1, v_2, \ldots, v_n)$.

Multiplying both sides of this equality by $(-1)^\sigma$, we obtain

$$(-1)^\sigma f_\otimes\left(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}\right) = \underbrace{(-1)^\sigma (-1)^\sigma}_{=((-1)^\sigma)^2=1} f(v_1, v_2, \ldots, v_n) = f(v_1, v_2, \ldots, v_n).$$

Now, applying the map $f_\otimes$ to both sides of the equality $t = v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}$, we find

$$\begin{aligned} f_\otimes(t) &= f_\otimes\left(v_1 \otimes v_2 \otimes \cdots \otimes v_n - (-1)^\sigma v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}\right) \\ &= \underbrace{f_\otimes(v_1 \otimes v_2 \otimes \cdots \otimes v_n)}_{=f(v_1,v_2,\ldots,v_n)} - \underbrace{(-1)^\sigma f_\otimes\left(v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(n)}\right)}_{=f(v_1,v_2,\ldots,v_n)} \\ &\qquad \text{(since the map } f_\otimes \text{ is } k\text{-linear)} \\ &= f(v_1, v_2, \ldots, v_n) - f(v_1, v_2, \ldots, v_n) = 0. \end{aligned}$$

In other words, $t \in \operatorname{Ker}(f_\otimes)$.

Now, forget that we fixed t. We thus have proven that $t \in \operatorname{Ker}(f_\otimes)$ for each $t \in T$. In other words, $T \subseteq \operatorname{Ker}(f_\otimes)$.

Proof of Lemma 93. The proof of Lemma 93 is completely analogous to the proof of Lemma 89, and thus is omitted. □

Next, we state a definition (which is analogous to Definition 90, but works in a greater generality, since W no longer needs to be a $\mathbb{Z}$-module):

Definition 94. Let $n \in \mathbb{N}$. Let V be a set.

Let W be a set. Let $f : V^n \to W$ be a map. We say that the map f is *symmetric* if and only if each $(v_1, v_2, \ldots, v_n) \in V^n$ and $\gamma \in S_n$ satisfy

$$f\left(v_{\gamma(1)}, v_{\gamma(2)}, \ldots, v_{\gamma(n)}\right) = f(v_1, v_2, \ldots, v_n).$$

Now, we can state a universal property for the symmetric powers $\operatorname{Sym}^n V$:

Corollary 95. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V be a k-module.

Let W be any k-module, and let $f : V^n \to W$ be a symmetric multilinear map. Then, there exists a unique k-linear map $f_{\mathrm{Sym}} : \operatorname{Sym}^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\mathrm{Sym}}\left(\mathrm{sym}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\right) = f(v_1, v_2, \ldots, v_n)$.

Proof of Corollary 95. The proof of Corollary 95 is completely analogous to the proof of Corollary 91 (up to some replacing of $+$ signs by $-$ signs and some removal of powers of -1), and thus is omitted. □

We shall next derive similar results for exterior powers. First of all, we can again easily obtain an analogue to Lemma 89:

Lemma 96. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V be a k-module. Let W be any k-module, and let $f : V^n \to W$ be any map. Then, there exists **at most one** k-linear map $f_\wedge : \wedge^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_\wedge\left(\mathrm{wedge}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\right) = f(v_1, v_2, \ldots, v_n)$.

Proof of Lemma 96. The proof of Lemma 96 is completely analogous to the proof of Lemma 89, and thus is omitted. □

Next, we define a notion of "weakly alternating" which is (in some weak sense) similar to Definition 90 (but at this point, there is no direct analogy any more):

Definition 97. Let $n \in \mathbb{N}$. Let V be a set.

Let W be a $\mathbb{Z}$-module. Let $f : V^n \to W$ be a map. We say that the map f is *weakly alternating* if and only if each $i \in \{1, 2, \ldots, n-1\}$ and $(v_1, v_2, \ldots, v_n) \in V^n$ satisfying $v_i = v_{i+1}$ satisfy

$$f(v_1, v_2, \ldots, v_n) = 0.$$

Now, we can state a universal property for the pseudoexterior powers $\wedge^n V$:

Corollary 98. Let k be a commutative ring. Let $n \in \mathbb{N}$. Let V be a k-module.

Let W be any k-module, and let $f : V^n \to W$ be a weakly alternating multilinear map. (The notion of "weakly alternating" makes sense here because the k-module W is clearly a $\mathbb{Z}$-module.) Then, there exists a unique k-linear map $f_\wedge : \wedge^n V \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_\wedge\left(\mathrm{wedge}_{V,n}(v_1 \otimes v_2 \otimes \cdots \otimes v_n)\right) = f(v_1, v_2, \ldots, v_n)$.

Proof of Corollary 98. Corollary 88 shows that there exists a unique k-linear map $f_{\otimes} : V^{\otimes n} \to W$ such that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies $f_{\otimes}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n)$. Consider this $f_{\otimes}$. The map $f_{\otimes}$ is k-linear; thus, $\operatorname{Ker}(f_{\otimes})$ is a k-submodule of $V^{\otimes n}$.

We know that every $(v_1, v_2, \ldots, v_n) \in V^n$ satisfies

$$f_{\otimes}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n). \tag{75}$$

The map f is weakly alternating. In other words, each $i \in \{1, 2, \ldots, n-1\}$ and $(v_1, v_2, \ldots, v_n) \in V^n$ satisfying $v_i = v_{i+1}$ satisfy

$$f(v_1, v_2, \ldots, v_n) = 0. \tag{76}$$

(by the definition of "weakly alternating").

Fix $i \in \{1, 2, \ldots, n-1\}$. Define a subset T of $V^{\otimes n}$ by

$$T = \{v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1}\}.$$

Thus,

$$\begin{aligned} &\langle T \rangle \\ &= \langle \{v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1}\} \rangle \\ &= \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle. \end{aligned} \tag{77}$$

Recall that $\operatorname{wedge}_{V,n}$ is the canonical projection $V^{\otimes n} \to V^{\otimes n} / R_n(V)$.

Now, $T \subseteq \operatorname{Ker}(f_{\otimes})$ [33]. Thus, $f_{\otimes}\left(\underbrace{T}_{\subseteq \operatorname{Ker}(f_{\otimes})}\right) \subseteq f_{\otimes}(\operatorname{Ker}(f_{\otimes})) = 0$, so that $f_{\otimes}(T) = 0$. But Proposition 35 **(b)** (applied to $V^{\otimes n}$, T, W and $f_{\otimes}$ instead of M, S, R and f) yields $f_{\otimes}(\langle T \rangle) = \left\langle \underbrace{f_{\otimes}(T)}_{=0} \right\rangle = \langle 0 \rangle = 0$. In view of (77), this rewrites as

$$f_{\otimes}(\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle) = 0. \tag{78}$$

Now, forget that we fixed i. We thus have proven (78) for each $i \in \{1, 2, \ldots, n-1\}$. But Proposition 67 yields

$$R_n(V) = \sum_{i=1}^{n-1} \langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1} \rangle.$$

[33] *Proof.* Let $t \in T$. Then,

$$t \in T = \{v_1 \otimes v_2 \otimes \cdots \otimes v_n \mid (v_1, v_2, \ldots, v_n) \in V^n;\ v_i = v_{i+1}\}.$$

In other words, t has the form $t = v_1 \otimes v_2 \otimes \cdots \otimes v_n$ for some $(v_1, v_2, \ldots, v_n) \in V^n$ satisfying $v_i = v_{i+1}$. Consider this $(v_1, v_2, \ldots, v_n)$.

From (75), we obtain $f_{\otimes}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = f(v_1, v_2, \ldots, v_n) = 0$ (by (76)).

Now, applying the map $f_{\otimes}$ to both sides of the equality $t = v_1 \otimes v_2 \otimes \cdots \otimes v_n$, we find $f_{\otimes}(t) = f_{\otimes}(v_1 \otimes v_2 \otimes \cdots \otimes v_n) = 0$. In other words, $t \in \operatorname{Ker}(f_{\otimes})$.

Now, forget that we fixed t. We thus have proven that $t \in \operatorname{Ker}(f_{\otimes})$ for each $t \in T$. In other words, $T \subseteq \operatorname{Ker}(f_{\otimes})$.

Applying the map $f_{\otimes}$ to both sides of this equality, we obtain

$$\begin{aligned}
& f_{\otimes}\left(R_n\left(V\right)\right) \\
& = f_{\otimes}\left(\sum_{i=1}^{n-1} \left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \ \mid \ \left(v_1, v_2, \ldots, v_n\right) \in V^n; \ v_i = v_{i+1} \right\rangle\right) \\
& = \sum_{i=1}^{n-1} \underbrace{f_{\otimes}\left(\left\langle v_1 \otimes v_2 \otimes \cdots \otimes v_n \ \mid \ \left(v_1, v_2, \ldots, v_n\right) \in V^n; \ v_i = v_{i+1} \right\rangle\right)}_{\substack{=0 \\ \text{(by (78))}}} \\
& \qquad \text{(since the map } f_{\otimes} \text{ is } k\text{-linear)} \\
& = \sum_{i=1}^{n-1} 0 = 0.
\end{aligned}$$

Hence, Proposition 92 (applied to $V^{\otimes n}$, $R_n\left(V\right)$, $\operatorname{wedge}_{V,n}$, W and $f_{\otimes}$ instead of V, I, π_I, W and f) yields that there exists a unique k-linear map $f' : V^{\otimes n} \diagup R_n\left(V\right) \to W$ satisfying $f_{\otimes} = f' \circ \operatorname{wedge}_{V,n}$. Consider this f'.

The map f' is a k-linear map $V^{\otimes n} \diagup R_n\left(V\right) \to W$. In other words, the map f' is a k-linear map $\wedge^n V \to W$ (since $V^{\otimes n} \diagup R_n\left(V\right) = \wedge^n V$). Every $\left(v_1, v_2, \ldots, v_n\right) \in V^n$ satisfies

$$\begin{aligned}
& f'\left(\operatorname{wedge}_{V,n}\left(v_1 \otimes v_2 \otimes \cdots \otimes v_n\right)\right) \\
& = \underbrace{\left(f' \circ \operatorname{wedge}_{V,n}\right)}_{= f_{\otimes}}\left(v_1 \otimes v_2 \otimes \cdots \otimes v_n\right) = f_{\otimes}\left(v_1 \otimes v_2 \otimes \cdots \otimes v_n\right) \\
& = f\left(v_1, v_2, \ldots, v_n\right) \qquad \text{(by (75))}.
\end{aligned}$$

Thus, f' is a k-linear map $\wedge^n V \to W$ and has the property that every $\left(v_1, v_2, \ldots, v_n\right) \in V^n$ satisfies $f'\left(\operatorname{wedge}_{V,n}\left(v_1 \otimes v_2 \otimes \cdots \otimes v_n\right)\right) = f\left(v_1, v_2, \ldots, v_n\right)$. Hence, there exists **at least one** k-linear map $f_{\wedge} : \wedge^n V \to W$ such that every $\left(v_1, v_2, \ldots, v_n\right) \in V^n$ satisfies $f_{\wedge}\left(\operatorname{wedge}_{V,n}\left(v_1 \otimes v_2 \otimes \cdots \otimes v_n\right)\right) = f\left(v_1, v_2, \ldots, v_n\right)$ (namely, $f_{\wedge} = f'$). Since we also know that there exists **at most one** such map (in fact, this follows from Lemma 96), we can therefore conclude that there exists a **unique** k-linear map $f_{\wedge} : \wedge^n V \to W$ such that every $\left(v_1, v_2, \ldots, v_n\right) \in V^n$ satisfies $f_{\wedge}\left(\operatorname{wedge}_{V,n}\left(v_1 \otimes v_2 \otimes \cdots \otimes v_n\right)\right) = f\left(v_1, v_2, \ldots, v_n\right)$. This proves Corollary 98. □

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