Witt vectors. Part 1

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Witt#4b: A combinatorial identity proven using symmetric functions identities

[absolutely not completed (proof of Thm 2 is very sloppy), not proofread]

The point of this note is to use the results of [2] (more precisely, its Theorem 5 **(b)**) in order to verify a combinatorial identity from [3]:

Theorem 1. Let $n \in \mathbb{N}$ and $x \in \mathbb{R}$. Then,

$$\sum_{\sigma \in S_n} \operatorname{sign} \sigma \cdot x^{\operatorname{cycle} \sigma} = n! \binom{x}{n}.$$

Here, for every permutation $\sigma \in S_n$, we let $\operatorname{cycle} \sigma$ denote the number of all cycles (including cycles of length 1) in the cycle decomposition of the permutation σ.

Note that x has been called k in [2].

In order to prove Theorem 1, we are going to use some of the notations of [2]; namely, we will use the Definitions 1, 2, 3, 4, 5, 10, 11, 12, 13 of [2].

First, let us find an alternative formula for the number z_λ defined in Definition 13 of [2]. In fact, let us recall that in Definition 13 of [2], the number z_λ was defined by

$$z_\lambda = \prod_{n=1}^{\infty} n^{m_n(\lambda)} (m_n(\lambda))! \qquad \text{for every } \lambda \in \operatorname{Par}.$$

Thus,

$$z_\lambda = \prod_{n=1}^{\infty} n^{m_n(\lambda)} (m_n(\lambda))! = \prod_{n=1}^{\infty} n^{m_n(\lambda)} \prod_{n=1}^{\infty} (m_n(\lambda))!. \tag{1}$$

But if we write the partition λ in the form $\lambda = (\lambda_1, \lambda_2, ..., \lambda_u)$ for some $u \in \mathbb{N}$ such that $\lambda_u \neq 0$ (clearly we can write the partition λ in this way, because every partition has only finitely many nonzero terms), then

$$\prod_{i=1}^{u} \lambda_i = \prod_{i \in \{1,2,...,u\}} \lambda_i = \prod_{n=1}^{\infty} \underbrace{\prod_{\substack{i \in \{1,2,...,u\}; \\ \lambda_i = n}} n}_{= n^{\left|\{i \in \{1,2,...,u\} \mid \lambda_i = n\}\right|} = n^{m_n(\lambda)}} = \prod_{n=1}^{\infty} n^{m_n(\lambda)},$$

so that (1) becomes

$$z_\lambda = \underbrace{\prod_{n=1}^{\infty} n^{m_n(\lambda)}}_{= \prod_{i=1}^{u} \lambda_i} \prod_{n=1}^{\infty} (m_n(\lambda))! = \prod_{i=1}^{u} \lambda_i \cdot \prod_{n=1}^{\infty} (m_n(\lambda))! = \prod_{i=1}^{u} \lambda_i \cdot \prod_{k=1}^{\infty} (m_k(\lambda))! \tag{2}$$

(here, we substituted k for n in the second product).

Next, let us define the *cycle type* of a permutation:

Definition 1. Let $\sigma \in S_n$ be a permutation. For every $i \in \{1, 2, 3, ...\}$, let us denote by $\operatorname{cycle}_i \sigma$ the number of all cycles of length i in the cycle decomposition of the permutation σ. Clearly, $(\operatorname{cycle}_1 \sigma, \operatorname{cycle}_2 \sigma, \operatorname{cycle}_3 \sigma, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ and

$$\sum_{i=1}^{\infty} \underbrace{\operatorname{cycle}_i \sigma}_{=(\text{the number of all cycles of length } i \text{ in the cycle decomposition of the permutation } \sigma)}$$
$$= \sum_{i=1}^{\infty} (\text{the number of all cycles of length } i \text{ in the cycle decomposition of the permutation } \sigma)$$
$$= (\text{the number of all cycles in the cycle decomposition of the permutation } \sigma) = \operatorname{cycle} \sigma$$

[1].

Now, the *cycle type* $\operatorname{cyc} \sigma \in \operatorname{Par}$ of the permutation σ is defined as the partition

$$m^{-1} (\operatorname{cycle}_1 \sigma, \operatorname{cycle}_2 \sigma, \operatorname{cycle}_3 \sigma, ...) ,$$

where $m : \operatorname{Par} \to \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$ is the bijection defined by

$$m (\lambda) = (m_1 (\lambda) , m_2 (\lambda) , m_3 (\lambda) , ...) \qquad \text{for all } \lambda \in \operatorname{Par} .$$

Hence,

$$(\operatorname{cycle}_1 \sigma, \operatorname{cycle}_2 \sigma, \operatorname{cycle}_3 \sigma, ...) = m (\operatorname{cyc} \sigma) = (m_1 (\operatorname{cyc} \sigma) , m_2 (\operatorname{cyc} \sigma) , m_3 (\operatorname{cyc} \sigma) , ...) .$$

Thus, $\operatorname{cycle}_i \sigma = m_i (\operatorname{cyc} \sigma)$ for every $i \in \{1, 2, 3, ...\}$.

It is clear that

$$\operatorname{wt} (\operatorname{cyc} \sigma) = \sum_{k=1}^{\infty} k \underbrace{m_k (\operatorname{cyc} \sigma)}_{=\operatorname{cycle}_k \sigma}$$
$$\left(\text{by the formula } \operatorname{wt} \lambda = \sum_{k=1}^{\infty} k m_k (\lambda) , \text{ which holds for every partition } \lambda \right)$$
$$= \sum_{k=1}^{\infty} k \operatorname{cycle}_k \sigma$$

[1] This sum $\sum_{i=1}^{\infty} \operatorname{cycle}_i \sigma$ is an infinite sum, but it contains only finitely many nonzero summands (since $(\operatorname{cycle}_1 \sigma, \operatorname{cycle}_2 \sigma, \operatorname{cycle}_3 \sigma, ...) \in \mathbb{N}_{\text{fin}}^{\{1,2,3,...\}}$), and thus has a well-defined value.

and

$$n=\sum_{k\in\{1,2,\ldots,n\}}1=\sum_{\substack{Z\text{ is a cycle}\\ \text{in the cycle}\\ \text{decomposition of}\\ \text{the permutation }\sigma}}\underbrace{\sum_{\substack{k\in\{1,2,\ldots,n\};\\ k\in Z}}1}_{\substack{=(\text{length of the cycle }Z)\cdot 1\\ =(\text{length of the cycle }Z)}}$$

$$\left(\begin{array}{c}\text{since every element of }\{1,2,\ldots,n\}\text{ lies in one and only one}\\ \text{cycle in the cycle decomposition of the permutation }\sigma\end{array}\right)$$

$$=\sum_{\substack{Z\text{ is a cycle}\\ \text{in the cycle}\\ \text{decomposition of}\\ \text{the permutation }\sigma}}(\text{length of the cycle }Z)=\sum_{k=1}^{\infty}\sum_{\substack{Z\text{ is a cycle of}\\ \text{length }k\text{ in the cycle}\\ \text{decomposition of}\\ \text{the permutation }\sigma}}\underbrace{(\text{length of the cycle }Z)}_{=k}$$

$$=\sum_{k=1}^{\infty}\underbrace{\sum_{\substack{Z\text{ is a cycle of}\\ \text{length }k\text{ in the cycle}\\ \text{decomposition of}\\ \text{the permutation }\sigma}}k}_{\substack{=(\text{the number of all cycles of length }k\text{ in the cycle decomposition of the permutation }\sigma)\cdot k\\ =\operatorname{cycle}_k\sigma\cdot k=k\operatorname{cycle}_k\sigma}}=\sum_{k=1}^{\infty}k\operatorname{cycle}_k\sigma,$$

and therefore

$$\operatorname{wt}(\operatorname{cyc}\sigma)=n\qquad\text{for every permutation }\sigma\in S_n.$$

The following simple property connects this notion of cycle types with the numbers z_λ defined above:

Theorem 2. Let $\lambda\in\operatorname{Par}$ and let $n=\operatorname{wt}\lambda$. Then,

$$\left|\{\sigma\in S_n\ \mid\ \operatorname{cyc}\sigma=\lambda\}\right|=\frac{n!}{z_\lambda}.$$

Proof of Theorem 2. We write the partition λ in the form $\lambda=(\lambda_1,\lambda_2,\ldots,\lambda_u)$ for some $u\in\mathbb{N}$ such that $\lambda_u\neq 0$ (clearly we can write the partition λ in this way, because every partition has only finitely many nonzero terms). Clearly, $\lambda_1+\lambda_2+\ldots+\lambda_u=\sum\limits_{i\in\{1,2,\ldots,u\}}\lambda_i=\operatorname{wt}\lambda=n$.

Let us introduce a notation: A *λ-partition* will mean a family $(I_1,I_2,\ldots,I_u)\in(\mathcal{P}(\{1,2,\ldots,n\}))^u$ of pairwise disjoint subsets of $\{1,2,\ldots,n\}$ satisfying $(|I_k|=\lambda_k$ for every $k\in\{1,2,\ldots,u\})$. The number of all λ-partitions is the multinomial coefficient $\dbinom{n}{\lambda_1,\lambda_2,\ldots,\lambda_u}=\dfrac{n!}{\prod\limits_{i=1}^{u}\lambda_i!}$ (since $\lambda_1+\lambda_2+\ldots+\lambda_u=n$).

For every finite set U, let S_U denote the set of all permutations of the set U. A permutation π of a nonempty finite set U is said to be *cyclic* if and only if there exists a bijection $\nu:\{1,2,\ldots,|U|\}\to U$ such that $\pi=\left(\nu_1,\nu_2,\ldots,\nu_{|U|}\right)$. In other words, a permutation π of a nonempty finite set U is said to be *cyclic* if and only if its cycle decomposition consists only of one cycle of length $|U|$. In other words, a permutation

π of a nonempty finite set U is said to be *cyclic* if and only if the cycle type of π is $(|U|)$. Clearly, for every nonempty finite set U, the number of cyclic permutations of U is $(|U|-1)!$ [2]. In other words, $\left|S_U^C\right| = (|U|-1)!$, where S_U^C denotes the set of all cyclic permutations of the set U.

If U is a subset of a finite set V, then we consider S_U as a subset of S_V (in fact, we identify every element π of S_U with the element π' of S_V defined by

$$\left(\pi'(v) = \begin{cases} \pi(v), & \text{if } v \in U; \\ v, & \text{if } v \notin U \end{cases} \qquad \text{for all } v \in V\right)$$

). In particular, if U is a subset of $\{1,2,...,n\}$, then S_U is thus considered as a subset of $S_{\{1,2,...,n\}} = S_n$.

For every λ-partition $(I_1, I_2, ..., I_u)$ and every family $(\pi_i)_{i\in\{1,2,...,u\}} \in \prod_{i=1}^{u} S_{I_i}^C$ of cyclic permutations of the sets I_i, we can define a permutation $\sigma \in S_n$ by $\sigma = \prod_{i=1}^{u} \pi_i$ (note that order doesn't matter in this product $\prod_{i=1}^{u} \pi_i$, because the permutations π_1, π_2, ..., π_u are disjoint cycles and therefore commute). This permutation σ has cycle decomposition $\pi_1 \circ \pi_2 \circ ... \circ \pi_u$, and thus for every $i \in \{1,2,3,...\}$, we have

$$\begin{aligned}
& m_i(\operatorname{cyc}\sigma) \\
& = \operatorname{cycle}_i(\sigma) \\
& = (\text{the number of all cycles of length } i \text{ in the cycle decomposition of the permutation } \sigma) \\
& = \left(\text{the number of all } k \in \{1,2,...,u\} \text{ such that } \underbrace{\text{the length of the cycle } \pi_k}_{=|I_k|=\lambda_k} \text{ is } i\right) \\
& \qquad (\text{since the cycle decomposition of the permutation } \sigma \text{ is } \pi_1 \circ \pi_2 \circ ... \circ \pi_u) \\
& = (\text{the number of all } k \in \{1,2,...,u\} \text{ such that } \lambda_k = i) = |\{k \in \{1,2,...,u\} \mid \lambda_k = i\}| \\
& = m_i(\lambda).
\end{aligned}$$

Consequently, $(m_1(\operatorname{cyc}\sigma), m_2(\operatorname{cyc}\sigma), m_3(\operatorname{cyc}\sigma), ...) = (m_1(\lambda), m_2(\lambda), m_3(\lambda), ...)$. This rewrites as $m(\operatorname{cyc}\sigma) = m(\lambda)$. Hence, $\operatorname{cyc}\sigma = \lambda$ (since m is a bijection).

Thus, for every λ-partition $(I_1, I_2, ..., I_u)$ and every family $(\pi_i)_{i\in\{1,2,...,u\}} \in \prod_{i=1}^{u} S_{I_i}^C$, we have defined a permutation $\sigma \in S_n$ by $\sigma = \prod_{i=1}^{u} \pi_i$, and this permutation σ satisfies $\operatorname{cyc}\sigma = \lambda$. Conversely, for every permutation $\sigma \in S_n$ satisfying $\operatorname{cyc}\sigma = \lambda$,

[2] *Proof.* Every bijection $\nu : \{1,2,...,|U|\} \to U$ induces a cyclic permutation $(\nu_1, \nu_2, ..., \nu_{|U|})$ of U, and conversely, every cyclic permutation of U can be written in the form $\pi = (\nu_1, \nu_2, ..., \nu_{|U|})$ for exactly $|U|$ different choices of a bijection $\nu : \{1,2,...,|U|\} \to U$. Hence,

$$\begin{aligned}
& (\text{the number of all cyclic permutations of } U) \\
& = \frac{1}{|U|} \cdot \underbrace{(\text{the number of all bijections } \nu : \{1,2,...,|U|\} \to U)}_{=|U|!} \\
& = \frac{1}{|U|} \cdot |U|! = (|U|-1)!.
\end{aligned}$$

we can find a λ-partition $(I_1, I_2, ..., I_u)$ and a family $(\pi_i)_{i\in\{1,2,...,u\}} \in \prod_{i=1}^{u} S_{I_i}^C$ such that $\sigma = \prod_{i=1}^{u} \pi_i$: In fact, the permutations π_1, π_2, ..., π_u must be chosen as the cycles in the cycle decomposition of σ (ordered by decreasing length), and the sets I_1, I_2, ..., I_u are the respective subsets of $\{1, 2, ..., n\}$ on which these cycles operate. The choice of the permutations π_1, π_2, ..., π_u involves an actual choice: For each $k \in \{1, 2, ..., n\}$, the order of the $\operatorname{cycle}_k \sigma$ cycles of length k can be chosen in $(\operatorname{cycle}_k \sigma)!$ different ways, each of them leading to a different λ-partition $(I_1, I_2, ..., I_u)$ and a different family $(\pi_i)_{i\in\{1,2,...,u\}} \in \prod_{i=1}^{u} S_{I_i}^C$ (though they only differ in their order). Hence, for every permutation $\sigma \in S_n$ satisfying $\operatorname{cyc} \sigma = \lambda$, we can choose a λ-partition $(I_1, I_2, ..., I_u)$ and a family $(\pi_i)_{i\in\{1,2,...,u\}} \in \prod_{i=1}^{u} S_{I_i}^C$ such that $\sigma = \prod_{i=1}^{u} \pi_i$ in $\prod_{k=1}^{\infty} (\operatorname{cycle}_k \sigma)!$ different ways. Since $\prod_{k=1}^{\infty} (\operatorname{cycle}_k \sigma)! = \prod_{k=1}^{\infty} m_k(\lambda)!$ (since $\operatorname{cycle}_i(\sigma) = m_i(\lambda)$ for every $i \in \{1, 2, 3, ...\}$ as shown above), this rewrites as follows: For every permutation $\sigma \in S_n$ satisfying $\operatorname{cyc} \sigma = \lambda$, we can choose a λ-partition $(I_1, I_2, ..., I_u)$ and a family $(\pi_i)_{i\in\{1,2,...,u\}} \in \prod_{i=1}^{u} S_{I_i}^C$ such that $\sigma = \prod_{i=1}^{u} \pi_i$ in $\prod_{k=1}^{\infty} m_k(\lambda)!$ different ways.

Thus,

$$\begin{aligned}
&(\text{the number of all permutations } \sigma \in S_n \text{ satisfying } \operatorname{cyc}\sigma = \lambda) \\
&= \frac{1}{\prod_{k=1}^{\infty} m_k(\lambda)!} \left(\text{number of all possible choices of a } \lambda\text{-partition } (I_1, I_2, \ldots, I_u) \right. \\
&\qquad\qquad \left. \text{and a family } (\pi_i)_{i\in\{1,2,\ldots,u\}} \in \prod_{i=1}^{u} S_{I_i}^C \right) \\
&= \frac{1}{\prod_{k=1}^{\infty} m_k(\lambda)!} \sum_{\substack{(I_1,I_2,\ldots,I_u) \text{ is a} \\ \lambda\text{-partition}}} \underbrace{\left(\text{number of all possible choices of a family } (\pi_i)_{i\in\{1,2,\ldots,u\}} \in \prod_{i=1}^{u} S_{I_i}^C \right)}_{\substack{= \left| \prod_{i=1}^{u} S_{I_i}^C \right| = \prod_{i=1}^{u} \left| S_{I_i}^C \right| = \prod_{i=1}^{u} (\lambda_i - 1)! \\ (\text{since each } i\in\{1,2,\ldots,u\} \text{ satisfies} \\ \left| S_{I_i}^C \right| = (|I_i|-1)! = (\lambda_i-1)!)}} \\
&= \frac{1}{\prod_{k=1}^{\infty} m_k(\lambda)!} \underbrace{\sum_{\substack{(I_1,I_2,\ldots,I_u) \text{ is a} \\ \lambda\text{-partition}}} \prod_{i=1}^{u} (\lambda_i - 1)!}_{= (\text{number of all } \lambda\text{-partitions}) \cdot \prod_{i=1}^{u} (\lambda_i - 1)!} \\
&= \frac{1}{\prod_{k=1}^{\infty} m_k(\lambda)!} \underbrace{(\text{number of all } \lambda\text{-partitions})}_{= \frac{n!}{\prod_{i=1}^{u} \lambda_i!}} \cdot \prod_{i=1}^{u} (\lambda_i - 1)! \\
&= \frac{1}{\prod_{k=1}^{\infty} m_k(\lambda)!} \cdot \frac{n!}{\prod_{i=1}^{u} \lambda_i!} \cdot \prod_{i=1}^{u} (\lambda_i - 1)! = \frac{n!}{\prod_{k=1}^{\infty} m_k(\lambda)!} \Big/ \underbrace{\frac{\prod_{i=1}^{u} \lambda_i!}{\prod_{i=1}^{u} (\lambda_i - 1)!}}_{= \prod_{i=1}^{u} \left(\frac{\lambda_i!}{(\lambda_i - 1)!} \right) = \prod_{i=1}^{u} \lambda_i} \\
&= \frac{n!}{\prod_{i=1}^{u} \lambda_i \cdot \prod_{k=1}^{\infty} m_k(\lambda)!} = \frac{n!}{z_\lambda}
\end{aligned}$$

(by (2)). This proves Theorem 2.

Now, we quote Theorem 5 **(b)** from [2]:

Theorem 3. Let I and J be two countable sets. In the ring $\left(\left(\left(\mathbb{Q}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty\right)[[T]]\right)[[S]]$, we have

$$\sum_{\lambda \in \operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} p_\lambda(\xi) \, p_\lambda(\eta) \, T^{\operatorname{wt}\lambda} = \prod_{(i,j)\in I\times J} \left(\frac{1}{1 - \xi_i \eta_j T} \right)^S, \qquad (3)$$

where the function msum : Par $\to \mathbb{N}$ is defined by

$$\operatorname{msum}\lambda = m_1(\lambda)+m_2(\lambda)+m_3(\lambda)+\ldots = \sum_{k=1}^{\infty} m_k(\lambda) \qquad \text{for every partition } \lambda.$$

Here, for any power series $P \in \left(\left(\left(\mathbb{Q}\left[\xi_i \mid i \in I\right]_{\infty}\right)\left[\eta_j \mid j \in J\right]_{\infty}\right)[[T]]\right)[[S]]$ with constant term 1, the power series $P^S \in \left(\left(\left(\mathbb{Q}\left[\xi_i \mid i \in I\right]_{\infty}\right)\left[\eta_j \mid j \in J\right]_{\infty}\right)[[T]]\right)[[S]]$ is defined by $P^S = \exp(S \log P)$ (where $\log P$ is computed using the $\log(1+X) = \sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k-1}}{k} X^k$ formula).

We are going to apply this theorem to the case when $I = J = \{1\}$. In this case,

$$\begin{aligned}
&\left(\mathbb{Q}\left[\xi_i \mid i \in I\right]_{\infty}\right)\left[\eta_j \mid j \in J\right]_{\infty} \\
&= \left(\mathbb{Q}\left[\xi_i \mid i \in I\right]\right)\left[\eta_j \mid j \in J\right] \qquad \text{(since the sets } I \text{ and } J \text{ are both finite)} \\
&= \left(\mathbb{Q}\left[\xi_1\right]\right)\left[\eta_1\right] \qquad \left(\text{since } I = \{1\} \text{ and } J = \{1\}\right).
\end{aligned}$$

Besides, every $n \in \{1, 2, 3, \ldots\}$ satisfies $p_n = \sum\limits_{i \in I} \xi_i^n = \xi_1^n$ (since we are in the case $I = \{1\}$), and thus

$$p_\lambda(\xi) = p_\lambda = \prod_{n=1}^{\infty} \Bigg(\underbrace{p_n}_{=\xi_1^n}\Bigg)^{m_n(\lambda)} = \prod_{n=1}^{\infty} \left(\xi_1^n\right)^{m_n(\lambda)} = \prod_{n=1}^{\infty} \xi_1^{n m_n(\lambda)} = \xi_1^{\sum\limits_{n=1}^{\infty} n m_n(\lambda)} = \xi_1^{\operatorname{wt}\lambda}.$$

If we replace ξ_1 by η_1 in this equation, it becomes $p_\lambda(\eta) = \eta_1^{\operatorname{wt}\lambda}$. Thus,

$$\begin{aligned}
\sum_{\lambda \in \operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} \underbrace{p_\lambda(\xi)}_{=\xi_1^{\operatorname{wt}\lambda}} \underbrace{p_\lambda(\eta)}_{=\eta_1^{\operatorname{wt}\lambda}} T^{\operatorname{wt}\lambda} &= \sum_{\lambda \in \operatorname{Par}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} \xi_1^{\operatorname{wt}\lambda} \eta_1^{\operatorname{wt}\lambda} T^{\operatorname{wt}\lambda} \\
&= \sum_{\ell=0}^{\infty} \sum_{\substack{\lambda \in \operatorname{Par}; \\ \operatorname{wt}\lambda = \ell}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} \xi_1^{\ell} \eta_1^{\ell} T^{\ell}. \qquad (4)
\end{aligned}$$

Finally, $I = J = \{1\}$ yields $I \times J = \{1\} \times \{1\} = \{(1,1)\}$ and thus

$$\begin{aligned}
\prod_{(i,j) \in I \times J} \left(\frac{1}{1 - \xi_i \eta_j T}\right)^S &= \left(\frac{1}{1 - \xi_1 \eta_1 T}\right)^S = \left(1 - \xi_1 \eta_1 T\right)^{-S} \\
&= \sum_{\ell=0}^{\infty} \binom{-S}{\ell} \left(-\xi_1 \eta_1 T\right)^{\ell} \qquad \text{(by the binomial formula)} \\
&= \sum_{\ell=0}^{\infty} \binom{-S}{\ell} \left(-\xi_1 \eta_1\right)^{\ell} T^{\ell}.
\end{aligned}$$

Using this and using (4), we can rewrite the identity (3) as

$$\sum_{\ell=0}^{\infty} \sum_{\substack{\lambda \in \operatorname{Par}; \\ \operatorname{wt}\lambda = \ell}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} \xi_1^{\ell} \eta_1^{\ell} T^{\ell} = \sum_{\ell=0}^{\infty} \binom{-S}{\ell} \left(-\xi_1 \eta_1\right)^{\ell} T^{\ell}. \qquad (5)$$

This is an identity in the ring

$$\left(\left(\left(\mathbb{Q}\left[\xi_i \mid i \in I\right]_\infty\right)\left[\eta_j \mid j \in J\right]_\infty\right)[[T]]\right)[[S]] = (((\mathbb{Q}[\xi_1])[\eta_1])[[T]])[[S]],$$

but it can also be considered an identity in the subring $(((\mathbb{Q}[\xi_1])[\eta_1])[S])[[T]]$ (since both sides of the identity (5) lie in this subring), i. e. as an identity between two power series in the indeterminate T over the ring $((\mathbb{Q}[\xi_1])[\eta_1])[S]$. Hence, comparing coefficients before T^n in this identity, we obtain

$$\sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} S^{\operatorname{msum}\lambda} \xi_1^n \eta_1^n = \binom{-S}{n} (-\xi_1\eta_1)^n .$$

This is an identity in the polynomial ring $((\mathbb{Q}[\xi_1])[\eta_1])[S] \cong \mathbb{Q}[\xi_1, \eta_1, S]$. Evaluating both sides at $\xi_1 = 1$, $\eta_1 = 1$ and $S = -x$, we obtain

$$\sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} (-x)^{\operatorname{msum}\lambda} 1^n 1^n = \binom{-(-x)}{n} (-1\cdot 1)^n .$$

This simplifies to

$$\sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} (-x)^{\operatorname{msum}\lambda} = \binom{x}{n} (-1)^n .$$

Multiplying this by $n!$ yields

$$n! \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} (-x)^{\operatorname{msum}\lambda} = n! \binom{x}{n} (-1)^n .$$

Since

$$\begin{aligned} n! \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} z_\lambda^{-1} (-x)^{\operatorname{msum}\lambda} &= \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} \underbrace{\frac{n!}{z_\lambda}}_{\substack{=\left|\{\sigma\in S_n \ \mid \ \operatorname{cyc}\sigma=\lambda\}\right| \\ \text{(by Theorem 2)}}} (-x)^{\operatorname{msum}\lambda} \\ &= \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} \underbrace{\left|\{\sigma \in S_n \ \mid \ \operatorname{cyc}\sigma = \lambda\}\right| (-x)^{\operatorname{msum}\lambda}}_{= \sum_{\substack{\sigma\in S_n;\\ \operatorname{cyc}\sigma=\lambda}} (-x)^{\operatorname{msum}\lambda} = \sum_{\substack{\sigma\in S_n;\\ \operatorname{cyc}\sigma=\lambda}} (-x)^{\operatorname{msum}(\operatorname{cyc}\sigma)}} \\ &= \sum_{\substack{\lambda\in\operatorname{Par};\\ \operatorname{wt}\lambda=n}} \sum_{\substack{\sigma\in S_n;\\ \operatorname{cyc}\sigma=\lambda}} (-x)^{\operatorname{msum}(\operatorname{cyc}\sigma)} = \sum_{\sigma\in S_n} (-x)^{\operatorname{msum}(\operatorname{cyc}\sigma)} \end{aligned}$$

(because for every $\sigma \in S_n$, there exists one and only one $\lambda \in \operatorname{Par}$ such that $\operatorname{wt}\lambda = n$ and $\operatorname{cyc}\sigma = \lambda$ (because $\operatorname{wt}(\operatorname{cyc}\sigma) = n$)), this rewrites as

$$\sum_{\sigma\in S_n} (-x)^{\operatorname{msum}(\operatorname{cyc}\sigma)} = n! \binom{x}{n} (-1)^n .$$

Now, every permutation $\sigma \in S_n$ satisfies

$$\operatorname{msum}(\operatorname{cyc}\sigma) = \sum_{k=1}^{\infty} m_k(\operatorname{cyc}\sigma) = \sum_{i=1}^{\infty} \underbrace{m_i(\operatorname{cyc}\sigma)}_{=\operatorname{cycle}_i \sigma} \qquad \text{(here, we substituted } i \text{ for } k \text{ in the sum)}$$
$$= \sum_{i=1}^{\infty} \operatorname{cycle}_i \sigma = \operatorname{cycle}\sigma,$$

and thus this becomes

$$\sum_{\sigma \in S_n} (-x)^{\operatorname{cycle}\sigma} = n! \binom{x}{n} (-1)^n . \tag{6}$$

Now,

$$\text{(the number of all even cycles in the cycle decomposition of the permutation } \sigma\text{)}$$
$$= \sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is even}}} \text{(the number of all cycles of length } i \text{ in the cycle decomposition of the permutation } \sigma\text{)}$$
$$= \sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is even}}} \operatorname{cycle}_i \sigma = \underbrace{\sum_{i \in \{1,2,3,\ldots\}} \operatorname{cycle}_i \sigma}_{= \sum_{i=1}^{\infty} \operatorname{cycle}_i \sigma = \operatorname{cycle}\sigma} - \sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is odd}}} \operatorname{cycle}_i \sigma = \operatorname{cycle}\sigma - \sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is odd}}} \operatorname{cycle}_i \sigma,$$

which, in view of

$$n = \sum_{k=1}^{\infty} k \operatorname{cycle}_k \sigma = \sum_{i=1}^{\infty} i \operatorname{cycle}_i \sigma \qquad \text{(here, we substituted } i \text{ for } k \text{ in the sum)}$$
$$= \sum_{i \in \{1,2,3,\ldots\}} i \operatorname{cycle}_i \sigma = \sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is even}}} \underbrace{i}_{\substack{\equiv 0 \bmod 2\\ \text{(since } i \text{ is even)}}} \operatorname{cycle}_i \sigma + \sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is odd}}} \underbrace{i}_{\substack{\equiv 1 \bmod 2\\ \text{(since } i \text{ is odd)}}} \operatorname{cycle}_i \sigma$$
$$\equiv \underbrace{\sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is even}}} 0 \operatorname{cycle}_i \sigma}_{=0} + \sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is odd}}} 1 \operatorname{cycle}_i \sigma = \sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is odd}}} 1 \operatorname{cycle}_i \sigma$$
$$= \sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is odd}}} \operatorname{cycle}_i \sigma \bmod 2,$$

becomes

$$\text{(the number of all even cycles in the cycle decomposition of the permutation } \sigma\text{)}$$
$$= \operatorname{cycle}\sigma - \underbrace{\sum_{\substack{i \in \{1,2,3,\ldots\};\\ i \text{ is odd}}} \operatorname{cycle}_i \sigma}_{\equiv n \bmod 2} \equiv \operatorname{cycle}\sigma - n \bmod 2,$$

so that

$$(-1)^{\text{(the number of all even cycles in the cycle decomposition of the permutation } \sigma\text{)}} = (-1)^{\operatorname{cycle}\sigma - n} .$$

Hence, the signum $\operatorname{sign}\sigma$ of the permutation σ satisfies

$$\operatorname{sign}\sigma = (-1)^{(\text{the number of all even cycles in the cycle decomposition of the permutation } \sigma)} = (-1)^{\operatorname{cycle}\sigma - n}.$$

Thus,

$$\begin{aligned}\sum_{\sigma\in S_n} \operatorname{sign}\sigma \cdot x^{\operatorname{cycle}\sigma} &= \sum_{\sigma\in S_n} (-1)^{\operatorname{cycle}\sigma - n} \cdot x^{\operatorname{cycle}\sigma} = (-1)^{-n} \sum_{\sigma\in S_n} \underbrace{(-1)^{\operatorname{cycle}\sigma} \cdot x^{\operatorname{cycle}\sigma}}_{=(-x)^{\operatorname{cycle}\sigma}} \\ &= (-1)^{-n} \sum_{\sigma\in S_n} (-x)^{\operatorname{cycle}\sigma} \\ &= (-1)^{-n}\, n! \binom{x}{n} (-1)^n \qquad \text{(by (6))} \\ &= n! \binom{x}{n}.\end{aligned}$$

This proves Theorem 1.

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