Shuffle-compatibility of the descent set

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/urbana18a.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf
project: https://github.com/darijgr/gzshuf

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- I will sketch the proofs of Theorem 2.8 and of Theorem 6.1 from their paper.
- Unlike that paper, I will avoid any extraneous notation and theory here.

Permutations and descents

- Let $\mathbb{N} = \{0, 1, 2, \ldots\}$.
- For $n \in \mathbb{N}$, an *n-permutation* means a tuple of n distinct positive integers.
  Example: $(3, 1, 7)$ is a 3-permutation, but $(2, 1, 2)$ is not.
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- If π is an n-permutation and $i \in \{1, 2, \ldots, n\}$, then π_i denotes the i-th entry of π.
- If π is an n-permutation, then a *descent* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$.
- The *descent set* $\operatorname{Des} \pi$ of an n-permutation π is the set of all descents of π.
  Example: $\operatorname{Des}(3, 1, 5, 2, 4) = \{1, 3\}$.

Shuffles of permutations

- Let $m \in \mathbb{N}$, and let π be an m-permutation.
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- Assume that π and σ are disjoint. An $(m+n)$-permutation τ is called a *shuffle* of π and σ if both π and σ appear as subsequences of τ.
  (And thus, no other letters can appear in τ.)
- **Example:** The shuffles of $(4,1)$ and $(2,5)$ are

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- Observe that π and σ have $\binom{m+n}{m}$ shuffles, in bijection with m-element subsets of $\{1, 2, \ldots, m+n\}$.

Weak compositions

- The set $\mathbb{N}^k$ of k-tuples is an additive monoid. (Keep in mind: $0 \in \mathbb{N}$.)
- If $\alpha = (a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$, then $|\alpha|$ is defined to be $a_1 + a_2 + \cdots + a_k$.

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- For any $(a_1, a_2, \ldots, a_k) \in \mathbb{N}^k$, we define a set $\operatorname{PS}(a_1, a_2, \ldots, a_k)$ to be

$$\begin{aligned} &\{a_1 + a_2 + \cdots + a_i \mid 1 \leq i \leq k-1\} \\ &= \{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{k-1}\}. \end{aligned}$$

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$$= \{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{k-1}\}.$$

  (PS stands for "partial sums".)
  (**Note:** $\operatorname{PS}(\alpha) \subseteq \{0, 1, \ldots, |\alpha|\}$.)
- Let $n \in \mathbb{N}$. A *weak composition of n* means an $\alpha \in \mathbb{N}^k$ satisfying $|\alpha| = n$.

- Let $m \in \mathbb{N}$, and let π be an m-permutation.
  Let $n \in \mathbb{N}$, and let σ be an n-permutation.
  Assume that π and σ are disjoint.

Shuffle-compatibility of Des: statement

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- How many shuffles τ of π and σ satisfy $\operatorname{Des} \tau \subseteq A$?
- The following theorem by Gessel and Zhuang gives the answer.

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- Let A be a subset of $[m+n-1]$.
  Here, $[k]$ means $\{1, 2, \ldots, k\}$ for each $k \in \mathbb{N}$.
- Let L be a weak composition of $m+n$ such that $\operatorname{PS}(L) = A$.
  (Such L can easily be constructed.)
  Let k be such that $L \in \mathbb{N}^k$.
- **Theorem (Gessel & Zhuang, arXiv:1706.00750, Theorem 2.8).**
  The number of shuffles τ of π and σ satisfying $\operatorname{Des} \tau \subseteq A$
  equals the number of pairs $(J, K) \in \mathbb{N}^k \times \mathbb{N}^k$ such that
  - J is a weak composition of m satisfying $\operatorname{Des} \pi \subseteq \operatorname{PS}(J)$;
  - K is a weak composition of n satisfying $\operatorname{Des} \sigma \subseteq \operatorname{PS}(K)$;
  - we have $J + K = L$ (in the monoid $\mathbb{N}^k$).

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  Their descent sets $\operatorname{Des} \tau$ are

$$\{1\}, \qquad \{1, 2\}, \qquad \{1, 3\},$$
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  Pick $A = \{3\}$. Then, the number of shuffles τ of π and σ satisfying $\operatorname{Des} \tau \subseteq A$ is 1.
  What about the other number?

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  What about the other number? We must pick a weak composition L of $m + n = 4$ such that $\operatorname{PS}(L) = A = \{3\}$.
  We can take $L = (3, 1)$ (or $L = (3, 0, 0, \ldots, 0, 1)$ for any number of 0's). Let's pick $L = (3, 1)$.

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  Let $n = 2$ and $\sigma = (2, 5)$.
  So we have $A = \{3\}$ and $L = (3, 1)$.
  We want to find the number of pairs (J, K) such that
  - J is a weak composition of m satisfying $\operatorname{Des} \pi \subseteq \operatorname{PS}(J)$;
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  Let's solve this:

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J	?	?	$\lvert J \rvert = m$, $\operatorname{PS} J \supseteq \operatorname{Des} \pi$
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J	1	1	$\lvert J\rvert = 2, \quad \operatorname{PS} J \supseteq \{1\}$
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Thus, there is exactly 1 solution, as the Theorem predicts.

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  Pick $A = \{2, 3\}$. Then, the number of shuffles τ of π and σ satisfying $\operatorname{Des} \tau \subseteq A$ is 3.
  What about the other number?

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  We can take $L = (2, 1, 1)$ (or
  $L = (2, 0, 0, \ldots, 0, 1, 0, 0, \ldots, 0, 1)$ for any number of 0's).
  Let's pick $L = (2, 1, 1)$.

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$= L$	2	1	1	

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$+ K$	?	?	?	$\lvert K\rvert = 2, \quad \operatorname{PS} K \supseteq \{\}$
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$+ K$	1	0	1	$\lvert K \rvert = 2, \quad \operatorname{PS} K \supseteq \{\}$
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  Thus, there are 3 solutions, as the Theorem predicts.

Shuffle-compatibility of Des: consequence

- Let $m \in \mathbb{N}$, and let π be an m-permutation.
  Let $n \in \mathbb{N}$, and let σ be an n-permutation.
  Assume that π and σ are disjoint.
- Let A be a subset of $[m+n-1]$.
- Let L be a weak composition of $m+n$ such that $\operatorname{PS}(L) = A$.
  Let k be such that $L \in \mathbb{N}^k$.
- **Theorem (Gessel & Zhuang, from previous slide).**
  The number of shuffles τ of π and σ satisfying $\operatorname{Des}\tau \subseteq A$ **equals** the number of pairs $(J, K) \in \mathbb{N}^k \times \mathbb{N}^k$ such that
  - J is a weak composition of m satisfying $\operatorname{Des}\pi \subseteq \operatorname{PS}(J)$;
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Shuffle-compatibility of Des: consequence

- Let $m \in \mathbb{N}$, and let π be an m-permutation.
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  Assume that π and σ are disjoint.
- Let A be a subset of $[m+n-1]$.
- Let L be a weak composition of $m+n$ such that $\operatorname{PS}(L) = A$.
  Let k be such that $L \in \mathbb{N}^k$.
- **Corollary.**
  The number of shuffles τ of π and σ satisfying $\operatorname{Des} \tau \subseteq A$ depends only on m, n, $\operatorname{Des} \pi$, $\operatorname{Des} \sigma$ and A (but not on π and σ themselves).

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- **Corollary.**
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  (Follows from previous corollary by induction on $|A|$.)

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- **Corollary.**
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  (Follows from previous corollary by induction on $|A|$.)
  Gessel and Zhuang say that this makes Des *shuffle-compatible*. See the next talk for more about this.

Shuffle-compatibility of Des: proof, 1

- Let $m \in \mathbb{N}$, and let π be an m-permutation.
  Let $n \in \mathbb{N}$, and let σ be an n-permutation.
  Assume that π and σ are disjoint.
- Let A be a subset of $[m+n-1]$.
- Let L be a weak composition of $m+n$ such that $\operatorname{PS}(L) = A$.
  Let k be such that $L \in \mathbb{N}^k$.
- To prove the Theorem, let us restate it using shorthands:

Shuffle-compatibility of Des: proof, 1

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  Let $n \in \mathbb{N}$, and let σ be an n-permutation.
  Assume that π and σ are disjoint.
- Let A be a subset of $[m + n - 1]$.
- Let L be a weak composition of $m + n$ such that $\operatorname{PS}(L) = A$.
  Let k be such that $L \in \mathbb{N}^k$.
- A *good shuffle* shall mean a shuffle τ of π and σ satisfying $\operatorname{Des} \tau \subseteq A$.
- A *good pair* shall mean a pair $(J, K) \in \mathbb{N}^k \times \mathbb{N}^k$ such that
  - J is a weak composition of m satisfying $\operatorname{Des} \pi \subseteq \operatorname{PS}(J)$;
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- **Theorem (Gessel & Zhuang, from previous slide).**
  The number of good shuffles equals the number of good pairs.

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- **Theorem (Gessel & Zhuang, from previous slide).**
  The number of good shuffles equals the number of good pairs.
- For a proof, we need bijections

$$\{\text{good shuffles}\} \rightleftarrows \{\text{good pairs}\}.$$

Shuffle-compatibility of Des: proof, 2: $\leftarrow$

- We construct the map $\{\text{good pairs}\} \to \{\text{good shuffles}\}$:
- Let (J, K) be a good pair. Thus, $(J, K) \in \mathbb{N}^k \times \mathbb{N}^k$ and
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- Write J as $J = (j_1, j_2, \ldots, j_k)$,
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- Write J as $J = (j_1, j_2, \ldots, j_k)$,
  and K as $K = (k_1, k_2, \ldots, k_k)$ (sorry).
- For each $p \in [k-1]$, insert a bar ("$|$") between the $(j_1 + j_2 + \cdots + j_p)$-th letter of π and the next one.
  Example: If $m = 8$ and $J = (3, 2, 0, 2, 1, 0)$, then we get
  $\pi_1 \pi_2 \pi_3 \mid \pi_4 \pi_5 \mid\mid \pi_6 \pi_7 \mid \pi_8 \mid$.

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- These bars subdivide π into k blocks (some empty), each increasing (since $\operatorname{Des} \pi \subseteq \operatorname{PS}(J)$).

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- Similarly, subdivide σ into k increasing blocks using K.

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- Similarly, subdivide σ into k increasing blocks using K.
- Now, for each $i \in [k]$, let
  - $\pi^{(i)}$ be the i-th block of π;
  - $\sigma^{(i)}$ be the i-th block of σ;
  - $\tau^{(i)}$ be the unique increasing shuffle of $\pi^{(i)}$ and $\sigma^{(i)}$.

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  Then, the concatenation $\pi^{(1)} \pi^{(2)} \cdots \pi^{(k)}$ is a good shuffle.

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  So we have found a map {good pairs} $\to$ {good shuffles}.

Shuffle-compatibility of Des: proof, 3: $\rightarrow$

- We now construct the map $\{$good shuffles$\} \rightarrow \{$good pairs$\}$:
- Let τ be a good shuffle. Thus, τ is a shuffle of π and σ satisfying $\operatorname{Des} \tau \subseteq A$.

Shuffle-compatibility of Des: proof, 3: $\rightarrow$

- We now construct the map $\{\text{good shuffles}\} \rightarrow \{\text{good pairs}\}$:
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- Write L as $L = (l_1, l_2, \ldots, l_k)$.

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- Write L as $L = (l_1, l_2, \ldots, l_k)$.
- For each $p \in [k-1]$, insert a bar ("|") between the $(l_1 + l_2 + \cdots + l_p)$-th letter of τ and the next one.
  (The positions of these bars are the elements of A, though they might have multiplicities.)

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- Similarly define K.

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- Then, (J, K) is a good pair.
  So we have found a map $\{\text{good shuffles}\} \rightarrow \{\text{good pairs}\}$.
- The two maps constructed are mutually inverse bijections

$$\{\text{good shuffles}\} \rightleftarrows \{\text{good pairs}\};$$

so the theorem is proven.

The hollowed-out descent sets $\operatorname{Des}_{i,j}\pi$

- Fix $i \in \mathbb{N}$ and $j \in \mathbb{N}$.
  For any n and any n-permutation π, we define the *hollowed-out descent set* $\operatorname{Des}_{i,j}\pi$ by

  $$\operatorname{Des}_{i,j}\pi = (\operatorname{Des}\pi) \cap (\{1, 2, \ldots, i\} \cup \{n-1, n-2, \ldots, n-j\}) .$$

The hollowed-out descent sets $\operatorname{Des}_{i,j} \pi$

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  Thus, $\operatorname{Des}_{i,j} \pi$ is the set of all descents of π that are among the i first or j last possible positions for a descent to be in.

Shuffle-compatibility of $\operatorname{Des}_{i,j}$: statement

- Let $m \in \mathbb{N}$, and let π be an m-permutation.
  Let $n \in \mathbb{N}$, and let σ be an n-permutation.
  Assume that π and σ are disjoint.
- Let B be a subset of
  $\{1, 2, \ldots, i\} \cup \{m+n-1, m+n-2, \ldots, m+n-j\}$.
- Let $A = B \cup \{i+1, i+2, \ldots, m+n-j-1\}$.
- Let L be a weak composition of $m+n$ such that $\operatorname{PS}(L) = A$.
  Let k be such that $L \in \mathbb{N}^k$.
- **Theorem (Gessel & Zhuang, arXiv:1706.00750, Theorem 6.1).**
  The number of shuffles τ of π and σ satisfying $\operatorname{Des}_{i,j} \tau \subseteq B$ **equals** the number of pairs $(J, K) \in \mathbb{N}^k \times \mathbb{N}^k$ such that
  - J is a weak composition of m satisfying $\operatorname{Des}_{i,j} \pi \subseteq \operatorname{PS}(J)$;
  - K is a weak composition of n satisfying $\operatorname{Des}_{i,j} \sigma \subseteq \operatorname{PS}(K)$;
  - we have $J + K = L$ (in the monoid $\mathbb{N}^k$).

Shuffle-compatibility of $\operatorname{Des}_{i,j}$: proof

- We can derive this Theorem from the previous Theorem. This relies on the following three observations:
    - We have $\operatorname{Des}_{i,j} \tau \subseteq B$ if and only if $\operatorname{Des} \tau \subseteq A$.
    - For any weak composition J of m satisfying $J \leq L$ (that is, each entry of J is $\leq$ to the corresponding entry of L), we have $\operatorname{Des}_{i,j} \pi \subseteq \operatorname{PS}(J)$ if and only if $\operatorname{Des} \pi \subseteq \operatorname{PS}(J)$.
    - A similar statement about weak compositions K of n.

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  - A similar statement about weak compositions K of n.
- The first observation is obvious.

Shuffle-compatibility of $\operatorname{Des}_{i,j}$: proof

- We can derive this Theorem from the previous Theorem. This relies on the following three observations:
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  - A similar statement about weak compositions K of n.
- Proof of the second observation: Since $\operatorname{PS}(L) = A \supseteq \{i+1, i+2, \ldots, m+n-j-1\}$, the composition L has the form

$$L = \Big((\text{some numbers with sum } \leq i+1), \\ (\text{a sequence of 0's and 1's}), \\ (\text{some numbers with sum } \leq j+1) \Big).$$

Since $J \leq L$, it follows that J also has this form. In other words, $\operatorname{PS}(J) \supseteq \{i+1, i+2, \ldots, m-j-1\}$. Hence, the second observation follows.

Thanks

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And thanks to you for attending!

slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/urbana18a.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf
project: https://github.com/darijgr/gzshuf