

# A quotient of the ring of symmetric functions generalizing quantum cohomology

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**slides:** [http:](http://www.cip.ifi.lmu.de/~grinberg/algebra/uconn2018.pdf)

[//www.cip.ifi.lmu.de/~grinberg/algebra/uconn2018.pdf](http://www.cip.ifi.lmu.de/~grinberg/algebra/uconn2018.pdf)

**paper:** [http:](http://www.cip.ifi.lmu.de/~grinberg/algebra/basisquot.pdf)

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## What is this about?

- From a modern point of view, **Schubert calculus** is about two cohomology rings:

$$H^* \left( \underbrace{\text{Gr}(k, n)}_{\text{Grassmannian}} \right) \text{ and } H^* \left( \underbrace{\text{Fl}(n)}_{\text{flag variety}} \right)$$

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- In this talk, we are concerned with the first.
- Classical result: as rings,

$$\begin{aligned} H^*(\text{Gr}(k, n)) \\ \cong (\text{symmetric polynomials in } x_1, x_2, \dots, x_k \text{ over } \mathbb{Z}) \\ / (h_{n-k+1}, h_{n-k+2}, \dots, h_n)_{\text{ideal}} \end{aligned}$$

(where the  $h_i$  are complete homogeneous symmetric polynomials).

- (Small) **Quantum cohomology** is a deformation of cohomology from the 1980–90s. For the Grassmannian, it is

$$QH^*(Gr(k, n))$$

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- Many properties from classical cohomology still hold. In particular:  $\text{QH}^*(\text{Gr}(k, n))$  has a  $\mathbb{Z}[q]$ -module basis  $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$  of (projected) Schur polynomials, with  $\lambda$  ranging over all partitions with  $\leq k$  parts and each part  $\leq n - k$ . The structure constants are the **Gromov–Witten invariants**.
- References:
  - Aaron Bertram, Ionut Ciocan-Fontanine, William Fulton, *Quantum multiplication of Schur polynomials*, 1999.
  - Alexander Postnikov, *Affine approach to quantum Schubert calculus*, 2005.

## A more general setting: $\mathcal{P}$ and $\mathcal{S}$

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- **Theorem (Artin  $\leq 1944$ ):** The  $\mathcal{S}$ -module  $\mathcal{P}$  is free with basis

$$(x^\alpha)_{\alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i}$$

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- Let  $a_1, a_2, \dots, a_k \in \mathcal{P}$  such that  $\deg a_i < n - k + i$  for all  $i$ .  
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- For each  $m \in \mathbb{Z}$ , we let  $h_m$  denote the  $m$ -th **complete homogeneous symmetric polynomial**:

$$h_m = \sum_{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq k} x_{i_1} x_{i_2} \cdots x_{i_m} = \sum_{\substack{\alpha \in \mathbb{N}^k; \\ |\alpha| = m}} x^\alpha \in \mathcal{S}.$$

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- **Theorem (G.):** The  $\mathbf{k}$ -module  $\mathcal{P}/J$  is free with basis

$$\overline{(x^\alpha)}_{\alpha \in \mathbb{N}^k; \alpha_i < n - k + i \text{ for each } i},$$

where the overline  $\overline{\phantom{x^\alpha}}$  means “projection” onto whatever quotient we need (here: from  $\mathcal{P}$  onto  $\mathcal{P}/J$ ).

(This basis has  $n(n-1)\cdots(n-k+1)$  elements.)

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  - If  $\mathbf{k} = \mathbb{Z}$  and  $a_1 = a_2 = \dots = a_k = 0$ , then  $\mathcal{S}/I$  becomes the cohomology ring  $H^*(\text{Gr}(k, n))$ ; the basis  $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$  corresponds to the Schubert classes.
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- The above theorem lets us work in these rings (and more generally) without relying on geometry.

- Recall that  $(\overline{s_\lambda})_{\lambda \in P_{k,n}}$  is a basis of the  $\mathbf{k}$ -module  $\mathcal{S}/I$ .

## $S_3$ -symmetry of the Gromov–Witten invariants

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- Equivalent restatement:** Each  $\nu \in P_{k,n}$  and  $f \in \mathcal{S}/I$  satisfy  $\text{coeff}_\omega (\overline{s_\nu} f) = \text{coeff}_{\nu^\vee} (f)$ .

## The $h$ -basis

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- **Proposition (G.):** Let  $m$  be a positive integer. Then,

$$\overline{h_{n+m}} = \sum_{j=0}^{k-1} (-1)^j a_{k-j} \overline{s_{(m, 1^j)}},$$

where  $(m, 1^j) := (m, \underbrace{1, 1, \dots, 1}_{j \text{ ones}})$  (a hook-shaped partition).

- Theorem (G.):** Let  $\lambda \in P_{k,n}$ . Let  $j \in \{0, 1, \dots, n - k\}$ .  
 Then,

$$\overline{s_\lambda h_j} = \sum_{\substack{\mu \in P_{k,n}; \\ \mu/\lambda \text{ is a} \\ \text{horizontal } j\text{-strip}}} \overline{s_\mu} - \sum_{i=1}^k (-1)^i a_i \sum_{\nu \subseteq \lambda} c_{(n-k-j+1, 1^{i-1}), \nu}^\lambda \overline{s_\nu},$$

where  $c_{\alpha, \beta}^\gamma$  are the usual Littlewood–Richardson coefficients.

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- Example:

$$\begin{aligned} \overline{s_{(4,3,2)} h_2} &= \overline{s_{(4,4,3)}} + a_1 (\overline{s_{(4,2)}} + \overline{s_{(3,2,1)}} + \overline{s_{(3,3)}}) \\ &\quad - a_2 (\overline{s_{(4,1)}} + \overline{s_{(2,2,1)}} + \overline{s_{(3,1,1)}} + 2\overline{s_{(3,2)}}) \\ &\quad + a_3 (\overline{s_{(2,2)}} + \overline{s_{(2,1,1)}} + \overline{s_{(3,1)}}). \end{aligned}$$

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- Multiplying by  $e_j$  appears harder:

$$\overline{s_{(2,2,1)} e_2} = a_1 \overline{s_{(2,2)}} - 2a_2 \overline{s_{(2,1)}} + a_3 (\overline{s_{(2)}} + \overline{s_{(1,1)}}) + a_1^2 \overline{s_{(1)}} - 2a_1 a_2 \overline{s_{()}}.$$

- **Conjecture:** Let  $b_i = (-1)^{n-k-1} a_i$  for each  $i \in \{1, 2, \dots, k\}$ . Let  $\lambda, \mu, \nu \in P_{k,n}$ . Then,  $(-1)^{|\lambda|+|\mu|-|\nu|} \text{coeff}_\nu(\overline{s_\lambda s_\mu})$  is a polynomial in  $b_1, b_2, \dots, b_k$  with coefficients in  $\mathbb{N}$ .
- Verified for all  $n \leq 7$  using SageMath.
- This would generalize positivity of Gromov–Witten invariants.



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- **Question:** Is there an analogous generalization of  $\text{QH}^*(\text{Fl}(n))$ ? Is it connected to Fulton’s “universal Schubert polynomials”?
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- **Question:** “Straightening rules” for  $\overline{s_\lambda}$  when  $\lambda \notin P_{k,n}$ , similar to the Bertram/Ciocan-Fontanine/Fulton “rim hook algorithm”?

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- What is the  $S_k$ -module structure on  $\mathcal{P}/J$ ?
- **Almost-theorem (G., needs to be checked):** Assume that  $\mathbf{k}$  is a  $\mathbb{Q}$ -algebra. Then, as  $S_k$ -modules,

$$\mathcal{P}/J \cong (\mathcal{P}/\mathcal{PS}^+)^{\times \binom{n}{k}} \cong \left( \underbrace{\mathbf{k}S_k}_{\text{regular rep}} \right)^{\times \binom{n}{k}},$$

where  $\mathcal{PS}^+$  is the ideal of  $\mathcal{P}$  generated by symmetric polynomials with constant term 0.



## Deforming symmetric functions, 1

- Let us recall symmetric **functions** (not polynomials) now; we'll need them soon anyway.

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- So why not replace the  $\mathbf{e}_j$  by  $\mathbf{e}_j - b_j$  too?

- **Theorem (G.):** Assume that  $a_1, a_2, \dots, a_k$  as well as  $b_1, b_2, b_3, \dots$  are elements of  $\mathbf{k}$ . Then,

$$\Lambda / (\mathbf{h}_{n-k+1} - a_1, \mathbf{h}_{n-k+2} - a_2, \dots, \mathbf{h}_n - a_k, \mathbf{e}_{k+1} - b_1, \mathbf{e}_{k+2} - b_2, \mathbf{e}_{k+3} - b_3, \dots)_{\text{ideal}}$$

is a free  $\mathbf{k}$ -module with basis  $(\overline{\mathbf{s}}_\lambda)_{\lambda \in P_{k,n}}$ .

- Proofs of all the above (except for the  $S_k$ -action) can be found in
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  - As for the rest, compute in  $\Lambda$ ... a lot.

## On the proofs, 2: the Gröbner basis argument

- The Gröbner basis argument relies on the easy identity

$$h_p(x_{i..k}) = \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) h_{p-t}(x_{1..k})$$

for all  $i \in \{1, 2, \dots, k+1\}$  and  $p \in \mathbb{N}$ .

Here,  $x_{a..b}$  means  $x_a, x_{a+1}, \dots, x_b$ .

- Use it to show that

$$\left( h_{n-k+i}(x_{i..k}) - \sum_{t=0}^{i-1} (-1)^t e_t(x_{1..i-1}) a_{i-t} \right)_{i \in \{1, 2, \dots, k\}}$$

is a Gröbner basis of the ideal  $J$  wrt the degree-lexicographic term order, where the variables are ordered by

$$x_1 > x_2 > \dots > x_k.$$

- This Gröbner basis leads to a basis of  $\mathcal{P}/J$ , which is precisely our  $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_j < n-k+i \text{ for each } i}$ .

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- Thus,  $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n}; \alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i}$  is a basis of  $\mathcal{P}/J$ .

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- On the other hand,  $(x^\alpha)_{\alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i}$  spans  $\mathcal{P}$  as an  $\mathcal{S}$ -module (Artin).
- Combining these yields that  $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n}; \alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i}$  spans  $\mathcal{P}/I\mathcal{P} = \mathcal{P}/J$ .
- But we also know that the family  $(\overline{x^\alpha})_{\alpha \in \mathbb{N}^k; \alpha_i < n-k+i \text{ for each } i}$  is a basis of  $\mathcal{P}/J$ .
- What can you say if a  $\mathbf{k}$ -module has a basis  $(a_v)_{v \in V}$  and a spanning family  $(b_u)_{u \in U}$  of the same finite size ( $|U| = |V| < \infty$ )?  
Easy exercise: You can say that  $(b_u)_{u \in U}$  is also a basis.
- Thus,  $(\overline{s_\lambda x^\alpha})_{\lambda \in P_{k,n}; \alpha \in \mathbb{N}^k; \alpha_i < i \text{ for each } i}$  is a basis of  $\mathcal{P}/J$ .
- $\implies (\overline{s_\lambda})_{\lambda \in P_{k,n}}$  is a basis of  $\mathcal{S}/I$ .

## On the proofs, 4: Bernstein's identity

- The rest of the proofs are long computations inside  $\Lambda$ , using various identities for symmetric functions.

- The rest of the proofs are long computations inside  $\Lambda$ , using various identities for symmetric functions.
- Maybe the most important one:  
**Bernstein's identity:** Let  $\lambda$  be a partition. Let  $m \in \mathbb{Z}$  be such that  $m \geq \lambda_1$ . Then,

$$\sum_{i \in \mathbb{N}} (-1)^i \mathbf{h}_{m+i} (\mathbf{e}_i)^\perp \mathbf{s}_\lambda = \mathbf{s}_{(m, \lambda_1, \lambda_2, \lambda_3, \dots)}.$$

Here,  $\mathbf{f}^\perp \mathbf{g}$  means “ $\mathbf{g}$  skewed by  $\mathbf{f}$ ” (so that  $(\mathbf{s}_\mu)^\perp \mathbf{s}_\lambda = \mathbf{s}_{\lambda/\mu}$ ).

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