Vertex algebras by Victor Kac. *Lecture 3*: Fundamentals of formal distributions

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Contents

Introduction

These notes introduce some algebraic analogues of notions from classical analysis such as Laurent series, differential operators, formal distributions and the δ-function. We believe that everything written below is well-known ("folklore"), but a good deal of it is not easy to find in the literature. The notes are self-contained and should be understandable for anyone who has prior experience with (multivariate) formal power series, binomial coefficients and localization of commutative rings.

Chapters 3 and 4 are scribe notes of a lecture in Victor Kac's 18.276 class at MIT in Spring 2015.[1] Chapters 1 and 2 are meant to set the stage and define the requisite objects beforehand.

Please let me (`darijgrinberg@gmail.com`) know of any mistakes!

1. Basics on polynomial-like objects

1.1. Polynomial-like objects

Let us first define the objects which we are going to deal with. These objects will be certain analogues of polynomials and power series. Some of them are well-known, but some are apocryphal. We assume that the reader has some familiarity with polynomials and formal power series; this will allow us to be brief about them and also about the more baroque objects that we introduce when they behave similarly to polynomials and formal power series.

In the following, $\mathbb{F}$ denotes a commutative ring[2], and $\mathbb{N}$ denotes $\{0, 1, 2, \ldots\}$. When R is a ring (for instance, $\mathbb{F}$), the word "R-algebra" means "associative central R-algebra with 1", unless the word "algebra" is qualified by additional adjectives (such as in "superalgebra", "Lie algebra" or "vertex algebra"). We use the notation $\delta_{i,j}$ for $\begin{cases} 1, & \text{if } i = j; \\ 0, & \text{if } i \neq j \end{cases}$ whenever i and j are any two objects.

We shall work with the usual notion of $\mathbb{F}$-modules in the following, but we shall keep in the back of our mind that all of our arguments can be adapted (using the Koszul-Quillen sign convention) to the setting of $\mathbb{F}$-supermodules (at the cost of sometimes having to require that 2 is invertible in $\mathbb{F}$).

If U is a $\mathbb{F}$-module and z is a symbol, then we can consider the following five $\mathbb{F}$-modules:

- The $\mathbb{F}$-module $U[z]$ of polynomials in z with coefficients in U. This $\mathbb{F}$-module consists of all families $(u_i)_{i\in\mathbb{N}} \in U^{\mathbb{N}}$ such that all but finitely many $i \in \mathbb{N}$ satisfy $u_i = 0$. We write such families $(u_i)_{i\in\mathbb{N}}$ in the forms $\sum\limits_{i=0}^{\infty} u_i z^i$

[1]The class is about vertex algebras, but no vertex algebras appear in these notes.
[2]Here and in the following, "ring" always means "ring with 1".

and $\sum_{i \in \mathbb{N}} u_i z^i$, and refer to them as "polynomials" (despite U not necessarily being a ring).

- The $\mathbb{F}$-module $U\left[z, z^{-1}\right]$ of Laurent polynomials in z with coefficients in U. This $\mathbb{F}$-module consists of all families $(u_i)_{i \in \mathbb{Z}} \in U^{\mathbb{Z}}$ such that all but finitely many $i \in \mathbb{Z}$ satisfy $u_i = 0$. We write such families $(u_i)_{i \in \mathbb{Z}}$ in the forms $\sum_{i=-\infty}^{\infty} u_i z^i$ and $\sum_{i \in \mathbb{Z}} u_i z^i$, and refer to them as "Laurent polynomials".
- The $\mathbb{F}$-module $U[[z]]$ of formal power series in z with coefficients in U. This $\mathbb{F}$-module consists of all families $(u_i)_{i \in \mathbb{N}} \in U^{\mathbb{N}}$. We write such families $(u_i)_{i \in \mathbb{N}}$ in the forms $\sum_{i=0}^{\infty} u_i z^i$ and $\sum_{i \in \mathbb{N}} u_i z^i$, and refer to them as "power series" (or "formal power series").
- The $\mathbb{F}$-module $U((z))$ of Laurent series in z with coefficients in U. This $\mathbb{F}$-module consists of all families $(u_i)_{i \in \mathbb{Z}} \in U^{\mathbb{Z}}$ such that all but finitely many **negative** $i \in \mathbb{Z}$ satisfy $u_i = 0$. We write such families $(u_i)_{i \in \mathbb{Z}}$ in the forms $\sum_{i=-\infty}^{\infty} u_i z^i$ and $\sum_{i \in \mathbb{Z}} u_i z^i$, and refer to them as "Laurent series".
- The $\mathbb{F}$-module $U\left[\left[z, z^{-1}\right]\right]$ of U-valued formal distributions. This $\mathbb{F}$-module consists of all families $(u_i)_{i \in \mathbb{Z}} \in U^{\mathbb{Z}}$. We write such families $(u_i)_{i \in \mathbb{Z}}$ in the forms $\sum_{i=-\infty}^{\infty} u_i z^i$ and $\sum_{i \in \mathbb{Z}} u_i z^i$, and refer to them as "$U$-valued formal distributions" (in analogy to the distributions of analysis, although these formal distributions are much more elementary than the latter).

The $\mathbb{F}$-module structure on each of these five $\mathbb{F}$-modules $U[z]$, $U\left[z, z^{-1}\right]$, $U[[z]]$, $U((z))$ and $U\left[\left[z, z^{-1}\right]\right]$ is defined to be componentwise (i.e., we have $(u_i)_{i \in \mathbb{N}} + (v_i)_{i \in \mathbb{N}} = (u_i + v_i)_{i \in \mathbb{N}}$ and $\lambda \cdot (u_i)_{i \in \mathbb{N}} = (\lambda u_i)_{i \in \mathbb{N}}$ for $U[z]$, and similar rules with $\mathbb{N}$ replaced by $\mathbb{Z}$ for the other four $\mathbb{F}$-modules). Of course, we have

$$U \subseteq U[z] \subseteq U\left[z, z^{-1}\right] \subseteq U((z)) \subseteq U\left[\left[z, z^{-1}\right]\right] \qquad \text{and}$$
$$U[z] \subseteq U[[z]] \subseteq U\left[\left[z, z^{-1}\right]\right].$$

We refer to the elements of any of the five $\mathbb{F}$-modules $U[z]$, $U\left[z, z^{-1}\right]$, $U[[z]]$, $U((z))$ and $U\left[\left[z, z^{-1}\right]\right]$ as "polynomial-like objects". Given such an object – say, $\sum_{i \in \mathbb{Z}} u_i z^i$ –, we shall refer to the elements u_i of U as its *coefficients*. More precisely, if $u = \sum_{i \in \mathbb{Z}} u_i z^i$, then u_i is called the *i-th coefficient* of u (or the *coefficient of z^i in u*).

Often, $U\left[z, z^{-1}\right]$ is denoted by $U\left[z^{\pm 1}\right]$, and $U\left[\left[z, z^{-1}\right]\right]$ is denoted by $U\left[\left[z^{\pm 1}\right]\right]$.

We shall use standard notations for elements of $U[z]$, $U[z,z^{-1}]$, $U[[z]]$, $U((z))$ and $U[[z,z^{-1}]]$. For instance, for given $u \in U$ and $m \in \mathbb{Z}$, we let uz^m denote the element $(u\delta_{i,m})_{i\in\mathbb{Z}}$ of $U[z,z^{-1}]$ (that is, the Laurent polynomial whose m-th coefficient is u and whose all other coefficients are 0). This is also an element of $U[z]$ when $m \in \mathbb{N}$. When $U = \mathbb{F}$ and $m \in \mathbb{Z}$, we write z^m for $1z^m$. We abbreviate z^1 by z.

When U has additional structure (such as a multiplication, or a module structure over some $\mathbb{F}$-algebra), we can endow some of the five $\mathbb{F}$-modules $U[z]$, $U[z,z^{-1}]$, $U[[z]]$, $U((z))$ and $U[[z,z^{-1}]]$ with additional structure as well. Here are some examples:

- If U is a $\mathbb{F}$-algebra[3], then $U[z]$, $U[z,z^{-1}]$, $U[[z]]$ and $U((z))$ become $\mathbb{F}$-algebras, with multiplication given by the rule

$$\left(\sum_i u_i z^i\right) \cdot \left(\sum_i v_i z^i\right) = \sum_i \left(\sum_j u_j v_{i-j}\right) z^i \tag{1}$$

  (and unity defined to be $\sum\limits_i \delta_{i,0} z^i$). Here, the sums range over $\mathbb{N}$ in the case of $U[z]$, and over $\mathbb{Z}$ in the cases of $U[z,z^{-1}]$, $U[[z]]$ and $U((z))$. When U is commutative, then these four $\mathbb{F}$-algebras are commutative U-algebras (with the action of U being componentwise). The case of $U = \mathbb{F}$ is the one most frequently encountered.

  However, $U[[z,z^{-1}]]$ does not become a $\mathbb{F}$-algebra in this way, not even for $U = \mathbb{F}$. In fact, attempting to compute $\left(\sum\limits_{i\in\mathbb{Z}} z^i\right) \cdot \left(\sum\limits_{i\in\mathbb{Z}} z^i\right)$ according to (1) would lead to the result $\sum\limits_{i\in\mathbb{Z}} \left(\sum\limits_{j\in\mathbb{Z}} 1 \cdot 1\right) z^i$, which makes no sense (as the inner sum $\sum\limits_{j\in\mathbb{Z}} 1 \cdot 1$ diverges in any meaningful topology). This is probably the reason why you rarely see $U[[z,z^{-1}]]$ studied in literature; its elements cannot be multiplied[4]. However, it is at least possible to multiply elements of $U[[z,z^{-1}]]$ with Laurent polynomials (i.e., elements of $U[z,z^{-1}]$); this multiplication again is defined according to (1), and it is well-defined because of the "all but finitely many $i \in \mathbb{N}$ satisfy $u_i = 0$" condition in the definition of $U[z,z^{-1}]$. This multiplication makes $U[[z,z^{-1}]]$ into a $U[z,z^{-1}]$-module. This module usually contains torsion, however:

[3]This includes $U = \mathbb{F}$ as a particular case.

[4]This is similar to the lack of a reasonable notion of product of distributions in analysis.

we have

$$(1-z)\cdot\left(\sum_{i\in\mathbb{Z}} z^i\right) = \sum_{i\in\mathbb{Z}} z^i - \sum_{i\in\mathbb{Z}} \underbrace{z\cdot z^i}_{=z^{i+1}} = \sum_{i\in\mathbb{Z}} z^i - \underbrace{\sum_{i\in\mathbb{Z}} z^{i+1}}_{=\sum_{i\in\mathbb{Z}} z^i}$$
$$= \sum_{i\in\mathbb{Z}} z^i - \sum_{i\in\mathbb{Z}} z^i = 0.$$

- More generally, if V is a $\mathbb{F}$-algebra and U is a V-module, then $U[z]$ becomes a $V[z]$-module, $U[z,z^{-1}]$ becomes a $V[z,z^{-1}]$-module, $U[[z]]$ becomes a $V[[z]]$-module, $U((z))$ becomes a $V((z))$-module, and $U[[z,z^{-1}]]$ becomes a $V[z,z^{-1}]$-module. Again, the actions are given by (1). This is particularly useful in the case when $V=\mathbb{F}$. This particular case shows that whenever U is an $\mathbb{F}$-module, the $\mathbb{F}$-module $U[z]$ becomes an $\mathbb{F}[z]$-module, $U[z,z^{-1}]$ becomes an $\mathbb{F}[z,z^{-1}]$-module, $U[[z]]$ becomes an $\mathbb{F}[[z]]$-module, $U((z))$ becomes an $\mathbb{F}((z))$-module, and $U[[z,z^{-1}]]$ becomes an $\mathbb{F}[z,z^{-1}]$-module. In particular, all five $\mathbb{F}$-modules $U[z]$, $U[z,z^{-1}]$, $U[[z]]$, $U((z))$ and $U[[z,z^{-1}]]$ thus become $\mathbb{F}[z]$-modules.

- The product of an element of $\mathbb{F}[z,z^{-1}]$ with an element of $U[[z,z^{-1}]]$ is well-defined and belongs to $U[[z,z^{-1}]]$ (since $U[[z,z^{-1}]]$ is an $\mathbb{F}[z,z^{-1}]$-module). The product of an element of $\mathbb{F}[[z,z^{-1}]]$ with an element of $U[z,z^{-1}]$ is also well-defined, and also belongs to $U[[z,z^{-1}]]$; but this notion of product makes neither $U[z,z^{-1}]$ nor $U[[z,z^{-1}]]$ into an $\mathbb{F}[[z,z^{-1}]]$-module (yet it is useful nevertheless).

- We defined $\mathbb{F}$-algebra structures on $U[z]$, $U[z,z^{-1}]$, $U[[z]]$ and $U((z))$ for any $\mathbb{F}$-algebra U. Here, we only used the associativity of U to ensure that these new $\mathbb{F}$-algebra structures are associative, and we only used the unity of U to construct a unity for these new $\mathbb{F}$-algebra structures. Thus, in the same way, we can obtain nonassociative nonunital $\mathbb{F}$-algebra structures on $U[z]$, $U[z,z^{-1}]$, $U[[z]]$ and $U((z))$ whenever U is a nonassociative nonunital $\mathbb{F}$-algebra. In particular, this construction works for Lie algebras: If $\mathfrak{g}$ is a $\mathbb{F}$-Lie algebra, then $\mathfrak{g}[z]$, $\mathfrak{g}[z,z^{-1}]$, $\mathfrak{g}[[z]]$ and $\mathfrak{g}((z))$ become $\mathbb{F}$-Lie algebras, with Lie bracket defined by the rule

$$\left[\sum_i u_i z^i, \sum_i v_i z^i\right] = \sum_i\left(\sum_j [u_j, v_{i-j}]\right) z^i.$$

[5]

[5]**Exercise:** Check that $\mathfrak{g}[z]$, $\mathfrak{g}[z,z^{-1}]$, $\mathfrak{g}[[z]]$ and $\mathfrak{g}((z))$ actually become $\mathbb{F}$-Lie algebras using this definition. (This is not completely obvious, because the $[a,a]=0$ axiom does not directly get inherited from $\mathfrak{g}$. But it is still easy to check.)

- Given any $\mathbb{Z}$-grading on an $\mathbb{F}$-module U and any integer d, we can define a $\mathbb{Z}$-grading on $U[z]$ by giving each uz^i (for $u \in U$ homogeneous and $i \in \mathbb{N}$) the degree $\deg u + id$. Similarly, we can define a $\mathbb{Z}$-grading on $U[z, z^{-1}]$ (but not on $U[[z]]$, $U((z))$ or $U[[z, z^{-1}]]$, unless we content ourselves with an "almost-grading" [6]). These gradings turn $U[z]$ and $U[z, z^{-1}]$ into graded $\mathbb{F}$-algebras. When the $\mathbb{Z}$-grading on U is trivial (i.e., everything in U is homogeneous of degree 0) and $d = 0$, these gradings are the "grading by degree" (i.e., each uz^i has degree i).

- The $\mathbb{F}$-modules $U[z]$, $U[z, z^{-1}]$, $U[[z]]$, $U((z))$ and $U[[z, z^{-1}]]$ are automatically endowed with topologies, which are defined as follows: Endow $U^{\mathbb{Z}}$ (a direct product of infinitely many copies of U) with the direct-product topology (where each copy of U is given the discrete topology), and pull back this topology onto $U[[z, z^{-1}]]$ via the $\mathbb{F}$-module isomorphism $U[[z, z^{-1}]] \to U^{\mathbb{Z}}$, $\sum\limits_{i \in \mathbb{Z}} u_i z^i \mapsto (u_i)_{i \in \mathbb{Z}}$. This defines a topology on $U[[z, z^{-1}]]$. Topologies on the four $\mathbb{F}$-modules $U[z]$, $U[z, z^{-1}]$, $U[[z]]$ and $U((z))$ are defined by restricting this topology (since these four $\mathbb{F}$-modules are $\mathbb{F}$-submodules of $U[[z, z^{-1}]]$). These topologies are called "topologies of coefficientwise convergence", due to the following fact (which we state for $U[[z, z^{-1}]]$ as an example): A sequence $\left(u_{(n)}\right)_{u \in \mathbb{N}}$ of formal distributions $u_{(n)} \in U[[z, z^{-1}]]$ converges to a formal distribution $u \in U[[z, z^{-1}]]$ with respect to this topology if and only if for every $i \in \mathbb{Z}$, we have

$$\left(\text{the } i\text{-th coefficient of } u_{(n)}\right) = (\text{the } i\text{-th coefficient of } u) \qquad \text{for all sufficiently high } n \in \mathbb{N}.$$

  The topologies that we introduced are Hausdorff and respect the $\mathbb{F}$-module structures (i.e., they turn our $\mathbb{F}$-modules into topological $\mathbb{F}$-modules). They also respect multiplication when U is an $\mathbb{F}$-algebra[7]. Furthermore, the set $U[z]$ is dense $U[[z]]$, and the set $U[z, z^{-1}]$ is dense in each of $U((z))$ and $U[[z, z^{-1}]]$. This ensures that, when we are proving certain kinds of identities in $U[[z]]$, $U((z))$ or $U[[z, z^{-1}]]$ (namely, the kind where both sides depend continuously on the inputs), we can WLOG assume that the inputs are polynomials (for $U[[z]]$) resp. Laurent polynomials (for $U((z))$ and for $U[[z, z^{-1}]]$).

[6] By an "almost-grading" of a topological $\mathbb{F}$-module P, I mean a family $(P_n)_{n \in \mathbb{Z}}$ of $\mathbb{F}$-submodules $P_n \subseteq P$ such that the internal direct sum $\bigoplus\limits_{n \in \mathbb{Z}} P_n$ is well-defined and is dense in P. Such almost-gradings exist on $U[[z]]$, $U((z))$ and $U[[z, z^{-1}]]$.

[7] Here is what we mean by this: If U is an $\mathbb{F}$-algebra, then the multiplication maps of the $\mathbb{F}$-algebras $U[z]$, $U[z, z^{-1}]$, $U[[z]]$ and $U((z))$ are continuous with respect to these topologies, and so are the maps $U[z, z^{-1}] \times U[[z, z^{-1}]] \to U[[z, z^{-1}]]$ and $U[[z, z^{-1}]] \times U[z, z^{-1}] \to U[[z, z^{-1}]]$ that send every (f, g) to fg.

1.2. Polynomial-like objects in multiple variables

We have so far studied the case of a single variable z; but it is possible to define similar structures in multiple variables. Let us briefly sketch how they are defined. Let U be a $\mathbb{F}$-module, and let $\left(x_j\right)_{j\in J}$ be a family of symbols. We briefly denote the family $\left(x_j\right)_{j\in J}$ as $\mathbf{x}$.

- The $\mathbb{F}$-module $U\left[\mathbf{x}\right] = U\left[x_j \mid j \in J\right]$ consists of all families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{N}^J} \in U^{\mathbb{N}^J}$ such that all but finitely many $\mathbf{i} \in \mathbb{N}^J$ satisfy $u_{\mathbf{i}} = 0$. We write such families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{N}^J}$ in the form $\sum\limits_{\mathbf{i}\in\mathbb{N}^J} u_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, and refer to them as "polynomials" in the x_j with coefficients in U.

- The $\mathbb{F}$-module $U\left[\mathbf{x},\mathbf{x}^{-1}\right] = U\left[x_j, x_j^{-1} \mid j \in J\right]$ consists of all families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^J} \in U^{\mathbb{Z}^J}$ such that all but finitely many $\mathbf{i} \in \mathbb{Z}^J$ satisfy $u_{\mathbf{i}} = 0$. We write such families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^J}$ in the form $\sum\limits_{\mathbf{i}\in\mathbb{Z}^J} u_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, and refer to them as "Laurent polynomials" in the x_j with coefficients in U.

- The $\mathbb{F}$-module $U\left[\left[\mathbf{x}\right]\right] = U\left[\left[x_j \mid j \in J\right]\right]$ consists of all families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{N}^J} \in U^{\mathbb{N}^J}$. We write such families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{N}^J}$ in the form $\sum\limits_{\mathbf{i}\in\mathbb{N}^J} u_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, and refer to them as "formal power series" in the x_j with coefficients in U.

- The $\mathbb{F}$-module $U\left(\left(\mathbf{x}\right)\right) = U\left(\left(x_j \mid j \in J\right)\right)$ consists of all families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^J} \in U^{\mathbb{Z}^J}$ for which there exists an $N \in \mathbb{Z}$ such that all $\mathbf{i} \in \mathbb{Z}^J \setminus \left\{N, N+1, N+2, \ldots\right\}^J$ satisfy $u_{\mathbf{i}} = 0$. We write such families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^J}$ in the form $\sum\limits_{\mathbf{i}\in\mathbb{Z}^J} u_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, and refer to them as "Laurent series" in the x_j with coefficients in U. (This is the only definition you might not have immediately guessed.)

- The $\mathbb{F}$-module $U\left[\left[\mathbf{x},\mathbf{x}^{-1}\right]\right] = U\left[\left[x_j, x_j^{-1} \mid j \in J\right]\right]$ consists of all families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^J} \in U^{\mathbb{Z}^J}$. We write such families $\left(u_{\mathbf{i}}\right)_{\mathbf{i}\in\mathbb{Z}^J}$ in the form $\sum\limits_{\mathbf{i}\in\mathbb{Z}^J} u_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}$, and refer to them as "$U$-valued formal distributions" in the x_j with coefficients in U.

Again, the first four of these five $\mathbb{F}$-modules become $\mathbb{F}$-algebras when U itself is a $\mathbb{F}$-algebra. The multiplication rule, like (1), is "what you would expect" if you are told that $\mathbf{x}^{\mathbf{i}}$ stands for the monomial $\prod\limits_{j\in J} x_j^{i_j}$ (where $\mathbf{i} = \left(i_j\right)_{j\in J}$). Explicitly:

$$\left(\sum_{\mathbf{i}} u_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\right) \cdot \left(\sum_{\mathbf{i}} v_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\right) = \sum_{\mathbf{i}} \left(\sum_{\mathbf{j}} u_{\mathbf{j}} v_{\mathbf{i}-\mathbf{j}}\right) \mathbf{x}^{\mathbf{i}},$$

where the subtraction on $\mathbb{N}^J$ and on $\mathbb{Z}^J$ is componentwise. Again, the same formula can be used to make $U\left[\left[\mathbf{x},\mathbf{x}^{-1}\right]\right]$ into a $U\left[\mathbf{x},\mathbf{x}^{-1}\right]$-module.

Again, $U\left[\mathbf{x},\mathbf{x}^{-1}\right]$ and $U\left[\left[\mathbf{x},\mathbf{x}^{-1}\right]\right]$ are often called $U\left[\mathbf{x}^{\pm 1}\right]$ and $U\left[\left[\mathbf{x}^{\pm 1}\right]\right]$, respectively.

We will often be working with polynomials and power series (and similar objects) in two variables. For instance, $U\left[z,w\right]$ (with z and w being two symbols) means $U\left[\mathbf{x}\right]$, where $\mathbf{x}$ is the two-element family (z,w). Similarly, $U\left[\left[z,z^{-1},w,w^{-1}\right]\right]$ means $U\left[\left[\mathbf{x},\mathbf{x}^{-1}\right]\right]$ for the same family $\mathbf{x}$. Explicitly,

$$U\left[z,w\right] = \left\{ \sum_{(i,j)\in\mathbb{N}^2} u_{(i,j)} z^i w^j \ \mid \ \text{all but finitely many } (i,j)\in\mathbb{N}^2 \text{ satisfy } u_{(i,j)} = 0 \right\}$$

and

$$U\left[\left[z,z^{-1},w,w^{-1}\right]\right] = \left\{ \sum_{(i,j)\in\mathbb{Z}^2} u_{(i,j)} z^i w^j \right\}$$

(where the $u_{(i,j)}$ are supposed to live in U both times). Similarly,

$$U\left[z,z^{-1},w,w^{-1}\right] = \left\{ \sum_{(i,j)\in\mathbb{Z}^2} u_{(i,j)} z^i w^j \ \mid \ \text{all but finitely many } (i,j)\in\mathbb{Z}^2 \text{ satisfy } u_{(i,j)} = 0 \right\},$$

$$U\left[\left[z,w\right]\right] = \left\{ \sum_{(i,j)\in\mathbb{N}^2} u_{(i,j)} z^i w^j \right\}$$

and

$$U\left(\left(z,w\right)\right) = \left\{ \sum_{(i,j)\in\mathbb{Z}^2} u_{(i,j)} z^i w^j \ \mid \ \text{there exists an } N\in\mathbb{Z} \text{ such that all } (i,j)\in\mathbb{Z}^2 \setminus \{N,N+1,N+2,\ldots\}^2 \text{ satisfy } u_{(i,j)} = 0 \right\}. \qquad (2)$$

There are obvious isomorphisms $U\left[z,w\right] \cong \left(U\left[z\right]\right)\left[w\right] \cong \left(U\left[w\right]\right)\left[z\right]$ which preserve the $\mathbb{F}$-module structure and, if U is an $\mathbb{F}$-algebra, also the $\mathbb{F}$-algebra structure. Similarly, there are obvious isomorphisms $U\left[\left[z,w\right]\right] \cong \left(U\left[\left[z\right]\right]\right)\left[\left[w\right]\right] \cong \left(U\left[\left[w\right]\right]\right)\left[\left[z\right]\right]$ which also preserve said structures.[8] Similar isomorphisms exist for Laurent polynomials and for formal distributions, but not for Laurent series.

[8] **Caveat:** These isomorphisms do not preserve the topology! The sequence $\left((z+w)^n\right)_{n\in\mathbb{N}}$ converges to 0 in the topology on $U\left[z,w\right]$, but not in the topology on $\left(U\left[z\right]\right)\left[w\right]$ (unless the latter topology is defined more subtly).

Sometimes we will also study "intermediate" $\mathbb{F}$-modules such as $U\left[\left[z, z^{-1}, w\right]\right]$ (intermediate between $U\left[\left[z, z^{-1}\right]\right]$ and $U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$); as you would expect, it is defined by

$$U\left[\left[z, z^{-1}, w\right]\right] = \left\{ \sum_{(i,j) \in \mathbb{Z} \times \mathbb{N}} u_{(i,j)} z^i w^j \right\}.$$

(It can also be viewed as $\left(U\left[\left[z, z^{-1}\right]\right]\right)[[w]]$, or as $(U[[w]])\left[\left[z, z^{-1}\right]\right]$, both times by canonical isomorphism.) Some authors write $U\left[\left[z^{\pm 1}, w\right]\right]$ for $U\left[\left[z, z^{-1}, w\right]\right]$.

As we said, $U\left[\left[z, z^{-1}\right]\right]$ is usually not a $\mathbb{F}$-algebra even if U is a $\mathbb{F}$-algebra. However, products of "formal distributions" can be well-defined in various specific cases for particular reasons. Most importantly, two formal distributions in distinct variables can be multiplied: for example, if U is a $\mathbb{F}$-algebra, then we can multiply $\sum\limits_{i \in \mathbb{Z}} u_i z^i \in U\left[\left[z, z^{-1}\right]\right]$ with $\sum\limits_{i \in \mathbb{Z}} v_i w^i \in U\left[\left[w, w^{-1}\right]\right]$ to obtain

$$\left(\sum_{i \in \mathbb{Z}} u_i z^i \right) \cdot \left(\sum_{i \in \mathbb{Z}} v_i w^i \right) = \sum_{(i,j) \in \mathbb{Z}^2} u_i v_j z^i w^j \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right].$$

Later (in the proof of Proposition 3.6 **(a)**) we will encounter a different case in which two formal distributions can be multiplied[9].

1.3. Substitution

In many cases, it is also possible to substitute things for variables into polynomial-like objects, although, the less "tame" the objects are, the more we need to require of what we substitute into them. Here are some examples:

- If U is a commutative $\mathbb{F}$-algebra, and if $f \in U[z]$, then any element of a U-algebra can be substituted for z in f.
- If U is a commutative $\mathbb{F}$-algebra, and if $f \in U\left[z, z^{-1}\right]$, then any invertible element of a U-algebra can be substituted for z in f.
- If U is a commutative $\mathbb{F}$-algebra, and if $f \in U[[z]]$, and if I is an ideal of a topological U-algebra P such that the I-adic topology on P is complete, then any element of I can be substituted for z in f. In particular, this shows that any nilpotent element of a U-algebra can be substituted for z in f, and also that any power series with zero constant term can be substituted for z in f.

[9] Namely, we will see a situation in which one power series belongs to $U\left[\left[z, z^{-1}\right]\right]$, while the other belongs to $\left(U\left[z, z^{-1}\right]\right)\left[\left[w, w^{-1}\right]\right]$ (which is embedded in $U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ via the identification of $U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ with $\left(U\left[\left[z, z^{-1}\right]\right]\right)\left[\left[w, w^{-1}\right]\right]$). The reader can come up with some even more general situations in which multiplication is possible.

- If U is a commutative $\mathbb{F}$-algebra, and if $f \in U((z))$, then we can substitute any positive power of z or of any other formal indeterminate for z in f.
- If U is a commutative $\mathbb{F}$-algebra, and if $f \in U[[z, z^{-1}]]$, then we can substitute any nonzero power of z or of any other formal indeterminate for z in f.

As usual, the result of substituting any object p for z in f will be denoted by $f(p)$.

In the case of multiple variables, some of these rules need additional conditions. For example, we can only substitute *commuting* elements of a U-algebra for the variables in a polynomial. We can substitute two distinct indeterminates for z and w in an element $f \in U[[z, z^{-1}, w, w^{-1}]]$, but not two equal indeterminates unless f has special properties. In particular, if $f \in U[[z, z^{-1}, w, w^{-1}]]$, then $f(z, w)$ and $f(w, z)$ are well-defined (of course, $f(z, w) = f$), but $f(z, z)$ (in general) is not.

2. Derivatives

2.1. Derivatives

Let us now define derivative operators on polynomial-like objects. We shall be brief, as we assume that the reader knows how to (formally – i.e., without recourse to analysis) define derivatives of polynomials, and much of the theory will be analogous for the other objects.[10]

First, let U be a $\mathbb{F}$-module, and z a symbol. Then, we can define an endomorphism ∂_z of the $\mathbb{F}$-module $U[z]$ by setting

$$\partial_z \left(\sum_{i \in \mathbb{N}} u_i z^i \right) = \sum_{i \in \mathbb{N}} i u_i z^{i-1}$$

for every $\sum\limits_{i \in \mathbb{N}} u_i z^i \in U[z]$ (with $u_i \in U$) [11]. This endomorphism ∂_z is often

[10] Our treatment of derivatives will, however, be complicated by the fact that we are allowing $\mathbb{F}$ to be an arbitrary commutative ring, not necessarily a field of characteristic 0.

[11] Let us see why this is well-defined. Fix $\sum\limits_{i \in \mathbb{N}} u_i z^i \in U[z]$ (with $u_i \in U$). It is clear that $\sum\limits_{i \in \mathbb{N}} i u_i z^{i-1} \in U[z, z^{-1}]$ (since all but finitely many $i \in \mathbb{N}$ satisfy $u_i = 0$), but we need to prove that $\sum\limits_{i \in \mathbb{N}} i u_i z^{i-1} \in U[z]$. For this it is clearly sufficient to show that $i u_i z^{i-1} \in U[z]$ for every $i \in \mathbb{N}$. So let $i \in \mathbb{N}$. We need to show that $i u_i z^{i-1} \in U[z]$. But:

- for $i \geq 1$, this is obvious (because for $i \geq 1$, we have $i - 1 \in \mathbb{N}$).
- for $i < 1$, this follows from $\underbrace{i}_{\substack{=0 \\ (\text{since } i<1)}} u_i z^{i-1} = 0$.

denoted by $\dfrac{\partial}{\partial z}$. Analogous endomorphisms ∂_z of $U\left[z,z^{-1}\right]$, of $U\left[\left[z\right]\right]$, of $U\left(\left(z\right)\right)$ and of $U\left[\left[z,z^{-1}\right]\right]$ are defined in the same way (except that $\mathbb{N}$ is replaced by $\mathbb{Z}$ when we are considering $U\left[z,z^{-1}\right]$, $U\left(\left(z\right)\right)$ and $U\left[\left[z,z^{-1}\right]\right]$). These endomorphisms are all continuous; they are furthermore U-linear derivations when U is a commutative $\mathbb{F}$-algebra.[12] Finally, these endomorphisms ∂_z form commutative diagrams with each other – e.g., the restriction of the endomorphism ∂_z of $U\left[z,z^{-1}\right]$ to $U\left[z\right]$ is precisely the endomorphism ∂_z of $U\left[z\right]$.

When there are multiple variables x_j (for $j \in J$) instead of a single variable z, an endomorphism ∂_{x_j} is defined for each of them[13]; it is given by

$$\partial_{x_j}\left(\sum_{\mathbf{i}} u_{\mathbf{i}}\mathbf{x}^{\mathbf{i}}\right) = \sum_{\mathbf{i}} \mathbf{i}_j u_{\mathbf{i}-\mathbf{e}_j}\mathbf{x}^{\mathbf{i}},$$

where we are using the following notations: $\mathbf{i}_j$ denotes the j-th entry of the family $\mathbf{i}$, and $\mathbf{e}_j$ denotes the family $\left(\delta_{k,j}\right)_{k\in J}$ (and the subtraction in $\mathbf{i}-\mathbf{e}_j$ is componentwise). The endomorphisms ∂_{x_j} for different j commute.

2.2. Hasse-Schmidt derivatives $\partial_z^{(n)}$

We can also generalize the endomorphisms ∂_z of $U\left[z\right]$, $U\left[z,z^{-1}\right]$, $U\left[\left[z\right]\right]$ and $U\left(\left(z\right)\right)$ in another direction:

Definition 2.1. Let $n \in \mathbb{N}$. Let U be an $\mathbb{F}$-module. We define an endomorphism $\partial_z^{(n)}$ of the $\mathbb{F}$-module $U\left[z\right]$ by setting

$$\partial_z^{(n)}\left(\sum_{i\in\mathbb{N}} u_i z^i\right) = \sum_{i\in\mathbb{N}} \binom{i}{n} u_i z^{i-n}$$

for every $\sum\limits_{i\in\mathbb{N}} u_i z^i \in U\left[z\right]$ (with $u_i \in U$) [14]. Again, we can define endomorphisms $\partial_z^{(n)}$ of $U\left[z,z^{-1}\right]$, of $U\left[\left[z\right]\right]$, of $U\left(\left(z\right)\right)$ and of $U\left[\left[z,z^{-1}\right]\right]$ according to the same rule.

Thus, $iu_iz^{i-1} \in U\left[z\right]$ is proven, qed.

[12] In the case of $U\left[\left[z,z^{-1}\right]\right]$, this means that ∂_z is U-linear and satisfies $\partial_z\left(fg\right) = \left(\partial_z f\right)g + f\left(\partial_z g\right)$ for any $f \in U\left[z,z^{-1}\right]$ and $g \in U\left[\left[z,z^{-1}\right]\right]$.

[13] The reader probably knows these endomorphisms on polynomial rings, and will not be surprised that they behave similarly for other polynomial-like objects.

[14] Let us see why this is well-defined. Fix $\sum\limits_{i\in\mathbb{N}} u_i z^i \in U\left[z\right]$ (with $u_i \in U$). It is clear that $\sum\limits_{i\in\mathbb{N}} \dbinom{i}{n} u_i z^{i-n} \in U\left[z,z^{-1}\right]$ (since all but finitely many $i \in \mathbb{N}$ satisfy $u_i = 0$), but we need to prove that $\sum\limits_{i\in\mathbb{N}} \dbinom{i}{n} u_i z^{i-n} \in U\left[z\right]$. For this it is clearly sufficient to show that

These endomorphisms $\partial_n^{(z)}$ form commutative diagrams with each other – e.g., the restriction of the endomorphism $\partial_z^{(n)}$ of $U\left[z,z^{-1}\right]$ to $U[z]$ is precisely the endomorphism $\partial_z^{(n)}$ of $U[z]$. Moreover, they satisfy the following properties:

Proposition 2.2. Let U be an $\mathbb{F}$-module. Let A be any of the five $\mathbb{F}$-modules $U[z]$, $U\left[z,z^{-1}\right]$, $U[[z]]$, $U((z))$ and $U\left[\left[z,z^{-1}\right]\right]$. Then,

$$\partial_z^{(0)} = \mathrm{id}; \tag{3}$$

$$\partial_z^{(1)} = \partial_z; \tag{4}$$

$$n!\partial_z^{(n)} = (\partial_z)^n \qquad \text{for all } n \in \mathbb{N}; \tag{5}$$

$$\partial_z^{(n)} \circ \partial_z^{(m)} = \binom{n+m}{n} \partial_z^{(n+m)} \qquad \text{for all } n \in \mathbb{N} \text{ and } m \in \mathbb{N}. \tag{6}$$

Proof of Proposition 2.2. We WLOG assume that $A = U\left[\left[z,z^{-1}\right]\right]$ (since the other four $\mathbb{F}$-modules $U[z]$, $U\left[z,z^{-1}\right]$, $U[[z]]$ and $U((z))$ are $\mathbb{F}$-submodules of $U\left[\left[z,z^{-1}\right]\right]$, and their respective endomorphisms $\partial_z^{(n)}$ are restrictions of the endomorphism $\partial_z^{(n)}$ of $U\left[\left[z,z^{-1}\right]\right]$).

For every $\sum\limits_{i\in\mathbb{N}} u_i z^i \in A$ (with all $u_i \in U$), we have

$$\partial_z^{(0)}\left(\sum_{i\in\mathbb{N}} u_i z^i\right) = \sum_{i\in\mathbb{N}} \underbrace{\binom{i}{0}}_{=1} u_i \underbrace{z^{i-0}}_{=z^i} \qquad \left(\text{by the definition of } \partial_z^{(0)}\right)$$
$$= \sum_{i\in\mathbb{N}} u_i z^i = \mathrm{id}\left(\sum_{i\in\mathbb{N}} u_i z^i\right).$$

Hence, $\partial_z^{(0)} = \mathrm{id}$. A similar argument shows that $\partial_z^{(1)} = \partial_z$ (here we use the fact that $\binom{i}{1} = i$ for each $i \in \mathbb{N}$).

$\binom{i}{n} u_i z^{i-n} \in U[z]$ for every $i \in \mathbb{N}$. So let $i \in \mathbb{N}$. We need to show that $\binom{i}{n} u_i z^{i-n} \in U[z]$. But:

- for $i \geq n$, this is obvious (because for $i \geq n$, we have $i - n \in \mathbb{N}$).
- for $i < n$, this follows from $\underbrace{\binom{i}{n}}_{\substack{=0 \\ (\text{since } i<n \text{ and } i\in\mathbb{N})}} u_i z^{i-n} = 0.$

Thus, $\binom{i}{n} u_i z^{i-n} \in U[z]$ is proven, qed.

Let us now prove (6). Indeed, let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. We need to prove that $\partial_z^{(n)} \circ \partial_z^{(m)} = \dbinom{n+m}{n} \partial_z^{(n+m)}$. It is clearly enough to show that

$$\left(\partial_z^{(n)} \circ \partial_z^{(m)}\right)(a) = \binom{n+m}{n} \partial_z^{(n+m)}(a) \tag{7}$$

for every $a \in A$.

Proof of (7): Let $a \in A$. We need to prove the equality (7). Both sides of this equality are continuous with respect to a (where "continuous" is to be understood with respect to the topology we introduced on A). Hence, we can WLOG assume that $a \in U\left[z, z^{-1}\right]$ (since $U\left[z, z^{-1}\right]$ is dense in $U\left[\left[z, z^{-1}\right]\right] = A$). Assume this. Now, both sides of the equality (7) are $\mathbb{F}$-linear in a. Hence (and since $a \in U\left[z, z^{-1}\right]$), we can WLOG assume that a belongs to the family $(uz^q)_{u \in U;\ q \in \mathbb{Z}}$ (since the family $(uz^q)_{u \in U;\ q \in \mathbb{Z}}$ spans the $\mathbb{F}$-module $U\left[z, z^{-1}\right]$). Assume this. Then, $a = uz^p$ for some $u \in U$ and $p \in \mathbb{Z}$. Consider these u and p. Now, the definition of $\partial_z^{(n)}$ yields

$$\partial_z^{(m)}(uz^p) = \binom{p}{m} uz^{p-m}; \qquad \partial_z^{(n)}\left(z^{p-m}\right) = \binom{p-m}{n} uz^{(p-m)-n};$$
$$\partial_z^{(n+m)}(uz^p) = \binom{p}{n+m} uz^{p-(n+m)}.$$

But it is easy to see that $\dbinom{p}{m}\dbinom{p-m}{n} = \dbinom{n+m}{n}\dbinom{p}{n+m}$ [15].

[15] *Proof.* The definition of $\dbinom{p}{m}$ yields

$$\binom{p}{m} = \frac{p(p-1)\cdots(p-m+1)}{m!}. \tag{8}$$

The definition of $\dbinom{p-m}{n}$ yields

$$\binom{p-m}{n} = \frac{(p-m)((p-m)-1)\cdots((p-m)-n+1)}{n!}$$
$$= \frac{(p-m)(p-m-1)\cdots(p-m-n+1)}{n!}.$$

Multiplying (8) with this equality, we obtain

$$\begin{aligned}
\binom{p}{m}\binom{p-m}{n} &= \frac{p\,(p-1)\cdots(p-m+1)}{m!}\cdot\frac{(p-m)\,(p-m-1)\cdots(p-m-n+1)}{n!} \\
&= \frac{1}{n!m!}\underbrace{(p\,(p-1)\cdots(p-m+1))\cdot((p-m)\,(p-m-1)\cdots(p-m-n+1))}_{=p(p-1)\cdots(p-m-n+1)} \\
&= \frac{1}{n!m!}p\,(p-1)\cdots\left(\underbrace{p-m-n+1}_{=p-(n+m)+1}\right) = \frac{1}{n!m!}p\,(p-1)\cdots(p-(n+m)+1) \\
&= \frac{(n+m)!}{n!m!}\cdot\frac{p\,(p-1)\cdots(p-(n+m)+1)}{(n+m)!}.
\end{aligned}$$

Comparing this with

$$\begin{aligned}
\overbrace{\frac{(n+m)!}{n!\,((n+m)-n)!}}^{\binom{n+m}{n}} &= \frac{(n+m)!}{n!m!} \overbrace{=}^{\binom{p}{n+m}} \underbrace{\frac{p\,(p-1)\cdots(p-(n+m)+1)}{(n+m)!}}_{\text{(by the definition of } \binom{p}{n+m}\text{)}} \\
&= \frac{(n+m)!}{n!m!}\cdot\frac{p\,(p-1)\cdots(p-(n+m)+1)}{(n+m)!},
\end{aligned}$$

we obtain $\binom{p}{m}\binom{p-m}{n} = \binom{n+m}{n}\binom{p}{n+m}$, qed.

It might seem that a simpler proof could be obtained using the identity $\binom{p}{m} = \frac{p!}{m!\,(p-m)!}$; but this identity only holds for $p \geq 0$. This type of argument can be salvaged, but we prefer the proof given above.

Now,

$$\begin{aligned}
&\left(\partial_z^{(n)} \circ \partial_z^{(m)}\right) \Bigg(\underbrace{a}_{=uz^p} \Bigg) \\
&= \left(\partial_z^{(n)} \circ \partial_z^{(m)}\right) (uz^p) = \partial_z^{(n)} \left(\underbrace{\partial_z^{(m)} (uz^p)}_{= \binom{p}{m} uz^{p-m}} \right) = \binom{p}{m} \quad \underbrace{\partial_z^{(n)} \left(uz^{p-m}\right)}_{= \binom{p-m}{n} uz^{(p-m)-n}} \\
&= \underbrace{\binom{p}{m}\binom{p-m}{n}}_{= \binom{n+m}{n}\binom{p}{n+m}} \quad u \underbrace{z^{(p-m)-n}}_{=z^{p-(n+m)}} = \binom{n+m}{n} \underbrace{\binom{p}{n+m} uz^{p-(n+m)}}_{= \partial_z^{(n+m)}(uz^p)} \\
&= \binom{n+m}{n} \partial_z^{(n+m)} \Bigg(\underbrace{uz^p}_{=a} \Bigg) = \binom{n+m}{n} \partial_z^{(n+m)} (a) .
\end{aligned}$$

This proves (7). Thus, (6) is proven.

It remains to prove (5). But this can be shown straightforwardly by induction over n: The induction base ($n = 0$) is dealt with by (3), while the induction step follows from

$$\begin{aligned}
\underbrace{\partial_z}_{\substack{=\partial_z^{(1)} \\ \text{(by (4))}}} \circ \partial_z^{(n-1)} = \partial_z^{(1)} \circ \partial_z^{(n-1)} &= \underbrace{\binom{1+(n-1)}{1}}_{=n} \underbrace{\partial_z^{(1+(n-1))}}_{=\partial_z^{(n)}} \\
&\qquad \text{(by (6), applied to 1 and } n-1 \text{ instead of } n \text{ and } m\text{)} \\
&= \partial_z^{(n)} .
\end{aligned}$$

This completes the proof of Proposition 2.2. □

The equality (5) yields that $\partial_z^{(n)} = \dfrac{(\partial_z)^n}{n!}$ in the case when $\mathbb{F}$ is a $\mathbb{Q}$-algebra; thus, the endomorphisms $\partial_z^{(n)}$ are of little significance in this case. But they become useful when one wants to prove results over arbitrary fields (or commutative rings) – where they, for instance, allow one to construct a Taylor formula without denominators. Endomorphisms akin to $\partial_z^{(n)}$ can also be defined in the case of multiple variables; they will be denoted by $\partial_{x_j}^{(n)}$.

Proposition 2.3. Let U be an $\mathbb{F}$-module. Let A be any of the five $\mathbb{F}$-modules $U[z]$, $U\left[z, z^{-1}\right]$, $U[[z]]$, $U((z))$ and $U\left[\left[z, z^{-1}\right]\right]$. Let L_z be the $\mathbb{F}$-linear map $A \to A$ which sends every $a \in A$ to $z \cdot a \in A$. [16]

Define $\partial_z^{(-1)}$ as the zero map $0 : A \to A$. Then, $\left[L_z, \partial_z^{(n)}\right] = -\partial_z^{(n-1)}$ for every $n \in \mathbb{N}$. Here, $[\cdot,\cdot]$ denotes the commutator of two endomorphisms of A; thus, $\left[L_z, \partial_z^{(n)}\right] = L_z \circ \partial_z^{(n)} - \partial_z^{(n)} \circ L_z$.

Proof of Proposition 2.3. Let $n \in \mathbb{N}$. We need to prove that $\left[L_z, \partial_z^{(n)}\right] = -\partial_z^{(n-1)}$. If $n = 0$, then this is obvious (because $\partial_z^{(0)} = \mathrm{id}$ and $\partial_z^{(-1)} = 0$). Hence, for the rest of this proof, we WLOG assume that $n \neq 0$.

We also WLOG assume that $A = U\left[\left[z, z^{-1}\right]\right]$, because the other four $\mathbb{F}$-modules $U[z]$, $U\left[z, z^{-1}\right]$, $U[[z]]$ and $U((z))$ are $\mathbb{F}$-submodules of $U\left[\left[z, z^{-1}\right]\right]$ and their respective $\partial_z^{(m)}$ maps are restrictions of the $\partial_z^{(m)}$ map of $U\left[\left[z, z^{-1}\right]\right]$.

We need to prove that $\left[L_z, \partial_z^{(n)}\right] = -\partial_z^{(n-1)}$. In other words, we need to prove that

$$\left[L_z, \partial_z^{(n)}\right](a) = -\partial_z^{(n-1)}(a) \tag{9}$$

for every $a \in A$. So let us fix $a \in A$.

We want to prove the equality (9). Both sides of this equality are continuous with respect to a (where "continuous" is to be understood with respect to the topology we introduced on A). Hence, we can WLOG assume that $a \in U\left[z, z^{-1}\right]$ (since $U\left[z, z^{-1}\right]$ is dense in $U\left[\left[z, z^{-1}\right]\right] = A$). Assume this. Now, both sides of the equality (9) are $\mathbb{F}$-linear in a. Hence (and since $a \in U\left[z, z^{-1}\right]$), we can WLOG assume that a belongs to the family $(uz^q)_{u \in U;\ q \in \mathbb{Z}}$ (since the family $(uz^q)_{u \in U;\ q \in \mathbb{Z}}$ spans the $\mathbb{F}$-module $U\left[z, z^{-1}\right]$). Assume this. Then, $a = uz^q$ for some $u \in U$ and $q \in \mathbb{Z}$. Consider these u and q. Now, the definition of $\partial_z^{(n)}$ yields $\partial_z^{(n)}(uz^q) = \dbinom{q}{n} uz^{q-n}$. Also, the definition of $\partial_z^{(n)}$ yields

$$\begin{aligned}
\partial_z^{(n)}\left(uz^{q+1}\right) &= \underbrace{\dbinom{q+1}{n}}_{\substack{=\binom{q}{n}+\binom{q}{n-1} \\ \text{(by the recursion of} \\ \text{the binomial coefficients)}}} uz^{(q+1)-n} = \left(\dbinom{q}{n} + \dbinom{q}{n-1}\right) uz^{(q+1)-n} \\
&= \left(\dbinom{q}{n} + \dbinom{q}{n-1}\right) uz^{q-n+1} = \dbinom{q}{n} uz^{q-n+1} + \dbinom{q}{n-1} uz^{q-n+1}.
\end{aligned}$$

Finally, the definition of $\partial_z^{(n-1)}$ yields $\partial_z^{(n-1)}(uz^q) = \dbinom{q}{n-1} uz^{q-(n-1)} = \dbinom{q}{n-1} uz^{q-n+1}$.

[16]Here, $z \cdot a$ is defined because A is an $\mathbb{F}[z]$-module.

Now,

$$
\begin{aligned}
&\underbrace{\left[L_z, \partial_z^{(n)}\right]}_{=L_z \circ \partial_z^{(n)} - \partial_z^{(n)} \circ L_z} \left(\underbrace{a}_{=uz^q} \right) \\
&= \left(L_z \circ \partial_z^{(n)} - \partial_z^{(n)} \circ L_z \right) (uz^q) \\
&= \left(L_z \circ \partial_z^{(n)} \right) (uz^q) - \left(\partial_z^{(n)} \circ L_z \right) (uz^q) \\
&= L_z \left(\underbrace{\partial_z^{(n)} (uz^q)}_{= \binom{q}{n} uz^{q-n}} \right) - \partial_z^{(n)} \left(\underbrace{L_z (uz^q)}_{\substack{=zuz^q \\ \text{(by the definition of } L_z)}} \right) \\
&= \underbrace{L_z \left(\binom{q}{n} uz^{q-n} \right)}_{\substack{=z\binom{q}{n} uz^{q-n} \\ \text{(by the definition of } L_z)}} - \partial_z^{(n)} \left(\underbrace{zuz^q}_{=uz^{q+1}} \right) \\
&= \underbrace{z \binom{q}{n} uz^{q-n}}_{= \binom{q}{n} uz^{q-n+1}} - \underbrace{\partial_z^{(n)} \left(uz^{q+1} \right)}_{= \binom{q}{n} uz^{q-n+1} + \binom{q}{n-1} uz^{q-n+1}} \\
&= \binom{q}{n} uz^{q-n+1} - \left(\binom{q}{n} uz^{q-n+1} + \binom{q}{n-1} uz^{q-n+1} \right) \\
&= - \underbrace{\binom{q}{n-1} uz^{q-n+1}}_{= \partial_z^{(n-1)} (uz^q)} = -\partial_z^{(n-1)} \left(\underbrace{uz^q}_{=a} \right) = -\partial_z^{(n-1)} (a) .
\end{aligned}
$$

This proves (9). Thus, Proposition 2.3 is proven. □

Remark 2.4. It is a common abuse of notation to write z for what was denoted by L_z in Proposition 2.3. This notation (which we will too use later in these notes) must be used with caution, however, since it can lead to confusion in certain cases (i.e., when $U = \mathbb{F}$, then $\partial_z z$ might be understood either as the element $\partial_z (z)$ of $\mathbb{F}[z]$ or as the endomorphism $\partial_z \circ z = \partial_z \circ L_z$ of $\mathbb{F}[z]$). We hope that context will clarify all resulting ambiguities.

When U is a commutative $\mathbb{F}$-algebra, the endomorphisms $\partial_z^{(n)}$ of $U[z]$, $U\left[z, z^{-1}\right]$, of $U[[z]]$, of $U((z))$ and of $U\left[\left[z, z^{-1}\right]\right]$ are U-linear. Moreover, in this case, the sequence $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ is an example of what is known as a "Hasse-Schmidt derivation" (or "higher derivation"). Let us now define this notion.

2.3. Hasse-Schmidt derivations in general

Definition 2.5. Let A and B be two $\mathbb{F}$-algebras. A *Hasse-Schmidt derivation* from A to B means a sequence $(D_0, D_1, D_2, \ldots)$ of $\mathbb{F}$-linear maps $A \to B$ such that $D_0(1) = 1$ and such that every $n \in \mathbb{N}$, $a \in A$ and $b \in A$ satisfy

$$D_n(ab) = \sum_{i=0}^{n} D_i(a) D_{n-i}(b). \tag{10}$$

We abbreviate the notion "Hasse-Schmidt derivation" by "*HSD*".

Definition 2.6. Let A be an $\mathbb{F}$-algebra. An HSD $(D_0, D_1, D_2, \ldots)$ from A to A is said to be *divided-powers* (we use this word as an adjective) if and only if it satisfies $D_0 = \mathrm{id}$ and

$$D_n \circ D_m = \binom{n+m}{n} D_{n+m} \qquad \text{for all } n \in \mathbb{N} \text{ and } m \in \mathbb{N}. \tag{11}$$

We notice that the term "Hasse-Schmidt derivation" is used by different authors in slightly different meanings, and the notion of a "divided-powers Hasse-Schmidt derivation" is the scribe's invention.

Now, we claim:

Proposition 2.7. Let U be a commutative $\mathbb{F}$-algebra.

(a) The sequence $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ is a divided-powers HSD from A to A whenever A is one of the $\mathbb{F}$-algebras $U[z]$, $U\left[z, z^{-1}\right]$, $U[[z]]$ and $U((z))$.

(b) Also,

$$\partial_z^{(n)}(ab) = \sum_{i=0}^{n} \partial_z^{(i)}(a) \partial_z^{(n-i)}(b)$$

for any $n \in \mathbb{N}$, any $a \in U\left[z, z^{-1}\right]$ and any $b \in U\left[\left[z, z^{-1}\right]\right]$.

(c) We have $\partial_z(ab) = \partial_z(a) b + a\partial_z(b)$ for any $a \in U\left[z, z^{-1}\right]$ and any $b \in U\left[\left[z, z^{-1}\right]\right]$.

Proof of Proposition 2.7. **(a)** Let A be one of the four $\mathbb{F}$-algebras $U[z]$, $U\left[z, z^{-1}\right]$, $U[[z]]$ and $U((z))$. Let us first show that $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ is a HSD from A to A. In order to do so, we need to show that $\partial_z^{(0)}(1) = 1$ and that every $n \in \mathbb{N}$, $a \in A$ and $b \in A$ satisfy

$$\partial_z^{(n)}(ab) = \sum_{i=0}^{n} \partial_z^{(i)}(a) \partial_z^{(n-i)}(b). \tag{12}$$

It is clear that $\partial_z^{(0)}(1) = 1$ (due to (3)), so it remains to prove (12).

Proof of (12): Let $n \in \mathbb{N}$, $a \in A$ and $b \in A$. We WLOG assume that $A = U((z))$, since the three other $\mathbb{F}$-algebras $U[z]$, $U[z, z^{-1}]$, $U[[z]]$ are subalgebras of $U((z))$. We need to prove the equality (12). Both sides of this equality are continuous with respect to each of a and b (where "continuous" is to be understood with respect to the topology we introduced on A). Hence, we can WLOG assume that each of a and b belongs to $U[z, z^{-1}]$ (since $U[z, z^{-1}]$ is dense in $U((z))$). Assume this. Now, both sides of the equality (12) are U-linear in each of a and b. Hence (and since a and b belong to $U[z, z^{-1}]$), we can WLOG assume that each of a and b belongs to the family $(z^m)_{m\in\mathbb{Z}}$ (since $(z^m)_{m\in\mathbb{Z}}$ is a basis of the U-module $U[z, z^{-1}]$). Assume this. Then, $a = z^p$ and $b = z^q$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{Z}$. Consider these p and q.

Now, the left hand side of (12) simplifies to

$$\partial_z^{(n)}\Big(\underbrace{a}_{=z^p}\underbrace{b}_{=z^q}\Big) = \partial_z^{(n)}\Big(\underbrace{z^p z^q}_{=z^{p+q}}\Big) = \partial_z^{(n)}\left(z^{p+q}\right) = \binom{p+q}{n} z^{p+q-n} \tag{13}$$

(by the definition of $\partial_z^{(n)}$), while the right hand side of (12) simplifies to

$$\begin{aligned}
&\sum_{i=0}^{n} \partial_z^{(i)}\Big(\underbrace{a}_{=z^p}\Big)\,\partial_z^{(n-i)}\Big(\underbrace{b}_{=z^q}\Big) \\
&= \sum_{i=0}^{n} \underbrace{\partial_z^{(i)}(z^p)}_{\substack{=\binom{p}{i} z^{p-i} \\ \text{(by the definition of } \partial_z^{(i)})}} \underbrace{\partial_z^{(n-i)}(z^q)}_{\substack{=\binom{q}{n-i} z^{q-(n-i)} \\ \text{(by the definition of } \partial_z^{(n-i)})}} \\
&= \sum_{i=0}^{n} \binom{p}{i} z^{p-i} \binom{q}{n-i} z^{q-(n-i)} = \sum_{i=0}^{n} \binom{p}{i}\binom{q}{n-i} \underbrace{z^{p-i} z^{q-(n-i)}}_{=z^{(p-i)+(q-(n-i))}=z^{p+q-n}} \\
&= \sum_{i=0}^{n} \binom{p}{i}\binom{q}{n-i} z^{p+q-n}.
\end{aligned} \tag{14}$$

But the Vandermonde convolution identity says that

$$\binom{p+q}{n} = \sum_{i=0}^{n} \binom{p}{i}\binom{q}{n-i}.$$

Hence, (13) becomes

$$\begin{aligned}\partial_z^{(n)}(ab) &= \underbrace{\binom{p+q}{n}}_{=\sum_{i=0}^{n}\binom{p}{i}\binom{q}{n-i}} z^{p+q-n} \\ &= \sum_{i=0}^{n}\binom{p}{i}\binom{q}{n-i} z^{p+q-n} = \sum_{i=0}^{n}\partial_z^{(i)}(a)\,\partial_z^{(n-i)}(b)\end{aligned}$$

(by (14)). Hence, (12) is proven.

This shows that $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ is a HSD from A to A. It remains to show that this HSD is divided-powers. For this, we need to check that $\partial_z^{(0)} = \mathrm{id}$ and

$$\partial_z^{(n)} \circ \partial_z^{(m)} = \binom{n+m}{n}\partial_z^{(n+m)} \qquad \text{for all } n \in \mathbb{N} \text{ and } m \in \mathbb{N}.$$

But this follows immediately from (3) and (6). This completes the proof of Proposition 2.7 **(a)**.

(b) This is proven similarly to our above proof of (12).

(c) Let $a \in U\left[z, z^{-1}\right]$ and any $b \in U\left[\left[z, z^{-1}\right]\right]$. Applying Proposition 2.7 **(b)** to $n = 1$, we obtain

$$\begin{aligned}\partial_z^{(1)}(ab) &= \sum_{i=0}^{1}\partial_z^{(i)}(a)\,\partial_z^{(1-i)}(b) = \underbrace{\partial_z^{(0)}}_{=\mathrm{id}}(a)\,\underbrace{\partial_z^{(1-0)}}_{=\partial_z^{(1)}=\partial_z}(b) + \underbrace{\partial_z^{(1)}}_{=\partial_z}(a)\,\underbrace{\partial_z^{(1-1)}}_{=\partial_z^{(0)}=\mathrm{id}}(b) \\ &= \mathrm{id}(a)\,\partial_z(b) + \partial_z(a)\,\mathrm{id}(b) = a\partial_z(b) + \partial_z(a)\,b = \partial_z(a)\,b + a\partial_z(b).\end{aligned}$$

Since $\partial_z^{(1)} = \partial_z$, this rewrites as $\partial_z(ab) = \partial_z(a)\,b + a\partial_z(b)$. This proves Proposition 2.7 **(c)**. □

We notice in passing that when $\mathbb{F}$ is a $\mathbb{Q}$-algebra[17], and $\partial : A \to A$ is any derivation of A, then $\left(\mathrm{id}, \frac{1}{1!}\partial, \frac{1}{2!}\partial^2, \frac{1}{3!}\partial^3, \ldots\right)$ is a divided-powers HSD from A to A. (This is very easy to prove.) A kind of converse holds as well (and not just in characteristic 0):

Remark 2.8. Let A be an $\mathbb{F}$-algebra. Let $(D_0, D_1, D_2, \ldots)$ be a divided-powers HSD from A to A.

(a) We have $n!D_n = (D_1)^n$ for every positive $n \in \mathbb{N}$.

(b) Assume that $D_0 = \mathrm{id}$. Then, $n!D_n = (D_1)^n$ for every $n \in \mathbb{N}$.

(c) Assume that $D_0 = \mathrm{id}$, and that $\mathbb{F}$ is a $\mathbb{Q}$-algebra. Then, $(D_0, D_1, D_2, \ldots)$ can be reconstructed from D_1 alone.

[17]for example, a field of characteristic 0

Proof of Remark 2.8. **(a)** We prove Remark 2.8 **(a)** by induction over n. The induction base ($n = 1$) is obvious (because if $n = 1$, then $n!D_n = 1!D_1 = D_1 = (D_1)^1 = (D_1)^n$). For the induction step, fix a positive $N \in \mathbb{N}$. Assume that Remark 2.8 **(a)** holds for $n = N$. We now need to show that Remark 2.8 **(a)** holds for $n = N + 1$. We have

$$\begin{aligned} D_N \circ D_1 &= \underbrace{\binom{N+1}{N}}_{=N+1} D_{N+1} \qquad \text{(by (11), applied to } n = N \text{ and } m = 1) \\ &= (N+1)\, D_{N+1}, \end{aligned}$$

so that $N!D_N \circ D_1 = \underbrace{N!\,(N+1)}_{=(N+1)!} D_{N+1} = (N+1)!D_{N+1}$. Compared with $\underbrace{N!D_N}_{\substack{=(D_1)^N \\ \text{(since Remark 2.8 (a)} \\ \text{holds for } n=N)}} \circ D_1 = (D_1)^N \circ D_1 = (D_1)^{N+1}$, this yields $(N+1)!D_{N+1} = (D_1)^{N+1}$. In other words, Remark 2.8 **(a)** holds for $n = N + 1$. This completes the induction step, and thus Remark 2.8 **(a)** is proven.

(b) Remark 2.8 **(a)** yields that $n!D_n = (D_1)^n$ for every positive $n \in \mathbb{N}$. It thus remains to show that $n!D_n = (D_1)^n$ for $n = 0$. This follows immediately from the assumption that $D_0 = \mathrm{id}$.

(c) This follows from Remark 2.8 **(b)**, since $n!$ is invertible in $\mathbb{F}$ for every $n \in \mathbb{N}$. $\square$

Proposition 2.7 **(a)** shows that the sequence $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ is a divided-powers HSD from A to A whenever A is one of the $\mathbb{F}$-algebras $U[z]$, $U[z, z^{-1}]$, $U[[z]]$ and $U((z))$. Similar statements can be proven (in the same way) for polynomial-like objects in multiple variables: For example, if A is one of the $\mathbb{F}$-algebras $U[z, w]$, $U[z, z^{-1}, w, w^{-1}]$, $U[[z, w]]$ and $U((z, w))$, then both sequences $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ and $\left(\partial_w^{(0)}, \partial_w^{(1)}, \partial_w^{(2)}, \ldots\right)$ are divided-powers HSDs from A to A.

To summarize, the notions "HSD" and "divided-powers HSD" abstract some properties of the families $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ of operators on the four $\mathbb{F}$-algebras $U[z]$, $U[z, z^{-1}]$, $U[[z]]$ and $U((z))$. [18] We will see the use of this generality when we prove a general fact (Corollary 2.17) that allows us to extend a divided-powers HSD to a localization of its domain. This will allows us, for example, to extend the operators $\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots$ to any localization of $U[z, w]$ or $U[[z, w]]$. This gives us a purely algebraic reason why derivatives are well-defined for

[18] We could also define a notion of "HSD on modules" to abstract properties of the family $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ of operators on the $\mathbb{F}$-module $U\left[\left[z, z^{-1}\right]\right]$, such as Proposition 2.7 **(b)**. We won't need this generality, however (or so I hope).

rational functions in two variables; this is a fact that is commonly used but rarely proven in literature.

Before we can do this, we will introduce another viewpoint on HSDs, which will make them easier to deal with.

2.4. Hasse-Schmidt derivations from A to B as algebra maps $A \to B[[t]]$

A sequence $(D_0, D_1, D_2, \ldots)$ of linear maps from one $\mathbb{F}$-module P to another $\mathbb{F}$-module Q can be described by a single map $P \to Q[[t]]$, and it turns out that the properties of the former sequence to be an HSD or a divided-powers HSD translate into some rather simple conditions on the latter map. Let us see this in some detail.

Definition 2.9. Let P and Q be two $\mathbb{F}$-modules. Let t be a symbol. Let $\mathbf{D} = (D_0, D_1, D_2, \ldots)$ be a sequence of $\mathbb{F}$-linear maps from P to Q. Then, we define an $\mathbb{F}$-linear map $\widetilde{\mathbf{D}}_{\langle t \rangle} : P \to Q[[t]]$ by

$$\widetilde{\mathbf{D}}_{\langle t \rangle}(p) = \sum_{i \in \mathbb{N}} D_i(p) t^i \qquad \text{for all } p \in P.$$

Theorem 2.10. Let A and B be two $\mathbb{F}$-algebras. Let t be a symbol. Let $\mathbf{D} = (D_0, D_1, D_2, \ldots)$ be a sequence of $\mathbb{F}$-linear maps from A to B. Then, $\mathbf{D}$ is an HSD if and only if $\widetilde{\mathbf{D}}_{\langle t \rangle} : A \to B[[t]]$ is an $\mathbb{F}$-algebra homomorphism.

Proof of Theorem 2.10. $\Longrightarrow$: Assume that $\mathbf{D}$ is an HSD. We need to show that $\widetilde{\mathbf{D}}_{\langle t \rangle} : A \to B[[t]]$ is an $\mathbb{F}$-algebra homomorphism.

Let $a \in A$ and $b \in A$. We are going to show that $\widetilde{\mathbf{D}}_{\langle t \rangle}(ab) = \widetilde{\mathbf{D}}_{\langle t \rangle}(a) \cdot \widetilde{\mathbf{D}}_{\langle t \rangle}(b)$. Indeed, the definition of $\widetilde{\mathbf{D}}_{\langle t \rangle}(ab)$ yields

$$\widetilde{\mathbf{D}}_{\langle t \rangle}(ab) = \sum_{i \in \mathbb{N}} D_i(ab) t^i = \sum_{n \in \mathbb{N}} \underbrace{D_n(ab)}_{\substack{= \sum_{i=0}^{n} D_i(a) D_{n-i}(b) \\ \text{(by (10), since } \mathbf{D} \\ \text{is an HSD)}}} t^n \tag{15}$$

$$= \sum_{n \in \mathbb{N}} \left(\sum_{i=0}^{n} D_i(a) D_{n-i}(b) \right) t^n.$$

Compared with

$$\underbrace{\widetilde{\mathbf{D}}_{\langle t\rangle}(a)}_{\substack{=\sum\limits_{i\in\mathbb{N}} D_i(a)t^i \\ \text{(by the definition of } \widetilde{\mathbf{D}}_{\langle t\rangle}(a))}} \cdot \underbrace{\widetilde{\mathbf{D}}_{\langle t\rangle}(b)}_{\substack{=\sum\limits_{i\in\mathbb{N}} D_i(b)t^i \\ \text{(by the definition of } \widetilde{\mathbf{D}}_{\langle t\rangle}(b))}} = \left(\sum_{i\in\mathbb{N}} D_i(a)\,t^i\right)\left(\sum_{i\in\mathbb{N}} D_i(b)\,t^i\right)$$
$$= \sum_{n\in\mathbb{N}} \left(\sum_{i=0}^{n} D_i(a)\,D_{n-i}(b)\right) t^n, \tag{16}$$

this yields $\widetilde{\mathbf{D}}_{\langle t\rangle}(ab) = \widetilde{\mathbf{D}}_{\langle t\rangle}(a)\cdot\widetilde{\mathbf{D}}_{\langle t\rangle}(b)$. Thus, $\widetilde{\mathbf{D}}_{\langle t\rangle}$ preserves multiplication.

But we are not done yet – an $\mathbb{F}$-algebra homomorphism has to also preserve the unity. So let us show that $\widetilde{\mathbf{D}}_{\langle t\rangle}(1) = 1$. The definition of $\widetilde{\mathbf{D}}_{\langle t\rangle}(1)$ yields $\widetilde{\mathbf{D}}_{\langle t\rangle}(1) = \sum\limits_{i\in\mathbb{N}} D_i(1)\,t^i$. This is a power series in t whose constant coefficient is $D_0(1) = 1$ (since $\mathbf{D}$ is an HSD); every such power series is invertible. Thus, $\widetilde{\mathbf{D}}_{\langle t\rangle}(1)$ is invertible. But since $\widetilde{\mathbf{D}}_{\langle t\rangle}$ preserves multiplication, we have $\widetilde{\mathbf{D}}_{\langle t\rangle}(1\cdot 1) = \widetilde{\mathbf{D}}_{\langle t\rangle}(1)\cdot\widetilde{\mathbf{D}}_{\langle t\rangle}(1)$. Hence, $\widetilde{\mathbf{D}}_{\langle t\rangle}(1)\cdot\widetilde{\mathbf{D}}_{\langle t\rangle}(1) = \widetilde{\mathbf{D}}_{\langle t\rangle}(1\cdot 1) = \widetilde{\mathbf{D}}_{\langle t\rangle}(1)$. We can cancel $\widetilde{\mathbf{D}}_{\langle t\rangle}(1)$ out of this equality (since $\widetilde{\mathbf{D}}_{\langle t\rangle}(1)$ is invertible), and obtain $\widetilde{\mathbf{D}}_{\langle t\rangle}(1) = 1$. Combined with the fact that $\widetilde{\mathbf{D}}_{\langle t\rangle}$ preserves multiplication, this yields that $\widetilde{\mathbf{D}}_{\langle t\rangle}$ is an $\mathbb{F}$-algebra homomorphism. This proves the $\Longrightarrow$ direction of Theorem 2.10.

$\Longleftarrow$: Assume that $\widetilde{\mathbf{D}}_{\langle t\rangle} : A \to B[[t]]$ is an $\mathbb{F}$-algebra homomorphism. Then, $\widetilde{\mathbf{D}}_{\langle t\rangle}(1) = 1$. Compared with $\widetilde{\mathbf{D}}_{\langle t\rangle}(1) = \sum\limits_{i\in\mathbb{N}} D_i(1)\,t^i$ (by the definition of $\widetilde{\mathbf{D}}_{\langle t\rangle}(1)$), this yields $\sum\limits_{i\in\mathbb{N}} D_i(1)\,t^i = 1$. Comparing the constant coefficients on both sides of this equality, we obtain $D_0(1) = 1$.

Also, let $a \in A$ and $b \in A$ satisfy

$$\begin{aligned}\sum_{n\in\mathbb{N}} D_n(ab)\,t^n &= \widetilde{\mathbf{D}}_{\langle t\rangle}(ab) && \text{(by (15))} \\ &= \widetilde{\mathbf{D}}_{\langle t\rangle}(a)\cdot\widetilde{\mathbf{D}}_{\langle t\rangle}(b) && \left(\text{since } \widetilde{\mathbf{D}}_{\langle t\rangle} \text{ is an } \mathbb{F}\text{-algebra homomorphism}\right) \\ &= \sum_{n\in\mathbb{N}} \left(\sum_{i=0}^{n} D_i(a)\,D_{n-i}(b)\right) t^n && \text{(by (16))}.\end{aligned}$$

Comparing coefficients on both sides of this equality, we obtain

$$D_n(ab) = \sum_{i=0}^{n} D_i(a)\,D_{n-i}(b)$$

for every $a \in A$, every $b \in A$ and every $n \in \mathbb{N}$. Combined with $D_0(1) = 1$, this yields that $\mathbf{D}$ is an HSD. This completes the proof of the $\Longleftarrow$ direction of Theorem 2.10. $\square$

Theorem 2.11. Let A be an $\mathbb{F}$-algebra. Let t and u be two distinct symbols. Let $\mathbf{D} = (D_0, D_1, D_2, \ldots)$ be an HSD from A to A.

Recall that whenever P and Q are two $\mathbb{F}$-modules, any $\mathbb{F}$-linear map $\phi : P \to Q$ gives rise to an $\mathbb{F}$-linear map $\phi\left[[t]\right] : P\left[[t]\right] \to Q\left[[t]\right]$ (which just applies the map ϕ to every coefficient). Thus, the map $\widetilde{\mathbf{D}}_{\langle u \rangle} : A \to A\left[[u]\right]$ gives rise to a map $\widetilde{\mathbf{D}}_{\langle u \rangle}\left[[t]\right] : A\left[[t]\right] \to \left(A\left[[u]\right]\right)\left[[t]\right]$.

We also define a map $\widetilde{\mathbf{D}}_{\langle u+t \rangle} : A \to \left(A\left[[u]\right]\right)\left[[t]\right]$ by

$$\widetilde{\mathbf{D}}_{\langle u+t \rangle}\left(p\right) = \sum_{i \in \mathbb{N}} D_i\left(p\right)\left(u+t\right)^i \qquad \text{for all } p \in A.$$

(This is well-defined since $\left(A\left[[u]\right]\right)\left[[t]\right] \cong A\left[[u,t]\right]$.)

Let $\pi_t : A\left[[t]\right] \to A$ be the projection map which sends every formal power series to its constant coefficient.

(a) The maps $\widetilde{\mathbf{D}}_{\langle t \rangle}$, $\widetilde{\mathbf{D}}_{\langle u \rangle}$, $\widetilde{\mathbf{D}}_{\langle u \rangle}\left[[t]\right]$, $\widetilde{\mathbf{D}}_{\langle u \rangle}\left[[t]\right] \circ \widetilde{\mathbf{D}}_{\langle t \rangle}$ and $\widetilde{\mathbf{D}}_{\langle u+t \rangle}$ are $\mathbb{F}$-algebra homomorphisms.

(b) The HSD $\mathbf{D}$ is divided-powers if and only if

$$\pi_t \circ \widetilde{\mathbf{D}}_{\langle t \rangle} = \mathrm{id} \qquad \text{and} \qquad \widetilde{\mathbf{D}}_{\langle u \rangle}\left[[t]\right] \circ \widetilde{\mathbf{D}}_{\langle t \rangle} = \widetilde{\mathbf{D}}_{\langle u+t \rangle}.$$

Proof of Theorem 2.11. **(a)** We know that $\mathbf{D}$ is an HSD. Thus, the map $\widetilde{\mathbf{D}}_{\langle t \rangle}$ is an $\mathbb{F}$-algebra homomorphism (according to the $\Longrightarrow$ direction of Theorem 2.10). Similarly, $\widetilde{\mathbf{D}}_{\langle u \rangle}$ and $\widetilde{\mathbf{D}}_{\langle u+t \rangle}$ are $\mathbb{F}$-algebra homomorphisms[19]. Since $\widetilde{\mathbf{D}}_{\langle u \rangle}$ is an $\mathbb{F}$-algebra homomorphism, we see that the map $\widetilde{\mathbf{D}}_{\langle u \rangle}\left[[t]\right]$ is an $\mathbb{F}$-algebra homomorphism, and thus the composition $\widetilde{\mathbf{D}}_{\langle u \rangle}\left[[t]\right] \circ \widetilde{\mathbf{D}}_{\langle t \rangle}$ is an $\mathbb{F}$-algebra homomorphism (being the composition of the $\mathbb{F}$-algebra homomorphisms $\widetilde{\mathbf{D}}_{\langle u \rangle}\left[[t]\right]$ and $\widetilde{\mathbf{D}}_{\langle t \rangle}$). This proves Theorem 2.11 **(a)**.

(b) $\Longrightarrow$: Assume that $\mathbf{D}$ is divided-powers. Then, $D_0 = \mathrm{id}$ and the equality

[19] To be completely fair, the proof for $\widetilde{\mathbf{D}}_{\langle u+t \rangle}$ is slightly different from the proof of $\widetilde{\mathbf{D}}_{\langle t \rangle}$ because $u+t$ is not a single variable like t but a sum of two variables. But the proof still goes through with the obvious changes, because we still have $\left(\sum\limits_{i \in \mathbb{N}} \alpha_i \left(u+t\right)^i\right)\left(\sum\limits_{i \in \mathbb{N}} \beta_i \left(u+t\right)^i\right) = \sum\limits_{n \in \mathbb{N}} \left(\sum\limits_{i=0}^{n} \alpha_i \beta_{n-i}\right)\left(u+t\right)^n$ for any two families $\left(\alpha_i\right)_{i \in \mathbb{N}} \in A^{\mathbb{N}}$ and $\left(\beta_i\right)_{i \in \mathbb{N}} \in A^{\mathbb{N}}$.

(11) holds (due to the definition of "divided-powers"). Now, let $p \in A$. Then,

$$\begin{aligned}
&\left(\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle}\right)(p) \\
&= \left(\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]]\right)\left(\underbrace{\widetilde{\mathbf{D}}_{\langle t\rangle}(p)}_{\substack{=\sum_{i\in\mathbb{N}} D_i(p)t^i \\ \text{(by the definition of } \widetilde{\mathbf{D}}_{\langle t\rangle})}}\right) = \left(\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]]\right)\left(\sum_{i\in\mathbb{N}} D_i(p)\,t^i\right) \\
&= \sum_{i\in\mathbb{N}} \widetilde{\mathbf{D}}_{\langle u\rangle}(D_i(p))\,t^i \qquad \left(\text{by the definition of } \widetilde{\mathbf{D}}_{\langle u\rangle}[[t]]\right) \\
&= \sum_{m\in\mathbb{N}} \underbrace{\widetilde{\mathbf{D}}_{\langle u\rangle}(D_m(p))}_{\substack{=\sum_{i\in\mathbb{N}} D_i(D_m(p))u^i \\ \text{(by the definition of } \widetilde{\mathbf{D}}_{\langle u\rangle})}} t^m = \sum_{m\in\mathbb{N}}\sum_{i\in\mathbb{N}} D_i(D_m(p))\,u^i t^m \\
&= \underbrace{\sum_{m\in\mathbb{N}}\sum_{n\in\mathbb{N}}}_{=\sum_{(n,m)\in\mathbb{N}^2}} \underbrace{D_n(D_m(p))}_{=(D_n\circ D_m)(p)} u^n t^m = \sum_{(n,m)\in\mathbb{N}^2} \underbrace{(D_n\circ D_m)}_{\substack{=\binom{n+m}{n}D_{n+m} \\ \text{(by (11))}}}(p)\,u^n t^m \qquad (17) \\
&= \sum_{(n,m)\in\mathbb{N}^2} \binom{n+m}{n} D_{n+m}(p)\,u^n t^m.
\end{aligned}$$

Compared with

$$
\begin{aligned}
&\widetilde{\mathbf{D}}_{\langle u+t\rangle}(p) \\
&= \sum_{i\in\mathbb{N}} D_i(p) \underbrace{(u+t)^i}_{\substack{=\sum_{n=0}^{i}\binom{i}{n}u^n t^{i-n} \\ \text{(by the binomial formula)}}} = \sum_{i\in\mathbb{N}} D_i(p) \underbrace{\sum_{n=0}^{i}\binom{i}{n}u^n t^{i-n}}_{\substack{=\sum_{\substack{(n,m)\in\mathbb{N}^2;\\ n+m=i}}\binom{i}{n}u^n t^m \\ \text{(here, we substituted } (n,m) \text{ for } (n,i-n))}} \\
&= \sum_{i\in\mathbb{N}} D_i(p) \sum_{\substack{(n,m)\in\mathbb{N}^2;\\ n+m=i}}\binom{i}{n}u^n t^m = \sum_{i\in\mathbb{N}} \sum_{\substack{(n,m)\in\mathbb{N}^2;\\ n+m=i}} \underbrace{\binom{i}{n}}_{\substack{=\binom{n+m}{n} \\ \text{(since } i=n+m)}} \underbrace{D_i(p)}_{\substack{=D_{n+m}(p) \\ \text{(since } i=n+m)}} u^n t^m \\
&= \underbrace{\sum_{i\in\mathbb{N}} \sum_{\substack{(n,m)\in\mathbb{N}^2;\\ n+m=i}}}_{=\sum_{(n,m)\in\mathbb{N}^2}} \binom{n+m}{n} D_{n+m}(p)\, u^n t^m \\
&= \sum_{(n,m)\in\mathbb{N}^2} \binom{n+m}{n} D_{n+m}(p)\, u^n t^m, \qquad (18)
\end{aligned}
$$

this yields $\left(\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle}\right)(p) = \widetilde{\mathbf{D}}_{\langle u+t\rangle}(p)$. Since we have proven this for every $p \in A$, we thus conclude that $\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \widetilde{\mathbf{D}}_{\langle u+t\rangle}$.

We shall now show that $\pi_t \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \mathrm{id}$. Every $p \in A$ satisfies

$$\left(\pi_t \circ \widetilde{\mathbf{D}}_{\langle t\rangle}\right)(p) = \pi_t \left(\underbrace{\widetilde{\mathbf{D}}_{\langle t\rangle}(p)}_{\substack{=\sum_{i\in\mathbb{N}} D_i(p)t^i \\ \text{(by the definition of } \widetilde{\mathbf{D}}_{\langle t\rangle})}} \right) = \pi_t\left(\sum_{i\in\mathbb{N}} D_i(p)\, t^i\right) = D_0(p)$$

(by the definition of π_t). Thus, $\pi_t \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = D_0 = \mathrm{id}$. Hence, we have shown

$$\pi_t \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \mathrm{id} \qquad \text{and} \qquad \widetilde{\mathbf{D}}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \widetilde{\mathbf{D}}_{\langle u+t\rangle}.$$

This proves the $\Longrightarrow$ direction of Theorem 2.11 **(b)**.

$\Longleftarrow$: Assume that $\pi_t \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \mathrm{id}$ and $\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \widetilde{\mathbf{D}}_{\langle u+t\rangle}$. Let $p \in A$.

Then, (17) yields

$$\sum_{(n,m)\in\mathbb{N}^2} (D_n \circ D_m)(p)\, u^n t^m$$
$$= \underbrace{\left(\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle}\right)}_{=\widetilde{\mathbf{D}}_{\langle u+t\rangle}}(p) = \widetilde{\mathbf{D}}_{\langle u+t\rangle}(p) = \sum_{(n,m)\in\mathbb{N}^2} \binom{n+m}{n} D_{n+m}(p)\, u^n t^m$$

(by (18)). This can be viewed as an identity between power series in $\mathbb{F}[[u,t]]$. Thus, comparing coefficients before $u^n t^m$ in this identity, we obtain

$$(D_n \circ D_m)(p) = \binom{n+m}{n} D_{n+m}(p) \qquad \text{for all } n \in \mathbb{N} \text{ and } m \in \mathbb{N}. \tag{19}$$

Let us now forget that we fixed p. We thus have shown (19) for every $p \in A$. In other words, $D_n \circ D_m = \binom{n+m}{n} D_{n+m}$ for all $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Thus, (11) holds. We shall now show that $D_0 = \mathrm{id}$. Indeed, we have $\pi_t \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = D_0$ (this can be proven just as in the proof of the $\Longrightarrow$ direction). Thus, $D_0 = \pi_t \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \mathrm{id}$. Thus, we have shown that $D_0 = \mathrm{id}$ and that (11) holds. In other words, $\mathbf{D}$ is divided-powers. This proves the $\Longleftarrow$ direction of Theorem 2.11 **(b)**. Theorem 2.11 **(b)** is thus proven. □

Theorem 2.10 shows that HSDs are just algebra homomorphisms (to a different target) in disguise; this makes dealing with them a lot easier. Here is one sample application:

Corollary 2.12. Let A and B be two $\mathbb{F}$-algebras. Let $\mathbf{D} = (D_0, D_1, D_2, \ldots)$ and $\mathbf{E} = (E_0, E_1, E_2, \ldots)$ be two HSDs from A to B.

(a) If a is an invertible element of A such that every $n \in \mathbb{N}$ satisfies $D_n(a) = E_n(a)$, then every $n \in \mathbb{N}$ satisfies $D_n(a^{-1}) = E_n(a^{-1})$.

(b) Let G be a subset of A such that G generates A as an $\mathbb{F}$-algebra. Assume that $D_n(g) = E_n(g)$ for every $g \in G$ and every $n \in \mathbb{N}$. Then, $\mathbf{D} = \mathbf{E}$.

Proof of Corollary 2.12. Let t be a new symbol. Consider the $\mathbb{F}$-linear maps $\widetilde{\mathbf{D}}_{\langle t\rangle} : A \to B[[t]]$ and $\widetilde{\mathbf{E}}_{\langle t\rangle} : A \to B[[t]]$ according to Definition 2.9. Then, Theorem 2.10 yields that $\widetilde{\mathbf{D}}_{\langle t\rangle}$ is an $\mathbb{F}$-algebra homomorphism (since $\mathbf{D}$ is an HSD). Similarly, $\widetilde{\mathbf{E}}_{\langle t\rangle}$ is an $\mathbb{F}$-algebra homomorphism.

(a) Let a be an invertible element of A such that every $n \in \mathbb{N}$ satisfies $D_n(a) = E_n(a)$. The definition of $\widetilde{\mathbf{D}}_{\langle t\rangle}$ yields

$$\widetilde{\mathbf{D}}_{\langle t\rangle}(a) = \sum_{i\in\mathbb{N}} \underbrace{D_i(a)}_{\substack{=E_i(a)\\ \text{(since every } n\in\mathbb{N}\\ \text{satisfies } D_n(a)=E_n(a))}} t^i = \sum_{i\in\mathbb{N}} E_i(a)\, t^i = \widetilde{\mathbf{E}}_{\langle t\rangle}(a)$$

(by the definition of $\widetilde{\mathbf{E}}_{\langle t\rangle}$). But since $\widetilde{\mathbf{D}}_{\langle t\rangle}$ is an $\mathbb{F}$-algebra homomorphism, we have $\widetilde{\mathbf{D}}_{\langle t\rangle}\left(a^{-1}\right)=\left(\widetilde{\mathbf{D}}_{\langle t\rangle}\left(a\right)\right)^{-1}$. Similarly, $\widetilde{\mathbf{E}}_{\langle t\rangle}\left(a^{-1}\right)=\left(\widetilde{\mathbf{E}}_{\langle t\rangle}\left(a\right)\right)^{-1}$. Thus,

$$\begin{aligned}\widetilde{\mathbf{D}}_{\langle t\rangle}\left(a^{-1}\right)&=\left(\widetilde{\mathbf{D}}_{\langle t\rangle}\left(a\right)\right)^{-1}=\left(\widetilde{\mathbf{E}}_{\langle t\rangle}\left(a\right)\right)^{-1}\qquad\left(\text{since }\widetilde{\mathbf{D}}_{\langle t\rangle}\left(a\right)=\widetilde{\mathbf{E}}_{\langle t\rangle}\left(a\right)\right)\\&=\widetilde{\mathbf{E}}_{\langle t\rangle}\left(a^{-1}\right).\end{aligned}$$

Since $\widetilde{\mathbf{D}}_{\langle t\rangle}\left(a^{-1}\right)=\sum\limits_{i\in\mathbb{N}}D_i\left(a^{-1}\right)t^i$ (by the definition of $\widetilde{\mathbf{D}}_{\langle t\rangle}$) and $\widetilde{\mathbf{E}}_{\langle t\rangle}\left(a^{-1}\right)=\sum\limits_{i\in\mathbb{N}}E_i\left(a^{-1}\right)t^i$ (similarly), this rewrites as $\sum\limits_{i\in\mathbb{N}}D_i\left(a^{-1}\right)t^i=\sum\limits_{i\in\mathbb{N}}E_i\left(a^{-1}\right)t^i$. Comparing coefficients in this equality, we conclude that every $n\in\mathbb{N}$ satisfies $D_n\left(a^{-1}\right)=E_n\left(a^{-1}\right)$. This proves Corollary 2.12 **(a)**.

(b) For every $g\in G$, we have

$$\widetilde{\mathbf{D}}_{\langle t\rangle}\left(g\right)=\sum_{i\in\mathbb{N}}\underbrace{D_i\left(g\right)}_{\substack{=E_i(g)\\\text{(since every }n\in\mathbb{N}\\\text{satisfies }D_n(g)=E_n(g))}}t^i=\sum_{i\in\mathbb{N}}E_i\left(g\right)t^i=\widetilde{\mathbf{E}}_{\langle t\rangle}\left(g\right)$$

(by the definition of $\widetilde{\mathbf{E}}_{\langle t\rangle}$). Hence, the two $\mathbb{F}$-algebra homomorphisms $\widetilde{\mathbf{D}}_{\langle t\rangle}$ and $\widetilde{\mathbf{E}}_{\langle t\rangle}$ are equal to each other on S. Since the set S generates the $\mathbb{F}$-algebra A, this yields that these two homomorphisms are identical. That is, we have $\widetilde{\mathbf{D}}_{\langle t\rangle}=\widetilde{\mathbf{E}}_{\langle t\rangle}$. Now, let $a\in A$. Then, the definition of $\widetilde{\mathbf{D}}_{\langle t\rangle}$ yields

$$\sum_{i\in\mathbb{N}}D_i\left(a\right)t^i=\underbrace{\widetilde{\mathbf{D}}_{\langle t\rangle}}_{=\widetilde{\mathbf{E}}_{\langle t\rangle}}\left(a\right)=\widetilde{\mathbf{E}}_{\langle t\rangle}\left(a\right)=\sum_{i\in\mathbb{N}}E_i\left(a\right)t^i$$

(by the definition of $\widetilde{\mathbf{E}}_{\langle t\rangle}$). Comparing coefficients, we obtain $D_n\left(a\right)=E_n\left(a\right)$ for every $n\in\mathbb{N}$. Since this holds for every $a\in A$, we can thus conclude that $D_n=E_n$ for every $n\in\mathbb{N}$. In other words, $\mathbf{D}=\mathbf{E}$. This proves Corollary 2.12 **(b)**. □

As another application of Theorem 2.10, let us show a (denominator-free!) formal version of Taylor's expansion formula:

Theorem 2.13. Let U be a commutative $\mathbb{F}$-algebra. Let z and t be two distinct symbols. Then,

$$a\left(z+t\right)=\sum_{n\in\mathbb{N}}t^n\partial_z^{(n)}a\left(z\right)\qquad\text{in }U\left[\left[z,t\right]\right]$$

for any $a\left(z\right)\in U\left[\left[z\right]\right]$.

Proof of Theorem 2.13. Set $A = U[[z]]$. We let $\mathbf{D}$ denote the sequence $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ of endomorphisms of A. Proposition 2.7 **(a)** shows that this sequence $\mathbf{D}$ is an HSD from A to A. Theorem 2.10 (applied to $B = A$) thus yields that $\widetilde{\mathbf{D}}_{\langle t\rangle} : A \to A[[t]]$ is an $\mathbb{F}$-algebra homomorphism.

We define a map $\mathbf{e} : A \to U[[z,t]]$ by

$$(\mathbf{e}(a(z)) = a(z+t) \qquad \text{for every } a(z) \in A).$$

It is easy to see that this map $\mathbf{e}$ is an $\mathbb{F}$-algebra homomorphism[20]. Moreover, $\mathbf{e}$ is continuous (with respect to the usual topologies on A and $U[[z,t]]$).

Due to the definition of $\widetilde{\mathbf{D}}_{\langle t\rangle}$, we see that every $a(z) \in A$ satisfies

$$\widetilde{\mathbf{D}}_{\langle t\rangle}(a(z)) = \sum_{i\in\mathbb{N}} \partial_z^{(i)}(a(z))\, t^i = \sum_{n\in\mathbb{N}} \partial_z^{(n)}(a(z))\, t^n = \sum_{n\in\mathbb{N}} t^n \partial_z^{(n)} a(z), \qquad (20)$$

Applying this to $a(z) = z$, we obtain

$$\widetilde{\mathbf{D}}_{\langle t\rangle}(z) = \sum_{n\in\mathbb{N}} t^n \partial_z^{(n)} z = \underbrace{t^0}_{=1} \underbrace{\partial_z^{(0)} z}_{\substack{=z \\ \text{(by the} \\ \text{definition of } \partial_z^{(0)})}} + \underbrace{t^1}_{=t} \underbrace{\partial_z^{(1)}(z)}_{\substack{=1 \\ \text{(by the} \\ \text{definition of } \partial_z^{(1)})}} + \sum_{\substack{n\in\mathbb{N};\\ n\geq 2}} t^n \underbrace{\partial_z^{(n)} z}_{\substack{=0 \\ \text{(by the} \\ \text{definition of } \partial_z^{(n)}, \\ \text{since } n\geq 2)}}$$

$$= z + t + \underbrace{\sum_{\substack{n\in\mathbb{N};\\ n\geq 2}} t^n 0}_{=0} = z + t = \mathbf{e}(z)$$

(since the definition of $\mathbf{e}$ yields $\mathbf{e}(z) = z + t$).

Recall that $U[z]$ is an $\mathbb{F}$-subalgebra of $U[[z]] = A$. The restrictions of the maps $\widetilde{\mathbf{D}}_{\langle t\rangle}$ and $\mathbf{e}$ to this $\mathbb{F}$-subalgebra $U[z]$ are $\mathbb{F}$-algebra homomorphisms[21], and are equal to each other on the generating set $\{z\}$ of the $\mathbb{F}$-algebra $U[z]$ (since $\widetilde{\mathbf{D}}_{\langle t\rangle}(z) = \mathbf{e}(z)$). Hence, these restrictions are equal.

We recall that $\widetilde{\mathbf{D}}_{\langle t\rangle}$ is a map $A \to A[[t]]$. Using the canonical isomorphism $\underbrace{A}_{=U[[z]]}[[t]] = (U[[z]])[[t]] \cong U[[z,t]]$, we regard $\widetilde{\mathbf{D}}_{\langle t\rangle}$ as a map $A \to U[[z,t]]$. It is easy to see that this map $\widetilde{\mathbf{D}}_{\langle t\rangle}$ is continuous[22]. Now, the two maps $\widetilde{\mathbf{D}}_{\langle t\rangle}$

[20] Indeed, this is because $\mathbf{e}$ is a substitution homomorphism. Explicitly, for instance, we see that the map $\mathbf{e}$ respects multiplication because any $a(z) \in A$ and $b(z) \in A$ satisfy

$$\begin{aligned} \mathbf{e}(a(z)\,b(z)) &= (a(z)\,b(z))(z+t) \qquad \text{(by the definition of } \mathbf{e}) \\ &= \underbrace{a(z+t)}_{\substack{=\mathbf{e}(a(z)) \\ \text{(by the definition of } \mathbf{e})}} \underbrace{b(z+t)}_{\substack{=\mathbf{e}(b(z)) \\ \text{(by the definition of } \mathbf{e})}} = \mathbf{e}(a(z))\,\mathbf{e}(b(z)). \end{aligned}$$

[21] since $\widetilde{\mathbf{D}}_{\langle t\rangle}$ and $\mathbf{e}$ are $\mathbb{F}$-algebra homomorphisms

[22] This is because the $z^m t^n$-coefficient of $\widetilde{\mathbf{D}}_{\langle t\rangle}(a(z))$ depends only on the $z^{m+n} t^n$-coefficient of $a(z)$. (This follows from (20).)

and $\mathbf{e}$ are continuous, and their restrictions to the $\mathbb{F}$-subalgebra $U[z]$ are equal. This yields that the two maps $\widetilde{\mathbf{D}}_{\langle t \rangle}$ and $\mathbf{e}$ must be equal (since $U[z]$ is a dense subset of $U[[z]] = A$). In other words, $\widetilde{\mathbf{D}}_{\langle t \rangle} = \mathbf{e}$. Thus, every $a(z) \in A$ satisfies $\underbrace{\widetilde{\mathbf{D}}_{\langle t \rangle}}_{=\mathbf{e}}(a(z)) = \mathbf{e}(a(z)) = a(z+t)$. Since $A = U[[z]]$ and $\widetilde{\mathbf{D}}_{\langle t \rangle}(a(z)) = \sum_{n\in\mathbb{N}} t^n \partial_z^{(n)} a(z)$, this rewrites as follows: Every $a(z) \in A$ satisfies $\sum_{n\in\mathbb{N}} t^n \partial_z^{(n)} a(z) = a(z+t)$. This proves Theorem 2.13. $\square$

2.5. Extending Hasse-Schmidt derivations to localizations

A more interesting application of Theorem 2.10 allows us to extend Hasse-Schmidt derivations to localizations of commutative $\mathbb{F}$-algebras. A similar result holds for usual (not Hasse-Schmidt) derivations, but we shall only state the one for HSDs:

Corollary 2.14. Let A and B be two commutative $\mathbb{F}$-algebras, and let $\mathbf{D} = (D_0, D_1, D_2, \ldots)$ be an HSD from A to B. Let S be a multiplicatively closed subset of A. Assume that $D_0(s)$ is an invertible element of B for every $s \in S$. Then, there exists a unique HSD $\mathbf{D}' = (D'_0, D'_1, D'_2, \ldots)$ from $S^{-1}A$ to B such that every $a \in A$ and $n \in \mathbb{N}$ satisfy $D'_n\left(\frac{a}{1}\right) = D_n(a)$.

Before we prove this corollary, let us recall some basic properties of localization:

Proposition 2.15. Let A be a commutative $\mathbb{F}$-algebra. Let S be a multiplicatively closed subset of A. Let $\iota : A \to S^{-1}A$ be the canonical $\mathbb{F}$-algebra homomorphism from A to $S^{-1}A$ sending each $a \in A$ to $\frac{a}{1} \in S^{-1}A$. (This ι might and might not be injective.)

Let B be any commutative $\mathbb{F}$-algebra.

(a) If $\phi : A \to B$ is an $\mathbb{F}$-algebra homomorphism with the property that $(\phi(a)$ is an invertible element of B for every $s \in S)$, then there exists a unique $\mathbb{F}$-algebra homomorphism $\phi' : S^{-1}A \to B$ such that $\phi' \circ \iota = \phi$.

(b) If ϕ and ψ are two $\mathbb{F}$-algebra homomorphisms $S^{-1}A \to B$ satisfying $\phi \circ \iota = \psi \circ \iota$, then $\phi = \psi$.

Notice that Proposition 2.15 **(a)** is the universal property of the localization $S^{-1}A$ as a commutative $\mathbb{F}$-algebra. Proposition 2.15 **(b)** says (in categorical language) that $\iota : A \to S^{-1}A$ is an epimorphism in the category of commutative $\mathbb{F}$-algebras.[23]

[23]It is the simplest example of epimorphisms in this category which are not surjective – or at least not usually surjective.

Proof of Proposition 2.15. **(a)** Proposition 2.15 **(a)** is well-known (as we said, it is a standard universal property), so we omit its proof.

(b) Let ϕ and ψ be two $\mathbb{F}$-algebra homomorphisms $A \to B$ satisfying $\phi \circ \iota = \psi \circ \iota$. Let $x \in S^{-1}A$. Then, $x = \dfrac{p}{q}$ for some $p \in A$ and $q \in S$. Consider these p and q. We have $x = \dfrac{p}{q} = \underbrace{\dfrac{p}{1}}_{=\iota(p)} \cdot \left(\underbrace{\dfrac{q}{1}}_{=\iota(q)} \right)^{-1} = \iota(p) \cdot (\iota(q))^{-1}$, thus

$$\begin{aligned}
\phi(x) &= \phi\left(\iota(p) \cdot (\iota(q))^{-1}\right) = \underbrace{\phi(\iota(p))}_{\substack{=(\phi\circ\iota)(p)=(\psi\circ\iota)(p) \\ \text{(since } \phi\circ\iota=\psi\circ\iota)}} \cdot \left(\underbrace{\phi(\iota(q))}_{\substack{=(\phi\circ\iota)(q)=(\psi\circ\iota)(q) \\ \text{(since } \phi\circ\iota=\psi\circ\iota)}} \right)^{-1} \\
&\qquad (\text{since } \phi \text{ is an } \mathbb{F}\text{-algebra homomorphism}) \\
&= (\psi \circ \iota)(p) \cdot ((\psi \circ \iota)(q))^{-1}.
\end{aligned}$$

Compared with

$$\begin{aligned}
\psi(x) &= \psi\left(\iota(p) \cdot (\iota(q))^{-1}\right) = \underbrace{\psi(\iota(p))}_{=(\psi\circ\iota)(p)} \cdot \left(\underbrace{\psi(\iota(q))}_{=(\psi\circ\iota)(q)} \right)^{-1} \\
&\qquad (\text{since } \phi \text{ is an } \mathbb{F}\text{-algebra homomorphism}) \\
&= (\psi \circ \iota)(p) \cdot ((\psi \circ \iota)(q))^{-1},
\end{aligned}$$

this yields $\phi(x) = \psi(x)$. Let us now forget that we fixed x. We thus have proven that $\phi(x) = \psi(x)$ for every $x \in S^{-1}A$. In other words, $\phi = \psi$. This proves Proposition 2.15 **(b)**. $\square$

Proof of Corollary 2.12. Let $\iota : A \to S^{-1}A$ be the canonical $\mathbb{F}$-algebra homomorphism from A to $S^{-1}A$ sending each $a \in A$ to $\dfrac{a}{1} \in S^{-1}A$. (This ι might and might not be injective.)

Let t be a new symbol. We must show that there exists a unique HSD $\mathbf{D}' = (D'_0, D'_1, D'_2, \ldots)$ from $S^{-1}A$ to B such that every $a \in A$ and $n \in \mathbb{N}$ satisfy $D'_n\left(\dfrac{a}{1}\right) = D_n(a)$. In other words, we must show that there exists a unique HSD

$\mathbf{D}' = (D'_0, D'_1, D'_2, \ldots)$ from $S^{-1}A$ to B such that $\widetilde{\mathbf{D}'}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}$ [24]. We will now prove the uniqueness and the existence parts of this statement separately.

Uniqueness: We will show that there exists at most one HSD $\mathbf{D}' = (D'_0, D'_1, D'_2, \ldots)$ from $S^{-1}A$ to B such that $\widetilde{\mathbf{D}'}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}$. Indeed, let $\mathbf{E}$ and $\mathbf{F}$ be two such HSDs $\mathbf{D}'$. We are going to show that $\mathbf{E} = \mathbf{F}$.

We know that $\mathbf{E}$ and $\mathbf{F}$ are two HSDs $\mathbf{D}' = (D'_0, D'_1, D'_2, \ldots)$ from $S^{-1}A$ to B such that $\widetilde{\mathbf{D}'}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}$. In other words, $\mathbf{E}$ and $\mathbf{F}$ are two HSDs from $S^{-1}A$ to B such that $\widetilde{\mathbf{E}}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}$ and $\widetilde{\mathbf{F}}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}$.

Theorem 2.10 shows that $\widetilde{\mathbf{D}}_{\langle t\rangle} : A \to B[[t]]$ is an $\mathbb{F}$-algebra homomorphism (since $\mathbf{D}$ is an HSD). Similarly, $\widetilde{\mathbf{E}}_{\langle t\rangle} : S^{-1}A \to B[[t]]$ and $\widetilde{\mathbf{F}}_{\langle t\rangle} : S^{-1}A \to B[[t]]$ are $\mathbb{F}$-algebra homomorphisms. Thus, Proposition 2.15 **(b)** (applied to $B[[t]]$, $\widetilde{\mathbf{E}}_{\langle t\rangle}$ and $\widetilde{\mathbf{F}}_{\langle t\rangle}$ instead of B, ϕ and ψ) yields $\widetilde{\mathbf{E}}_{\langle t\rangle} = \widetilde{\mathbf{F}}_{\langle t\rangle}$ (since $\widetilde{\mathbf{E}}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle} = \widetilde{\mathbf{F}}_{\langle t\rangle} \circ \iota$).

Now, let $x \in S^{-1}A$. Then, the definition of $\widetilde{\mathbf{E}}_{\langle t\rangle}$ yields

$$\sum_{n\in\mathbb{N}} E_n(x)\, t^n = \underbrace{\widetilde{\mathbf{E}}_{\langle t\rangle}}_{=\widetilde{\mathbf{F}}_{\langle t\rangle}}(x) = \widetilde{\mathbf{F}}_{\langle t\rangle}(x) = \sum_{n\in\mathbb{N}} F_n(x)\, t^n \qquad \left(\text{by the definition of } \widetilde{\mathbf{F}}_{\langle t\rangle}\right).$$

[24] This is because of the following the logical equivalence:

$$\begin{aligned}
&\left(\text{every } a \in A \text{ and } n \in \mathbb{N} \text{ satisfy } D'_n\left(\frac{a}{1}\right) = D_n(a)\right) \\
\iff &\left(\text{every } a \in A \text{ satisfies } \underbrace{\left(\text{every } n \in \mathbb{N} \text{ satisfies } D'_n\left(\frac{a}{1}\right) = D_n(a)\right)}_{\iff \left(\sum_{n\in\mathbb{N}} D'_n\left(\frac{a}{1}\right)t^n = \sum_{n\in\mathbb{N}} D_n(a)t^n \text{ in } B[[t]]\right)}\right) \\
\iff &\left(\text{every } a \in A \text{ satisfies } \underbrace{\sum_{n\in\mathbb{N}} D'_n\left(\frac{a}{1}\right)t^n}_{\substack{=\widetilde{\mathbf{D}'}_{\langle t\rangle}\left(\frac{a}{1}\right) \\ (\text{by the definition of } \widetilde{\mathbf{D}'}_{\langle t\rangle})}} = \underbrace{\sum_{n\in\mathbb{N}} D_n(a)\, t^n}_{\substack{=\widetilde{\mathbf{D}}_{\langle t\rangle}(a) \\ (\text{by the definition of } \widetilde{\mathbf{D}}_{\langle t\rangle})}} \text{ in } B[[t]]\right) \\
\iff &\left(\text{every } a \in A \text{ satisfies } \widetilde{\mathbf{D}'}_{\langle t\rangle}\left(\underbrace{\frac{a}{1}}_{\substack{=\iota(a) \\ (\text{by the definition of } \iota)}}\right) = \widetilde{\mathbf{D}}_{\langle t\rangle}(a)\right) \\
\iff &\left(\text{every } a \in A \text{ satisfies } \widetilde{\mathbf{D}'}_{\langle t\rangle}(\iota(a)) = \widetilde{\mathbf{D}}_{\langle t\rangle}(a)\right) \iff \left(\widetilde{\mathbf{D}'}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}\right).
\end{aligned}$$

Comparing coefficients in this equality, we obtain $E_n(x) = F_n(x)$ for every $n \in \mathbb{N}$.

Now, let us forget that we fixed x. We thus have shown that $E_n(x) = F_n(x)$ for every $n \in \mathbb{N}$ and every $x \in S^{-1}A$. In other words, $\mathbf{E} = \mathbf{F}$. This completes the proof of the uniqueness statement of Corollary 2.14.

Existence: We will now show that there exists at least one HSD $\mathbf{D}' = (D'_0, D'_1, D'_2, \ldots)$ from $S^{-1}A$ to B such that $\widetilde{\mathbf{D}'}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}$. Indeed, we are going to construct such an HSD $\mathbf{D}'$ explicitly.

Theorem 2.10 shows that $\widetilde{\mathbf{D}}_{\langle t\rangle} : A \to B[[t]]$ is an $\mathbb{F}$-algebra homomorphism (since $\mathbf{D}$ is an HSD). Moreover, $\widetilde{\mathbf{D}}_{\langle t\rangle}(s)$ is an invertible element of $B[[t]]$ for every $s \in S$ [25]. Thus, we can apply Proposition 2.15 **(a)** to $B[[t]]$ and $\widetilde{\mathbf{D}}_{\langle t\rangle}$ instead of B and ϕ. As a consequence, we conclude that there exists a unique $\mathbb{F}$-algebra homomorphism $\phi' : S^{-1}A \to B[[t]]$ such that $\phi' \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}$. Consider this ϕ'. Now, for every $n \in \mathbb{N}$, we define an $\mathbb{F}$-linear map $D'_n : S^{-1}A \to B$ as follows: For every $x \in S^{-1}A$, let $D'_n(x)$ be the coefficient of t^n in the power series $\phi'(x)$. Then, for every $x \in S^{-1}A$, we have

$$\phi'(x) = \sum_{n\in\mathbb{N}} D'_n(x)\, t^n. \qquad (21)$$

Now, let $\mathbf{D}'$ denote the sequence $(D'_0, D'_1, D'_2, \ldots)$ of $\mathbb{F}$-linear maps from $S^{-1}A$ to B. We shall now show that $\mathbf{D}'$ is an HSD from $S^{-1}A$ to B such that $\widetilde{\mathbf{D}'}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}$. Once this is proven, the existence statement of Corollary 2.14 will be proven, and thus the proof of Corollary 2.14 will be complete.

For every $x \in S^{-1}A$, we have

$$\begin{aligned}\widetilde{\mathbf{D}'}_{\langle t\rangle}(x) &= \sum_{n\in\mathbb{N}} D'_n(x)\, t^n \qquad \left(\text{by the definition of } \widetilde{\mathbf{D}'}_{\langle t\rangle}\right)\\ &= \phi'(x) \qquad (\text{by (21)}).\end{aligned}$$

Hence, $\widetilde{\mathbf{D}'}_{\langle t\rangle} = \phi'$. Thus, $\widetilde{\mathbf{D}'}_{\langle t\rangle} : S^{-1}A \to B[[t]]$ is an $\mathbb{F}$-algebra homomorphism. Thus, Theorem 2.10 (applied to $S^{-1}A$, D'_i and $\mathbf{D}'$ instead of A, D_i and $\mathbf{D}$) yields that $\mathbf{D}'$ is an HSD. Also, that $\underbrace{\widetilde{\mathbf{D}'}_{\langle t\rangle}}_{=\phi'} \circ \iota = \phi' \circ \iota = \widetilde{\mathbf{D}}_{\langle t\rangle}$. This completes the proof of Corollary 2.14. □

[25] *Proof.* Let $s \in S$. Then, $D_0(s)$ is an invertible element of B (by one of the hypotheses of Corollary 2.14). Now, the definition of $\widetilde{\mathbf{D}}_{\langle t\rangle}$ yields $\widetilde{\mathbf{D}}_{\langle t\rangle}(s) = \sum_{n\in\mathbb{N}} D_n(s)\, t^n$. This is a power series whose constant coefficient $D_0(s)$ is invertible. It is known that every such power series is invertible. Thus, this power series $\widetilde{\mathbf{D}}_{\langle t\rangle}(s)$ is invertible in $B[[t]]$, qed.

Definition 2.16. Let A, B and C be $\mathbb{F}$-algebras.

(a) If $\mathbf{D} = (D_0, D_1, D_2, \ldots)$ is an HSD from A to B, and if $\phi : B \to C$ is an $\mathbb{F}$-algebra homomorphism, then we denote by $\phi \circ \mathbf{D}$ the sequence $(\phi \circ D_0, \phi \circ D_1, \phi \circ D_2, \ldots)$ of $\mathbb{F}$-linear maps $A \to C$. It is easy to see that this sequence $\phi \circ \mathbf{D}$ is an HSD from A to C.

(b) If $\mathbf{D} = (D_0, D_1, D_2, \ldots)$ is an HSD from B to C, and if $\psi : A \to B$ is an $\mathbb{F}$-algebra homomorphism, then we denote by $\mathbf{D} \circ \psi$ the sequence $(D_0 \circ \psi, D_1 \circ \psi, D_2 \circ \psi, \ldots)$ of $\mathbb{F}$-linear maps $A \to C$. It is easy to see that this sequence $\mathbf{D} \circ \psi$ is an HSD from A to C.

Corollary 2.17. Let A be a commutative $\mathbb{F}$-algebra, and let $\mathbf{D} = (D_0, D_1, D_2, \ldots)$ be a divided-powers HSD from A to A. Let S be a multiplicatively closed subset of A.

(a) There exists a unique HSD $\mathbf{D}' = (D'_0, D'_1, D'_2, \ldots)$ from $S^{-1}A$ to $S^{-1}A$ such that every $a \in A$ and $n \in \mathbb{N}$ satisfy $D'_n\left(\frac{a}{1}\right) = \frac{D_n(a)}{1}$.

(b) This HSD $\mathbf{D}'$ is divided-powers.

(c) Let $\iota : A \to S^{-1}A$ be the canonical $\mathbb{F}$-algebra homomorphism from A to $S^{-1}A$ sending each $a \in A$ to $\frac{a}{1} \in S^{-1}A$. Then, $\mathbf{D}' \circ \iota = \iota \circ \mathbf{D}$.

Proof of Corollary 2.17. Let $\iota : A \to S^{-1}A$ be the canonical $\mathbb{F}$-algebra homomorphism from A to $S^{-1}A$ sending each $a \in A$ to $\frac{a}{1} \in S^{-1}A$. (This ι might and might not be injective.)

Recall that $\iota \circ \mathbf{D}$ denotes the sequence $(\iota \circ D_0, \iota \circ D_1, \iota \circ D_2, \ldots)$ of maps from A to $S^{-1}A$. It is easy to see that $\iota \circ \mathbf{D}$ is an HSD from A to $S^{-1}A$ (since $\mathbf{D}$ is an HSD from A to A, and since ι is an $\mathbb{F}$-algebra homomorphism $A \to S^{-1}A$).

We have $D_0 = \mathrm{id}$ (since the HSD $\mathbf{D}$ is divided-powers).

(a) We know that $(\iota \circ D_0)(s)$ is an invertible element of $S^{-1}A$ for every $s \in S$ [26]. Hence, Corollary 2.14 (applied to $S^{-1}A$, $\iota \circ \mathbf{D}$ and $\iota \circ D_i$ instead of B, $\mathbf{D}$ and D_i) yields that there exists a unique HSD $\mathbf{D}' = (D'_0, D'_1, D'_2, \ldots)$ from $S^{-1}A$ to $S^{-1}A$ such that every $a \in A$ and $n \in \mathbb{N}$ satisfy $D'_n\left(\frac{a}{1}\right) = (\iota \circ D_n)(a)$. In other words, there exists a unique HSD $\mathbf{D}' = (D'_0, D'_1, D'_2, \ldots)$ from $S^{-1}A$ to $S^{-1}A$ such that every $a \in A$ and $n \in \mathbb{N}$ satisfy $D'_n\left(\frac{a}{1}\right) = \frac{D_n(a)}{1}$ (because every $a \in A$ and $n \in \mathbb{N}$ satisfies $(\iota \circ D_n)(a) = \iota(D_n(a)) = \frac{D_n(a)}{1}$). This proves Corollary 2.17 **(a)**.

[26] *Proof.* Let $s \in S$. Then, $D_0 = \mathrm{id}$, so that $D_0(s) = \mathrm{id}(s) = s$. Thus, $(\iota \circ D_0)(s) = \iota\left(\underbrace{D_0(s)}_{=s}\right) = \iota(s) = \frac{s}{1}$. This is clearly invertible in $S^{-1}A$ (since $s \in S$). Thus, $(\iota \circ D_0)(s)$ is invertible in $S^{-1}A$, qed.

(b) Consider this HSD $\mathbf{D}'$. Let us write it in the form $(D'_0, D'_1, D'_2, \ldots)$. Then, every $a \in A$ and $n \in \mathbb{N}$ satisfy

$$D'_n \left(\frac{a}{1}\right) = \frac{D_n(a)}{1} \tag{22}$$

(according to the definition of $\mathbf{D}'$). In other words,

$$D'_n \circ \iota = \iota \circ D_n \qquad \text{for every } n \in \mathbb{N} \tag{23}$$

[27].

Whenever P and Q are two $\mathbb{F}$-modules, t is a symbol, and $\phi : P \to Q$ is an $\mathbb{F}$-linear map, we shall denote by $\phi[[t]]$ the $\mathbb{F}$-linear map $P[[t]] \to Q[[t]]$ which sends every $\sum\limits_{n\in\mathbb{N}} p_n t^n \in P[[t]]$ (with $p_n \in P$) to $\sum\limits_{n\in\mathbb{N}} q_n t^n \in Q[[t]]$. Notice that if P and Q are $\mathbb{F}$-algebras and ϕ is an $\mathbb{F}$-algebra homomorphism, then $\phi[[t]]$ is an $\mathbb{F}$-algebra homomorphism as well.

Let t and u be two distinct symbols. Define the maps $\widetilde{\mathbf{D}}_{\langle t\rangle}$, $\widetilde{\mathbf{D}}_{\langle u\rangle}$ and $\widetilde{\mathbf{D}}_{\langle u+t\rangle}$ as in Theorem 2.11. Furthermore, define the maps $\widetilde{\mathbf{D}'}_{\langle t\rangle}$, $\widetilde{\mathbf{D}'}_{\langle u\rangle}$ and $\widetilde{\mathbf{D}'}_{\langle u+t\rangle}$ in the same way (but for $\mathbf{D}'$, D'_i and $S^{-1}A$ instead of $\mathbf{D}$, D_i and A).

Theorem 2.11 **(a)** yields that the maps $\widetilde{\mathbf{D}}_{\langle t\rangle}$, $\widetilde{\mathbf{D}}_{\langle u\rangle}$, $\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]]$, $\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle}$ and $\widetilde{\mathbf{D}}_{\langle u+t\rangle}$ are $\mathbb{F}$-algebra homomorphisms. The same Theorem 2.11 **(a)** (but applied to $\mathbf{D}'$, D'_i and $S^{-1}A$ instead of $\mathbf{D}$, D_i and A) yields that the maps $\widetilde{\mathbf{D}'}_{\langle t\rangle}$, $\widetilde{\mathbf{D}'}_{\langle u\rangle}$, $\widetilde{\mathbf{D}'}_{\langle u\rangle}[[t]]$, $\widetilde{\mathbf{D}'}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D}'}_{\langle t\rangle}$ and $\widetilde{\mathbf{D}'}_{\langle u+t\rangle}$ are $\mathbb{F}$-algebra homomorphisms.

We will use the notation π_t to denote both the projection map $A[[t]] \to A$ which sends every formal power series to its constant coefficient, and the projection map $(S^{-1}A)[[t]] \to S^{-1}A$ which sends every formal power series to its constant coefficient. This will not cause any confusion, because the two maps have distinct domains. Both of these two maps π_t are $\mathbb{F}$-algebra homomorphisms.

Recall that $\mathbf{D}$ is divided-powers. Hence, Theorem 2.11 **(b)** yields that

$$\pi_t \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \mathrm{id} \qquad \text{and} \qquad \widetilde{\mathbf{D}}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \widetilde{\mathbf{D}}_{\langle u+t\rangle}. \tag{24}$$

Our goal is to prove that $\mathbf{D}'$ is divided-powers. According to Theorem 2.11 **(b)** (applied to $\mathbf{D}'$, D'_i and $S^{-1}A$ instead of $\mathbf{D}$, D_i and A), this boils down to proving

[27] *Proof.* Let $n \in \mathbb{N}$. Every $a \in A$ satisfies $\iota(a) = \frac{a}{1}$ (by the definition of ι) and $\iota(D_n(a)) = \frac{D_n(a)}{1}$ (by the definition of ι). Hence, every $a \in A$ satisfies

$$\begin{aligned}(D'_n \circ \iota)(a) &= D'_n \left(\underbrace{\iota(a)}_{=\frac{a}{1}}\right) = D'_n\left(\frac{a}{1}\right) = \frac{D_n(a)}{1} \qquad (\text{by (22)}) \\ &= \iota(D_n(a)) = (\iota \circ D_n)(a).\end{aligned}$$

Thus, $D'_n \circ \iota = \iota \circ D_n$. This proves (23).

that

$$\pi_t \circ \widetilde{\mathbf{D'}}_{\langle t\rangle} = \operatorname{id} \qquad \text{and} \qquad \widetilde{\mathbf{D'}}_{\langle u\rangle}[[t]] \circ \widetilde{\mathbf{D'}}_{\langle t\rangle} = \widetilde{\mathbf{D'}}_{\langle u+t\rangle}. \tag{25}$$

So let us now prove (25).

We have $\widetilde{\mathbf{D'}}_{\langle t\rangle} \circ \iota = \iota[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle}$ [28]. Similarly, $\widetilde{\mathbf{D'}}_{\langle u\rangle} \circ \iota = (\iota[[u]]) \circ \widetilde{\mathbf{D}}_{\langle u\rangle}$ and

[28] *Proof.* For every $a \in A$, we have

$$\begin{aligned}
\left(\widetilde{\mathbf{D'}}_{\langle t\rangle} \circ \iota\right)(a) &= \widetilde{\mathbf{D'}}_{\langle t\rangle}(\iota(a)) = \sum_{n\in\mathbb{N}} \underbrace{D'_n(\iota(a))}_{=(D'_n\circ\iota)(a)} t^n \qquad \left(\text{by the definition of } \widetilde{\mathbf{D'}}_{\langle t\rangle}\right) \\
&= \sum_{n\in\mathbb{N}} \underbrace{\left(D'_n \circ \iota\right)}_{\substack{=\iota\circ D_n \\ \text{(by (23))}}}(a)\, t^n = \sum_{n\in\mathbb{N}} \underbrace{(\iota \circ D_n)(a)}_{=\iota(D_n(a))} t^n \\
&= \sum_{n\in\mathbb{N}} \iota(D_n(a))\, t^n = (\iota[[t]]) \left(\underbrace{\sum_{n\in\mathbb{N}} D_n(a)\, t^n}_{\substack{=\widetilde{\mathbf{D}}_{\langle t\rangle}(a) \\ \text{(by the definition of } \widetilde{\mathbf{D}}_{\langle t\rangle})}} \right) = (\iota[[t]]) \left(\widetilde{\mathbf{D}}_{\langle t\rangle}(a)\right) \\
&= \left(\iota[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle}\right)(a).
\end{aligned}$$

Thus, $\widetilde{\mathbf{D'}}_{\langle t\rangle} \circ \iota = \iota[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle}$, qed.

$\widetilde{\mathbf{D}'}_{\langle u+t\rangle} \circ \iota = (\iota\,[[u]])\,[[t]] \circ \widetilde{\mathbf{D}}_{\langle u+t\rangle}$ [29]. Thus,

$$\begin{aligned}
\widetilde{\mathbf{D}'}_{\langle u\rangle}\,[[t]] \circ \underbrace{\widetilde{\mathbf{D}'}_{\langle t\rangle} \circ \iota}_{=\iota[[t]]\circ\widetilde{\mathbf{D}}_{\langle t\rangle}} &= \underbrace{\widetilde{\mathbf{D}'}_{\langle u\rangle}\,[[t]] \circ \iota\,[[t]]}_{=\left(\widetilde{\mathbf{D}'}_{\langle u\rangle}\circ\iota\right)[[t]]} \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \underbrace{\left(\widetilde{\mathbf{D}'}_{\langle u\rangle} \circ \iota\right)}_{=(\iota[[u]])\circ\widetilde{\mathbf{D}}_{\langle u\rangle}} [[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle} \\
&= \underbrace{\left((\iota\,[[u]]) \circ \widetilde{\mathbf{D}}_{\langle u\rangle}\right)[[t]]}_{=(\iota[[u]])[[t]]\circ\widetilde{\mathbf{D}}_{\langle u\rangle}[[t]]} \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = (\iota\,[[u]])\,[[t]] \circ \underbrace{\widetilde{\mathbf{D}}_{\langle u\rangle}\,[[t]] \circ \widetilde{\mathbf{D}}_{\langle t\rangle}}_{\substack{=\widetilde{\mathbf{D}}_{\langle u+t\rangle} \\ \text{(by (24))}}} \\
&= (\iota\,[[u]])\,[[t]] \circ \widetilde{\mathbf{D}}_{\langle u+t\rangle} = \widetilde{\mathbf{D}'}_{\langle u+t\rangle} \circ \iota.
\end{aligned}$$

Therefore, Proposition 2.15 **(b)** (applied to $B = \left(\left(S^{-1}A\right)[[u]]\right)[[t]]$, $\phi = \widetilde{\mathbf{D}'}_{\langle u\rangle}\,[[t]] \circ \widetilde{\mathbf{D}'}_{\langle t\rangle}$ and $\psi = \widetilde{\mathbf{D}'}_{\langle u+t\rangle}$) yields that $\widetilde{\mathbf{D}'}_{\langle u\rangle}\,[[t]] \circ \widetilde{\mathbf{D}'}_{\langle t\rangle} = \widetilde{\mathbf{D}'}_{\langle u+t\rangle}$ (since we know that $\widetilde{\mathbf{D}'}_{\langle u\rangle}\,[[t]] \circ \widetilde{\mathbf{D}'}_{\langle t\rangle}$ and $\widetilde{\mathbf{D}'}_{\langle u+t\rangle}$ are $\mathbb{F}$-algebra homomorphisms). This proves the second claim of (25).

Also, $\pi_t \circ \widetilde{\mathbf{D}'}_{\langle t\rangle}$ is an $\mathbb{F}$-algebra homomorphism (since π_t and $\widetilde{\mathbf{D}'}_{\langle t\rangle}$ are $\mathbb{F}$-algebra homomorphisms), and we have

$$\begin{aligned}
\pi_t \circ \underbrace{\widetilde{\mathbf{D}'}_{\langle t\rangle} \circ \iota}_{=\iota[[t]]\circ\widetilde{\mathbf{D}}_{\langle t\rangle}} &= \underbrace{\pi_t \circ \iota\,[[t]]}_{\substack{=\iota\circ\pi_t \\ \text{(this follows from the} \\ \text{definitions of } \pi_t \text{ and } \iota[[t]])}} \circ \widetilde{\mathbf{D}}_{\langle t\rangle} = \iota \circ \underbrace{\pi_t \circ \widetilde{\mathbf{D}}_{\langle t\rangle}}_{\substack{=\mathrm{id} \\ \text{(by (24))}}} \\
&= \iota = \mathrm{id} \circ \iota.
\end{aligned}$$

---

[29] When proving the latter equality, we need to use

$$\sum_{n\in\mathbb{N}} \iota\left(D_n(a)\right)(u+t)^n = \left((\iota\,[[u]])\,[[t]]\right)\left(\sum_{n\in\mathbb{N}} D_n(a)\,(u+t)^n\right) \tag{26}$$

instead of $\sum\limits_{n\in\mathbb{N}} \iota\left(D_n(a)\right)t^n = (\iota\,[[t]])\left(\sum\limits_{n\in\mathbb{N}} D_n(a)\,t^n\right)$. If you do not find (26) obvious, you can check it easily as follows: Regard $S^{-1}A$ as an A-algebra via the map $\iota : A \to S^{-1}A$; then $\left(S^{-1}A\right)[[u]]$ and $\left(\left(S^{-1}A\right)[[u]]\right)[[t]]$ become A-algebras as well. The map ι is an A-algebra homomorphism, and therefore so are the maps $\iota\,[[u]]$ and $(\iota\,[[u]])\,[[t]]$. Moreover, the map $(\iota\,[[u]])\,[[t]]$ is continuous when regarded as a map $A\,[[u,t]] \to \left(S^{-1}A\right)[[u,t]]$ (since it just acts on each coefficient separately). Thus, $(\iota\,[[u]])\,[[t]]$ is a continuous A-algebra homomorphism, and therefore we have

$$\begin{aligned}
\left((\iota\,[[u]])\,[[t]]\right)\left(\sum_{n\in\mathbb{N}} D_n(a)\,(u+t)^n\right) &= \sum_{n\in\mathbb{N}} D_n(a)\left(\underbrace{\left((\iota\,[[u]])\,[[t]]\right)(u+t)}_{=u+t}\right)^n \\
&= \sum_{n\in\mathbb{N}} \underbrace{D_n(a)\,(u+t)^n}_{\substack{=\iota(D_n(a))(u+t)^n \\ \text{(since } S^{-1}A \text{ becomes an } A\text{-module via } \iota)}} = \sum_{n\in\mathbb{N}} \iota\left(D_n(a)\right)(u+t)^n.
\end{aligned}$$

This proves (26).

Therefore, Proposition 2.15 **(b)** (applied to $B = S^{-1}A$, $\phi = \pi_t \circ \widetilde{\mathbf{D}'}_{\langle t \rangle}$ and $\psi = \mathrm{id}$) yields that $\pi_t \circ \widetilde{\mathbf{D}'}_{\langle t \rangle} = \mathrm{id}$ (since we know that $\pi_t \circ \widetilde{\mathbf{D}'}_{\langle t \rangle}$ and id are $\mathbb{F}$-algebra homomorphisms). This proves the first claim of (25). Hence, (25) is proven. As explained above, this shows that $\mathbf{D}'$ is divided-powers. This proves Corollary 2.17 **(b)**.

(c) The definition of $\mathbf{D}' \circ \iota$ yields

$$\begin{aligned}\mathbf{D}' \circ \iota &= \left(D'_0 \circ \iota, D'_1 \circ \iota, D'_2 \circ \iota, \ldots\right) = \left(\iota \circ D_0, \iota \circ D_1, \iota \circ D_2, \ldots\right) && \text{(due to (23))} \\ &= \iota \circ \mathbf{D} && \text{(by the definition of } \iota \circ \mathbf{D}).\end{aligned}$$

This proves Corollary 2.17 **(c)**. ☐

Definition 2.18. In the situation of Corollary 2.17, we shall refer to the unique HSD $\mathbf{D}'$ constructed in Corollary 2.17 **(a)** as the *lift of the HSD* $\mathbf{D}$ *to the localization* $S^{-1}A$.

Corollary 2.19. Let A and B be two commutative $\mathbb{F}$-algebras. Let S be a multiplicatively closed subset of A. Let $\iota : A \to S^{-1}A$ be the canonical $\mathbb{F}$-algebra homomorphism from A to $S^{-1}A$ sending each $a \in A$ to $\dfrac{a}{1} \in S^{-1}A$.

Let $\mathbf{E}$ and $\mathbf{F}$ be two HSDs from $S^{-1}A$ to B such that $\mathbf{E} \circ \iota = \mathbf{F} \circ \iota$. (Recall that $\mathbf{E} \circ \iota$ and $\mathbf{F} \circ \iota$ are defined according to Definition 2.16.) Then, $\mathbf{E} = \mathbf{F}$.

Proof of Corollary 2.19. Write $\mathbf{E}$ and $\mathbf{F}$ in the forms $\mathbf{E} = (E_0, E_1, E_2, \ldots)$ and $\mathbf{F} = (F_0, F_1, F_2, \ldots)$, respectively. We have $\mathbf{E} \circ \iota = \mathbf{F} \circ \iota$. Thus, $E_n \circ \iota = F_n \circ \iota$ for every $n \in \mathbb{N}$ [30]. Now, let t be a new symbol. Define $\widetilde{\mathbf{E}}_{\langle t \rangle}$ and $\widetilde{\mathbf{F}}_{\langle t \rangle}$ as in Theorem 2.10. Then, Theorem 2.10 (applied to $S^{-1}A$, $\mathbf{E}$ and E_i instead of A, $\mathbf{D}$ and D_i) yields that $\widetilde{\mathbf{E}}_{\langle t \rangle} : S^{-1}A \to B[[t]]$ is an $\mathbb{F}$-algebra homomorphism (since $\mathbf{E}$ is an HSD). Similarly, $\widetilde{\mathbf{F}}_{\langle t \rangle} : S^{-1}A \to B[[t]]$ is an $\mathbb{F}$-algebra homomorphism.

Every $x \in A$ satisfies

$$\begin{aligned}\left(\widetilde{\mathbf{F}}_{\langle t \rangle} \circ \iota\right)(x) &= \widetilde{\mathbf{F}}_{\langle t \rangle}(\iota(x)) = \sum_{n \in \mathbb{N}} \underbrace{F_n(\iota(x))}_{=(F_n \circ \iota)(x)} t^n && \left(\text{by the definition of } \widetilde{\mathbf{F}}_{\langle t \rangle}\right) \\ &= \sum_{n \in \mathbb{N}} (F_n \circ \iota)(x)\, t^n.\end{aligned}$$

Similarly, every $x \in A$ satisfies $\left(\widetilde{\mathbf{E}}_{\langle t \rangle} \circ \iota\right)(x) = \sum_{n \in \mathbb{N}} (E_n \circ \iota)(x)\, t^n$. Thus, every $x \in A$ satisfies

$$\left(\widetilde{\mathbf{E}}_{\langle t \rangle} \circ \iota\right)(x) = \sum_{n \in \mathbb{N}} \underbrace{(E_n \circ \iota)}_{=F_n \circ \iota}(x)\, t^n = \sum_{n \in \mathbb{N}} (F_n \circ \iota)(x)\, t^n = \left(\widetilde{\mathbf{F}}_{\langle t \rangle} \circ \iota\right)(x).$$

[30] *Proof.* The definition of $\mathbf{E} \circ \iota$ yields $\mathbf{E} \circ \iota = (E_0 \circ \iota, E_1 \circ \iota, E_2 \circ \iota, \ldots)$, so that

$$(E_0 \circ \iota, E_1 \circ \iota, E_2 \circ \iota, \ldots) = \mathbf{E} \circ \iota = \mathbf{F} \circ \iota = (F_0 \circ \iota, F_1 \circ \iota, F_2 \circ \iota, \ldots)$$

(by the definition of $\mathbf{F} \circ \iota$). In other words, $E_n \circ \iota = F_n \circ \iota$ for every $n \in \mathbb{N}$.

In other words, $\widetilde{\mathbf{E}}_{\langle t\rangle} \circ \iota = \widetilde{\mathbf{F}}_{\langle t\rangle} \circ \iota$. Thus, Proposition 2.15 **(b)** (applied to $B\left[\left[t\right]\right]$, $\widetilde{\mathbf{E}}_{\langle t\rangle}$ and $\widetilde{\mathbf{E}}_{\langle t\rangle}$ instead of B, ϕ and ψ) yields $\widetilde{\mathbf{E}}_{\langle t\rangle} = \widetilde{\mathbf{F}}_{\langle t\rangle}$. Now, every $x \in S^{-1}A$ satisfies

$$\begin{aligned}\sum_{n\in\mathbb{N}} E_n\left(x\right)t^n &= \underbrace{\widetilde{\mathbf{E}}_{\langle t\rangle}}_{=\widetilde{\mathbf{F}}_{\langle t\rangle}}\left(x\right) \qquad \left(\text{by the definition of } \widetilde{\mathbf{E}}_{\langle t\rangle}\right)\\ &= \widetilde{\mathbf{F}}_{\langle t\rangle}\left(x\right) = \sum_{n\in\mathbb{N}} F_n\left(x\right)t^n.\end{aligned}$$

Comparing coefficients in this equality, we conclude that $E_n\left(x\right) = F_n\left(x\right)$ for every $x \in S^{-1}A$ and $n \in \mathbb{N}$. In other words, $E_n = F_n$ for every $n \in \mathbb{N}$. In other words, $\mathbf{E} = \mathbf{F}$. This proves Corollary 2.19. $\square$

2.6. Residues

If U is a $\mathbb{F}$-module, and if $f \in U\left[\left[z, z^{-1}\right]\right]$, then we let $\operatorname{Res} f dz$ denote the (-1)-st coefficient of f. That is,

$$\operatorname{Res}\left(\sum_{i\in\mathbb{Z}} u_i z^i\right) dz = u_{-1}.$$

Notice that neither "d" not "dz" should be regarded as standalone objects in the expression "$\operatorname{Res} f dz$"; they are part of the notation.[31]

A similar notation can be defined for U-valued formal distributions in multiple variables. Indeed, the case of multiple variables can be reduced to the case of one variable as follows: Let U be a $\mathbb{F}$-module, and let $\mathbf{x} = \left(x_j\right)_{j\in J}$ be a family of indeterminates. Let $k \in J$, and let $\widetilde{\mathbf{x}}_k$ denote the family $\left(x_j\right)_{j\in J\setminus\{k\}}$ of indeterminates (that is, the family $\mathbf{x}$ with x_k removed). Then, we can identify $U\left[\left[\mathbf{x}, \mathbf{x}^{-1}\right]\right]$ with $\left(U\left[\left[\widetilde{\mathbf{x}}_k, \left(\widetilde{\mathbf{x}}_k\right)^{-1}\right]\right]\right)\left[\left[x_k, x_k^{-1}\right]\right]$ [32]. Hence, $f \in U\left[\left[\mathbf{x}, \mathbf{x}^{-1}\right]\right]$, an element $\operatorname{Res} f dx_k$ of $U\left[\left[\widetilde{\mathbf{x}}_k, \left(\widetilde{\mathbf{x}}_k\right)^{-1}\right]\right]$ is defined (because $f \in U\left[\left[\mathbf{x}, \mathbf{x}^{-1}\right]\right] = \left(U\left[\left[\widetilde{\mathbf{x}}_k, \left(\widetilde{\mathbf{x}}_k\right)^{-1}\right]\right]\right)\left[\left[x_k, x_k^{-1}\right]\right]$).

We notice a simple fact:

[31] This notation, of course, imitates the "residue at 0" notation from complex analysis. But it is purely formal and elementary and works over any commutative ring $\mathbb{F}$.

[32] This is similar to how we can identify $U\left[\mathbf{x}\right]$ with $\left(U\left[\widetilde{\mathbf{x}}_k\right]\right)\left[x_k\right]$ because a polynomial in the variables $\mathbf{x}$ can be regarded as a polynomial in the variable x_k whose coefficients are polynomials in the remaining variables $\widetilde{\mathbf{x}}_k$. Similarly it is possible to identify $U\left[\mathbf{x}, \mathbf{x}^{-1}\right]$ with $\left(U\left[\widetilde{\mathbf{x}}_k, \left(\widetilde{\mathbf{x}}_k\right)^{-1}\right]\right)\left[x_k, x_k^{-1}\right]$, and to identify $U\left[\left[\mathbf{x}\right]\right]$ with $\left(U\left[\left[\widetilde{\mathbf{x}}_k\right]\right]\right)\left[\left[x_k\right]\right]$. (But we cannot identify $U\left(\left(\mathbf{x}\right)\right)$ with $\left(U\left(\left(\widetilde{\mathbf{x}}_k\right)\right)\right)\left(\left(x_k\right)\right)$.)

Proposition 2.20. Let U be an $\mathbb{F}$-module. Let $f \in U\left[\left[z, z^{-1}\right]\right]$.

(a) We have $\operatorname{Res}\left(\partial_z^{(n)} f\right) dz = 0$ for every positive integer n.

(b) We have $\operatorname{Res}\left(\partial_z f\right) dz = 0$.

(c) For every $g \in U\left[z, z^{-1}\right]$, we have

$$\operatorname{Res} f \partial_z (g)\, dz = -\operatorname{Res} \partial_z (f)\, g dz \tag{27}$$

and

$$\operatorname{Res} g \partial_z (f)\, dz = -\operatorname{Res} \partial_z (g)\, f dz \tag{28}$$

Proof of Proposition 2.20. **(a)** Let n be a positive integer. Then, $n-1 \in \mathbb{N}$ and $n > n-1$. Hence, $\dbinom{n-1}{n} = 0$.

Let us write f in the form $f = \sum_{i \in \mathbb{Z}} u_i z^i$ for some $(u_i)_{i \in \mathbb{Z}} \in U^{\mathbb{Z}}$. Then, the definition of $\partial_z^{(n)}$ yields $\partial_z^{(n)} f = \sum_{i \in \mathbb{Z}} \dbinom{i}{n} u_i z^{i-n} = \sum_{i \in \mathbb{Z}} \dbinom{i+n}{n} u_{i+n} z^i$ (here, we have substituted $i+n$ for i in the sum). Hence, the definition of $\operatorname{Res}\left(\partial_z^{(n)} f\right) dz$ yields

$$\operatorname{Res}\left(\partial_z^{(n)} f\right) dz = \underbrace{\dbinom{-1+n}{n}}_{= \binom{n-1}{n} = 0} u_{-1+n} = 0.$$

This proves Proposition 2.20 **(a)**.

(b) Proposition 2.20 **(a)** (applied to $n = 1$) yields $\operatorname{Res}\left(\partial_z^{(1)} f\right) dz = 0$. Since $\partial_z^{(1)} = \partial_z$, this yields $\operatorname{Res}\left(\partial_z f\right) dz = 0$. This proves Proposition 2.20 **(b)**.

(c) Let $g \in U\left[z, z^{-1}\right]$. Proposition 2.7 **(c)** (applied to $a = g$ and $b = f$) yields $\partial_z (gf) = \partial_z (g) f + g \partial_z (f)$. Hence, $\partial_z (g) f = \partial_z (gf) - g \partial_z (f)$. Now,

$$\begin{aligned}
\operatorname{Res} \underbrace{f \partial_z (g)}_{= \partial_z (g) f = \partial_z (gf) - g \partial_z (f)} dz &= \operatorname{Res}\left(\partial_z (gf) - g \partial_z (f)\right) dz \\
&= \underbrace{\operatorname{Res} \partial_z (gf)\, dz}_{\substack{=0 \\ \text{(by Proposition 2.20 (b),} \\ \text{applied to } gf \text{ instead of } f)}} - \operatorname{Res} \underbrace{g \partial_z (f)}_{= \partial_z (f) g} dz \\
&= -\operatorname{Res} \partial_z (f)\, g dz.
\end{aligned}$$

This proves (27). Now,

$$\begin{aligned}
\operatorname{Res} \underbrace{g \partial_z (f)}_{= \partial_z (f) g} dz = \operatorname{Res} \partial_z (f)\, g dz &= -\operatorname{Res} \underbrace{f \partial_z (g)}_{= \partial_z (g) f} dz \qquad \text{(by (27))} \\
&= -\operatorname{Res} \partial_z (g)\, f dz.
\end{aligned}$$

This proves (28), and thus completes the proof of Proposition 2.20 **(c)**. □

2.7. Differential operators

Definition 2.21. If U is an $\mathbb{F}$-module, and if $h \in \mathbb{F}\left[\left[w, w^{-1}\right]\right]$, then we shall write L_h for the map $U\left[\left[w, w^{-1}\right]\right] \to U\left[\left[w, w^{-1}\right]\right]$, $a \mapsto ha$. (Notice that U is implicit in this notation; therefore it sometimes leads to several different maps being denoted by L_h. But this rarely makes any troubles, since these maps can be distinguished by their domains.)

Also, if U is an $\mathbb{F}$-module, and if $h \in U\left[\left[w, w^{-1}\right]\right]$, then we shall write L_h for the map $\mathbb{F}\left[\left[w, w^{-1}\right]\right] \to U\left[\left[w, w^{-1}\right]\right]$, $a \mapsto ha$. (Again, U is implicit, and we rely on context to clarify what we mean by L_h.)

When U is an $\mathbb{F}$-module, and when h belongs to either $\mathbb{F}\left[\left[w, w^{-1}\right]\right]$ or $U\left[\left[w, w^{-1}\right]\right]$, we are often going to abbreviate the map L_h by h (by abuse of notation). Thus, $h(a) = ha$ for every h. This is a generalization of the abuse of notation we mentioned in Remark 2.4.

We next define a (purely algebraic) notion of differential operators – one of many reasonable such notions.

If U is an $\mathbb{F}$-module, and if $\left(c_j(w)\right)_{j \in \mathbb{N}}$ is a family of formal distributions in $U\left[\left[w, w^{-1}\right]\right]$ with the property that all but finitely many $j \in \mathbb{N}$ satisfy $c_j(w) = 0$, then $\sum\limits_{j \in \mathbb{N}} c_j(w)\, \partial_w^{(j)}$ is a well-defined $\mathbb{F}$-linear map $\mathbb{F}\left[w, w^{-1}\right] \to U\left[\left[w, w^{-1}\right]\right]$ (where $c_j(w)\, \partial_w^{(j)}$ means $c_j(w) \circ \partial_w^{(j)}$, and where $c_j(w)$ abbreviates $L_{c_j(w)}$, according to Definition 2.21).[33] These maps will be called *differential operators of finite order into U*. In other words:

Definition 2.22. Let U be an $\mathbb{F}$-module. A *differential operator of finite order into U* means a map $\mathbb{F}\left[w, w^{-1}\right] \to U\left[\left[w, w^{-1}\right]\right]$ which has the form $\sum\limits_{j \in \mathbb{N}} c_j(w)\, \partial_w^{(j)}$ for a family $\left(c_j(w)\right)_{j \in \mathbb{N}}$ of formal distributions in $U\left[\left[w, w^{-1}\right]\right]$ with the property that all but finitely many $j \in \mathbb{N}$ satisfy $c_j(w) = 0$.

Remark 2.23. Maps of the form $\sum\limits_{j \in \mathbb{N}} c_j(w)\, \partial_w^{(j)}$ are well-defined even if we do not require that all but finitely many $j \in \mathbb{N}$ satisfy $c_j(w) = 0$. We shall not make use of them in this generality, however.

It is clear that (for given U) the differential operators of finite order into U form an $\mathbb{F}$-module. This $\mathbb{F}$-module contains the operator $c(w)$ for each $w \in U\left[\left[w, w^{-1}\right]\right]$ (recall that "the operator $c(w)$" really means the map $L_{c(w)}$:

[33]This follows from the fact that $U\left[\left[w, w^{-1}\right]\right]$ is an $\mathbb{F}\left[w, w^{-1}\right]$-module.

$\mathbb{F}\left[w,w^{-1}\right] \to U\left[\left[w,w^{-1}\right]\right]$), and when $U=\mathbb{F}$, it also contains the operator $\partial_w^{(j)}$ for every $j \in \mathbb{N}$.

We first notice that the "coefficients" $c_j(w)$ of a differential operator $\sum_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)}$ are uniquely determined by the operator. More precisely:

Proposition 2.24. Let U be an $\mathbb{F}$-module. Let D be a differential operator of finite order into U. Then, there exists a unique family $\left(c_j(w)\right)_{j\in\mathbb{N}}$ of formal distributions in $U\left[\left[w,w^{-1}\right]\right]$ with the properties that

- all but finitely many $j \in \mathbb{N}$ satisfy $c_j(w) = 0$;
- we have $D = \sum_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)}$.

Proof of Proposition 2.24. The existence of such a family follows from the fact that D is a differential operator of finite order into U. It thus remains to prove its uniqueness. In other words, it remains to prove that if $\left(c_j^{\langle 1\rangle}(w)\right)_{j\in\mathbb{N}}$ and $\left(c_j^{\langle 2\rangle}(w)\right)_{j\in\mathbb{N}}$ are two families $\left(c_j(w)\right)_{j\in\mathbb{N}}$ with the two properties stated in Proposition 2.24, then $\left(c_j^{\langle 1\rangle}(w)\right)_{j\in\mathbb{N}} = \left(c_j^{\langle 2\rangle}(w)\right)_{j\in\mathbb{N}}$.

Thus, let $\left(c_j^{\langle 1\rangle}(w)\right)_{j\in\mathbb{N}}$ and $\left(c_j^{\langle 2\rangle}(w)\right)_{j\in\mathbb{N}}$ be two families $\left(c_j(w)\right)_{j\in\mathbb{N}}$ with the two properties stated in Proposition 2.24.

For every $j \in \mathbb{N}$, define a formal distribution $s_j(w) \in U\left[\left[w,w^{-1}\right]\right]$ by

$$s_j(w) = c_j^{\langle 1\rangle}(w) - c_j^{\langle 2\rangle}(w).$$

It is easy to show that

$$0 = \sum_{j\in\mathbb{N}} s_j(w)\,\partial_w^{(j)}. \tag{29}$$

[34]

[34]*Proof.* The family $\left(c_j^{\langle 1\rangle}(w)\right)_{j\in\mathbb{N}}$ satisfies the second of the two properties stated in Proposition 2.24. In other words, we have

$$D = \sum_{j\in\mathbb{N}} c_j^{\langle 1\rangle}(w)\,\partial_w^{(j)} \tag{30}$$

The same argument, applied to the family $\left(c_j^{\langle 2\rangle}(w)\right)_{j\in\mathbb{N}}$ instead of $\left(c_j^{\langle 1\rangle}(w)\right)_{j\in\mathbb{N}}$, yields

$$D = \sum_{j\in\mathbb{N}} c_j^{\langle 2\rangle}(w)\,\partial_w^{(j)}.$$

We will now show that every $n \in \mathbb{N}$ satisfies

$$s_n(w) = 0. \tag{31}$$

Proof of (31): We shall prove (31) by strong induction over n.

So let $N \in \mathbb{N}$. Assume that (31) holds for each $n \in \mathbb{N}$ satisfying $n < N$. We need to show that (31) holds for $n = N$ as well.

We know that (31) holds for each $n \in \mathbb{N}$ satisfying $n < N$. In other words,

$$s_n(w) = 0 \qquad \text{for each } n \in \mathbb{N} \text{ satisfying } n < N. \tag{32}$$

Applying both sides of the equality (29) to the Laurent polynomial $w^N \in \mathbb{F}\left[w, w^{-1}\right]$, we obtain

$$\begin{aligned}
0 &= \sum_{j\in\mathbb{N}} s_j(w)\,\partial_w^{(j)}\left(w^N\right) \\
&= \sum_{j=0}^{N-1} \underbrace{s_j(w)}_{\substack{=0 \\ \text{(by (32),} \\ \text{applied to } n=j)}} \partial_w^{(j)}\left(w^N\right) + \sum_{j\geq N} s_j(w) \underbrace{\partial_w^{(j)}\left(w^N\right)}_{\substack{=\binom{N}{j} w^{N-j} \\ \text{(by the definition of } \partial_w^{(j)})}} \\
&= \underbrace{\sum_{j=0}^{N-1} 0\partial_w^{(j)}\left(w^N\right)}_{=0} + \sum_{j\geq N} s_j(w)\binom{N}{j} w^{N-j} = \sum_{j\geq N} s_j(w)\binom{N}{j} w^{N-j} \\
&= s_N(w)\underbrace{\binom{N}{N}}_{=1}\underbrace{w^{N-N}}_{=1} + \sum_{j>N} s_j(w) \underbrace{\binom{N}{j}}_{\substack{=0 \\ \text{(since } N<j)}} w^{N-j} \\
&= s_N(w) + \underbrace{\sum_{j=0}^{N-1} s_j(w)\,0w^{N-j}}_{=0} = s_N(w).
\end{aligned}$$

Subtracting this equality from (30), we obtain

$$\begin{aligned}
0 &= \sum_{j\in\mathbb{N}} c_j^{\langle 1\rangle}(w)\,\partial_w^{(j)} - \sum_{j\in\mathbb{N}} c_j^{\langle 2\rangle}(w)\,\partial_w^{(j)} = D = \sum_{j\in\mathbb{N}} \underbrace{\left(c_j^{\langle 1\rangle}(w) - c_j^{\langle 2\rangle}(w)\right)}_{\substack{=s_j(w) \\ \text{(since } s_j(w) \text{ was defined as} \\ c_j^{\langle 1\rangle}(w) - c_j^{\langle 2\rangle}(w))}} \partial_w^{(j)} \\
&= \sum_{j\in\mathbb{N}} s_j(w)\,\partial_w^{(j)}.
\end{aligned}$$

This proves (29).

Thus, $s_N(w) = 0$. In other words, (31) holds for $n = N$. This completes the induction step.

Now, we have proven (31) by induction. Thus, every $n \in \mathbb{N}$ satisfies $s_n(w) = 0$. In other words, every $n \in \mathbb{N}$ satisfies $c_n^{\langle 1\rangle}(w) - c_n^{\langle 2\rangle}(w) = 0$ (since $s_n(w)$ is defined to be $c_n^{\langle 1\rangle}(w) - c_n^{\langle 2\rangle}(w)$). In other words, every $n \in \mathbb{N}$ satisfies $c_n^{\langle 1\rangle}(w) = c_n^{\langle 2\rangle}(w)$. Hence, $\left(c_j^{\langle 1\rangle}(w)\right)_{j\in\mathbb{N}} = \left(c_j^{\langle 2\rangle}(w)\right)_{j\in\mathbb{N}}$. This completes the proof of Proposition 2.24. $\square$

Next, we shall discuss composition of differential operators. Two differential operators of finite order into U do not necessarily have a well-defined composition, even if $U = \mathbb{F}$ (because they send Laurent polynomials to formal distributions, but they cannot be applied to formal distributions). However, a composition can be defined in a particular setting:

Definition 2.25. Let U be an $\mathbb{F}$-module.

(a) Let D be a differential operator of finite order into U. We can then write D in the form $\sum\limits_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)}$ for a family $\left(c_j(w)\right)_{j\in\mathbb{N}}$ of formal distributions in $U\left[\left[w, w^{-1}\right]\right]$ with the property that all but finitely many $j \in \mathbb{N}$ satisfy $c_j(w) = 0$. Consider this family. (Proposition 2.24 says that this family is unique.) We say that D is *Laurent* if all $c_j(w)$ are Laurent polynomials (i.e., belong to $\mathbb{F}\left[w, w^{-1}\right]$).

It is clear that if D is Laurent, then D has the following two properties:

- The image of D is contained in $U\left[w, w^{-1}\right]$.
- The operator D can be uniquely extended to a continuous $\mathbb{F}$-linear map $\mathbb{F}\left[\left[w, w^{-1}\right]\right] \to U\left[\left[w, w^{-1}\right]\right]$ (which is again defined as $\sum\limits_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)}$), since $U\left[\left[w, w^{-1}\right]\right]$ is an $\mathbb{F}\left[w, w^{-1}\right]$-module. We denote this extension by D again.

(b) Let D be any differential operator of finite order into U, and let E be any differential operator of finite order into $\mathbb{F}$. We cannot define a map $D \circ E$ in general.

However, if E is Laurent, then $D \circ E$ is a well-defined map from $\mathbb{F}\left[w, w^{-1}\right]$ to $U\left[\left[w, w^{-1}\right]\right]$ (since the image of E is contained in $\mathbb{F}\left[w, w^{-1}\right]$).

On the other hand, if D is Laurent, then $D \circ E$ is a well-defined map from $\mathbb{F}\left[w, w^{-1}\right]$ to $U\left[\left[w, w^{-1}\right]\right]$ (but here, the D in $D \circ E$ does not mean the original map $D : \mathbb{F}\left[w, w^{-1}\right] \to U\left[\left[w, w^{-1}\right]\right]$, but rather the extension of D to a continuous $\mathbb{F}$-linear map $\mathbb{F}\left[\left[w, w^{-1}\right]\right] \to U\left[\left[w, w^{-1}\right]\right]$).

Altogether, we thus know that there is a well-defined operator $D \circ E$ whenever at least one of D and E is Laurent.

We shall now prove that the composition of two differential operators is again a differential operator, provided that it is well-defined:

Proposition 2.26. Let U be an $\mathbb{F}$-module. Let D be any differential operator of finite order into U. Let E be any differential operator of finite order into $\mathbb{F}$.

(a) Assume that at least one of D and E is Laurent. Then, the map $D \circ E$ is a differential operator of finite order into U.

(b) If both D and E are Laurent, then $D \circ E$ is also Laurent.

Proof of Proposition 2.26. For every $e(w) \in \mathbb{F}\left[\left[w, w^{-1}\right]\right]$ and every $n \in \mathbb{N}$, we have

$$\partial_w^{(n)} \circ e(w) = \sum_{i=0}^{n} \left(\partial_w^{(n-i)}(e(w))\right) \circ \partial_w^{(i)} \tag{33}$$

(as maps from $\mathbb{F}\left[w, w^{-1}\right]$ to $\mathbb{F}\left[w, w^{-1}\right]$). [35]

Now, D is a differential operator of finite order into $\mathbb{F}$. Hence, we can write D in the form $D = \sum_{j\in\mathbb{N}} d_j(w)\, \partial_w^{(j)}$ for a family of $\left(d_j(w)\right)_{j\in\mathbb{N}}$ of formal distributions in $\mathbb{F}\left[\left[w, w^{-1}\right]\right]$ with the property that all but finitely many $j \in \mathbb{N}$ satisfy $d_j(w) = 0$. Consider this family.

Also, E is a differential operator of finite order into U. Hence, we can write E in the form $E = \sum_{j\in\mathbb{N}} e_j(w)\, \partial_w^{(j)}$ for a family of $\left(e_j(w)\right)_{j\in\mathbb{N}}$ of formal distributions in $U\left[\left[w, w^{-1}\right]\right]$ with the property that all but finitely many $j \in \mathbb{N}$ satisfy $e_j(w) = 0$. Consider this family.

For all $i \in \mathbb{N}$ and $j \in \mathbb{N}$, the product $d_i(w) \cdot e_j(w)$ in $U\left[\left[w, w^{-1}\right]\right]$ is well-defined[36].

[35] *Proof of (33):* Let $e(w) \in \mathbb{F}\left[\left[w, w^{-1}\right]\right]$ and $n \in \mathbb{N}$. Let $g \in \mathbb{F}\left[w, w^{-1}\right]$ be arbitrary. Applying Proposition 2.7 **(b)** to $\mathbb{F}$ instead of U (and renaming the indeterminate z as w), we conclude that

$$\partial_w^{(n)}(ab) = \sum_{i=0}^{n} \partial_w^{(i)}(a)\, \partial_w^{(n-i)}(b)$$

for any $a \in \mathbb{F}\left[w, w^{-1}\right]$ and $b \in \mathbb{F}\left[w, w^{-1}\right]$. Applying this to $a = g$ and $b = e(w)$, we obtain

$$\partial_w^{(n)}(e(w) \cdot g) = \sum_{i=0}^{n} \partial_w^{(i)}(g)\, \partial_w^{(n-i)}(e(w)) = \left(\sum_{i=0}^{n} \left(\partial_w^{(n-i)}(e(w))\right) \circ \partial_w^{(i)}\right)(g).$$

Hence,

$$\left(\partial_w^{(n)} \circ e(w)\right)(g) = \partial_w^{(n)}(e(w) \cdot g) = \left(\sum_{i=0}^{n} \left(\partial_w^{(n-i)}(e(w))\right) \circ \partial_w^{(i)}\right)(g). \tag{34}$$

Let us now forget that we fixed g. We thus have proven (34) for every $g \in \mathbb{F}\left[w, w^{-1}\right]$. In other words, we have $\partial_w^{(n)} \circ e(w) = \sum_{i=0}^{n} \left(\partial_w^{(n-i)}(e(w))\right) \circ \partial_w^{(i)}$. This proves (33).

[36] *Proof.* We assumed that at least one of D and E is Laurent. We WLOG assume that D is

The operator $D \circ E$ is well-defined (since at least one of D and E is Laurent).

Laurent (since the proof in the case when E is Laurent is similar). Then, all $d_j(w)$ are Laurent polynomials. That is, $d_i(w) \in U[w, w^{-1}]$ for every $i \in \mathbb{N}$. Hence, $d_i(w) \cdot e_j(w)$ is well-defined for all $i \in \mathbb{N}$ and $j \in \mathbb{N}$ (since the product of an element of $U[w, w^{-1}]$ with an element of $\mathbb{F}[[w, w^{-1}]]$ is always well-defined). Qed.

We have $D = \sum_{j\in\mathbb{N}} d_j(w)\,\partial_w^{(j)} = \sum_{n\in\mathbb{N}} d_n(w)\,\partial_w^{(n)}$ and thus

$$\begin{aligned}
&\underbrace{D}_{=\sum_{n\in\mathbb{N}} d_n(w)\partial_w^{(n)}} \circ \underbrace{E}_{=\sum_{j\in\mathbb{N}} e_j(w)\partial_w^{(j)}} \\
&= \left(\sum_{n\in\mathbb{N}} d_n(w)\,\partial_w^{(n)}\right) \circ \left(\sum_{j\in\mathbb{N}} e_j(w)\,\partial_w^{(j)}\right) \\
&= \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}} d_n(w) \circ \underbrace{\partial_w^{(n)} \circ e_j(w)}_{\substack{=\sum_{i=0}^{n}\left(\partial_w^{(n-i)}\left(e_j(w)\right)\right)\circ\partial_w^{(i)} \\ \text{(by (33), applied to } e(w)=e_j(w)\text{)}}} \circ\partial_w^{(j)} \\
&= \sum_{n\in\mathbb{N}}\sum_{j\in\mathbb{N}} d_n(w) \circ \left(\sum_{i=0}^{n}\left(\partial_w^{(n-i)}\left(e_j(w)\right)\right)\circ\partial_w^{(i)}\right)\circ\partial_w^{(j)} \\
&= \sum_{n\in\mathbb{N}}\sum_{i=0}^{n}\sum_{j\in\mathbb{N}} \underbrace{d_n(w)\circ\left(\partial_w^{(n-i)}\left(e_j(w)\right)\right)}_{=d_n(w)\cdot\left(\partial_w^{(n-i)}\left(e_j(w)\right)\right)} \circ \underbrace{\partial_w^{(i)}\circ\partial_w^{(j)}}_{\substack{=\binom{i+j}{i}\partial_w^{(i+j)} \\ \text{(by (6), applied to } i \text{ and } j \text{ instead of} \\ n \text{ and } m\text{, and with } z \text{ renamed as } w\text{)}}} \\
&= \sum_{n\in\mathbb{N}}\sum_{i=0}^{n}\sum_{j\in\mathbb{N}} d_n(w)\cdot\left(\partial_w^{(n-i)}\left(e_j(w)\right)\right)\circ\left(\binom{i+j}{i}\partial_w^{(i+j)}\right) \\
&= \sum_{n\in\mathbb{N}}\sum_{i=0}^{n}\sum_{j\in\mathbb{N}} \binom{i+j}{i} d_n(w)\cdot\left(\partial_w^{(n-i)}\left(e_j(w)\right)\right)\circ\partial_w^{(i+j)} \\
&= \underbrace{\sum_{n\in\mathbb{N}}\sum_{i=0}^{n}\sum_{\substack{m\in\mathbb{N};\\ m\geq i}}}_{=\sum_{m\in\mathbb{N}}\sum_{i=0}^{m}\sum_{\substack{n\in\mathbb{N};\\ n\geq i}}} \binom{m}{i} d_n(w)\cdot\left(\partial_w^{(n-i)}\left(e_{m-i}(w)\right)\right)\circ\partial_w^{(m)} \\
&\qquad\text{(here, we have substituted } m \text{ for } i+j \text{ in the third sum)} \\
&= \sum_{m\in\mathbb{N}}\sum_{i=0}^{m}\sum_{\substack{n\in\mathbb{N};\\ n\geq i}} \binom{m}{i} d_n(w)\cdot\left(\partial_w^{(n-i)}\left(e_{m-i}(w)\right)\right)\circ\partial_w^{(m)} \\
&= \sum_{m\in\mathbb{N}} \underbrace{\sum_{i=0}^{m}\binom{m}{i}\left(\sum_{\substack{n\in\mathbb{N};\\ n\geq i}} d_n(w)\cdot\left(\partial_w^{(n-i)}\left(e_{m-i}(w)\right)\right)\right)}_{\substack{\in\mathbb{F}\left[\left[w,w^{-1}\right]\right] \\ \text{(because the inner sum has only finitely many nonzero addends} \\ \text{(since all but finitely many } j\in\mathbb{N} \text{ satisfy } d_j(w)=0\text{))}}} \circ\partial_w^{(m)}. \qquad (35)
\end{aligned}$$

Since all but finitely many $m \in \mathbb{N}$ satisfy $\sum\limits_{i=0}^{m} \dbinom{m}{i} \left(\sum\limits_{\substack{n \in \mathbb{N}; \\ n \geq i}} d_n(w) \cdot \left(\partial_w^{(n-i)} (e_{m-i}(w)) \right) \right) = 0$ [37], this yields that $D \circ E$ is a differential operator of finite order into U. Proposition 2.26 **(a)** is thus proven.

(b) Assume that D and E Laurent. Then, all $d_j(w)$ are Laurent polynomials, and all $e_j(w)$ are Laurent polynomials. Now, it is easy to see that

$$\sum_{i=0}^{m} \binom{m}{i} \left(\sum_{\substack{n \in \mathbb{N}; \\ n \geq i}} d_n(w) \cdot \left(\partial_w^{(n-i)} (e_{m-i}(w)) \right) \right)$$

is a Laurent polynomial for each $m \in \mathbb{N}$ (since the inner sum $\sum\limits_{\substack{n \in \mathbb{N}; \\ n \geq i}}$ in this expression has only finitely many nonzero terms). Thus, (35) shows that $D \circ E$ is Laurent. This proves Proposition 2.26 **(b)**. □

We now define the notion of an adjoint differential operator:

Definition 2.27. Let U be an $\mathbb{F}$-module. Let D be a differential operator of finite order into U. We can then write D in the form $\sum\limits_{j \in \mathbb{N}} c_j(w) \partial_w^{(j)}$ for a family $(c_j(w))_{j \in \mathbb{N}}$ of formal distributions in $U\left[\left[w, w^{-1}\right]\right]$ with the property that all but finitely many $j \in \mathbb{N}$ satisfy $c_j(w) = 0$. Consider this family. (Proposition 2.24 says that this family is unique.)

We denote by D^* the map

$$\sum_{j \in \mathbb{N}} (-1)^j \partial_w^{(j)} \circ c_j(w) : \mathbb{F}\left[w, w^{-1}\right] \to U\left[\left[w, w^{-1}\right]\right],$$

which is again a differential operator of finite order into U (due to Proposition 2.26). This operator D^* is called the *adjoint differential operator* to D. If D is Laurent, then D^* is Laurent (this follows easily from Proposition 2.26 **(b)**).

We end this section with some results whose proofs we leave to the reader:

Proposition 2.28. Let U be an $\mathbb{F}$-module. Let D be a Laurent differential operator of finite order into U. Then, any $f \in \mathbb{F}\left[w, w^{-1}\right]$ and $g \in \mathbb{F}\left[w, w^{-1}\right]$ satisfy

$$\operatorname{Res} (D^* f) \, g dw = \operatorname{Res} f \, (Dg) \, dw.$$

[37] This is easily derived from the facts that all but finitely many $j \in \mathbb{N}$ satisfy $d_j(w) = 0$, and all but finitely many $j \in \mathbb{N}$ satisfy $e_j(w) = 0$.

In view of the fact that we regard $\operatorname{Res} fgdw$ as a sort of bilinear pairing between f and g (see Section 3.1 below), this Proposition 2.28 likely explains where the name "adjoint differential operator" for D^* comes from.

Definition 2.29. Let U be an $\mathbb{F}$-module. Let D be any differential operator of finite order into $\mathbb{F}$. We can then write D in the form $\sum_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)}$ for a family $(c_j(w))_{j\in\mathbb{N}}$ of formal distributions in $\mathbb{F}[[w,w^{-1}]]$ with the property that all but finitely many $j\in\mathbb{N}$ satisfy $c_j(w)=0$. Consider this family. (Proposition 2.24 says that this family is unique.) Then, we define a map $D_{(U)} : U[w,w^{-1}] \to U[[w,w^{-1}]]$ by $D_{(U)} = \sum_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)}$ (this is the same expression as for D, but it is now acting on $U[w,w^{-1}]$).

Proposition 2.30. Let U be an $\mathbb{F}$-module. Let D be any differential operator of finite order into U. Let E be any differential operator of finite order into $\mathbb{F}$. Assume that at least one of D and E is Laurent. We can canonically lift the differential operator $D : \mathbb{F}[w,w^{-1}] \to \mathbb{F}[[w,w^{-1}]]$ to a map $\widetilde{D} : U[w,w^{-1}] \to U[[w,w^{-1}]]$ by writing D in the form $\sum_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)}$ (with $(c_j(w))_{j\in\mathbb{N}}$ being as usual) and defining $\widetilde{D}$ to be the map $\sum_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)}$ (but now acting on $U[w,w^{-1}]$). (This is well-defined because of Proposition 2.24.) Notice that if D is Laurent, then so is $\widetilde{D}$.

Now, we have
$$(D\circ E)^* = (E^*)_U \circ D^*.$$

The proof of Proposition 2.30 can be obtained easily using Proposition 2.28, once one reduces to the case of D and E both being Laurent (by $\mathbb{F}$-linearity and continuity) and of $U=\mathbb{F}$ (by U-linearity).

3. Locality and the formal δ-function

3.1. Pairing between distributions and polynomials

In analysis, distributions are often defined as linear functionals on the space of test functions. Similarly, formal distributions can be regarded as linear maps acting on polynomials. Specifically, let U be a $\mathbb{F}$-module. For every $f\in\mathbb{F}[z,z^{-1}]$ and $a\in U[[z,z^{-1}]]$, we can set
$$\langle f,a\rangle = \operatorname{Res} fadz \in U. \tag{36}$$
When $U=\mathbb{F}$, this value belongs to $\mathbb{F}$, so we have defined a pairing $\langle\cdot,\cdot\rangle : \mathbb{F}[z,z^{-1}]\times\mathbb{F}[[z,z^{-1}]] \to \mathbb{F}$. This pairing leads to a linear map $\mathbb{F}[[z,z^{-1}]] \to (\mathbb{F}[z,z^{-1}])^*$, which is easily seen to be an isomorphism.

The equality (36) motivates a seemingly weird notation:

Definition 3.1. If $a \in U\left[\left[z, z^{-1}\right]\right]$ and $n \in \mathbb{Z}$, then we denote by $a_{[n]}$ the $(-n-1)$-st coefficient of a. (Thus, $a = \sum\limits_{n\in\mathbb{Z}} a_{[n]} z^{-n-1}$ for every $a \in U\left[\left[z, z^{-1}\right]\right]$.)

Why did we choose to have $a_{[n]}$ mean the $(-n-1)$-st coefficient rather than the n-th coefficient? The purpose was to ensure that

$$\langle z^n, f \rangle = f_{[n]} \qquad \text{for every } n \in \mathbb{Z}.$$

Similarly, if $a \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ and $(m, n) \in \mathbb{Z}^2$, then $a_{[m,n]}$ shall denote the coefficient of a before $z^{-m-1} w^{-n-1}$.

[Notice that I am using the notations $a_{[n]}$ and $a_{[m,n]}$ instead of the more common notations $a_{(n)}$ and $a_{(m,n)}$ to avoid conflict with the "$_{(n)}$" notation for vertex-algebra products.]

3.2. Local formal distributions

Let U be an $\mathbb{F}$-module.

Definition 3.2. A formal distribution $a \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ is said to be *local* if $(z-w)^N a(z, w) = 0$ for some $N \in \mathbb{N}$.

(Recall once again that $a(z, w)$ is just another way to say a, and that the $\mathbb{F}[z, w]$-module $U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ has torsion, so that $(z-w)^N a(z, w) = 0$ does not force $a(z, w) = 0$.)

Proposition 3.3. Let $a = a(z, w) \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ be local. Then:
(a) The formal distribution $a(w, z)$ is also local.
(b) The formal distributions $\partial_z a(z, w)$ and $\partial_w a(z, w)$ are local as well.

Proof of Proposition 3.3. We know that a is local. Thus, $(z-w)^N a(z, w) = 0$ for some $N \in \mathbb{N}$. Consider this N.

(a) Substituting w and z for z and w in $(z-w)^N a(z, w) = 0$, we obtain $(w-z)^N a(w, z) = 0$. Thus,

$$\underbrace{(z-w)^N}_{=(-(w-z))^N = (-1)^N (w-z)^N} a(w, z) = (-1)^N \underbrace{(w-z)^N a(w, z)}_{=0} = 0.$$

This shows that $a(w, z)$ is local. Proposition 3.3 **(a)** is proven.

(b) We have

$$\partial_z \left(\underbrace{(z-w)^{N+1}}_{=(z-w)(z-w)^N} a(z,w) \right) = \partial_z \left((z-w) \underbrace{(z-w)^N a(z,w)}_{=0} \right) = 0,$$

thus

$$\begin{aligned} 0 &= \partial_z \left((z-w)^{N+1} a(z,w) \right) \\ &= \underbrace{\left(\partial_z (z-w)^{N+1} \right)}_{=(N+1)(z-w)^N} a(z,w) + (z-w)^{N+1} \partial_z a(z,w) \qquad \text{(since } \partial_z \text{ is a derivation)} \\ &= (N+1) \underbrace{(z-w)^N a(z,w)}_{=0} + (z-w)^{N+1} \partial_z a(z,w) = (z-w)^{N+1} \partial_z a(z,w). \end{aligned}$$

Hence, $(z-w)^{N+1} \partial_z a(z,w) = 0$. This shows that $\partial_z a(z,w)$ is local. Similarly, $\partial_w a(z,w)$ is local. Proposition 3.3 **(b)** is thus proven. $\square$

3.3. The formal δ-function

Example 3.4. We define an element $\delta(z-w)$ of $\mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ as follows:

$$\delta(z-w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n.$$

(The notation "$\delta(z-w)$" is slightly confusing; the minus should be interpreted as a symbolic piece of the notation rather than as a subtraction sign. Of course, it is supposed to resemble the subtraction sign, as the whole formal distribution $\delta(z-w)$ is supposed to resemble the delta function of $z-w$ as a distribution on the z-w-plane. Two other notations for $\delta(z-w)$ are $\delta(z,w)$ and $\delta(z/w)$.)

The $\mathbb{F}$-valued formal distribution $\delta(z-w)$ is called the *formal δ-function*. Notice that $\delta(z-w)$ is symmetric in z and w: We have

$$\begin{aligned} \delta(z-w) &= \sum_{n \in \mathbb{Z}} z^{-n-1} w^n = \sum_{n \in \mathbb{Z}} z^n w^{-n-1} \\ &\qquad \text{(here, we have substituted } -n-1 \text{ for } n \text{ in the sum)} \\ &= \sum_{n \in \mathbb{Z}} w^{-n-1} z^n = \delta(w-z). \end{aligned}$$

The next proposition (whose proof is immediate) shows some motivation for the study of $\delta(z-w)$ and for its name:

Proposition 3.5. **(a)** For every $m \in \mathbb{Z}$, we have

$$\delta(z-w) z^m = \left(\sum_{n \in \mathbb{Z}} z^{-n-1} w^n \right) z^m = \sum_{n \in \mathbb{Z}} z^{-n+m-1} w^n = \sum_{n \in \mathbb{Z}} z^{-n+1} w^{n+m}$$

(here, we have substituted $n+m$ for n in the sum)

$$= \left(\sum_{n \in \mathbb{Z}} z^{-n-1} w^n \right) w^m = \delta(z-w) w^m. \tag{37}$$

(b) We have

$$\operatorname{Res} \delta(z-w) \, dz = 1. \tag{38}$$

(c) For every $\varphi \in \mathbb{F}[z, z^{-1}]$, we have

$$\delta(z-w) \varphi(z) = \delta(z-w) \varphi(w). \tag{39}$$

(This follows from (37), because linearity allows us to WLOG assume that φ is a monomial.)

(d) For every $\varphi \in \mathbb{F}[z, z^{-1}]$, we have

$$\operatorname{Res} \underbrace{\delta(z-w) \varphi(z)}_{\substack{= \delta(z-w)\varphi(w) \\ \text{(by (39))}}} dz = \operatorname{Res} \delta(z-w) \varphi(w) \, dz$$

$$= \varphi(w) \underbrace{\operatorname{Res} \delta(z-w) \, dz}_{=1} = \varphi(w). \tag{40}$$

One can view this equality as a defining property for the formal δ-function.

(e) Applying (37) to $m = 1$, we obtain $\delta(z-w) z = \delta(z-w) w$, so that $(z-w)\delta(z-w) = 0$. Thus, the formal distribution $\delta(z-w)$ is local.

The formal distribution $\delta(z-w)$ is interesting in that it can be multiplied not only with Laurent polynomials, but also with univariate formal distributions:

Proposition 3.6. Let U be a $\mathbb{F}$-module. Let $a(z) \in U[[z, z^{-1}]]$.

(a) The formal distributions $\delta(z-w) a(z) \in U[[z, z^{-1}, w, w^{-1}]]$ and $\delta(z-w) a(w) \in U[[z, z^{-1}, w, w^{-1}]]$ are well-defined (although there is no general notion of a product of a formal distribution in $U[[z, z^{-1}, w, w^{-1}]]$ with a formal distribution in $U[[z, z^{-1}]]$).

(b) We have

$$\delta(z-w) a(z) = \delta(z-w) a(w). \tag{41}$$

(In other words, the equality (39) holds not only for $\varphi \in \mathbb{F}[z, z^{-1}]$, but also more generally for $\varphi \in U[[z, z^{-1}]]$.)

(c) We have

$$\operatorname{Res} \delta(z-w) a(z) \, dz = a(w). \tag{42}$$

Proof of Proposition 3.6. **(a)** There are essentially two ways to prove that $\delta(z-w)\,a(z)$ is well-defined. One is to directly compute it:

$$\underbrace{\delta(z-w)}_{=\sum_{n\in\mathbb{Z}} z^{-n-1}w^n}\ a(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n a(z) = \sum_{n\in\mathbb{Z}} \underbrace{z^{-n-1}a(z)}_{\substack{\text{a well-defined element}\\ \text{of } U[[z,z^{-1}]]}} w^n.$$

The other way is to argue that $\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n$ belongs not only to $U[[z,z^{-1},w,w^{-1}]]$, but also to the smaller $\mathbb{F}$-module $(U[z,z^{-1}])[[w,w^{-1}]]$ (which is embedded in $U[[z,z^{-1},w,w^{-1}]]$ via the identification of $U[[z,z^{-1},w,w^{-1}]]$ with $(U[[z,z^{-1}]])[[w,w^{-1}]]$) [38], and that any element of the latter $\mathbb{F}$-module can be multiplied with any element of $U[[z,z^{-1}]]$.

Either way, we have shown that $\delta(z-w)\,a(z)$ is well-defined. A similar argument shows that $\delta(z-w)\,a(w)$ is well-defined. Proposition 3.6 **(a)** is proven.

(b) Essentially, we can obtain (41) by breaking up $a(z)$ into an infinite sum of monomials and applying (37). Let us give some details on this argument to make sure that everything is well-defined: Write $a(z) = \sum_{m\in\mathbb{Z}} a_m z^m$. Then, $a(w) = \sum_{m\in\mathbb{Z}} a_m w^m$ and

$$\delta(z-w)\ \underbrace{a(z)}_{=\sum_{m\in\mathbb{Z}} a_m z^m} = \delta(z-w)\cdot\left(\sum_{m\in\mathbb{Z}} a_m z^m\right) = \sum_{m\in\mathbb{Z}} a_m \underbrace{\delta(z-w)\,z^m}_{\substack{=\delta(z-w)w^m\\ \text{(by (37))}}}$$

(check that distributivity does hold here!)

$$= \sum_{m\in\mathbb{Z}} a_m \delta(z-w)\,w^m = \delta(z-w)\underbrace{\left(\sum_{m\in\mathbb{Z}} a_m w^m\right)}_{=a(w)} = \delta(z-w)\,a(w).$$

This proves Proposition 3.6 **(b)**.

(c) Proposition 3.6 **(c)** follows from Proposition 3.6 **(b)** in the same way as (40) follows from (39). ☐

We shall soon (in Section 3.5) see another way to construct $\delta(z-w)$. But before we get there, we need to introduce a few more $\mathbb{F}$-algebras.

[38]This $\mathbb{F}$-module $(U[z,z^{-1}])[[w,w^{-1}]]$ can be characterized as the set of all $u = \sum_{(i,j)\in\mathbb{Z}^2} u_{i,j} z^i w^j$ (with all $u_{i,j} \in U$) having the property that, for every $j \in \mathbb{Z}$, all but finitely many $i \in \mathbb{Z}$ satisfy $u_{i,j} = 0$. Keep in mind that how many these "finitely many" are can depend on j. Thus, $(U[z,z^{-1}])[[w,w^{-1}]]$ is not the same as $(U[[w,w^{-1}]])[z,z^{-1}]$.

3.4. The rings $U((z, w/z))$ and $U((w, z/w))$

Recall that, in Section 1.2, we defined an $\mathbb{F}$-algebra $U((z, w))$ of Laurent series in z and w for every $\mathbb{F}$-algebra U; it is explicitly given by (2). Now, for every $\mathbb{F}$-algebra U, let us define an $\mathbb{F}$-algebra $U((z, w/z))$, which will satisfy

$$U((z, w)) \subseteq U((z, w/z)) \subseteq U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$$

(and these containments are strict, unless $U = 0$). In analytical terms, $U((z, w/z))$ is the $\mathbb{F}$-algebra of "U-valued formal Laurent series expanded in the domain $|z| > |w|$" (or, maybe better said, in the domain $1 > |z| > |w|$). Rather than dwelling on what this means, we are going to define this $\mathbb{F}$-algebra purely algebraically.

Definition 3.7. Let U be an $\mathbb{F}$-module. Let a and b be two new (unrelated and distinct) symbols. Then, $U((z, w/z))$ denotes the image of the continuous $\mathbb{F}$-linear map

$$\begin{aligned} U((a, b)) &\to U\left[\left[z, z^{-1}, w, w^{-1}\right]\right], \\ a^i b^j &\mapsto z^i (w/z)^j = z^i w^j z^{-j} = z^{i-j} w^j. \end{aligned}$$

Informally, this means that $U((z, w/z))$ can be obtained as follows: Imagine for a moment that w/z is a formal variable. and consider all Laurent series in z and w/z. Then, transform these series into elements of $U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ by replacing each $(w/z)^j$ by $w^j z^{-j}$. The set of all possible results is $U((z, w/z))$.

It is easy to see that

$$U((z, w/z)) = \left\{ \sum_{(i,j) \in \mathbb{Z}^2} u_{(i,j)} z^i w^j \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right] \mid \text{all } u_{(i,j)} \in U, \text{ and there exists an } N \in \mathbb{Z} \text{ such that all } (i, j) \in \mathbb{Z}^2 \text{ satisfying } \min\{i + j, j\} < N \text{ satisfy } u_{(i,j)} = 0 \right\} \tag{43}$$

as subsets of $U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$. From this, it follows easily that $U((z, w)) \subseteq U((z, w/z))$.

It is easy to see that

$$\partial_z (U((z, w/z))) \subseteq U((z, w/z)) \qquad \text{and} \qquad \partial_w (U((z, w/z))) \subseteq U((z, w/z))$$

[39]. Hence, the endomorphisms ∂_z and ∂_w of $U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ restrict to endomorphisms ∂_z and ∂_w of $U((z, w/z))$.

[39] *Proof.* Let us introduce a new notation: If $N \in \mathbb{Z}$, and if $u = \sum_{(i,j) \in \mathbb{Z}^2} u_{(i,j)} z^i w^j$ is an element of

More generally, we can see that every $n \in \mathbb{N}$ satisfies

$$\partial_z^{(n)} \left(U\left(\left(z, w/z\right)\right)\right) \subseteq U\left(\left(z, w/z\right)\right) \qquad \text{and} \qquad \partial_w^{(n)} \left(U\left(\left(z, w/z\right)\right)\right) \subseteq U\left(\left(z, w/z\right)\right).$$

Hence, for every $n \in \mathbb{N}$, the endomorphisms $\partial_z^{(n)}$ and $\partial_w^{(n)}$ of $U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ restrict to endomorphisms $\partial_z^{(n)}$ and $\partial_w^{(n)}$ of $U\left(\left(z, w/z\right)\right)$.

When U is a $\mathbb{F}$-algebra, then any two elements of $U\left(\left(z, w/z\right)\right)$ can be multiplied[40]. Thus, in this case, $U\left(\left(z, w/z\right)\right)$ becomes an $\mathbb{F}$-algebra. Clearly, $U\left(\left(z, w\right)\right)$ is an $\mathbb{F}$-subalgebra of $U\left(\left(z, w/z\right)\right)$. The endomorphisms $\partial_z^{(n)}$ and $\partial_w^{(n)}$ of $U\left(\left(z, w/z\right)\right)$ satisfy the following analogue of Proposition 2.7 **(a)**:

Proposition 3.8. Let U be a commutative $\mathbb{F}$-algebra. The sequence $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ is a divided-powers HSD from $U\left(\left(z, w/z\right)\right)$ to $U\left(\left(z, w/z\right)\right)$. Also, the sequence $\left(\partial_w^{(0)}, \partial_w^{(1)}, \partial_w^{(2)}, \ldots\right)$ is a divided-powers HSD from $U\left(\left(z, w/z\right)\right)$ to $U\left(\left(z, w/z\right)\right)$.

Proof of Proposition 3.8. The proof of Proposition 3.8 is analogous to that of Proposition 2.7 **(a)**; the fact that we now have two variables instead of one does not require any significant changes (since the $\partial_z^{(n)}$ commute with multiplication by w, and the $\partial_w^{(n)}$ commute with multiplication by z). We leave the details to the reader. $\square$

$U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ (with all $u_{(i,j)} \in U$), then we say that u is *N-downbounded* if all $(i, j) \in \mathbb{Z}^2$ satisfying $\min\{i + j, j\} < N$ satisfy $u_{(i,j)} = 0$. Then, (43) says that

$$U\left(\left(z, w/z\right)\right) = \left\{u \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right] \mid u \text{ is } N\text{-downbounded for some } N \in \mathbb{Z}\right\}.$$

Hence, $\partial_z \left(U\left(\left(z, w/z\right)\right)\right) \subseteq U\left(\left(z, w/z\right)\right)$ follows from the trivial observation that for every $N \in \mathbb{Z}$ and every N-downbounded element u of $U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$, the element $\partial_z u$ is $(N-1)$-downbounded. Similarly, $\partial_w \left(U\left(\left(z, w/z\right)\right)\right) \subseteq U\left(\left(z, w/z\right)\right)$ can be shown.

[40]The multiplication, of course, is defined as usual:

$$\left(\sum_{(i,j)\in\mathbb{Z}^2} u_{(i,j)} z^i w^j\right) \left(\sum_{(i,j)\in\mathbb{Z}^2} v_{(i,j)} z^i w^j\right) = \sum_{(i,j)\in\mathbb{Z}^2} \left(\sum_{(i',j')\in\mathbb{Z}^2} u_{(i',j')} v_{(i-i',j-j')}\right) z^i w^j.$$

What needs to be checked is that this definition makes sense, i.e., that the inner sum $\sum_{(i',j')\in\mathbb{Z}^2} u_{(i',j')} v_{(i-i',j-j')}$ is well-defined and the outer sum $\sum_{(i,j)\in\mathbb{Z}^2} \left(\sum_{(i',j')\in\mathbb{Z}^2} u_{(i',j')} v_{(i-i',j-j')}\right) z^i w^j$ is an element of $U\left(\left(z, w/z\right)\right)$. This can be done either directly (using (43)), or using the fact that the continuous $\mathbb{F}$-linear map

$$U\left(\left(a, b\right)\right) \to U\left(\left(z, w/z\right)\right),$$
$$a^i b^j \mapsto z^{i-j} w^j$$

is an $\mathbb{F}$-module isomorphism and preserves multiplication. We leave this to the reader.

When U is a commutative $\mathbb{F}$-algebra, the $\mathbb{F}$-algebra $U((z,w/z))$ is commutative and actually a U-algebra.

We finally notice that we can define an $\mathbb{F}$-module $U((w,z/w))$ analogously to how we defined $U((z,w/z))$ above. Again, this is a commutative $\mathbb{F}$-algebra if U is a commutative $\mathbb{F}$-algebra, and satisfies similar properties as $U((z,w/z))$. It is easy to see that $U((w,z/w)) \cap U((z,w/z)) = U((z,w))$.

3.5. Another point of view on $\delta(z-w)$: the i_z and i_w operators

Definition 3.9. Let R be the localization of the ring $\mathbb{F}\left[z,z^{-1},w,w^{-1}\right]$ at the multiplicatively closed subset $\left\{(z-w)^k \mid k \in \mathbb{N}\right\}$. In other words, R is the localization of the ring $\mathbb{F}[z,w]$ at the multiplicatively closed subset $\left\{z^a w^b (z-w)^c \mid a,b,c \in \mathbb{N}\right\}$. In other words, R is the ring of rational functions in z and w whose denominators are of the form $z^a w^b (z-w)^c$ with $a,b,c \in \mathbb{N}$. (In geometrical language, R is the ring of all rational functions in z and w with poles only at $z=0$, at $w=0$ and at $z=w$.)

It is easy to see that the derivative operators ∂_z and ∂_w, and also the operators $\partial_z^{(n)}$ and $\partial_w^{(n)}$ for each $n \in \mathbb{N}$, can be lifted to R from $\mathbb{F}\left[z,z^{-1},w,w^{-1}\right]$. In more precise terms, this means the following:

Recall that R is a localization of the ring $\mathbb{F}\left[z,z^{-1},w,w^{-1}\right]$: namely, $R = \left\{(z-w)^k \mid k \in \mathbb{N}\right\}^{-1} \mathbb{F}\left[z,z^{-1},w,w^{-1}\right]$. Hence, the lift of the divided-powers HSD $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ to the localization $R = \left\{(z-w)^k \mid k \in \mathbb{N}\right\}^{-1} \mathbb{F}\left[z,z^{-1},w,w^{-1}\right]$ is well-defined (according to Definition 2.18). We shall denote this lift again by $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$, since the restrictions of these maps $\partial_z^{(n)}$ to $\mathbb{F}\left[z,z^{-1},w,w^{-1}\right]$ are the $\partial_z^{(n)}$ we already know. Corollary 2.17 **(b)** shows that this lift $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ is a divided-powers HSD. Similarly, there is a well-defined lift of the divided-powers HSD $\left(\partial_w^{(0)}, \partial_w^{(1)}, \partial_w^{(2)}, \ldots\right)$ to the localization $R = \left\{(z-w)^k \mid k \in \mathbb{N}\right\}^{-1} \mathbb{F}\left[z,z^{-1},w,w^{-1}\right]$; this lift will be denoted by $\left(\partial_w^{(0)}, \partial_w^{(1)}, \partial_w^{(2)}, \ldots\right)$ and is again a divided-powers HSD.

We thus have defined operators $\partial_z^{(0)}$, $\partial_z^{(1)}$, $\partial_z^{(2)}$, ... and operators $\partial_w^{(0)}$, $\partial_w^{(1)}$, $\partial_w^{(2)}$, ... on R. We denote the operator $\partial_z^{(1)}$ by ∂_z, and we denote the operator $\partial_w^{(1)}$ by ∂_w. (Of course, this is analogous to the fact that $\partial_z^{(1)} = \partial_z$ and $\partial_w^{(1)} = \partial_w$ for all the other domains we have introduced these operators on.)

The operators $\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots$ on R satisfy the equalities (3), (4), (5) and (6)[41]. An analogous statement holds for the operators $\partial_w^{(0)}, \partial_w^{(1)}, \partial_w^{(2)}, \ldots$ on R. When $\mathbb{F}$ is a $\mathbb{Q}$-algebra, we can use (5) to rewrite all $\partial_z^{(n)}$ in terms of ∂_z, but in the general case the $\partial_z^{(n)}$ carry some information that is not in ∂_z.

We shall now define an $\mathbb{F}$-algebra homomorphism $i_z : R \to \mathbb{F}((z, w/z))$ which will serve as an algebraic analogue of what would be called "expanding a rational function in the domain $|z| > |w|$" in complex analysis. Namely, we define an $\mathbb{F}$-algebra homomorphism $i_z : R \to \mathbb{F}((z, w/z))$ by sending

$$z \mapsto z, \qquad w \mapsto w.$$

In order to see that this is well-defined (according to the universal property of localization), we need to show that the elements z, w and $z - w$ of the ring $\mathbb{F}((z, w/z))$ are invertible. This can be directly checked: The element z is invertible since $z^{-1} \in \mathbb{F}((z, w/z))$. The element w is invertible since $w^{-1} = z^{-1}(w/z)^{-1} \in \mathbb{F}((z, w/z))$. The element $z - w$ is invertible since $\dfrac{1}{z}\dfrac{1}{1-\frac{w}{z}} \in \mathbb{F}((z, w/z))$ is its inverse (because $z - w = z\left(1 - \dfrac{w}{z}\right)$). So the well-definedness of i_z is proven. Since $\mathbb{F}((z, w/z)) \subseteq \mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$, we can regard i_z as an $\mathbb{F}$-linear map $i_z : R \to \mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$, but this point of view prevents us from speaking of i_z as an $\mathbb{F}$-algebra homomorphism (since $\mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ is not an $\mathbb{F}$-algebra).

Explicitly, the map i_z sends Laurent polynomials in $\mathbb{F}\left[z, z^{-1}, w, w^{-1}\right]$ to themselves (regarded as elements of $\mathbb{F}((z, w/z)) \subseteq \mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$), while sending $\dfrac{1}{z-w}$ to

$$\begin{aligned} i_z \frac{1}{z-w} &= \frac{1}{z}\frac{1}{1-\frac{w}{z}} \qquad \left(\text{since } \frac{1}{z}\frac{1}{1-\frac{w}{z}} \text{ is the inverse of } z - w \text{ in } \mathbb{F}((z, w/z))\right) \\ &= \frac{1}{z}\left(1 + \frac{w}{z} + \frac{w^2}{z^2} + \cdots\right) = \sum_{n \geq 0} z^{-n-1} w^n \qquad (44) \\ &\in \mathbb{F}((z, w/z)) \subseteq \mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right]. \end{aligned}$$

The map i_z is an $\mathbb{F}\left[z, z^{-1}, w, w^{-1}\right]$-algebra homomorphism from R to $\mathbb{F}((z, w/z))$, and thus an $\mathbb{F}\left[z, z^{-1}, w, w^{-1}\right]$-module homomorphism from R to $\mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$.

[41] *Proof.* The operators $\partial_z^{(0)}$, $\partial_z^{(1)}$, $\partial_z^{(2)}$, $\ldots$ on R satisfy the equalities (3) and (6) because the HSD $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ is divided-powers. They satisfy the equality (4) by definition of ∂_z. Finally, Remark 2.8 **(b)** (applied to R and $\partial_z^{(i)}$ instead of A and D_i) yields that $n!\partial_z^{(n)} = \left(\partial_z^{(1)}\right)^n$ for every $n \in \mathbb{N}$. Since $\partial_z^{(1)} = \partial_z$, this rewrites as $n!\partial_z^{(n)} = (\partial_z)^n$ for every $n \in \mathbb{N}$. Thus, the operators $\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots$ on R satisfy the equality (5) as well.

We can similarly define an $\mathbb{F}$-algebra homomorphism $i_w : R \to \mathbb{F}((w, z/w))$ which is analogous to "expansion in the domain $|w| > |z|$". It sends Laurent polynomials in $\mathbb{F}[z, z^{-1}, w, w^{-1}]$ to themselves, while sending $\dfrac{1}{z-w}$ to

$$\begin{aligned} i_w \frac{1}{z-w} &= \frac{1}{w} \frac{1}{\frac{z}{w} - 1} = -\frac{1}{w} \frac{1}{1 - \frac{z}{w}} \\ &= \frac{1}{w}\left(1 + \frac{z}{w} + \frac{z^2}{w^2} + \cdots\right) = -\sum_{n \geq 0} z^n w^{-n-1} = -\sum_{n<0} z^{-n-1} w^n \qquad (45) \\ &\qquad \text{(here, we substituted } -n-1 \text{ for } n \text{ in the sum)} \\ &\in \mathbb{F}((w, z/w)) \subseteq \mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right]. \end{aligned}$$

The map i_w is an $\mathbb{F}[z, z^{-1}, w, w^{-1}]$-algebra homomorphism from R to $\mathbb{F}((w, z/w))$, and thus an $\mathbb{F}[z, z^{-1}, w, w^{-1}]$-module homomorphism from R to $\mathbb{F}[[z, z^{-1}, w, w^{-1}]]$.

Subtracting (45) from (44), we obtain

$$\begin{aligned} i_z \frac{1}{z-w} - i_w \frac{1}{z-w} &= \sum_{n \geq 0} z^{-n-1} w^n - \left(-\sum_{n<0} z^{-n-1} w^n\right) \\ &= \sum_{n \geq 0} z^{-n-1} w^n + \sum_{n<0} z^{-n-1} w^n = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n = \delta(z-w). \end{aligned}$$

Thus,

$$\delta(z-w) = i_z \frac{1}{z-w} - i_w \frac{1}{z-w}. \qquad (46)$$

Proposition 3.10. **(a)** The maps i_z and i_w commute with the operators ∂_z and ∂_w. In other words,

$$\begin{aligned} i_z(\partial_z q) &= \partial_z(i_z q); & i_z(\partial_w q) &= \partial_w(i_z q); \\ i_w(\partial_z q) &= \partial_z(i_w q); & i_w(\partial_w q) &= \partial_w(i_w q) \end{aligned}$$

for every $q \in R$.

(b) The maps i_z and i_w commute with the operators $\partial_z^{(n)}$ and $\partial_w^{(n)}$ for every $n \in \mathbb{N}$. (These operators were defined in Section 2.1.) In other words,

$$\begin{aligned} i_z\left(\partial_z^{(n)} q\right) &= \partial_z^{(n)}(i_z q); & i_z\left(\partial_w^{(n)} q\right) &= \partial_w^{(n)}(i_z q); \\ i_w\left(\partial_z^{(n)} q\right) &= \partial_z^{(n)}(i_w q); & i_w\left(\partial_w^{(n)} q\right) &= \partial_w^{(n)}(i_w q) \end{aligned}$$

for every $q \in R$ and $n \in \mathbb{N}$.

(c) The maps i_z and i_w are $\mathbb{F}[z, z^{-1}, w, w^{-1}]$-linear. In other words, they commute with multiplication by elements of $\mathbb{F}[z, z^{-1}, w, w^{-1}]$. In other words,

$$i_z(pq) = p i_z(q); \qquad i_w(pq) = p i_w(q)$$

for every $q \in R$ and $p \in \mathbb{F}[z, z^{-1}, w, w^{-1}]$.

Proof of Proposition 3.10. **(c)** The map i_z is an $\mathbb{F}\left[z, z^{-1}, w, w^{-1}\right]$-algebra homomorphism (since it is an $\mathbb{F}$-algebra homomorphism which sends z to z and w to w), thus $\mathbb{F}\left[z, z^{-1}, w, w^{-1}\right]$-linear. This proves Proposition 3.10 **(c)**.

(b) Let $q \in R$ and $n \in \mathbb{N}$. We shall only prove the identity $i_z\left(\partial_z^{(n)} q\right) = \partial_z^{(n)}\left(i_z q\right)$. The other three identities that are claimed in Proposition 3.10 can be shown in an analogous manner.

Let $A = \mathbb{F}\left[z, z^{-1}, w, w^{-1}\right]$, and let S be the multiplicatively closed subset $\left\{(z-w)^k \mid k \in \mathbb{N}\right\}$ of A. Then, $R = S^{-1}A$ (by the definition of R). Let $\iota : A \to S^{-1}A$ be the canonical $\mathbb{F}$-algebra homomorphism from A to $S^{-1}A$ sending each $a \in A$ to $\dfrac{a}{1} \in S^{-1}A$. Thus, ι is an $\mathbb{F}$-algebra homomorphism from A to R (since $R = S^{-1}A$).

We define three HSDs:

- Let $\mathbf{D}_P$ be the HSD $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ from A to A.
- Let $\mathbf{D}_R$ be the HSD $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ from R to R.
- Let $\mathbf{D}_L$ be the HSD $\left(\partial_z^{(0)}, \partial_z^{(1)}, \partial_z^{(2)}, \ldots\right)$ from $\mathbb{F}((z, w/z))$ to $\mathbb{F}((z, w/z))$.

We recall that the HSD $\mathbf{D}_R$ was defined as the lift of the HSD $\mathbf{D}_P$ to the localization $S^{-1}A = R$. Hence, we can apply Corollary 2.17 **(c)** to $\mathbf{D}_P$ and $\mathbf{D}_R$ instead of $\mathbf{D}$ and $\mathbf{D}'$. As the result, we obtain $\mathbf{D}_R \circ \iota = \iota \circ \mathbf{D}_P$ (where $\mathbf{D}_R \circ \iota$ and $\iota \circ \mathbf{D}_P$ are defined according to Definition 2.16).

Recall that $i_z : R \to \mathbb{F}((z, w/z))$ is an $\mathbb{F}$-algebra homomorphism. Hence, two HSDs $i_z \circ \mathbf{D}_R$ and $\mathbf{D}_L \circ i_z$ from R to $\mathbb{F}((z, w/z))$ are defined (according to Definition 2.16). We shall now show that these two HSDs are equal.

Let inc denote the canonical inclusion map $A \to \mathbb{F}((z, w/z))$. Then, $i_z \circ \iota = \operatorname{inc}$ (since the map i_z sends Laurent polynomials in $\mathbb{F}\left[z, z^{-1}, w, w^{-1}\right]$ to themselves). Moreover, inc is an $\mathbb{F}$-algebra homomorphism, so that two HSDs $\operatorname{inc} \circ \mathbf{D}_P$ and $\mathbf{D}_L \circ \operatorname{inc}$ are well-defined. It is easy to see that $\operatorname{inc} \circ \mathbf{D}_P = \mathbf{D}_L \circ \operatorname{inc}$ [42]. Now,

$$i_z \circ \underbrace{\mathbf{D}_R \circ \iota}_{= \iota \circ \mathbf{D}_P} = \underbrace{i_z \circ \iota}_{= \operatorname{inc}} \circ \mathbf{D}_P = \operatorname{inc} \circ \mathbf{D}_P = \mathbf{D}_L \circ \underbrace{\operatorname{inc}}_{= i_z \circ \iota} = \mathbf{D}_L \circ i_z \circ \iota.$$

Thus, Corollary 2.19 (applied to $\mathbb{F}((z, w/z))$, $i_z \circ \mathbf{D}_R$ and $\mathbf{D}_L \circ i_z$ instead of B, $\mathbf{E}$ and $\mathbf{F}$) yields $i_z \circ \mathbf{D}_R = \mathbf{D}_L \circ i_z$. But the definition of $i_z \circ \mathbf{D}_R$ yields

$$\left(i_z \circ \partial_z^{(0)}, i_z \circ \partial_z^{(1)}, i_z \circ \partial_z^{(2)}, \ldots\right) = i_z \circ \mathbf{D}_R = \mathbf{D}_L \circ i_z = \left(\partial_z^{(0)} \circ i_z, \partial_z^{(1)} \circ i_z, \partial_z^{(2)} \circ i_z, \ldots\right)$$

[42] *Proof.* Proving that $\operatorname{inc} \circ \mathbf{D}_P = \mathbf{D}_L \circ \operatorname{inc}$ is tantamount to showing that $\operatorname{inc} \circ \partial_z^{(n)} = \partial_z^{(n)} \circ \operatorname{inc}$ for every $n \in \mathbb{N}$ (because of how $\operatorname{inc} \circ \mathbf{D}_P$ and $\mathbf{D}_L \circ \operatorname{inc}$ are defined). This is equivalent to showing that the restriction of the endomorphism $\partial_z^{(n)}$ of $\mathbb{F}((z, w/z))$ to A is precisely the endomorphism $\partial_z^{(n)}$ of A. But the latter claim is obvious (just remember that both endomorphisms $\partial_z^{(n)}$ were defined by the same formula).

(by the definition of $\mathbf{D}_L \circ i_z$). Hence, $i_z \circ \partial_z^{(n)} = \partial_z^{(n)} \circ i_z$. Thus, $i_z\left(\partial_z^{(n)} q\right) = \underbrace{\left(i_z \circ \partial_z^{(n)}\right)}_{=\partial_z^{(n)} \circ i_z}(q) = \left(\partial_z^{(n)} \circ i_z\right)(q) = \partial_z^{(n)}(i_z q)$. This completes our proof of Proposition 3.10 **(b)**.

(a) Proposition 3.10 **(a)** follows from Proposition 3.10 **(b)** (applied to $n = 1$), since $\partial_z^{(1)} = \partial_z$ and $\partial_w^{(1)} = \partial_w$. $\square$

3.6. Further properties of $\delta(z-w)$

Next, we observe:

Proposition 3.11. **(a)** For every $k \in \mathbb{N}$, we have $\partial_w^{(k)} \dfrac{1}{z-w} = \dfrac{1}{(z-w)^{k+1}}$ in R.

(b) For every $k \in \mathbb{N}$, we have

$$\partial_w^{(k)} \delta(z-w) = i_z \frac{1}{(z-w)^{k+1}} - i_w \frac{1}{(z-w)^{k+1}} \tag{47}$$

$$= \sum_{n \in \mathbb{Z}} \binom{n}{k} w^{n-k} z^{-n-1} \tag{48}$$

in $\mathbb{F}\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$.

Proof of Proposition 3.11. **(a)** When $\mathbb{F}$ is a $\mathbb{Q}$-algebra, one can easily prove Proposition 3.11 **(a)** by induction over k using (5). However, we want to prove it in general. So let us take a different approach.

Let $\mathbf{D}$ denote the HSD $\left(\partial_w^{(0)}, \partial_w^{(1)}, \partial_w^{(2)}, \ldots\right)$ of R. Let t be a new symbol. Define an $\mathbb{F}$-linear map $\widetilde{\mathbf{D}}_{\langle t\rangle} : R \to R[[t]]$ as in Definition 2.9. Theorem 2.10 then yields that $\widetilde{\mathbf{D}}_{\langle t\rangle}$ is an $\mathbb{F}$-algebra homomorphism (since $\mathbf{D}$ is an HSD). But the definition of $\widetilde{\mathbf{D}}_{\langle t\rangle}$ yields

$$\begin{aligned}
\widetilde{\mathbf{D}}_{\langle t\rangle}(z-w) &= \sum_{n \in \mathbb{N}} \partial_w^{(n)}(z-w)\, t^n = (z-w) + 1t \\
&\qquad \left(\text{since } \partial_w^{(n)}(z-w) = \begin{cases} z-w, & \text{if } n=0; \\ 1, & \text{if } n=1; \\ 0, & \text{otherwise} \end{cases} \quad \text{for all } n \in \mathbb{N}\right) \\
&= (z-w) + t.
\end{aligned}$$

Since $\widetilde{\mathbf{D}}_{\langle t\rangle}(z-w)$ is an $\mathbb{F}$-algebra homomorphism, we have

$$\begin{aligned}
\widetilde{\mathbf{D}}_{\langle t\rangle}\left((z-w)^{-1}\right) &= \left(\underbrace{\widetilde{\mathbf{D}}_{\langle t\rangle}(z-w)}_{=(z-w)+t}\right)^{-1} = ((z-w)+t)^{-1} \\
&= (z-w)^{-1}\underbrace{\left(1+\frac{t}{z-w}\right)^{-1}}_{=\sum_{n\in\mathbb{N}}\left(\frac{t}{z-w}\right)^{n}} = (z-w)^{-1}\sum_{n\in\mathbb{N}}\left(\frac{t}{z-w}\right)^{n} \\
&= \sum_{n\in\mathbb{N}}\underbrace{(z-w)^{-1}\left(\frac{t}{z-w}\right)^{n}}_{=\frac{1}{(z-w)^{n+1}}t^{n}} = \sum_{n\in\mathbb{N}}\frac{1}{(z-w)^{n+1}}t^{n}.
\end{aligned}$$

Compared with

$$\widetilde{\mathbf{D}}_{\langle t\rangle}\left((z-w)^{-1}\right) = \sum_{n\in\mathbb{N}}\partial_w^{(n)}\left((z-w)^{-1}\right)t^{n},$$

this yields $\sum_{n\in\mathbb{N}}\partial_w^{(n)}\left((z-w)^{-1}\right)t^{n} = \sum_{n\in\mathbb{N}}\frac{1}{(z-w)^{n+1}}t^{n}$. Comparing coefficients in this equality, we obtain $\partial_w^{(n)}\left((z-w)^{-1}\right) = \frac{1}{(z-w)^{n+1}}$ for all $n\in\mathbb{N}$. In other words, $\partial_w^{(k)}\left((z-w)^{-1}\right) = \frac{1}{(z-w)^{k+1}}$ for all $k\in\mathbb{N}$. This proves Proposition 3.11 **(a)**.

(b) Let $k\in\mathbb{N}$. Applying the map $\partial_w^{(k)}$ to both sides of (46), we obtain

$$\begin{aligned}
\partial_w^{(k)}\delta(z-w) &= \underbrace{\partial_w^{(k)}\left(i_z\frac{1}{z-w}\right)}_{\substack{=i_z\left(\partial_w^{(k)}\frac{1}{z-w}\right)\\ \text{(by Proposition 3.10 } \mathbf{(b)}\text{)}}} - \underbrace{\partial_w^{(k)}\left(i_w\frac{1}{z-w}\right)}_{\substack{=i_w\left(\partial_w^{(k)}\frac{1}{z-w}\right)\\ \text{(by Proposition 3.10 } \mathbf{(b)}\text{)}}} \\
&= i_z\underbrace{\left(\partial_w^{(k)}\frac{1}{z-w}\right)}_{\substack{=\frac{1}{(z-w)^{k+1}}\\ \text{(by Proposition 3.11 } \mathbf{(a)}\text{)}}} - i_w\underbrace{\left(\partial_w^{(k)}\frac{1}{z-w}\right)}_{\substack{=\frac{1}{(z-w)^{k+1}}\\ \text{(by Proposition 3.11 } \mathbf{(a)}\text{)}}} \\
&= i_z\frac{1}{(z-w)^{k+1}} - i_w\frac{1}{(z-w)^{k+1}}.
\end{aligned}$$

On the other hand, applying the map $\partial_w^{(k)}$ to both sides of the equality $\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n$, we obtain

$$\partial_w^{(k)}\delta(z-w) = \partial_w^{(k)}\sum_{n\in\mathbb{Z}} z^{-n-1}w^n = \sum_{n\in\mathbb{Z}}\binom{n}{k}z^{-n-1}w^{n-k} = \sum_{n\in\mathbb{Z}}\binom{n}{k}w^{n-k}z^{-n-1}.$$

This completes the proof of Proposition 3.11 **(b)**. $\square$

Proposition 3.12. **(a)** We have $\operatorname{Res}\partial_w^{(n)}\delta(z-w)\,dz = \delta_{n,0}$ for every $n\in\mathbb{N}$.

(b) Any $m\in\mathbb{N}$ and $n\in\mathbb{N}$ satisfy

$$(z-w)^m\,\partial_w^{(n)}\delta(z-w) = \begin{cases}\partial_w^{(n-m)}\delta(z-w), & \text{if } m\le n;\\ 0, & \text{if } m>n\end{cases}.$$

(c) We have $\delta(z-w) = \delta(w-z)$. (Here, $\delta(w-z)$ means $(\delta(z-w))(w,z)$, that is, the result of switching z with w in $\delta(z-w)$.)

(d) We have $\partial_z\delta(z-w) = -\partial_w\delta(z-w)$.

(e) We have $\partial_z^{(j)}\delta(z-w) = (-1)^j\,\partial_w^{(j)}\delta(z-w)$ for every $j\in\mathbb{N}$.

Proof of Proposition 3.12. **(a)** For every $k\in\mathbb{N}$, we have

$$\begin{aligned}\operatorname{Res}\underbrace{\partial_w^{(k)}\delta(z-w)}_{\substack{=\sum_{n\in\mathbb{Z}}\binom{n}{k}w^{n-k}z^{-n-1}\\ \text{(by (48))}}}dz &= \operatorname{Res}\left(\sum_{n\in\mathbb{Z}}\binom{n}{k}w^{n-k}z^{-n-1}\right)dz = \underbrace{\binom{0}{k}}_{=\delta_{k,0}}w^{0-k}\\ &= \delta_{k,0}w^{0-k} = \delta_{k,0}.\end{aligned}$$

In other words, for every $n\in\mathbb{N}$, we have $\operatorname{Res}\partial_w^{(n)}\delta(z-w)\,dz = \delta_{n,0}$. This proves Proposition 3.12 **(a)**.

(b) Let $m\in\mathbb{N}$ and $n\in\mathbb{N}$. Applying (47) to $k=n$, we obtain

$$\partial_w^{(n)}\delta(z-w) = i_z\frac{1}{(z-w)^{n+1}} - i_w\frac{1}{(z-w)^{n+1}}.$$

Multiplying this equality with $(z-w)^m$ from the left, we obtain

$$(z-w)^m \partial_w^{(n)} \delta(z-w) = \underbrace{(z-w)^m i_z \frac{1}{(z-w)^{n+1}}}_{\substack{=i_z\left((z-w)^m \cdot \frac{1}{(z-w)^{n+1}}\right) \\ \text{(since } i_z \text{ is an } \mathbb{F}[z,z^{-1},w,w^{-1}]\text{-module} \\ \text{homomorphism)}}} - \underbrace{(z-w)^m i_w \frac{1}{(z-w)^{n+1}}}_{\substack{=i_w\left((z-w)^m \cdot \frac{1}{(z-w)^{n+1}}\right) \\ \text{(since } i_w \text{ is an } \mathbb{F}[z,z^{-1},w,w^{-1}]\text{-module} \\ \text{homomorphism)}}}$$

$$= i_z \left(\underbrace{(z-w)^m \cdot \frac{1}{(z-w)^{n+1}}}_{=\frac{1}{(z-w)^{(n-m)+1}}} \right) - i_w \left(\underbrace{(z-w)^m \cdot \frac{1}{(z-w)^{n+1}}}_{=\frac{1}{(z-w)^{(n-m)+1}}} \right)$$

$$= i_z \frac{1}{(z-w)^{(n-m)+1}} - i_w \frac{1}{(z-w)^{(n-m)+1}}. \tag{49}$$

Now, we must be in one of the following two cases:

Case 1: We have $m \leq n$.

Case 2: We have $m > n$.

Let us first consider Case 1. In this case, we have $m \leq n$. Thus, applying (47) to $k = n - m$, we obtain

$$\partial_w^{(n-m)} \delta(z-w) = i_z \frac{1}{(z-w)^{(n-m)+1}} - i_w \frac{1}{(z-w)^{(n-m)+1}}.$$

Compared with (49), this yields

$$(z-w)^m \partial_w^{(n)} \delta(z-w) = \partial_w^{(n-m)} \delta(z-w) = \begin{cases} \partial_w^{(n-m)} \delta(z-w), & \text{if } m \leq n; \\ 0, & \text{if } m > n \end{cases}$$

(since $m \leq n$). Thus, Proposition 3.12 **(b)** is proven in Case 1.

Let us next consider Case 2. In this case, we have $m > n$, so that $m - n - 1 \geq 0$. Now, $\frac{1}{(z-w)^{(n-m)+1}} = (z-w)^{m-n-1} \in \mathbb{F}[z, z^{-1}, w, w^{-1}]$ (since $m - n - 1 \geq 0$), so that $i_z \frac{1}{(z-w)^{(n-m)+1}} = \frac{1}{(z-w)^{(n-m)+1}}$ (since i_z sends Laurent polynomials in $\mathbb{F}[z, z^{-1}, w, w^{-1}]$ to themselves). Similarly, $i_w \frac{1}{(z-w)^{(n-m)+1}} =$

$\dfrac{1}{(z-w)^{(n-m)+1}}$. Hence, (49) becomes

$$(z-w)^m \partial_w^{(n)} \delta(z-w) = i_z \underbrace{\frac{1}{(z-w)^{(n-m)+1}}}_{= \frac{1}{(z-w)^{(n-m)+1}}} - i_w \underbrace{\frac{1}{(z-w)^{(n-m)+1}}}_{= \frac{1}{(z-w)^{(n-m)+1}}}$$
$$= 0 = \begin{cases} \partial_w^{(n-m)} \delta(z-w), & \text{if } m \leq n; \\ 0, & \text{if } m > n \end{cases}$$

(since $m > n$). Thus, Proposition 3.12 **(b)** is proven in Case 2. The proof of Proposition 3.12 **(b)** is now complete.

(c) We have $\delta(z-w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n$, so that

$$\begin{aligned} \delta(w-z) &= \sum_{n \in \mathbb{Z}} w^{-n-1} z^n = \sum_{n \in \mathbb{Z}} w^n z^{-n-1} && \left(\begin{array}{c}\text{here, we have substituted} \\ -n-1 \text{ for } n \text{ in the sum}\end{array}\right) \\ &= \sum_{n \in \mathbb{Z}} z^{-n-1} w^n = \delta(z-w), \end{aligned}$$

so that Proposition 3.12 **(c)** is proven.

(d) Applying ∂_z to both sides of $\delta(z-w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n$, we obtain

$$\partial_z \delta(z-w) = \partial_z \sum_{n \in \mathbb{Z}} z^{-n-1} w^n = \sum_{n \in \mathbb{Z}} (-n-1) z^{-n-2} w^n = \sum_{n \in \mathbb{Z}} (-n) z^{-n-1} w^{n-1}$$

(here, we have substituted $n-1$ for n in the sum). Applying ∂_w to both sides of $\delta(z-w) = \sum_{n \in \mathbb{Z}} z^{-n-1} w^n$, we obtain

$$\partial_w \delta(z-w) = \partial_w \sum_{n \in \mathbb{Z}} z^{-n-1} w^n = \sum_{n \in \mathbb{Z}} n z^{-n-1} w^{n-1}.$$

Adding these two inequalities together, we obtain

$$\begin{aligned} \partial_z \delta(z-w) + \partial_w \delta(z-w) &= \sum_{n \in \mathbb{Z}} (-n) z^{-n-1} w^{n-1} + \sum_{n \in \mathbb{Z}} n z^{-n-1} w^{n-1} \\ &= \sum_{n \in \mathbb{Z}} \underbrace{(-n+n)}_{=0} z^{-n-1} w^{n-1} = 0. \end{aligned}$$

This yields Proposition 3.12 **(d)**.

(e) Let $j \in \mathbb{N}$. Every $n \in \mathbb{Z}$ satisfies

$$\binom{j-1-n}{j} = (-1)^j \binom{n}{j}. \tag{50}$$

[43] But applying (48) to $k = j$, we obtain

$$\partial_w^{(j)} \delta(z - w) = \sum_{n \in \mathbb{Z}} \binom{n}{j} w^{n-j} z^{-n-1}.$$

Renaming the indeterminates z and w as w and z in this equality, we obtain

$$\partial_z^{(j)} \delta(w - z) = \sum_{n \in \mathbb{Z}} \binom{n}{j} z^{n-j} w^{-n-1} = \sum_{n \in \mathbb{Z}} \underbrace{\binom{j-1-n}{j}}_{\substack{= (-1)^j \binom{n}{j} \\ \text{(by (50))}}} \underbrace{z^{-n-1} w^{n-j}}_{= w^{n-j} z^{-n-1}}$$

$$\text{(here, we substituted } j-1-n \text{ for } n \text{ in the sum)}$$

$$= \sum_{n \in \mathbb{Z}} (-1)^j \binom{n}{j} w^{n-j} z^{-n-1} = (-1)^j \underbrace{\sum_{n \in \mathbb{Z}} \binom{n}{j} w^{n-j} z^{-n-1}}_{= \partial_w^{(j)} \delta(z-w)}$$

$$= (-1)^j \partial_w^{(j)} \delta(z - w).$$

This proves Proposition 3.12 **(e)**. □

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[43] *Proof of (50):* Let $n \in \mathbb{Z}$. Then,

$$\begin{aligned}
&(j-1-n)((j-1-n)-1)\cdots((j-1-n)-j+1) \\
&= (j-1-n)(j-2-n)\cdots(j-j-n) \\
&= (-(n-j+1))(-(n-j+2))\cdots(-(n-j+j)) \\
&= (-1)^j \cdot \underbrace{(n-j+1)(n-j+2)\cdots(n-j+j)}_{\substack{=(n-j+j)(n-j+(j-1))\cdots(n-j+1) \\ =n(n-1)\cdots(n-j+1)}} \\
&= (-1)^j \cdot n(n-1)\cdots(n-j+1). \qquad (51)
\end{aligned}$$

Now, the definition of binomial coefficients shows that

$$\begin{aligned}
\binom{j-1-n}{j} &= \frac{(j-1-n)((j-1-n)-1)\cdots((j-1-n)-j+1)}{j!} \\
&= \frac{(-1)^j \cdot n(n-1)\cdots(n-j+1)}{j!} \qquad \text{(by (51))} \\
&= (-1)^j \cdot \underbrace{\frac{n(n-1)\cdots(n-j+1)}{j!}}_{\substack{=\binom{n}{j} \\ \text{(by the definition of } \binom{n}{j})}} = (-1)^j \binom{n}{j}.
\end{aligned}$$

This proves (50).

3.7. The decomposition theorem

The next fact is known as the *decomposition theorem*:

Theorem 3.13. Let U be an $\mathbb{F}$-module. Let $a(z,w) \in U\left[\left[z,z^{-1},w,w^{-1}\right]\right]$ be a local U-valued formal distribution. Then, $a(z,w)$ uniquely decomposes as a sum

$$\sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$$

with $\left(c^j(w)\right)_{j\in\mathbb{N}}$ being a family of formal distributions $c^j(w) \in U\left[\left[w,w^{-1}\right]\right]$ having the property that all but finitely many $j \in \mathbb{N}$ satisfy $c^j(w) = 0$.

The formal distributions $c^j(w)$ in this decomposition are given by

$$c^j(w) = \operatorname{Res}(z-w)^j a(z,w)\,dz \qquad \text{for all } j \in \mathbb{N}. \tag{52}$$

Before we prove this theorem, let us show a lemma:

Lemma 3.14. Let U be an $\mathbb{F}$-module. Let $b(z,w) \in U\left[\left[z,z^{-1},w,w^{-1}\right]\right]$. Assume that

$$\operatorname{Res}(z-w)^n b(z,w)\,dz = 0 \qquad \text{for all } n \in \mathbb{N}. \tag{53}$$

(a) Then, $b(z,w) \in \left(U\left[\left[w,w^{-1}\right]\right]\right)[[z]]$.

(b) Assume that $b(z,w)$ is local. Then, $b(z,w) = 0$.

Proof of Lemma 3.14. We have $b(z,w) \in U\left[\left[z,z^{-1},w,w^{-1}\right]\right] = \left(U\left[\left[w,w^{-1}\right]\right]\right)\left[\left[z,z^{-1}\right]\right]$. Hence, we can write $b(z,w)$ in the form $b(z,w) = \sum_{j\in\mathbb{Z}} z^j b_j(w)$ for some $b_j(w) \in U\left[\left[w,w^{-1}\right]\right]$. Consider these $b_j(w)$. Every $n \in \mathbb{N}$ satisfies

$$\operatorname{Res} \underbrace{z^n}_{\substack{=((z-w)+w)^n \\ =\sum_{k=0}^{n}\binom{n}{k}(z-w)^k w^{n-k}}} b(z,w)\,dz = \sum_{k=0}^{n}\binom{n}{k} \underbrace{\operatorname{Res}(z-w)^k b(z,w)\,dz}_{\substack{=0 \\ \text{(by (53), applied to } k \text{ instead of } n)}} w^{n-k} = 0.$$

Compared with

$$\operatorname{Res} z^n \underbrace{b(z,w)}_{=\sum_{j\in\mathbb{Z}} z^j b_j(w)} dz = \operatorname{Res} z^n \sum_{j\in\mathbb{Z}} z^j b_j(w)\,dz = \operatorname{Res} \sum_{j\in\mathbb{Z}} z^{n+j} b_j(w)\,dz = b_{-n-1}(w),$$

this yields that $b_{-n-1}(w) = 0$ for every $n \in \mathbb{N}$. In other words, $b_j(w) = 0$ for every negative $j \in \mathbb{Z}$. Hence,

$$b(z,w) = \sum_{j\in\mathbb{Z}} z^j b_j(w) = \sum_{j\in\mathbb{N}} z^j b_j(w) + \sum_{\substack{j\in\mathbb{Z}; \\ j \text{ is negative}}} z^j \underbrace{b_j(w)}_{=0} = \sum_{j\in\mathbb{N}} z^j b_j(w)$$
$$\in \left(U\left[\left[w,w^{-1}\right]\right]\right)[[z]].$$

This proves Lemma 3.14 **(a)**.

(b) Lemma 3.14 **(a)** yields $b(z,w) \in \left(U\left[\left[w,w^{-1}\right]\right]\right)[[z]]$. But $\left(U\left[\left[w,w^{-1}\right]\right]\right)[[z]]$ is an $\mathbb{F}[z,w]$-module, and the element $z-w$ of $\mathbb{F}[z,w]$ acts as a non-zero-divisor on this module (i.e., every element $\alpha \in \left(U\left[\left[w,w^{-1}\right]\right]\right)[[z]]$ satisfies $(z-w)\alpha = 0$ must itself satisfy $\alpha = 0$) [44]. But $b(z,w)$ is local. In other words, $(z-w)^N b(z,w) = 0$ for some $N \in \mathbb{N}$. Consider this N. We can cancel the $(z-w)^N$ from the equality $(z-w)^N b(z,w) = 0$ (since $z-w$ acts as a non-zero-divisor on the module $\left(U\left[\left[w,w^{-1}\right]\right]\right)[[z]]$, and $b(z,w)$ belongs to said module). As a result, we obtain $b(z,w) = 0$. This proves Lemma 3.14 **(b)**. $\square$

Proof of Theorem 3.13. Uniqueness: Let us first show the uniqueness of the family $\left(c^j(w)\right)_{j\in\mathbb{N}}$ satisfying $a(z,w) = \sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$. For this, we let $\left(c^j(w)\right)_{j\in\mathbb{N}}$ be any family of formal distributions $c^j(w) \in U\left[\left[w,w^{-1}\right]\right]$ having the property that all but finitely many $j \in \mathbb{N}$ satisfy $c^j(w) = 0$. Assume that $a(z,w) = \sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$. We need to prove that (52) holds.

For every $n \in \mathbb{N}$, we have

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} c^j(w)\operatorname{Res}\underbrace{(z-w)^n\,\partial_w^{(j)}\delta(z-w)}_{\substack{=\begin{cases}\partial_w^{(j-n)}\delta(z-w), & \text{if } n\le j;\\ 0, & \text{if } n>j\end{cases}\\ \text{(by Proposition 3.12 (b), applied to } n \text{ and } j\\ \text{instead of } m \text{ and } n)}} dz \\
&= \sum_{j\in\mathbb{N}} c^j(w)\operatorname{Res}\begin{cases}\partial_w^{(j-n)}\delta(z-w), & \text{if } n\le j;\\ 0, & \text{if } n>j\end{cases} dz \\
&= \sum_{\substack{j\in\mathbb{N};\\ j\ge n}} c^j(w)\underbrace{\operatorname{Res}\partial_w^{(j-n)}\delta(z-w)\,dz}_{\substack{=\delta_{j-n,0}\\ \text{(by Proposition 3.12 (a), applied to } j-n \text{ instead of } n)}} \\
&= \sum_{\substack{j\in\mathbb{N};\\ j\ge n}} c^j(w)\,\delta_{j-n,0} = c^n(w).
\end{aligned} \tag{54}$$

Hence, for every $j \in \mathbb{N}$, we have

$$\begin{aligned}
&\operatorname{Res}(z-w)^n\underbrace{a(z,w)}_{=\sum_{j\in\mathbb{N}} c^j(w)\partial_w^{(j)}\delta(z-w)} dz = \sum_{j\in\mathbb{N}} c^j(w)\operatorname{Res}(z-w)^n\,\partial_w^{(j)}\delta(z-w)\,dz \\
&= c^n(w).
\end{aligned}$$

[44] *Proof.* Actually, $\left(U\left[\left[w,w^{-1}\right]\right]\right)[[z]]$ is not just an $\mathbb{F}[z,w]$-module, but also an $\left(\mathbb{F}\left[w,w^{-1}\right]\right)[[z]]$-module (since $U\left[\left[w,w^{-1}\right]\right]$ is an $\mathbb{F}\left[w,w^{-1}\right]$-module). The element $z-w$ of $\left(\mathbb{F}\left[w,w^{-1}\right]\right)[[z]]$ is invertible (since $z-w = w\left(zw^{-1}-1\right)$), and thus acts as a non-zero-divisor on the $\left(\mathbb{F}\left[w,w^{-1}\right]\right)[[z]]$-module $\left(U\left[\left[w,w^{-1}\right]\right]\right)[[z]]$, qed.

Renaming n as j and interchanging the two sides of this equality, we obtain precisely (52). Thus, (52) holds, and so the uniqueness of the family $\left(c^j(w)\right)_{j\in\mathbb{N}}$ is proven.

Existence: Now, we shall show that the family $\left(c^j(w)\right)_{j\in\mathbb{N}}$ defined by (52) actually satisfies $a(z,w) = \sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$ and has the property that all but finitely many $j\in\mathbb{N}$ satisfy $c^j(w)=0$.

Indeed, the latter property obviously holds (since $a(z,w)$ is local). So it remains to prove that the family $\left(c^j(w)\right)_{j\in\mathbb{N}}$ defined by (52) satisfies

$$a(z,w) = \sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w). \tag{55}$$

Consider this family $\left(c^j(w)\right)_{j\in\mathbb{N}}$. Since $a(z,w)$ is local, we know that every sufficiently high $j\in\mathbb{N}$ satisfies $(z-w)^j a(z,w)=0$, and therefore

$$\text{every sufficiently high } j\in\mathbb{N} \text{ satisfies } c^j(w)=0. \tag{56}$$

Thus, the sum $\sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$ is well-defined. We set

$$b(z,w) = a(z,w) - \sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w).$$

Then, our goal is to show $b(z,w)=0$ (because this will immediately yield (55)).

We know that $a(z,w)$ is local. Also, for every $j\in\mathbb{N}$, the formal distribution $\partial_w^{(j)}\delta(z-w)$ is local (since Proposition 3.12 **(b)** yields $(z-w)^{j+1}\partial_w^{(j)}\delta(z-w)=0$), and therefore the formal distribution $c^j(w)\,\partial_w^{(j)}\delta(z-w)$ is local as well. Hence, the sum $\sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$ is local as well (since it has only finitely many nonzero addends – due to (56) – and these addends are local). Thus, $b(z,w)$ is local (since we have defined $b(z,w)$ as a difference between the local formal distributions $a(z,w)$ and $\sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$).

For every $n\in\mathbb{N}$, we have

$$\begin{aligned}
&\operatorname{Res}(z-w)^n \underbrace{b(z,w)}_{=a(z,w)-\sum_{j\in\mathbb{N}} c^j(w)\partial_w^{(j)}\delta(z-w)} dz \\
&= \underbrace{\operatorname{Res}(z-w)^n a(z,w)\,dz}_{\substack{=c^n(w)\\ \text{(by (52), applied to } j=n)}} - \underbrace{\sum_{j\in\mathbb{N}} c^j(w)\operatorname{Res}(z-w)^n\,\partial_w^{(j)}\delta(z-w)\,dz}_{\substack{=c^n(w)\\ \text{(by (54))}}} \\
&= c^n(w) - c^n(w) = 0.
\end{aligned}$$

Hence, Lemma 3.14 **(b)** yields $b(z,w) = 0$ (since $b(z,w)$ is local). This completes the proof of (55). Thus, the existence part of Theorem 3.13 is proven. The proof of Theorem 3.13 is thus complete. $\square$

For the next corollary, we shall use the notations of Section 2.7.

Corollary 3.15. Let U be an $\mathbb{F}$-module. To every $a = a(z,w) \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$, let us associate the $\mathbb{F}$-linear operator

$$D_a : \mathbb{F}\left[w, w^{-1}\right] \to U\left[\left[w, w^{-1}\right]\right],$$
$$\varphi(w) \mapsto \operatorname{Res} \varphi(z)\, a(z,w)\, dz.$$

(a) For every $c(w) \in U\left[\left[w, w^{-1}\right]\right]$ and $j \in \mathbb{N}$, we have

$$D_{c(w)\partial_w^{(j)}\delta(z-w)} = c(w)\,\partial_w^{(j)} \tag{57}$$

as maps $\mathbb{F}\left[w, w^{-1}\right] \to U\left[\left[w, w^{-1}\right]\right]$. (Here, $c(w)\,\partial_w^{(j)}$ means the operator $\partial_w^{(j)}$, followed by multiplication with $c(w)$.) In particular, if $U = \mathbb{F}$, then $D_{\delta(z-w)} = 1$ (the identity map, or, rather, the canonical inclusion $\mathbb{F}\left[w, w^{-1}\right] \to \mathbb{F}\left[\left[w, w^{-1}\right]\right]$).

(b) Let $a = a(z,w) \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$. Prove that $a(z,w)$ is local if and only if D_a is a differential operator of finite order. (See Definition 2.22 for the definition of a differential operator of finite order.)

(c) Let $a = a(z,w) \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ be local. Then, show that $a(w,z)$ is local and satisfies

$$D_{a(w,z)} = \left(D_{a(z,w)}\right)^*. \tag{58}$$

(See Definition 2.27 for the meaning of $\left(D_{a(z,w)}\right)^*$.)

(d) Let $a = a(z,w) \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$. Show that

$$D_{\psi(w)a(z,w)} = \psi(w) \circ D_{a(z,w)} \qquad \text{for all } \psi(w) \in \mathbb{F}\left[w, w^{-1}\right]; \tag{59}$$
$$D_{\partial_w a(z,w)} = \partial_w \circ D_{a(z,w)}; \tag{60}$$
$$D_{\psi(z)a(z,w)} = D_{a(z,w)} \circ \psi(w) \qquad \text{for all } \psi(z) \in \mathbb{F}\left[z, z^{-1}\right]; \tag{61}$$
$$D_{\partial_z a(z,w)} = -D_{a(z,w)} \circ \partial_w. \tag{62}$$

Proof of Corollary 3.15. **(a)** Let $c(w) \in U\left[\left[w, w^{-1}\right]\right]$ and $j \in \mathbb{N}$. Let $\varphi(w) \in \mathbb{F}\left[w, w^{-1}\right]$. We write $\varphi(w)$ in the form $\varphi(w) = \sum_{m \in \mathbb{Z}} \varphi_m w^m$, where all φ_m belong to $\mathbb{F}$ and all but finitely many of these φ_m are zero. Then, $\partial_w^{(j)} \varphi(w) = \sum_{m \in \mathbb{Z}} \binom{m}{j} \varphi_m w^{m-j}$ (by the definition of $\partial_w^{(j)}$).

Now,

$$\begin{aligned} D_{c(w)\partial_w^{(j)}\delta(z-w)}\left(\varphi\left(w\right)\right) &= \operatorname{Res}\varphi\left(z\right)c\left(w\right)\partial_w^{(j)}\delta\left(z-w\right)dz \\ &\qquad \left(\text{by the definition of } D_{c(w)\partial_w^{(j)}\delta(z-w)}\right) \\ &= c\left(w\right)\operatorname{Res}\varphi\left(z\right)\partial_w^{(j)}\delta\left(z-w\right)dz. \end{aligned}$$

Since

$$\begin{aligned} &\operatorname{Res} \underbrace{\varphi\left(z\right)}_{\substack{=\sum\limits_{m\in\mathbb{Z}}\varphi_m z^m \\ (\text{since } \varphi(w)=\sum\limits_{m\in\mathbb{Z}}\varphi_m w^m)}} \partial_w^{(j)}\delta\left(z-w\right)dz \\ &= \operatorname{Res}\left(\sum_{m\in\mathbb{Z}}\varphi_m z^m\right)\partial_w^{(j)}\delta\left(z-w\right)dz = \sum_{m\in\mathbb{Z}}\varphi_m \operatorname{Res} z^m \underbrace{\partial_w^{(j)}\delta\left(z-w\right)}_{\substack{=\sum\limits_{n\in\mathbb{Z}}\binom{n}{j}w^{n-j}z^{-n-1} \\ (\text{by (48), applied to } k=j)}} dz \\ &= \sum_{m\in\mathbb{Z}}\varphi_m \operatorname{Res} \underbrace{z^m\left(\sum_{n\in\mathbb{Z}}\binom{n}{j}w^{n-j}z^{-n-1}\right)}_{=\sum\limits_{n\in\mathbb{Z}}\binom{n}{j}w^{n-j}z^{m-n-1}} dz = \sum_{m\in\mathbb{Z}}\varphi_m \underbrace{\operatorname{Res}\sum_{n\in\mathbb{Z}}\binom{n}{j}w^{n-j}z^{m-n-1}dz}_{=\binom{m}{j}w^{m-j}} \\ &= \sum_{m\in\mathbb{Z}}\varphi_m\binom{m}{j}w^{m-j} = \sum_{m\in\mathbb{Z}}\binom{m}{j}\varphi_m w^{m-j} = \partial_w^{(j)}\varphi\left(w\right), \end{aligned}$$

this rewrites as

$$D_{c(w)\partial_w^{(j)}\delta(z-w)}\left(\varphi\left(w\right)\right) = c\left(w\right)\underbrace{\operatorname{Res}\varphi\left(z\right)\partial_w^{(j)}\delta\left(z-w\right)dz}_{=\partial_w^{(j)}\varphi(w)} = c\left(w\right)\partial_w^{(j)}\varphi\left(w\right).$$

Now, we forget that we fixed $\varphi\left(w\right)$. We thus have shown that $D_{c(w)\partial_w^{(j)}\delta(z-w)}\left(\varphi\left(w\right)\right) = c\left(w\right)\partial_w^{(j)}\varphi\left(w\right)$ for every $\varphi\left(w\right)\in\mathbb{F}\left[w,w^{-1}\right]$. Hence, $D_{c(w)\partial_w^{(j)}\delta(z-w)} = c\left(w\right)\partial_w^{(j)}$ is proven. It remains to show that $D_{\delta(z-w)} = 1$ if $U=\mathbb{F}$. But this follows from (57), applied to $c\left(w\right)=1$ and $j=0$. Proposition 3.15 **(a)** is proven.

Before we start proving Proposition 3.15 **(b)**, let us show two auxiliary results:

- Every $a = a\left(z,w\right)\in U\left[\left[z,z^{-1},w,w^{-1}\right]\right]$ satisfies

$$D_{(z-w)a(z,w)} = \left[D_a, w\right]. \tag{63}$$

  Here, w denotes the continuous $\mathbb{F}$-linear map $U\left[\left[z,z^{-1},w,w^{-1}\right]\right]\to U\left[\left[z,z^{-1},w,w^{-1}\right]\right]$ which sends every

$b \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ to $w \cdot b$. (In analogy to Proposition 2.3, we should have called it L_w instead of w, but the notation w is shorter.)]

Proof of (63): Let $a = a(z,w) \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$. Then, every $\varphi(w) \in \mathbb{F}\left[w, w^{-1}\right]$ satisfies

$$\begin{aligned}
& D_{(z-w)a(z,w)}\left(\varphi(w)\right) \\
&= \operatorname{Res} \varphi(z)\,(z-w)\,a(z,w)\,dz \qquad \left(\text{by the definition of } D_{(z-w)a(z,w)}\right) \\
&= \operatorname{Res} \varphi(z)\,za(z,w)\,dz - w \operatorname{Res} \varphi(z)\,a(z,w)\,dz \\
&= \operatorname{Res} z\varphi(z)\,a(z,w)\,dz - w \operatorname{Res} \varphi(z)\,a(z,w)\,dz.
\end{aligned} \tag{64}$$

Hence, every $\varphi(w) \in \mathbb{F}\left[w, w^{-1}\right]$ satisfies

$$\begin{aligned}
& \underbrace{\left[D_a, w\right]}_{=D_a \circ w - w \circ D_a} \left(\varphi(w)\right) \\
&= \left(D_a \circ w - w \circ D_a\right)\left(\varphi(w)\right) = \underbrace{D_a\left(w\varphi(w)\right)}_{\substack{=\operatorname{Res} z\varphi(z)a(z,w)dz \\ \text{(by the definition of } D_a)}} - w \underbrace{D_a\left(\varphi(w)\right)}_{\substack{=\operatorname{Res} \varphi(z)a(z,w)dz \\ \text{(by the definition of } D_a)}} \\
&= \operatorname{Res} z\varphi(z)\,a(z,w)\,dz - w \operatorname{Res} \varphi(z)\,a(z,w)\,dz \\
&= D_{(z-w)a(z,w)}\left(\varphi(w)\right) \qquad \text{(by (64))}.
\end{aligned}$$

In other words, $\left[D_a, w\right] = D_{(z-w)a(z,w)}$. This proves (63).

- If an $a \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ satisfies $D_a = 0$, then

$$a = 0. \tag{65}$$

Proof of (65): Let $a \in U\left[\left[z, z^{-1}, w, w^{-1}\right]\right]$ be such that $D_a = 0$. Write a in the form $a = \sum\limits_{(n,m) \in \mathbb{Z}^2} a_{n,m} z^n w^m$ with $a_{n,m} \in U$. Then, every $i \in \mathbb{Z}$ satisfies

$$\begin{aligned}
D_a\left(w^i\right) &= \operatorname{Res} z^i \underbrace{a(z,w)}_{=a=\sum\limits_{(n,m)\in\mathbb{Z}^2} a_{n,m}z^n w^m} dz \qquad \text{(by the definition of } D_a) \\
&= \operatorname{Res} z^i \sum_{(n,m)\in\mathbb{Z}^2} a_{n,m} z^n w^m dz = \sum_{m \in \mathbb{Z}} a_{-i-1,m} w^m.
\end{aligned}$$

Hence, every $i \in \mathbb{Z}$ satisfies $\sum\limits_{m\in\mathbb{Z}} a_{-i-1,m} w^m = \underbrace{D_a}_{=0}\left(w^i\right) = 0$. Comparing coefficients in this equality, we conclude that every $i \in \mathbb{Z}$ and $m \in \mathbb{Z}$ satisfy $a_{-i-1,m} = 0$. In other words, every $(n,m) \in \mathbb{Z}^2$ satisfy $a_{n,m} = 0$. Hence, $a = 0$. This completes our proof of (65).

(b) $\Longrightarrow$: Assume that $a(z,w)$ is local. Theorem 3.13 thus yields that $a(z,w)$ can be written in the form

$$\sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$$

with $\left(c^j(w)\right)_{j\in\mathbb{N}}$ being a family of formal distributions $c^j(w) \in U\left[\left[w,w^{-1}\right]\right]$ having the property that all but finitely many $j\in\mathbb{N}$ satisfy $c^j(w)=0$. Consider this $\left(c^j(w)\right)_{j\in\mathbb{N}}$. Since $a(z,w)=\sum\limits_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$, we have

$$\begin{aligned} D_a &= \sum_{j\in\mathbb{N}} \underbrace{D_{c^j(w)\partial_w^{(j)}\delta(z-w)}}_{\substack{=c^j(w)\partial_w^{(j)} \\ \text{(by (57), applied to } c(w)=c^j(w))}} \qquad \text{(since } D_a \text{ depends } \mathbb{F}\text{-linearly on } a) \\ &= \sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}. \end{aligned} \tag{66}$$

Thus, D_a is a differential operator of finite order (since all but finitely many $j\in\mathbb{N}$ satisfy $c^j(w)=0$). This proves the $\Longrightarrow$ direction of Proposition 3.15 **(b)**.

$\Longleftarrow$: Assume that D_a is a differential operator of finite order. In other words, $D_a=\sum\limits_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)}$ for a family $\left(c_j(w)\right)_{j\in\mathbb{N}}$ of formal distributions in $U\left[\left[w,w^{-1}\right]\right]$ with the property that all but finitely many $j\in\mathbb{N}$ satisfy $c_j(w)=0$. Consider this family $\left(c_j(w)\right)_{j\in\mathbb{N}}$.

Let us define the $\mathbb{F}$-linear map $w: U\left[\left[z,z^{-1},w,w^{-1}\right]\right]\to U\left[\left[z,z^{-1},w,w^{-1}\right]\right]$ as in (63). We have

$$\left[w,\partial_w^{(n)}\right] = -\partial_w^{(n-1)} \qquad \text{for every } n\in\mathbb{N}. \tag{67}$$

[45]

All but finitely many $j\in\mathbb{N}$ satisfy $c_j(w)=0$. Thus, there exists a $J\in\mathbb{N}$ such that every $j\in\mathbb{N}$ satisfying $j\geq J$ satisfies $c_j(w)=0$. Consider this J. Then,

$$D_a=\sum_{j\in\mathbb{N}} c_j(w)\,\partial_w^{(j)} = \sum_{j=0}^{J-1} c_j(w)\,\partial_w^{(j)} \tag{68}$$

[45] Indeed, this is similar to Proposition 2.3; there are only two differences:

- We now have two variables z and w instead of a single variable z. But this does not change much because only the variable w is "active". (If we identify $U\left[\left[z,z^{-1},w,w^{-1}\right]\right]$ with $\left(U\left[\left[z,z^{-1}\right]\right]\right)\left[\left[w,w^{-1}\right]\right]$, then we are back in the single-variable case.)
- We are now denoting by w the map that would be denoted by L_w in the terminology of Proposition 2.3.

These two differences are insubstantial, and the proof of Proposition 2.3 can be easily adapted to prove (67).

(by the definition of J). We are going to prove that $(z-w)^J a(z,w) = 0$.

Indeed, let us show that

$$D_{(z-w)^k a(z,w)} = \sum_{j=k}^{J-1} c_j(w) \partial_w^{(j-k)} \qquad \text{for every } k \in \mathbb{N} \tag{69}$$

(where any sum whose lower limit is larger than its upper limit is understood to be empty).

Proof of (69): We shall prove (69) by induction over k.

Induction base: For $k = 0$, the equality (69) follows immediately from (68). Thus, the induction base is complete.

Induction step: Let $K \in \mathbb{N}$. Assume that (69) holds for $k = K$. We need to show that (69) holds for $k = K+1$.

We have

$$\begin{aligned}
& D_{(z-w)^{K+1} a(z,w)} \\
&= D_{(z-w)(z-w)^K a(z,w)} = \Bigg[\underbrace{D_{(z-w)^K a(z,w)}}_{\substack{= \sum_{j=K}^{J-1} c_j(w) \partial_w^{(j-K)} \\ \text{(since (69) holds for } k=K)}}, w \Bigg] \\
&\qquad \left(\text{by (63), applied to } (z-w)^K a(z,w) \text{ instead of } a\right) \\
&= \left[\sum_{j=K}^{J-1} c_j(w) \partial_w^{(j-K)}, w \right] = \sum_{j=K}^{J-1} \underbrace{\left[c_j(w) \partial_w^{(j-K)}, w \right]}_{\substack{= c_j(w) \left[\partial_w^{(j-K)}, w \right] \\ \text{(since } c_j(w) \text{ commutes with } w)}} = \sum_{j=K}^{J-1} c_j(w) \underbrace{\left[\partial_w^{(j-K)}, w \right]}_{= -\left[w, \partial_w^{(j-K)} \right]} \\
&= - \sum_{j=K}^{J-1} c_j(w) \underbrace{\left[w, \partial_w^{(j-K)} \right]}_{\substack{= -\partial_w^{(j-K-1)} \\ \text{(by (67), applied to } n=j-K)}} = \sum_{j=K}^{J-1} c_j(w) \partial_w^{(j-K-1)} \\
&= c_K(w) \underbrace{\partial_w^{(K-K-1)}}_{= \partial_w^{(-1)} = 0} + \sum_{j=K+1}^{J-1} c_j(w) \partial_w^{(j-K-1)} = \sum_{j=K+1}^{J-1} c_j(w) \partial_w^{(j-K-1)}.
\end{aligned}$$

In other words, (69) holds for $k = K+1$. This completes the induction step. Thus, (69) is proven.

Now, applying (69) to $k = J$, we obtain

$$D_{(z-w)^J a(z,w)} = \sum_{j=J}^{J-1} c_j(w) \partial_w^{(j-J)} = (\text{empty sum}) = 0.$$

Thus, (65) (applied to $(z-w)^J a(z,w)$ instead of a) yields that $(z-w)^J a(z,w) = 0$. Hence, $a(z,w)$ is local. This proves the $\Longleftarrow$ direction of Proposition 3.15 **(b)**.

(c) Clearly, $a(w,z)$ is local.[46] It remains to prove that $D_{a(w,z)} = \left(D_{a(z,w)}\right)^*$. In other words, it remains to prove that $D_{a(w,z)} = (D_a)^*$ (since $a(z,w) = a$).

Theorem 3.13 yields that $a(z,w)$ can be written in the form

$$\sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$$

with $\left(c^j(w)\right)_{j\in\mathbb{N}}$ being a family of formal distributions $c^j(w) \in U\left[\left[w,w^{-1}\right]\right]$ having the property that all but finitely many $j\in\mathbb{N}$ satisfy $c^j(w) = 0$. Consider this $\left(c^j(w)\right)_{j\in\mathbb{N}}$. Theorem 3.13 furthermore shows that these $c^j(w)$ are given by (52).

We have $a(z,w) = \sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}\delta(z-w)$. Renaming the indeterminates z and w as w and z in this equality, we obtain

$$\begin{aligned}
a(w,z) &= \sum_{j\in\mathbb{N}} c^j(z)\,\partial_z^{(j)} \underbrace{\delta(w-z)}_{\substack{=\delta(z-w)\\ \text{(by Proposition 3.12 (c))}}} = \sum_{j\in\mathbb{N}} c^j(z) \underbrace{\partial_z^{(j)}\delta(z-w)}_{\substack{=(-1)^j\partial_w^{(j)}\delta(z-w)\\ \text{(by Proposition 3.12 (e))}}} \\
&= \sum_{j\in\mathbb{N}} c^j(z)(-1)^j\,\partial_w^{(j)}\delta(z-w) = \sum_{j\in\mathbb{N}} (-1)^j \underbrace{c^j(z)\,\partial_w^{(j)}}_{\substack{=\partial_w^{(j)}c^j(z)\\ \text{(since } c^j(z) \text{ is a polynomial in } z\\ \text{and thus commutes with } \partial_w^{(j)})}} \delta(z-w) \\
&= \sum_{j\in\mathbb{N}} (-1)^j\,\partial_w^{(j)}\left(c^j(z)\,\delta(z-w)\right).
\end{aligned}$$

From (66), we have $D_a = \sum_{j\in\mathbb{N}} c^j(w)\,\partial_w^{(j)}$, so that $(D_a)^* = \sum_{j\in\mathbb{N}} (-1)^j\,\partial_w^{(j)} c^j(w)$.

[46] *Proof.* We know that $a(z,w)$ is local. Thus, there exists an $N\in\mathbb{N}$ such that $(z-w)^N a(z,w) = 0$. Consider this N. Switching the indeterminates z and w in $(z-w)^N a(z,w) = 0$, we obtain $(w-z)^N a(w,z) = 0$. Hence, $\underbrace{(z-w)^N}_{=(-1)^N(w-z)^N} a(w,z) = (-1)^N \underbrace{(w-z)^N a(w,z)}_{=0} = 0$. Thus, $a(w,z)$ is local, qed.

Now, for every $\varphi(w) \in \mathbb{F}[w, w^{-1}]$, we have

$$\begin{aligned}
& D_{a(w,z)}(\varphi(w)) \\
&= \operatorname{Res} \varphi(z) \underbrace{a(w,z)}_{= \sum_{j \in \mathbb{N}} (-1)^j \partial_w^{(j)} \left(c^j(z) \delta(z-w)\right)} dz \qquad \left(\text{by the definition of } D_{a(w,z)}\right) \\
&= \operatorname{Res} \varphi(z) \left(\sum_{j \in \mathbb{N}} (-1)^j \partial_w^{(j)} \left(c^j(z) \delta(z-w) \right) \right) dz \\
&= \sum_{j \in \mathbb{N}} (-1)^j \underbrace{\operatorname{Res} \varphi(z) \partial_w^{(j)} \left(c^j(z) \delta(z-w) \right) dz}_{\substack{= \partial_w^{(j)} \operatorname{Res} \varphi(z) c^j(z) \delta(z-w) dz \\ \text{(since both multiplication with } \varphi(z) \text{ and} \\ \text{the map } q \mapsto \operatorname{Res} q dz \text{ commute with } \partial_w^{(j)})}} \\
&= \sum_{j \in \mathbb{N}} (-1)^j \partial_w^{(j)} \underbrace{\operatorname{Res} \varphi(z) c^j(z) \delta(z-w) dz}_{\substack{= \operatorname{Res} \delta(z-w) \varphi(z) c^j(z) dz \\ = \varphi(w) c^j(w) \\ \text{(by Proposition 3.6 (c), applied} \\ \text{to } \varphi(z) c^j(z) \text{ instead of } a(z))}} \\
&= \sum_{j \in \mathbb{N}} (-1)^j \partial_w^{(j)} \left(\varphi(w) c^j(w) \right) = \underbrace{\left(\sum_{j \in \mathbb{N}} (-1)^j \partial_w^{(j)} c^j(w) \right)}_{= (D_a)^*} (\varphi(w)) = (D_a)^* (\varphi(w)).
\end{aligned}$$

Hence, $D_{a(w,z)} = (D_a)^*$. This completes the proof of Proposition 3.15 **(c)**.

(d) *Proof of (59):* For every $\psi(w) \in \mathbb{F}[w, w^{-1}]$ and $\varphi(w) \in \mathbb{F}[w, w^{-1}]$, we have

$$\begin{aligned}
D_{\psi(w) a(z,w)}(\varphi(w)) &= \operatorname{Res} \varphi(z) \psi(w) a(z,w) dz \qquad \left(\text{by the definition of } D_{\psi(w) a(z,w)}\right) \\
&= \psi(w) \underbrace{\operatorname{Res} \varphi(z) a(z,w) dz}_{\substack{= D_{a(z,w)}(\varphi(w)) \\ \text{(by the definition of } D_{a(z,w)})}} = \psi(w) D_{a(z,w)}(\varphi(w)) \\
&= \left(\psi(w) \circ D_{a(z,w)} \right) (\varphi(w)).
\end{aligned}$$

This yields (59).

Proof of (60): For every $\varphi(w) \in \mathbb{F}\left[w, w^{-1}\right]$, we have

$$\begin{aligned} D_{\partial_w a(z,w)}(\varphi(w)) &= \operatorname{Res} \varphi(z)\, \partial_w a(z,w)\, dz && \left(\text{by the definition of } D_{\partial_w a(z,w)}\right) \\ &= \partial_w \underbrace{\operatorname{Res} \varphi(z)\, a(z,w)\, dz}_{\substack{=D_{a(z,w)}(\varphi(w)) \\ \text{(by the definition of } D_{a(z,w)})}} \\ &\qquad \left(\text{since the map } q \mapsto \operatorname{Res} q dz \text{ commutes with } \partial_w\right) \\ &= \partial_w D_{a(z,w)}(\varphi(w)) = \left(\partial_w \circ D_{a(z,w)}\right)(\varphi(w)). \end{aligned}$$

This yields (60).

Proof of (61): For every $\varphi(w) \in \mathbb{F}\left[w, w^{-1}\right]$, we have

$$\begin{aligned} D_{\psi(z)a(z,w)}(\varphi(w)) &= \operatorname{Res} \varphi(z)\, \psi(z)\, a(z,w)\, dz && \left(\text{by the definition of } D_{\psi(w)a(z,w)}\right) \\ &= D_{a(z,w)}(\varphi(w)\, \psi(w)) && \left(\text{by the definition of } D_{a(z,w)}\right) \\ &= D_{a(z,w)}(\psi(w)\, \varphi(w)) = \left(D_{a(z,w)} \circ \psi(w)\right)(\varphi(w)). \end{aligned}$$

This yields (61).

Proof of (62): For every $\varphi(w) \in \mathbb{F}\left[w, w^{-1}\right]$, we have

$$\begin{aligned} D_{\partial_z a(z,w)}(\varphi(w)) &= \operatorname{Res} \varphi(z)\, \partial_z a(z,w)\, dz && \left(\text{by the definition of } D_{\partial_z a(z,w)}\right) \\ &= -\underbrace{\operatorname{Res} \partial_z(\varphi(z))\, a(z,w)\, dz}_{\substack{=D_{a(z,w)}(\partial_w(\varphi(w))) \\ \text{(by the definition of } D_{a(z,w)})}} \\ &\qquad \left(\begin{array}{c} \text{by (28), applied to } U\left[\left[w, w^{-1}\right]\right], \varphi(z) \text{ and } a(z,w) \\ \text{instead of } U, g \text{ and } f \end{array}\right) \\ &= -D_{a(z,w)}(\partial_w(\varphi(w))) = \left(-D_{a(z,w)} \circ \partial_w\right)(\varphi(w)). \end{aligned}$$

This yields (62). $\square$

4. j-th products over Lie (super)algebras

4.1. Local pairs over Lie (super)algebras

Now, assume that 2 and 3 are invertible in the ground ring $\mathbb{F}$ (for instance, this holds when $\mathbb{F}$ is a field of characteristic $\notin \{2, 3\}$). Let $\mathfrak{g}$ be a Lie (super)algebra. We remind the reader that a *Lie superalgebra* is defined as an $\mathbb{F}$-supermodule $\mathfrak{g} = \mathfrak{g}_{\overline{0}} \oplus \mathfrak{g}_{\overline{1}}$ endowed with a Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which is even (i.e., satisfies $\left[\mathfrak{g}_\alpha, \mathfrak{g}_\beta\right] \subseteq \mathfrak{g}_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}/2\mathbb{Z}$) and satisfies the following axioms (which are obtained from the axioms in the definition of a Lie algebra by the Koszul-Quillen rule):[47]

- We have $[a,b] = -(-1)^{p(a)p(b)}[b,a]$ for any two homogeneous elements a and b of $\mathfrak{g}$. Here, as usual, $p(c)$ denotes the parity of the homogeneous element c.
- We have $[a,[b,c]] = [[a,b],c] + (-1)^{p(a)p(b)}[b,[a,c]]$ for any three homogeneous elements a, b and c of $\mathfrak{g}$.

(Keep in mind that $[a,a] = 0$ holds for any $a \in \mathfrak{g}_0$, as can be easily derived from the first of these two axioms; but in general, $a \in \mathfrak{g}_1$ do not satisfy $[a,a] = 0$.)

Example 4.1. **(a)** If A is any associative superalgebra (i.e., any $\mathbb{Z}/2\mathbb{Z}$-graded associative algebra), then A becomes a Lie superalgebra when equipped with the supercommutator bracket

$$[a,b] := ab - (-1)^{p(a)p(b)} ba \qquad \text{for all } a,b \in A.$$

(b) If $V = V_{\overline{0}} \oplus V_{\overline{1}}$ is an $\mathbb{F}$-supermodule, then $\operatorname{End} V$ (the $\mathbb{F}$-superalgebra of **all** $\mathbb{F}$-linear endomorphisms of V, not just the even ones) becomes a Lie superalgebra when equipped with the supercommutator bracket

$$[a,b] := ab - (-1)^{p(a)p(b)} ba \qquad \text{for all } a,b \in \operatorname{End} V.$$

This Lie superalgebra structure is, of course, a particular case of part **(a)** of this example.

Definition 4.2. Let $\mathfrak{g}$ be a Lie (super)algebra.

(a) Two $\mathfrak{g}$-valued formal distributions $a(z) \in \mathfrak{g}\left[\left[z,z^{-1}\right]\right]$ and $b(z) \in \mathfrak{g}\left[\left[z,z^{-1}\right]\right]$ are said to *form a local pair* if the formal distribution $[a(z), b(w)] \in \mathfrak{g}\left[\left[z,z^{-1},w,w^{-1}\right]\right]$ is local.

(b) Let $(a(z), b(z))$ be a local pair. According to Theorem 3.13, we have

$$[a(z), b(w)] = \sum_{j\in\mathbb{N}} c^j(w)\, \partial_w^{(j)} \delta(z-w), \tag{70}$$

where the formal distributions $c^j(w) \in \mathfrak{g}\left[\left[w,w^{-1}\right]\right]$ are given by

$$c^j(w) = \operatorname{Res}(z-w)^j [a(z), b(w)]\, dz \qquad \text{for all } j \in \mathbb{N}. \tag{71}$$

[47]We remark that Lie superalgebras could also be defined more generally, without assuming that 2 and 3 are invertible in $\mathbb{F}$. But in this generality, the two axioms below would not be sufficient to obtain a good theory. For instance, if $\mathbb{F}$ is a field of characteristic 2, it is important to require $[a,a] = 0$ for all even elements a of $\mathfrak{g}$ as an extra axiom; when 2 is invertible, this follows easily from the antisymmetry.

For each $j \in \mathbb{N}$, we shall denote by $a(w)_{(j)} b(w)$ the formal distribution $c^j(w) \in \mathfrak{g}\left[\left[w, w^{-1}\right]\right]$ defined by (71). Notice that the notation "$a(w)_{(j)} b(w)$" is a ternary operator with three arguments: $a(w)$, $b(w)$ and j. We shall often use a curried version of this operator: Namely, given an $a(z) \in \mathfrak{g}\left[\left[z, z^{-1}\right]\right]$ and a $j \in \mathbb{N}$, we write $a(w)_{(j)}$ for the operator which takes a $b(w)$ for which $(a(z), b(z))$ is a local pair, and returns the formal distribution $a(w)_{(j)} b(w)$.

As one would expect, $a(z)_{(j)} b(z)$ means the result of substituting z for w in the formal distribution $a(w)_{(j)} b(w)$ (and similarly for every other indeterminate instead of z).

We refer to $a(z)_{(j)} b(z)$ as the *j-th product of the formal distributions* $a(z)$ and $b(z)$. For given $j \in \mathbb{N}$, we refer to the operator which sends every local pair $(a(z), b(z))$ to $a(z)_{(j)} b(z)$ as the *j-th multiplication*.

The formula (70) is called the *singular part of the OPE (operator-product expansion)*.

Let us make an important remark.

Remark 4.3. Let $\mathfrak{g}$ be a Lie (super)algebra, and let $(a(z), b(z))$ be a local pair. We have denoted by $a(w)_{(j)} b(w)$ the formal distribution $c^j(w) \in \mathfrak{g}\left[\left[w, w^{-1}\right]\right]$

defined by (71). Thus, (70) becomes

$$\begin{aligned}
[a(z), b(w)] &= \sum_{j\in\mathbb{N}} \underbrace{\left(a(w)_{(j)} b(w)\right)}_{\substack{=\sum_{n\in\mathbb{Z}}\left(a(w)_{(j)}b(w)\right)_{[n]} w^{-n-1} \\ \text{(here, we are using the notation of} \\ \text{Definition 3.1, for } w \text{ instead of } z)}} \partial_w^{(j)} \delta(z-w) \\
&= \sum_{j\in\mathbb{N}} \sum_{n\in\mathbb{Z}} \left(a(w)_{(j)} b(w)\right)_{[n]} w^{-n-1} \partial_w^{(j)} \delta(z-w) \\
&= \sum_{j\in\mathbb{N}} \sum_{r\in\mathbb{Z}} \left(a(w)_{(j)} b(w)\right)_{[r]} w^{-r-1} \underbrace{\partial_w^{(j)} \delta(z-w)}_{\substack{=\sum_{n\in\mathbb{Z}} \binom{n}{j} w^{n-j} z^{-n-1} \\ \text{(by (48))}}} \\
&\qquad \text{(here, we renamed the summation index } n \text{ as } r) \\
&= \sum_{j\in\mathbb{N}} \sum_{r\in\mathbb{Z}} \left(a(w)_{(j)} b(w)\right)_{[r]} w^{-r-1} \sum_{n\in\mathbb{Z}} \binom{n}{j} w^{n-j} z^{-n-1} \\
&= \sum_{j\in\mathbb{N}} \sum_{r\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \binom{n}{j} \left(a(w)_{(j)} b(w)\right)_{[r]} \underbrace{w^{-r-1} w^{n-j} z^{-n-1}}_{=w^{n-j-r-1} z^{-n-1}} \\
&= \sum_{j\in\mathbb{N}} \sum_{r\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} \binom{n}{j} \left(a(w)_{(j)} b(w)\right)_{[r]} w^{n-j-r-1} z^{-n-1} \\
&= \sum_{j\in\mathbb{N}} \sum_{r\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} \binom{m}{j} \left(a(w)_{(j)} b(w)\right)_{[r]} w^{m-j-r-1} z^{-m-1} \\
&\qquad \text{(here, we renamed the summation index } m \text{ as } n).
\end{aligned}$$

Comparing coefficients in front of $z^{-m-1}w^{-n-1}$ on both sides of this equality, we conclude that

$$\left[a_{[m]}, b_{[n]}\right] = \sum_{j\in\mathbb{N}} \binom{m}{j} \left(a(w)_{(j)} b(w)\right)_{[m+n-j]}. \tag{72}$$

This equality is an equivalent rewriting of (70).