Ideals of QSym, shuffle-compatibility and exterior peaks

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/urbana18b.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf
project: https://github.com/darijgr/gzshuf

Section 1

Shuffle-compatibility

Reference:

- Ira M. Gessel, Yan Zhuang, *Shuffle-compatible permutation statistics*, arXiv:1706.00750.
- See also the previous talk for a combinatorial introduction.

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  If π is an n-permutation, then $|\pi| := n$.
  We say that π is *nonempty* if $n > 0$.
- If π is an n-permutation and $i \in \{1, 2, \ldots, n\}$, then π_i denotes the i-th entry of π.

- Two n-permutations α and β (with the same n) are *order-equivalent* if all $i, j \in \{1, 2, \ldots, n\}$ satisfy $(\alpha_i < \alpha_j) \iff (\beta_i < \beta_j)$.
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  Intuition: A statistic computes some "fingerprint" of a permutation that only depends on the relative order of its letters.

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  Note: The values of a statistic can be anything (integers, sets, etc.).

Examples of permutation statistics, 1: descents et al

- If π is an n-permutation, then a *descent* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_i > \pi_{i+1}$.
- The *descent set* $\operatorname{Des} \pi$ of a permutation π is the set of all descents of π.
  Thus, Des is a statistic.
  Example: $\operatorname{Des}(3, 1, 5, 2, 4) = \{1, 3\}$.

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- The *descent number* $\operatorname{des} \pi$ of a permutation π is the number of all descents of π: that is, $\operatorname{des} \pi = |\operatorname{Des} \pi|$.
  Thus, des is a statistic.
  Example: $\operatorname{des}(3, 1, 5, 2, 4) = 2$.

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- The *major index* $\operatorname{maj} \pi$ of a permutation π is the **sum** of all descents of π.
  Thus, maj is a statistic.
  Example: $\operatorname{maj}(3, 1, 5, 2, 4) = 4$.

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- The *major index* $\operatorname{maj} \pi$ of a permutation π is the **sum** of all descents of π.
  Thus, maj is a statistic.
  Example: $\operatorname{maj}(3, 1, 5, 2, 4) = 4$.
- The *Coxeter length* inv (i.e., *number of inversions*) and the *set of inversions* are statistics, too.

Examples of permutation statistics, 2: peaks

- If π is an n-permutation, then a *peak* of π means an $i \in \{2, 3, \ldots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$.
  (Thus, peaks can only exist if $n \geq 3$.
  The name refers to the plot of π, where peaks are local maxima.)
- The *peak set* $\operatorname{Pk} \pi$ of a permutation π is the set of all peaks of π.
  Thus, Pk is a statistic.
  Examples:
  - $\operatorname{Pk}(3, 1, 5, 2, 4) = \{3\}$.
  - $\operatorname{Pk}(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
  - $\operatorname{Pk}(3, 2) = \{\}$.

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  - $\operatorname{Pk}(3, 2) = \{\}$.
- The *peak number* $\operatorname{pk} \pi$ of a permutation π is the number of all peaks of π: that is, $\operatorname{pk} \pi = |\operatorname{Pk} \pi|$.
  Thus, pk is a statistic.
  Example: $\operatorname{pk}(3, 1, 5, 2, 4) = 1$.

Examples of permutation statistics, 3: left peaks

- If π is an n-permutation, then a *left peak* of π means an $i \in \{1, 2, \ldots, n-1\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$.
  (Thus, left peaks are the same as peaks, except that 1 counts as a left peak if $\pi_1 > \pi_2$.)
- The *left peak set* $\operatorname{Lpk} \pi$ of a permutation π is the set of all left peaks of π.
  Thus, Lpk is a statistic.
  Examples:
  - $\operatorname{Lpk}(3, 1, 5, 2, 4) = \{1, 3\}$.
  - $\operatorname{Lpk}(1, 3, 2, 5, 4, 6) = \{2, 4\}$.
  - $\operatorname{Lpk}(3, 2) = \{1\}$.
- The *left peak number* $\operatorname{lpk} \pi$ of a permutation π is the number of all left peaks of π: that is, $\operatorname{lpk} \pi = |\operatorname{Lpk} \pi|$.
  Thus, lpk is a statistic.
  Example: $\operatorname{lpk}(3, 1, 5, 2, 4) = 2$.

Examples of permutation statistics, 4: right peaks

- If π is an n-permutation, then a *right peak* of π means an $i \in \{2, 3, \ldots, n\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_{n+1} = 0$.
  (Thus, right peaks are the same as peaks, except that n counts as a right peak if $\pi_{n-1} < \pi_n$.)
- The *right peak set* $\operatorname{Rpk} \pi$ of a permutation π is the set of all right peaks of π.
  Thus, Rpk is a statistic.
  Examples:
  - $\operatorname{Rpk}(3, 1, 5, 2, 4) = \{3, 5\}$.
  - $\operatorname{Rpk}(1, 3, 2, 5, 4, 6) = \{2, 4, 6\}$.
  - $\operatorname{Rpk}(3, 2) = \{\}$.
- The *right peak number* $\operatorname{rpk} \pi$ of a permutation π is the number of all right peaks of π: that is, $\operatorname{rpk} \pi = |\operatorname{Rpk} \pi|$.
  Thus, rpk is a statistic.
  Example: $\operatorname{rpk}(3, 1, 5, 2, 4) = 2$.

Examples of permutation statistics, 5: exterior peaks

- If π is an n-permutation, then an *exterior peak* of π means an $i \in \{1, 2, \ldots, n\}$ such that $\pi_{i-1} < \pi_i > \pi_{i+1}$, where we set $\pi_0 = 0$ *and* $\pi_{n+1} = 0$.
  (Thus, exterior peaks are the same as peaks, except that 1 counts if $\pi_1 > \pi_2$, and n counts if $\pi_{n-1} < \pi_n$.)
- The *exterior peak set* $\operatorname{Epk} \pi$ of a permutation π is the set of all exterior peaks of π.
  Thus, Epk is a statistic.
  Examples:
  - $\operatorname{Epk}(3, 1, 5, 2, 4) = \{1, 3, 5\}$.
  - $\operatorname{Epk}(1, 3, 2, 5, 4, 6) = \{2, 4, 6\}$.
  - $\operatorname{Epk}(3, 2) = \{1\}$.
- Thus, $\operatorname{Epk} \pi = \operatorname{Lpk} \pi \cup \operatorname{Rpk} \pi$ if $n \geq 2$.
- The *exterior peak number* $\operatorname{epk} \pi$ of a permutation π is the number of all exterior peaks of π: that is, $\operatorname{epk} \pi = |\operatorname{Epk} \pi|$.
  Thus, epk is a statistic.
  Example: $\operatorname{epk}(3, 1, 5, 2, 4) = 3$.

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(And thus, no other letters can appear in τ.)
- We let $S(\pi, \sigma)$ be the set of all shuffles of π and σ.
- **Example:**

$$S((4,1),(2,5)) = \{(4,1,2,5),(4,2,1,5),(4,2,5,1),$$
$$(2,4,1,5),(2,4,5,1),(2,5,4,1)\}.$$

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- Observe that π and σ have $\binom{m+n}{m}$ shuffles, in bijection with m-element subsets of $\{1,2,\ldots,m+n\}$.

Shuffle-compatible statistics: definition

- A statistic st is said to be *shuffle-compatible* if for any two disjoint permutations π and σ, the multiset

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  In particular, it has to stay unchanged if π and σ are replaced by two permutations order-equivalent to them: e.g., st must have the same distribution on the three sets

$$S((4,1),(2,5)), \qquad S((2,1),(3,5)), \qquad S((9,8),(2,3)).$$

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- Statistics that are **not shuffle-compatible**: inv, $\operatorname{des} + \operatorname{maj}$, maj_2 (sending π to the sum of the squares of its descents), $(\operatorname{Pk}, \operatorname{des})$ (sending π to $(\operatorname{Pk} \pi, \operatorname{des} \pi)$), and others.

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- Their proofs use a mixture of enumerative combinatorics (including some known formulas of MacMahon, Stanley, ...), quasisymmetric functions, Hopf algebra theory, P-partitions (and variants by Stembridge and Petersen), Eulerian polynomials (based on earlier work by Zhuang, and even earlier work by Foata and Strehl).

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- The shuffle-compatibility of Epk is left unproven in Gessel/Zhuang. Proving this is our first goal.

Left- and right-shuffle-compatibility

- We further begin the study of a finer version of shuffle-compatibility: "left- and right-shuffle-compatibility".
- Given two disjoint nonempty permutations π and σ,
  - a *left shuffle* of π and σ is a shuffle of π and σ that starts with a letter of π;
  - a *right shuffle* of π and σ is a shuffle of π and σ that starts with a letter of σ.
- We let $S_{\prec}(\pi, \sigma)$ be the set of all left shuffles of π and σ. We let $S_{\succ}(\pi, \sigma)$ be the set of all right shuffles of π and σ.

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- A statistic st is said to be *left-shuffle-compatible* if for any two disjoint nonempty permutations π and σ such that

  the first entry of π is greater than the first entry of σ,

  the multiset

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  depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, $|\pi|$ and $|\sigma|$.

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- We'll show that Des, des, Lpk and Epk are left- and right-shuffle-compatible.

Section 2

The algebraic approach: QSym and kernels

Reference:

- Ira M. Gessel, Yan Zhuang, *Shuffle-compatible permutation statistics*, arXiv:1706.00750.
- Darij Grinberg, Victor Reiner, *Hopf Algebras in Combinatorics*, arXiv:1409.8356, and various other texts on combinatorial Hopf algebras.

- Gessel and Zhuang prove **most** of their shuffle-compatibilities algebraically. Their methods involve combinatorial Hopf algebras (QSym and NSym).
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Descent statistics

- Gessel and Zhuang prove **most** of their shuffle-compatibilities algebraically. Their methods involve combinatorial Hopf algebras (QSym and NSym).
- These methods work for **descent statistics** only. What is a descent statistic?
- A *descent statistic* is a statistic st such that $\operatorname{st}\pi$ depends only on $|\pi|$ and $\operatorname{Des}\pi$ (in other words: if π and σ are two n-permutations with $\operatorname{Des}\pi = \operatorname{Des}\sigma$, then $\operatorname{st}\pi = \operatorname{st}\sigma$).
  Intuition: A descent statistic is a statistic which "factors through Des in each size".

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(1,2,1,1), & (1,2,2), & (1,3,1), & (1,4), \\
(2,1,1,1), & (2,1,2), & (2,2,1), & (2,3), \\
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  Thus, a composition of n is the same as a composition of size n.
- For each positive integer n, there are exactly 2^{n-1} compositions of n. Why?

Compositions vs. subsets: The Des and Comp bijections

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- Let n be a positive integer.

  Then, there are mutually inverse bijections

$$\{\text{compositions of } n\} \underset{\text{Comp}}{\overset{\text{Des}}{\rightleftarrows}} \{\text{subsets of } [n-1]\},$$

$$(i_1, i_2, \ldots, i_k) \mapsto \{i_1 + i_2 + \cdots + i_j \mid j \in [k-1]\},$$

$$(s_1 - s_0, s_2 - s_1, \ldots, s_{k+1} - s_k) \hookleftarrow \{s_1 < s_2 < \cdots < s_k\}$$

  (using the notations $s_0 = 0$ and $s_{k+1} = n$).

  Caveat lector:

$$\text{Des}\left((1,5,2) \text{ the composition}\right) = \{1, 6\};$$

$$\text{Des}\left((1,5,2) \text{ the permutation}\right) = \{2\}.$$

  Context must disambiguate.

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- Thus, a descent statistic is a statistic st that factors through Comp (that is, $\operatorname{st}\pi$ depends only on $\operatorname{Comp}\pi$).

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- If π is an n-permutation, then $\text{Comp}(\text{Des}\,\pi)$ is called the *descent composition* of π, and is written $\text{Comp}\,\pi$.
- If st is a descent statistic, then we use the notation $\text{st}\,\alpha$ (where α is a composition) for $\text{st}\,\pi$, where π is any permutation with $\text{Comp}\,\pi = \alpha$.
  (Again, this notation is ambiguous if compositions are not distinguished from permutations.)

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$$\operatorname{Pk}\pi = (\operatorname{Des}\pi) \setminus ((\operatorname{Des}\pi \cup \{0\}) + 1),$$

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- Similarly, Lpk, Rpk and Epk are descent statistics.
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- **Question (Gessel & Zhuang).** Is every shuffle-compatible statistic a descent statistic?

Power series & symmetric functions

- Let's now talk about power series, which are crucial to the algebraic approach to shuffle-compatibility.
- Consider the ring $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ of formal power series in countably many indeterminates.

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- A formal power series f is said to be *bounded-degree* if the monomials it contains are bounded (from above) in degree.
- A formal power series f is said to be *symmetric* if it is invariant under permutations of the indeterminates. Equivalently, if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1, i_2, \ldots, i_k$ are distinct and $j_1, j_2, \ldots, j_k$ are distinct.
- For example:
  - $1 + x_1 + x_2^3$ is bounded-degree but not symmetric.
  - $(1 + x_1)(1 + x_2)(1 + x_3) \cdots$ is symmetric but not bounded-degree.

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- The symmetric bounded-degree power series form a $\mathbb{Q}$-subalgebra Sym of $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$, called the *ring of symmetric functions* over $\mathbb{Q}$ (often denoted by Λ). This talk is not about it.

Quasisymmetric functions, part 1: definition

- We shall now define the quasisymmetric functions – a bigger algebra than Sym, but still with many of its nice properties.
- A formal power series f (still in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$) is said to be *quasisymmetric* if its coefficients in front of $x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_k}^{a_k}$ and $x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_k}^{a_k}$ are equal whenever $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.
- For example:
  - Every symmetric power series is quasisymmetric.
  - $\sum\limits_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + x_1^2 x_4 + \cdots$ is quasisymmetric, but not symmetric.

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- Let QSym be the set of all quasisymmetric bounded-degree power series in $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. This is a $\mathbb{Q}$-subalgebra, called the *ring of quasisymmetric functions* over $\mathbb{Q}$. (Gessel, 1980s.)

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- We have $\mathrm{Sym} \subseteq \mathrm{QSym} \subseteq \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$.

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- The $\mathbb{Q}$-vector space QSym has several combinatorial bases. We will use two of them: the monomial basis and the fundamental basis.

- For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, define

$$M_\alpha = \sum_{i_1 < i_2 < \cdots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}$$

$=$ sum of all monomials whose nonzero exponents are $\alpha_1, \alpha_2, \ldots, \alpha_k$ in **this** order.

This is a homogeneous power series of degree $|\alpha|$.

- Examples:
  - $M_{()} = 1$.
  - $M_{(1,1)} = \sum_{i<j} x_i x_j = x_1 x_2 + x_1 x_3 + x_2 x_3 + x_1 x_4 + x_2 x_4 + \cdots$.
  - $M_{(2,1)} = \sum_{i<j} x_i^2 x_j = x_1^2 x_2 + x_1^2 x_3 + x_2^2 x_3 + \cdots$.
  - $M_{(3)} = \sum_i x_i^3 = x_1^3 + x_2^3 + x_3^3 + \cdots$.

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  This is a homogeneous power series of degree $|\alpha|$.
- The family $(M_\alpha)_{\alpha \text{ is a composition}}$ is a basis of the $\mathbb{Q}$-vector space QSym, called the *monomial basis* (or M-basis).

- For every composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, define

$$F_\alpha = \sum_{\substack{i_1 \leq i_2 \leq \cdots \leq i_n; \\ i_j < i_{j+1} \text{ for all } j \in \operatorname{Des} \alpha}} x_{i_1} x_{i_2} \cdots x_{i_n}$$

$$= \sum_{\substack{\beta \text{ is a composition of } n; \\ \operatorname{Des} \beta \supseteq \operatorname{Des} \alpha}} M_\beta, \qquad \text{where } n = |\alpha| .$$

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- Examples:
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  - $F_{(2,1)} = \sum_{i \leq j < k} x_i x_j x_k$.
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  This is a homogeneous power series of degree $|\alpha|$ again.
- The family $(F_\alpha)_{\alpha \text{ is a composition}}$ is a basis of the $\mathbb{Q}$-vector space QSym, called the *fundamental basis* (or F-basis). Sometimes, F_α is also denoted L_α.

The product formula for the F_α

- What connects QSym with shuffles of permutations is the following fact:

  Theorem. If π and σ are two disjoint permutations, then

$$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp}\tau}.$$

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  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $|\pi| = |\pi'|$ and $|\sigma| = |\sigma'|$ and $\operatorname{Des}\pi = \operatorname{Des}\pi'$ and $\operatorname{Des}\sigma = \operatorname{Des}\sigma'$. We must prove that
  $$\{\operatorname{Des}\tau \mid \tau \in S(\pi,\sigma)\}_{\text{multiset}} = \left\{\operatorname{Des}\tau \mid \tau \in S\left(\pi',\sigma'\right)\right\}_{\text{multiset}}.$$

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    (this is equivalent to what we just said, since $\operatorname{Comp}\pi$ encodes the same data as $\operatorname{Des}\pi$ and $|\pi|$ together).

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    (this is equivalent to what we just said, since the F_α for α ranging over all compositions are linearly independent).

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    $$F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma} = F_{\operatorname{Comp}\pi'} \cdot F_{\operatorname{Comp}\sigma'}$$
    (this is equivalent to what we just said, by the Theorem above).
    But this follows from assumptions.

Shuffle-compatibility of des

- The same technique works for some other statistics. For example, we can show that des is shuffle-compatible.

- For any $n \in \mathbb{N}$ and $k \in \mathbb{N}$, define the polynomial

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- **Corollary (of preceding Theorem).** If π and σ are two disjoint permutations, with $n = |\pi|$ and $m = |\sigma|$, then
$$f_{n,\operatorname{des}\pi} \cdot f_{m,\operatorname{des}\sigma} = \sum_{\tau \in S(\pi,\sigma)} f_{n+m,\operatorname{des}\tau}.$$

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- **Proof idea (from Gessel/Zhuang).** There is a $\mathbb{Q}$-algebra homomorphism $\operatorname{QSym} \to \mathbb{Q}[p, x]$ sending each $g \in \operatorname{QSym}$ to

$$g\left(\underbrace{x, x, \ldots, x}_{p \text{ times}}, 0, 0, 0, \ldots\right)$$ (yes, this can be made sense of).

This is a variant of the *(generic) principal specialization*.

Shuffle-compatibility of des

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    where $n = |\pi| = |\pi'|$ and $m = |\sigma| = |\sigma'|$ (this is equivalent to what we just said, since the $f_{n,k}$ for $n, k \in \mathbb{N}$ are linearly independent).

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$$f_{n, \operatorname{des} \pi} \cdot f_{m, \operatorname{des} \sigma} = \sum_{\tau \in S(\pi, \sigma)} f_{n+m, \operatorname{des} \tau}.$$

- This corollary yields that des is shuffle-compatible. Why?
  - Let $\pi, \pi', \sigma, \sigma'$ be permutations with $|\pi| = |\pi'|$ and $|\sigma| = |\sigma'|$ and $\operatorname{des} \pi = \operatorname{des} \pi'$ and $\operatorname{des} \sigma = \operatorname{des} \sigma'$.
    We must prove that

    $$f_{n, \operatorname{des} \pi} \cdot f_{m, \operatorname{des} \sigma} = f_{n, \operatorname{des} \pi'} \cdot f_{m, \operatorname{des} \sigma'}$$

    (this is equivalent to what we just said, by the Corollary above).
    But this follows from assumptions.

The kernel criterion for shuffle-compatibility, 1

- The above arguments can be abstracted into a general criterion for shuffle-compatibility of a descent statistic (Gessel and Zhuang, in arXiv:1706.00750v2, Section 4.1). QSym and $\mathbb{Q}[p, x]$ get replaced by a "shuffle algebra" with an algebra homomorphism from QSym.
- We shall give our own variant of the criterion.

The kernel criterion for shuffle-compatibility, 2

- If st is a descent statistic, then two compositions α and β are said to be *st-equivalent* if $|\alpha| = |\beta|$ and $\operatorname{st} \alpha = \operatorname{st} \beta$.
  (Remember: $\operatorname{st} \alpha$ means $\operatorname{st} \pi$ for any permutation π satisfying $\operatorname{Comp} \pi = \alpha$.)

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- The *kernel* $\mathcal{K}_{\mathrm{st}}$ of a descent statistic st is the $\mathbb{Q}$-vector subspace of QSym spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st-equivalent compositions:

$$\mathcal{K}_{\mathrm{st}} = \langle F_\alpha - F_\beta \ \mid \ |\alpha| = |\beta| \text{ and } \operatorname{st}\alpha = \operatorname{st}\beta \rangle_{\mathbb{Q}} .$$

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- **Theorem.** The descent statistic st is shuffle-compatible if and only if $\mathcal{K}_{\text{st}}$ is an ideal of QSym.

Section 3

The exterior peak set

References:

- Darij Grinberg, *Shuffle-compatible permutation statistics II: the exterior peak set*, draft.
- John R. Stembridge, *Enriched P-partitions*, Trans. Amer. Math. Soc. 349 (1997), no. 2, pp. 763–788.
- T. Kyle Petersen, *Enriched P-partitions and peak algebras*, Adv. in Math. 209 (2007), pp. 561–610.

Roadmap to Epk

- We will now outline our proof that Epk is shuffle-compatible.
- The main idea is to imitate the above proof for Des, but instead of $F_{\operatorname{Comp}\pi}$ we'll now have some different power series (not in QSym).

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- The idea is simple, but the proof has technical parts I am not showing.

Labeled posets

- A *labeled poset* means a pair (P, γ) consisting of a finite poset $P = (X, \leq)$ and an injective map $\gamma : X \to A$ into some totally ordered set A. The injective map γ is called the *labeling* of the labeled poset (P, γ).

$\mathcal{N}$ and $\mathcal{Z}$: definitions

- Fix a totally ordered set $\mathcal{N}$, and denote its strict order relation by $\prec$.
- Let $+$ and $-$ be two distinct symbols.
  Let $\mathcal{Z}$ be a subset of the set $\mathcal{N} \times \{+, -\}$.
- **Intuition:** $\mathcal{N}$ is a set of letters that will index our indeterminates.
  $\mathcal{Z}$ is a set of "signed letters", which are pairs of a letter in $\mathcal{N}$ and a sign in $\{+, -\}$. (Not all such pairs must lie in $\mathcal{Z}$.)

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- Let us totally order the set $\mathcal{Z}$ in such a way that the (strict) order relation $\prec$ satisfies

$$(n, s) \prec (n', s') \text{ if and only if either } n \prec n' \\ \text{or } \left(n = n' \text{ and } s = - \text{ and } s' = +\right).$$

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- Let $\operatorname{Pow} \mathcal{N}$ be the ring of all power series over $\mathbb{Q}$ in the indeterminates x_n for $n \in \mathcal{N}$.

$\mathcal{N}$ and $\mathcal{Z}$: example

- For an example of the setting just introduced, take $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Then,

$$\mathcal{Z} \subseteq \mathbb{N} \times \{+, -\} = \{-0, +0, -1, +1, -2, +2, \ldots\}.$$

Caveat lector: $-0 \neq +0$, since these are shorthands for pairs, not numbers.

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- $\operatorname{Pow} \mathcal{N} = \mathbb{Q}\left[\left[x_0, x_1, x_2, \ldots\right]\right]$.

$\mathcal{Z}$-enriched (P, γ)-partitions: definition

- Now, let (P, γ) be a labeled poset. A *$\mathcal{Z}$-enriched (P, γ)-partition* means a map $f : P \to \mathcal{Z}$ such that for all $x < y$ in P, the following conditions hold:

  (i) We have $f(x) \preccurlyeq f(y)$.

  (ii) If $f(x) = f(y) = +n$ for some $n \in \mathcal{N}$, then $\gamma(x) < \gamma(y)$.

  (iii) If $f(x) = f(y) = -n$ for some $n \in \mathcal{N}$, then $\gamma(x) > \gamma(y)$.

  (Keep in mind: $\mathcal{N}$ and $\mathcal{Z}$ are fixed.)

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  (Keep in mind: $\mathcal{N}$ and $\mathcal{Z}$ are fixed.)

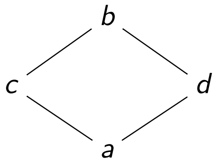
- **(Attempt at) intuition:** A $\mathcal{Z}$-enriched (P, γ)-partition is a map $f : P \to \mathcal{Z}$ (that is, assigning a signed letter to each poset element) which

  (i) is weakly increasing on P;

  (ii) + (iii) is occasionally strictly increasing, when γ and the sign of the f-value "are out of alignment".

$\mathcal{Z}$-enriched (P, γ)-partitions: example

- Let P be the poset with the following Hasse diagram:



and let $\gamma : P \to \mathbb{Z}$ be a labeling that satisfies $\gamma(a) < \gamma(b) < \gamma(c) < \gamma(d)$ (for example, γ could be the map that sends a, b, c, d to $2, 3, 5, 7$, respectively). Then, a $\mathcal{Z}$-enriched (P, γ)-partition is a map $f : P \to \mathcal{Z}$ satisfying the following conditions:

(i) We have $f(a) \preccurlyeq f(c) \preccurlyeq f(b)$ and $f(a) \preccurlyeq f(d) \preccurlyeq f(b)$.

(ii) We cannot have $f(c) = f(b) = +n$ with $n \in \mathcal{N}$. Also, we cannot have $f(d) = f(b) = +n$ with $n \in \mathcal{N}$.

(iii) We cannot have $f(a) = f(c) = -n$ with $n \in \mathcal{N}$. Also, we cannot have $f(a) = f(d) = -n$ with $n \in \mathcal{N}$.

- Consider again the case when $\mathcal{N} = \mathbb{N}$ with $\prec$ being the usual order. Let us see what $\mathcal{Z}$-enriched (P, γ)-partitions are, depending on $\mathcal{Z}$.

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- If $\mathcal{Z} = \mathbb{N} \times \{+\} = \{+0 \prec +1 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are just the (usual) (P, γ)-partitions into $\mathbb{N}$ (up to renaming n as $+n$).

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- If $\mathcal{Z}=\mathbb{N}\times\{+,-\}=$ $\{-0\prec+0\prec-1\prec+1\prec-2\prec+2\prec\cdots\}$, then the $\mathcal{Z}$-enriched (P,γ)-partitions are Stembridge's enriched (P,γ)-partitions (up to renaming n as $n-1$).

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- If $\mathcal{Z} = \mathbb{N} \times \{+, -\} = \{-0 \prec +0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are Stembridge's enriched (P, γ)-partitions (up to renaming n as $n - 1$).
- If $\mathcal{Z} = (\mathbb{N} \times \{+, -\}) \setminus \{-0\} = \{+0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P, γ)-partitions are Petersen's left enriched (P, γ)-partitions.

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- If $\mathcal{Z} = (\mathbb{N} \times \{+,-\}) \setminus \{-0\} =$ $\{+0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots\}$, then the $\mathcal{Z}$-enriched (P,γ)-partitions are Petersen's left enriched (P,γ)-partitions.
- We shall later focus on the case when $\mathcal{N} = \mathbb{N} \cup \{\infty\}$ and $\mathcal{Z} = (\mathcal{N} \times \{+,-\}) \setminus \{-0,+\infty\}$.

$\mathcal{E}(P,\gamma)$ and $\mathcal{L}(P)$

- A few more notations are needed.
- If (P,γ) is a labeled poset, then $\mathcal{E}(P,\gamma)$ shall denote the set of all $\mathcal{Z}$-enriched (P,γ)-partitions.

$\mathcal{E}(P,\gamma)$ and $\mathcal{L}(P)$

- A few more notations are needed.
- If (P,γ) is a labeled poset, then $\mathcal{E}(P,\gamma)$ shall denote the set of all $\mathcal{Z}$-enriched (P,γ)-partitions.
- If P is any poset, then $\mathcal{L}(P)$ shall denote the set of all linear extensions of P.
  A linear extension of P shall be understood simultaneously as a totally ordered set extending P and as a list $(w_1, w_2, \ldots, w_n)$ of all elements of P such that no two integers $i < j$ satisfy $w_i \geq w_j$ in P.

Any $\mathcal{E}(P,\gamma)$-partition has its favorite linear extension

- **Proposition.** For any labeled poset (P,γ), we have

$$\mathcal{E}(P,\gamma) = \bigsqcup_{w \in \mathcal{L}(P)} \mathcal{E}(w,\gamma).$$

- This is a generalization of a standard result on P-partitions ("Stanley's main lemma"), and is proven by the same reasoning.

The power series $\Gamma_{\mathcal{Z}}(P, \gamma)$

- Let (P, γ) be a labeled poset. We define a power series $\Gamma_{\mathcal{Z}}(P, \gamma) \in \operatorname{Pow} \mathcal{N}$ by

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{f \in \mathcal{E}(P, \gamma)} \prod_{p \in P} x_{|f(p)|}.$$

  Here, $|f(p)| \in \mathcal{N}$ is defined to be the first entry of $f(p)$ (recall: $f(p)$ is a pair of an element of $\mathcal{N}$ and a sign in $\{+, -\}$).

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- This generalizes the classical quasisymmetric P-partition enumerators (which give the fundamental basis F_α when P is totally ordered).
- **Corollary.** For any labeled poset (P, γ), we have

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{w \in \mathcal{L}(P)} \Gamma_{\mathcal{Z}}(w, \gamma).$$

- Let (P, γ) be a labeled poset. We define a power series $\Gamma_{\mathcal{Z}}(P, \gamma) \in \operatorname{Pow} \mathcal{N}$ by

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- This generalizes the classical quasisymmetric P-partition enumerators (which give the fundamental basis F_α when P is totally ordered).
- **Question.** Where do these $\Gamma_{\mathcal{Z}}(P, \gamma)$ live (other than in $\operatorname{Pow} \mathcal{N}$) ?
  I don't know a good answer; it should be a generalization of QSym.
  Jia Huang's work (`arXiv:1506.02962v2`) looks relevant.

Disjoint unions give product of Γ's

- Let P be any set. Let A be a totally ordered set. Let $\gamma : P \to A$ and $\delta : P \to A$ be two maps. We say that γ and δ are *order-equivalent* if the following holds: For every pair $(p, q) \in P \times P$, we have $\gamma(p) \leq \gamma(q)$ if and only if $\delta(p) \leq \delta(q)$.

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- **Proposition.** Let (P, γ) and (Q, δ) be two labeled posets. Let $(P \sqcup Q, \varepsilon)$ be the labeled poset
  - for which $P \sqcup Q$ is the disjoint union of P and Q, and
  - whose labeling ε is such that the restriction of ε to P is order-equivalent to γ and such that the restriction of ε to Q is order-equivalent to δ.

  Then,
  $$\Gamma_{\mathcal{Z}}(P, \gamma) \cdot \Gamma_{\mathcal{Z}}(Q, \delta) = \Gamma_{\mathcal{Z}}(P \sqcup Q, \varepsilon).$$
- Again, the proof is simple.

From a permutation to a labeled poset

- Let $n \in \mathbb{N}$. Write $[n]$ for $\{1, 2, \ldots, n\}$.
  Let π be any n-permutation. Consider π as an injective map $[n] \to \{1, 2, 3, \ldots\}$ (sending i to π_i). Thus, $([n], \pi)$ is a labeled poset. We define $\Gamma_{\mathcal{Z}}(\pi)$ to be the power series $\Gamma_{\mathcal{Z}}([n], \pi)$.

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- Explicitly:

$$\Gamma_{\mathcal{Z}}(\pi) = \sum x_{|j_1|} x_{|j_2|} \cdots x_{|j_n|},$$

  where the sum is over all n-tuples $(j_1, j_2, \ldots, j_n) \in \mathcal{Z}^n$ having the properties that:

  (i) $j_1 \preccurlyeq j_2 \preccurlyeq \cdots \preccurlyeq j_n$;

  (ii) if $j_k = j_{k+1} = +s$ for some $s \in \mathcal{N}$, then $\pi_k < \pi_{k+1}$;

  (iii) if $j_k = j_{k+1} = -s$ for some $s \in \mathcal{N}$, then $\pi_k > \pi_{k+1}$.
- This $\Gamma_{\mathcal{Z}}(\pi)$ will serve as an analogue of $F_{\operatorname{Comp} \pi}$.

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- **Proposition.** Let w be a finite totally ordered set with ground set W. Let $n = |W|$. Let $\overline{w}$ be the unique poset isomorphism $w \to [n]$. Let $\gamma : W \to \{1, 2, 3, \ldots\}$ be any injective map. Then, $\Gamma_{\mathcal{Z}}(w, \gamma) = \Gamma_{\mathcal{Z}}\left(\gamma \circ \overline{w}^{-1}\right)$.
- Again, this follows the roadmap of classical P-partition theory.

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- Again, this follows the roadmap of classical P-partition theory.
- **Corollary.** Let (P, γ) be a labeled poset. Let $n = |P|$. Then,

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{\substack{x : P \to [n] \\ \text{bijective poset} \\ \text{homomorphism}}} \Gamma_{\mathcal{Z}}\left(\gamma \circ x^{-1}\right).$$

From a permutation to a labeled poset

- Let $n \in \mathbb{N}$. Write $[n]$ for $\{1, 2, \ldots, n\}$. Let π be any n-permutation. Consider π as an injective map $[n] \to \{1, 2, 3, \ldots\}$ (sending i to π_i). Thus, $([n], \pi)$ is a labeled poset. We define $\Gamma_{\mathcal{Z}}(\pi)$ to be the power series $\Gamma_{\mathcal{Z}}([n], \pi)$.
- **Proposition.** Let w be a finite totally ordered set with ground set W. Let $n = |W|$. Let $\overline{w}$ be the unique poset isomorphism $w \to [n]$. Let $\gamma : W \to \{1, 2, 3, \ldots\}$ be any injective map. Then, $\Gamma_{\mathcal{Z}}(w, \gamma) = \Gamma_{\mathcal{Z}}\left(\gamma \circ \overline{w}^{-1}\right)$.
- Again, this follows the roadmap of classical P-partition theory.
- **Corollary.** Let (P, γ) be a labeled poset. Let $n = |P|$. Then,

$$\Gamma_{\mathcal{Z}}(P, \gamma) = \sum_{\substack{x:P\to[n] \\ \text{bijective poset} \\ \text{homomorphism}}} \Gamma_{\mathcal{Z}}\left(\gamma \circ x^{-1}\right).$$

- Thus, the $\Gamma_{\mathcal{Z}}$ of any labeled poset can be described in terms of the $\Gamma_{\mathcal{Z}}(\pi)$.

The product formula for the $\Gamma_{\mathcal{Z}}(P, \gamma)$

- Combining the above results, we see:
  Theorem. Let π and σ be two disjoint permutations. Then,

$$\Gamma_{\mathcal{Z}}(\pi) \cdot \Gamma_{\mathcal{Z}}(\sigma) = \sum_{\tau \in S(\pi,\sigma)} \Gamma_{\mathcal{Z}}(\tau).$$

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$$\Gamma_{\mathcal{Z}}(\pi) \cdot \Gamma_{\mathcal{Z}}(\sigma) = \sum_{\tau \in S(\pi,\sigma)} \Gamma_{\mathcal{Z}}(\tau).$$

- This generalizes the

$$F_{\operatorname{Comp} \pi} \cdot F_{\operatorname{Comp} \sigma} = \sum_{\tau \in S(\pi,\sigma)} F_{\operatorname{Comp} \tau}$$

  formula in QSym (which you can recover by setting $\mathcal{N} = \mathbb{N}$ and $\mathcal{Z} = \mathbb{N} \times \{+\} = \{+0 \prec +1 \prec +2 \prec \cdots\}$).
- Likewise, you can recover similar results by Stembridge and Petersen from this.

Customizing the setting for Epk

- Remember: we want to show Epk is shuffle-compatible.
- Specialize the above setting as follows:
  - Set $\mathcal{N} = \{0, 1, 2, \ldots\} \cup \{\infty\}$, with total order given by $0 \prec 1 \prec 2 \prec \cdots \prec \infty$.
  - Set

$$\begin{aligned} \mathcal{Z} &= (\mathcal{N} \times \{+, -\}) \setminus \{-0, +\infty\} \\ &= \{+0\} \cup \{+n \mid n \in \{1, 2, 3, \ldots\}\} \\ &\qquad \cup \{-n \mid n \in \{1, 2, 3, \ldots\}\} \cup \{-\infty\}. \end{aligned}$$

  Recall that the total order on $\mathcal{Z}$ has

$$+0 \prec -1 \prec +1 \prec -2 \prec +2 \prec \cdots \prec -\infty.$$

Fiber-ends

- Let $n \in \mathbb{N}$. Let $g : [n] \to \mathcal{N}$ be any map. We define a subset $\operatorname{FE}(g)$ of $[n]$ by

$$\operatorname{FE}(g) = \left\{\min\left(g^{-1}(h)\right) \mid h \in \{1, 2, 3, \ldots, \infty\}\right\} \cup \left\{\max\left(g^{-1}(h)\right) \mid h \in \{0, 1, 2, 3, \ldots\}\right\}$$

(ignore the maxima/minima of empty fibers).
In other words, $\operatorname{FE}(g)$ is the set comprising
    - the smallest elements of all nonempty fibers of g except for $g^{-1}(0)$ as well as
    - the largest elements of all nonempty fibers of g except for $g^{-1}(\infty)$.

- Let $n \in \mathbb{N}$. If Λ is any subset of $[n]$, then we define a power series $K^{\mathcal{Z}}_{n,\Lambda} \in \operatorname{Pow} \mathcal{N}$ by

$$K^{\mathcal{Z}}_{n,\Lambda} = \sum_{\substack{g:[n]\to\mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda\subseteq \operatorname{FE}(g)}} 2^{|g([n])\cap\{1,2,3,\ldots\}|} x_{g(1)} x_{g(2)} \cdots x_{g(n)}.$$

- **Proposition.** Let $n \in \mathbb{N}$. Let π be an n-permutation. Then,

$$\Gamma_{\mathcal{Z}}(\pi) = K^{\mathcal{Z}}_{n,\operatorname{Epk}\pi}.$$

  This is proven by a counting argument (if a map g comes from an $([n], \pi)$-partition, then the fibers of g subdivide $[n]$ into intervals on which π is "V-shaped"; a peak can only occur at a border between two such intervals).

- Let $n \in \mathbb{N}$. If Λ is any subset of $[n]$, then we define a power series $K_{n,\Lambda}^{\mathcal{Z}} \in \operatorname{Pow} \mathcal{N}$ by

$$K_{n,\Lambda}^{\mathcal{Z}} = \sum_{\substack{g:[n]\to\mathcal{N} \text{ is} \\ \text{weakly increasing;} \\ \Lambda\subseteq \operatorname{FE}(g)}} 2^{|g([n])\cap\{1,2,3,\ldots\}|} x_{g(1)} x_{g(2)} \cdots x_{g(n)}.$$

- **Proposition.** Let $n \in \mathbb{N}$. Let π be an n-permutation. Then,

$$\Gamma_{\mathcal{Z}}(\pi) = K_{n,\operatorname{Epk}\pi}^{\mathcal{Z}}.$$

- Thus, the product formula above specializes to

$$K_{n,\operatorname{Epk}\pi}^{\mathcal{Z}} \cdot K_{m,\operatorname{Epk}\sigma}^{\mathcal{Z}} = \sum_{\tau\in S(\pi,\sigma)} K_{n+m,\operatorname{Epk}\tau}^{\mathcal{Z}}.$$

- To prove that Epk is shuffle-compatible, we need this formula, but we **also** need to show that the "relevant" $K_{n,\Lambda}^{\mathcal{Z}}$ are linearly independent.

- Let $n \in \mathbb{N}$. If Λ is any subset of $[n]$, then we define a power series $K^{\mathcal{Z}}_{n,\Lambda} \in \operatorname{Pow} \mathcal{N}$ by

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- **Proposition.** Let $n \in \mathbb{N}$. Let π be an n-permutation. Then,

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- Thus, the product formula above specializes to

$$K^{\mathcal{Z}}_{n,\operatorname{Epk}\pi} \cdot K^{\mathcal{Z}}_{m,\operatorname{Epk}\sigma} = \sum_{\tau \in S(\pi,\sigma)} K^{\mathcal{Z}}_{n+m,\operatorname{Epk}\tau}.$$

- To prove that Epk is shuffle-compatible, we need this formula, but we **also** need to show that the "relevant" $K^{\mathcal{Z}}_{n,\Lambda}$ are linearly independent.
- Not all $K^{\mathcal{Z}}_{n,\Lambda}$ are linearly independent. Rather, we need to pick the right subset.

Lacunar subsets and linear independence

- A set S of integers is called *lacunar* if it contains no two consecutive integers.
- **Well-known fact:** The number of lacunar subsets of $[n]$ is the Fibonacci number f_{n+1}.

Lacunar subsets and linear independence

- A set S of integers is called *lacunar* if it contains no two consecutive integers.
- **Well-known fact:** The number of lacunar subsets of $[n]$ is the Fibonacci number f_{n+1}.
- **Lemma.** For each permutation π, the set $\operatorname{Epk} \pi$ is a nonempty lacunar subset of $[n]$.
  (And conversely – although we won't need it –, any such subset has the form $\operatorname{Epk} \pi$ for some π.)

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  (And conversely – although we won't need it –, any such subset has the form $\operatorname{Epk} \pi$ for some π.)
- **Lemma.** The family

  $$\left(K_{n,\Lambda}^{\mathcal{Z}}\right)_{n\in\mathbb{N};\ \Lambda\subseteq[n] \text{ is lacunar and nonempty}}$$

  is $\mathbb{Q}$-linearly independent.
- This actually takes work to prove. But once proven, it completes the argument for the shuffle-compatibility of Epk.

The kernel $\mathcal{K}_{\mathsf{Epk}}$

- Recall: The *kernel* $\mathcal{K}_{\mathsf{st}}$ of a descent statistic st is the $\mathbb{Q}$-vector subspace of QSym spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st-equivalent compositions:

$$\mathcal{K}_{\mathsf{st}} = \langle F_\alpha - F_\beta \mid |\alpha| = |\beta| \text{ and } \mathsf{st}\,\alpha = \mathsf{st}\,\beta \rangle_{\mathbb{Q}} .$$

- Recall: The *kernel* $\mathcal{K}_{\mathrm{st}}$ of a descent statistic st is the $\mathbb{Q}$-vector subspace of QSym spanned by all differences of the form $F_\alpha - F_\beta$, with α and β being two st-equivalent compositions:

$$\mathcal{K}_{\mathrm{st}} = \langle F_\alpha - F_\beta \ \mid \ |\alpha| = |\beta| \text{ and } \mathrm{st}\,\alpha = \mathrm{st}\,\beta \rangle_{\mathbb{Q}} .$$

- Since Epk is shuffle-compatible, its kernel $\mathcal{K}_{\mathrm{Epk}}$ is an ideal of QSym. How can we describe it?
- Two ways: using the F-basis and using the M-basis.

The kernel $\mathcal{K}_{\text{Epk}}$ in terms of the F-basis

- If $J = (j_1, j_2, \ldots, j_m)$ and K are two compositions, then we write $J \to K$ if there exists an $\ell \in \{2, 3, \ldots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \ldots, j_{\ell-1}, 1, j_\ell - 1, j_{\ell+1}, j_{\ell+2}, \ldots, j_m)$. (In other words, we write $J \to K$ if K can be obtained from J by "splitting" some non-initial entry $j_\ell > 2$ into two consecutive entries 1 and $j_\ell - 1$.)
- **Example.** Here are all instances of the $\to$ relation on compositions of size ≤ 5:

$$(1,3) \to (1,1,2), \qquad (1,4) \to (1,1,3),$$
$$(1,3,1) \to (1,1,2,1), \qquad (1,1,3) \to (1,1,1,2),$$
$$(2,3) \to (2,1,2).$$

- **Proposition.** The ideal $\mathcal{K}_{\text{Epk}}$ of QSym is spanned (as a $\mathbb{Q}$-vector space) by all differences of the form $F_J - F_K$, where J and K are two compositions satisfying $J \to K$.

The kernel $\mathcal{K}_{\text{Epk}}$ in terms of the M-basis

- If $J = (j_1, j_2, \ldots, j_m)$ and K are two compositions, then we write $J \underset{M}{\rightarrow} K$ if there exists an $\ell \in \{2, 3, \ldots, m\}$ such that $j_\ell > 2$ and $K = (j_1, j_2, \ldots, j_{\ell-1}, 2, j_\ell - 2, j_{\ell+1}, j_{\ell+2}, \ldots, j_m)$.
  (In other words, we write $J \underset{M}{\rightarrow} K$ if K can be obtained from J by "splitting" some non-initial entry $j_\ell > 2$ into two consecutive entries 2 and $j_\ell - 2$.)
- **Example.** Here are all instances of the $\underset{M}{\rightarrow}$ relation on compositions of size ≤ 5:

$$(1,3) \underset{M}{\rightarrow} (1,2,1), \qquad (1,4) \underset{M}{\rightarrow} (1,2,2),$$
$$(1,3,1) \underset{M}{\rightarrow} (1,2,1,1), \qquad (1,1,3) \underset{M}{\rightarrow} (1,1,2,1),$$
$$(2,3) \underset{M}{\rightarrow} (2,2,1).$$

- **Proposition.** The ideal $\mathcal{K}_{\text{Epk}}$ of QSym is spanned (as a $\mathbb{Q}$-vector space) by all sums of the form $M_J + M_K$, where J and K are two compositions satisfying $J \underset{M}{\rightarrow} K$.

What about other statistics?

- **Question.** Do other descent statistics allow for similar descriptions of $\mathcal{K}_{\mathrm{st}}$?

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- **Question.** Do other descent statistics allow for similar descriptions of $\mathcal{K}_{\mathrm{st}}$?
- **Example.**

$$\begin{aligned}\mathcal{K}_{\mathrm{des}} &= \langle F_I - F_J \ \mid \ |I| = |J| \text{ and } \ell(I) = \ell(J) \rangle_{\mathbb{Q}} \\ &= \langle M_I - M_J \ \mid \ |I| = |J| \text{ and } \ell(I) = \ell(J) \rangle_{\mathbb{Q}}\end{aligned}$$

(where $\ell(\alpha)$ denotes the length of a composition α).

Section 4

Left-/right-shuffle-compatibility

References:

- Darij Grinberg, *Shuffle-compatible permutation statistics II: the exterior peak set*, draft.
- Darij Grinberg, *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions*, Canad. J. Math. 69 (2017), pp. 21–53.

Left/right-shuffle-compatibility (repeated)

- We further begin the study of a finer version of shuffle-compatibility: "left/right-shuffle-compatibility".
- Given two disjoint nonempty permutations π and σ,
  - a *left shuffle* of π and σ is a shuffle of π and σ that starts with a letter of π;
  - a *right shuffle* of π and σ is a shuffle of π and σ that starts with a letter of σ.
- We let $S_{\prec}(\pi, \sigma)$ be the set of all left shuffles of π and σ. We let $S_{\succ}(\pi, \sigma)$ be the set of all right shuffles of π and σ.
- A statistic st is said to be *left-shuffle-compatible* if for any two disjoint nonempty permutations π and σ such that

  the first entry of π is greater than the first entry of σ,

  the multiset
  $$\{\operatorname{st}\tau \mid \tau \in S_{\prec}(\pi, \sigma)\}_{\text{multiset}}$$
  depends only on $\operatorname{st}\pi$, $\operatorname{st}\sigma$, $|\pi|$ and $|\sigma|$.
- We show that Des, des, Lpk and Epk are left- and right-shuffle-compatible. (But not maj or Rpk.)

Dendriform structure on QSym, introduction

- This proof will use a **dendriform algebra** structure on QSym, as well as two other operations and a bit of the Hopf algebra structure.
  I don't know of a combinatorial proof.
- This structure first appeared in:
  Darij Grinberg, *Dual immaculate creation operators and a dendriform algebra structure on the quasisymmetric functions*, Canad. J. Math. 69 (2017), pp. 21–53.
  But the ideas go back to:
  - Glânffrwd P. Thomas, *Frames, Young tableaux, and Baxter sequences*, Advances in Mathematics, Volume 26, Issue 3, December 1977, Pages 275–289.
  - Jean-Christophe Novelli, Jean-Yves Thibon, *Construction of dendriform trialgebras*, arXiv:math/0510218.

  Something similar also appeared in: Aristophanes Dimakis, Folkert Müller-Hoissen, *Quasi-symmetric functions and the KP hierarchy*, Journal of Pure and Applied Algebra, Volume 214, Issue 4, April 2010, Pages 449–460.

Dendriform structure on QSym, part 1

- For any monomial $\mathfrak{m}$, let $\operatorname{Supp} \mathfrak{m}$ denote the set $\{i \mid x_i \text{ appears in } \mathfrak{m}\}$.
- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.

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- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.
- We define a binary operation $\prec$ on the $\mathbb{Q}$-vector space $\mathbb{Q}\left[\left[x_1, x_2, x_3, \ldots\right]\right]$ as follows:
  - On monomials, it should be given by

    $$\mathfrak{m} \prec \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min\left(\operatorname{Supp}\mathfrak{m}\right) < \min\left(\operatorname{Supp}\mathfrak{n}\right); \\ 0, & \text{if } \min\left(\operatorname{Supp}\mathfrak{m}\right) \geq \min\left(\operatorname{Supp}\mathfrak{n}\right) \end{cases}$$

    for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbb{Q}$-bilinear.
  - It should be continuous (i.e., its $\mathbb{Q}$-bilinearity also applies to infinite $\mathbb{Q}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $\left(x_2^2 x_4\right) \prec \left(x_3^2 x_5\right) = x_2^2 x_3^2 x_4 x_5$, but $\left(x_2^2 x_4\right) \prec \left(x_2^2 x_5\right) = 0$.

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- **Example.** $\operatorname{Supp}\left(x_3^5 x_6 x_8\right) = \{3, 6, 8\}$.
- We define a binary operation $\succeq$ on the $\mathbb{Q}$-vector space $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ as follows:
  - On monomials, it should be given by
    $$\mathfrak{m} \succeq \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) \geq \min(\operatorname{Supp}\mathfrak{n}); \\ 0, & \text{if } \min(\operatorname{Supp}\mathfrak{m}) < \min(\operatorname{Supp}\mathfrak{n}) \end{cases}$$
    for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbb{Q}$-bilinear.
  - It should be continuous (i.e., its $\mathbb{Q}$-bilinearity also applies to infinite $\mathbb{Q}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $\left(x_2^2 x_4\right) \succeq \left(x_3^2 x_5\right) = 0$, but $\left(x_2^2 x_4\right) \succeq \left(x_2^2 x_5\right) = x_2^4 x_4 x_5$.

Dendriform structure on QSym, part 2

- We now have defined two binary operations $\prec$ and $\succeq$ on $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$. They satisfy:

$$a \prec b + a \succeq b = ab;$$
$$(a \prec b) \prec c = a \prec (bc);$$
$$(a \succeq b) \prec c = a \succeq (b \prec c);$$
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- This says that $(\mathbb{Q}[[x_1, x_2, x_3, \ldots]], \prec, \succeq)$ is a *dendriform algebra* in the sense of Loday (see, e.g., Zinbiel, *Encyclopedia of types of algebras 2010*, arXiv:1101.0267).

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- QSym is closed under both operations $\prec$ and $\succeq$. Thus, QSym becomes a dendriform subalgebra of $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$.

The kernel criterion for left/right-shuffle-compatibility

- Recall the **Theorem:** The descent statistic st is shuffle-compatible if and only if $\mathcal{K}_{\text{st}}$ is an ideal of QSym.

The kernel criterion for left/right-shuffle-compatibility

- Similarly, we have:
  - **Theorem.** The descent statistic st is left-shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is a $\prec$-ideal of QSym (that is: $\mathrm{QSym} \prec \mathcal{K}_{\mathrm{st}} \subseteq \mathcal{K}_{\mathrm{st}}$ and $\mathcal{K}_{\mathrm{st}} \prec \mathrm{QSym} \subseteq \mathcal{K}_{\mathrm{st}}$).
  - **Theorem.** The descent statistic st is right-shuffle-compatible if and only if $\mathcal{K}_{\mathrm{st}}$ is a $\succeq$-ideal of QSym (that is: $\mathrm{QSym} \succeq \mathcal{K}_{\mathrm{st}} \subseteq \mathcal{K}_{\mathrm{st}}$ and $\mathcal{K}_{\mathrm{st}} \succeq \mathrm{QSym} \subseteq \mathcal{K}_{\mathrm{st}}$).

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- **Corollary.** Let st be a descent statistic. If st has 2 of the 3 properties "shuffle-compatible", "left-shuffle-compatible" and "right-shuffle-compatible", then it has all 3.
(To prove this, recall $ab = a \prec b + a \succeq b$.)

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- **Question.** Are there non-shuffle-compatible but left-shuffle-compatible descent statistics? (I don't know of any, but haven't looked far.)

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  (To prove this, recall $ab = a \prec b + a \succeq b$.)
- Okay, but how do we actually prove that $\mathcal{K}_{\mathrm{st}}$ is a $\prec$-ideal of QSym ?

- An analogue of the product formula for $F_{\operatorname{Comp}\pi} \cdot F_{\operatorname{Comp}\sigma}$:
Theorem. Let π and σ be two disjoint nonempty permutations. Assume that

  the first entry of π is greater than the first entry of σ.

  Then,

$$F_{\operatorname{Comp}\pi} \prec F_{\operatorname{Comp}\sigma} = \sum_{\tau \in S_{\prec}(\pi,\sigma)} F_{\operatorname{Comp}\tau}$$

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- This theorem yields that Des is left-shuffle-compatible and right-shuffle-compatible, just as the product formula showed that Des is shuffle-compatible.
- Can we play the same game with Epk, using our $K_{n,\Lambda}^{\mathcal{Z}}$ series instead of F_α ?
  I don't know how. Instead, I use a different approach.

The Φ and $\divideontimes$ operations

- I need two other operations on quasisymmetric functions.
- We define a binary operation Φ on the $\mathbb{Q}$-vector space $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ as follows:
  - On monomials, it should be given by
    $$\mathfrak{m} \Phi \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \leq \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) > \min(\operatorname{Supp} \mathfrak{n}) \end{cases}$$
    for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbb{Q}$-bilinear.
  - It should be continuous (i.e., its $\mathbb{Q}$-bilinearity also applies to infinite $\mathbb{Q}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $\left(x_2^2 x_4\right) \Phi \left(x_4^2 x_5\right) = x_2^2 x_4^3 x_5$ and $\left(x_2^2 x_4\right) \Phi \left(x_3^2 x_5\right) = 0$.

- I need two other operations on quasisymmetric functions.
- We define a binary operation $\between$ on the $\mathbb{Q}$-vector space $\mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ as follows:
  - On monomials, it should be given by
    $$\mathfrak{m} \between \mathfrak{n} = \begin{cases} \mathfrak{m} \cdot \mathfrak{n}, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) < \min(\operatorname{Supp} \mathfrak{n}); \\ 0, & \text{if } \max(\operatorname{Supp} \mathfrak{m}) \geq \min(\operatorname{Supp} \mathfrak{n}) \end{cases}$$
    for any two monomials $\mathfrak{m}$ and $\mathfrak{n}$.
  - It should be $\mathbb{Q}$-bilinear.
  - It should be continuous (i.e., its $\mathbb{Q}$-bilinearity also applies to infinite $\mathbb{Q}$-linear combinations).
- Well-definedness is pretty clear.
- **Example.** $(x_2^2 x_4) \between (x_4^2 x_5) = 0$, but $(x_2^2 x_4) \between (x_5^2 x_6) = x_2^2 x_4 x_5^2 x_6$.

The ᛄ and ᛯ operations

- QSym is closed under both operations ᛄ and ᛯ.
- Belgthor (ᛄ) and Tvimadur (ᛯ) are calendar runes (for two of the 19 years of the Metonic cycle).
  I sought two (unused) symbols that (roughly) look like "stacking one thing (monomial) atop another", allowing overlap (ᛄ) and disallowing overlap (ᛯ).

A crucial identity

- **Proposition.** For any $a \in \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ and $b \in \mathrm{QSym}$, we have
$$\sum_{(b)} \left(S\left(b_{(1)}\right) \mathbin{\phi} a \right) b_{(2)} = a \prec b,$$
where we use the Hopf algebra structure on QSym and the following notations:
    - S for the antipode of QSym;
    - Sweedler's notation $\sum_{(b)} b_{(1)} \otimes b_{(2)}$ for $\Delta(b)$.

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- **Proposition.** For any $a \in \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$ and $b \in \operatorname{QSym}$, we have
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where we use the Hopf algebra structure on QSym .
Restatement without using Hopf algebras:
$$\sum_{p=0}^{k} (-1)^p \left(M'_{\alpha_p, \alpha_{p-1}, \ldots, \alpha_1} \between a \right) M_{\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_k} = a \prec M_\alpha$$
for any composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ and any $a \in \mathbb{Q}[[x_1, x_2, x_3, \ldots]]$, where
$$M'_{\alpha_p, \alpha_{p-1}, \ldots, \alpha_1} = \sum_{i_1 \leq i_2 \leq \cdots \leq i_p} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_p}^{\alpha_p}.$$

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$$\sum_{(b)} \left(S\left(b_{(1)}\right) \,\Phi\, a\right) b_{(2)} = a \prec b,$$
where we use the Hopf algebra structure on QSym .
- This proposition was important in my study of "dual immaculate creation operators"; it is equally helpful here.
Corollary. Let M be an ideal of QSym. If $\operatorname{QSym} \,\Phi\, M \subseteq M$, then $M \prec \operatorname{QSym} \subseteq M$.

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Corollary. Let M be an ideal of QSym. If $\mathrm{QSym} ⚹ M \subseteq M$, then $\mathrm{QSym} \succeq M \subseteq M$.

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where we use the Hopf algebra structure on QSym .
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Corollary. Let M be an ideal of QSym. If $\operatorname{QSym} \Phi\, M \subseteq M$, then $M \prec \operatorname{QSym} \subseteq M$.
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Corollary. Let M be an ideal of QSym. If $\operatorname{QSym} ⚹ M \subseteq M$, then $\operatorname{QSym} \succeq M \subseteq M$.
- **Corollary.** Let M be an ideal of QSym that is a left Φ-ideal (that is, $\operatorname{QSym} \Phi\, M \subseteq M$) and a left $⚹$-ideal (that is, $\operatorname{QSym} ⚹ M \subseteq M$). Then, M is a $\prec$-ideal and a $\succeq$-ideal of QSym.

“Runic calculus”

- The operations ᛰ and ᛯ are associative and unital (with unity 1).

“Runic calculus”

- The operations Φ and ᚼ are associative and unital (with unity 1).
- For any two nonempty (i.e., $\neq ()$) compositions α and β, we have

$$\begin{aligned} M_\alpha \mathbin{\Phi} M_\beta &= M_{[\alpha,\beta]} + M_{\alpha\odot\beta}; \\ M_\alpha \mathbin{\text{ᚼ}} M_\beta &= M_{[\alpha,\beta]}; \\ F_\alpha \mathbin{\Phi} F_\beta &= F_{\alpha\odot\beta}; \\ F_\alpha \mathbin{\text{ᚼ}} F_\beta &= F_{[\alpha,\beta]}, \end{aligned}$$

where $[\alpha, \beta]$ and $\alpha \odot \beta$ are two compositions defined by

$$\begin{aligned} &[(\alpha_1, \alpha_2, \ldots, \alpha_\ell), (\beta_1, \beta_2, \ldots, \beta_m)] \\ &= (\alpha_1, \alpha_2, \ldots, \alpha_\ell, \beta_1, \beta_2, \ldots, \beta_m) \end{aligned}$$

and

$$\begin{aligned} &(\alpha_1, \alpha_2, \ldots, \alpha_\ell) \odot (\beta_1, \beta_2, \ldots, \beta_m) \\ &= (\alpha_1, \alpha_2, \ldots, \alpha_{\ell-1}, \alpha_\ell + \beta_1, \beta_2, \beta_3, \ldots, \beta_m). \end{aligned}$$

“Runic calculus”

- The operations Φ and ᚼ are associative and unital (with unity 1).
- They satisfy

$$(a \mathbin{\Phi} b) \mathbin{\text{ᚼ}} c - a \mathbin{\Phi} (b \mathbin{\text{ᚼ}} c) = \varepsilon(b)(a \mathbin{\text{ᚼ}} c - a \mathbin{\Phi} c);$$
$$(a \mathbin{\text{ᚼ}} b) \mathbin{\Phi} c - a \mathbin{\text{ᚼ}} (b \mathbin{\Phi} c) = \varepsilon(b)(a \mathbin{\Phi} c - a \mathbin{\text{ᚼ}} c),$$

  where $\varepsilon : \mathbb{Q}[[x_1, x_2, x_3, \ldots]] \to \mathbb{Q}$ sends f to $f(0, 0, 0, \ldots)$.
- As a consequence,

$$(a \mathbin{\Phi} b) \mathbin{\text{ᚼ}} c + (a \mathbin{\text{ᚼ}} b) \mathbin{\Phi} c = a \mathbin{\Phi} (b \mathbin{\text{ᚼ}} c) + a \mathbin{\text{ᚼ}} (b \mathbin{\Phi} c).$$

  This says that (QSym, Φ, ᚼ) is a $As^{\langle 2 \rangle}$*-algebra* (in the sense of Loday).
- **Question.** What other identities do Φ, ᚼ, $\prec$ and $\succeq$ satisfy?

How to check left-/right-shuffle-compatibility

- Recall the **Corollary:** Let M be an ideal of QSym that is a left Φ-ideal (that is, $\mathrm{QSym} \Phi M \subseteq M$) and a left $Ж$-ideal (that is, $\mathrm{QSym}\ Ж\ M \subseteq M$). Then, M is a $\prec$-ideal and a $\succeq$-ideal of QSym.

How to check left-/right-shuffle-compatibility

- Recall the **Corollary:** Let M be an ideal of QSym that is a left Φ-ideal (that is, $\operatorname{QSym} \Phi M \subseteq M$) and a left $\Join$-ideal (that is, $\operatorname{QSym} \Join M \subseteq M$). Then, M is a $\prec$-ideal and a $\succeq$-ideal of QSym.
- Given a shuffle-compatible descent statistic st, we thus conclude that if $\mathcal{K}_{\mathrm{st}}$ is a left Φ-ideal and a left $\Join$-ideal, then st is left-shuffle-compatible and right-shuffle-compatible.

How to check left-/right-shuffle-compatibility

- Recall the **Corollary:** Let M be an ideal of QSym that is a left Φ-ideal (that is, QSym Φ $M \subseteq M$) and a left $\between$-ideal (that is, QSym $\between$ $M \subseteq M$). Then, M is a $\prec$-ideal and a $\succeq$-ideal of QSym.
- Given a shuffle-compatible descent statistic st, we thus conclude that if $\mathcal{K}_{\text{st}}$ is a left Φ-ideal and a left $\between$-ideal, then st is left-shuffle-compatible and right-shuffle-compatible.
- Fortunately, this is easy to apply:

  Proposition. Let st be a descent statistic.

  - $\mathcal{K}_{\text{st}}$ is a left Φ-ideal of QSym if and only if st has the following property: If J and K are two st-equivalent nonempty compositions, and if G is any nonempty composition, then $G \odot J$ and $G \odot K$ are st-equivalent.
  - $\mathcal{K}_{\text{st}}$ is a left $\between$-ideal of QSym if and only if st has the following property: If J and K are two st-equivalent nonempty compositions, and if G is any nonempty composition, then $[G, J]$ and $[G, K]$ are st-equivalent.

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  - $\mathcal{K}_{\text{st}}$ is a left Φ-ideal of QSym if and only if for each fixed nonempty composition A, the value $\operatorname{st}(A \odot B)$ (for a nonempty composition B) is uniquely determined by $|B|$ and $\operatorname{st} B$.
  - $\mathcal{K}_{\text{st}}$ is a left $\between$-ideal of QSym if and only if for each fixed nonempty composition A, the value $\operatorname{st}([A, B])$ (for a nonempty composition B) is uniquely determined by $|B|$ and $\operatorname{st} B$.

Why Epk is left- and right-shuffle-compatible

- Thus, proving that Epk is left- and right-shuffle-compatible requires showing that $\operatorname{Epk}(A \odot B)$ and $\operatorname{Epk}([A, B])$ (for nonempty compositions A and B) are uniquely determined by $|B|$ and $\operatorname{Epk} B$ when A is fixed.

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- This is not hard:

$$\operatorname{Epk}(A \odot B) = ((\operatorname{Epk} A) \setminus \{n\}) \cup (\operatorname{Epk} B + n);$$
$$\operatorname{Epk}([A, B]) = (\operatorname{Epk} A) \cup ((\operatorname{Epk} B + n) \setminus \{n+1\}),$$

where $n = |A|$.

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  where $n = |A|$.
- Similarly,
  - Des is left- and right-shuffle-compatible (again);
  - des is left- and right-shuffle-compatible;
  - maj is **not** left- or right-shuffle-compatible ($\operatorname{maj}(A \odot B)$ and $\operatorname{maj}([A, B])$ depend not just on $|A|$, $|B|$, $\operatorname{maj} A$ and $\operatorname{maj} B$, but also on $\operatorname{des} B$).

Why Epk is left- and right-shuffle-compatible

- Thus, proving that Epk is left- and right-shuffle-compatible requires showing that $\operatorname{Epk}(A \odot B)$ and $\operatorname{Epk}([A, B])$ (for nonempty compositions A and B) are uniquely determined by $|B|$ and $\operatorname{Epk} B$ when A is fixed.
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  where $n = |A|$.
- Similarly,
  - $(\operatorname{des}, \operatorname{maj})$ is left- and right-shuffle-compatible;
  - Lpk is left- and right-shuffle-compatible;
  - Rpk is **not** left- or right-shuffle-compatible;
  - Pk is **not** left- or right-shuffle-compatible.
- More statistics remain to be analyzed.

- **Question (repeated).** Can a statistic be shuffle-compatible without being a descent statistic?
  (Would FQSym help in studying such statistics?)
- **Question (repeated).** Can a descent statistic be left-shuffle-compatible without being shuffle-compatible?
- **Question.** What mileage do we get out of $\mathcal{Z}$-enriched (P, γ)-partitions for other choices of $\mathcal{N}$ and $\mathcal{Z}$?
- **Question (repeated).** Where do the $\Gamma_{\mathcal{Z}}(P, \gamma)$ live?
- **Question.** Hsiao and Petersen have generalized enriched (P, γ)-partitions to "colored (P, γ)-partitions" (with $\{+, -\}$ replaced by an m-element set). Does this generalize our results?

Thanks

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slides: http://www.cip.ifi.lmu.de/~grinberg/algebra/urbana18b.pdf
paper: http://www.cip.ifi.lmu.de/~grinberg/algebra/gzshuf2.pdf
project: https://github.com/darijgr/gzshuf